The Local Cohomology of the Jacobian Ring

Edoardo Sernesi

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Abstract. We study the 0-th local cohomology module $H^0_{\mathbf{m}}(R(f))$ of the jacobian ring $R(f)$ of a *singular* reduced complex projective hypersurface X, by relating it to the sheaf of logarithmic vector fields along X. We investigate the analogies between $H^0_{\mathbf{m}}(R(f))$ and the well known properties of the jacobian ring of a nonsingular hypersurface. In particular we study self-duality, Hodge theoretic and Torelli type questions for $H^0_{\mathbf{m}}(R(f))$.

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1. Introduction

In this paper we focus on the relation existing between a (singular) projective hypersurface and the 0-th local cohomology of its jacobian ring. Most of the results we will present are well known to the experts, and perhaps the only novelty here is a unifying approach obtained by relating the local cohomology to the sheaf of logarithmic vector fields along X. We will take the opportunity to introduce what seem to us some interesting open problems on this subject.
Consider the polynomial ring $P = \mathbb{C}[X_0, \dots, X_r]$ in $r+1$ variables, $r \geq 2$, with coefficients in $\mathbb{C}$. Given a reduced polynomial $f \in P$ homogeneous of degree d let $X := V(f) \subset \mathbb{P}^r$ be the hypersurface defined by f. The *jacobian ring* of f is defined as

$$R = R(f) := P/J(f)$$

where

$$J(f) := \left(\frac{\partial f}{\partial X_0}, \dots, \frac{\partial f}{\partial X_r} \right)$$

is the *gradient ideal* of f.
If X is nonsingular then $J(f)$ is generated by a regular sequence, and $R(f)$ is a Gorenstein artinian ring with socle in degree $\sigma := (r+1)(d-2)$. It carries information on the geometry of X and on its period map. This classical case has been studied by Griffiths and his school. In [25] Griffiths has shown the relation existing between the jacobian ring of a nonsingular projective hypersurface and

the Hodge decomposition of its primitive cohomology in middle dimension, and studied the relation of $R(f)$ with the period map (see also [44] for details and [7] for a survey).

Assume now that $X \subset \mathbb{P}^r$ is singular, but reduced. In this case the jacobian ring is not of finite length, in particular it is not artinian Gorenstein any more. It contains information on the structure of the singularities and on the global geometry of X. This situation has been studied extensively, both from the point of view of singularity theory (see e.g. [24, 35, 40, 45, 46]) and in relation with the (mixed) Hodge theory of $U := \mathbb{P}^r \setminus X$ (see [8, 6, 9, 10, 15]). Our main purpose is to indicate a method to distinguish the global information contained in $R(f)$ from the local one coming from the nature of the singularities.

Our starting point is the observation that, if X is nonsingular, we have a canonical identification of P-modules

$$R(f) = \bigoplus_{j \in \mathbb{Z}} H^1(T\langle X\rangle(j-d))$$

where $T\langle X\rangle$ is the subsheaf of $T_{\mathbb{P}^r}$ of logarithmic vector fields along X. If X is singular this identification does not hold, but the P-module on the right hand side is the 0-th local cohomology of $R(f)$. We will see that this object contains relevant global informations about X.

To any finite type graded P-module one can associate a coherent sheaf $M^\sim$ on $\mathbb{P}^r$ and there is a well-known exact sequence involving the local cohomology graded modules (see [26], Prop. 2.1.5):

$$0 \longrightarrow H^0_{\mathbf{m}}(M) \longrightarrow M \longrightarrow H^0_*(M^\sim) \longrightarrow H^1_{\mathbf{m}}(M) \longrightarrow 0 \tag{1}$$

where we used the notation $H^i_*(\mathcal{F}) = \bigoplus_{\nu \in \mathbb{Z}} H^i(\mathcal{F}(\nu))$ for a coherent sheaf $\mathcal{F}$. In case $M = R(f)$ with X singular both $H^i_{\mathbf{m}}(R(f))$ are finite length modules that carry interesting information about the hypersurface X. In particular, $H^0_{\mathbf{m}}(R(f))$ contains global information about X, while $H^1_{\mathbf{m}}(R(f))$ is related with the singularities of X. We want to collect evidence supporting the following principle:

Most properties of the jacobian ring $R(f)$ in the nonsingular case are transferred to the local cohomology module $H^0_{\mathbf{m}}(R(f))$ if X is a singular hypersurface.

In particular one expects the following in some generality:

(a) *Self-duality*, extending the analogous property of Artinian Gorenstein algebras.
(b) *Existence of a connection with moduli of X*, in particular with first order locally trivial deformations of X.
(c) *Existence of a relation with the Hodge decomposition* of the middle dimension primitive cohomology a nonsingular model of X.
(d) *Torelli type results*, stating the possibility of reconstructing X from $H^0_{\mathbf{m}}(R(f))$, under some hypothesis.

Question (a) has already attracted the attention of several authors and some results are known. One is led naturally to consider more generally the 0-th local cohomology of algebras of the form P/I where $I = (f_0, \dots, f_r)$ is an ideal generated by $r+1$ homogeneous polynomials, of degrees $d_0, \dots, d_r$. The following result is a special case of [38], Theorem 4.7:

Theorem 3.2. *Assume that* $\dim[\mathrm{Proj}(P/I)] = 0$. *Then there is a natural isomorphism:*

$$H^0_{\mathbf{m}}(P/I) \cong [H^0_{\mathbf{m}}(P/I)(\sigma)]^\vee$$

where $\sigma = \sum_{i=0}^r d_i - r - 1$. *In particular we have natural isomorphisms*

$$H^0_{\mathbf{m}}(P/I)_k \cong H^0_{\mathbf{m}}(P/I)^\vee_{\sigma-k}, \quad 0 \le k \le \sigma$$

We include an independent proof of Theorem 3.2, more related with our point of view, which uses a spectral sequence argument and is an adaptation of the standard proof of Macaulay's Theorem (see e.g. [44]). I am also aware of work in progress of H. Hassanzadeh and A. Simis about extensions of Theorem 3.2 to a local algebra situation. Taking $I = J(f)$, as a special case we obtain:

Theorem 3.4. *Assume that the hypersurface X has only isolated singularities. Then:*

$$H^0_{\mathbf{m}}(R(f))_k \cong H^0_{\mathbf{m}}(R(f))^\vee_{\sigma-k}, \quad 0 \le k \le \sigma$$

where $\sigma = (r+1)(d-2)$.

This is a generalization of Macaulay's Theorem, that states the self-duality of $R(f)$ in the case X nonsingular. The theorem, in an equivalent form, appeared already in [12], Theorem 1. A similar result for hypersurfaces with isolated quasi-homogeneous singularities is proved in [20]. We also refer the reader to the recent preprint [19] where all these duality results are reconsidered and further generalized. For recent related work see [36, 37].

As mentioned before, we interpret $H^0_{\mathbf{m}}(R(f))$ by means of the sheaf $T\langle X\rangle$, also denoted by $\mathrm{Der}(-\log X)$, associated to any hypersurface X in a smooth variety M (see §2 where we recall its definition). Precisely we show that there is an identification:

$$H^0_{\mathbf{m}}(R(f)) = H^1_*(T\langle X\rangle(-d)) \tag{2}$$

(Proposition 2.1). In particular:

$$H^0_{\mathbf{m}}(R(f))_d = H^1(T\langle X\rangle) \tag{3}$$

The right hand side is the space of first-order locally trivial deformations of X in $\mathbb{P}^r$ (see [31], §3.4.4). Therefore (3) generalizes what happens in the nonsingular case, when we have the identification of $R(f)_d$ with the space of first order deformations of X in $\mathbb{P}^r$ modulo projective automorphisms [7]. Thus (3) gives an answer to (b).

In passing note that Theorem 3.4 and (2) together imply the self-duality of $H^1_*(T\langle X\rangle(-d))$ in the case when X has isolated singularities. This fact is quite straightforward when $r = 2$ but it is not so when $r \ge 3$, since $T\langle X\rangle$ is not even locally free.

