Partial Classification of the Baumslag-Solitar Group Von Neumann Algebras

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Abstract. We prove that the rational number $|n/m|$ is an invariant of the group von Neumann algebra of the Baumslag-Solitar group $\mathrm{BS}(n,m)$. More precisely, if $L(\mathrm{BS}(n,m))$ is isomorphic with $L(\mathrm{BS}(n',m'))$, then $|n'/m'| = |n/m|^{\pm 1}$. We obtain this result by associating to abelian, but not maximal abelian, subalgebras of a II_1 factor, an equivalence relation that can be of type III. In particular, we associate to $L(\mathrm{BS}(n,m))$ a canonical equivalence relation of type $\mathrm{III}_{|n/m|}$.

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1. Introduction and statement of the main result

Some of the deepest open problems in functional analysis center around the classification of group von Neumann algebras $L(G)$ associated with certain natural families of countable groups G. In the case of the free groups, this becomes the famous free group factor problem asking whether $L(\mathbb{F}_n) \cong L(\mathbb{F}_m)$ when $n, m \geq 2$ and $n \neq m$. For property (T) groups with infinite conjugacy classes (icc), this leads to Connes's rigidity conjecture ([Co80]) asserting that an isomorphism $L(G) \cong L(\Lambda)$ between the property (T) factors entails an isomorphism $G \cong \Lambda$ of the groups.

As a consequence of Connes's uniqueness theorem of injective II_1 factors ([Co75]), the group von Neumann algebra $L(G)$ of an amenable icc group G is

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isomorphic with the unique hyperfinite II_1 factor R. In the nonamenable case, many nonisomorphic groups G are known to have nonisomorphic group von Neumann algebras $L(G)$. Nevertheless, concerning the classification of group von Neumann algebras of natural families of groups, e.g. lattices in simple Lie groups, little is known. A notable exception however is [CH88] where it is shown that for $n \neq m$, lattices in $\mathrm{Sp}(n,1)$, respectively $\mathrm{Sp}(m,1)$, have nonisomorphic group von Neumann algebras.

Since 2001, Popa has been developing a new arsenal of techniques to study II_1 factors, called deformation/rigidity theory. This theory has provided several classes $\mathcal{G}$ of groups such that an isomorphism $L(G) \cong L(\Lambda)$ with both $G, \Lambda \in \mathcal{G}$ entails the isomorphism $G \cong \Lambda$. By [Po04], this holds in particular when $\mathcal{G}$ is the class of wreath product groups of the form $(\mathbb{Z}/2\mathbb{Z}) \wr \Gamma$ with Γ an icc property (T) group.

In [IPV10], the first W*-superrigidity theorems for group von Neumann algebras were discovered, yielding icc groups G such that an isomorphism $L(G) \cong L(\Lambda)$ with Λ an *arbitrary* countable group, implies that $G \cong \Lambda$. The groups G discovered in [IPV10] are generalized wreath products of a special form. In [BV12], it was then shown that one can actually take $G = (\mathbb{Z}/2\mathbb{Z})^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$ with Γ ranging over a large family of nonamenable groups including the free groups $\mathbb{F}_n$, $n \geq 2$.

In this article, we apply Popa's deformation/rigidity theory to partially classify the group von Neumann algebras of the Baumslag-Solitar groups $\mathrm{BS}(n,m)$. Recall that for all $n, m \in \mathbb{Z} - \{0\}$, this group is defined as the group generated by a and b subject to the relation $ba^n b^{-1} = a^m$. So,

$$\mathrm{BS}(n,m) := \langle a, b \mid ba^n b^{-1} = a^m \rangle \ .$$

The Baumslag-Solitar groups were introduced in [BS62] as the first examples of finitely presented non-Hopfian groups. Ever since, they have been used as examples and counterexamples for numerous group theoretic phenomena. Therefore, it is a natural problem to classify the group von Neumann algebras $L(\mathrm{BS}(n,m))$.

Whenever $|n| = 1$ or $|m| = 1$, the group $\mathrm{BS}(n,m)$ is solvable, hence amenable. So we always assume that $|n| \geq 2$ and $|m| \geq 2$. In that case, $\mathrm{BS}(n,m)$ contains a copy of the free group $\mathbb{F}_2$ and hence, is nonamenable. In [Mo91], the Baumslag-Solitar groups were classified up to isomorphism: $\mathrm{BS}(n,m) \cong \mathrm{BS}(n',m')$ if and only if $\{n,m\} = \{\varepsilon n', \varepsilon m'\}$ for some $\varepsilon \in \{-1,1\}$. So, up to isomorphism, we only consider $2 \leq n \leq |m|$. Finally by [St05, Exemple 2.4], the group $\mathrm{BS}(n,m)$ is icc if and only if $|n| \neq |m|$. Therefore, we always assume that $2 \leq n < |m|$.

Using Popa's deformation/rigidity theory and in particular his spectral gap rigidity ([Po06]) and the work on amalgamated free products ([IPP05]), several structural properties of the II_1 factors $M = L(\mathrm{BS}(n,m))$ were proven. In particular, it was shown in [Fi10] that M is not solid, that M is prime and that M has no Cartan subalgebra. More generally, it is proven in [Fi10] that any amenable regular von Neumann subalgebra of M must have a nonamenable relative commutant.

Our main result is the following partial classification theorem for the Baumslag-Solitar group von Neumann algebras $L(\mathrm{BS}(n,m))$. Whenever M is a II_1 factor and $t > 0$, we denote by M^t the *amplification* of M. Up to unitary conjugacy, M^t is defined as $p(\mathrm{M}_n(\mathbb{C}) \otimes M)p$ where p is a projection satisfying $(\mathrm{Tr} \otimes \tau)(p) = t$. The II_1 factors M and N are called *stably isomorphic* if there exists a $t > 0$ such that $M \cong N^t$.

Theorem A. *Let $n, m, n', m' \in \mathbb{Z}$ such that $2 \leq n < |m|$ and $2 \leq n' < |m'|$. If $L(\mathrm{BS}(n,m))$ is stably isomorphic with $L(\mathrm{BS}(n',m'))$, then $\frac{n}{|m|} = \frac{n'}{|m'|}$.*

Note that Theorem A formally resembles, but is independent of, the results in [Ki11] on orbit equivalence relations of essentially free ergodic probability measure preserving actions of Baumslag-Solitar groups, especially [Ki11, Proposition B.2 and Theorem 1.2]. It would be very interesting to find a framework that unifies both types of results.

We prove our Theorem A by associating a canonical equivalence relation to $L(\mathrm{BS}(n,m))$ and proving that it is of type $\mathrm{III}_{n/|m|}$. More precisely, assume that (M, τ) is a von Neumann algebra with separable predual, equipped with a faithful normal tracial state. Whenever $A \subset M$ is an abelian von Neumann subalgebra, the normalizer

$$\mathcal{N}_M(A) := \{u \in \mathcal{U}(M) \mid uAu^* = A\}$$

induces a group of trace preserving automorphisms of A. Writing $A = L^\infty(X, \mu)$ with μ being induced by $\tau_{|A}$, the corresponding orbit equivalence relation is a countable probability measure preserving (pmp) equivalence relation on (X, μ).

