Big De Rham-Witt Cohomology: Basic Results

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Abstract. Let X be a smooth projective R-scheme, where R is a smooth $\mathbb{Z}$-algebra. As constructed by Hesselholt, we have the absolute big de Rham-Witt complex $\mathbb{W}\Omega^*_X$ of X at our disposal. There is also a relative version $\mathbb{W}\Omega^*_{X/R}$ with $\mathbb{W}(R)$-linear differential. In this paper we study the hypercohomology of the relative (big) de Rham-Witt complex after truncation with finite truncation sets S. We show that it is a projective $\mathbb{W}_S(R)$-module, provided that the de Rham cohomology is a flat R-module. In addition, we establish a Poincaré duality theorem. explicit description of the relative de Rham-Witt complex of a smooth λ-ring, which may be of independent interest.

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Introduction

Let X be a scheme over a perfect field k of characteristic $p > 0$. The de Rham-Witt complex $W\Omega^*_{X/k}$ was defined by Illusie [Ill79] relying on ideas of Bloch, Deligne and Lubkin. It is a projective system of complexes of $W(k)$-modules on X, which is indexed by the positive integers. If X is smooth then the hypercohomology of $W_n\Omega^*_{X/k}$ admits a natural comparison isomorphism to the crystalline cohomology of X with respect to $W_n(k)$.

Langer and Zink have extended Illusie's definition of the de Rham-Witt complex to a relative situation, where X is a scheme over $\mathrm{Spec}(R)$ and R is a $\mathbb{Z}_{(p)}$-algebra [LZ04]. If p is nilpotent in R and X is smooth, then they construct a functorial comparison isomorphism

$$H^*(X, W_n\Omega^*_{X/R}) \cong H^*_{crys}(X/W_n(R)).$$

The big de Rham-Witt complex $\mathbb{W}\Omega^*_A$ was introduced, for any commutative ring A, by Hesselholt and Madsen [HM01]. The original construction relied on the adjoint functor theorem and has been replaced by a direct and explicit method due to Hesselholt [Hes].

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Again, it is a projective system of graded sheaves $[S \mapsto \mathbb{W}_S\Omega^*_A]$, but the index set consists of finite truncation sets; that is, finite subsets S of $\mathbb{N}_{>0}$ having the property that whenever $n \in S$, all (positive) divisors of n are also contained in S. For the ring of integers, $\mathbb{W}\Omega^*_{\mathbb{Z}}$ has been computed by Hesselholt [Hes]. It vanishes in degree ≥ 2, but $\mathbb{W}\Omega^1_{\mathbb{Z}}$ is non-zero.
Let X be an R-scheme. In this paper we will consider the relative version

$$S \mapsto \mathbb{W}_S\Omega^*_{X/R}$$

of the (big) de Rham-Witt complex, which is constructed from $\mathbb{W}\Omega^*_X$ by killing the ideal generated by $\mathbb{W}\Omega^1_R$. The relation with the de Rham-Witt complex of Langer-Zink is given in Proposition 1.2.7: if R is a $\mathbb{Z}_{(p)}$-algebra then

$$\mathbb{W}_{\{1,p,\ldots,p^{n-1}\}}\Omega^*_{A/R} = W_n\Omega^*_{A/R}.$$

In the following we will use the notation $W_n = \mathbb{W}_{\{1,p,\ldots,p^{n-1}\}}$, assuming that a prime p has been fixed.
It is natural to consider $\mathbb{W}_S\Omega^*_{X/R}$ as a sheaf of complexes on the scheme $\mathbb{W}_S(X)$, which is obtained by gluing $\mathrm{Spec}(\mathbb{W}_S(A_i))$ for an affine covering $X = \bigcup_i \mathrm{Spec}(A_i)$. Then the components $\mathbb{W}_S\Omega^q_{X/R}$ form quasi-coherent sheaves, and are coherent under suitable finiteness conditions.
Our purpose is to show that the de Rham-Witt cohomology

$$H^i_{dRW}(X/\mathbb{W}_S(R)) \stackrel{\text{def}}{=} H^i(\mathbb{W}_S(X), \mathbb{W}_S\Omega^*_{X/R})$$

is as well-behaved as the usual de Rham cohomology. The main theorem of the paper is the following.

Theorem 1 (cf. Theorem 2.2.1). *Let R be a smooth $\mathbb{Z}$-algebra. Let X be a smooth and proper R-scheme. Suppose that the de Rham cohomology $H^*_{dR}(X/R)$ of X is a flat R-module. Then $H^*_{dRW}(X/\mathbb{W}_S(R))$ is a finitely generated projective $\mathbb{W}_S(R)$-module for all finite truncation sets S. Moreover, for an inclusion of finite truncation sets $T \subset S$, the induced map*

$$H^*_{dRW}(X/\mathbb{W}_S(R)) \otimes_{\mathbb{W}_S(R)} \mathbb{W}_T(R) \xrightarrow{\cong} H^*_{dRW}(X/\mathbb{W}_T(R)) \tag{0.0.1}$$

is an isomorphism.

If R is a smooth $\mathbb{Z}$-algebra and X/R is smooth and proper, then there is a non-zero integer N such that the assumptions of the theorem hold for the base change $X \otimes_R R[N^{-1}]/R[N^{-1}]$. Curves and abelian varieties are examples where the de Rham cohomology is flat (see Remark 2.2.2).
In order to prove Theorem 1, we will construct for all maximal ideals $\mathfrak{m}$ of R and $n, j > 0$, a natural quasi-isomorphism:

$$R\Gamma(W_n\Omega^*_{X/R}) \otimes^{\mathbb{L}}_{W_n(R)} W_n(R/\mathfrak{m}^j) \xrightarrow{\text{q-iso}} R\Gamma(W_n\Omega^*_{X\otimes R/\mathfrak{m}^j/(R/\mathfrak{m}^j)}),$$

where $p = \mathrm{char}(R/\mathfrak{m})$. The right hand side is $R\Gamma$ of the de Rham-Witt complex defined by Langer and Zink. Thus it computes the crystalline cohomology, which in our case is a free $W_n(R/\mathfrak{m}^j)$-module. Taking the limit $\varprojlim_j$, this will yield the flatness of

$$H^*_{dRW}(X/W_n(R)) \otimes_{W_n(R)} W_n(\varprojlim_j R/\mathfrak{m}^j)$$

as $W_n(\varprojlim_j R/\mathfrak{m}^j)$-module for all maximal ideals $\mathfrak{m}$, which is sufficient in order to conclude the flatness of the de Rham-Witt cohomology.
Concerning Poincaré duality we will show the following theorem.

Theorem 2 (cf. Corollary 3.3.7). *Let R be a smooth $\mathbb{Z}$-algebra. Let $X \to \mathrm{Spec}(R)$ be a smooth projective morphism such that $H^*_{dR}(X/R)$ is a flat R-module. Suppose that X is connected of relative dimension d. If the canonical map*

$$H^i_{dR}(X/R) \to \mathrm{Hom}_R(H^{2d-i}_{dR}(X/R), R)$$

is an isomorphism, then the same holds for the de Rham-Witt cohomology:

$$H^i_{dRW}(X/\mathbb{W}_S(R)) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{W}_S(R)}(H^{2d-i}_{dRW}(X/\mathbb{W}_S(R)), \mathbb{W}_S(R)),$$

for all finite truncation sets S.

In fact, de Rham-Witt cohomology is equipped with a richer structure than the $\mathbb{W}(R)$-module structure, coming from the Frobenius operators

$$\phi_n : H^*_{dRW}(X/\mathbb{W}_S(R)) \to H^*_{dRW}(X/\mathbb{W}_{S/n}(R)),$$

for all positive integers n, and where $S/n := \{s \in S \mid ns \in S\}$. These are Frobenius linear maps satisfying $\phi_n \circ \phi_m = \phi_{nm}$.
The relationship with the Frobenius action on the crystalline cohomology of the fibers is as follows. Let $\mathfrak{m}$ be a maximal ideal of R, set $k = R/\mathfrak{m}$ and $p = \mathrm{char}(k)$. If $H^*_{dR}(X/R)$ is torsion-free then there is a natural isomorphism

$$H^i_{dRW}(X/W_n(R)) \otimes_{W_n(R)} W_n(k) \cong H^i_{crys}(X \otimes_R k/W_n(k)),$$

and $\phi_p \otimes F_p$ corresponds via this isomorphism to the composition of $H^i_{crys}(\mathrm{Frob})$ with the projection.
As will be made precise in Section 3, the projective system

$$H^i_{dRW}(X/\mathbb{W}(R)) \stackrel{\mathrm{def}}{=} [S \mapsto H^i_{dRW}(X/\mathbb{W}_S(R))],$$

together with the Frobenius morphisms $\{\phi_n\}_{n\in\mathbb{N}_{>0}}$, defines an object in a rigid $\otimes$-category $\mathcal{C}_R$. Maybe the most important property of $\mathcal{C}_R$ is the existence of a conservative, faithful $\otimes$-functor

$$T : \mathcal{C}_R \to (R\text{-modules}), \qquad T(H^i_{dRW}(X/\mathbb{W}(R))) = H^i_{dR}(X/R).$$

Moreover, $\mathcal{C}_R$ has Tate objects $\mathbf{1}(m)$, $m \in \mathbb{Z}$, and the first step towards Poincaré duality will be to prove the existence of a natural morphism in $\mathcal{C}_R$:

$$H^{2d}_{dRW}(X/\mathbb{W}(R)) \to \mathbf{1}(-d) \qquad (d = \text{relative dimension of } X/R).$$

Then it will follow easily that

$$H^i_{dRW}(X/\mathbb{W}(R)) \xrightarrow{\cong} \underline{\mathrm{Hom}}(H^{2d-i}_{dRW}(X/\mathbb{W}(R)), \mathbf{1}(-d)),$$

provided that the assumptions of Theorem 2 are satisfied. Taking the underlying $\mathbb{W}(R)$-modules one obtains Theorem 2.

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1. Relative de Rham-Witt complexes

1.1. Witt vectors. For the definition and the basic properties of the ring of Witt vectors we refer to [Hes, §1]. We briefly recall the notions in this section.
A subset $S \subset \mathbb{N} = \{1, 2, \dots\}$ is called a *truncation set* if $n \in S$ implies that all positive divisors of n are contained in S. For a truncation set S and $n \in S$, we define $S/n := \{s \in S \mid sn \in S\}$.
Let A be a commutative ring. For all truncation sets S we have the ring of Witt vectors $\mathbb{W}_S(A)$ at our disposal. The ghost map is the functorial ring homomorphism

$$gh = (gh_n)_{n \in S} : \mathbb{W}_S(A) \to \prod_{n \in S} A, \quad gh_n((a_s)_{s \in S}) := \sum_{d \mid n} d \cdot a_d^{n/d}.$$

It is injective provided that A is $\mathbb{Z}$-torsion-free.
For all positive integers n, there is a functorial morphism of rings

$$F_n : \mathbb{W}_S(A) \to \mathbb{W}_{S/n}(A),$$

called the *Frobenius*. Moreover there is a functorial morphism of $\mathbb{W}_S(A)$-modules, the *Verschiebung*,

$$V_n : \mathbb{W}_{S/n}(A) \to \mathbb{W}_S(A),$$

where the source is a $\mathbb{W}_S(A)$-module via F_n. For all coprime positive integers $n, m \in \mathbb{N}$ we have

$$F_n \circ V_n = n, \quad F_n \circ V_m = V_m \circ F_n \qquad ((m,n) = 1).$$

We have a multiplicative Teichmüller map

$$[-] : A \to \mathbb{W}_S(A), \quad a \mapsto [a] := (a, 0, 0, \dots) \in \mathbb{W}_S(A),$$

and if S is finite then every element $a \in \mathbb{W}_S(A)$ can be written as

$$a = \sum_{s \in S} V_s([a_s])$$

with unique elements $(a_s)_{s \in S}$ in A.
Let $T \subset A$ be a multiplicative set and suppose that S is a finite truncation set. We can consider T via the Teichmüller map as multiplicative set in $\mathbb{W}_S(A)$. Then the natural ring homomorphism

$$T^{-1}\mathbb{W}_S(A) \to \mathbb{W}_S(T^{-1}A).$$

is an isomorphism. If $T \subset \mathbb{Z}$ is a multiplicative set then

$$\mathbb{W}_S(A) \otimes_{\mathbb{Z}} T^{-1}\mathbb{Z} \to \mathbb{W}_S(T^{-1}A)$$

is an isomorphism.
Let S be a truncation set, and let n be a positive integer; set $T := S \backslash \{s \in S; n \mid s\}$. Then T is a truncation set and we have a short exact sequence of $\mathbb{W}_S(A)$-modules:

$$0 \to \mathbb{W}_{S/n}(A) \xrightarrow{V_n} \mathbb{W}_S(A) \xrightarrow{R^S_T} \mathbb{W}_T(A) \to 0. \tag{1.1.1}$$

Example 1.1.1. We have $\mathbb{W}_S(\mathbb{Z}) = \prod_{n \in S} \mathbb{Z} \cdot V_n(1)$, and the product is given by $V_m(1) \cdot V_n(1) = c \cdot V_{mn/c}(1)$, where $c = (m,n)$ is the greatest common divisor [Hes, Proposition 1.6].

1.1.2. The following theorem will be very useful throughout the paper.

Theorem 1.1.3. *(Borger-van der Kallen) Let S be a finite truncation set, and let n be a positive integer. Let $\rho : A \to B$ be an étale ring homomorphism. The following hold.*

(1) *The induced ring homomorphism $\mathbb{W}_S(A) \to \mathbb{W}_S(B)$ is étale.*
(2) *The morphism*

$$\mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A), F_n} \mathbb{W}_{S/n}(A) \to \mathbb{W}_{S/n}(B), \quad b \otimes a \mapsto F_n(b) \cdot \mathbb{W}_{S/n}(\rho)(a),$$

is an isomorphism.