As of Question (c), one expects that there exists a relation between the local cohomology of $R(f)$ and the Hodge decomposition of the middle primitive cohomology of a nonsingular model X' of X. We collect some evidence that this relation exists at least for strictly normal crossing hypersurfaces. In particular we show that for such hypersurfaces one has an isomorphism

$$H^0_{\mathbf{m}}(R(f))_{d-r-1} \cong \bigoplus_{i=1}^{s} H^0(X_i, \Omega^{r-1}_{X_i})$$

where $X_1, \ldots, X_s$ are the irreducible components of X (see Theorem 5.1 for a precise statement). This result and duality imply a result completely analogous to Griffiths' for strictly normal crossing plane curves (Corollary 5.2). We also prove a result for surfaces in $\mathbb{P}^3$ indicating that the local cohomology contains information on how the various components intersect (Theorem 5.3).

Question (d) is related to interesting issues that have been widely considered in the case of arrangements of hyperplanes and of hypersurfaces, but from a different point of view. Several authors have investigated the problem of reconstructing certain arrangements of hyperplanes and of hypersurfaces from their sheaf of logarithmic differentials (see [2, 16, 22, 42, 43]). Our Question (d) is quite different, at least when $r \geq 3$, while it is essentially equivalent to it when $r = 2$. We discuss the problem and we give a few examples.

In the paper we also consider the question of freeness of the sheaf $T\langle X\rangle$, which is a special case of the condition $H^0_{\mathbf{m}}(R(f)) = 0$. We overview some of the known results in the case $r = 2$.

In detail the paper is organized as follows. §2 is devoted to the relation between local cohomology of the jacobian ring of X and the sheaf $T\langle X\rangle$. In §3 we consider the self duality properties. §4 is devoted to generalities on sheaves of logarithmic differentials and §5 to the Hodge theoretic properties of the local cohomology. In the next §6 we discuss the Torelli problem (d) above, and its relations with related reconstruction problems. §7 treats the freeness of $T\langle X\rangle$.

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After posting the first version of this paper I became aware of references [12] and [38]. I am thankful to D. Van Straten, and M. Saito for calling my attention on them and for some helpful remarks. Finally it is a pleasure to thank A. Dimca for his correspondence and for bringing Example 5.7 to my attention.

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2. Logarithmic derivations and local cohomology

We will adopt the following standard notation and terminology. Consider the graded polynomial ring $P = \bigoplus_{k\geq 0} P_k = \mathbb{C}[X_0, \ldots, X_r]$, in $r+1$ variables, $r \geq 2$, with coefficients in $\mathbb{C}$, and denote by $\mathbf{m} = \bigoplus_{k\geq 1} P_k$ its irrelevant maximal ideal. A graded P-module $M = \bigoplus_k M_k$ is *TF-finite* if $M_{\geq k_0} := \bigoplus_{k\geq k_0} M_k$ is of finite type for some k_0. If M is TF-finite we let

$$M^\vee = \bigoplus_k (M^\vee)_k = \bigoplus_k M^\vee_{-k} = \bigoplus_k \mathrm{Hom}_{\mathbb{C}}(M_{-k}, \mathbb{C})$$

For any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^r$ and $0 \leq i \leq r$ we let

$$H^i_*(\mathcal{F}) = \bigoplus_{k\in\mathbb{Z}} H^i(\mathbb{P}^r, \mathcal{F}(k))$$

which is a graded P-module.

Consider a reduced polynomial $f \in P$ homogeneous of degree d. Let $X := V(f) \subset \mathbb{P}^r$ be the hypersurface defined by f and let

$$R(f) := P/J(f)$$

be the *jacobian ring* of f (or of X) where

$$J(f) := \left(\frac{\partial f}{\partial X_0}, \dots, \frac{\partial f}{\partial X_r}\right)$$

is the *gradient ideal* of f. The scheme $\mathrm{Proj}(R(f))$ is called the *jacobian scheme* of f, or the *singular scheme* of X (see [1]), and also denoted by $\mathrm{Sing}(X)$. We denote by $\mathcal{J}_f = J(f)^\sim \subset \mathcal{O}_{\mathbb{P}^r}$ the ideal sheaf associated to $J(f)$, and by

$$\mathcal{J}_{f/X} = \mathcal{J}_f/\mathcal{I}_X \subset \mathcal{O}_X$$

its image in $\mathcal{O}_X$. Then $\mathcal{J}_{f/X}$ is called *the jacobian ideal sheaf of* X. Note that $\mathcal{O}_X/\mathcal{J}_{f/X} = \mathcal{O}_{\mathrm{Sing}(X)} = T^1_X(-d)$ where T^1_X is *the first cotangent sheaf of* X, .

A more useful description of the jacobian ring is the following. Consider the diagram of sheaf homomorphisms:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K} \arrow[r] \arrow[d, equal] & \mathcal{O}_{\mathbb{P}^r}(-d+1)^{r+1} \arrow[r, "\partial f"] \arrow[d, equal] & \mathcal{O}_{\mathbb{P}^r} \arrow[r] & T^1_X(-d) \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{K} \arrow[r] & \mathcal{O}_{\mathbb{P}^r}(-d+1)^{r+1} \arrow[r, "\partial f"] & \mathcal{J}_f \arrow[u, hook] \arrow[rr] & & 0
\end{tikzcd} \tag{4}$$

where ∂f is defined by the partials of f, and $\mathcal{K} = \ker(\partial f)$. It induces

$$\begin{tikzcd}
H^0_*(\mathcal{O}_{\mathbb{P}^r}(-d+1))^{r+1} \arrow[r] \arrow[d, equal] & P \arrow[r] & R(f) \arrow[r] & 0 \\
H^0_*(\mathcal{O}_{\mathbb{P}^r}(-d+1))^{r+1} \arrow[r] & J(f)^{sat} \arrow[r] \arrow[u, hook] & H^1_*(\mathcal{K}) \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

where $J(f)^{sat}$ is the saturation of $J(f)$. The following are clearly equivalent conditions:

(a) X is nonsingular.
(b) $T^1_X = 0$.
(c) $R(f)$ has finite length.
(d) $R(f) = H^1_*(\mathcal{K})$.

When they are not satisfied then $H^1_*(\mathcal{K})$ is just a submodule of finite length of $R(f)$ and we have an identification:

$$H^1_*(\mathcal{K}) = \frac{J(f)^{sat}}{J(f)} = H^0_{\mathbf{m}}(R(f)) \tag{5}$$

where $H^0_{\mathbf{m}}(M)$ denotes the 0-*th local cohomology* of a graded P-module with respect to $\mathbf{m}$.
We also have the exact sequence:

$$0 \longrightarrow T\langle X\rangle \longrightarrow T_{\mathbb{P}^r} \xrightarrow{\eta} \mathcal{J}_{f/X}(d) \longrightarrow 0 \tag{6}$$

where $T\langle X\rangle := \ker(\eta)$ is the *sheaf of logarithmic vector fields along* X and η is defined as:

$$\eta\left(\sum_i A_i(X)\frac{\partial}{\partial X_i}\right) = \sum_i A_i \frac{\partial f}{\partial X_i}$$

the sheaf $T\langle X\rangle$ is also denoted by $\mathrm{Der}(-\mathrm{log}X)$ in the literature [30]. We then have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & T\langle X\rangle \arrow[r] & T_{\mathbb{P}^r} \arrow[u] \arrow[r, "\eta"] & \mathcal{J}_{f/X}(d) \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{K}(d) \arrow[u, "\cong"] \arrow[r] & \mathcal{O}_{\mathbb{P}^r}(1)^{r+1} \arrow[u] \arrow[r, "\partial f"] & \mathcal{J}_f(d) \arrow[u] \arrow[r] & 0 \\
 & & \mathcal{O}_{\mathbb{P}^r} \arrow[u] \arrow[r, "f"] & \mathcal{I}_X(d) \arrow[u] \arrow[r] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

where the middle vertical is the Euler sequence. From this diagram we deduce the isomorphisms:

$$T\langle X\rangle \cong \mathcal{K}(d) \tag{7}$$

$$H^1_*(\mathcal{J}_{f/X}) \cong H^1_*(\mathcal{J}_f) \ (= H^2_*(\mathcal{K}) \text{ if } r \geq 3) \tag{8}$$

Now we can prove the following:

PROPOSITION 2.1. *In the above situation we have a canonical isomorphism:*

$$H^0_{\mathbf{m}}(R(f)) \cong H^1_*(T\langle X\rangle(-d)) \tag{9}$$

In particular

$$R(f) \cong H^1_*(T\langle X\rangle(-d))$$

if X is nonsingular.