More generally, we can consider the set of partial isometries

$$\{u \in M \mid u^*u \text{ and } uu^* \text{ are projections in } A' \cap M \text{ and } uAu^* = Auu^*\} . \quad (1)$$

Every such partial isometry induces a partial automorphism of A and hence a partial automorphism of (X, μ). We denote by $\mathcal{R}(A \subset M)$ the equivalence relation generated by all these partial automorphisms. When $A \subset M$ is maximal abelian, i.e. $A' \cap M = A$, then $\mathcal{R}(A \subset M)$ coincides with the orbit equivalence relation induced by the normalizer $\mathcal{N}_M(A)$. In particular, in that case the equivalence relation $\mathcal{R}(A \subset M)$ preserves the probability measure μ.

If however $A \subset M$ is not maximal abelian, the partial automorphisms of A induced by the partial isometries in the set (1) need not be trace preserving. So in general, $\mathcal{R}(A \subset M)$ can be an equivalence relation of type III.

Our main technical result is Theorem 3.3 below, roughly saying the following. If $A, B \subset M$ are abelian subalgebras such that $\mathcal{Z}(A' \cap M) = A$ and $\mathcal{Z}(B' \cap M) = B$, and if there exist intertwining bimodules $A \prec B$ and $B \prec A$ (in the sense of Popa, see [Po03] and Theorem 2.3 below), then the equivalence relations $\mathcal{R}(A \subset M)$ and $\mathcal{R}(B \subset M)$ must be stably isomorphic. In particular, their types must be the same.

In Section 4, we apply this to $M = L(\mathrm{BS}(n,m))$ and A equal to the abelian von Neumann subalgebra generated by the unitary u_a. We prove that $\mathcal{R}(A \subset M)$ is the unique hyperfinite ergodic equivalence relation of type $\mathrm{III}_{n/|m|}$.
The proof of Theorem A can then be outlined as follows. First we note that the von Neumann algebra $A' \cap M$ is nonamenable. Conversely if $Q \subset M$ is a nonamenable subalgebra, it was proven in [CH08], using spectral gap rigidity ([Po06]) and the structure theory of amalgamated free product factors ([IPP05]), that $Q' \cap M \prec A$. So, up to intertwining-by-bimodules, the position of A inside M is "canonical". Therefore a stable isomorphism between $L(\mathrm{BS}(n,m))$ and $L(\mathrm{BS}(n',m'))$ will preserve, up to intertwining-by-bimodules, these canonical abelian subalgebras. Hence their associated equivalence relations are stably isomorphic and, in particular, have the same type. This gives us the equality $n/|m| = n'/|m'|$.

2. Preliminaries

We denote by (M,τ) a von Neumann algebra equipped with a faithful normal tracial state τ. We always assume that M has a *separable predual*. If B is a von Neumann subalgebra of (M,τ), we denote by E_B the unique trace preserving *conditional expectation* of M onto B.
Whenever $x \in M$ is a normal element, we denote by $\mathrm{supp}(x)$ its support, i.e. the smallest projection $p \in M$ that satisfies $xp = x$ (or equivalently, $px = x$).
Let $\mathcal{R}$ be a countable nonsingular (i.e. measure class preserving) equivalence relation on a standard probability space (X,μ). We denote by $[[\mathcal{R}]]$ the *full pseudogroup* of $\mathcal{R}$, i.e. the pseudogroup of all partial nonsingular automorphisms φ of X such that the graph of φ is contained in $\mathcal{R}$. We denote the domain of φ by $\mathrm{dom}(\varphi)$ and its range by $\mathrm{ran}(\varphi)$. We denote by $[x]$ the equivalence class of $x \in X$.
Assume that also $\mathcal{R}'$ is a countable nonsingular equivalence relation on the standard probability space (X',μ'). The equivalence relations $\mathcal{R}$ and $\mathcal{R}'$ are called

- *isomorphic*, if there exists a nonsingular isomorphism $\Delta : X \to X'$ such that $\Delta([x]) = [\Delta(x)]$ for almost every $x \in X$;
- *stably isomorphic*, if there exist Borel subsets $Z \subset X$ and $Z' \subset X'$ that meet almost every orbit and a nonsingular isomorphism $\Delta : Z \to Z'$ such that $\Delta([x] \cap Z) = [\Delta(x)] \cap Z'$ for almost every $x \in Z$.

2.1. HNN extensions and Baumslag-Solitar groups

Let G be a group, $H < G$ a subgroup and $\theta : H \to G$ an injective group homomorphism. The *HNN extension* $\mathrm{HNN}(G,H,\theta)$ is defined as the group generated by G and an additional element t subject to the relation $\theta(h) = tht^{-1}$ for all $h \in H$. So,

$$\mathrm{HNN}(G,H,\theta) = \langle G, t \mid \theta(h) = tht^{-1} \text{ for all } h \in H \rangle \, .$$

Elements of $\mathrm{HNN}(G,H,\theta)$ can be canonically written as "reduced words" using as letters the elements of G and the letters $t^{\pm 1}$. More precisely, we have the following lemma.

Lemma 2.1 (Britton's lemma, [Br63]). *Consider the expression $g = g_0 t^{n_1} g_1 t^{n_2} \cdots t^{n_k} g_k$ with $k \geq 0$, $g_0, g_k \in G$, $g_1, \dots, g_{k-1} \in G - \{e\}$ and $n_1, \dots, n_k \in \mathbb{Z} - \{0\}$. We call this expression reduced if the following two conditions hold:*

- *for every $i \in \{1, \dots, k-1\}$ with $n_i > 0$ and $n_{i+1} < 0$, we have $g_i \notin H$,*
- *for every $i \in \{1, \dots, k-1\}$ with $n_i < 0$ and $n_{i+1} > 0$, we have $g_i \notin \theta(H)$.*

If the above expression for g is reduced, then $g \neq e$ in the group $\mathrm{HNN}(G,H,\theta)$, unless $k = 0$ and $g_0 = e$. In particular, the natural homomorphism of G to $\mathrm{HNN}(G,H,\theta)$ is injective.

Recall from the introduction that the Baumslag-Solitar group $\mathrm{BS}(n,m)$ is defined for all $n, m \in \mathbb{Z} - \{0\}$ as

$$\mathrm{BS}(n,m) := \langle a, b \mid b a^n b^{-1} = a^m \rangle .$$

It is one of the easiest examples of an HNN extension. We also recall from the introduction that the $\mathrm{BS}(n,m)$ with $2 \leq n < |m|$ form a complete list of all nonamenable icc Baumslag-Solitar groups up to isomorphism. Since we only want to consider the case where $L(\mathrm{BS}(n,m))$ is a nonamenable II_1 factor, we always assume that $2 \leq n < |m|$.

2.2. Hilbert bimodules and intertwining-by-bimodules

If M and N are tracial von Neumann algebras, then a *left M-module* is a Hilbert space $\mathcal{H}$ endowed with a normal $*$-homomorphism $\pi : M \to \mathrm{B}(\mathcal{H})$. A *right N-module* is a left N^{op}-module. An *M-N-bimodule* is a Hilbert space $\mathcal{H}$ endowed with commuting normal $*$-homomorphisms $\pi : M \to \mathrm{B}(\mathcal{H})$ and $\varphi : N^{\mathrm{op}} \to \mathrm{B}(\mathcal{H})$. For $x \in M, y \in N$ and $\xi \in \mathcal{H}$, we write $x\xi y$ instead of $\pi(x)\varphi(y^{\mathrm{op}})(\xi)$. We denote an M-N-bimodule $\mathcal{H}$ by ${}_M\mathcal{H}_N$. We call an M-N-bimodule *bifinite* if it is finitely generated both as a left Hilbert M-module and a right Hilbert N-module.