The references for this theorem are [Bor11a, Theorem B] [Bor11b, Corollary 15.4] and [vdK86, Theorem 2.4] (cf. [Hes, Theorem 1.22]).
By using Theorem 1.1.3, the exact sequence (1.1.1), and induction on the length of S, we easily obtain the following corollary.

COROLLARY 1.1.4. *Let $\rho : A \to B$ be an étale ring homomorphism. Let S be a finite truncation set.*

(i) *For an inclusion of truncation sets $T \subset S$, the map*

$$\mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_T(A) \to \mathbb{W}_T(B)$$

is an isomorphism.

(ii) *Let n be a positive integer. For any A-algebra C, the natural ring homomorphism*

$$\mathbb{W}_{S/n}(C) \otimes_{F_n, \mathbb{W}_S(A)} \mathbb{W}_S(B) \to \mathbb{W}_{S/n}(C \otimes_A B), \quad c \otimes b \mapsto c \cdot F_n(b)$$

is an isomorphism.

NOTATION 1.1.5. If a prime p has been fixed then we set $W_n := \mathbb{W}_{\{1,p,p^2,\dots,p^{n-1}\}}$.

1.1.6. Let p be a prime. Let R be a $\mathbb{Z}_{(p)}$-algebra. Since all primes different from p are invertible in R, the same holds in $\mathbb{W}_S(R)$. The category of $\mathbb{W}_S(R)$-modules, for a finite truncation set S, factors in the following way. Set

$$\epsilon_{1,S} := \prod_{\substack{\text{primes } \ell \neq p \\ S/\ell \neq \emptyset}} (1 - \frac{1}{\ell} V_\ell(1)) \in \mathbb{W}_S(R),$$

and $\epsilon_{n,S} := \frac{1}{n} V_n \left(\epsilon_{1,S/n}\right)$ for all positive integers n with $(n,p) = 1$. Of course, if $S/n = \emptyset$ then $\epsilon_{S,n} = 0$. In the following we will simply write ϵ_n for $\epsilon_{n,S}$. For all positive integers $n \neq n'$ with $(n,p) = 1 = (n',p)$ the equalities

$$\epsilon_n^2 = \epsilon_n, \qquad \epsilon_n \epsilon_{n'} = 0,$$

hold. Moreover, if $(m,p) = 1 = (n,p)$ then

$$F_m(\epsilon_n) = \begin{cases} \epsilon_{n/m} & \text{if } m \mid n, \\ 0 & \text{if } m \nmid n. \end{cases}$$

Since $\sum_{(n,p)=1} \epsilon_n = 1$ we obtain a decomposition of rings

$$\mathbb{W}_S(R) = \prod_{n \geq 1, (n,p)=1} \epsilon_n \mathbb{W}_S(R). \tag{1.1.2}$$

NOTATION 1.1.7. For a finite truncation set S we denote by S_p the elements in S that are p-powers, that is $S_p = S \cap \{p^i \mid i \geq 0\}$.

The map

$$R^{S/n}_{(S/n)_p} \circ F_n : \mathbb{W}_S(R) \to \mathbb{W}_{(S/n)_p}(R)$$

induces an isomorphism $\epsilon_n \mathbb{W}_S(R) \cong \mathbb{W}_{(S/n)_p}(R)$. Thus

$$M \mapsto \bigoplus_{n \geq 1, (n,p)=1} \epsilon_n M$$

defines an equivalence of categories

$$(\mathbb{W}_S(R)\text{-modules}) \xrightarrow{\cong} \prod_{n \geq 1, (n,p)=1} (\mathbb{W}_{(S/n)_p}(R)\text{-modules}). \tag{1.1.3}$$

1.1.8. The following two lemmas are concerned with maximal ideals in $\mathbb{W}_S(R)$.

Lemma 1.1.9. *Let R be a ring. Let S be a finite truncation set. For every maximal ideal $\mathfrak{m} \subset \mathbb{W}_S(R)$ there exists a maximal ideal $\mathfrak{p} \subset R$ such that $\mathbb{W}_S(R) \to \mathbb{W}_S(R)/\mathfrak{m}$ factors through $\mathbb{W}_S(R_{\mathfrak{p}})$.*

Proof. Set $k = \mathbb{W}_S(R)/\mathfrak{m}$, we distinguish two cases:

(1) k has characteristic 0,
(2) k has characteristic $p > 0$.

In the first case we can factor

$$\mathbb{W}_S(R) \to \mathbb{W}_S(R) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{=} \mathbb{W}_S(R \otimes_{\mathbb{Z}} \mathbb{Q}) \to k.$$

Since $\mathbb{W}_S(R \otimes_{\mathbb{Z}} \mathbb{Q}) \xrightarrow{gh, \cong} \prod_{s \in S} R \otimes \mathbb{Q}$, the claim follows.
Suppose now that k has characteristic $p > 0$. We have a factorization

$$\mathbb{W}_S(R) \to \mathbb{W}_S(R) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \xrightarrow{=} \mathbb{W}_S(R \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}) \to k.$$

By decomposing

$$\begin{aligned} \mathbb{W}_S(R \otimes \mathbb{Z}_{(p)}) \xrightarrow{=} & \prod_{n \geq 1, (n,p)=1} \epsilon_n \mathbb{W}_S(R \otimes \mathbb{Z}_{(p)}) \\ & \xrightarrow{\cong, \prod_n R^{S/n}_{(S/n)_p} \circ F_n} \prod_{n \geq 1, (n,p)=1} \mathbb{W}_{(S/n)_p}(R \otimes \mathbb{Z}_{(p)}), \end{aligned}$$

we can reduce to the case where S consists only of p-powers. Finally, $V_p(a)^2 = pV_p(a^2)$, for all $a \in \mathbb{W}_{S/p}(R \otimes \mathbb{Z}_{(p)})$, hence $V_p(a)$ maps to zero in k. Therefore $\mathbb{W}_S(R \otimes \mathbb{Z}_{(p)}) \to k$ factors through $\mathbb{W}_S(R \otimes \mathbb{Z}_{(p)}) \to \mathbb{W}_{\{1\}}(R \otimes \mathbb{Z}_{(p)}) = R \otimes \mathbb{Z}_{(p)} \xrightarrow{\rho} k$. In this case we can take $\mathfrak{p} = \ker(R \to R \otimes \mathbb{Z}_{(p)} \xrightarrow{\rho} k)$. $\square$

Lemma 1.1.10. *Let p be a prime. Let R be a ring such that every maximal ideal $\mathfrak{p}$ satisfies $\mathrm{char}(R/\mathfrak{p}) = p > 0$. Let S be a p-typical finite truncation set. Then every maximal ideal $\mathfrak{m}$ of $\mathbb{W}_S(R)$ is of the form $\ker(\mathbb{W}_S(R) \xrightarrow{R^S_{\{1\}}} R \to R/\mathfrak{p})$, for a unique maximal ideal $\mathfrak{p}$ of R.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{W}_S(R)$, set $k = \mathbb{W}_S(R)/\mathfrak{m}$. We claim that $\mathrm{char}(k) = p$. Suppose that $\mathrm{char}(k) \neq p$. From the commutative diagram

$$\begin{tikzcd} \mathbb{W}_S(R) \arrow[r] \arrow[d, "gh"] & \mathbb{W}_S(R)\otimes\mathbb{Z}[p^{-1}] \arrow[r] \arrow[d, "gh", "\cong"'] & k \\ \prod_{s\in S} R \arrow[r] & \prod_{s\in S} R\otimes\mathbb{Z}[p^{-1}] \arrow[ur] & \end{tikzcd}$$

we conclude that there is a factorization $\mathbb{W}_S(R) \xrightarrow{gh_i} R \to k$, but there are no epimorphism $R \to k$ to a field of characteristic $\neq p$.
Thus we may suppose that $\mathrm{char}(k) = p$. Because $V_p(a)^2 = pV_p(a^2)$ for all $a \in \mathbb{W}_{S/p}(R)$, we obtain a factorization $\mathbb{W}_S(R) \xrightarrow{R^S_{\{1\}}} R \to k$, which defines $\mathfrak{p} := \ker(R \to k)$. □

1.2. Relative de Rham-Witt complex. For every commutative ring A we have the absolute de Rham-Witt complex

$$S \mapsto \mathbb{W}_S\Omega^*_A$$

constructed by Hesselholt [Hes], at our disposal. The absolute de Rham-Witt complex is the initial object in the category of Witt complexes [Hes, §4]. In this section we will define the relative version, which is studied in this paper.

Definition 1.2.1. Let A be an R-algebra. Let S be a truncation set and $q \geq 0$. We define

$$\mathbb{W}_S\Omega^q_{A/R} = \varprojlim_{\substack{T\subset S\\ T \text{ finite}}} \mathbb{W}_T\Omega^q_A/\left(\mathbb{W}_T\Omega^1_R \cdot \mathbb{W}_T\Omega^{q-1}_A\right)$$

For $q = 0$, the definition means $\mathbb{W}_S\Omega^0_{A/R} = \mathbb{W}_S(A)$.

We get an induced anti-symmetric graded algebra structure on $\mathbb{W}_S\Omega^*_{A/R}$, that is, $\omega_1 \cdot \omega_2 = (-1)^{\deg(\omega_1)\deg(\omega_2)}\omega_2 \cdot \omega_1$.
Recall that by construction of $\mathbb{W}_S\Omega^*_A$, there is, for all finite truncation sets S, a surjective morphism of graded $\mathbb{W}_S(A)$-algebras

$$\pi : T^*_{\mathbb{W}_S(A)}\Omega^1_{\mathbb{W}_S(A)} \to \mathbb{W}_S\Omega^*_A, \tag{1.2.1}$$

such that $\pi(da) = da$ for all $a \in \mathbb{W}_S(A)$.

Lemma 1.2.2. *Let S be a finite truncation set.*

(1) *The morphism* (1.2.1) *induces a surjective morphism of anti-symmetric graded algebras*

$$\pi : \Omega^*_{\mathbb{W}_S(A)/\mathbb{W}_S(R)} \to \mathbb{W}_S\Omega^*_{A/R}, \tag{1.2.2}$$

which by abuse of notation is called π again.

(2) *$\mathbb{W}_S\Omega^*_{A/R}$ is a differential graded algebra and (1.2.2) is compatible with the differential.*

Proof. For (1). This follows from $\pi(da \otimes da) \in d\log[-1] \cdot \mathbb{W}_S\Omega^1_A$ [Hes, §3] and $d\log[-1] \in \mathbb{W}_S\Omega^1_R$.
For (2). The differential $d : \mathbb{W}_S\Omega^*_{A/R} \to \mathbb{W}_S\Omega^*_{A/R}$ is well-defined, because $\mathbb{W}_S\Omega^*_R$ is generated by $\mathbb{W}_S\Omega^1_R$. It satisfies $d \circ d = 0$, because $d\log[-1] \in \mathbb{W}_S\Omega^1_R$. The compatibility of π with d follows from $\pi(da) = da$ for all $a \in \mathbb{W}_S(A)$. □

1.2.3. Induced from the absolute de Rham-Witt complex, we obtain for all positive integers n:

$$F_n : \mathbb{W}_S\Omega^q_{A/R} \to \mathbb{W}_{S/n}\Omega^q_{A/R}, \tag{1.2.3}$$

$$V_n : \mathbb{W}_{S/n}\Omega^q_{A/R} \to \mathbb{W}_S\Omega^q_{A/R}, \tag{1.2.4}$$

and $S \mapsto \mathbb{W}_S\Omega^*_{A/R}$ forms a Witt complex. Note that, computed in the absolute de Rham-Witt complex, we have

$$V_n(da \cdot \omega) = V_n(F_n dV_n(a) \cdot \omega) = dV_n(a) \cdot V_n(\omega),$$

hence $V_n(\mathbb{W}_{S/n}\Omega^1_R \cdot \mathbb{W}_{S/n}\Omega^{q-1}_A) \subset \mathbb{W}_S\Omega^1_R \cdot \mathbb{W}_S\Omega^{q-1}_A$.
The following equalities hold for the maps (1.2.3), (1.2.4):

$$V_nF_nd = dV_nF_n, \quad dV_nd = 0.$$

Proposition 1.2.4. *The Witt complex $S \mapsto \mathbb{W}_S\Omega^*_{A/R}$ is the initial object in the category of Witt complexes over A with $\mathbb{W}(R)$-linear differential.*

Proof. Let $S \mapsto E^*_S$ be a Witt complex over A with $\mathbb{W}(R)$-linear differential, that is, $d(a\omega) = ad(\omega)$ for $a \in \mathbb{W}_S(R)$ and $\omega \in E^*_S$. We only need to show that the canonical morphism

$$[S \mapsto \mathbb{W}_S\Omega^*_A] \to [S \mapsto E^*_S]$$

factors through $[S \mapsto \mathbb{W}_S\Omega^*_{A/R}]$. It is enough to check this for finite truncation sets. Because π (1.2.1) is surjective, we conclude that $\mathbb{W}_S\Omega^1_R$ is generated by elements of the form da with $a \in \mathbb{W}_S(R)$, which implies the claim. □

As a corollary we obtain the following statement.

Corollary 1.2.5. *Let A be an R-algebra, let p be a prime, and set $R' := R \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}, A' := A \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. There is a unique isomorphism*

$$[S \mapsto \mathbb{W}_S\Omega^*_{A'/R'}] \to [S \mapsto \varprojlim_{\substack{T \subset S \\ T \text{ finite}}} \mathbb{W}_T\Omega^*_{A/R} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}]$$

of Witt complexes over A'.