Proof. It follow directly from (5) and (7). The last assertion is obvious. $\square$

Corollary 2.2. *The vector space $H^0_{\mathbf{m}}(R(f))_d$ is naturally identified with the space of first order locally trivial deformations of X in $\mathbb{P}^r$ modulo the action of* $\mathrm{PGL}(r+1)$.

Proof. The proposition identifies $H^0_{\mathbf{m}}(R(f))_d$ with $H^1(T\langle X\rangle)$ which is the space of first order locally trivial deformations of the inclusion $X \subset \mathbb{P}^r$ (see [31], §3.4.4 p. 176). □

Remarks 2.3. (i) It is easy to compute that for $X \subset \mathbb{P}^2$ the *Chern classes of* $T\langle X\rangle(k)$ are:

$$c_1(T\langle X\rangle(k)) = 3 - d + 2k, \quad c_2(T\langle X\rangle(k)) = d^2 - (3+k)d + 3 + 3k + k^2 - t^1_X$$

where $t^1_X = h^0(T^1_X) = h^0(\mathcal{O}_{\mathrm{Sing}(X)})$. Moreover:

$$-\chi(T\langle X\rangle) = \frac{1}{2}d(d+3) - t^1_X - 8$$

which is the expected dimension of the family of locally trivial deformation of X modulo PGL(3). This is explained by the fact that $T\langle X\rangle$ is the sheaf controlling the locally trivial deformation theory of X in $\mathbb{P}^2$ (see [31]).

(ii) If X is a normal crossing arrangement of $d \geq r+2$ hyperplanes then $T\langle X\rangle$ is the dual of a Steiner bundle [16], in particular it is locally free, and these bundles are known to be stable [3]. In the special case $d = r+2$ we have $T\langle X\rangle = \Omega(1)$. If $1 \leq d \leq r+1$ then

$$T\langle X\rangle = \mathcal{O}^{d-1}_{\mathbb{P}^r} \bigoplus \mathcal{O}_{\mathbb{P}^r}(1)^{r+1-d}$$

and these bundles are not stable.
(iii) If $X \subset \mathbb{P}^2$ is nonsingular then $T\langle X\rangle$ is stable ([41], Lemma 3).

In the case of plane curves we have more generally:

Proposition 2.4. *Let $X \subset \mathbb{P}^2$ be of degree $d \geq 4$. Then $T\langle X\rangle$ is stable if and only if (f_0, f_1, f_2), where $f_i = \frac{\partial f}{\partial X_i}$, has no syzygies of degree $[(d-1)/2]$. In particular $T\langle X\rangle$ is stable if X is nonsingular.*

Proof. Twist $T\langle X\rangle$ by $k = [(d-3)/2]$. Then $c_1(T\langle X\rangle(k)) = 0, -1$ according to whether d is odd or even, and $T\langle X\rangle$ is stable if and only if $H^0(T\langle X\rangle(k)) = 0$ ([29], Lemma 1.2.5 p. 165). The exact sequence

$$0 \longrightarrow T\langle X\rangle(k) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(k+1) \xrightarrow{(f_0,f_1,f_2)} \mathcal{J}_f(d+k) \longrightarrow 0$$

identifies $H^0(T\langle X\rangle(k))$ with the space of syzygies of (f_0, f_1, f_2) of degree $k+1 = [(d-1)/2]$.
In the nonsingular case (f_0, f_1, f_2) has no syzygies of degree less than $d-1$ because they form a regular sequence. □

Example 2.5. Let $f = X_1^\alpha X_0^{d-\alpha} - X_2^d$, with $2 \leq \alpha < d$, and $d \geq 4$. Then $T\langle X\rangle$ is not stable because (f_0, f_1, f_2) has the linear syzygy $(\alpha X_0, -(d-\alpha)X_1, 0)$. Additional interesting informations concerning the syzygies of (f_0, f_1, f_2) for a singular plane curve are in [11].

3. Self-duality of the local cohomology

In this section we will consider a situation slightly more general than before. Let

$$I = (\mathbf{f}) = (f_0, \dots, f_s) \subset P$$

be a proper homogeneous ideal, whose generators have degrees $d_0, \dots, d_s$ respectively, and let $R = P/I$. Denote by $Y = \mathrm{Proj}(R)$ and by $\mathcal{I} = I^{\sim} \subset \mathcal{O}_{\mathbb{P}^r}$. We have an exact sequence:

$$0 \longrightarrow \mathcal{K} \longrightarrow \bigoplus_{j=0,\dots,s} \mathcal{O}_{\mathbb{P}^r}(-d_j) \xrightarrow{\mathbf{f}} \mathcal{I} \longrightarrow 0 \tag{10}$$

where $\mathcal{K} := \ker(\mathbf{f})$. The 0*-th and* 1*-st local cohomology modules of* R (with respect to $\mathbf{m}$) are defined respectively as:

$$\begin{aligned} H^0_{\mathbf{m}}(R) &:= H^1_*(\mathcal{K}) \\ H^1_{\mathbf{m}}(R) &:= H^1_*(\mathcal{I}) \ (= H^2_*(\mathcal{K}) \ \text{ if } r \geq 3) \end{aligned}$$

They are graded P-modules of finite length. In case $\mathbf{m}^k \subset I$ for some $k > 0$, i.e. $Y = \emptyset$, we have

$$H^0_{\mathbf{m}}(R) = R, \quad H^1_{\mathbf{m}}(R) = (0)$$

There is a standard exact sequence:

$$0 \longrightarrow H^0_{\mathbf{m}}(R) \longrightarrow R \longrightarrow H^0_*(\mathcal{O}_Y) \longrightarrow H^1_{\mathbf{m}}(R) \longrightarrow 0 \tag{11}$$

Assume now that $s = r$. Denote by

$$\mathcal{E} := \bigoplus_{j=0,\dots,r} \mathcal{O}_{\mathbb{P}^r}(-d_j)$$

and let

$$\sigma := \sum_j (d_j - 1) = \sum_j d_j - r - 1$$

Consider the Koszul complex:

$$\mathcal{E}^\bullet : \ 0 \longrightarrow \mathcal{E}_{-r-1} \longrightarrow \mathcal{E}_{-r} \longrightarrow \cdots \longrightarrow \mathcal{E}_{-1} \xrightarrow{\mathbf{f}} \mathcal{E}_0 \longrightarrow 0$$

where $\mathcal{E}_{-p} = \bigwedge^p \mathcal{E}$. For every $k \in \mathbb{Z}$ we can consider the twist $\mathcal{E}^\bullet(k)$ and the two corresponding spectral sequences of hypercohomology. Taking direct sums over all k we can collect them in the following two spectral sequences of hypercohomology:

$$\begin{aligned} A_1^{pq} &= H^q_*(\mathcal{E}_p) \\ B_2^{pq} &= H^p_*(\mathcal{H}^q(\mathcal{E}^\bullet)) \end{aligned}$$

where $\mathcal{H}^q(\mathcal{E}^\bullet)$ is the q-th cohomology sheaf of $\mathcal{E}^\bullet$. In particular $\mathcal{H}^0(\mathcal{E}^\bullet) = \mathcal{O}_Y$. In the A-spectral sequence we have in particular:

$$A_2^{00} = \cdots = A_{r+1}^{00} = \mathrm{coker}[H^0_*(\mathcal{E}_{-1}) \longrightarrow H^0_*(\mathcal{E}_0)] = R \tag{12}$$

$$A_2^{-r-1r} = \cdots = A_{r+1}^{-r-1r} = \ker[H^r_*(\mathcal{E}_{-r-1}) \longrightarrow H^r_*(\mathcal{E}_{-r})] = [R(\sigma)]^\vee \tag{13}$$

and

$$\mathbf{d}_{r+1} : [R(\sigma)]^\vee = A_{r+1}^{-r-1r} \longrightarrow A_{r+1}^{00} = R$$

We denote by $\mathbf{H}^i_*(\mathcal{E}^\bullet)$ the i-th hypercohomology of $\mathcal{E}^\bullet$.