Let A and B be abelian von Neumann algebras. We denote by $\mathrm{PIso}(A,B)$ the set of all partial isomorphisms from A to B, i.e. isomorphisms $\alpha : Aq \to Bp$, where $q \in A$ and $p \in B$ are projections. We write $\mathrm{PAut}(A)$ instead of $\mathrm{PIso}(A,A)$. Note that to every $\alpha \in \mathrm{PIso}(A,B)$ we can associate an A-B-bimodule ${}_A\mathcal{H}(\alpha)_B$ given by $\mathcal{H}(\alpha) = L^2(Bp)$ and $a\xi b = \alpha(aq)\,\xi\, bp$. The composition of two partial isomorphisms is defined as follows: if $\alpha \in \mathrm{PIso}(B,C)$ and $\beta \in \mathrm{PIso}(A,B)$ are given by $\alpha : Bp \to Cr$ and $\beta : Aq \to Bp'$ for projections $q \in A$, $p, p' \in B$ and $r \in C$, then the composition $\alpha \circ \beta \in \mathrm{PIso}(A,C)$ is defined by $x \mapsto \alpha(\beta(x))$ for all $x \in Aq\beta^{-1}(pp')$.

The following is a well known result.

LEMMA 2.2. *Let A and B be abelian von Neumann algebras. Then every bifinite A-B-bimodule ${}_A\mathcal{H}_B$ is isomorphic to a direct sum of bimodules of the form ${}_A\mathcal{H}(\alpha)_B$ with $\alpha \in \mathrm{PIso}(A,B)$.*

We finally recall Popa's *intertwining-by-bimodules* theorem.

THEOREM 2.3 ([Po03, Theorem 2.1 and Corollary 2.3]). *Let (M,τ) be a tracial von Neumann algebra and let $A, B \subset M$ be possibly nonunital von Neumann subalgebras. Denote their respective units by 1_A and 1_B. The following three conditions are equivalent:*

1. *$1_A L^2(M) 1_B$ admits a nonzero A-B-subbimodule that is finitely generated as a right B-module.*
2. *There exist nonzero projections $p \in A$, $q \in B$, a normal unital $*$-homomorphism $\psi : pAp \to qBq$ and a nonzero partial isometry $v \in pMq$ such that $av = v\psi(a)$ for all $a \in pAp$.*
3. *There is no sequence of unitaries $u_n \in \mathcal{U}(A)$ satisfying $\|E_B(x u_n y^*)\|_2 \to 0$ for all $x, y \in 1_B M 1_A$.*

If one of these equivalent conditions holds, we write $A \prec_M B$.

2.3. QUASI-REGULARITY

Let (M,τ) be a tracial von Neumann algebra and $N \subset M$ a von Neumann subalgebra. We denote by $\mathrm{QN}_M(N)$ the *quasi-normalizer* of N inside M, i.e. the unital $*$-algebra defined by

$$\Big\{ a \in M \;\Big|\; \exists b_1, \ldots, b_k \in M, \exists d_1, \ldots, d_r \in M \quad \text{such that } \; Na \subset \sum_{i=1}^{k} b_i N \; \text{ and } \; aN \subset \sum_{j=1}^{r} N d_j \Big\} \, .$$

We call $N \subset M$ *quasi-regular* if $\mathrm{QN}_M(N)'' = M$.
If $A, B \subset M$ are abelian von Neumann subalgebras, we define $\mathrm{Q}_M(A,B)$ as

$$\mathrm{Q}_M(A,B) := \{ v \in M \mid vv^* \in A' \cap M \, , \; v^*v \in B' \cap M \; \text{ and } \; Av = vB \} \, .$$

Whenever $v \in \mathrm{Q}_M(A,B)$, we define $q_v = \mathrm{supp}(E_A(vv^*))$ and $p_v = \mathrm{supp}(E_B(v^*v))$, and we denote by $\alpha_v : Aq_v \to Bp_v$ the unique $*$-isomorphism satisfying $av = v\alpha_v(a)$ for all $a \in Aq_v$.
Note that the set $\mathrm{Q}_M(A,B)$ can be $\{0\}$. In Lemma 2.4, we will see that $\mathrm{Q}_M(A,B) \neq \{0\}$ if and only if there exists a bifinite A-B-subbimodule ${}_A\mathcal{H}_B$ of ${}_A L^2(M)_B$.
We denote $\mathrm{Q}_M(A,A)$ by $\mathrm{Q}_M(A)$.

LEMMA 2.4. *Let (M,τ) be a tracial von Neumann algebra and $A, B \subset M$ abelian von Neumann subalgebras. Then the following statements hold.*

1. *If* $\alpha \in \mathrm{PIso}(A,B)$ *and if* $\theta : {}_A\mathcal{H}(\alpha)_B \to {}_AL^2(M)_B$ *is an* A-B*-bimodular isometry, then there exists a partial isometry* $v \in \mathrm{Q}_M(A,B)$ *such that* $\alpha = \alpha_v$ *and such that*

$$\theta(\mathcal{H}(\alpha)) \subset \overline{v(B' \cap M)}^{\|\cdot\|_2} \subset \overline{\mathrm{span}}^{\|\cdot\|_2}\, \mathrm{Q}_M(A,B)\ .$$

2. *Every bifinite* A-B*-subbimodule* ${}_A\mathcal{H}_B$ *of* ${}_AL^2(M)_B$ *is contained in* $\overline{\mathrm{span}}^{\|\cdot\|_2}\, \mathrm{Q}_M(A,B)$.

3. $\mathrm{Q}_M(A)'' = \mathrm{QN}_M(A)''$.

4. *We have* $\mathrm{Q}_M(A,B) \neq \{0\}$ *if and only if* ${}_AL^2(M)_B$ *admits a nonzero bifinite* A-B*-subbimodule.*

Proof. 1. Let $\alpha : Aq \to Bp$ be an element of $\mathrm{PIso}(A,B)$. Define $\xi := \theta(p) \in L^2(M)$ and let $\xi = v|\xi|$ be its polar decomposition. For all $a \in A$, we have $a\xi = \xi\alpha(a)$ and hence, $av = v\alpha(a)$. Furthermore $p = \mathrm{supp}(E_B(v^*v))$ and $q = \alpha^{-1}(p) = \mathrm{supp}(E_A(vv^*))$. So we find that $v \in \mathrm{Q}_M(A,B)$ and $\alpha = \alpha_v$. Because $|\xi| \in L^2(B' \cap M)$, we have that $\xi = v|\xi|$ is an element of $\overline{v(B' \cap M)}^{\|\cdot\|_2}$. Since p generates ${}_A\mathcal{H}(\alpha)_B$ as a right Hilbert B-module, we have proven the first inclusion $\theta(\mathcal{H}(\alpha)) \subset \overline{v(B' \cap M)}^{\|\cdot\|_2}$. Since $v \in \mathrm{Q}_M(A,B)$, also $v(B' \cap M) \subset \mathrm{Q}_M(A,B)$ and the second inclusion in statement 1 is proven as well.
2. Let ${}_A\mathcal{H}_B$ be a bifinite A-B-subbimodule of ${}_AL^2(M)_B$. By Lemma 2.2, ${}_A\mathcal{H}_B$ is isomorphic to a direct sum of bimodules of the form ${}_A\mathcal{H}(\alpha_i)_B$ with $\alpha_i \in \mathrm{PIso}(A,B)$. Using statement 1 of the lemma, we find that $\mathcal{H}$ is generated by subspaces of $\overline{\mathrm{span}}^{\|\cdot\|_2}\, \mathrm{Q}_M(A,B)$. This proves statement 2.
3. By definition, we have $\mathrm{Q}_M(A)'' \subset \mathrm{QN}_M(A)''$. On the other hand, ${}_AL^2(\mathrm{QN}_M(A)'')_A$ is a direct sum of bifinite A-A-subbimodules of ${}_AL^2(M)_A$. So by statement 2, we have that $L^2(\mathrm{QN}_M(A)'') \subset \overline{\mathrm{span}}^{\|\cdot\|_2}(\mathrm{Q}_M(A))$. Therefore we conclude that $\mathrm{QN}_M(A)'' = \mathrm{Q}_M(A)''$.
Finally, 4 is an immediate consequence of 2. □

We end this subsection with the following lemma, clarifying why later, we will consider abelian subalgebras $A \subset M$ satisfying $\mathcal{Z}(A' \cap M) = A$. Note that since A is abelian, the condition $\mathcal{Z}(A' \cap M) = A$ is equivalent with the "bicommutant" property $(A' \cap M)' \cap M = A$. Also note that the composition of two partial isomorphisms was defined before Lemma 2.2.

