Semistable Modules over Lie Algebroids in Positive Characteristic

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Abstract. We study Lie algebroids in positive characteristic and moduli spaces of their modules. In particular, we show a Langton's type theorem for the corresponding moduli spaces. We relate Langton's construction to Simpson's construction of gr-semistable Griffiths transverse filtration. We use it to prove a recent conjecture of Lan-Sheng-Zuo that semistable systems of Hodge sheaves on liftable varieties in positive characteristic are strongly semistable.

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Introduction

In this paper we give a general approach to relative moduli spaces of modules over Lie algebroids. As a special case one recovers Simpson's "non-abelian Hodge filtration" moduli space (see [Si4] and [Si5]). This allows to consider Higgs sheaves and sheaves with integrable connections at the same time as objects corresponding to different fibers of the relative moduli space of modules over a deformation of a Lie algebroid over an affine line.
A large part of the paper is devoted to generalizing various facts concerning vector bundles with connections to modules over Lie algebroids. In particular, we introduce restricted Lie algebroids, which generalize Ekedahl's 1-foliations [Ek]. In positive characteristic we define a p-curvature for modules over restricted Lie algebroids. This leads to a deformation of the morphism given by p-curvature on the moduli space of

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modules to the Hitchin morphism corresponding to the trivial Lie algebroid structure. In the special case of bundles with connections on curves this deformation was already studied by Y. Laszlo and Ch. Pauly [LP].
We prove Langton's type theorem for the moduli spaces of modules over Lie algebroids. We compare it via Rees' construction with Simpson's inductive construction of gr-semistable Griffiths transverse filtration (see [Si5]), concluding that the latter must finish.
This leads to the main application of our results. Namely, we obtain a canonical gr-semistable Griffiths transverse filtration on a module over a Lie algebroid. This implies a recent conjecture of Lan-Sheng-Zuo that semistable systems of Hodge sheaves on liftable varieties in positive characteristic are strongly semistable.
The rank 2 case of this conjecture was proven in [LSZ], the rank 3 case in [Li]. Recently, independently of the author Lan, Sheng, Yang and Zuo [LSYZ] also proved the Lan-Sheng-Zuo conjecture using a similar approach. However, they give a different proof that Simpson's inductive construction must finish. They also obtain a slightly weaker result proving their conjecture only for an algebraic closure of a finite field.

The results of this paper are used in [La3] to prove Bogomolov's type inequality for Higgs sheaves on varieties liftable modulo p^2.

0.1 Notation

If X is a scheme and E is a quasi-coherent $\mathcal{O}_X$-module then we set $E^* = \mathcal{H}om_{\mathcal{O}_X}(E, \mathcal{O}_X)$ and $\mathbb{V}(E) = \mathrm{Spec}\,(S^\bullet E)$.
Let S be a scheme of characteristic p (i.e., $\mathcal{O}_S$ is an $\mathbb{F}_p$-algebra). By $F_S^r : S \to S$ we denote the r-th *absolute Frobenius morphism* of S which corresponds to the p^r-th power mapping on $\mathcal{O}_S$. If X is an S-scheme, we denote by $X^{(1/S)}$ the fiber product of X and S over the (1-st) absolute Frobenius morphism of S. The absolute Frobenius morphism of X induces the *relative Frobenius morphism* $F_{X/S} : X \to X^{(1/S)}$.

Let X be a projective scheme over some algebraically closed field k. Let $\mathcal{O}_X(1)$ be an ample line bundle on X. For any coherent sheaf E on X we define its *Hilbert polynomial* by $P(E)(n) = \chi(X, E(n))$ for $n \in \mathbb{Z}$. If d is the dimension of the support of E then we can write

$$P(E)(n) = \frac{r(E)n^d}{d!} + \text{lower order terms in } n.$$

The (rational) number $r = r(E)$ is called the *generalized rank* of E (note that if X is not integral then the generalized rank of a sheaf depends on the polarization). The quotient $p(E) = \frac{P(E)}{r(E)}$ is called the *normalized Hilbert polynomial* of E.
In case X is a variety then for a torsion free sheaf E the generalized rank $r(E)$ is a product of the degree of X with respect to $\mathcal{O}_X(1)$ and of the usual rank.
If X is normal and E is a rank r torsion free sheaf on X then we define the *slope* $\mu(E)$ of E as the quotient of the degree of $\det E = (\bigwedge^r E)^{**}$ with respect to $\mathcal{O}_X(1)$ by the rank r. In some cases we consider generalized slopes defined with respect to a fixed 1-cycle class, coming from a collection of nef divisors on X.

Let us recall that E is *slope semistable* if for every subsheaf $E' \subset E$ we have $\mu(E') \leq \mu(E)$.

1 Moduli spaces of modules over sheaves of rings of differential operators

In this section we recall some definitions and the theorem on existence of moduli spaces of modules over sheaves of rings of differential operators. This combines the results of Simpson [Si2] with the results of [La1] and [La2].

Let S be a locally noetherian scheme and let $f: X \to S$ be a scheme of finite type over S. A *sheaf of (associative and unital) $\mathcal{O}_S$-algebras $\mathcal{A}$ on X* is a sheaf $\mathcal{A}$ on X of (possibly non-commutative) rings of $\mathcal{O}_X$-bimodules such that the image of $f^{-1}\mathcal{O}_S \to \mathcal{A}$ is contained in the center of $\mathcal{A}$.

Let us recall after [Si2] that a *sheaf of rings of differential operators on X over S* is a sheaf Λ of $\mathcal{O}_S$-algebras on X, with a filtration $\Lambda_0 \subset \Lambda_1 \subset ...$ by subsheaves of abelian subgroups satisfying the following properties:

1. $\Lambda = \bigcup_{i=0}^{\infty} \Lambda_i$ and $\Lambda_i \cdot \Lambda_j \subset \Lambda_{i+j}$,
2. the image of $\mathcal{O}_X \to \Lambda$ is equal to Λ_0,
3. the left and right $\mathcal{O}_X$-module structures on $\mathrm{Gr}_i(\Lambda) := \Lambda_i/\Lambda_{i-1}$ coincide and the $\mathcal{O}_X$-modules $\mathrm{Gr}_i(\Lambda)$ are coherent,
4. the sheaf of graded $\mathcal{O}_X$-algebras $\mathrm{Gr}(\Lambda) := \bigoplus_{i=0}^{\infty} \mathrm{Gr}_i(\Lambda)$ is generated in degree 1, i.e., the canonical graded morphism from the tensor $\mathcal{O}_X$-algebra $T^{\bullet}\,\mathrm{Gr}_1(\Lambda)$ of $\mathrm{Gr}_1(\Lambda)$ to $\mathrm{Gr}(\Lambda)$ is surjective.

Note that in positive characteristic, the sheaf of rings of crystalline differential operators (see [BMR] or Subsection 2.2) is a sheaf of rings of differential operators, but the sheaf of rings of usual differential operators is not as it almost never is generated in degree 1.

Assume that S is a scheme of finite type over a universally Japanese ring R. Let $f: X \to S$ be a projective morphism of R-schemes of finite type with geometrically connected fibers and let $\mathcal{O}_X(1)$ be an f-very ample line bundle. Let Λ be a sheaf of rings of differential operators on X over S.

A Λ-*module* is a sheaf of (left) Λ-modules on X which is quasi-coherent with respect to the induced $\mathcal{O}_X$-module structure.

Let $T \to S$ be a morphism of R-schemes with T locally noetherian over S. Let us set $X_T = X \times_S T$ and let p be the projection of X_T onto X. Then $\Lambda_T = \mathcal{O}_{X_T} \otimes_{p^{-1}\mathcal{O}_X} p^{-1}\Lambda$ has a natural structure of a sheaf of rings of differential operators on X_T over T. Moreover, if E is a Λ-module on X then the pull back $E_T = p^*E$ has a natural structure of a Λ_T-module.

Note that if E is a Λ-module and $E' \subset E$ is a quasi-coherent $\mathcal{O}_X$-submodule such that $\Lambda_1 \cdot E' \subset E'$ then E' has a unique structure of Λ-module compatible with the Λ-module structure on E (i.e., such that E' is a Λ-submodule of E).

Let Y be a projective scheme over an algebraically closed field k (with fixed polarization) and let Λ_Y be a sheaf of rings of differential operators on Y. Let E be a Λ_Y-module which is coherent as an $\mathscr{O}_Y$-module. E is called *Gieseker (semi)stable* if it is of pure dimension as an $\mathscr{O}_Y$-module (i.e., all its associated points have the same dimension) and for any Λ_Y-submodule $F \subset E$ we have inequality $p(F) < p(E)$ ($p(F) \leq p(E)$, respectively) of normalized Hilbert polynomials.
Every Gieseker semistable Λ_Y-module E has a filtration $0 = E_0 \subset E_1 \subset \ldots \subset E_m = E$ by Λ_Y-submodules such that the associated graded $\oplus_{i=0}^{m} E_i/E_{i-1}$ is a *Gieseker polystable* Λ_Y-module (i.e., it is a direct sum of Gieseker stable Λ_Y-modules with the same normalized Hilbert polynomial). Such a filtration is called a *Jordan–Hölder filtration* of this Λ_Y-module.

Now let us go back to the relative situation, i.e., Λ on X over S (over R).
A *family of Gieseker semistable* Λ*-modules on the fibres of* $p_T : X_T = X \times_S T \to T$ is a Λ_T-module E on X_T which is T-flat (as an $\mathscr{O}_{X_T}$-module) and such that for every geometric point t of T the restriction of E to the fibre X_t is pure and Gieseker semistable as a Λ_t-module.
We introduce an equivalence relation $\sim$ on such families by saying that $E \sim E'$ if and only if there exists an invertible $\mathscr{O}_T$-module L such that $E' \simeq E \otimes p_T^* L$.
Let us define the moduli functor

$$\underline{M}^{\Lambda}(X/S,P) : (\mathrm{Sch}/S)^o \to \mathrm{Sets}$$

from the category of locally noetherian schemes over S to the category of sets by

$$\underline{M}^{\Lambda}(X/S,P)(T) = \left\{ \begin{array}{l} \sim \text{equivalence classes of families of Gieseker} \\ \text{semistable } \Lambda\text{-modules on the fibres of } X_T \to T, \\ \text{which have Hilbert polynomial } P. \end{array} \right\}.$$

Then we have the following theorem summing up the results of Simpson and the author (see [Si2, Theorem 4.7], [La1, Theorem 0.2] and [La2, Theorem 4.1]).

THEOREM 1.1. *Let us fix a polynomial P. Then there exists a quasi-projective S-scheme $M^{\Lambda}(X/S,P)$ of finite type over S and a natural transformation of functors*

$$\varphi : \underline{M}^{\Lambda}(X/S,P) \to \mathrm{Hom}_S(\cdot, M^{\Lambda}(X/S,P)),$$

which uniformly corepresents the functor $\underline{M}^{\Lambda}(X/S,P)$.
For every geometric point $s \in S$ the induced map $\varphi(s)$ is a bijection. Moreover, there is an open scheme $M^{\Lambda,s}(X/S,P) \subset M^{\Lambda}(X/S,P)$ that universally corepresents the subfunctor of families of geometrically Gieseker stable Λ-modules.