Proposition 3.1. *In the above situation, suppose that* $\dim(Y) \leq 0$. *Then*

$$\mathbf{H}^0_*(\mathcal{E}^\bullet) = H^0_*(\mathcal{O}_Y)$$
$$\mathrm{Im}(\mathbf{d}_{r+1}) = H^0_{\mathbf{m}}(R), \quad A^{00}_\infty = R/H^0_{\mathbf{m}}(R), \quad A^{-rr}_\infty = H^1_{\mathbf{m}}(R)$$

and the exact sequence of edge homomorphisms

$$0 \longrightarrow A^{00}_\infty \longrightarrow \mathbf{H}^0_*(\mathcal{E}^\bullet) \longrightarrow A^{-rr}_\infty \longrightarrow 0$$

coincides with the sequence:

$$0 \longrightarrow R/H^0_{\mathbf{m}}(R) \longrightarrow H^0_*(\mathcal{O}_Y) \longrightarrow H^1_{\mathbf{m}}(R) \longrightarrow 0 \tag{14}$$

Proof. Let $x \in \mathbb{P}^r$. Then

$$\mathrm{depth}_x(\mathcal{I}_Y) \begin{cases} \geq r & \text{if } x \in Y \\ =\text{r+1} & \text{otherwise} \end{cases}$$

Therefore, by [18], Thm. 17.4 p. 424, $(\mathcal{H}^q)_x = 0$ if $q \leq -2$ for all $x \in \mathbb{P}^r$, and $(\mathcal{H}^{-1})_x = 0$ if $x \notin Y$. Therefore $\mathcal{H}^q = 0$ if $q \leq -2$ and $\mathcal{H}^{-1}$ is supported on Y. It follows that $H^p(\mathcal{H}^{-1}) = 0$ for all $p > 0$. Now we decompose $\mathcal{E}^\bullet$ into short exact sequences of sheaves as follows:

$$0 \longrightarrow \mathcal{E}_{-r-1} \longrightarrow \mathcal{E}_{-r} \longrightarrow I_{-r+1} \longrightarrow 0 \tag{15}$$

$$0 \longrightarrow I_{-r+1} \longrightarrow \mathcal{E}_{-r+1} \longrightarrow I_{-r+2} \longrightarrow 0 \tag{16}$$

etc., up to:

$$0 \longrightarrow I_{-2} \longrightarrow \mathcal{E}_{-2} \longrightarrow I_{-1} \longrightarrow 0$$

$$0 \longrightarrow I_{-1} \longrightarrow K_{-1} \longrightarrow \mathcal{H}^{-1} \longrightarrow 0 \tag{17}$$

$$0 \longrightarrow K_{-1} \longrightarrow \mathcal{E}_{-1} \longrightarrow \mathcal{I}_Y \longrightarrow 0 \tag{18}$$

The map $\mathbf{d}_{r+1}$ is obtained from a diagram chasing out of these sequences. Since the $\mathcal{E}_i$'s are direct sums of $\mathcal{O}(k)$'s, from (15) and comparing with (13) we deduce

$$A^{-r-1r}_{r+1} \cong H^{r-1}_*(I_{-r+1})$$

and from (16), etc, we have isomorphisms

$$A^{-r-1r}_{r+1} \cong H^{r-1}_*(I_{-r+1}) \cong \cdots \cong H^1_*(I_{-1})$$

Now we use (17) and we obtain a surjective map:

$$H^1_*(I_{-1}) \longrightarrow H^1_*(K_{-1}) \longrightarrow 0$$

But from sequence (18) it follows that

$$H^1_*(K_{-1}) = H^0_{\mathbf{m}}(R)$$

and this proves that $\mathrm{Im}(\mathbf{d}_{r+1}) = H^0_{\mathbf{m}}(R)$. Therefore it also follows that

$$A^{00}_\infty = A^{00}_{r+1}/\mathrm{Im}(\mathbf{d}_{r+1}) = R/H^0_{\mathbf{m}}(R)$$

Now observe that $A_\infty^{-rr} = H^r_*(I_{-r+1})$. A diagram chasing similar to the previous one shows that

$$H^r_*(I_{-r+1}) \cong H^1_*(\mathcal{I}_Y)$$

Since $H^1_*(\mathcal{I}_Y) = H^1_{\mathbf{m}}(R)$ we obtain the identification $A_\infty^{-rr} = H^1_{\mathbf{m}}(R)$.
Noting that the B-spectral sequence degenerates at B_2, we get in particular that

$$\mathbf{H}^0_*(\mathcal{E}^\bullet) = H^0_*(\mathcal{H}^0(\mathcal{E}^\bullet)) = H^0_*(\mathcal{O}_Y)$$

Therefore the edge exact sequence is (14). □

As a consequence we can now derive the following:

THEOREM 3.2. *Let* $I = (f_0, \dots, f_r)$ *with* $deg(f_j) = d_j$, $R = P/I$ *and* $Y = \mathrm{Proj}(R)$. *Assume that* $\dim(Y) \leq 0$. *Then there is a natural isomorphism:*

$$H^0_{\mathbf{m}}(R) \cong [H^0_{\mathbf{m}}(R)(\sigma)]^\vee$$

where $\sigma = \sum_{j=0}^r d_j - r - 1$. *Therefore we have natural isomorphisms*

$$H^0_{\mathbf{m}}(R)_k \cong H^0_{\mathbf{m}}(R)^\vee_{\sigma-k}, \quad 0 \leq k \leq \sigma$$

Proof. The surjective map:

$$\mathbf{d}_{r+1} : \ [R(\sigma)]^\vee \longrightarrow H^0_{\mathbf{m}}(R)$$

dualizes as an injective map:

$$\mathbf{d}^\vee_{r+1} : \ H^0_{\mathbf{m}}(R)^\vee \longrightarrow R(\sigma)$$

whose image must be contained in $H^0_{\mathbf{m}}(R)(\sigma)$ because it consists of elements which are killed by $\mathbf{m}^{\sigma+1}$. But then $\mathrm{Im}(\mathbf{d}^\vee_{r+1}) = H^0_{\mathbf{m}}(R)(\sigma)$ because $H^0_{\mathbf{m}}(R)^\vee$ and $H^0_{\mathbf{m}}(R)(\sigma)$ have the same dimension as vector spaces. □

REMARK 3.3. As already stated in the Introduction, Corollary 3.2 is a special case of [38], Theorem 4.7. The case $Y = \emptyset$ of course corresponds to the situation when the elements $f_0, \dots, f_r$ form a regular sequence, and this happens if and only if $\mathcal{H}^q(\mathcal{E}^\bullet) = 0$ for all q. In this case the hypercohomology $\mathbf{H}^\bullet_*(\mathcal{E}^\bullet)$ is zero in all dimensions, because the B_2-spectral sequence is zero. It follows that the map:

$$\mathbf{d}_{r+1} : A^{-r-1r}_{r+1} \longrightarrow A^{00}_{r+1}$$

is an isomorphism, which means that we have an isomorphism $[R(\sigma)]^\vee \cong R$. This is the well known duality theorem of Macaulay for Gorenstein artinian algebras ([44], Th. II6.19, p. 172).

As a special case of Theorem 3.2 we obtain the following (see also [12], Theorem 1):

THEOREM 3.4. *Assume that the hypersurface* X *has at most isolated singularities. Then:*

$$H^0_{\mathbf{m}}(R(f))_k \cong H^0_{\mathbf{m}}(R(f))^\vee_{\sigma-k}, \quad 0 \leq k \leq \sigma$$

where $\sigma = (r+1)(d-2)$.