Lemma 2.5. *Let* (M,τ) *be a tracial von Neumann algebra and* $A,B,C \subset M$ *abelian von Neumann subalgebras. If* $v \in \mathrm{Q}_M(A,B)$, $w \in \mathrm{Q}_M(B,C)$ *and if* $\mathcal{Z}(B' \cap M) = B$, *then there exists an element* $u \in \mathrm{Q}_M(A,C)$ *such that* $\alpha_w \circ \alpha_v = \alpha_u$.

Proof. Choose $v \in \mathrm{Q}_M(A,B)$ and $w \in \mathrm{Q}_M(B,C)$. Note that $vbw \in \mathrm{Q}_M(A,C)$ for every $b \in B' \cap M$ and $\alpha_{vbw} = \alpha_w \circ \alpha_{v|Aq_{vbw}}$. We claim that $\bigvee_{b \in B' \cap M} q_{vbw} = \alpha_v^{-1}(q_w p_v)$.

Denote $\alpha_v^{-1}(q_w p_v) - \bigvee_{b \in B' \cap M} q_{vbw}$ by r. We need to prove that r is zero. Since $(B' \cap M)' \cap M = B$, we have that for every $x \in M$, the projection $\operatorname{supp}(E_B(xx^*))$ equals the projection of $L^2(M)$ onto the closed linear span of $(B' \cap M)xM \subset L^2(M)$. Since $w^* b v^* r = 0$ for every $b \in B' \cap M$, it follows that $w^* q_{v^* r} = 0$. Therefore q_w is orthogonal to $q_{v^* r}$. Because $q_{v^* r} = \alpha_v(r)$ and $\alpha_v(r) \leq q_w$, it follows that $\alpha_v(r) = 0$. Hence $r = 0$ and our claim that $\bigvee_{b \in B' \cap M} q_{vbw} = \alpha_v^{-1}(q_w p_v)$ is proven.
By cutting down with appropriate projections, we find $b_n \in B' \cap M$ such that the projections $q_{vb_n w}$ are orthogonal and sum up to $\alpha_v^{-1}(q_w p_v)$. In particular, the left supports, resp. right supports, of the elements $vb_n w$ are orthogonal. So we can define $u = \sum_n vb_n w$. It follows that $u \in \mathrm{Q}_M(A, C)$ and $\alpha_w \circ \alpha_v = \alpha_u$. $\square$

2.4. The type of an ergodic nonsingular countable equivalence relation

Let $\mathcal{R}$ be a nonsingular ergodic countable Borel equivalence relation on a standard probability space (X, μ). Using the map $\pi : \mathcal{R} \to X : \pi(x, y) = x$, we define the measure $\mu^{(1)}$ on $\mathcal{R}$ given by

$$\mu^{(1)}(U) = \int_X \#(U \cap \pi^{-1}(x))\, d\mu(x) \quad \text{for all Borel sets } U \subset \mathcal{R} .$$

We define $\mathcal{R}^{(2)} := \{(x, y, z) \in X^3 \mid (x, y), (y, z) \in \mathcal{R}\}$. Similarly, using the map $\rho : \mathcal{R}^{(2)} \to X : \rho(x, y, z) = x$, we define the measure $\mu^{(2)}$ on $\mathcal{R}^{(2)}$ given by

$$\mu^{(2)}(V) = \int_X \#(V \cap \rho^{-1}(x))\, d\mu(x) \quad \text{for all Borel sets } V \subset \mathcal{R}^{(2)} .$$

The *Radon-Nikodym 1-cocycle* of $\mathcal{R}$ is the $\mu^{(1)}$-a.e. uniquely defined Borel map $\omega : \mathcal{R} \to \mathbb{R}$ such that

$$\omega(\varphi(x), x) = \log\Big(\frac{d\mu \circ \varphi}{d\mu}(x)\Big) \quad \text{for all } \varphi \in [[\mathcal{R}]] \text{ and almost every } x \in \operatorname{dom} \varphi .$$

Note that ω satisfies the 1-cocycle relation $\omega(x, z) = \omega(x, y) + \omega(y, z)$ for $\mu^{(2)}$-a.e. $(x, y, z) \in \mathcal{R}^{(2)}$. One then defines the *Maharam extension* $\widetilde{\mathcal{R}}$ of $\mathcal{R}$ as the equivalence relation on $(X \times \mathbb{R}, \mu \times \exp(-t)dt)$ defined by

$$(x, t) \sim (y, s) \quad \text{if and only if} \quad (x, y) \in \mathcal{R} \text{ and } t - s = \omega(x, y).$$

Note that $\mu \times \exp(-t)dt$ is an infinite invariant measure for $\widetilde{\mathcal{R}}$. Denote the von Neumann algebra of all $\widetilde{\mathcal{R}}$-invariant functions in $L^\infty(X \times \mathbb{R})$ by $L^\infty(X \times \mathbb{R})^{\widetilde{\mathcal{R}}}$. Since $\mathcal{R}$ was assumed to be ergodic, one can easily check that the action of $\mathbb{R}$ on $L^\infty(X \times \mathbb{R})^{\widetilde{\mathcal{R}}}$ given by translation of the second variable, is also ergodic. Depending on how this action of $\mathbb{R}$ looks like, we define as follows the type of $\mathcal{R}$.

- I or II, if the action is conjugate with $\mathbb{R} \curvearrowright \mathbb{R}$;

- III_λ ($0 < \lambda < 1$), if the action is conjugate with $\mathbb{R} \curvearrowright \mathbb{R}/\mathbb{Z}\log(\lambda)$;
- III_1, if the action is on one point ;
- III_0, if the action is properly ergodic, i.e. is ergodic and has orbits of measure zero.

Remark 2.6. Denote by $L(\mathcal{R})$ the von Neumann algebra associated with $\mathcal{R}$. Denote by φ the normal semifinite faithful state on $L(\mathcal{R})$ that is induced by μ. Finally denote by $(\sigma_t^\varphi)_{t\in\mathbb{R}}$ its modular automorphism group. There is a canonical identification $L(\widetilde{\mathcal{R}}) \cong L(\mathcal{R}) \rtimes_{\sigma^\varphi} \mathbb{R}$. Under this identification, the dual action of $\mathbb{R}$ on $L(\mathcal{R}) \rtimes_{\sigma^\varphi} \mathbb{R}$ corresponds to the action of $\mathbb{R}$ on $L(\widetilde{\mathcal{R}})$ that we defined above. Also, the center of $L(\mathcal{R}) \rtimes_{\sigma^\varphi} \mathbb{R}$ corresponds to $L^\infty(X\times\mathbb{R})^{\widetilde{\mathcal{R}}}$. Altogether it follows that the type of the equivalence relation $\mathcal{R}$ coincides with the type of the factor $L(\mathcal{R})$.