In general, for every locally noetherian S-scheme T we have a well defined morphism $M^{\Lambda}(X/S,P) \times_S T \to M^{\Lambda_T}(X_T/T,P)$ which is a bijection of sets if T is a geometric point of S.
Let us recall that a scheme $M^{\Lambda}(X/S,P)$ *uniformly corepresents* $\underline{M}^{\Lambda}(X/S,P)$ if for every flat base change $T \to S$ the fiber product $M^{\Lambda}(X/S,P) \times_S T$ corepresents the fiber product functor $\mathrm{Hom}_S(\cdot,T) \times_{\mathrm{Hom}_S(\cdot,S)} \underline{M}^{\Lambda}(X/S,P)$.

2 Lie algebroids

2.1 Lie algebroids and de Rham complexes

Let $f: X \to S$ be a morphism of schemes. A *sheaf of $\mathscr{O}_S$-Lie algebras on X* is a pair $(L, [\cdot,\cdot]_L)$ consisting of a (left) $\mathscr{O}_X$-module L (which is an $f^{-1}\mathscr{O}_S$-bimodule) with a morphism of $f^{-1}\mathscr{O}_S$-modules $[\cdot,\cdot]_L : L \otimes_{f^{-1}\mathscr{O}_S} L \to L$, which is alternating and which satisfies the Jacobi identity. A homomorphism of sheaves of $\mathscr{O}_S$-Lie algebras on X is an $\mathscr{O}_X$-linear morphism $L \to L'$ which preserves the Lie bracket. As usual for $x \in L(U)$ we define $\mathrm{ad}\, x : L(U) \to L(U)$ by $(\mathrm{ad}\, x)(y) = [x,y]_L$.

Let $T_{\mathscr{O}_S}(L) = \bigoplus_{n\geq 0} \overbrace{L \otimes_{f^{-1}\mathscr{O}_S} \dots \otimes_{f^{-1}\mathscr{O}_S} L}^{n}$ be the tensor algebra of L over $f^{-1}\mathscr{O}_S$ (it is a non-commutative $f^{-1}\mathscr{O}_S$-algebra). Let us recall that the *universal enveloping algebra* $\mathscr{U}_{\mathscr{O}_S}(L)$ of a Lie algebra sheaf $(L, [\cdot,\cdot]_L)$ is defined as the quotient of $T_{\mathscr{O}_S}(L)$ by the two-sided ideal generated by $x \otimes y - y \otimes x - [x,y]_L$ for all local sections $x, y \in L$.
The most important example of a sheaf of $\mathscr{O}_S$-Lie algebras on X is the relative tangent sheaf $T_{X/S} = \mathscr{D}er_{\mathscr{O}_S}(\mathscr{O}_X, \mathscr{O}_X)$ with a natural bracket given by $[D_1, D_2] = D_1D_2 - D_2D_1$ for local $\mathscr{O}_S$-derivations D_1, D_2 of $\mathscr{O}_X$.

Definition 2.1. An *$\mathscr{O}_S$-Lie algebroid on X* is a triple $(L, [\cdot,\cdot]_L, \alpha)$ consisting of a sheaf of $\mathscr{O}_S$-Lie algebras $(L, [\cdot,\cdot]_L)$ on X and a homomorphism $\alpha : L \to T_{X/S}$, $x \to \alpha_x$, of sheaves of $\mathscr{O}_S$-Lie algebras on X, which satisfies the following Leibniz rule

$$[x, fy]_L = \alpha_x(f)y + f[x,y]_L$$

for all local sections $f \in \mathscr{O}_X$ and $x, y \in L$ (in the formula we treat α_x as an $\mathscr{O}_S$-derivation of $\mathscr{O}_X$). We say that L is *smooth* if it is coherent and locally free as an $\mathscr{O}_X$-module. L is *quasi-smooth* if it is coherent and torsion free as an $\mathscr{O}_X$-module.

The map α in the above definition is usually called *the anchor*. A Lie algebroid is a sheaf of Lie-Rinehart algebras (see [Ri]). It is also a special case of the more general notion of a Lie algebra in a topos defined by Illusie (see [Il, Chapitre VIII, Definition 1.1.5]).
A *homomorphism of $\mathscr{O}_S$-Lie algebroids* L and L' on X is a homomorphism $L \to L'$ of sheaves of $\mathscr{O}_S$-Lie algebras on X which commutes with the anchors.
Note that an $\mathscr{O}_S$-Lie algebroid on X with the zero anchor map corresponds to a sheaf of $\mathscr{O}_X$-Lie algebras.

Definition 2.2. A *de Rham complex on X over S* is a pair $(\bigwedge^\bullet M, d_M^\bullet)$ consisting of the exterior algebra $\bigwedge^\bullet M := \bigwedge^\bullet_{\mathscr{O}_X} M$ of an $\mathscr{O}_X$-module M and an $\mathscr{O}_S$-anti-derivation $d_M^\bullet : \bigwedge^\bullet M \to \bigwedge^\bullet M$ of degree 1 (i.e., $d_M^\bullet(x \wedge y) = (d_M^\bullet x) \wedge y + (-1)^j x \wedge d_M^\bullet y$ for all local sections $x \in \bigwedge^j M$ and $y \in \bigwedge^\bullet M$) such that $(d_M^\bullet)^2 = 0$. We say that $(\bigwedge^\bullet M, d_M^\bullet)$ is *smooth* if M is coherent and locally free.

A de Rham complex is a special case of a sheaf of graded-commutative differential graded algebras. A special case of a de Rham complex is the de Rham complex $(\Omega^\bullet_{X/S}, d^\bullet_{X/S})$, which is the unique de Rham complex extending the canonical $\mathscr{O}_S$-derivation $d_{X/S} : \mathscr{O}_X \to \Omega_{X/S}$ (uniqueness follows because $\Omega_{X/S}$ is generated by $d_{X/S}\mathscr{O}_X$ as a left $\mathscr{O}_X$-module). By the universal property of $d_{X/S}$ we have $\mathrm{Der}_{\mathscr{O}_S}(\mathscr{O}_X, M) \simeq \mathrm{Hom}_{\mathscr{O}_X}(\Omega_{X/S}, M)$ and hence for every de Rham complex $(\bigwedge^\bullet M, d^\bullet_M)$ we have a unique morphism of de Rham complexes $(\Omega^\bullet_{X/S}, d^\bullet_{X/S}) \to (\bigwedge^\bullet M, d^\bullet_M)$. This morphism induces a well defined map on the hypercohomology groups:

$$H^i_{DR}(X/S) := \mathbb{H}^\bullet(\Omega^\bullet_{X/S}) \to \mathbb{H}^\bullet(\bigwedge^\bullet M).$$

To every $\mathscr{O}_S$-Lie algebroid $(L, [\cdot,\cdot]_L, \alpha)$ on X we can associate a de Rham complex $(\bigwedge^\bullet M, d^\bullet_M)$ on X over S for $M = L^*$. This is done by the following well known formula generalizing the usual exterior differential:

$$\begin{array}{rcl}(d_M m)(l_1,\dots,l_{k+1}) & = & \sum_{i=1}^{k+1}(-1)^{i+1}\alpha_{l_i}(m(l_1,\dots,\hat{l}_i,\dots,l_{k+1})) \\ & + & \sum_{1\le i<j\le k+1}(-1)^{i+j} m([l_i,l_j]_L, l_1,\dots,\hat{l}_i,\dots,\hat{l}_j,\dots,l_{k+1})\end{array}$$

for $m \in \bigwedge^k M$ and $l_1,\dots,l_{k+1} \in L$. This gives a functor from the category of Lie algebroids to the category of de Rham complexes.
On the other hand, to every de Rham complex $(\bigwedge^\bullet M, d^\bullet_M)$ on X over S we can associate a Lie algebroid structure on $L = M^*$. The anchor $L \to T_{X/S} = (\Omega_{X/S})^*$ is obtained as the transpose of the $\mathscr{O}_X$-homomorphism $\Omega_{X/S} \to M$ corresponding to the $\mathscr{O}_S$-derivation $d_M : \mathscr{O}_X \to M$. The bracket on L can be read off the above formula defining $d_M : M \to \bigwedge^2 M$. This provides a functor in the opposite direction: from the category of de Rham complexes to the category of Lie algebroids. These functors are quasi-inverse on subcategories of smooth objects.

If L is a smooth $\mathscr{O}_S$-Lie algebroid on X then the corresponding de Rham complex is denoted by $(\Omega^\bullet_L, d^\bullet_L)$. In this case we set

$$H^i_{DR}(L) := \mathbb{H}^i(\Omega^\bullet_L, d^\bullet_L).$$

We have the following standard spectral sequence associated to the de Rham complex of L:

$$E_1^{ij} = H^j(X/S, \Omega^i_L) \Rightarrow H^{i+j}_{DR}(L).$$

2.2 Universal enveloping algebra of differential operators

DEFINITION 2.3. A *sheaf of $\mathscr{O}_S$-Poisson algebras on X* is a pair $(\mathscr{A}, \{\cdot,\cdot\})$ consisting of a sheaf $\mathscr{A}$ of commutative, associative and unital $\mathscr{O}_X$-algebras with a Poisson bracket $\{\cdot,\cdot\}$ such that $(\mathscr{A}, \{\cdot,\cdot\})$ is a sheaf of $\mathscr{O}_S$-Lie algebras on X satisfying the Leibniz rule

$$\{x, y\cdot z\} = \{x,y\}\cdot z + y\cdot\{x,z\}$$

for all $x,y,z \in \mathscr{A}$.

Let Λ be a sheaf of rings of differential operators on X over S such that $\Lambda_0 = \mathscr{O}_X$. Let us assume that Λ is *almost commutative*, i.e., the associated graded $\mathrm{Gr}(\Lambda)$ is a sheaf of commutative $\mathscr{O}_X$-algebras. Then $\mathrm{Gr}(\Lambda)$ has a natural structure of a sheaf of $\mathscr{O}_S$-Poisson algebras on X with the Poisson bracket given by

$$\{[x],[y]\} := \big(xy - yx \mod \Lambda_{i+j-2}\big) \in \mathrm{Gr}_{i+j-1}(\Lambda),$$

where $[x] \in \mathrm{Gr}_i(\Lambda)$ is the class of $x \in \Lambda_i$ and $[y] \in \mathrm{Gr}_j(\Lambda)$ is the class of $y \in \Lambda_j$. The Poisson bracket induces an $\mathscr{O}_S$-Lie algebroid structure on $\mathrm{Gr}_1(\Lambda)$. The Lie bracket on $\mathrm{Gr}_1(\Lambda)$ is equal to the Poisson bracket and the anchor map $\alpha : \mathrm{Gr}_1(\Lambda) \to T_{X/S}$ is given by sending $[x]$ to the $\mathscr{O}_S$-derivation $y \to \{[x], y\}$, $y \in \mathscr{O}_X = \mathrm{Gr}_0(\Lambda)$.
On the other hand, if L is an $\mathscr{O}_S$-Lie algebroid on X then we can associate to L a sheaf of rings of differential operators on X over S in the following way. We define an $\mathscr{O}_S$-Lie algebra structure on $\tilde{L} = \mathscr{O}_X \oplus L$ by setting