Proposition 1.2.6. *Let R be a $\mathbb{Z}_{(p)}$-algebra and let A, B be R-algebras. Let S be a finite truncation set.*

(1) *Via the equivalence from* (1.1.3) *we have*

$$\mathbb{W}_S\Omega^*_{A/R} \mapsto \bigoplus_{n \geq 1, (n,p)=1} \mathbb{W}_{(S/n)_p}\Omega^*_{A/R}. \tag{1.2.5}$$

(2) *For a morphism $f : A \to B$ the induced morphism $f_S : \mathbb{W}_S\Omega^*_{A/R} \to \mathbb{W}_S\Omega^*_{B/R}$ maps to*

$$f_S \mapsto \bigoplus_{n\geq 1,(n,p)=1} f_{(S/n)_p}$$

via the equivalence from (1.1.3).

Proof. For (1). The claim follows from [HM01, Proposition 1.2.5]. In the notation of loc. cit. the right hand side (1.2.5) equals $i_! i^* \mathbb{W}\Omega^*_{A/R}$, and $i^*, i_!$ preserve initial objects, since both functors admit a right adjoint.
For (2). Follows immediately from the construction in (1). $\square$

Proposition 1.2.7. *Let R be a $\mathbb{Z}_{(p)}$-algebra, let A be an R-algebra. Then*

$$n \mapsto \mathbb{W}_{\{1,p,\ldots,p^{n-1}\}}\Omega^*_{A/R}$$

is the relative de Rham-Witt complex $n \mapsto W_n\Omega^*_{A/R}$ defined by Langer and Zink [LZ04].

Proof. We have a restriction functor

$$\begin{aligned} i^* : (\text{Witt systems over } A \text{ with } \mathbb{W}(R)\text{-linear differential}) \to \\ (F\text{-}V\text{-procomplexes over the } R\text{-algebra } A), \end{aligned}$$

where we use the definition of [Hes, §4] for the source category and the definition of [LZ04, Introduction] for the target category. The functor i^* admits a right adjoint functor $i_!$ defined in [HM01, §1.2]. Therefore $i^*([S \mapsto \mathbb{W}_S\Omega^*_{A/R}])$ is the initial object in the category of F-V-procomplexes as is the relative de Rham-Witt complex constructed by Langer and Zink [LZ04]. $\square$

1.2.8. Let S be a finite truncation set. Let $A \to B$ be an étale morphism of R-algebras. For all $q \geq 0$ the induced morphism of $\mathbb{W}_S(B)$-modules

$$\mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_S\Omega^q_{A/R} \xrightarrow{\cong} \mathbb{W}_S\Omega^q_{B/R} \tag{1.2.6}$$

is an isomorphism. Indeed, this follows immediately from the analogous fact for the absolute de Rham-Witt complex [Hes, Theorem C].

Lemma 1.2.9. *Let $R' \to R$ be an étale ring homomorphism. Let A be an R-algebra. Then, for all truncation sets S,*

$$\mathbb{W}_S\Omega^*_{A/R'} \to \mathbb{W}_S\Omega^*_{A/R}$$

is an isomorphism.

Proof. We may assume that S is finite. The assertion follows from

$$\begin{aligned} \mathbb{W}_S\Omega^1_{R'} \otimes_{\mathbb{W}_S(R')} \mathbb{W}_S(A) &\xrightarrow{=} \mathbb{W}_S\Omega^1_{R'} \otimes_{\mathbb{W}_S(R')} \mathbb{W}_S(R) \otimes_{\mathbb{W}_S(R)} \mathbb{W}_S(A) \\ &\xrightarrow{\cong} \mathbb{W}_S\Omega^1_R \otimes_{\mathbb{W}_S(R)} \mathbb{W}_S(A). \end{aligned}$$

$\square$

1.2.10. For every truncation set S we have a functor

$$\mathbb{W}_S : (\text{Schemes}) \to (\text{Schemes}), \quad X \mapsto \mathbb{W}_S(X).$$

This functor has been studied by Borger [Bor11b], our notation differs slightly: the notation is W^* in [Bor11b].
For an affine scheme $U = \operatorname{Spec}(A)$, we have $\mathbb{W}_S(U) = \operatorname{Spec}(\mathbb{W}_S(A))$. If X is separated and $(U_i)_{i\in I}$ is an affine covering of X, then $\mathbb{W}_S(X)$ is obtained by gluing $\mathbb{W}_S(U_i)$ along $\mathbb{W}_S(U_i \times_X U_j)$. In particular, $(\mathbb{W}_S(U_i))_{i\in I}$ is an affine covering of $\mathbb{W}_S(X)$. The functor is extended to non-separated schemes in the usual way.
If $T \subset S$ is an inclusion of finite truncation sets then

$$\imath_{T,S} : \mathbb{W}_T(X) \to \mathbb{W}_S(X)$$

is a closed immersion and functorial in X.

Proposition 1.2.11. *Let X be an R-scheme and let S be a finite truncations set. There is a unique quasi-coherent sheaf of $\mathbb{W}_S(\mathcal{O})$-modules $\mathbb{W}_S\Omega^q_{X/R}$ for the étale topology of $\mathbb{W}_S(X)$ such that $\Gamma(\mathbb{W}_S(\operatorname{Spec}(A)), \mathbb{W}_S\Omega^q_{X/R}) = \mathbb{W}_S\Omega^q_{A/R}$ for every étale map $\operatorname{Spec}(A) \to X$ with the evident restriction maps.*

Proof. Let us glue a quasi-coherent sheaf $\mathbb{W}_S\Omega^q_{X/R}$ on $\mathbb{W}_S(X)$. Indeed, suppose first that X is separated. Let $(\operatorname{Spec}(A_i))_{i\in I}$ be an affine covering and set $\operatorname{Spec}(A_{ij}) = \operatorname{Spec}(A_i)\times_X \operatorname{Spec}(A_j)$. For every i, the $\mathbb{W}_S(A_i)$-module $\mathbb{W}_S\Omega^q_{A_i/R}$ defines a quasi-coherent sheaf $\mathbb{W}_S\Omega^q_{\operatorname{Spec}(A_i)/R}$ on $\mathbb{W}_S(\operatorname{Spec}(A_i))$. Since

$$\Gamma(\mathbb{W}_S(\operatorname{Spec}(A_{ij})), \mathbb{W}_S\Omega^q_{\operatorname{Spec}(A_i)/R}) = \mathbb{W}_S\Omega^q_{A_i/R}\otimes_{\mathbb{W}_S(A_i)}\mathbb{W}_S(A_{ij}) = \mathbb{W}_S\Omega^q_{A_{ij}/R},$$

by using (1.2.6), we can glue to a quasi-coherent sheaf $\mathbb{W}_S\Omega_{X/R}$ on $\mathbb{W}_S(X)$. Independence of the covering and $\mathbb{W}_S(\jmath)^*\mathbb{W}_S\Omega^q_{X/R} = \mathbb{W}_S\Omega^q_{U/R}$, for every étale map $\jmath : U \to X$, can be checked by using (1.2.6) again. □

Proposition 1.2.12. *If $\mathbb{W}_S(X) \to \mathbb{W}_S(\operatorname{Spec}(R))$ is of finite type and $\mathbb{W}_S(X)$ is noetherian, then $\mathbb{W}_S\Omega^q_{X/R}$ is coherent.*

Proof. We have a surjective morphism $\Omega^j_{\mathbb{W}_S(X)/\mathbb{W}_S(R)} \to \mathbb{W}_S\Omega^j_{X/R}$ and the assumptions imply that $\Omega^j_{\mathbb{W}_S(X)/\mathbb{W}_S(R)}$ is coherent. □

1.2.13. If $f : X \to Y$ is a morphism of R-schemes then we get

$$\mathbb{W}_S\Omega^q_{Y/R} \to \mathbb{W}_S(f)_*\mathbb{W}_S\Omega^q_{X/R}.$$

For an inclusion of truncation sets $T \subset S$, we obtain

$$\mathbb{W}_S\Omega^q_{X/R} \to \imath_{T,S*}\mathbb{W}_T\Omega^q_{X/R}.$$

The following diagram is commutative:

$$\begin{array}{ccccc} \mathbb{W}_S\Omega^q_{Y/R} & \longrightarrow & & & \mathbb{W}_S(f)_*\mathbb{W}_S\Omega^q_{X/R} \\ \downarrow & & & & \downarrow \\ \imath_{T,S*}\mathbb{W}_T\Omega^q_{Y/R} & \longrightarrow & \imath_{T,S*}\mathbb{W}_T(f)_*\mathbb{W}_T\Omega^q_{X/R} & \xrightarrow{=} & \mathbb{W}_S(f)_*\imath_{T,S*}\mathbb{W}_T\Omega^q_{X/R}. \end{array}$$

The differential, the Frobenius and the Verschiebung operations are defined in the evident way:

$$d : \mathbb{W}_S\Omega^q_{X/R} \to \mathbb{W}_S\Omega^{q+1}_{X/R},$$
$$F_n : \mathbb{W}_S\Omega^q_{X/R} \to \imath_{S/n,S*}\mathbb{W}_{S/n}\Omega^q_{X/R},$$
$$V_n : \imath_{S/n,S*}\mathbb{W}_{S/n}\Omega^q_{X/R} \to \mathbb{W}_S\Omega^q_{X/R}.$$

Definition 1.2.14. Let X be an R-scheme, let S be a finite truncation set. We define

$$H^i_{dRW}(X/\mathbb{W}_S(R)) := H^i(\mathbb{W}_S(X), \mathbb{W}_S\Omega^*_{X/R}),$$

where the right hand side is the hypercohomology for the Zariski topology.

1.2.15. Note that F_n and V_n are not morphisms of complexes. For all positive integers n and all finite truncation sets we set

$$\phi_n = n^q F_n : \mathbb{W}_S\Omega^q_{X/R} \to \imath_{S/n,S*}\mathbb{W}_{S/n}\Omega^q_{X/R}, \tag{1.2.7}$$

to get a morphism of complexes

$$\mathbb{W}_S\Omega^*_{X/R} \xrightarrow{\phi_n} \imath_{S/n,S*}\mathbb{W}_{S/n}\Omega^*_{X/R}.$$

Suppose that X is smooth over R of relative dimension d. Then we set

$$\beta_n = n^{d-q}V_n : \imath_{S/n,S*}\mathbb{W}_{S/n}\Omega^q_{X/R} \to \mathbb{W}_S\Omega^q_{X/R}$$

(we will prove $\mathbb{W}_S\Omega^q_{X/R} = 0$ if $q > d$ in Proposition 1.2.17(ii)). We obtain a morphism of complexes

$$\beta_n : \imath_{S/n,S*}\mathbb{W}_{S/n}\Omega^*_{X/R} \to \mathbb{W}_S\Omega^*_{X/R},$$

satisfying the equalities:

$$\phi_n \circ \beta_n = n^{d+1},$$
$$\beta_n(\lambda \cdot \phi_n(x)) = n^d V_n(\lambda) \cdot x \quad \text{for all } x \in \mathbb{W}_S\Omega^*_{X/R} \text{ and } \lambda \in \imath_{S/n,S*}\mathbb{W}_{S/n}\Omega^*_{X/R}.$$

In Section 3 we will study the $\{\phi_n\}_{n\geq 1}$ operations induced on the de Rham-Witt cohomology.

1.2.16. Note that the Hodge to de Rham spectral sequence and the quasi-coherence of $\mathbb{W}_S\Omega^q_{X/R}$ imply the following fact. Assume X is separated and $\mathbb{W}_S(X)$ is a noetherian scheme. Let (U_i) be an open affine covering for X, we denote by $\mathfrak{U} = (\mathbb{W}_S(U_i))$ the induced covering of $\mathbb{W}_S(X)$. Then we can compute $H^i_{dRW}(X/\mathbb{W}_S(R))$ by using the Čech complex for $\mathfrak{U}$:

$$H^i(C(\mathfrak{U}, \mathbb{W}_S\Omega^*_{X/R})) \xrightarrow{\cong} H^i_{dRW}(X/\mathbb{W}_S(R)).$$

In the derived category we have a quasi-isomorphism:

$$C(\mathfrak{U}, \mathbb{W}_S\Omega^*_{X/R}) \xrightarrow{\text{q-iso}} R\Gamma(\mathbb{W}_S\Omega^*_{X/R}).$$

Proposition 1.2.17. *Let R be a flat $\mathbb{Z}$-algebra. Let X be a smooth R-scheme. Let S be a finite truncation set.*

(i) *For all non-negative integers q, $\mathbb{W}_S\Omega^q_{X/R}$ is $\mathbb{Z}$-torsion-free, that is, multiplication by a non-zero integer is injective.*
(ii) *Let d be the relative dimension of X/R. Then $\mathbb{W}_S\Omega^q_{X/R} = 0$ for all $q > d$.*

Proof. For (i) it suffices to prove that $\mathbb{W}_S\Omega^q_{X/R} \otimes \mathbb{Z}_{(p)} = \mathbb{W}_S\Omega^q_{X'/R'}$ is p-torsion-free for all primes p, where $X' = X \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ and $R' = R \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. For (ii) it suffices to show that $\mathbb{W}_S\Omega^q_{X'/R'}$ vanishes.
Via the decomposition 1.2.5 we may suppose that $S = \{1, p, \ldots, p^{n-1}\}$. Certainly we may assume that $X' = \mathrm{Spec}(B)$ and that there exists an étale ring homomorphism $R'[x_1, \ldots, x_d] \to B$. By using (1.2.6) we are reduced to the case $B = R'[x_1, \ldots, x_d]$. The claim follows in this case from the explicit description of the de Rham-Witt complex in [LZ04, §2], more precisely [LZ04, Proposition 2.17]. $\square$

1.3. Finiteness.

Proposition 1.3.1. *Let R be a flat and finitely generated $\mathbb{Z}$-algebra. Let X be a flat and proper scheme of relative dimension d over R. Let S be a finite truncation set. The following hold.*

(i) *For all non-negative integers i, j the cohomology group $H^i(\mathbb{W}_S(X), \mathbb{W}_S\Omega^j_{X/R})$ is a finitely generated $\mathbb{W}_S(R)$-module.*
(ii) *For all $i > d$ and $j \geq 0$, we have $H^i(\mathbb{W}_S(X), \mathbb{W}_S\Omega^j_{X/R}) = 0$.*
(iii) *For all i, the de Rham-Witt cohomology $H^i_{dRW}(X/\mathbb{W}_S(R))$ (Definition 1.2.14) is a finitely generated $\mathbb{W}_S(R)$-module.*
(iv) *Suppose X/R is smooth. Then $H^i_{dRW}(X/\mathbb{W}_S(R)) = 0$ for all $i > 2d$.*

Proof. For (i). We denote by $f : X \to \mathrm{Spec}(R)$ the structure morphism. The scheme $\mathbb{W}_S(X)$ is noetherian, because it is of finite type over $\mathrm{Spec}(\mathbb{Z})$. By [Bor11b, Proposition 16.13] the induced morphism $\mathbb{W}_S(f) : \mathbb{W}_S(X) \to \mathbb{W}_S(R)$ is proper. Moreover, $\mathbb{W}_S\Omega^j_{X/R}$ defines a coherent sheaf on $\mathbb{W}_S(X)$ (see Proposition 1.2.12).