Lemma 2.7. *Let $\mathcal{R}$ be a nonsingular ergodic countable Borel equivalence relation on the standard probability space (X,μ). Denote by ω its Radon-Nikodym 1-cocycle. If the essential image* $\mathrm{Im}(\omega)$ *of ω equals* $\log(\lambda)\mathbb{Z}$ *for some* $0 < \lambda < 1$ *and if the kernel* $\mathrm{Ker}(\omega)$ *of ω is an ergodic equivalence relation, then $\mathcal{R}$ is of type III_λ.*

Proof. Since $\mathrm{Ker}(\omega)$ is an ergodic equivalence relation on (X,μ), we have

$$L^\infty(X\times\mathbb{R})^{\widetilde{\mathcal{R}}} \subset L^\infty(X\times\mathbb{R})^{\mathrm{Ker}(\omega)} = 1 \otimes L^\infty(\mathbb{R}) \ .$$

For a given $F \in L^\infty(\mathbb{R})$, we have that $1\otimes F$ is $\widetilde{\mathcal{R}}$-invariant if and only if F is invariant under translation by the essential image of ω. So,

$$L^\infty(X\times\mathbb{R})^{\widetilde{\mathcal{R}}} = 1 \otimes L^\infty(\mathbb{R}/\log(\lambda)\mathbb{Z}) \ .$$

□

3. Equivalence relations associated to subalgebras that are abelian, but not maximal abelian

Throughout this section, we fix a tracial von Neumann algebra (M,τ) with separable predual. We also fix an abelian von Neumann subalgebra $A \subset M$ satisfying $\mathcal{Z}(A' \cap M) = A$. Choose a standard probability space (X,μ) such that $A = L^\infty(X,\mu)$. For every nonsingular partial automorphism φ of (X,μ), we denote by α_φ the corresponding partial automorphism of A.
We first prove that $\mathrm{Q}_M(A)$ induces a nonsingular countable Borel equivalence relation $\mathcal{R}(A\subset M)$ on (X,μ). For this, we introduce the notation

$$\mathcal{G}(A\subset M) := \{\alpha_v \mid v \in \mathrm{Q}_M(A)\} \ . \tag{2}$$

Proposition 3.1. *There exists a nonsingular countable Borel equivalence relation $\mathcal{R}$ on (X,μ) with the following property: a nonsingular partial automorphism φ of X satisfies $\alpha_\varphi \in \mathcal{G}(A\subset M)$ if and only if $(x,\varphi(x)) \in \mathcal{R}$ for a.e. $x \in \mathrm{dom}(\varphi)$.*

Moreover, $\mathcal{R}$ is essentially unique: if a nonsingular countable Borel equivalence relation $\mathcal{R}'$ on (X,μ) satisfies the same property, then there exists a Borel subset $X_0 \subset X$ with $\mu(X - X_0) = 0$ and $\mathcal{R}_{|X_0} = \mathcal{R}'_{|X_0}$.
We denote $\mathcal{R}(A \subset M) := \mathcal{R}$. The equivalence relation $\mathcal{R}(A \subset M)$ is ergodic if and only if $\mathrm{QN}_M(A)''$ is a factor.

Before proving Proposition 3.1, we introduce some terminology and a lemma. To every $\alpha \in \mathrm{PAut}(A)$ are associated the support projections $q_\alpha, p_\alpha \in A$ such that $\alpha : Aq_\alpha \to Ap_\alpha$ is a $*$-isomorphism. Assume that $\alpha \in \mathrm{PAut}(A)$ and $\mathcal{F} \subset \mathrm{PAut}(A)$. We say that α is a *gluing* of elements in $\mathcal{F}$, if there exists a sequence of elements $\alpha_n \in \mathcal{F}$ and projections $q_n \in A$ such that $q_\alpha = \sum_n q_n$ and such that $q_n \leq q_{\alpha_n}$ and $\alpha_{|Aq_n} = \alpha_{n|Aq_n}$ for all n.

Lemma 3.2. *Let $\mathcal{J} \subset \mathrm{Q}_M(A)$ and $v \in \mathrm{Q}_M(A)$ such that $v \in \overline{\mathrm{span}}^{\|\cdot\|_2}\mathcal{J}$. Then α_v is a gluing of elements in $\{\alpha_w \mid w \in \mathcal{J}\}$.*

Proof. By a standard maximality argument, it suffices to prove that for every nonzero projection $q \in Aq_v$, there exists a nonzero subprojection $q_0 \in Aq$ and a $w \in \mathcal{J}$ such that $q_0 \leq q_w$ and $\alpha_{v|Aq_0} = \alpha_{w|Aq_0}$.
So fix a nonzero projection $q \in Aq_v$. It follows that $qE_A(vv^*) \neq 0$. Since $v \in \overline{\mathrm{span}}^{\|\cdot\|_2}\mathcal{J}$, we can pick a $w \in \mathcal{J}$ such that $qE_A(vw^*) \neq 0$. Define $q_1 := \mathrm{supp}(E_A(vw^*))$ and note that $q_1 \in q_vAq_w = Aq_vq_w$. Also note that $qq_1 \neq 0$. For all $a \in A$, we have

$$\alpha_v^{-1}(ap_v)\, vw^* = v\, a\, w^* = vw^*\, \alpha_w^{-1}(ap_w)\ .$$

Applying the conditional expectation onto A and using that A is abelian, we find that

$$\alpha_v^{-1}(ap_v)\, q_1 = \alpha_w^{-1}(ap_w)\, q_1 \quad \text{for all } \ a \in A\ .$$

This means that $\alpha_{v|Aq_1} = \alpha_{w|Aq_1}$. We put $q_0 := qq_1$. We already showed that $q_0 \neq 0$. Since $q_0 \leq q_1$, we have that $\alpha_{v|Aq_0} = \alpha_{w|Aq_0}$. □

Proof of Proposition 3.1. We say that a subpseudogroup $\mathcal{G} \subset \mathrm{PAut}(A)$ is *of countable type* if there exists a countable subset $\mathcal{J} \subset \mathcal{G}$ such that every $\alpha \in \mathcal{G}$ is a gluing of elements in $\mathcal{J}$. To prove the first part of the proposition, we must show that $\mathcal{G}(A \subset M)$ is a subpseudogroup of countable type of $\mathrm{PAut}(A)$. From Lemma 2.5, it follows that $\mathcal{G}(A \subset M)$ is indeed a subpseudogroup. Since M has a separable predual, we can choose a countable $\|\cdot\|_2$-dense subset $\mathcal{J} \subset \mathrm{Q}_M(A)$. By Lemma 3.2, every $\alpha \in \mathcal{G}(A \subset M)$ is a gluing of elements in $\{\alpha_w \mid w \in \mathcal{J}\}$. Hence $\mathcal{G}(A \subset M)$ is of countable type. So the first part of the proposition is proven and we can essentially uniquely define the nonsingular countable Borel equivalence relation $\mathcal{R}$ on (X,μ).
Since $A' \cap M \subset \mathrm{QN}_M(A)''$ and since we assumed that $(A' \cap M)' \cap M = A$, the center of $\mathrm{QN}_M(A)''$ is a subalgebra of A. By Lemma 2.4.3, we have $\mathrm{QN}_M(A)'' = \mathrm{Q}_M(A)''$. Therefore,

$$\mathcal{Z}(\mathrm{QN}_M(A)'') = \{a \in A \mid av = va \ \text{ for all } \ v \in \mathrm{Q}_M(A)\}\ .$$

The right hand side equals $A^{\mathcal{R}}$, the subalgebra of $\mathcal{R}$-invariant functions in A. So $\mathcal{R}$ is ergodic if and only if $\mathrm{QN}_M(A)''$ is a factor. $\square$

For our application, the following theorem is crucial. It says that $\mathcal{R}(A \subset M)$ remains the same, up to stable isomorphism, if we replace A by an abelian subalgebra B that has a mutual intertwining bimodule into A.