$$[f + x, g + y]_{\tilde{L}} = \alpha_x(g) - \alpha_y(f) + [x,y]_L$$

for all local sections $f, g \in \mathscr{O}_X$ and $x, y \in L$. Let $\mathscr{U}_{\mathscr{O}_S}(\tilde{L})$ be the universal enveloping algebra of $\tilde{L}$ and let $\tilde{\mathscr{U}}_{\mathscr{O}_S}(\tilde{L})$ be the sheaf of subalgebras (without unit!) generated by the image of the canonical map $i_{\tilde{L}} : \tilde{L} \to \mathscr{U}_{\mathscr{O}_S}(\tilde{L})$ (note that in general this map need not be injective). We define Λ_L as the quotient of $\tilde{\mathscr{U}}_{\mathscr{O}_S}(\tilde{L})$ by the two-sided ideal generated by all elements of the form $i_{\tilde{L}}(f) i_{\tilde{L}}(x) - i_{\tilde{L}}(fx)$ for all $f \in \mathscr{O}_X$ and $x \in \tilde{L}$. Let $\Lambda_{L,j}$ be the left $\mathscr{O}_X$-submodule of Λ_L generated by products of at most j elements of the image of L in Λ_L. This defines a filtration of Λ_L equipping it with structure of sheaf of rings of differential operators (since the canonical graded morphism $S^\bullet \mathrm{Gr}_1(\Lambda_L) \to \mathrm{Gr}(\Lambda_L)$ is surjective, the constructed Λ_L is almost commutative). We call Λ_L *the universal enveloping algebra of differential operators associated to L.*
By the Poincare-Birkhoff-Witt theorem, if the Lie algebroid L is smooth then $L \to \mathrm{Gr}_1(\Lambda_L)$ is an isomorphism and the canonical epimorphism $S^\bullet L \to \mathrm{Gr}(\Lambda_L)$ is an isomorphism of sheaves of graded $\mathscr{O}_X$-algebras (see [Ri, Theorem 3.1]). This implies that if L is quasi-smooth then the canonical map $L \to \Lambda_L$ is injective.
If $L = T_{X/S}$ and the anchor map is identity, then Λ_L is denoted by $\mathscr{D}_{X/S}$ and it is called *the sheaf of crystalline differential operators* (see [BMR]). In [BO] the authors call it the sheaf of PD differential operators. In the characteristic zero case the sheaf Λ_L and the correspondence between Lie algebroids and sheaves of rings of differential operators was studied by Simpson in [Si2, Theorem 2.11] with subsequent corrections by Tortella in [To, Theorem 4.4].

We can also consider twisted versions of sheaves of rings of differential operators associated to a Lie algebroid (see [BB] and [To]).
Let Λ be an almost commutative sheaf of rings of differential operators on X over S such that $\Lambda_0 = \mathscr{O}_X$. Then Λ_1 has an $\mathscr{O}_S$-Lie algebra structure on X given by the usual Lie bracket $[\cdot,\cdot]$ coming from Λ and the anchor map given by sending $x \in \Lambda_1$ to $f \to [x, f]$. Then $\Lambda_1 \to \mathrm{Gr}_1(\Lambda)$ is a homomorphism of $\mathscr{O}_S$-Lie algebras with kernel being the sheaf $\mathscr{O}_X$ (with a trivial $\mathscr{O}_S$-Lie algebroid structure).
The following definition is motivated by [BB, Definition 2.1.3]:

Definition 2.4. A *generalized $\mathcal{O}_S$-Picard Lie algebroid* on X is an $\mathcal{O}_S$-Lie algebroid $\tilde{L}$ equipped with a section $1_{\tilde{L}}$ of $\tilde{L}$ inducing an exact sequence of $\mathcal{O}_S$-Lie algebroids

$$0 \to \mathcal{O}_X \to \tilde{L} \to L \to 0,$$

where $\mathcal{O}_X$ is taken with the trivial $\mathcal{O}_S$-Lie algebroid structure.

To any generalized $\mathcal{O}_S$-Picard Lie algebroid $\tilde{L}$ we can associate an almost commutative sheaf of rings of differential operators $\tilde{\Lambda}_{\tilde{L}}$ on X over S such that $\tilde{\Lambda}_{\tilde{L},0} = \mathcal{O}_X$ and $\tilde{\Lambda}_{\tilde{L},1} = \tilde{L}$. $\tilde{\Lambda}_{\tilde{L}}$ is constructed as a quotient of the universal enveloping algebra of differential operators $\Lambda_{\tilde{L}}$ by the two-sided ideal generated by $1_{\tilde{L}} - 1$. As in [BB, Lemma 2.1.4], this defines a fully faithful functor from the category of generalized Picard Lie algebroids to the category of almost commutative sheaves of rings of differential operators.
The analogous construction can be also found in [To], where the author constructs $\tilde{\Lambda}_{\tilde{L}}$ by gluing local pieces.

3 Modules over Lie algebroids

3.1 Modules with generalized connections

Let X be an S-scheme. Let M be a coherent $\mathcal{O}_X$-module with an $\mathcal{O}_S$-derivation $d_M : \mathcal{O}_X \to M$. A *$d_M$-connection* on a coherent $\mathcal{O}_X$-module E is an $\mathcal{O}_S$-linear morphism $\nabla : E \to E \otimes_{\mathcal{O}_X} M$ satisfying the following Leibniz rule

$$\nabla(fe) = f\nabla(e) + e \otimes d_M(f)$$

for all local sections $f \in \mathcal{O}_X$ and $e \in E$.
Note that notion of d_M-connection depends on the choice of derivation d_M and not only the sheaf M. For example if $M = \Omega_{X/S}$ then the standard derivation $d_{X/S}$ leads to a sheaf with a usual connection whereas the zero derivation leads to a Higgs sheaf (but without any integrability condition).

3.2 Generalized Higgs sheaves

Assume that $(\bigwedge^{\bullet} M, d_M^{\bullet})$ is a de Rham complex and let E be a coherent $\mathcal{O}_X$-module. Then a d_M-connection $\nabla : E \to E \otimes M$ can be extended to a morphism $\nabla_i : E \otimes_{\mathcal{O}_X} \bigwedge^i M \to E \otimes_{\mathcal{O}_X} \bigwedge^{i+1} M$ by setting

$$\nabla_i(e \otimes \omega) = e \otimes d_M \omega + (-1)^i \nabla(e) \wedge \omega,$$

where $e \in E$ and $\omega \in \bigwedge^i M$ are local sections. As usually one can check that the *curvature* $K = \nabla_1 \circ \nabla$ is $\mathcal{O}_X$-linear and $\nabla_{i+1} \circ \nabla_i(e \otimes \omega) = K(e) \wedge \omega$. We say that (E, ∇) is *integrable* if the curvature $K = 0$. If (E, ∇) is integrable then the sequence

$$0 \to E \stackrel{\nabla}{\to} E \otimes M \stackrel{\nabla_1}{\to} E \otimes \bigwedge^2 M \to \dots$$

becomes a complex. The hypercohomology groups of this complex are denoted by $H^i_{DR}(X,E) := \mathbb{H}^i(E\otimes \bigwedge^\bullet M, \nabla)$.

Let $\bigwedge^\bullet M$ be the de Rham complex corresponding to the exterior $\mathscr{O}_S$-algebra of M with zero anti-derivation d_M. Then a coherent $\mathscr{O}_X$-module with an integrable d_M-connection $\theta : E\to E\otimes_{\mathscr{O}_X} M$ is called an *M-Higgs sheaf.* The corresponding homomorphism θ is $\mathscr{O}_X$-linear and it is called an *M-Higgs field* (or just a Higgs field). A *system of M-Hodge sheaves* is an M-Higgs sheaf (E,θ) with decomposition $E=\bigoplus E^j$ such that $\theta : E^j\to E^{j-1}\otimes M$. For $M=\Omega_{X/S}$ we recover the usual notions of a Higgs sheaf and a system of Hodge sheaves.

To be consistent with notation in the characteristic zero case, the hypercohomology groups $\mathbb{H}^i(E\otimes \bigwedge^\bullet M,\theta)$ of the complex associated to an M-Higgs sheaf are denoted by $H^i_{Dol}(X,E)$. The following lemma can be proven in the same way as [Si1, Lemma 2.5]:

Lemma 3.1. *Let X be a smooth d-dimensional projective variety over an algebraically closed field k and let (E,θ) be an M-Higgs sheaf. Then we have $\chi_{Dol}(X,E) = \mathrm{rk} E\cdot \chi_{Dol}(X,\mathscr{O}_X)$. Moreover, if E is locally free then we have a perfect pairing*

$$H^i_{Dol}(X,E)\otimes H^{2d-i}_{Dol}(X,E^*)\to k$$

induced by Serre's duality.

3.3 Modules over Lie algebroids and coHiggs sheaves

Let L be an $\mathscr{O}_S$-Lie algebroid on X and let E be an $\mathscr{O}_X$-module. Let us recall that a (left) Λ_L-module structure on E is the same as an L-module structure, i.e., a homomorphism $\nabla : L\to \mathscr{E}nd_{\mathscr{O}_S} E$ of sheaves of $\mathscr{O}_S$-Lie algebras on X (in particular, ∇ is $\mathscr{O}_X$-linear) satisfying Leibniz's rule

$$\nabla(x)(fe) = \alpha_x(f)e + \nabla(fx)(e)$$

for all local sections $f\in\mathscr{O}_X$, $x\in L$ and $e\in E$. One can also look at L-modules E as modules E over the sheaf of $\mathscr{O}_S$-Lie algebras $\tilde{L} = \mathscr{O}_X\oplus L$ on X defined in Subsection 2.2, which satisfy equality $(fy)e = f(ye)$ for all local sections $f\in\mathscr{O}_X$, $y\in L'$ and $e\in E$.

Proof of the following easy lemma is left to the reader:

Lemma 3.2. *Let L be a smooth $\mathscr{O}_S$-Lie algebroid L and let $(\bigwedge^\bullet \Omega_L, d^\bullet_{\Omega_L})$ be the associated de Rham complex. Then we have an equivalence of categories between the category of L-modules and coherent $\mathscr{O}_X$-modules with integrable d_{Ω_L}-connection.*

Let L be a coherent $\mathscr{O}_X$-module. Let us provide it with the trivial $\mathscr{O}_S$-Lie algebroid structure, i.e., we take zero bracket and zero anchor map. In this case we say that L is *a trivial Lie algebroid.* For a trivial Lie algebroid the corresponding sheaf of rings of differential operators Λ_L is equal to the (commutative) symmetric $\mathscr{O}_X$-algebra $S^\bullet(L)$. In this case *an L-coHiggs sheaf* is a (left) Λ_L-module, coherent as an $\mathscr{O}_X$-module. If L is smooth then giving an L-coHiggs sheaf is equivalent to giving an Ω_L-Higgs sheaf.