Remark 3.5. In case X is nonsingular the jacobian ring $R = R(f)$ is Gorenstein artinian with socle in degree σ. The self duality of $R(f)$ is induced by a pairing

$$R_k \times R_{\sigma-k} \longrightarrow R_\sigma \cong \mathbb{C}$$

where the first map is induced by multiplication of polynomials and the last isomorphism is obtained from the trace map for local duality.

Corollary 3.6. *Assume that X has only isolated singularities. Then there are natural isomorphisms:*

$$H^1(T\langle X\rangle(-d+k)) \cong H^1(T\langle X\rangle(\sigma - d - k))^\vee$$

for all k.

Proof. Use (9) and Theorem (3.4). □

Observe that, in case the hypersurface X is singular with isolated singularities and $r \geq 3$, the sheaf $T\langle X\rangle(-d)$ is reflexive of rank r but not locally free (see [30]). Therefore the duality statement of Corollary 3.6 is not a consequence of standard properties of locally free sheaves.

On the other hand if $r = 2$ then $T\langle X\rangle(-d)$ is locally free of rank two and its first Chern class is given by:

$$c_1(T\langle X\rangle(-d)) = 3 - 3d$$

Then Corollary 3.6 follows directly from the straightforward fact that for every locally free sheaf E of rank two on $\mathbb{P}^2$ we have

$$H^1(E(k)) \cong H^1(E(\sigma - k))^\vee$$

where $\sigma = -c_1(E) - 3$.

It is not clear how far one can go relaxing the hypothesis of Theorem 3.4, as the next two examples show.

Example 3.7. The *ruled cubic surface* $X \subset \mathbb{P}^3$ has equation

$$XT^2 - YZ^2 = 0$$

and is singular along the line $T = Z = 0$. The local cohomology has only one non-zero term in degree 2, and:

$$h^0_{\mathbf{m}}(R(f))_2 = 1$$

Since $\sigma = 4$, the symmetry condition $H^0_{\mathbf{m}}(R(f))_k \cong H^0_{\mathbf{m}}(R(f))_{\sigma-k}$ is fullfilled even though X doesn't satisfy the hypothesis of Theorem 3.4.

Example 3.8. A *quartic surface with a double conic* $X \subset \mathbb{P}^3$ has equation:

$$(ZT - XY)^2 + (X + Y + Z + T)^2(X^2 + Y^2 + Z^2 + T^2) = 0$$

The table of its local cohomology dimensions is:

j	$h^0_{\mathbf{m}}(R)_j$
0	0
1	0
2	1
3	4
4	5
5	1
6	0
7	0
8	0

Since $\sigma = 10$, we see that self-duality does not hold in this case.

4. Logarithmic differentials

Let's restrict for a moment to the case when our $X \subset \mathbb{P}^r$ of degree d is nonsingular. Then Griffiths' Theorem identifies:

$$\bigoplus_{p=1}^{r} H^{r-p,p-1}(X)_0 = \bigoplus_{p=1}^{r} H^1(T\langle X\rangle(K_{\mathbb{P}^r} + (p-1)X) \tag{19}$$

thanks to Proposition 2.1, which identifies

$$\bigoplus_{p=1}^{r} H^1(T\langle X\rangle(K_{\mathbb{P}^r} + (p-1)X) = \bigoplus_{p=1}^{r} R(f)_{pd-r-1}$$

The right hand side of (19) is well defined if X is just a reduced hypersurface in a projective manifold Z of dimension r, after replacing $\mathbb{P}^r$ with Z. In such a situation it is convenient to consider, together with $T_Z\langle X\rangle$, the *sheaves of logarithmic differentials along* X which are defined as follows:

$$\Omega^k_Z(\log X) := \{\omega \in \Omega^k_Z(X) : d\omega \in \Omega^{k+1}_Z(X)\}, \quad k = 0, \dots, r$$

In particular $\Omega^0_Z(\log X) = \mathcal{O}_Z$ and $\Omega^r_Z(\log X) = K_Z + X$. For $k \neq 0, r$ these sheaves are not locally free in general. For $k = 1$ one has:

$$\Omega^1_Z(\log X) := Hom_Z(T_Z\langle X\rangle, \mathcal{O}_Z)$$

and this sheaf is reflexive ([30], n. 1.7). By definition we have inclusions

$$\Omega^k_Z \subset \Omega^k_Z(\log X) \subset \Omega^k_Z(X)$$

which in turn induce the inclusions:

$$\Omega^k_Z(\log X)(-X) \subset \Omega^k_Z \subset \Omega^k_Z(\log X) \tag{20}$$

We collect in the following Lemmas the properties we need about the sheaves of logarithmic differentials.

Lemma 4.1. *The following conditions are equivalent:*

(i) $T_Z\langle X\rangle$ *is locally free.*

(ii)

$$\Omega^k_Z(\log X) = \bigwedge^k \Omega^1_Z(\log X)$$

for all $k = 1, \ldots, r$.

(iii) $\bigwedge^r \Omega^1_Z(\log X) = \Omega^r_Z(\log X)(= K_Z + X)$

If the above conditions are satisfied then we have a canonical identification:

$$T_Z\langle X\rangle(K_Z + X) = \Omega^{r-1}_Z(\log X) \tag{21}$$

Proof. The equivalence of the conditions stated is Theorem 1.8 of [30]. From (iii) we obtain $c_1(T_Z\langle X\rangle) = -(K_Z + X)$. Therefore:

$$\begin{aligned} T_Z\langle X\rangle(K_Z + X) &= T_Z\langle X\rangle c_1(T_Z\langle X\rangle^\vee) \\ &= \bigwedge^{r-1} T_Z\langle X\rangle^\vee \\ &= \bigwedge^{r-1} \Omega^1_Z(\log X) \\ \text{by (i)} &= \Omega^{r-1}_Z(\log X) \end{aligned}$$

□

The following are examples such that $T_Z\langle X\rangle$ is locally free (see [30]):

- X nonsingular.
- Z is a surface ($r = 2$).
- X has normal crossing singularities at every point (it is a normal crossing divisor). Recall that this means that for each $x \in X$ the local ring $\mathcal{O}_{X,x}$ is formally, or etale, equivalent to $\mathcal{O}_{Z,x}/(t_1 \cdots t_k)$ for some $1 \le k \le r-1$, where $t_1, \ldots, t_k$ are part of a local system of coordinates.

Recall that $X \subset Z$ is a *strictly normal crossing divisor* if it is a normal crossing divisor whose irreducible components $X_1, \ldots, X_s$ are nonsingular.

Lemma 4.2. *Assume that* $X = X_1 \cup \cdots \cup X_s \subset Z$ *is a strictly normal crossing divisor. Denote by* $\widehat{X}_1 = X_2 \cap \cdots \cap X_s$, *and by* $Y_1 = X_1 \cap \widehat{X}_1$. *Then there are exact sequences, for* $a = 1, \ldots, r = \dim(Z)$*:*

$$0 \longrightarrow \Omega^1_Z \longrightarrow \Omega^1_Z(\log X) \longrightarrow \bigoplus_{i=1}^s \mathcal{O}_{X_i} \longrightarrow 0 \tag{22}$$

$$0 \longrightarrow \Omega^a_Z(\log \widehat{X}_1) \longrightarrow \Omega^a_Z(\log X) \xrightarrow{R} \Omega^{a-1}_{X_1}(\log Y_1) \longrightarrow 0 \tag{23}$$

$$0 \longrightarrow \Omega^a_Z(\log X)(-X_1) \longrightarrow \Omega^a_Z(\log \widehat{X}_1) \longrightarrow \Omega^a_{X_1}(\log Y_1) \longrightarrow 0 \tag{24}$$

where R *is the* residue operator.