Theorem 3.3. *Let M be a II_1 factor with separable predual. Let $A, B \subset M$ be abelian, quasi-regular von Neumann subalgebras satisfying $\mathcal{Z}(A' \cap M) = A$ and $\mathcal{Z}(B' \cap M) = B$. If $A \prec_M B$ and $B \prec_M A$, then the equivalence relations $\mathcal{R}(A \subset M)$ and $\mathcal{R}(B \subset M)$ are stably isomorphic.*

Proof. Since A, B are quasi-regular and since $A \prec_M B$ as well as $B \prec_M A$, there exists a nonzero bifinite A-B-subbimodule of $L^2(M)$. So by Lemma 2.4.4, there exists a nonzero element $v \in \mathrm{Q}_M(A, B)$ with corresponding $\alpha_v \in \mathrm{PAut}(A, B)$. Using the notation in (2) and using Lemma 2.5, we find that

$$\begin{aligned} &\alpha_v \circ \beta \circ \alpha_v^{-1} \in \mathcal{G}(B \subset M) \quad \text{for all } \beta \in \mathcal{G}(A \subset M) \quad \text{and} \\ &\alpha_v^{-1} \circ \gamma \circ \alpha_v \in \mathcal{G}(A \subset M) \quad \text{for all } \gamma \in \mathcal{G}(B \subset M)\ . \end{aligned}$$

So α_v implements a stable isomorphism between $\mathcal{R}(A \subset M)$ and $\mathcal{R}(B \subset M)$. $\square$

The following lemma will allow us to easily compute $\mathcal{R}(A \subset M)$ in concrete examples.

Lemma 3.4. *Let (M, τ) be a tracial von Neumann algebra and $A \subset M$ an abelian von Neumann subalgebra satisfying $\mathcal{Z}(A' \cap M) = A$. Let $\mathcal{F} \subset M$ be a subset such that*

- *$M = (\mathcal{F} \cup \mathcal{F}^* \cup (A' \cap M))''$,*
- *as an A-A-bimodule, $\overline{\mathrm{span}}^{\|\cdot\|_2} A\mathcal{F}A$ is isomorphic to a direct sum of bimodules of the form ${}_A\mathcal{H}(\alpha_n)_A$ with $\alpha_n \in \mathrm{PAut}(A)$.*

Choose nonsingular partial automorphisms φ_n of (X, μ) such that $\alpha_n = \alpha_{\varphi_n}$ for all n. Up to measure zero, $\mathcal{R}(A \subset M)$ is generated by the graphs of the partial automorphisms φ_n.

Proof. We again use the notation (2). By Lemma 2.4.1, we find $v_n \in \mathrm{Q}_M(A)$ such that $\alpha_n = \alpha_{v_n}$ and

$$\overline{\mathrm{span}}^{\|\cdot\|_2} A\mathcal{F}A \subset \overline{\mathrm{span}}^{\|\cdot\|_2} \{v_n(A' \cap M) \mid n \in \mathbb{N}\}\ . \tag{3}$$

In particular, we have $\alpha_n \in \mathcal{G}(A \subset M)$. Choose nonsingular partial automorphisms φ_n of (X, μ) such that $\alpha_n = \alpha_{\varphi_n}$ for all n.
Denote by $\mathcal{R}$ the smallest (up to measure zero) equivalence relation on (X, μ) that contains the graphs of all the partial automorphisms φ_n. By the previous

paragraph, we know that $\mathcal{R}$ is a subequivalence relation of $\mathcal{R}(A \subset M)$. Denote by $\mathcal{J}$ the set of all products of elements in

$$\{v_n \mid n \in \mathbb{N}\} \cup \{v_n^* \mid n \in \mathbb{N}\} \cup (A' \cap M) .$$

By construction, the graph of every α_w, $w \in \mathcal{J}$, belongs to $\mathcal{R}$. Combining our assumption that $M = (\mathcal{F} \cup \mathcal{F}^* \cup (A' \cap M))''$ with (3), it follows that $\operatorname{span} \mathcal{J}$ is $\|\cdot\|_2$-dense in $L^2(M)$. By Lemma 3.2, every $\alpha \in \mathcal{G}(A \subset M)$ is a gluing of elements in $\{\alpha_w \mid w \in \mathcal{J}\}$. So the graph of every $\alpha \in \mathcal{G}(A \subset M)$ belongs to $\mathcal{R}$ a.e. Hence $\mathcal{R}$ equals $\mathcal{R}(A \subset M)$ almost everywhere. □

We finally note in the following proposition that every nonsingular countable Borel equivalence relation $\mathcal{R}$ arises as $\mathcal{R}(A \subset M)$.

Proposition 3.5. *Let $\mathcal{R}$ be a nonsingular countable Borel equivalence relation. Then there exists a quasi-regular inclusion of an abelian von Neumann algebra A in a tracial von Neumann algebra (M, τ) satisfying $\mathcal{Z}(A' \cap M) = A$ and such that $\mathcal{R} \cong \mathcal{R}(A \subset M)$.*

Proof. Let $\mathcal{R}$ be a nonsingular countable Borel equivalence relation on a standard probability space (X, μ). Denote by (P, Tr) the unique hyperfinite II_∞ factor and choose a trace-scaling action $(\alpha_t)_{t \in \mathbb{R}}$ of $\mathbb{R}$ on P. This means that $\mathrm{Tr} \circ \alpha_t = e^{-t} \mathrm{Tr}$. The corresponding action of $\mathbb{R}$ on $L^2(P)$ will also be denoted by (α_t). We denote by $\omega : \mathcal{R} \to \mathbb{R}$ the Radon-Nikodym 1-cocycle of $\mathcal{R}$ (see Section 2.4).