For (ii). The fibers of $\mathbb{W}_S(f)$ at closed points of $\operatorname{Spec}(\mathbb{W}_S(R))$ have dimension d. In fact, as topological spaces they are disjoint unions of the corresponding fibers of f. This implies the claim.
For (iii). Follows from (i) via the Hodge to de Rham spectral sequence.
For (iv). Again this follows from the Hodge to de Rham spectral sequence, statement (ii), and Proposition 1.2.17(ii). □

2. De Rham-Witt cohomology

2.1. Reduction modulo an ideal.

2.1.1. Recall that $W_n = \mathbb{W}_{\{1,p,\ldots,p^{n-1}\}}$ whenever a prime p has been fixed (Notation 1.1.5). The goal of this section is to prove the following theorem.

Theorem 2.1.2. *Let R be a flat $\mathbb{Z}_{(p)}$-algebra, let B be a smooth R-algebra, and let n be a positive integer. Let $I \subset R$ be an ideal such that $p^m \in I$ for some m. Choose a $W_n(R)$-free resolution*

$$T := \ldots \to T^{-2} \to T^{-1} \to T^0$$

of $W_n(R/I)$. There exists a functorial quasi-isomorphism of complexes of $W_n(R)$-modules

$$W_n\Omega^*_{B/R} \otimes_{W_n(R)} T \to W_n\Omega^*_{(B/IB)/(R/I)}. \tag{2.1.1}$$

In particular, we obtain an isomorphism

$$W_n\Omega^*_{B/R} \otimes^{\mathbb{L}}_{W_n(R)} W_n(R/I) \xrightarrow{\cong} W_n\Omega^*_{(B/IB)/(R/I)}, \tag{2.1.2}$$

in the derived category of $W_n(R)$-modules.

More precisely, functoriality means that for any morphism $A \to B$ of smooth R-algebras, the diagram

$$\begin{tikzcd} W_n\Omega^*_{B/R} \otimes_{W_n(R)} T \arrow[r] & W_n\Omega^*_{(B/IB)/(R/I)} \\ W_n\Omega^*_{A/R} \otimes_{W_n(R)} T \arrow[r] \arrow[u] & W_n\Omega^*_{(A/IA)/(R/I)} \arrow[u] \end{tikzcd}$$

is commutative.

Remark 2.1.3. The proof of Theorem 2.1.2 does not go beyond the methods of [LZ04], so that the theorem may be well-known but we couldn't provide a reference.

Proof of Theorem 2.1.2. We define the morphism (2.1.1) by

$$W_n\Omega^*_{B/R} \otimes_{W_n(R)} T \to W_n\Omega^*_{B/R} \otimes_{W_n(R)} W_n(R/I) \to W_n\Omega^*_{(B/I)/(R/I)},$$

so that the functoriality of (2.1.1) is obvious.
1.Step: The first step is the reduction to $B = R[x_1, \ldots, x_d]$. We can use the Čech complex (see 1.2.16) in order to reduce to the case where there exists an étale morphism $A = R[x_1, \ldots, x_d] \to B$.

Note that $p^{nm} = 0$ in $W_n(R/I)$. Since $W_n\Omega^*_{B/R}$ is p-torsion-free (Proposition 1.2.17), we see that

$$(2.1.3)\quad W_n\Omega^*_{B/R} \otimes^{\mathbb{L}}_{W_n(R)} W_n(R/I) \to W_n\Omega^*_{B/R}/p^{nm} \otimes^{\mathbb{L}}_{W_n(R)/p^{nm}} W_n(R/I)$$

is a quasi-isomorphism. Clearly, morphism (2.1.2) factors through (2.1.3). It will be easier to work modulo p^{nm}, because $dF_p^{nm} = p^{nm}F_p^{nm}d$ vanishes modulo p^{nm}.

Set $c = nm + n$, we claim that

(2.1.4)

$$\Big(W_c(B)/p^{nm} \otimes_{W_c(A)/p^{nm}} W_n\Omega^*_{A/R}/p^{nm}, id \otimes d\Big) \to (W_n\Omega^*_{B/R}/p^{nm}, d)$$
$$b \otimes \omega \mapsto F_p^{nm}(b) \cdot \omega,$$

is an isomorphism of complexes. Note that $W_c(A)$ acts on $W_n\Omega^*_{A/\mathbb{Z}}/p^{nm}$ via $W_c(A) \xrightarrow{F_p^{nm}} W_n(A)$, and therefore (2.1.4) is a morphism of complexes. Theorem 1.1.3 implies that

$$W_c(B) \otimes_{W_c(A)} M \xrightarrow{\cong} W_n(B) \otimes_{W_n(A)} M, \quad b \otimes m \mapsto F_p^{nm}(b) \otimes m,$$

is an isomorphism for all $W_n(A)$-modules M. Thus the claim follows from (1.2.6).

On the other hand, Corollary 1.1.4 shows that for every $W_n(A/I)$-module M the map

$$W_c(B)/p^{nm} \otimes_{W_c(A)/p^{nm}} M \to W_n(B/I) \otimes_{W_n(A/I)} M, \quad b \otimes m \mapsto F_p^{nm}(b) \otimes m,$$

is an isomorphism. This yields an isomorphism of complexes

$$\Big(W_c(B)/p^{nm} \otimes_{W_c(A)/p^{nm}} W_n\Omega^*_{(A/IA)/(R/I)}, id \otimes d\Big) \to (W_n\Omega^*_{(B/IB)/(R/I)}, d).$$

Finally, since $W_c(B)/p^{nm}$ is étale over $W_c(A)/p^{nm}$, we are reduced to proving that

$$W_n\Omega^*_{A/R}/p^{nm} \otimes^{\mathbb{L}}_{W_n(R)/p^{nm}} W_n(R/I) \to W_n\Omega^*_{(A/IA)/(R/I)}$$

is a quasi-isomorphism.

2.Step: Proof of the case $B = R[x_1, \dots, x_d]$. In this case it follows from [LZ04, §2] and the proof of [LZ04, Theorem 3.5] that

$$\Omega^*_{W_n(R)[x_1,\dots,x_d]/W_n(R)} \to \Omega^*_{W_n(B)/W_n(R)} \xrightarrow{\pi} W_n\Omega^*_{B/R}$$

is a quasi-isomorphism, where the first morphism is induced by $x_i \mapsto [x_i]$. The same statement holds for R/I, hence the assertion follows from the quasi-isomorphism

$$\Omega^*_{W_n(R)[x_1,\dots,x_d]/W_n(R)} \otimes^{\mathbb{L}}_{W_n(R)} W_n(R/I) \to \Omega^*_{W_n(R/I)[x_1,\dots,x_d]/W_n(R/I)}.$$

$\square$

Corollary 2.1.4. *Let R be a flat and finitely generated $\mathbb{Z}$-algebra, and let $\mathfrak{m} \subset R$ be a maximal ideal; set $p = \mathrm{char}(R/\mathfrak{m})$. Let X be a smooth and proper R-scheme, let n, j be positive integers. There is a natural quasi-isomorphism of complexes of $W_n(R)$-modules:*

$$R\Gamma(W_n\Omega^*_{X/R}) \otimes^{\mathbb{L}}_{W_n(R)} W_n(R/\mathfrak{m}^j) \to R\Gamma(W_n\Omega^*_{X\otimes_R R/\mathfrak{m}^j/(R/\mathfrak{m}^j)}).$$

Proof. The claim follows from Theorem 2.1.2 by using Čech complexes (see 1.2.16). □

2.2. Flatness.

Theorem 2.2.1. *Let R be a smooth $\mathbb{Z}$-algebra. Let X be a smooth and proper R-scheme. Suppose that the de Rham cohomology $H^*_{dR}(X/R)$ of X is a flat R-module. Then $H^*_{dRW}(X/\mathbb{W}_S(R))$ is a finitely generated projective $\mathbb{W}_S(R)$-module for all finite truncation sets S. Moreover, for an inclusion of finite truncation sets $T \subset S$, the induced map*

$$H^*_{dRW}(X/\mathbb{W}_S(R)) \otimes_{\mathbb{W}_S(R)} \mathbb{W}_T(R) \xrightarrow{\cong} H^*_{dRW}(X/\mathbb{W}_T(R)) \tag{2.2.1}$$

is an isomorphism.

Remark 2.2.2. Let R be a smooth $\mathbb{Z}$-algebra. Let X be a smooth and proper R-scheme. We know that $H^*_{dR}(X/R)$ is a coherent R-module and $H^*_{dR}(X/R)\otimes_{\mathbb{Z}}\mathbb{Q}$ is a flat $R\otimes_{\mathbb{Z}}\mathbb{Q}$-module. The first assertion follows from the Hodge to de Rham spectral sequence, the second assertion follows from the existence of a Gauss – Manin connection. Therefore we can find an integer $N > 0$ such that the base change $X \otimes_R R[N^{-1}]$ has $R[N^{-1}]$-flat de Rham cohomology.
If all $H^i(X, \Omega^j_{X/R})$ are flat R-modules then $H^*_{dR}(X/R)$ is a flat R-module, because in this case the Hodge to de Rham spectral sequence degenerates at E_1. Examples include curves or abelian varieties over R.

Since $\mathbb{W}_S(R)$ is a noetherian ring and we know that $H^*_{dRW}(X/\mathbb{W}_S(R))$ is a finitely generated $\mathbb{W}_S(R)$-module (Proposition 1.3.1), it remains to show that it is flat. This is a local property and can be checked after localization at maximal ideals of $\mathbb{W}_S(R)$. Our proof relies on Theorem 2.1.2 or, more precisely, Corollary 2.1.4.

Lemma 2.2.3. *Let R be a finitely generated $\mathbb{Z}$-algebra. Let $\mathfrak{m}$ be a maximal ideal of R, let n be a positive integer, and set $p = \mathrm{char}(R/\mathfrak{m})$. Then $W_n(R_\mathfrak{m}) \to W_n(\varprojlim_i R/\mathfrak{m}^i)$ is faithfully flat.*

Proof. By Lemma 1.1.10, both rings are local. Thus we only need to prove flatness.
We note that $W_n(R)$ is a noetherian ring, because R is a finitely generated $\mathbb{Z}$-algebra. Indeed, let $x_1, \dots, x_d$ be generators for R, we claim that $W_n(R)$ is a finitely generated module over the subring S_n of $W_n(R)$ generated by $[x_1], \dots, [x_d]$. By using induction on n, we only need to show that the ideal $V_p^{n-1}(R)$ is a finitely generated S_n-module. We have

$$[x_i] \cdot V_p^{n-1}(r) = V_p^{n-1}(x_i^{p^{n-1}} \cdot r) \quad \text{for all } r \in R,$$

hence $\{V_p^{n-1}(x_1^{i_1}\cdots x_d^{i_d}) \mid 0\le i_k\le p^{n-1}-1 \text{ for all } k\}$ is a set of generators. Therefore $W_n(R_{\mathfrak{m}})$, being a localization of $W_n(R)$, is a noetherian ring. Obviously, we have the equalities

$$W_n(\varprojlim_i R/\mathfrak{m}^i) = \varprojlim_i W_n(R/\mathfrak{m}^i) = \varprojlim_i W_n(R_{\mathfrak{m}})/W_n(\mathfrak{m}^i R_{\mathfrak{m}}).$$

Moreover, it is easy to check that $(W_n(\mathfrak{m}^i R_{\mathfrak{m}}))_i$ and $(W_n(\mathfrak{m}R_{\mathfrak{m}})^i)_i$ induce the same topology on $W_n(R_{\mathfrak{m}})$. Therefore

$$\varprojlim_i W_n(R_{\mathfrak{m}})/W_n(\mathfrak{m}R_{\mathfrak{m}})^i \xrightarrow{\cong} \varprojlim_i W_n(R_{\mathfrak{m}})/W_n(\mathfrak{m}^i R_{\mathfrak{m}}), \tag{2.2.2}$$

which implies flatness. □

Lemma 2.2.4. *Let R be a finitely generated $\mathbb{Z}$-algebra. Let $\mathfrak{m}$ be a maximal ideal of R, let n be a positive integer, and set $p = \mathrm{char}(R/\mathfrak{m})$. Let C be a bounded complex of $W_n(R_{\mathfrak{m}})$-modules such that $H^i(C)$ is a finitely generated $W_n(R_{\mathfrak{m}})$-module for all i. Then, for all i,*

$$H^i(C)\otimes_{W_n(R_{\mathfrak{m}})} W_n(\varprojlim_j R/\mathfrak{m}^j) \cong \varprojlim_j H^i\left(C\otimes^{\mathbb{L}}_{W_n(R_{\mathfrak{m}})} W_n(R/\mathfrak{m}^j)\right).$$