Proof. see [21], §2.3. □

Note that, by twisting (24) by $\mathcal{O}_Z(-\widehat{X}_1)$ we obtain the following exact sequence:

$$0 \longrightarrow \Omega^a_Z(\log X)(-X) \longrightarrow \Omega^a_Z(\log \widehat{X}_1)(-\widehat{X}_1) \longrightarrow \Omega^a_{X_1}(\log Y_1)(-Y_1) \longrightarrow 0 \tag{25}$$

For future reference it is worth emphasizing that when $X = X_1$ is irreducible and nonsingular then the sequences (23) and (25) become respectively:

$$0 \longrightarrow \Omega^a_Z \longrightarrow \Omega^a_Z(\log X) \xrightarrow{R} \Omega^{a-1}_X \longrightarrow 0 \tag{26}$$

$$0 \longrightarrow \Omega^a_Z(\log X)(-X) \longrightarrow \Omega^a_Z \longrightarrow \Omega^a_X \longrightarrow 0 \tag{27}$$

LEMMA 4.3. *Assume that $X \subset Z$ is an irreducible and nonsingular divisor. For each $k = 0, \dots, r-1$ consider the composition:*

$$\lambda: \ H^k(\Omega^k_X) \xrightarrow{\delta} H^{k+1}(\Omega^{k+1}_Z) \xrightarrow{\nu^*_{k+1}} H^{k+1}(\Omega^{k+1}_X)$$

where δ is a coboundary map of the sequence (26) and ν^*_{k+1} is induced by the second homomorphism in the sequence (27). Then λ is the map defined by the Lefschetz operator corresponding to the Kahler metric on X associated to $\mathcal{O}_X(X)$.

Proof. The Lefschetz operator $L : H^k(X, \mathbb{C}) \longrightarrow H^{k+2}(X, \mathbb{C})$ is the composition:

$$H^k(X, \mathbb{C}) \xrightarrow{\gamma} H^{k+2}(Z, \mathbb{C}) \xrightarrow{\nu^*} H^{k+2}(X, \mathbb{C})$$

where γ is the Gysin map and ν^* is induced by the inclusion

$$X \overset{\nu}{\hookrightarrow} Z$$

([44], v. II, (2.11) p. 57). Moreover γ is the cokernel of the map:

$$\rho : H^{k+1}(U, \mathbb{C}) \longrightarrow H^k(X, \mathbb{C})$$

induced by the residue operator, where $U = Z \setminus X$. More precisely, we have an isomorphism ([44], Corollary I.8.19 p. 198)

$$H^\bullet(U, \mathbb{C}) \cong \mathbb{H}^\bullet(\Omega^\bullet(\log X))$$

(where $\mathbb{H}$ denotes hypercohomology) and the map ρ is induced by the residue operators R of the exact sequences (26). Therefore the restriction of γ to $H^k(\Omega^k_X)$ is identified with δ (see [44], Prop. I.8.34 p. 210). On the other hand ν^*_{k+1} is the restriction of ν^* to $H^{k+1}(\Omega^{k+1}_Z)$. $\square$

5. LOCAL COHOMOLOGY AND HODGE THEORY

We now come back to the original situation of a reduced hypersurface $X = V(f) \subset \mathbb{P}^r$ of degree d. By Proposition 2.1 for $p = 1, \dots, r$ we can identify

$$H^0_{\mathbf{m}}(R(f))_{pd-r-1} = H^1_*(T\langle X\rangle(K_{\mathbb{P}^r} + (p-1)X)) \tag{28}$$

Moreover, *if $T\langle X\rangle$ is locally free* then, by Lemma 4.1, we also have:

$$H^0_{\mathbf{m}}(R(f))_{pd-r-1} = H^1(\Omega^{r-1}(\log X)(p-2)X) \tag{29}$$

Our first result is the following:

Theorem 5.1. *Assume that $X \subset \mathbb{P}^r$ is a strictly normal crossing hypersurface, with irreducible components $X_1, \dots, X_s$. Then we have:*

$$H^0_{\mathbf{m}}(R(f))_{d-r-1} \cong \bigoplus_{i=1}^{s} H^0(X_i, \Omega^{r-1}_{X_i})$$

Proof. Since $T\langle X\rangle$ is locally free we have the identification (29) for $p = 1$:

$$H^0_{\mathbf{m}}(R(f))_{d-r-1} = H^1(\Omega^{r-1}(\log X)(-X))$$

Assume first $r \geq 3$. Consider the exact sequence (25) for $a = r-1$. Since

$$h^0(\mathbb{P}^r, \Omega^{r-1}_{\mathbb{P}^r}(\log \widehat{X}_1)(-\widehat{X}_1)) = 0$$

we obtain the exact sequence:

$$0 \longrightarrow H^0(\Omega^{r-1}_{X_1}) \longrightarrow H^0_{\mathbf{m}}(R(f))_{d-r-1} \longrightarrow H^1(\Omega^{r-1}_{\mathbb{P}^r}\log \widehat{X}_1)(-\widehat{X}_1)) \longrightarrow 0$$

where the zero on the right is $H^1(\Omega^{r-1}_{X_1})$. Now the conclusion follows by induction on s.
If $r = 2$ and $s = 1$ use (27) and Lemma 4.3. If $s \geq 2$ use (25) and induction. □

Corollary 5.2. *Let $X = X_1 + \cdots + X_s \subset \mathbb{P}^2$ be a strictly normal crossing plane curve. Then*

$$H^0_{\mathbf{m}}(R(f))_{d-3} \cong \bigoplus_{i=1}^{s} H^0(X_i, \omega_{X_i}), \quad H^0_{\mathbf{m}}(R(f))_{2d-3} \cong \bigoplus_{i=1}^{s} H^1(X_i, \mathcal{O}_{X_i})$$

Proof. It follows from the theorem, from the self duality theorem 3.4, and Serre duality applied to each component X_i. □
When $r \geq 3$ the relation between the other graded pieces $H^0_{\mathbf{m}}(R(f))_{pd-r-1}$, $p = 2, \dots, r$, of the local cohomology and the primitive middle cohomology of the components of X is more complicated because the intersections of the components contribute non-trivially. As an example we compute the dimension of the middle term in the case $r = 3$.

Theorem 5.3. *Let $X = X_1 + \cdots + X_s \subset \mathbb{P}^3$ be a strictly normal crossing surface, whose components have degrees $d_1, \dots, d_s$ respectively. Then:*

$$h^0_{\mathbf{m}}(R(f))_{2d-4} = \sum_{i=1}^{s} \dim[H^{1,1}(X_i)_0] + \sum_{1 \leq i < j \leq s} g(X_i \cap X_j)$$

where $g(X_i \cap X_j) = \frac{1}{2} d_i d_j (d_i + d_j - 4) + 1$ is the genus of the curve $X_i \cap X_j$.

Proof. By induction on s. If $s = 1$ the formula is true by Griffiths' Theorem. Assume $s \geq 2$. Then $H^0_{\mathbf{m}}(R(f))_{2d-4} = H^1(\Omega^2_{\mathbb{P}^3}(\log X))$, by (29). We let

$$\begin{aligned} \widehat{X}_1 &= X_2 + \cdots + X_s \\ Y_1 &= X_1 \cap (X_2 + \cdots + X_s) \\ \widehat{Y}_1 &= X_1 \cap (X_3 + \cdots + X_s) \end{aligned}$$

We have the following diagram of exact sequences:

(30)

$$\begin{tikzcd} & & & 0 \arrow[d] & \\ & & & \Omega^1_{X_1}(\log\widehat{Y}_1) \arrow[d] & \\ 0 \arrow[r] & \Omega^2_{\mathbb{P}^3}(\log\widehat{X}_1) \arrow[r] & \Omega^2_{\mathbb{P}^3}(\log X) \arrow[r] & \Omega^1_{X_1}(\log Y_1) \arrow[r] \arrow[d] & 0 \\ & & & \mathcal{O}_{X_1\cap X_2} \arrow[d] & \\ & & & 0 & \end{tikzcd}$$

We claim the following:

(a) $h^0(\Omega^1_{X_1}(\log Y_1)) = s-2$.
(b) $H^2(\Omega^2_{\mathbb{P}^3}(\log\widehat{X}_1)) = 0$.
(c) $H^2(\Omega^1_{X_1}(\log\widehat{Y}_1)) = 0$.