In the same way as with the Maharam extension of a nonsingular group action, the equivalence relation $\mathcal{R}$ admits a natural trace preserving action on $L^\infty(X) \overline{\otimes} P$. We denote by $(\mathcal{M}, \mathrm{Tr})$ the crossed product. For completeness, we recall the construction of $(\mathcal{M}, \mathrm{Tr})$. To every $\varphi \in [[\mathcal{R}]]$, we associate the operator W_φ on $L^2(\mathcal{R}, L^2(P))$ given by

$$(W_\varphi \xi)(x, y) = \begin{cases} \alpha_{\omega(x, \varphi^{-1}(x))}(\xi(\varphi^{-1}(x), y)) & \text{if } x \in \operatorname{dom}(\varphi^{-1}) , \\ 0 & \text{otherwise,} \end{cases}$$

for every $\xi \in L^2(\mathcal{R}, L^2(P))$. One checks that $W_\varphi W_\psi = W_{\varphi \circ \psi}$ and $W_\varphi^* = W_{\varphi^{-1}}$. We represent $L^\infty(X) \overline{\otimes} P = L^\infty(X, P)$ on $L^2(\mathcal{R}, L^2(P))$ by

$$(F\xi)(x, y) = F(x)\xi(x, y) \;\text{ for all }\; \xi \in L^2(\mathcal{R}, L^2(P)) \;\text{ and }\; F \in L^\infty(X, P) .$$

Note that the partial isometries W_φ, $\varphi \in [[\mathcal{R}]]$, normalize $L^\infty(X, P)$ and that

$$(W_\varphi^* F W_\varphi)(x) = \begin{cases} \alpha_{\omega(x, \varphi(x))}(F(\varphi(x))) & \text{if } x \in \operatorname{dom} \varphi \\ 0 & \text{otherwise.} \end{cases}$$

Define $\mathcal{M}$ as the von Neumann algebra generated by $L^\infty(X, P)$ and the partial isometries W_φ, $\varphi \in [[\mathcal{R}]]$. Denoting by $\Delta \subset \mathcal{R}$ the diagonal subset, the orthogonal projection onto $L^2(\Delta, L^2(P))$ implements a normal faithful conditional expectation $E : \mathcal{M} \to L^\infty(X) \overline{\otimes} P$ satisfying

$$E(W_\varphi) = \chi_{\{x \mid \varphi(x) = x\}} \otimes 1 \;\text{ for all }\; \varphi \in [[\mathcal{R}]] .$$

The formula $\mathrm{Tr} := (\mu \otimes \mathrm{Tr}) \circ E$ defines a normal semifinite faithful trace on $\mathcal{M}$. Fix a nonzero projection $q \in P$ with $\mathrm{Tr}(q) = 1$. Define the projection $p \in L^\infty(X) \mathbin{\overline{\otimes}} P$ given by $p = 1 \otimes q$. Write $A := L^\infty(X)p$ and $M := p\mathcal{M}p$. Then A is a quasi-regular abelian von Neumann subalgebra of M and the restriction of Tr to M gives a normal faithful tracial state τ on M. The relative commutant $L^\infty(X)' \cap \mathcal{M}$ equals $L^\infty(X) \mathbin{\overline{\otimes}} P$. Since P is a factor, it follows that $\mathcal{Z}(A' \cap M) = A$.

We finally prove that $\mathcal{R} \cong \mathcal{R}(A \subset M)$. Write $\mathcal{R} = \bigcup_k \mathrm{graph}(\varphi_k)$, with $\varphi_k \in [\mathcal{R}]$. Then φ_k induces an automorphism of $L^\infty(X)$ and hence of $A = L^\infty(X)p$ that we denote by $\beta_k \in \mathrm{Aut}(A)$. Since P is a II_∞ factor and $q \in P$ is a finite projection, we can choose partial isometries $w_n \in P$ such that $\sum_n w_n^* w_n = 1$ and $w_n w_n^* = q$ for all n. Define the elements

$$v_{n,k} := (1 \otimes w_n) W_{\varphi_k} p \; .$$

All $v_{n,k}$ belong to $\mathrm{Q}_M(A)$ and $\alpha_{v_{n,k}}$ equals the restriction of β_k to $Ap_{n,k}$ for projections $p_{n,k} \in A$. Since the sum of all $w_n^* w_n$ equals 1, we also have that $\bigvee_n p_{n,k} = p$. Therefore the graphs of the partial automorphisms $\alpha_{v_{n,k}}$ generate an equivalence relation that is isomorphic with $\mathcal{R}$. To conclude the proof, we put $\mathcal{F} := \{v_{n,k} \mid n, k \in \mathbb{N}\}$ and observe that $M = (\mathcal{F} \cup \mathcal{F}^* \cup (A' \cap M))''$. By Lemma 3.4, the equivalence relation $\mathcal{R}(A \subset M)$ is generated by the graphs of the partial automorphisms $\alpha_{v_{n,k}}$. ☐

4. Proof of Theorem A

Throughout this section, we assume that n and m are integers satisfying $2 \leq n < |m|$. As explained in the introduction, the corresponding groups $\mathrm{BS}(n,m)$ form a complete list of the nonamenable icc Baumslag-Solitar groups up to isomorphism.

Throughout this section, we write $M = L(\mathrm{BS}(n,m))$ and $A = \{u_a, u_a^*\}''$. We start with the following observation.

Proposition 4.1. *We have that $A \subset M$ is a quasi-regular abelian von Neumann subalgebra satisfying $\mathcal{Z}(A' \cap M) = A$. Moreover, $A' \cap M$ has no amenable direct summand.*

Proof. It is clear that $A \subset M$ is a quasi-regular abelian von Neumann subalgebra, because the element $a \in \mathrm{BS}(n,m)$ generates an almost normal abelian subgroup of $\mathrm{BS}(n,m)$: for every $g \in \mathrm{BS}(n,m)$, the group $g a^{\mathbb{Z}} g^{-1} \cap a^{\mathbb{Z}}$ has finite index in $a^{\mathbb{Z}}$.

To prove that $\mathcal{Z}(A' \cap M) = A$, we define the finite index subalgebra $A_0 := \{u_a^n, u_a^{-n}\}''$ of A. We will first prove that $\mathcal{Z}(A_0' \cap M) = A_0$. Afterwards we will show that this implies that $\mathcal{Z}(A' \cap M) = A$.

Define $G := \langle a^{\mathbb{Z}}, b^{-1} a^{\mathbb{Z}} b \rangle \subset \mathrm{BS}(n,m)$. Then $L(G)$ is a subalgebra of $A_0' \cap M$. So $\mathcal{Z}(A_0' \cap M) \subset L(G)' \cap M$. Using Lemma 2.1, one can easily see that $\{g \gamma g^{-1} \mid g \in G\}$ is an infinite set for every $\gamma \in \mathrm{BS}(n,m) - a^{n\mathbb{Z}}$. Therefore $L(G)' \cap M \subset A_0$. This shows that $\mathcal{Z}(A_0' \cap M) \subset A_0$. Since the converse inclusion is obvious, we find that $A_0 = \mathcal{Z}(A_0' \cap M)$.

Since $A_0 \subset A$ has finite index, there exist orthogonal projections $p_j \in A$ such that $Ap_j = A_0 p_j$ and $\sum_j p_j = 1\!\!1$. But then

$$\begin{aligned}\mathcal{Z}(A' \cap M)p_j &= \mathcal{Z}((A' \cap M)p_j) = \mathcal{Z}((Ap_j)' \cap p_j M p_j) \\ &= \mathcal{Z}((A_0 p_j)' \cap p_j M p_j) = \mathcal{Z}(p_j(A_0' \cap M)p_j) \\ &= \mathcal{Z}(A_0' \cap M)p_j = A_0 p_j \subset A\end{aligned}$$

Therefore $\mathcal{Z}(A' \cap M) \subset A$. The converse inclusion being obvious, we have proven that $A = \mathcal{Z}(A' \cap M)$.
Using Lemma 2.1, it follows that G is an amalgamated free product of two copies of $\mathbb{Z}$ over a copy of $\mathbb{Z}$ embedded as $n\mathbb{Z}$ and $m\mathbb{Z}$ respectively. In particular, G is nonamenable and $L(G)$ has no amenable direct summand. Since $L(G) \subset A_0' \cap M$, it follows that $A_0' \cap M$ has no amenable direct summand either. As above, we have that

$$(A' \cap M)p_j = p_j(A_0' \cap M)p_j$$

for all j. Hence $A' \cap M$ has no amenable direct summand. $\square$

We now identify the associated countable equivalence relation $\mathcal{R}(A \subset M)$.