Proof. Set $\hat{R} := \varprojlim_j R/\mathfrak{m}^j$. The map is induced by $C\to C\otimes^{\mathbb{L}}_{W_n(R_{\mathfrak{m}})} W_n(R/\mathfrak{m}^j)$ and the $W_n(\hat{R})$-module structure on the right hand side.
As a first step we will prove that $H^i\left(C\otimes^{\mathbb{L}}_{W_n(R_{\mathfrak{m}})} W_n(R/\mathfrak{m}^j)\right)$ is a finite group. Clearly, we may assume that $C = C_0$ is concentrated in degree 0. Since C_0 is finitely generated we conclude that $\mathrm{Tor}_i^{W_n(R_{\mathfrak{m}})}(C_0, W_n(R/\mathfrak{m}^j))$ is a finitely generated $W_n(R/\mathfrak{m}^j)$-module for all i. The ring $W_n(R/\mathfrak{m}^j)$ contains only finitely many elements, hence

$$H^{-i}(C\otimes^{\mathbb{L}}_{W_n(R_{\mathfrak{m}})} W_n(R/\mathfrak{m}^j)) = \mathrm{Tor}_i(C_0, W_n(R/\mathfrak{m}^j))$$

is finite.
By using Lemma 2.2.3 and the first step (all $R^1\varprojlim$ vanish) we can reduce the assertion to the case of a complex $C = C_0$ that is concentrated in degree zero (hence C_0 is finitely generated). In this case we need to show:

(a) $C_0\otimes_{W_n(R_{\mathfrak{m}})} W_n(\hat{R}) \xrightarrow{=} \varprojlim_j (C_0\otimes_{W_n(R_{\mathfrak{m}})} W_n(R/\mathfrak{m}^j))$,

(b) $\varprojlim_j \mathrm{Tor}_i(C_0, W_n(R/\mathfrak{m}^j)) = 0$ for all $i>0$.

Claim (a) follows from (2.2.2). Claim (b) follows from (a) and the flatness of $W_n(R_{\mathfrak{m}})\to W_n(\hat{R})$. □

Proposition 2.2.5. *Assumptions as in Corollary 2.1.4. Set $X_j := X\otimes_R R/\mathfrak{m}^j$, $R_j := R/\mathfrak{m}^j$, $\hat{R} = \varprojlim_j R_j$.*

(i) *For all i and n, we have a functorial isomorphism*

$$H^i_{dRW}(X/W_n(R))\otimes_{W_n(R)} W_n(\hat{R}) \xrightarrow{\cong} \varprojlim_j H^i(X_j, W_n\Omega^*_{X_j/R_j}). \tag{2.2.3}$$

(ii) *Suppose furthermore that the following conditions are satisfied:*

(1) *There exists a lifting $\phi : \hat{R} \to \hat{R}$ of the absolute Frobenius on $R/\mathfrak{m}$; let $\rho : \hat{R} \to W_n(\hat{R})$ be the induced ring homomorphism. By abuse of notation we will denote the restriction of ρ to R by ρ again.*
(2) *The de Rham cohomology $H^*_{dR}(X/R)$ is a locally free R-module.*
Then there is an isomorphism

$$H^i(X_j, W_n\Omega^*_{X_j/R_j}) \cong H^i_{dR}(X/R) \otimes_{R,\rho} W_n(R_j)$$

which is natural in the following sense. For all $l > j$ we have a commutative diagram

$$\begin{tikzcd} H^i(X_l, W_n\Omega^*_{X_l/R_l}) \arrow[r, "\cong"] \arrow[d] & H^i_{dR}(X/R) \otimes_{R,\rho} W_n(R_l) \arrow[d, "id\otimes W_n(R_l \to R_j)"] \\ H^i(X_j, W_n\Omega^*_{X_j/R_j}) \arrow[r, "\cong"] & H^i_{dR}(X/R) \otimes_{R,\rho} W_n(R_j). \end{tikzcd}$$

For a morphism of R-schemes $f : X \to Y$, where Y/R satisfies the same assumptions as X, the following diagram is commutative:

$$\begin{tikzcd} H^i(Y_j, W_n\Omega^*_{Y_j/R_j}) \arrow[r, "\cong"] \arrow[d, "f^*"] & H^i_{dR}(Y/R) \otimes_{R,\rho} W_n(R_j) \arrow[d, "f^*\otimes id"] \\ H^i(X_j, W_n\Omega^*_{X_j/R_j}) \arrow[r, "\cong"] & H^i_{dR}(X/R) \otimes_{R,\rho} W_n(R_j). \end{tikzcd}$$

Proof. For (i). Set $C = R\Gamma(W_n\Omega^*_{X/R}) \otimes_{W_n(R)} W_n(R_\mathfrak{m})$. In view of Proposition 1.3.1, the assumptions for Lemma 2.2.4 are satisfied. Applying the lemma and using Corollary 2.1.4 implies the claim.
For (ii). Consider the following cartesian squares

$$\begin{tikzcd} X_j \arrow[r] \arrow[d] & X_{n,j} \arrow[r] \arrow[d] & X \otimes_R \hat{R} \arrow[d] \\ \mathrm{Spec}(R_j) \arrow[r, "\mathrm{gh}_1"] & \mathrm{Spec}(W_n(R_j)) \arrow[r, "\rho"] & \mathrm{Spec}(\hat{R}), \end{tikzcd}$$

where $X_{n,j}$ is by definition the fibre product. Note that $\hat{R} \xrightarrow{\rho} W_n(\hat{R}) \xrightarrow{\mathrm{gh}_1} \hat{R}$ is the identity, which implies that the left hand square is cartesian.
By the comparison theorem [LZ04, Theorem 3.1] we have a functorial isomorphism

$$H^i(X_j, W_n\Omega^*_{X_j/R_j}) \cong H^i_{crys}(X_j/W_n(R_j)).$$

By the comparison isomorphism of crystalline cohomology with de Rham cohomology due to Berthelot-Ogus we get

$$\begin{aligned} H^i_{crys}(X_j/W_n(R_j)) &\cong H^i_{dR}(X_{n,j}/W_n(R_j)) \\ &\cong H^i_{dR}(X/R) \otimes_{R,\rho} W_n(R_j). \end{aligned}$$

For the last isomorphism we have used condition (2) on the de Rham cohomology of X. □

Proof of Theorem 2.2.1. Without loss of generality we may assume that R is integral. It suffices to show the flatness of $H^i_{dRW}(X/\mathbb{W}_S(R))$ when considered as a $\mathbb{W}_S(R)$-module. This can be checked after localizing at maximal ideals. By using Lemma 1.1.9 it suffices to prove that $H^i_{dRW}(X/\mathbb{W}_S(R)) \otimes_{\mathbb{W}_S(R)} \mathbb{W}_S(R_\mathfrak{m})$ is a flat $\mathbb{W}_S(R_\mathfrak{m})$-module for every maximal ideal $\mathfrak{m} \subset R$. Similarly, it is sufficient to prove (2.2.1) after tensoring with $\mathbb{W}_T(R_\mathfrak{m})$.
Let $\mathfrak{m} \subset R$ be a maximal ideal, and set $p = \mathrm{char}(R/\mathfrak{m})$. By using the decomposition of $\mathbb{W}_S\Omega^*_{X/R} \otimes \mathbb{Z}_{(p)}$ from Proposition 1.2.6 together with (1.1.3) we may assume that S is p-typical, say $S = \{1, p, \ldots, p^{n-1}\}$, and hence $T = \{1, p, \ldots, p^{m-1}\}$.
Since R is a smooth $\mathbb{Z}$-algebra, there is a lifting $\phi : \hat{R} \to \hat{R}$ of the absolute Frobenius of $R/\mathfrak{m}$, where $\hat{R} = \varprojlim_j R/\mathfrak{m}^j$. Therefore Proposition 2.2.5 implies

$$\begin{aligned} H^i_{dRW}(X/W_n(R)) \otimes_{W_n(R)} W_n(\hat{R}) &\xrightarrow{\cong} \varprojlim_j H^i(X_j, W_n\Omega^*_{X_j/R_j}) \\ &\xrightarrow{\cong} H^i_{dR}(X/R) \otimes_{R,\rho} W_n(\hat{R}), \end{aligned}$$

and we can prove the flatness by using Lemma 2.2.3.
Tensoring (2.2.1) with $W_m(\hat{R})$ (recall that $T = \{1, p, \ldots, p^{m-1}\}$) and by using Proposition 2.2.5(ii), we see that (2.2.1) $\otimes W_m(\hat{R})$ is induced by the identity on the de Rham cohomology. Hence it is an isomorphism by Lemma 2.2.3. □

3. Poincaré duality

3.1. A rigid $\otimes$-category.

Definition 3.1.1. Let R be a $\mathbb{Z}$-torsion-free ring and Q a non-empty truncation set. We denote by $\mathcal{C}'_{Q,R}$ the category with objects being contravariant functors $S \mapsto M_S$ from finite truncation sets contained in Q to sets, together with

- a $\mathbb{W}_S(R)$-module structure on M_S, for all truncation sets $S \subset Q$, such that the maps $M_S \to M_T$, for $T \subset S$, are morphisms of $\mathbb{W}_S(R)$-modules when M_T is considered as a $\mathbb{W}_S(R)$-module via the projection $\pi_T : \mathbb{W}_S(R) \to \mathbb{W}_T(R)$,
- for all positive integers n and all truncation sets $S \subset Q$, maps

$$\phi_n : M_S \to M_{S/n},$$

  such that
  - $\phi_n \circ \phi_m = \phi_{nm}$ for all n, m,
  - ϕ_n is a morphism of $\mathbb{W}_S(R)$-modules when $M_{S/n}$ is considered as a $\mathbb{W}_S(R)$ module via $F_n : \mathbb{W}_S(R) \to \mathbb{W}_{S/n}(R)$,

– for all truncation sets $T \subset S \subset Q$ the following diagram is commutative:

$$\begin{tikzcd} M_S \arrow[r, "\phi_n"] \arrow[d] & M_{S/n} \arrow[d] \\ M_T \arrow[r, "\phi_n"] & M_{T/n}. \end{tikzcd}$$

The functor $S \mapsto M_S$ is required to satisfy the following properties.

- For all truncation sets $S \subset Q$, the $\mathbb{W}_S(R)$-module M_S is finitely generated and projective.
- For all truncation sets $T \subset S \subset Q$:

$$\mathbb{W}_T(R) \otimes_{\mathbb{W}_S(R)} M_S \to M_T$$

  is an isomorphism.
- There is a positive integer a such that there exist morphisms

$$\beta_n : M_{S/n} \to M_S, \tag{3.1.1}$$

  for all positive integers n and all finite truncation sets $S \subset Q$, satisfying the following properties:
  – β_n is a morphism of $\mathbb{W}_S(R)$-modules when $M_{S/n}$ is considered as a $\mathbb{W}_S(R)$ module via $F_n : \mathbb{W}_S(R) \to \mathbb{W}_{S/n}(R)$,
  – $\beta_n(\lambda \cdot \phi_n(x)) = n^{a-1}V_n(\lambda) \cdot x$, for all $x \in M_S, \lambda \in \mathbb{W}_{S/n}(R)$,
  – $\phi_n \circ \beta_n = n^a$.

Morphisms between two objects in $\mathcal{C}'_{Q,R}$ are morphism of functors that are compatible with the $[S \mapsto \mathbb{W}_S(R)]$-module structure and commute with ϕ_n for all positive integers n. We simply write $\mathcal{C}'_R$ for $\mathcal{C}'_{\mathbb{N}_{>0},R}$.

Remark 3.1.2. Note that the β_n are not part of the datum; we can always change $\beta_n \mapsto n^b\beta_n$ for a non-negative integer b.

For an inclusion of truncation sets $Q \subset Q'$, we have an evident functor

$$\mathcal{C}'_{Q',R} \to \mathcal{C}'_{Q,R}.$$

PROPOSITION 3.1.3. *Let $M \in \mathrm{ob}(\mathcal{C}'_{Q,R})$. Let $S \subset Q$ be a finite truncation set. Fix $a > 0$ and β_n as in 3.1.1.*

(1) *For all positive integers n, m with $(n, m) = 1$ we have*

$$\phi_n \circ \beta_m = \beta_m \circ \phi_n,$$

considered as morphisms $M_{S/m} \to M_{S/n}$.

(2) *For all positive integers n, m we have*

$$\beta_n \circ \beta_m = \beta_{nm},$$

considered as morphisms $M_{S/nm} \to M_S$.