If L is smooth then $\mathbb{V}(L)\to X$ is a vector bundle and we can take its projective completion $\pi: Y=\mathbb{P}(L\oplus\mathscr{O}_X)\to X$. The divisor at infinity $D=Y-\mathbb{V}(L)$ is canonically isomorphic to $\mathbb{P}(L)$. On Y we have the tautological relatively ample line bundle $\mathscr{O}_{\mathbb{P}(L\oplus\mathscr{O}_X)}(1)$. If $\mathscr{O}_X(1)$ is an S-ample polarization on X then for sufficiently large n the line bundle $\mathscr{A}=\mathscr{O}_{\mathbb{P}(L\oplus\mathscr{O}_X)}(1)\otimes\pi^*(\mathscr{O}_X(n))$ is also S-ample.
By definition any L-coHiggs sheaf gives rise to a coherent $\mathscr{O}_{\mathbb{V}(L)}$-module. The following lemma describes image of the corresponding functor (cf. [Si3, Lemma 6.8 and Corollary 6.9]):

Lemma 3.3. *We have an equivalence of categories between L-coHiggs sheaves and coherent sheaves on Y, whose support does not intersect D. Under this equivalence pure sheaves correspond to pure sheaves of the same dimension and the notions of (semi)-stability are the same when considered with respect to polarizations $\mathscr{O}_X(1)$ on X and $\mathscr{A}$ on Y.*

This lemma suggests another construction of the moduli space $M^L_{\mathrm{Dol}}(X/S,P)=M^{\Lambda_L}(X/S,P)$ of Gieseker semistable L-coHiggs sheaves (with fixed Hilbert polynomial P) on X/S using construction of the moduli space $M(Y/S,P)$ of Gieseker semistable sheaves of pure dimension $n=\dim(X/S)$ on Y/S (with Hilbert polynomial P). Namely, $M(Y/S,P)$ is constructed as a GIT quotient $R/\!/G$, where R is some parameter space and G is a reductive group acting on R. Then $M^L_{\mathrm{Dol}}(X/S,P)$ can be constructed as the quotient $R'/\!/G$, where R' is the G-invariant subscheme of R corresponding to subsheaves whose support does not intersect D.

3.4 Modules on varieties over fields

In this subsection we take as S the spectrum of an algebraically closed field k. We also assume that X is normal and projective with fixed polarization $\mathscr{O}_X(1)$.
We say that a sheaf with an M-connection (E,∇) is *slope semistable* if E is torsion free as an $\mathscr{O}_X$-module and if for any $\mathscr{O}_X$-submodule $E'\subset E$ such that $\nabla(E')\subset E'\otimes_{\mathscr{O}_X}M$ we have

$$\mu(E')\le\mu(E).$$

We say that (E,∇) is *slope stable* if we have stronger inequality $\mu(E')<\mu(E)$ for every proper $\mathscr{O}_X$-submodule $E'\subset E$ preserved by ∇ and such that $\mathrm{rk}E'<\mathrm{rk}E$. In much the same way we can introduce notions of slope (semi)stability for M-Higgs sheaves and systems of M-Hodge sheaves. In each case to define (semi)stability we use only subobjects in the corresponding category.

Let us fix a smooth k-Lie algebroid L on X. We have a natural action of $\mathbb{G}_m$ on Ω_L-Higgs sheaves given by sending (E,θ) to $(E,t\theta)$ for $t\in\mathbb{G}_m$. The following lemma is a simple generalization of the well known fact in case of usual Higgs bundles (see, e.g., [Si1, Lemma 4.1]) but we include proof for completeness. The assertion in the positive characteristic case is slightly different to that of [Si1, Lemma 4.1]. The difference comes from the fact that for $k=\bar{\mathbb{F}}_p$ every $t\in k^*$ is a root of unity.

Lemma 3.4. *A rank r torsion free Ω_L-Higgs sheaf (E,θ) is a fixed point of the $\mathbb{G}_m$-action if and only if it has a structure of system of Ω_L-Hodge sheaves.*

Proof. Taking reflexivization we can assume that E is reflexive. By assumption for every $t \in \mathbb{G}_m$ there exists an isomorphism of $\mathcal{O}_X$-modules $f : E \to E$ (depending on t) such that $f\theta = t\theta f$. On the subset U where E is locally free, the coefficients of the characteristic polynomial of f define sections of $\mathcal{O}_X$. Since X is normal and projective we have $\mathcal{O}_X(U) = \mathcal{O}_X(X) = k$, so they are constant. Hence we can decompose E into eigensubsheaves $E = \bigoplus E_\lambda$, where $E_\lambda = \ker(f-\lambda)^r$ for $\lambda \in k^*$ (eigenvalue 0 does not occur as f is an isomorphism). Since $(f-t\lambda)^r\theta = t^r\theta(f-\lambda)^r$, the Higgs field θ maps E_λ to $E_{t\lambda}$. If we take t such that $t^j \neq 1$ for $j = 0,...,r$ then for every eigenvalue λ the elements $\lambda, t\lambda, ..., t^r\lambda$ are pairwise distinct. So there exists j_0 such that $t^{j_0}\lambda$ is an eigenvalue but $t^{j_0-1}\lambda$ is not an eigenvalue. Then $E^i = \bigoplus_{j_0 \le j \le i} E_{t^j\lambda}$ defines a system of Ω_L-Hodge sheaves which is a direct summand of (E,θ). So we can complete the proof by induction on the rank r of E. □

COROLLARY 3.5. *A system of Ω_L-Hodge sheaves (E,θ) is slope (or Gieseker) semistable if and only if it is slope (respectively, Gieseker) semistable as an Ω_L-Higgs sheaf.*

Proof. It is sufficient to prove that the maximal destabilizing Ω_L-Higgs subsheaf of a system of Ω_L-Hodge sheaves (E,θ) is a system of Ω_L-Hodge sheaves. This follows from the above lemma and the fact that the maximal destabilizing Ω_L-Higgs subsheaf is unique so it is preserved by the natural $\mathbb{G}_m$-action. □

3.5 HITCHIN'S MORPHISM FOR MODULI SPACES OF L-COHIGGS SHEAVES

Let G be a quasi-coherent $\mathcal{O}_S$-module. Consider the functor which to an S-scheme T associates $\mathrm{Hom}_{\mathcal{O}_T}(G_T, \mathcal{O}_T)$. It is representable by the S-scheme $\mathbb{V}(G)$. In particular, for $\pi : T = \mathbb{V}(G) \to S$ we get the tautological homomorphism

$$\lambda_G \in \mathrm{Hom}_{\mathcal{O}_{\mathbb{V}(G)}}(\pi^*G, \mathcal{O}_{\mathbb{V}(G)}) = \mathrm{Hom}_{\mathcal{O}_S}(G, \pi_*\mathcal{O}_{\mathbb{V}(G)}) = \mathrm{Hom}_{\mathcal{O}_S-\mathrm{alg}}(S^\bullet G, S^\bullet G)$$

corresponding to the identity on $S^\bullet G$.
If G is a locally free sheaf of finite rank then $\mathbb{V}(G) \to S$ is a vector bundle with sheaf of sections isomorphic to G^*.

The following lemma was explained to the author by C. Simpson:

LEMMA 3.6. *Let $f : X \to S$ be a flat projective morphism of noetherian schemes and let G be a locally free sheaf on X. Then the functor $\underline{H}^0(X/S, G)$ which to an S-scheme $h : T \to S$ associates $H^0(X_T/T, G_T)$ is representable by an S-scheme.*

Proof. Since certain twist of G^* by a relatively very ample line bundle is relatively globally generated, we can embed G as a subbundle into a direct sum K_1 of relatively very ample line bundles. Then we can again embed the quotient K_1/G into K_2 with K_2 a direct sum of relatively very ample bundles. Then for any S-scheme T we have an exact sequence

$$0 \to \underline{H}^0(X/S, G)(T) \to \underline{H}^0(X/S, K_1)(T) \to \underline{H}^0(X/S, K_2)(T).$$

But we can assume that all the higher direct images of K_1 vanish and then by the Grauert's theorem $\underline{H}^0(X/S,K_1)$ is representable by the bundle $\mathbb{V}(f_*K_1)\to S$. Similarly, $\underline{H}^0(X/S,K_1)$ is representable by the bundle $\mathbb{V}(f_*K_2)\to S$. Therefore $\underline{H}^0(X/S,G)$ is represented by the kernel of the map between bundles. This is a vector subscheme of $\mathbb{V}(f_*K_1)\to S$. ☐

We will also need the following well-known lemma:

Lemma 3.7. *Let $f:X\to S$ be a flat family of irreducible d-dimensional schemes satisfying Serre's condition (S_2). Let E be an S-flat coherent $\mathscr{O}_X$-module such that $E\otimes k(s)$ is pure of dimension d for every point $s\in S$. Then there exists a relatively big open subset $j:U\subset X$ such that $E^{**}\to j_*(E|_U)$ is an isomorphism.*

Consider a flat projective morphism $f:X\to S$ of noetherian schemes. Let L be a smooth $\mathscr{O}_S$-Lie algebroid on X and let us recall that $\Omega_L=L^*$. Consider the functor which to an S-scheme $h:T\to S$ associates

$$\bigoplus_{i=1}^{r}H^0(X_T/T,S^i\Omega_{L,T}).$$

By Lemma 3.6 this functor is representable by an S-scheme $\mathbb{V}^L(X/S,r)\to S$.
Let us also assume that X/S is a family of d-dimensional varieties satisfying Serre's condition (S_2). If T is an S-scheme then X_T/T is also a flat family of d-dimensional varieties satisfying Serre's condition (S_2).
Assume that L is a trivial $\mathscr{O}_S$-Lie algebroid and consider a family $(E,\theta:E\to E\otimes\Omega_{L,T})$ of L-coHiggs sheaves of pure dimension $d=\dim(X/S)$ on the fibres of $X_T\to T$. Then there exists an open subset $U\subset X_T$ such that E is locally free on U and the intersection of U with any fiber of $X_T\to T$ has a complement of codimension at least 2. Let us consider $\bigwedge^i\theta|_U:\bigwedge^i(E|_U)\to\bigwedge^i(E|_U\otimes_{\mathscr{O}_U}\Omega_{L,T}|_U)$. We have a well defined surjection $\bigwedge^i(E|_U\otimes_{\mathscr{O}_U}\Omega_{L,T}|_U)\to\bigwedge^iE|_U\otimes_{\mathscr{O}_U}S^i\Omega_{L,T}|_U$, given by

$$(e_1\otimes\lambda_1)\wedge...\wedge(e_i\otimes\lambda_i)\to(e_1\wedge...\wedge e_i)\otimes(\lambda_1...\lambda_i),$$

where $e_1,...,e_i\in E$ and $\lambda_1,...,\lambda_i\in\Omega_{L,T}$. So we get a morphism of sheaves

$$\mathscr{O}_U\to\mathscr{E}nd_{\mathscr{O}_X}(\bigwedge^iE)|_U\otimes_{\mathscr{O}_U}S^i\Omega_{L,T}|_U\stackrel{(-1)^i\mathrm{Tr}\otimes\mathrm{id}}{\longrightarrow}S^i\Omega_{L,T}|_U$$

The corresponding section $\sigma_i(\theta|_U)\in H^0(U,S^i\Omega_{L,T}|_U)$ is just an evaluation of the i-th elementary symmetric polynomial on $\theta|_U$. By Lemma 3.7 this section extends uniquely to section $\sigma_i(\theta)\in H^0(X_T/T,S^i\Omega_{L,T})$. In this way we can define a T-point $\sigma(E,\theta)=(\sigma_1(\theta),...,\sigma_r(\theta))$ of $\mathbb{V}^L(X/S,r)$.
Let P be a polynomial of degree $d=\dim(X/S)$ corresponding to (some) rank r torsion free sheaves on the fibres of $X\to S$. Consider the moduli space $M^L_{\mathrm{Dol}}(X/S,P)$ of Gieseker semistable L-coHiggs sheaves with Hilbert polynomial P. Then the above construction defines a morphism of functors inducing the corresponding morphism

of coarse moduli spaces $H_L : M^L_{\mathrm{Dol}}(X/S,P) \to \mathbb{V}^L(X/S,r)$. This morphism is called *Hitchin's morphism*.