Assume that (a),(b),(c) are proved. Then from the above diagram we deduce the exact sequence:
(31)

$$0 \longrightarrow H^1(\Omega^2_{\mathbb{P}^3}(\log\widehat{X}_1)) \longrightarrow H^1(\Omega^2_{\mathbb{P}^3}(\log X)) \longrightarrow H^1(\Omega^1_{X_1}(\log Y_1)) \longrightarrow 0$$

The term on the right in (31) can be computed using the vertical exact sequence of diagram (30). Assume first that $s=2$. In this case $Y_1 = X_1 \cap X_2$ and recalling (a) we obtain:

$$0 \to H^0(\mathcal{O}_{X_1\cap X_2}) \to H^1(\Omega^1_{X_1}) \to H^1(\Omega^1_{X_1}(\log(X_1\cap X_2)) \to H^1(\mathcal{O}_{X_1\cap X_2}) \to 0$$

whence:

$$H^1(\Omega^1_{X_1}(\log(X_1\cap X_2)) = \dim[H^{1,1}(X_1)_0] + g(X_1\cap X_2)$$

If $s \geq 3$ then the map

$$H^0(\mathcal{O}_{X_1\cap X_2}) \longrightarrow H^1(\Omega^1_{X_1}(\log\widehat{Y}_1))$$

is zero by (a). Therefore, applying induction, from the vertical exact sequence of diagram (30) we deduce:

$$\begin{aligned}\dim[H^1(\Omega^1_{X_1}(\log Y_1))] &= \dim[H^1(\Omega^1_{X_1}(\log \widehat{Y}_1))] + g(X_1 \cap X_2)\\ &= \dim[H^{1,1}(X_1)_0] + \sum_{i=3}^{s} g(X_1 \cap X_i) + g(X_1 \cap X_2)\\ &= \dim[H^{1,1}(X_1)_0] + \sum_{i=2}^{s} g(X_1 \cap X_i)\end{aligned}$$

By induction we have:

$$h^1(\Omega^2_{\mathbb{P}^3}(\log \widehat{X}_1)) = \sum_{i=2}^{s} \dim[H^{1,1}(X_i)_0] + \sum_{2 \le i < j \le s} g(X_i \cap X_j)$$

Therefore, putting all these computations together the claimed expression for $h^0_{\mathbf{m}}(R(f))_{2d-4}$ follows. We still have to prove (a),(b) and (c).

Proof of (a). Use the exact sequence (22) with $Z = X_1$ and $X = Y_1$, and the fact that the image of the coboundary map is the space generated by the Chern classes of $X_1 \cap X_2, \dots, X_1 \cap X_s$, which is 1-dimensional.

Proof of (c). Use the vertical sequence in (30) and induction on $s \ge 2$.

Proof of (b). Assume $s = 1$. The map $H^1(\Omega^1_{X_1}) \longrightarrow H^2(\Omega^2_{\mathbb{P}^3})$ coming from the sequence

$$0 \longrightarrow \Omega^2_{\mathbb{P}^3} \longrightarrow \Omega^2_{\mathbb{P}^3}(\log X_1) \longrightarrow \Omega^1_{X_1} \longrightarrow 0$$

is surjective (this follows from Lemma 4.3 and Hodge theory). Therefore since $H^2(\Omega^1_{X_1}) = 0$, it follows that $H^2(\Omega^2_{\mathbb{P}^3}(\log X_1)) = 0$. The general case of (b) now follows by induction, from (c) and from the exact row in (30). $\square$

We give a few examples illustrating these results.

Example 5.4. Let $f = X_0(X_0^3 + X_1^3 + X_2^3)$. Then $X = V(f) = \Lambda \cup C \subset \mathbb{P}^2$ is a *strictly normal crossing reducible plane quartic*, consisting of a line Λ and a nonsingular cubic C. One computes that:

$$H^0_{\mathbf{m}}(R)_1 \cong \mathbb{C} \cong H^{1,0}(X') = H^{1,0}(C)$$
$$H^0_{\mathbf{m}}(R)_5 \cong \mathbb{C} \cong H^{0,1}(X') = H^{0,1}(C)$$

The complete table is:

j	$h^0_{\mathbf{m}}(R)_j$	$\dim(R(f)_j)$	$h^0(\mathcal{O}_{\mathrm{Sing}(X)}(j))$
0	0	1	1
1	1	3	2
2	3	6	3
3	4	7	3
4	3	6	3
5	1	4	3
6	0	3	3
7	0	3	3
8	0	3	3

The conclusion of Corollary 5.2 fails even in the simplest cases if one weakens the assumptions about the singularities of X, as the following two examples show.

EXAMPLE 5.5. A *1-cuspidal plane quartic* $f = X_0^2X_1^2 + X_1^2X_2^2 + X_1^4 + X_2^4$. Here the table is:

j	$h^0_{\mathbf{m}}(R)_j$	$\dim(R(f)_j)$	$h^0(\mathcal{O}_{\mathrm{Sing}(X)}(j))$
0	0	1	1
1	1	3	2
2	4	6	2
3	5	7	2
4	4	6	2
5	1	3	2
6	0	1	2
7	0	1	2
8	0	1	2

Then X' has genus two, has self-dual local cohomology but $h^0_{\mathbf{m}}(R)_1 = 1 = h^0_{\mathbf{m}}(R)_5 < 2$.

EXAMPLE 5.6. A *reducible plane quartic* consisting of a nonsingular cubic and of an inflectional tangent:

$$f = X_0(X_0^2X_1 + X_0X_1^2 + X_2^3)$$

In this case the table is:

j	$h^0_{\mathbf{m}}(R)_j$	$\dim(R(f)_j)$	$h^0(\mathcal{O}_{\mathrm{Sing}(X)}(j))$
0	0	1	1
1	0	3	3
2	1	6	5
3	2	7	5
4	1	6	5
5	0	5	5
6	0	5	5
7	0	5	5
8	0	5	5

Example 5.7. *A strictly normal crossing quintic surface.* (This example has been kindly suggested by A. Dimca). As an illustration of Theorem 5.3 consider $X = V(f) \subset \mathbb{P}^3$, where

$$f(X_0, \dots, X_3) = (X_0^2 + X_1^2 + X_2^2 + X_3^2)(X_0^3 + X_1^3 + X_2^3 + X_3^3)$$

Then $X = X_1 + X_2$ is the union of a quadric and a cubic, and $C = X_1 \cap X_2$ is a canonical curve (of genus 4). The table of local cohomology is:

j	$h^0_{\mathbf{m}}(R)_j$	$\dim(R(f)_j)$	$h^0(\mathcal{O}_C(j))$
0	0	1	1
1	0	4	4
2	1	10	9
3	5	20	15
4	10	31	21
5	13	40	27
6	11	44	33
7	5	44	39
8	1	46	45
9	0	51	51
10	0	57	57
11	0	63	63
12	0	69	69
13	0	75	75

Note that

$$h^0_{\mathbf{m}}(R)_6 = 11 = (2-1) + (7-1) + 4 = h^{1,1}(X_1) + h^{1,1}(X_2) + g(C)$$

as expected.

6. Torelli-type questions

Following a terminology introduced in [16], a reduced hypersurface $X \subset \mathbb{P}^r$ is called *Torelli* in the sense of Dolgachev-Kapranov if it can be reconstructed from the sheaf $T\langle X\rangle$. In their paper [16] they studied the Torelli property of normal crossing arrangements of hyperplanes. Their main result has been later improved by Vallès in [43]. For arbitrary arrangements of hyperplanes the Torelli problem has been settled in [22]. In [42] it is proved that a smooth hypersurface is Torelli if and only if it is not of Sebastiani-Thom type. E. Angelini [2] studied certain normal crossing configurations of smooth hypersurfaces proving that they are Torelli in several cases.