(3) *For all truncation sets $T \subset S$ the following diagram is commutative:*

$$\begin{tikzcd} M_{S/n} \arrow[r, "\beta_n"] \arrow[d] & M_S \arrow[d] \\ M_{T/n} \arrow[r, "\beta_n"] & M_T. \end{tikzcd}$$

Proof. The ring $\mathbb{W}_S(R)$ is $\mathbb{Z}$-torsion-free, because it can be considered via the ghost map as a subring of $\prod_{s \in S} R$, and R is $\mathbb{Z}$-torsion-free by assumption. Since M_S is a flat $\mathbb{W}_S(R)$-module, it is $\mathbb{Z}$-torsion-free.
For (1). Since image$(\phi_m) \supset m^a M_{S/m}$ it is sufficient to prove

$$\phi_n \circ \beta_m \circ \phi_m = \beta_m \circ \phi_n \circ \phi_m.$$

This follows from $\beta_m \circ \phi_m = V_m(1)m^{a-1}$ and $\phi_n \circ \phi_m = \phi_m \circ \phi_n$.
For (2). We may argue as in (1) by composing with $\circ\phi_{nm}$.

$$\beta_n \circ \beta_m \circ \phi_{nm}(x) = \beta_n(V_m(1)m^{a-1}\phi_n(x)) = m^{a-1}n^{a-1}V_{nm}(1)x = \beta_{nm} \circ \phi_{nm}(x).$$

For (3). We may argue as in (1) by composing with $\circ\phi_n$. The computation is straightforward. □

Lemma 3.1.4. *Let $f : M \to N$ be a morphism in $\mathcal{C}'_{Q,R}$, and choose a positive integer a and $\beta_{M,n}$, $\beta_{N,n}$ as in (3.1.1). Then $f_S \circ \beta_{M,n} = \beta_{N,n} \circ f_{S/n}$ for all S, n. In particular, the choice of the β_n in Definition 3.1.1 depends only on the positive integer a.*

Proof. Again, we may use that M_S is $\mathbb{Z}$-torsion-free. Now,

$$\begin{aligned} n^a \beta_n f(x) &= \beta_n(f(n^a x)) = \beta_n(f(\phi_n \beta_n(x))) \\ &= \beta_n \phi_n f(\beta_n(x)) = n^{a-1}V_n(1)f(\beta_n(x)) \\ &= f(n^{a-1}V_n(1)\beta_n(x)) = f(\beta_n \phi_n \beta_n(x)) = n^a f(\beta_n(x)). \end{aligned}$$

□

Proposition 3.1.5 (Tensor products). *For two objects M, N in $\mathcal{C}'_{Q,R}$ we set*

$$(M \otimes N)_S := M_S \otimes_{\mathbb{W}_S(R)} N_S, \quad \phi_n := \phi_{M,n} \otimes \phi_{N,n}.$$

Then $M \otimes N$ defines an object in $\mathcal{C}'_{Q,R}$.

Proof. This is a straightforward calculation. We can take $\beta_{M \otimes N,n} = \beta_{M,n} \otimes \beta_{N,n}$. □

The tensor product equips $\mathcal{C}'_{Q,R}$ with the structure of a $\otimes$-category with identity object $\mathbf{1}$, where

$$\mathbf{1}_S := \mathbb{W}_S(R), \qquad \phi_{\mathbf{1},n} = F_n.$$

Definition 3.1.6. (Tate objects) Let b be a non-negative integer. We define the object $\mathbf{1}(-b)$ in $\mathcal{C}'_{Q,R}$ by

$$\mathbf{1}(-b)_S := \mathbb{W}_S(R), \qquad \phi_{\mathbf{1}(-b),n} = n^b F_n.$$

For an object M in $\mathcal{C}'_{Q,R}$, M_S is $\mathbb{Z}$-torsion-free, hence we get an isomorphism:

$$\mathrm{Hom}_{\mathcal{C}'_{Q,R}}(M,N) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}'_{Q,R}}(M\otimes \mathbf{1}(-b), N\otimes \mathbf{1}(-b))$$

Definition 3.1.7. We denote by $\mathcal{C}_{Q,R}$ the category with objects $M(b)$, where M is an object in $\mathcal{C}'_{Q,R}$ and $b\in\mathbb{Z}$. As morphisms we set

$$\mathrm{Hom}_{\mathcal{C}_{Q,R}}(M(b_1),N(b_2)) = \mathrm{Hom}_{\mathcal{C}'_{Q,R}}(M\otimes \mathbf{1}(b_1-c), N\otimes \mathbf{1}(b_2-c)),$$

where $c\in\mathbb{Z}$ is such that $b_1-c, b_2-c\leq 0$.

For two truncation sets $Q\subset Q'$, we have an obvious functor

$$\mathcal{C}_{Q',R}\to\mathcal{C}_{Q,R}.$$

The category $\mathcal{C}_{Q,R}$ is additive and via $M\mapsto M(0)$ the category $\mathcal{C}'_{Q,R}$ is a full subcategory of $\mathcal{C}_{Q,R}$. For $M\in\mathcal{C}'_{Q,R}$, we have $M(-b)=M\otimes\mathbf{1}(-b)$ if b is non-negative. For an integer b, the functor

$$\mathcal{C}_{Q,R}\to\mathcal{C}_{Q,R},\quad M(n)\mapsto M(n+b)$$

is an equivalence and has $M(n)\mapsto M(n-b)$ as inverse functor.
For $M(b_1),N(b_2)$ in $\mathcal{C}_{Q,R}$ we set

$$M(b_1)\otimes N(b_2) := (M\otimes N)(b_1+b_2).$$

The tensor product equips $\mathcal{C}_{Q,R}$ with the structure of a $\otimes$-category with identity object $\mathbf{1}$.

3.1.8. *Internal Hom.* The reason for introducing the new category $\mathcal{C}_{Q,R}$ is the internal Hom construction.
Let M,N be two objects in $\mathcal{C}'_{Q,R}$, fix positive integers a_M,a_N and $\beta_{n,M},\beta_{n,N}$ as in (3.1.1). In a first step we are going to define an object $\underline{\mathrm{Hom}}'(M,N)$ in $\mathcal{C}'_{Q,R}$ that depends on the choice of a_M. We set

$$\underline{\mathrm{Hom}}'(M,N)_S := \mathrm{Hom}_{\mathbb{W}_S(R)}(M_S,N_S).$$

We note that

$$\mathrm{Hom}_{\mathbb{W}_S(R)}(M_S,N_S)\otimes_{\mathbb{W}_S(R)}\mathbb{W}_T(R)\xrightarrow{\cong}\mathrm{Hom}_{\mathbb{W}_T(R)}(M_T,N_T),$$

since M_S is finitely generated and projective. We define

$$\phi_n : \mathrm{Hom}_{\mathbb{W}_S(R)}(M_S,N_S)\to\mathrm{Hom}_{\mathbb{W}_{S/n}(R)}(M_{S/n},N_{S/n})$$
$$\phi_n(f) := \phi_n\circ f\circ\beta_n.$$

This definition depends on a_M. It is easy to check that $\underline{\mathrm{Hom}}'(M,N)$ is an object in $\mathcal{C}'_{Q,R}$ (take $\beta_n(f):=\beta_n\circ f\circ\phi_n$ and $a=a_M+a_N$). We set

$$\underline{\mathrm{Hom}}(M,N) := \underline{\mathrm{Hom}}'(M,N)(a_M) \tag{3.1.2}$$

as an object in $\mathcal{C}_{Q,R}$. In view of Lemma 3.1.4 this definition is independent of any choices. For two objects $M(b_1),N(b_2)$ in $\mathcal{C}_{Q,R}$ we set

$$\underline{\mathrm{Hom}}(M(b_1),N(b_2)) := \underline{\mathrm{Hom}}(M,N)(b_2-b_1).$$

3.1.9. For three objects M, N, P in $\mathcal{C}_{Q,R}$ we have an obvious natural isomorphism

$$\underline{\mathrm{Hom}}(M \otimes N, P) = \underline{\mathrm{Hom}}(M, \underline{\mathrm{Hom}}(N, P)).$$

Proposition 3.1.10. *For objects M, N in $\mathcal{C}_{Q,R}$ we have a natural isomorphism*

$$\mathrm{Hom}(\mathbf{1}, \underline{\mathrm{Hom}}(M, N)) \to \mathrm{Hom}(M, N).$$

Proof. We may assume that $M, N \in \mathcal{C}'_{Q,R}$. Fix a_M and $\beta_{M,n}$ as in (3.1.1). We need to show that

$$\mathrm{Hom}(\mathbf{1}(-a_M), \underline{\mathrm{Hom}}'(M, N)) = \mathrm{Hom}(M, N),$$

and know that

$$\begin{aligned} &\mathrm{Hom}(\mathbf{1}(-a_M), \underline{\mathrm{Hom}}'(M, N)) = \\ &\quad = \{[S \mapsto f_S] \mid f_S \otimes_{\mathbb{W}_S(R)} \mathbb{W}_T(R) = f_T \text{ for } T \subset S \subset Q, \\ &\qquad\qquad \phi_{N,n} \circ f_S \circ \beta_{M,n} = n^{a_M} f_{S/n} \quad \text{for all } n, S \subset Q.\} \end{aligned}$$

Since $\phi_{M,n}(M_S) \supset n^{a_M} M_{S/n}$ we have

$$\begin{aligned} \phi_{N,n} \circ f_S \circ \beta_{M,n} = n^{a_M} f_{S/n} &\Leftrightarrow \phi_{N,n} \circ f_S \circ \beta_{M,n} \circ \phi_{M,n} = n^{a_M} f_{S/n} \circ \phi_{M,n} \\ &\Leftrightarrow \phi_{N,n} \circ f_S \circ n^{a_M - 1} V_n(1) = n^{a_M} f_{S/n} \circ \phi_{M,n} \\ &\Leftrightarrow n^{a_M} \phi_{N,n} \circ f_S = n^{a_M} f_{S/n} \circ \phi_{M,n} \\ &\Leftrightarrow \phi_{N,n} \circ f_S = f_{S/n} \circ \phi_{M,n}. \end{aligned}$$

□

For $M \in \mathcal{C}_{Q,R}$ we define the dual by

$$M^\vee := \underline{\mathrm{Hom}}(M, \mathbf{1}).$$

It equips $\mathcal{C}_{Q,R}$ with the structure of rigid $\otimes$-category. We have

$$M^\vee \otimes N = \underline{\mathrm{Hom}}(M, N).$$

3.1.11. *Functoriality.*

Proposition 3.1.12. *Let $R \to A$ be a ring homomorphism between $\mathbb{Z}$-torsion-free rings. The assignment*

$$\begin{aligned} &[S \mapsto M_S] \mapsto [S \mapsto M_S \otimes_{\mathbb{W}_S(R)} \mathbb{W}_S(A)], \quad [n \mapsto \phi_n] \mapsto [n \mapsto \phi_n \otimes F_n], \\ &[S \mapsto f_S] \mapsto [S \mapsto f_S \otimes id_{\mathbb{W}_S(A)}] \end{aligned}$$

defines a functor

$$\mathcal{C}'_{Q,R} \to \mathcal{C}'_{Q,A}.$$

The functor can be extended in the obvious way to a functor $\mathcal{C}_{Q,R} \to \mathcal{C}_{Q,A}$.

Proof. Straightforward. □

3.1.13. Our motivation for introducing $\mathcal{C}_{Q,R}$ comes from geometry.

Proposition 3.1.14. *Assumptions as in Theorem 2.2.1. Let Q be a non-empty truncation set. For all $i \geq 0$ the assignment*

$$S \mapsto H^i_{dRW}(X/\mathbb{W}_S(R)), \quad n \mapsto \phi_n,$$

defines an object in $\mathcal{C}'_{Q,R}$.

Proof. Theorem 2.2.1 implies that these modules are projective and finitely generated. For the construction of ϕ_n and β_n see Section 1.2.15. □

Definition 3.1.15. Let $X \to \operatorname{Spec}(R)$ be a morphism such that the assumptions of Theorem 2.2.1 are satisfied. For all i, we denote by $H^i_{dRW}(X/\mathbb{W}(R))$ the object in $\mathcal{C}_R$ that is given by $S \mapsto H^i_{dRW}(X/\mathbb{W}_S(R))$ (Proposition 3.1.14). We call $H^*_{dRW}(X/\mathbb{W}(R))$ the *de Rham-Witt cohomology* of X.

3.1.16. Let X, Y be smooth proper schemes over R such that the assumptions of Theorem 2.2.1 are satisfied for X and Y. The multiplication

$$R\Gamma(\mathbb{W}_S\Omega^*_{X/R}) \times R\Gamma(\mathbb{W}_S\Omega^*_{Y/R}) \to R\Gamma(\mathbb{W}_S\Omega^*_{X\times_R Y/R})$$

induces a morphism in $\mathcal{C}_R$:

$$H^i_{dRW}(X/\mathbb{W}(R)) \otimes H^j_{dRW}(Y/\mathbb{W}(R)) \to H^{i+j}_{dRW}(X \times_R Y/\mathbb{W}(R)). \tag{3.1.3}$$

3.2. The tangent space functor. We have a functor of rigid $\otimes$-categories

$$T : \mathcal{C}_{Q,R} \to (\text{finitely generated and projective } R\text{-modules})$$
$$T(M(n)) := M_{\{1\}}.$$

Proposition 3.2.1. *The functor T is conservative, i.e. if $T(f)$ is an isomorphism then f is an isomorphism.*

Proof. It is sufficient to consider a morphism $f : M \to N$ in $\mathcal{C}'_{Q,R}$. We need to show that $f_S : M_S \to N_S$ is an isomorphism provided that $f_{\{1\}}$ is an isomorphism. We may choose a positive integer a and $\beta_{M,n}$, $\beta_{N,n}$ as in (3.1.1). By Lemma 3.1.4 the morphism f commutes with β_n.
Let $n := \max\{s \mid s \in S\}$; by induction we know that f_T is an isomorphism for $T = S\backslash\{n\}$. Set $I = \ker(\mathbb{W}_S(R) \to \mathbb{W}_T(R))$, we know that $I = \{V_n(\lambda) \mid \lambda \in R\}$. It suffices to show that

$$IM_S \xrightarrow{f_S} IN_S \tag{3.2.1}$$

is an isomorphism. If $f_S(V_n(\lambda)x) = 0$ then $n^{a-1}V_n(\lambda)f_S(x) = 0$ and therefore $\beta_n(\lambda\cdot\phi_n f_S(x)) = \beta_n(\lambda\cdot f_{\{1\}}(\phi_n(x)))$ vanishes. Since β_n is injective, we conclude $\lambda \cdot \phi_n(x) = 0$, hence

$$0 = \beta_n(\lambda \cdot \phi_n(x)) = n^{a-1}V_n(\lambda)x,$$

which implies $V_n(\lambda)x = 0$.