There is also a stack theoretic version of Hitchin's morphism. The moduli stack of L-coHiggs sheaves is defined as a lax functor between 2-categories by

$$\begin{array}{cccc} \mathscr{M}^L_{\mathrm{Dol}}(X/S,P): & (\mathrm{Sch}/S) & \to & (\text{groupoids}) \\ & T & \to & \mathscr{M}(T), \end{array}$$

where $\mathscr{M}(T)$ is the category whose objects are T-flat families of pure d-dimensional L-coHiggs sheaves with Hilbert polynomial P on the fibres of $X_T \to T$, and whose morphisms are isomorphisms of coherent sheaves. Then $\mathscr{M}^L_{\mathrm{Dol}}(X/S,P)$ is an algebraic stack for the fppf topology on (Sch/S). As above we can construct Hitchin's morphism $\mathscr{M}^L_{\mathrm{Dol}}(X/S,P) \to \mathbb{V}^L(X/S,r)$. By abuse of notation, we also denote this morphism by H_L.

As in the usual Higgs bundle and characteristic zero case, one can construct the total spectral scheme $\mathbb{W}^L(X/S,r) \subset \mathbb{V}(L) \times_S \mathbb{V}^L(X/S,r)$, which is finite and flat over $X \times_S \mathbb{V}^L(X/S,r)$. This subscheme has the property that for any family $(E, \theta : E \to E \otimes \Omega_{L,T})$ of L-coHiggs sheaves of pure dimension d on the fibres of $X_T \to T$, the corresponding coherent sheaf on $\mathbb{V}(L_T)$ is set-theoretically supported on $\mathbb{W}^L(X/S,r) \times_{\mathbb{V}^L(X/S,r)} T$. This can be seen as follows. Let x be a geometric point of X at which E is locally free. Then $S^\bullet L \otimes k(x)$ acts on $V = E \otimes k(x)$ via $\theta(x)$. Let us recall that over an algebraically closed field any finitely dimensional vector space which is irreducible with respect to a set of commuting linear maps has dimension 1. Therefore V has a filtration $0 = V_0 \subset V_1 \subset \ldots \subset V_r = V$ with quotients $V^i = V_i/V_{i-1}$ of dimension 1 over $k(x)$ and such that $\theta(x)$ acts on V^i as multiplication by $\lambda_i \in (L \otimes k(x))^*$. It is clear from our definition that $\tau \in L \otimes k(x)$ acts on V via $\theta_\tau := \theta(x)^T(\tau)$ in such a way that in the characteristic polynomial

$$\det(t \cdot I - \theta_\tau) = t^r + \sigma_1(\theta_\tau)t^{r-1} + \ldots + \sigma_r(\theta_\tau)$$

we have $\sigma_i(\theta_\tau) = (-1)^i \sum_{1 \le j_1 < \ldots < j_i \le r} \lambda_{j_1} \ldots \lambda_{j_i}$. This and the Cayley–Hamilton theorem show that the coherent sheaf on $\mathbb{V}(L_T)$ corresponding to (E,θ) has a scheme-theoretic support contained in $\mathbb{W}^L(X/S,r) \times_{\mathbb{V}^L(X/S,r)} T$ and it coincides with it set-theoretically.

Note that in the curve case there exists a different interpretation of Higgs bundles using cameral covers. Such an approach allows to deal with general reductive groups (see [DG] for the characteristic zero case). In positive characteristic the analogous construction requires some restrictions on the characteristic of the base field.

The following theorem can be proven in a similar way as the usual characteristic zero version [Si3, Theorem 6.11]. It also follows from Langton's type Theorem 5.3.

THEOREM 3.8. *Hitchin's morphism $H_L : M^L_{\mathrm{Dol}}(X/S,P) \to \mathbb{V}^L(X/S,r)$ is proper.*

3.6 Deformation of a Lie algebroid over an affine line.

Let R be a commutative ring with unity. Let $f : X \to S$ be a morphism of R-schemes. Let $\mathbb{A}^1_R = \operatorname{Spec} R[t]$ and let $p_1 : X \times_R \mathbb{A}^1_R \to X$ be the projection onto the first factor. Let us consider an $\mathcal{O}_S$-Lie algebroid L on X and the morphism $f \times \mathrm{id} : X \times_R \mathbb{A}^1_R \to S \times_R \mathbb{A}^1_R$ of R-schemes. We can define an $\mathcal{O}_{S\times_R \mathbb{A}^1_R}$-Lie algebroid L^R on $X \times_R \mathbb{A}^1_R$ by taking $L^R := p_1^* L$ with Lie bracket given by $[\cdot,\cdot]_{L^R} := p_1^*[\cdot,\cdot]_L \otimes t$ and the anchor map given by $\alpha^R := p_1^*\alpha \otimes t$.
The universal enveloping algebra of differential operators $\Lambda^R_L := \Lambda_{L^R}$ associated to L^R can be constructed as a subsheaf of $p_1^*\Lambda_L$ generated by sections of the form $\sum t^i \lambda_i$, where λ_i are local sections of $\Lambda_{L,i}$.
If $R = k$ is a field and $s \in \mathbb{A}^1(k) - \{0\}$ then the restricted sheaf $\Lambda^R_L|_{X\times\{s\}}$ is naturally isomorphic to Λ_L. The sheaf $\Lambda^R_L|_{X\times\{0\}}$ is naturally isomorphic to the associated graded sheaf of algebras $\operatorname{Gr}\Lambda_L$. This gives a deformation of Λ_L to its associated graded sheaf of algebras (or a quantization of the commutative algebra $\operatorname{Gr}\Lambda_L$).

Let T be an S-scheme and let us fix $\lambda \in H^0(T/R, \mathcal{O}_T)$. Let E be a coherent $\mathcal{O}_{X_T}$-module and let p_X and p_T be the projections of $X \times_S T$ onto X and T, respectively. Let (M, d_M) be a coherent $\mathcal{O}_X$-module with an $\mathcal{O}_S$-derivation.
Then we set $\tilde{M} = p_X^* M$ and $d_{\tilde{M}} = p_X^* d_M \cdot p_T^*\lambda$. A $d_{\tilde{M}}$-connection on E is called a *λ-d_M-connection*. This generalizes the usual notion of λ-connection.
For the constant section $\lambda = 0 \in H^0(T/R, \mathcal{O}_T)$ an integrable λ-d_M-connection is just an M-Higgs field. Similarly, for $\lambda = 1 \in H^0(T/R, \mathcal{O}_T)$ we recover the notion of a d_M-connection.

Assume that L is a smooth $\mathcal{O}_S$-Lie algebroid on X. Let us fix a morphism of R-schemes $T \to S \times_R \mathbb{A}^1_R$ and let $\lambda \in H^0(T/R, \mathcal{O}_T)$ be the section corresponding to the composition of $T \to S \times_R \mathbb{A}^1_R$ with the canonical projection $S \times_R \mathbb{A}^1_R \to \mathbb{A}^1_R$. Since $T \times_{S\times_R \mathbb{A}^1_R} X \times_R \mathbb{A}^1_R = X_T$, an L^R-module structure on a coherent $\mathcal{O}_{X_T}$-module E is equivalent to giving an integrable λ-d_{Ω_L}-connection.

4 Lie algebroids in positive characteristic

4.1 Sheaves of restricted Lie algebras

Let R be a commutative ring (with unity) of characteristic p and let L be a Lie R-algebra. We define the universal Lie polynomials s_j by the formula

$$s_j(x_1, x_2) = -\frac{1}{j}\sum_{\sigma} \operatorname{ad} x_{\sigma(1)} \dots \operatorname{ad} x_{\sigma(p-1)}(x_2)$$

in which we sum over all $\sigma : \{1, ..., p-1\} \to \{1, 2\}$ taking j times value 1.
Let A be an associative R-algebra. For $x \in A$ we define $\operatorname{ad}(x) : A \to A$ by the formula $(\operatorname{ad}(x))(y) = xy - yx$ for $y \in A$. Then we have the following well known Jacobson's formulas:

$$\operatorname{ad}(x^p) = \operatorname{ad}(x)^p$$

$$(x+y)^p = x^p + y^p + \sum_{0<j<p} s_j(x,y).$$

Let X be a scheme over a scheme S of characteristic $p>0$. A *sheaf of restricted $\mathscr{O}_S$-Lie algebras on X* is a sheaf of $\mathscr{O}_S$-Lie algebras $(L,[\cdot,\cdot])$ on X equipped with a p-th power operation $L \to L$, $x \to x^{[p]}$, which satisfies the following conditions:

1. $(fx)^{[p]} = f^p x^{[p]}$ for all local sections $f \in \mathscr{O}_S$ and $x \in L$,
2. $\mathrm{ad}(x^{[p]}) = (\mathrm{ad}(x))^p$ for $x \in L$,
3. $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{0<j<p} s_j(x,y)$ for all $x,y \in L$.

A homomorphism of sheaves of restricted $\mathscr{O}_S$-Lie algebras $\varphi : L \to L'$ on X is such a homomorphism of sheaves of $\mathscr{O}_S$-Lie algebras on X that $\varphi(x^{[p]}) = \varphi(x)^{[p]}$ for all $x \in L$.

Let $\mathscr{A}$ be a sheaf of associative $\mathscr{O}_S$-algebras on X. It has a natural structure of a sheaf of restricted $\mathscr{O}_S$-Lie algebras on X with bracket $[x,y] = xy - yx$ and p-th power operation $x^{[p]} = x^p$ for local sections $x,y \in \mathscr{A}$.
Now let L be a sheaf of restricted $\mathscr{O}_S$-Lie algebras on X. For any homomorphism $\varphi : L \to \mathscr{A}$ of sheaves of $\mathscr{O}_S$-Lie algebras on X we can define $\psi : L \to \mathscr{A}$ by $\psi(x) = (\varphi(x))^p - \varphi(x^{[p]})$ for $x \in L$. The map ψ measures how far is φ from being a homomorphism of sheaves of restricted $\mathscr{O}_S$-Lie algebras on X.