Proposition 4.2. *The equivalence relation $\mathcal{R}(A \subset M)$ is isomorphic with the unique hyperfinite ergodic countable equivalence relation of type $III_{n/|m|}$.*

Proof. Let k be the greatest common divisor of n and $|m|$. Write $n = n_0 k$ and $m = m_0 k$. By our assumptions on n and m, we have that $1 \leq n_0 < |m_0|$. Define the countable Borel equivalence relation $\mathcal{R}_{n,m}$ on the circle $\mathbb{T}$ given by

$$\mathcal{R}_{n,m} := \big\{(y,z) \in \mathbb{T} \times \mathbb{T} \;\big|\; \exists a,b \in \mathbb{N} \text{ such that } a+b>0 \text{ and } y^{(n_0^a m_0^b k)} = z^{(m_0^a n_0^b k)}\big\} \; .$$

Equip $\mathbb{T}$ with its Lebesgue measure λ and note that $\mathcal{R}_{n,m}$ is a nonsingular countable Borel equivalence relation on $(\mathbb{T}, \lambda)$.
Define $\mathcal{R}_0 := \{(y,z) \in \mathbb{T} \times \mathbb{T} \mid y^m = z^n\}$. Note that $\mathcal{R}_0 \subset \mathcal{R}_{n,m}$ and that $\mathcal{R}_{n,m}$ is the smallest equivalence relation containing $\mathcal{R}_0$. Define $\pi : \mathcal{R}_0 \to \mathbb{T} : \pi(y,z) = y^m$. Note that π is $n|m|$-to-1. Define the probability measure μ on $\mathcal{R}_0$ given by

$$\mu(U) = \frac{1}{n|m|} \int_{\mathbb{T}} \#\big(U \cap \pi^{-1}(\{x\})\big)\, d\lambda(x) \; .$$

For all $k,l \in \mathbb{Z}$, we define the function $P_{k,l} : \mathcal{R}_0 \to \mathbb{T} : P_{k,l}(y,z) = y^k z^l$. A direct computation yields a unique unitary

$$T : L^2(\mathcal{R}_0, \mu) \to \overline{\text{span}}^{\|\cdot\|_2} A u_b A : P_{k,l} \mapsto u_a^k u_b u_a^l \; .$$

We turn $L^2(\mathcal{R}_0, \mu)$ into an $L^\infty(\mathbb{T})$-$L^\infty(\mathbb{T})$-bimodule by the formula

$$(F \cdot \xi \cdot F')(y,z) = F(y)\, \xi(y,z)\, F'(z) \; .$$

Under the natural identification of $L^\infty(\mathbb{T})$ and A, the unitary T is A-A-bimodular.
By construction ${}_AL^2(\mathcal{R}_0,\mu)_A$ is isomorphic with a direct sum of bimodules of the form ${}_A\mathcal{H}(\alpha_j)_A$ where the union of the graphs of the partial automorphisms α_j equals $\mathcal{R}_0$ and hence generates the equivalence relation $\mathcal{R}_{n,m}$. Applying Lemma 3.4 to $\mathcal{F}=\{u_b\}$, we conclude that $\mathcal{R}(A\subset M)\cong\mathcal{R}_{n,m}$ up to measure zero.
As in Section 2.4, denote by $\omega:\mathcal{R}_{n,m}\to\mathbb{R}$ the Radon-Nikodym 1-cocycle. Denote by $\Lambda\subset\mathbb{T}$ the subgroup given by

$$\Lambda := \Big\{\exp\Big(\frac{2\pi is}{(n_0m_0)^b}\Big)\ \Big|\ s\in\mathbb{Z}, b\in\mathbb{N}\Big\}\ .$$

For every $z\in\Lambda$, we denote by $\alpha_z:\mathbb{T}\to\mathbb{T}$ the rotation $\alpha_z(y)=zy$. We have $\operatorname{graph}\alpha_z\subset\mathcal{R}_{n,m}$ for all $z\in\Lambda$. Since all α_z are measure preserving, we actually have $\operatorname{graph}\alpha_z\subset\operatorname{Ker}(\omega)$. Since $\Lambda\subset\mathbb{T}$ is a dense subgroup, it follows that $\operatorname{Ker}(\omega)$ is an ergodic equivalence relation. In particular, $\mathcal{R}_{n,m}$ is ergodic. A direct computation shows that $\omega(y,z)=\log(n/|m|)$ for all $(y,z)\in\mathcal{R}_0$. Since $\mathcal{R}_0$ generates the equivalence relation $\mathcal{R}_{n,m}$, it follows that the essential image of ω equals $\log(n/|m|)\mathbb{Z}$. Using Lemma 2.7, we conclude that $\mathcal{R}_{n,m}$ is of type $\mathrm{III}_{n/|m|}$. By construction, $\mathcal{R}_{n,m}$ is amenable and hence, hyperfinite. □

We are now ready to prove our main theorem.

Proof of Theorem A. Fix for $i=1,2$, integers $n_i,m_i\in\mathbb{Z}$ with $2\le n_i<|m_i|$. Put $M_i=L(\mathrm{BS}(n_i,m_i))$ and denote by $A_i\subset M_i$ the abelian von Neumann subalgebra generated by u_a, where $a\in\mathrm{BS}(n_i,m_i)$ is the first canonical generator. Assume that M_1 and M_2 are stably isomorphic. We must prove that

$$\frac{n_1}{|m_1|}=\frac{n_2}{|m_2|}\ . \tag{4}$$

Interchanging if necessary the roles of M_1 and M_2, we can take a nonzero projection $p_1\in A_1$ and a $*$-isomorphism $\alpha:p_1M_1p_1\to M_2$.
We claim that inside M_2, we have $\alpha(A_1p_1)\prec A_2$. From Proposition 4.1, we know that

$$P:=\alpha(A_1p_1)'\cap M_2=\alpha((A_1'\cap M_1)p_1)$$

has no amenable direct summand. By Proposition 3.1 in [Ue07], the HNN extension M_2 can be viewed as the corner of an amalgamated free product of tracial von Neumann algebras. Since P has no amenable direct summand, it then follows from [CH08, Theorem 4.2] that $P'\cap M_2\prec A_2$. So our claim that $\alpha(A_1p_1)\prec A_2$ follows.
By symmetry, we also have the intertwining $\alpha^{-1}(A_2)\prec A_1p_1$ inside $p_1M_1p_1$. Applying α, we find that $A_2\prec\alpha(A_1p_1)$ inside M_2.
Having proven that inside M_2 we have the intertwining relations $\alpha(A_1p_1)\prec A_2$ and $A_2\prec\alpha(A_1p_1)$, it follows from Theorem 3.3 that the equivalence relations $\mathcal{R}(A_1p_1\subset p_1M_1p_1)$ and $\mathcal{R}(A_2\subset M_2)$ are stably isomorphic. By construction,

$\mathcal{R}(A_1p_1 \subset p_1M_1p_1)$ is the restriction of $\mathcal{R}(A_1 \subset M_1)$ to the support of p_1. So we conclude that the equivalence relations $\mathcal{R}(A_1 \subset M_1)$ and $\mathcal{R}(A_2 \subset M_2)$ are stably isomorphic. In particular, these ergodic nonsingular equivalence relations must have the same type. Using Proposition 4.2, we find (4). $\square$

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