For the surjectivity of (3.2.1) we note that, by induction, for every $y \in N_S$ there is $x \in M_S$ with $f_S(x) - y \in IN_S$. Therefore it suffices to show that $I^a N_S$ is contained in the image of f_S. Now,

$$V_n(\lambda_1) \cdots V_n(\lambda_a) = n^{a-1} V_n(\lambda_1 \cdots \lambda_a).$$

Thus

$$V_n(\lambda_1) \cdots V_n(\lambda_a) y = f_S(\beta_n f_{\{1\}}^{-1}(\lambda_1 \cdots \lambda_a \cdot \phi_n(y))).$$

□

Corollary 3.2.2. *Let X, Y be smooth proper schemes over R such that the assumptions of Theorem 2.2.1 are satisfied for X and Y. If*

$$\bigoplus_{i+j=n} H^i_{dR}(Y/R) \otimes_R H^j_{dR}(X/R) \to H^n_{dR}(X \times_R Y/R)$$

is an isomorphism then

$$\bigoplus_{i+j=n} H^i_{dRW}(X/\mathbb{W}(R)) \otimes H^j_{dRW}(Y/\mathbb{W}(R)) \to H^n_{dRW}(X \times_R Y/\mathbb{W}(R))$$

(see (3.1.3)) is an isomorphism in $\mathcal{C}_R$.

Proof. This is an application of Proposition 3.2.1, because

$$T(H^i_{dRW}(-/\mathbb{W}(R))) = H^i_{dR}(-/R).$$

□

Proposition 3.2.3. *The functor T is faithful.*

Proof. It is sufficient to consider a morphism $f : M \to N$ in $\mathcal{C}'_{Q,R}$. We need to show that $f_S : M_S \to N_S$ vanishes provided that $f_{\{1\}}$ is zero. We may choose a positive integer a and $\beta_{M,n}$, $\beta_{N,n}$ as in (3.1.1). By Lemma 3.1.4 the morphism f commutes with β_n.
Let $n := \max(S)$; by induction we know that $f_T = 0$ for $T = S \backslash \{n\}$, so that for all $x \in M_S$ the image $f_S(x)$ is of the form $f_S(x) = V_n(\lambda)y$. Since

$$0 = f_{\{1\}} \circ \phi_n(x) = \phi_n \circ f_S(x) = n \cdot \lambda \cdot \phi_n(y),$$

we conclude $\lambda \cdot \phi_n(y) = 0$ and $n^{a-1} V_n(\lambda) y = 0$, hence $f_S(x) = 0$. □

3.2.4. The following proposition shows that an object in $\mathcal{C}'_{Q,R}$, where R is a $\mathbb{Z}_{(p)}$-algebra, is determined by the p-typical part, that is, on its values for truncation sets consisting of p-powers. Recall the notation S_p from Notation 1.1.7.

Proposition 3.2.5. *Let R a $\mathbb{Z}$-torsion-free ring. Let Q be a truncation set. Suppose p is a prime such that $\ell^{-1} \in R$ for all primes $\ell \in Q \backslash \{p\}$. Let M, N be $\mathcal{C}'_{Q,R}$-modules.*

(1) *Via the equivalence of* (1.1.3)*:*

$$M_S \mapsto \bigoplus_{n \in S, (n,p)=1} M_{(S/n)_p}.$$

(2) *If* $f : M \to N$ *is a morphism in* $\mathcal{C}'_{Q,R}$ *then* $f_S \mapsto \bigoplus_{n\in S,(n,p)=1} f_{(S/n)_p}$ *via the equivalence* (1.1.3).

(3) *The restriction functor*

$$\mathcal{C}'_{Q,R} \to \mathcal{C}'_{Q_p,R}$$

is an equivalence of categories.

Proof. For (1). First one proves that the projection $\epsilon_1 M_S \to M_{S_p}$ is an isomorphism (see Notation 1.1.7 for S_p). The second step is the isomorphism

$$\phi_n : \epsilon_n M_S \to \epsilon_1 M_{S/n},$$

with $\frac{\epsilon_n}{n^a}\beta_n$ as inverse.
Statement (2) is obvious, and (3) follows from (1) and (2). □

Proposition 3.2.6. *Let R a $\mathbb{Z}$-torsion-free ring. Let Q be a truncation set. Suppose $p^{-1} \in R$ for all primes $p \in Q$. Then*

$$T : \mathcal{C}'_{Q,R} \to \text{(finitely generated and projective } R\text{-modules)}$$

defines an equivalence of categories.

Proof. Straightforward. □

3.2.7. Let P be a set of primes (maybe infinite). We set $\mathbb{Z}_P := \mathbb{Z}[p^{-1} \mid p \in P]$. Let A be a commutative ring. We denote by Mod_A the category of A-modules. We define the category $\mathrm{Mod}_{A,P}$ to be the category with objects

$$((M_p)_{p\in P}, (\alpha_{p,\ell})_{p,\ell\in P}),$$

where M_p is an $A\otimes_{\mathbb{Z}}\mathbb{Z}_{P\setminus\{p\}}$-module, and $\alpha_{p,\ell} : M_\ell \otimes_{\mathbb{Z}_{P\setminus\{\ell\}}} \mathbb{Z}_P \to M_p \otimes_{\mathbb{Z}_{P\setminus\{p\}}} \mathbb{Z}_P$ is an isomorphism of $A \otimes \mathbb{Z}_P$-modules such that

$$\alpha_{p_1,p_1} = id, \quad \alpha_{p_1,p_2} \circ \alpha_{p_2,p_3} = \alpha_{p_1,p_3} \quad \text{for all } p_1, p_2, p_3 \in P.$$

The morphisms of $\mathrm{Mod}_{A,P}$ are defined in the evident way.
If P is finite and non-empty, then the evident functor

$$R_P : \mathrm{Mod}_A \to \mathrm{Mod}_{A,P}$$

is an equivalence of categories, because we can glue quasi-coherent sheaves. If P is infinite then this may fail to be an equivalence, but we still have the following properties, whose proof is left to the reader.

Lemma 3.2.8. *Suppose $P \neq \emptyset$.*

(i) *R_P is faithful.*

(ii) *For every $N \in \mathrm{Mod}_A$ such that $N \to N \otimes_{\mathbb{Z}} \mathbb{Z}_P$ is injective, and every $M \in \mathrm{Mod}_A$ the following map is an isomorphism:*

$$\mathrm{Hom}_{\mathrm{Mod}_A}(M, N) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Mod}_{A,P}}(R_P(M), R_P(N)).$$

(iii) *Suppose that $A \to A \otimes \mathbb{Z}_P$ is injective. Let $\tilde{M} = ((\tilde{M}_p), (\alpha_{p,\ell})) \in \mathrm{Mod}_{A,P}$ be such that $\tilde{M}_p$ is a finitely generated and projective $A \otimes_{\mathbb{Z}} \mathbb{Z}_{P\setminus\{p\}}$-module for all $p \in P$. Then there exists a finitely generated and projective $M \in \mathrm{Mod}_A$ such that $R_P(M) \cong \tilde{M}$.*

For a positive integer a, we denote by $\mathcal{C}'_{Q,R,a}$ the full subcategory of $\mathcal{C}'_{Q,R}$ consisting of objects such that there exist $\{\beta_n\}_n$ as in (3.1.1) for a.

Definition 3.2.9. Let Q be a non-empty truncation set, and let P be the set of primes of Q. We denote by $\mathcal{LC}'_{Q,R,a}$ the category with objects

$$((M_p)_{p\in P}, (\alpha_{p,\ell})_{p,\ell\in P}),$$

where

- $M_p \in \mathrm{ob}(\mathcal{C}'_{Q_p,R\otimes\mathbb{Z}_{P\setminus\{p\}},a})$ for all $p\in P$,
- $\alpha_{p,\ell} : T(M_\ell)\otimes_{\mathbb{Z}_{P\setminus\{\ell\}}}\mathbb{Z}_P \xrightarrow{\cong} T(M_p)\otimes_{\mathbb{Z}_{P\setminus\{p\}}}\mathbb{Z}_P$ is an isomorphism such that

$$\alpha_{p_1,p_1}=id,\quad \alpha_{p_1,p_2}\circ\alpha_{p_2,p_3}=\alpha_{p_1,p_3}\quad \text{for all } p_1,p_2,p_3\in P.$$

The morphisms are defined in the evident way.

Broadly speaking the next proposition shows that the category $\mathcal{C}'_{Q,R,a}$ is glued from the local components via the functor T.

Proposition 3.2.10. *Let R be a $\mathbb{Z}$-torsion-free ring, and let a be a positive integer. For every non-empty truncation set Q the evident functor*

$$\mathcal{C}'_{Q,R,a}\to\mathcal{LC}'_{Q,R,a}$$

is an equivalence of categories.

Proof. The claim follows easily from Proposition 3.2.5, Proposition 3.2.6, and Lemma 3.2.8. □

3.3. Proof of Poincaré duality.

3.3.1. Let $f : X\to \mathrm{Spec}(R)$ be a smooth, projective morphism of relative dimension d between noetherian schemes such that $H^*_{dR}(X/R)$ is a flat R-module. Suppose furthermore that $\mathrm{Spec}(R)$ is integral and the field of fractions of R has characteristic zero.
We know that $H^0(X,\mathcal{O}_X)$ is a finite étale R-algebra and

$$H^0_{dR}(X/R)=H^0(X,\mathcal{O}_X).$$

Since $H^*_{dR}(X/R)$ is flat, we have

$$H^i_{dR}(X/R)\otimes_R k(y)\xrightarrow{\cong} H^i_{dR}(X_y/k(y)),$$

for every point $y\in\mathrm{Spec}(R)$, and X_y being the fibre of y. In particular, we obtain

$$H^0(X,\mathcal{O}_X)\otimes_R k(y)\xrightarrow{\cong} H^0(X_y,\mathcal{O}_{X_y}). \tag{3.3.1}$$

By Grothendieck-Serre duality we see that $y\mapsto \dim_{k(y)} H^d(X_y,\omega_{X_y})$ is a constant function, thus $H^d(X,\omega_{X/R})$ is a finitely generated projective R-module and we have

$$H^d(X,\omega_{X/R})\otimes_R k(y)\xrightarrow{\cong} H^d(X_y,\omega_{X_y})$$

for every point $y \in \operatorname{Spec}(R)$. Since the Hodge to de Rham spectral sequence degenerates at the generic point of $\operatorname{Spec}(R)$, we conclude:

$$H^d(X, \omega_{X/R}) \xrightarrow{\cong} H^{2d}_{dR}(X/R).$$

Recall that we have a trace map

$$\mathrm{Tr} : H^d(X, \omega_{X/R}) \to R;$$

we will also denote by Tr the induced map $H^{2d}_{dR}(X/R) \to R$. The duality pairing

$$H^0(X, \mathcal{O}_X) \times H^d(X, \omega_{X/R}) \to R$$

induces a duality pairing

$$H^0_{dR}(X/R) \times H^{2d}_{dR}(X/R) \to R.$$

Note that the fibres of f are connected if $H^0(X, \mathcal{O}_X) = R$. Moreover, the equality $H^0(X, \mathcal{O}_X) = R$ implies that the fibres are geometrically connected by using (3.3.1).

Suppose now that $H^0(X, \mathcal{O}_X) = R$, and set $c_X := \mathrm{Tr}^{-1}(1) \in H^d(X, \omega_{X/R}) = H^{2d}_{dR}(X/R)$. For a generically finite R-morphism $g : X \to Y$, where Y satisfies the same assumptions as X (in particular, Y/R is of relative dimension d), we have a pull-back map

$$g^* : H^{2d}_{dR}(Y/R) = H^d(Y, \omega_{Y/R}) \to H^d(X, \omega_{X/R}) = H^{2d}_{dR}(X/R)$$

which is dual to the trace map

$$g_* : H^0(X, \mathcal{O}_X) \to H^0(Y, \mathcal{O}_Y), \quad g_*(1) = \deg(g),$$

thus $g^*(c_Y) = \deg(g) \cdot c_X$.

Proposition 3.3.2. *Let R be a smooth $\mathbb{Z}$-algebra. Let X be a smooth projective scheme over R such that $H^*_{dR}(X/R)$ is a projective R-module. Suppose that X is connected of relative dimension d. There is an isomorphism*

$$H^{2d}_{dRW}(X/\mathbb{W}(R)) \cong H^0_{dRW}(X/\mathbb{W}(R)) \otimes \mathbf{1}(-d)$$

and a natural morphism in $\mathcal{C}_R$:

$$H^{2d}_{dRW}(X/\mathbb{W}(R)) \to \mathbf{1}(-d)$$

Proof. Certainly, we may suppose that $\operatorname{Spec}(R)$ is integral.

1.Step: Reduction to X/R has geometrically connected fibres.