Lemma 4.1. *The map $\psi : L \to \mathscr{A}$ is additive and its image commutes with the image of φ. In particular, $[\psi(L), \psi(L)] = 0$.*

Proof. Let us take sections $x,y \in L(U)$ for some open subset $U \subset X$. From Jacobson's formula in $\mathscr{A}$ we have

$$(\varphi(x+y))^p = \varphi(x)^p + \varphi(y)^p + \sum_{0<j<p} s_j(\varphi(x), \varphi(y))$$

On the other hand, from definition of a sheaf of restricted Lie algebras we have

$$\varphi((x+y)^{[p]}) = \varphi(x^{[p]}) + \varphi(y^{[p]}) + \sum_{0<j<p} s_j(\varphi(x), \varphi(y)),$$

so subtracting these equalities we get additivity of ψ.
Now we need to prove that $[\psi(x), \varphi(y)] = 0$. But we have

$$[\varphi(x)^p, \varphi(y)] = \mathrm{ad}(\varphi(x)^p)(\varphi(y)) = (\mathrm{ad}\,\varphi(x))^p(\varphi(y))$$

and

$$[\varphi(x^{[p]}), \varphi(y)] = \varphi([x^{[p]}, y]) = \varphi(\mathrm{ad}(x^{[p]})(y)) = \varphi(\mathrm{ad}(x)^p(y)) = (\mathrm{ad}\,\varphi(x))^p(\varphi(y)),$$

so subtracting yields the required equality. $\square$

The *restricted universal enveloping algebra* $\mathscr{U}^{[p]}_{\mathscr{O}_S}(L)$ of a sheaf of restricted $\mathscr{O}_S$-Lie algebras L on X is the quotient of the universal enveloping algebra $\mathscr{U}_{\mathscr{O}_S}(L)$ by the two-sided ideal generated by all elements of the form $x^p - x^{[p]}$ for local sections $x \in L$. If $S = X$ and L is locally free as an $\mathscr{O}_X$-module then L is contained in $\mathscr{U}^{[p]}_{\mathscr{O}_X}(L)$. Moreover, if $x_1, \dots, x_r$ are local generators of L as an $\mathscr{O}_X$-module then $x_1^{i_1} \dots x_r^{i_r}$ with $0 \le i_j < p$ for all j, form a local basis of $\mathscr{U}^{[p]}_{\mathscr{O}_X}(L)$ as an $\mathscr{O}_X$-module. In particular, $\mathscr{U}^{[p]}_{\mathscr{O}_X}(L)$ is locally free of rank $p^{\mathrm{rk}\, L}$. In this case for any sheaf $\mathscr{A}$ of associative algebras on X and any homomorphism $\varphi : L \to \mathscr{A}$ of sheaves of Lie algebras on X, the map $\psi : L \to \mathscr{A}$ is F_X^*-linear, i.e., $\psi(fx) = f^p \psi(x)$ for all $f \in \mathscr{O}_X$ and $x \in L$ (this follows from the first condition in the definition of a sheaf of restricted Lie algebras). So by adjunction ψ induces an $\mathscr{O}_X$-linear map $F_X^* L \to \mathscr{A}$ that by abuse of notation is also denoted by ψ. Then the restricted universal enveloping algebra $\mathscr{U}^{[p]}_{\mathscr{O}_X}(L)$ has the following universal property. For any sheaf $\mathscr{A}$ of associative $\mathscr{O}_X$-algebras and any homomorphism $\varphi : L \to \mathscr{A}$ of sheaves of $\mathscr{O}_X$-Lie algebras with $\psi : L \to \mathscr{A}$ equal to zero, there exists a unique homomorphism $\tilde{\varphi} : \mathscr{U}^{[p]}_{\mathscr{O}_X}(L) \to \mathscr{A}$ of sheaves of associative $\mathscr{O}_X$-algebras such that $\varphi : L \to \mathscr{A}$ is the composition of the natural map $L \to \mathscr{U}^{[p]}_{\mathscr{O}_X}(L)$ with $\tilde{\varphi}$.

4.2 Restricted Lie algebroids

Note that the relative tangent sheaf $T_{X/S}$ has a natural structure of a sheaf of restricted $\mathscr{O}_S$-Lie algebras on X in which the p-th power operation on $\mathscr{O}_S$-derivation $D : \mathscr{O}_X \to \mathscr{O}_X$ is defined as the derivation acting on functions as the p-th power differential operator D^p. In fact, $T_{X/S}$ with the usual Lie bracket and this p-th power operation is a sheaf of restricted $\mathscr{O}_S$-Lie subalgebras of the associative algebra $\mathscr{E}nd_{\mathscr{O}_S} \mathscr{O}_X$ taken with the natural structure of a sheaf of restricted $\mathscr{O}_S$-Lie algebras on X. This motivates the following definition:

Definition 4.2. A *restricted $\mathscr{O}_S$-Lie algebroid* on X is a quadruple $(L, [\cdot,\cdot], \cdot^{[p]}, \alpha)$ consisting of a sheaf of restricted $\mathscr{O}_S$-Lie algebras $(L, [\cdot,\cdot], \cdot^{[p]})$ on X and a homomorphism of sheaves of restricted $\mathscr{O}_S$-Lie algebras $\alpha : L \to T_{X/S}$ on X satisfying the Leibniz rule and the following formula:

$$(fx)^{[p]} = f^p x^{[p]} + \alpha_{fx}^{p-1}(f) x$$

for all $f \in \mathscr{O}_X$ and $x \in L$.

As in the non-restricted case we can define a *trivial restricted Lie algebroid* as a trivial Lie algebroid with the zero p-th power operation. $T_{X/S}$ with the usual Lie bracket and p-th power operation will be called the *standard restricted $\mathscr{O}_S$-Lie algebroid on X*.
The last condition in the definition requires certain compatibility of the p-th power operation on L with the anchor map and $\mathscr{O}_X$-module structure of L. It can be explained by the fact that, as expected, a restricted $\mathscr{O}_S$-Lie algebroid on X with the zero anchor map is a sheaf of restricted $\mathscr{O}_X$-Lie algebras. In fact, the formula in the definition comes from the following Hochschild's identity:

Lemma 4.3. (see [Ho, Lemma 1]) *Let A be an associative $\mathbb{F}_p$-algebra and $R \subset A$ a commutative subalgebra. If for an element $x \in A$ we have $(\mathrm{ad}\, x)(R) \subset R$ then for any element $r \in R$ we have*

$$(rx)^p = r^p x^p + (\mathrm{ad}(rx))^{p-1}(r)x.$$

A similar formula can be found as [Ka1, Proposition 5.3] (although with a sign error as pointed out by A. Ogus in [Og]).

The following criterion allows us to check when a submodule of a restricted Lie algebroid is a restricted Lie subalgebroid. It generalizes well known Ekedahl's criterion allowing to check when a submodule of the tangent bundle defines a 1-foliation (see [Ek, Lemma 4.2]).

Lemma 4.4.

1. *Let L' be an $\mathcal{O}_X$-submodule of an $\mathcal{O}_S$-Lie algebroid L on X. Then the Lie bracket on L induces an $\mathcal{O}_X$-linear map*
$$\bigwedge^2 L' \to L/L'$$
sending $x \wedge y$ to the class of $[x,y]$. If this map is the zero map then L' is an $\mathcal{O}_S$-Lie subalgebroid of L.
2. *If L' is an $\mathcal{O}_S$-Lie subalgebroid of a restricted $\mathcal{O}_S$-Lie algebroid L then the p-th power map induces an $\mathcal{O}_X$-linear morphism $F_X^* L' \to L/L'$. If this map is the zero map then L' is a restricted $\mathcal{O}_S$-Lie subalgebroid of L.*

Proof. Let us take $f \in \mathcal{O}_X$ and $x, y \in L'$. The first part follows from the equality

$$[x, fy] = f[x,y] + \alpha_x(f)y \equiv f[x,y] \mod L'.$$

To prove the second part note that

$$(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{0<j<p} s_j(x,y) \equiv x^{[p]} + y^{[p]} \mod L',$$

since $s_j(x,y) \in L'$, as the s_j are Lie polynomials. Therefore $F_X^* L' \to L/L'$ is additive. Hence to prove that it is $\mathcal{O}_X$-linear it is sufficient to note that

$$(fx)^{[p]} = f^p x^{[p]} + \alpha_{fx}^{p-1}(f)x \equiv f^p x^{[p]} \mod L'.$$

∎

Let us consider the following commutative diagram

$$\begin{tikzcd}
\mathbb{V}(L)^{(1/X)} \arrow[r, "="] \arrow[dr] & \mathbb{V}(F_{X/S}^* L') \arrow[r, "\tilde{F}_{X/S}"] \arrow[d, "\pi"] & \mathbb{V}(L') \arrow[r] \arrow[d, "\pi'"] & \mathbb{V}(L) \arrow[d] \\
 & X \arrow[r, "F_{X/S}"] \arrow[rr, bend right, "F_X"'] & X' \arrow[r] & X
\end{tikzcd}$$

in which L' is the pull back of L via $X' \to X$.
The following lemma is an analogue of [BMR, Lemma 1.3.2]:

LEMMA 4.5. *Let L be a restricted $\mathscr{O}_S$-Lie algebroid on X. Then the map $\iota : L \to \Lambda_L$ sending $x \in L$ to $\iota(x) := x^p - x^{[p]} \in \Lambda_L$ is F_X^*-linear and its image is contained in the center $Z(\Lambda_L)$ of Λ_L. In particular, if L smooth then ι extends to an $\mathscr{O}_{X'}$-linear inclusion $S^\bullet L' \hookrightarrow F_{X/S,*}Z(\Lambda_L)$.*

Proof. Lemma 4.3 proves that the p-th power operation satisfies $\alpha_{x^{[p]}} = (\alpha_x)^p$ and $(fx)^{[p]} - f^p x^{[p]} = (fx)^p - f^p x^p$ in Λ_L for all $f \in \mathscr{O}_X$ and $x \in L$. Hence ι is F_X^*-linear. Lemma 4.1 implies that its image is contained in $Z(\Lambda_L)$.
For any $f \in \mathscr{O}_X$ and $x \in L$ we have $xf^p - f^p x = \alpha_x(f^p) = 0$ in Λ_L, as α_x is an $\mathscr{O}_S$-derivation. Therefore $\mathscr{O}_{X'} \subset F_{X/S,*}Z(\Lambda_L)$ which together with the first part proves the required assertion. $\square$

Note that the above lemma shows that Λ_L contains a commutative subalgebra $S^\bullet(F_{X/S}^*L')$, so Λ_L defines a quasi-coherent sheaf $\tilde{\Lambda}_L$ on $\mathbb{V}(F_X^*L)$.