We want to consider a different reconstruction problem, namely we ask:

Question: Under which circumstances can X be reconstructed from

$$H^0_{\mathbf{m}}(R(f)) \cong H^1_*(T\langle X\rangle(-d))$$

In the nonsingular case this is merely the question of reconstructability of X from its jacobian ring. This question has been considered extensively in the literature, even in the singular case. The typical result one would like to generalize is the following:

THEOREM 6.1. (i) [17] *Let f and f' be homogeneous polynomials of degree d defining reduced hypersurfaces in $\mathbb{P}^r$. If $J(f)_d = J(f')_d$ then f and f' are projectively equivalent.*

(ii) [5] *Let $f \in P$ be a generic polynomial of degree $d \geq 3$. Then f is determined by $J(f)_{d-1}$, up to a constant factor.*

In this respect the following result is relevant:

THEOREM 6.2 ([27]). *A locally free sheaf $\mathcal{F}$ of rank two on $\mathbb{P}^2$ can be reconstructed from the P-module $H^1_*(\mathcal{F})$.*

Theorem 6.2 suggests that, at least in $\mathbb{P}^2$, the reconstructability of X from the module $H^1_*(T\langle X\rangle)$ is equivalent to the reconstructability of X from the sheaf $T\langle X\rangle$. In fact we have the following:

THEOREM 6.3. *A reduced plane curve is Torelli in the sense of Dolgachev-Kapranov if and only if it can be reconstructed from the local cohomology of its jacobian ring.*

Proof. It is an immediate consequence of Theorem 6.2 and of the fact that $T\langle X\rangle$ is locally free for reduced plane curves. □

Theorem 6.3 of course applies to Torelli arrangements of lines, that have been characterized as recalled above, and to normal crossing arrangements of sufficiently many nonsingular curves of the same degree n (see [2] for the precise statement). Much less is known in the irreducible case, even for plane curves. For partial results in this direction we refer the reader to [14]. The Torelli property is related with freeness, that we are going to discuss next.

7. FREENESS

According to Proposition 2.1 the vanishing of $H^0_{\mathbf{m}}(R(f))$ is equivalent to that of $H^1_*(T\langle X\rangle)$ and it is a necessary condition for the freeness of $T\langle X\rangle$. If X is nonsingular then $H^0_{\mathbf{m}}(R(f)) = R(f)$ is never zero, and therefore $T\langle X\rangle$ cannot be free. The same is true if $\mathrm{Sing}(X) \neq \emptyset$ and has codimension ≥ 2 in X, because then $T\langle X\rangle$ is not even locally free.

In general little seems to be known about the freeness of $T\langle X\rangle$, even in the case $r = 2$. We will mostly restrict to this case in the remaining of this section. Look at the exact sequence:

$$0 \longrightarrow T\langle X\rangle(-1) \longrightarrow \mathcal{O}^3_{\mathbb{P}^2} \xrightarrow{\partial f} \mathcal{J}_f(d-1) \longrightarrow 0 \tag{32}$$

Then

$$c_1(T\langle X\rangle(-1)) = 1 - d, \quad c_2(T\langle X\rangle(-1)) = (d-1)^2 - t^1_X$$

where $t^1_X = \dim(T^1_X)$. If $T\langle X\rangle(-1) = \mathcal{O}(-a) \oplus \mathcal{O}(-b)$ is free then

$$a + b = d - 1, \quad ab = (d-1)^2 - t^1_X \tag{33}$$

They imply together that:

$$a^2 + ab + b^2 = t^1_X \tag{34}$$

Observe also that, since under the restriction $a + b = d - 1$ the product ab attains its maximum when (a, b) is balanced, we deduce from (33) the following inequality:

$$(d-1)^2 - I \leq t^1_X \tag{35}$$

where:

$$I = \begin{cases} \frac{(d-1)^2}{4} & \text{if } d \text{ is odd} \\ \frac{d(d-2)}{4} & \text{if } d \text{ is even} \end{cases}$$

These conditions easily imply the following result, whose part (1) is proved in a different way in [33] and part (2) has been subsequently generalized in [14] (see Remark 7.2 below).

Proposition 7.1. *(1) If X is nodal then it is not free unless $f = X_0X_1X_2$.*
(2) If X is irreducible, has n nodes and κ ordinary cusps as its only singularities and it is free then $\kappa \geq \frac{d^2}{4}$.

Proof. 1) If X is nodal of degree $d = a + b + 1$ then $t^1_X \leq \binom{a+b+1}{2}$. It follows that

$$(d-1)^2 - t^1_X \geq (a+b)^2 - \binom{a+b+1}{2} = \binom{a+b}{2} = ab + \frac{1}{2}[a(a-1) + b(b-1)]$$

and this inequality is incompatible with the second condition (33) unless $a = b = 1$. This leaves space for the existence of only one free (reducible) nodal curve: the curve given by $f = X_0X_1X_2$, which is in fact free.
2) Recalling that $t^1_X = n + 2\kappa$ and combining the inequality $n + \kappa \leq \binom{d-1}{2}$ with (35) we obtain:

$$(d-1)^2 - I \leq \kappa + \binom{d-1}{2}$$

Now both possibilities for I give the desired inequality after an easy calculation. □

Remark 7.2. In the recent preprint [14] it has been proved that *all* curves of degree $d \geq 4$ having only nodes and cusps are not free (see loc.cit., Example 4.5(ii)). The method of proof is quite different, so we believe it can be useful to maintain the present weaker statement and its more elementary proof.

Several examples of free arrangements of lines are known. A notable example is the dual of the configuration of flexes of a nonsingular plane cubic. It consists of 9 lines meeting in 12 triple points. Another free arrangement is given by $f = X_0X_1X_2(X_0 - X_1)(X_1 - X_2)(X_0 - X_2)$: it has 4 triple points and 3 double points (see [39], Ex. 3.4).

The first example of free irreducible plane curve has been given by Simis in [33]. It is the sextic X given by the polynomial:

$$f = 4(X^2 + Y^2 + XZ)^3 - 27(X^2 + Y^2)^2 Z^2 \tag{36}$$

It has 4 distinct singular points, defined by the ideal

$$\mathrm{rad}(J) = (YZ, 2X^2 + 2Y^2 - XZ)$$

One of them is a node and the other three are E_6-singularities. This curve is dual to a rational quartic C with three nodes and three undulations (hyperflexes). The E_6-singularities of X are dual to the undulations of C. They have δ-invariant 3 and Tjurina number 6. Thus $t^1_X = 3 \cdot 6 + 1 = 19$. Therefore $a + b = 5$ and $ab = 25 - 19 = 6$ and necessarily

$$T\langle X\rangle(-1) = \mathcal{O}(-3) \oplus \mathcal{O}(-2)$$

An interesting example is the irreducible plane quintic curve X of equation $X_1^5 - X_0^2 X_2^3 = 0$. It has an E_8 and an A_4 singularity. They have respectively $\delta = 4, 2$ thus making the curve rational. On the other hand they have Milnor (equal to Tjurina) numbers equal to $8, 4$ respectively, thus making $t^1_X = 12$. The dual $X^\vee$ is again a quintic. According to (33), if X were free one should have

$$T\langle X\rangle(-1) = \mathcal{O}(-2) \oplus \mathcal{O}(-2)$$

But (f_0, f_1, f_2) has a linear syzygy (Example 2.5) and therefore this cannot be. Other series of free irreducible plane curves are given in [4, 28, 32, 34, 39]. For a detailed discussion of freeness and more examples in the case of plane curves we refer to [14].

EXAMPLE 7.3. The *Steiner quartic surface* in $\mathbb{P}^3$, has equation in normal (Weierstrass) form: $Z^2T^2 + T^2Y^2 + Y^2Z^2 = XYZT$. It is irreducible and singular along the three coordinate axes for the origin $(0, 0, 0, 1)$. The jacobian ideal is

$$J = (YZT, 2YZ^2 - XZT + 2YT^2, 2Y^2Z - XYT + 2ZT^2, XYZ - 2Y^2T - 2Z^2T)$$

and it turns out that $J^{sat} = J$. Therefore $H^0_{\mathbf{m}}(R(f)) = 0$. Nevertheless it can be computed that $T\langle X\rangle$ is not free.

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E. Sernesi
Dipartimento
di Matematica e Fisica
Università Roma Tre
Largo S. L. Murialdo 1
00146 Roma, Italy
sernesi@mat.uniroma3.it