Set $L = H^0(X, \mathcal{O}_X)$, L is a finite étale R-algebra. It suffices to show the existence of an isomorphism

$$H^{2d}_{dRW}(X/\mathbb{W}(L)) \xrightarrow{\tau} \mathbf{1}(-d) \tag{3.3.2}$$

in $\mathcal{C}_L$ such that $\tau_{\{1\}}$ is the trace map. In view of

$$H^{2d}_{dRW}(X/\mathbb{W}(L)) = H^{2d}_{dRW}(X/\mathbb{W}(R)),$$

(3.3.2) yields in $\mathcal{C}_R$:

$$H^{2d}_{dRW}(X/\mathbb{W}(R)) \xrightarrow{\cong} H^0_{dRW}(X/\mathbb{W}(R)) \otimes \mathbf{1}(-d) \xrightarrow{tr \otimes id} \mathbf{1}(-d), \tag{3.3.3}$$

with $tr : H^0_{dRW}(X/\mathbb{W}(R)) \to \mathbf{1}$ being defined by the usual trace map

$$H^0_{dRW}(X/\mathbb{W}_S(R)) = \mathbb{W}_S(L) \to \mathbb{W}_S(R).$$

The morphism (3.3.3) is functorial because it induces the usual trace map after evaluation at $\{1\}$. Therefore we may assume $R = L$ in the following.
2.Step: Proposition 3.2.6 implies the existence of a unique isomorphism

$$e : \mathbf{1}(-d) \otimes \mathbb{Q} \xrightarrow{\cong} H^{2d}_{dRW}(X/\mathbb{W}(R)) \otimes \mathbb{Q}$$

that induces Tr^{-1} after evaluation at $\{1\}$. In other words, there is a unique system $(e_S)_S$ with $e_S \in H^{2d}_{dRW}(X/\mathbb{W}_S(R)) \otimes \mathbb{Q}$ such that

(1) $\pi_{S,T}(e_S) = e_T$ for all $T \subset S$, where $\pi_{S,T}$ is induced by the projection
$$H^{2d}_{dRW}(X/\mathbb{W}_S(R)) \to H^{2d}_{dRW}(X/\mathbb{W}_T(R)),$$
(2) $\phi_n(e_S) = n^d \cdot e_{S/n}$ for all n, S,
(3) $e_{\{1\}} = \mathrm{Tr}^{-1}(1)$.

Our goal is to show

$$e_S \in H^{2d}_{dRW}(X/\mathbb{W}_S(R)) \tag{3.3.4}$$

for every finite truncation set S. The strategy of the proof will be to show this for $X = \mathbb{P}^d_R$ first. The next step will be to prove that (3.3.4) is local in $\mathrm{Spec}(R)$. Locally on $\mathrm{Spec}(R)$ we can find generically finite morphisms to $\mathbb{P}^d$, which can be used together with the explicit description of de Rham-Witt cohomology after completion (Proposition 2.2.5) to prove the claim.
3.Step: Suppose $X = \mathbb{P}^d_R$. For any finite S, we get a morphism of $\mathbb{W}_S(R)$-schemes

$$g_S : \mathbb{W}_S(\mathbb{P}^d_R) \to \mathbb{P}^d_{\mathbb{W}_S(R)}$$

induced by $\frac{x_i}{x_j} \mapsto [\frac{x_i}{x_j}]$ on the standard affine covering. The morphisms g_S are compatible with the Frobenius morphisms provided that the action on $\mathbb{P}^d_{\mathbb{W}_S(R)}$ is given by $\phi_n^*(x_i) = x_i^n$.
We obtain

$$\begin{aligned} g^* : H^d(\mathbb{P}^d_{\mathbb{W}_S(R)}, \omega_{\mathbb{P}^d_{\mathbb{W}_S(R)}/\mathbb{W}_S(R)}) &\to H^d(\mathbb{W}_S(\mathbb{P}^d_R), \Omega^d_{\mathbb{W}_S(\mathbb{P}^d_R)/\mathbb{W}_S(R)}) \\ &\to H^d(\mathbb{W}_S(\mathbb{P}^d_R), \mathbb{W}_S\Omega^d_{\mathbb{P}^d_R/R}) \to H^{2d}_{dRW}(\mathbb{P}^d_R/\mathbb{W}_S(R)). \end{aligned}$$

Note that $\mathrm{Tr} : H^d(\mathbb{P}^d_{\mathbb{W}_S(R)}, \omega_{\mathbb{P}^d_{\mathbb{W}_S(R)}/\mathbb{W}_S(R)}) \xrightarrow{\cong} \mathbb{W}_S(R)$ and $\delta_S := \mathrm{Tr}^{-1}(1)$ satisfies $\phi_n^*(\delta_S) = n^d \cdot \delta_{S/n}$. Therefore $e_S = g^*(\delta_S)$, which proves (3.3.4) in the case of a projective space.
4.Step: We claim that in order to prove (3.3.4) it is sufficient to prove

$$e_S \in H^{2d}_{dRW}(X/\mathbb{W}_S(R)) \otimes_{\mathbb{W}_S(R)} \mathbb{W}_S(R_{\mathfrak{m}}) \tag{3.3.5}$$

for every maximal ideal $\mathfrak{m}$. Indeed, let $\mathscr{F}$ be the coherent sheaf on $\mathrm{Spec}(\mathbb{W}_S(R))$ associated to $M := H^{2d}_{dRW}(X/\mathbb{W}_S(R))$. For every $\mathfrak{m}$, we can choose an open affine neighborhood $U_{\mathfrak{m}} \subset \mathrm{Spec}(R)$ and a section $e_{\mathfrak{m}} \in \mathscr{F}(\mathbb{W}_S(U_{\mathfrak{m}}))$ mapping to $e_S \in M \otimes_{\mathbb{W}_S(R)} \mathbb{W}_S(R_\mathfrak{m})$. The section $e_\mathfrak{m}$ is unique and the sections $(e_\mathfrak{m})_\mathfrak{m}$ glue to a section of $\mathscr{F}$ on $\bigcup_\mathfrak{m} \mathbb{W}_S(U_\mathfrak{m}) = \mathbb{W}_S(\mathrm{Spec}(R))$, which proves the claim. Let $\hat{R}$ be the completion $\varprojlim_j R/\mathfrak{m}^j$. For every integer n, we have

$$n\hat{R} \cap R_\mathfrak{m} = \bigcap_{j=1}^{\infty} (nR_\mathfrak{m} + \mathfrak{m}^j) = nR_\mathfrak{m},$$

and thus $(R_\mathfrak{m} \otimes_\mathbb{Z} \mathbb{Q}) \cap \hat{R} = R_\mathfrak{m}$ as intersection in $\hat{R} \otimes_\mathbb{Z} \mathbb{Q}$. Therefore

$$e_S \in H^{2d}_{dRW}(X/\mathbb{W}_S(R)) \otimes_{\mathbb{W}_S(R)} \mathbb{W}_S(\hat{R}) \tag{3.3.6}$$

implies (3.3.5).

5.Step: We will show (3.3.6). We may pass from $\mathrm{Spec}(R)$ to a neighborhood $\mathrm{Spec}(R')$ of $\mathfrak{m}$. Let R' be such that there exists a generically finite R'-morphism

$$f : X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R') \to \mathbb{P}^d_{R'}.$$

The existence is proved in Proposition 3.3.3 below. Then $e_S = \frac{1}{\deg(f)} f^*(e_S)$, because the classes $(\frac{1}{\deg(f)} f^*(e_S))_S$ satisfy the properties listed in the second step.

Set $p = \mathrm{char}(R/\mathfrak{m})$. To prove (3.3.6) we may assume that $S = \{1, p, \dots, p^{n-1}\}$. Then Proposition 2.2.5(ii) yields the claim, because for the de Rham cohomology we know that $f^*(\mathrm{Tr}^{-1}(1))$ is divisible by $\deg(f)$. □

Proposition 3.3.3. *Let Y be of finite type over* $\mathrm{Spec}(\mathbb{Z})$. *Let X/Y be smooth projective such that X is connected of relative dimension d. For every closed point $y \in Y$ there is open neighborhood U of y, and a generically finite U-morphism $X \times_Y U \to \mathbb{P}^d_U$.*

In order to prove Proposition 3.3.3 we will need a sequence of lemmas.

Lemma 3.3.4. *Let R be a local noetherian ring. Let X/R be a smooth projective R-scheme such that every connected component of X has relative dimension $d \geq 0$ over* $\mathrm{Spec}(R)$. *Let $\mathscr{L}$ be a relative ample line bundle. There is $n > 0$ satisfying the following property: for every $k \geq 1$ there is a section $s \in H^0(X, \mathscr{L}^{\otimes kn})$ such that $V(s)$ is smooth of relative dimension $d-1$ over R.*

Proof. Let $y \in \mathrm{Spec}(R)$ denote the closed point. For $n \gg 0$ we have $H^i(X_y, \mathscr{L}^{\otimes n}_{|X_y}) = 0$ for all $i > 0$. By semicontinuity we get $H^i(X, \mathscr{L}^{\otimes n}) = 0$ for all $i > 0$, and

$$H^0(X, \mathscr{L}^{\otimes n}) \to H^0(X_y, \mathscr{L}^{\otimes n}_{|X_y})$$

is surjective. Replace $\mathscr{L}$ by a power such that this holds for all $n \geq 1$.

If the residue field of R is infinite then we can find a section $s_y \in H^0(X_y, \mathscr{L}^{\otimes n}_{|X_y})$ such that $V(s_y)$ is smooth of dimension $d-1$. In the case of a finite residue field we have to use [Poo04] and may have to replace $\mathscr{L}$ by a high enough power again.

Let s be a lifting of s_y to $H^0(X, \mathscr{L}^{\otimes n})$, set $H := V(s)$. If $d = 0$ then H is empty, because it has empty intersection with the special fibre. For $d \geq 1$, H is flat by the local criterion for flatness, because it has transversal intersection with the special fibre. Since $H \to \mathrm{Spec}(R)$ is flat and the special fibre is smooth, we conclude that H is smooth. By Chevalley's theorem, H is of relative dimension $d-1$. $\square$

Remark 3.3.5. H is empty if and only if $d = 0$.

Lemma 3.3.6. *Assumptions as in Lemma 3.3.4. There is $n \geq 1$ and sections $s_0, \ldots, s_d \in H^0(X, \mathscr{L}^{\otimes n})$ such that*

(1) $\bigcap_{i=0}^d V(s_i)$ *is empty,*
(2) $\bigcap_{i=1}^d V(s_i)$ *is finite over R and non-empty.*

Proof. Let m and $s \in H^0(X, \mathscr{L}^{\otimes m})$ such that $H = V(s)$ is a smooth hypersurface as in Lemma 3.3.4. Without loss of generality $m = 1$. For $k \gg 0$, we get a surjective map

$$H^0(X, \mathscr{L}^{\otimes k}) \to H^0(H, \mathscr{L}_{|H}^{\otimes k}).$$

By induction on d we can find $s_{H,0}, \ldots, s_{H,d-1} \in H^0(H, \mathscr{L}_{|H}^{\otimes k})$, for some $k \geq 1$, satisfying the desired properties for H. Note that $s_{H,0}^j, \ldots, s_{H,d-1}^j$, for all $j \geq 1$, also satisfy the properties, hence we may suppose $k \gg 0$. Choose some liftings $s_0, s_1, \ldots, s_{d-1} \in H^0(X, \mathscr{L}^{\otimes k})$. Then $s_0, s_1, \ldots, s_{d-1}, s^k$ satisfy the required properties. $\square$

Proof of Proposition 3.3.3. Let $\mathscr{L}$ be a relative ample line bundle. Apply Lemma 3.3.6 to the local ring of Y at y. The sections $s_0, \ldots, s_d$ extend to $X \times_Y \mathrm{Spec}(U)$ for an open affine neighborhood U of y. After possibly shrinking U we have $\bigcap_{i=0}^d V(s_i) = \emptyset$ so that

$$X \times_Y \mathrm{Spec}(U) \to \mathbb{P}_U^d,$$

defined by $s_0, \ldots, s_d$, is well-defined. The second property of Lemma 3.3.6 implies that the morphism is generically finite. $\square$

Corollary 3.3.7. *Let R be a smooth $\mathbb{Z}$-algebra. Let $X \to \mathrm{Spec}(R)$ be a smooth projective morphism such that $H_{dR}^*(X/R)$ is a projective R-module. Suppose that X is connected of relative dimension d. If the canonical map*

$$H_{dR}^i(X/R) \to \mathrm{Hom}_R(H_{dR}^{2d-i}(X/R), R) \tag{3.3.7}$$

is an isomorphism, then

$$H_{dRW}^i(X/\mathbb{W}(R)) \xrightarrow{\cong} \underline{\mathrm{Hom}}(H_{dRW}^{2d-i}(X/\mathbb{W}(R)), \mathbf{1}(-d)). \tag{3.3.8}$$

Proof. In view of Proposition 3.3.2 and (3.1.3) we get a morphism in $\mathcal{C}_R$:

$$H_{dRW}^i(X/\mathbb{W}(R)) \otimes H_{dRW}^{2d-i}(X/\mathbb{W}(R)) \to H_{dRW}^{2d}(X/\mathbb{W}(R)) \to \mathbf{1}(-d)$$

inducing (3.3.8). Now, $T(3.3.8) = (3.3.7)$ proves the claim. $\square$

Remark 3.3.8. Note that the map

$$H^i_{dR}(X/R) \to \mathrm{Hom}_R(H^{2d-i}_{dR}(X/R), R)$$

induced by the pairing

$$H^i_{dR}(X/R) \otimes_R H^{2d-i}_{dR}(X/R) \to H^{2d}_{dR}(X/R) \to R$$

is an isomorphism if for every closed point $y \in \mathrm{Spec}(R)$ the Hodge-to-de-Rham spectral sequence for the fibre at y,

$$H^j(X_y, \Omega^i_{X_y/k(y)}) \Rightarrow H^{i+j}_{dR}(X_y/k(y)), \tag{3.3.9}$$

degenerates. Indeed, since the de Rham cohomology is locally free, it is also stable under base change and it suffices to show that, for every closed point $y \in \mathrm{Spec}(R)$, the Poincaré pairing for the fibre at y,

$$H^i_{dR}(X_y/k(y)) \otimes_{k(y)} H^{2d-i}_{dR}(X_y/k(y)) \to H^{2d}_{dR}(X_y/k(y)) \to k(y),$$

is non-degenerate. This follows easily from the degeneration of the Hodge-to-de-Rham spectral sequence and Serre duality.
The degeneration of the spectral sequence (3.3.9) is known in the following cases:

- $H^j(X, \Omega^i_{X/R})$ is torsion-free for all i, j,
- $\dim X_y \leq \mathrm{char}(k(y))$.

As an example, we have abelian schemes or curves over R.

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