Let $\Lambda_L^{[p]}$ be the quotient of Λ_L by the two-sided ideal generated by $\iota(x)$ for $x \in L$. We call it the *restricted universal enveloping algebra of differential operators of L.*

LEMMA 4.6. *Let L be smooth of rank m. Then $\tilde{\Lambda}_L$ is a locally free $\mathscr{O}_{\mathbb{V}(L)}$-module of rank p^m.*

Proof. The canonical embedding $j : L \to \Lambda_L$ induces an embedding $\tilde{j} : L \to \Lambda_L^{[p]}$. Let us take an open subset $U \subset X$ such that $L(U)$ is a free $\mathscr{O}_X(U)$-module with generators $x_1, \dots, x_m$. The kernel of $\Lambda_L(U) \to \Lambda_L^{[p]}(U)$ is generated by elements $\iota(x_1), \dots, \iota(x_m)$ which are in the center of $\Lambda_L(U)$. But $\iota(x_i) \equiv x_i^p \mod \Lambda_{L,p-1}$, so by the Poincare-Birkhoff-Witt theorem $\Lambda_L^{[p]}$ has local generators $\tilde{j}(x_1)^{i_1} \dots \tilde{j}(x_m)^{i_m}$ for $0 \le i_l < p$. Hence $j(x_1)^{i_1} \dots j(x_m)^{i_m}$ for $0 \le i_l < p$ locally generate Λ_L as an $S^\bullet(F_X^*L)$-module and $\tilde{\Lambda}_L$ is locally free of rank p^m. $\square$

Lemma 4.5 shows that if L is smooth then ι induces an $\mathscr{O}_{X'}$-linear map $L' \to F_{X/S,*}\Lambda_L$ and a homomorphism of sheaves of $\mathscr{O}_{X'}$-algebras

$$S^\bullet(L') \to F_{X/S,*}(Z(\Lambda_L)) \subset \Lambda_L' := F_{X/S,*}\Lambda_L.$$

In particular, it makes Λ_L' into a quasi-coherent sheaf of $S^\bullet(L')$-modules. This sheaf defines on $\mathbb{V}(L')$ a quasi-coherent sheaf of $\mathscr{O}_{\mathbb{V}(L')}$-algebras $\tilde{\Lambda}_L'$. Note that by construction

$$\pi'_*\tilde{\Lambda}_L' = F_{X/S,*}\Lambda_L = F_{X/S,*}\pi_*\tilde{\Lambda}_L = \pi'_*\tilde{F}_{X/S,*}\tilde{\Lambda}_L,$$

so we have

$$\tilde{\Lambda}_L' = \tilde{F}_{X/S,*}\tilde{\Lambda}_L.$$

By an explicit computation as in Lemma 4.6 one can prove the following theorem:

THEOREM 4.7. *Assume that X/S is smooth of relative dimension d and L is smooth of rank m. Then $\tilde{\Lambda}_L'$ is a locally free $\mathscr{O}_{\mathbb{V}(L')}$-module of rank p^{m+d}.*

By [BMR] in the special case when $L = T_{X/S}$ is the standard $\mathcal{O}_S$-Lie algebroid on X, the sheaf $\tilde{\Lambda}'_L$ is a sheaf of Azumaya $\mathcal{O}_{\mathbb{V}(L')}$-algebras. In this case we have a canonical splitting

$$\tilde{F}^*_{X/S}\tilde{\Lambda}'_L \simeq \mathcal{E}nd_{\mathcal{O}_{\mathbb{V}(F^*_{X/S}L')}}\tilde{\Lambda}_L.$$

4.3 Relation with groupoid schemes

This subsection contains a quick tour on relation between Lie algebroids and groupoid schemes of height ≤ 1. This is analogous to the well-known relation between restricted Lie algebras and group schemes of height ≤ 1.

Let us recall that a *groupoid* is a small category in which every morphism is an isomorphism. Let X and R be S-schemes. An *S-groupoid scheme* G is a quintuple of S-maps $s,t : R \to X$ ("source and target objects"), $c : R \times_{(s,t)} R \to R$ ("composition"), $e : X \to R$ ("identity map") and $i : R \to R$ ("inverse map") such that for every S-scheme T the quintuple $s(T)$, $t(T)$, $c(T)$, $e(T)$ and $i(T)$ defines in a functorial way a groupoid with morphisms $R(T)$ and objects $X(T)$.
For an S-groupoid scheme G we denote by $\mathcal{J}$ the kernel of $s_*\mathcal{O}_R \to \mathcal{O}_X$. We say that G is *infinitesimal* if s is an affine homeomorphism and $\mathcal{J}$ is a nilpotent ideal. An infinitesimal S-groupoid scheme is of *height* ≤ 1 if $(s,t) : R \to X \times_S X$ factors through the first Frobenius neighbourhood of the diagonal (i.e., through $X \times_{X^{(1/S)}} X$). An S-groupoid scheme is called *finite* (*flat*) if s is finite (respectively, flat).
If X is smooth over a perfect field k then restricted k-Lie subalgebras L of the standard k-Lie algebroid $T_{X/k}$ such that $T_{X/k}/L$ is locally free are in bijection with finite flat height 1 morphisms $X \to Y$ (see [Ek, Proposition 2.4]). Note that a sheaf of restricted k-Lie subalgebras of $T_{X/k}$ is automatically a restricted k-Lie subalgebroid of $T_{X/k}$. So the following proposition generalizes the above fact (and it corrects [Ek, Proposition 2.3]):

Proposition 4.8. *Let X/S be a smooth morphism. Assume that for every point $x \in X$ the set $t(s^{-1}(x))$ is contained in an affine open subset of X. Then there exists an equivalence of categories between the category of finite flat S-groupoid schemes of height ≤ 1 with X/S as a scheme of objects and with locally free "conormal sheaf" $\mathcal{J}/\mathcal{J}^2$ and the category of smooth restricted $\mathcal{O}_S$-Lie algebroids on X/S.*

Proof. We sketch the proof leaving details to the reader.
If G is a finite, flat, infinitesimal S-groupoid scheme then we define L as the Lie algebra of this groupoid, i.e., the dual of $\mathcal{J}/\mathcal{J}^2$. It has a natural structure of a sheaf of restricted $\mathcal{O}_S$-Lie algebras. Since G has height ≤ 1, L is equipped with the anchor map.
In the other direction, to a smooth restricted $\mathcal{O}_S$-Lie algebroid L on X/S we associate $\Lambda_L^{[p]}$ which comes with a canonical homomorphism of $\mathcal{O}_S$-algebras $\Lambda_L^{[p]} \to \Lambda_{T_{X/S}}^{[p]}$. But $\Lambda_{T_{X/S}}^{[p]}$ is an $\mathcal{O}_S$-subalgebra of the sheaf of rings of "true" differential operators and the "morphisms" R of the groupoid scheme can be defined as the spectrum of the dual of $\Lambda_L^{[p]}$. ∎

4.4 Modules over restricted Lie algebroids

If E is a module over a restricted $\mathscr{O}_S$-Lie algebroid L then $\nabla : L \to \mathscr{E}nd_{\mathscr{O}_S} E$ leads to a morphism

$$\psi : L \to \mathscr{E}nd_{\mathscr{O}_S} E$$

defined by sending x to $(\nabla(x))^p - \nabla(x^{[p]})$ for $x \in L$.
Let us set $\alpha_x^0(f) = f$ and $(\nabla(x))^0(e) = e$. Using Leibniz' rule one can easily see that

$$(\nabla(x))^m(fe) = \sum_{i=0}^{m} \binom{m}{i} \alpha_x^i(f)(\nabla(x))^{m-i}(e)$$

for any sections $f \in \mathscr{O}_X(U)$, $x \in L(U)$ and $e \in E(U)$ and any open subset $U \subset X$. In particular, we have

$$(\nabla(x))^p(fe) = \alpha_x^p(f)e + f(\nabla(x))^p(e).$$

Since

$$\nabla(x^{[p]})(fe) = \alpha_{x^{[p]}}(f)e + f\nabla(x^{[p]})(e)$$

and $\alpha_{x^{[p]}} = \alpha_x^p$ we see that for any $x \in L$ the image $\psi(x)$ is $\mathscr{O}_X$-linear. So we can consider ψ as the mapping $\psi : L \to \mathscr{E}nd_{\mathscr{O}_X} E$. This mapping is called the *p-curvature morphism* of the L-module E. The following lemma generalizes [Ka1, Proposition 5.2]:

Lemma 4.9. *The p-curvature morphism $\psi : L \to \mathscr{E}nd_{\mathscr{O}_X} E$ is F_X^*-linear and its image commutes with the image of ∇ in $\mathscr{E}nd_{\mathscr{O}_S} E$.*

Proof. By Lemma 4.1 we know that ψ is additive and its image commutes with the image of ∇. So it is sufficient to check that

$$\psi(fx) = f^p \psi(x)$$

for all local sections $f \in \mathscr{O}_X$ and $x \in L$. Applying Hochschild's identity to elements f and $\nabla(x)$ in $\mathscr{E}nd_{\mathscr{O}_S} E$ we obtain

$$(\nabla(fx))^p = f^p \nabla(x)^p + (\mathrm{ad}(f\nabla(x)))^{p-1}(f)\nabla(x) = f^p\nabla(x)^p + \alpha_{fx}^{p-1}(f)\nabla(x).$$

From the definition of a restricted $\mathscr{O}_S$-Lie algebroid and $\mathscr{O}_X$-linearity of $\nabla : L \to \mathscr{E}nd_{\mathscr{O}_S} E$ we have

$$\nabla((fx)^{[p]}) = f^p \nabla(x^{[p]}) + \alpha_{fx}^{p-1}(f)\nabla(x).$$

Subtracting these equalities we get the required identity. □

By the above lemma ψ defines an $\mathscr{O}_X$-linear map $L \to F_{X,*}\mathscr{E}nd_{\mathscr{O}_X} E$ and hence the adjoint $\mathscr{O}_X$-linear map

$$\psi_L : F_X^* L \to \mathscr{E}nd_{\mathscr{O}_X} E,$$

which will also be called the p-curvature morphism. Note that ψ_L makes E into an F_X^*L-coHiggs sheaf (integrability of the F_X^*L-coHiggs field follows from the lemma). Another way of seeing it is that if E is a Λ_L-module then by Lemma 4.5 it has a structure of $S^\bullet(F_X^*L)$-module given by the p-curvature ψ_L.

Example 4.10. Let L be a smooth trivial restricted $\mathscr{O}_S$-Lie algebroid on X. Then giving an L-module is equivalent to giving $S^\bullet L$-module structure on E. In this case the p-curvature morphism $\psi_L : F_X^* L \to \mathscr{E}nd_{\mathscr{O}_X} E$ is obtained by composing the canonical inclusion $F_X^* L \to S^p L$ with the action map $S^p L \to \mathscr{E}nd_{\mathscr{O}_X} E$.

Example 4.11. Let X be a smooth S-scheme and let us fix $\lambda \in H^0(\mathscr{O}_S)$. Let us denote by $T^\lambda_{X/S}$ the restricted $\mathscr{O}_S$-Lie algebroid structure on $T_{X/S}$ with Lie bracket $[\cdot,\cdot]_{T^\lambda_{X/S}} = \lambda \cdot [\cdot,\cdot]_{T_{X/S}}$, anchor map α given by multiplication by λ and the p-th power operation given by

$$x^{[p]}_{T^\lambda_{X/S}} = \lambda^{p-1} \cdot x^{[p]}_{T_{X/S}}$$

for $x \in T_{X/S}$. The apparently strange formula for the p-th power operation comes from the requirement

$$\alpha(x^{[p]}_{T^\lambda_{X/S}}) = \lambda \cdot x^{[p]}_{T^\lambda_{X/S}} = (\alpha(x))^{[p]} = \lambda^p \cdot x^{[p]}.$$

Giving a $T^\lambda_{X/S}$-module is equivalent to giving a coherent $\mathscr{O}_X$-module E with an integrable λ-connection $\nabla : E \to E \otimes_{\mathscr{O}_X} \Omega_{X/S}$. In this case the above defined p-curvature of the $T^\lambda_{X/S}$-module gives a more conceptual approach to the p-curvature of an $\mathscr{O}_X$-module with λ-connection (E,∇) defined in [LP, Definition 3.1].

Remark 4.12. If ∇_1 and ∇_2 are two L-module structures on E then $\varphi = \nabla_1 - \nabla_2 : L \to \mathscr{E}nd_{\mathscr{O}_S} E$ is $\mathscr{O}_X$-linear and its image lies in $\mathscr{E}nd_{\mathscr{O}_X} E$. In particular, if the p-curvatures $\psi_L(\nabla_1)$ and $\psi_L(\nabla_2)$ are equal then φ is zero on the kernel of $\Lambda_L \to \Lambda_L^{[p]}$ and hence it induces the homomorphism $\Lambda_L^{[p]} \to \mathscr{E}nd_{\mathscr{O}_X} E$ of $\mathscr{O}_X$-algebras.

Definition 4.13. We say that the p-curvature of (E,∇) is *nilpotent of level less than* l if (E,∇) satisfies one of the following equivalent conditions:

1. There exists a filtration $M^m = 0 \subset M^{m-1} \subset ... \subset M^0 = (E,\nabla)$ of length $m \le l$ such that the associated graded L-module has p-curvature 0.
2. For any open subset $U \subset X$ and any collection $\{x_1,...,x_l\}$ of sections of $L(U)$ we have $\psi_L(x_1)...\psi_L(x_l) = 0$.

We say that he p-curvature of (E,∇) is *nilpotent of level* l if it is nilpotent of level less than $(l+1)$ but not nilpotent of level less than l (for $l = 0$ we require simply that the p-curvature is nilpotent of level less than 1).

4.5 Deformation of Hitchin's morphism for restricted Lie algebroids

This subsection contains a partial generalization of the results of Laszlo and Pauly [LP] to higher dimensions. Note that in general, the direct analogue of their [LP, Proposition 3.2] is not expected to be true.

Let S be a noetherian scheme of characteristic p and let $X \to S$ be a flat, projective family of d-dimensional varieties satisfying Serre's condition (S_2). Let L be a smooth restricted $\mathscr{O}_S$-Lie algebroid on X. Let us fix a polynomial P and a relatively ample line bundle on X/S. We define the moduli stack as a lax functor between 2-categories by

$$\begin{array}{rccc} \mathscr{M}^L(X/S,P): & (\mathrm{Sch}/S) & \to & (\mathrm{groupoids}) \\ & T & \to & \mathscr{M}(T), \end{array}$$

where $\mathscr{M}(T)$ is the category whose objects are T-flat families of pure d-dimensional L-modules with Hilbert polynomial P on the fibres of $X_T \to T$, and whose morphisms are isomorphisms of coherent sheaves. One can prove that $\mathscr{M}^L(X/S,P)$ is an algebraic stack for the fppf topology on (Sch/S). If M is a coherent $\mathscr{O}_X$-module considered as an $\mathscr{O}_S$-Lie algebroid on X with the trivial structure, then the corresponding moduli stack is denoted by $\mathscr{M}^M_{\mathrm{Dol}}(X/S,P)$.
The p-curvature defines a morphism of stacks

$$\begin{array}{rccc} \Psi_L: & \mathscr{M}^L(X/S,P) & \to & \mathscr{M}^{F_X^*L}_{\mathrm{Dol}}(X/S,P) \\ & (E,\nabla) & \to & (E,\psi(\nabla)). \end{array}$$

Let us consider the deformation L^R of L over an affine line $\mathbb{A}^1$ over $\mathbb{F}_p$ (see Subsection 3.6). For simplicity of notation, in the following we skip writing $\mathbb{F}_p$. L^R has a natural structure of a smooth restricted $\mathscr{O}_{S\times\mathbb{A}^1}$-Lie algebroid on $X \times \mathbb{A}^1$ with the p-th power operation given by $\cdot^{[p]}_{L^R} = p_1^*\left(\cdot^{[p]}_L\right) \otimes t^{p-1}$. We can treat L^R as a family of restricted $\mathscr{O}_S$-Lie algebroids on X parameterized by $\mathbb{A}^1$. For example, if X/S is smooth and we fix $\lambda \in H^0(\mathscr{O}_S) = \mathrm{Hom}(S,\mathbb{A}^1)$ then for $L = T_{X/S}$ with the standard restricted $\mathscr{O}_S$-Lie algebroid structure, the pull-back of L^R along $(\mathrm{id}_S,\lambda): S \to S \times \mathbb{A}^1$ gives $T^\lambda_{X/S}$ from Example 4.11.
We have a commutative diagram

$$\begin{tikzcd} \mathscr{M}^{L^R}(X\times\mathbb{A}^1/S\times\mathbb{A}^1,P) \arrow[r, "\Psi_{L^R}"] & \mathscr{M}^{F^*_{X\times\mathbb{A}^1}L^R}_{\mathrm{Dol}}(X\times\mathbb{A}^1/S\times\mathbb{A}^1,P) \arrow[r, "H_{F^*_{X\times\mathbb{A}^1}L^R}"] & \mathbb{V}^{F^*_{X\times\mathbb{A}^1}L^R}(X\times\mathbb{A}^1/S\times\mathbb{A}^1,r) \\ \mathscr{M}^L_{\mathrm{Dol}}(X/S,P) \arrow[u] \arrow[r, "H_L"] & \mathbb{V}^L(X/S,r) \arrow[r] & \mathbb{V}^{F_X^*L}(X/S,r), \arrow[u] \end{tikzcd}$$

where the vertical arrows are induced by the base change via the zero section $0: S \to S \times \mathbb{A}^1$ and $\mathbb{V}^L(X/S,r) \to \mathbb{V}^{F_X^*L}(X/S,r)$ is the canonical morphism induced by the absolute Frobenius on X. Roughly speaking, this diagram says that the p-curvature morphism Ψ_L deforms to the p-th power of the Hitchin morphism.

Let $\mathscr{N}ilp^L(X/S,P)$ be the substack of $\mathscr{M}^L(X/S,P)$ of L-modules with nilpotent p-curvature. By definition $\tilde{\Psi}$ maps $\mathscr{N}ilp^L(X/S,P)$ into $\{0\}\times\mathbb{A}^1 = \mathbb{A}^1$ and the corresponding map will be still denoted by $\tilde{\Psi}$. The stacks $\mathscr{M}^L(X/S,P)$ and $\mathscr{N}ilp^L(X/S,P)$ contain open substacks $\mathscr{M}^{L,ss}(X/S,P)$ and $\mathscr{N}ilp^{L,ss}(X/S,P)$ parametrizing slope semi-stable objects (openness of semistability is a standard exercise left to the reader).

By boundedness theorem (see [La1]) these substacks are of finite type. Theorem 5.1 implies that the morphisms $\tilde{\Psi}^{ss}:\mathscr{M}^{L^R,ss}(X/S,P)\to\mathbb{V}^{F_X^*L}(X/S,r)\times\mathbb{A}^1$ and $\mathscr{N}ilp^{L,ss}(X/S,P)\to\mathbb{A}^1$ are universally closed.

Let $\mathscr{N}ilp_l^{L,ss}(X/S,P)$ be the substack of $\mathscr{N}ilp^{L,ss}(X/S,P)$ parametrizing objects with nilpotent p-curvature of level $<l$. Note that it is a closed substack, since nilpotence of level $<l$ is a closed condition. Therefore $\mathscr{N}ilp_l^{L,ss}(X/S,P)\to\mathbb{A}^1$ is universally closed (see [LP, Proposition 5.1] for a special case of this assertion).
Let us note that the fiber of $\mathscr{N}ilp_1^{L,ss}(X/S,P)\to\mathbb{A}^1$ over 0 is equal to the moduli stack of semistable L-coHiggs sheaves (E,θ) with vanishing p-curvature (see Example 4.10). In particular, [LP, Remark 5.1] is false.
On smooth projective curves of genus $g\geq 2$ the proof of [LP, Lemma 5.1] shows that a vector bundle with a λ-connection of level less than l can be extended to a Higgs bundle with the Higgs field θ satisfying $\theta^l=0$. In particular, for $l=1$ we get the zero Higgs field.

So one could hope that in this case, e.g., if $\widetilde{\mathscr{N}ilp_1}^{L,ss}(X/S,P)\to\mathbb{A}^1$ is the open substack of $\mathscr{N}ilp_1^{L,ss}(X/S,P)\to\mathbb{A}^1$, which over 0 is the moduli substack of semistable sheaves then $\widetilde{\mathscr{N}ilp_1}^{L,ss}(X/S,P)\to\mathbb{A}^1$ is also universally closed as suggested by [LP, Remark 5.1]. However, this expectation is false. In case of a smooth projective curve X of genus $g\geq 2$ there exists a semistable bundle E whose Frobenius pull back F_X^*E is not semistable. But F_X^*E carries a canonical connection ∇_{can} and $(F_X^*E,\nabla_{\mathrm{can}})$ is semistable. After pulling back via $X_K\to X$, where $K=k((t))$, and twisting by t, this provides a semistable vector bundle with a t-connection on X_K which cannot be extended to a semistable family on $X_{k[[t]]}$ so that the Higgs field at the special fibre vanishes. Otherwise, we would get a contradiction with openness of the usual semistability of vector bundles.

5 Deformations of semistable sheaves and the Lan-Sheng-Zuo conjecture

5.1 Langton's theorems

Let R be a discrete valuation ring with maximal ideal m generated by $\pi\in R$. Let K be the quotient field of R and let us assume that the residue field $k=R/m$ is algebraically closed.
Let $X\to S=\operatorname{Spec} R$ be a smooth projective morphism and let L be a smooth $\mathscr{O}_S$-Lie algebroid on X. Let us fix a collection $(D_0,D_1,\dots,D_{n-1})$ of n relatively nef divisors on X/S such that $D_0=D_1$. In the following stability of sheaves on the fibers of $X\to S$ is considered with respect to this fixed collection.
The following theorem generalizes well known Langton's theorem [Lt, Theorem 2)]. We recall the proof as it is not available in the generality that we need. The notation introduced in this proof will be also used in proof of Theorem 5.5.

Theorem 5.1. *Let F be an R-flat $\mathscr{O}_X$-coherent L-module of relative pure dimension n such that the L_K-module $F_K=F\otimes_R K$ is slope semistable. Then there exists an*

L-submodule $E \subset F$ such that $E_K = F_K$ and E_k is a slope semistable L_k-module on X_k.

Proof. First let us note that we can assume that F_k is torsion free as an $\mathcal{O}_{X_k}$-module (this follows, e.g., from [HL, Proposition 4.4.2] or can be proven using a similar method as below). We use without warning the fact that for an R-flat F the degrees of F_K and F_k with respect to $(D_1, \dots, D_{n-1})$ coincide. This follows from the fact that F has a finite locally free resolution on X and intersection products are compatible with specialization (see [SGA6, Expose X, Appendice]).
Let us set $F^0 := F$. If F_k^0 is not slope semistable then we take the maximal destabilizing L-submodule B^0 in F_k^0 and denote by F^1 the kernel of the composition $F^0 \to F_k^0 \to G^0 := F_k^0/B^0$. If F_k^1 is semistable then we get the required submodule of F. Otherwise, we repeat the same procedure for F^1. In this way we construct a sequence of L-modules $F = F^0 \supset F^1 \supset F^2 \supset \dots$ and the main point of the proof is to show that this process cannot continue indefinitely.
Let us assume otherwise. First, let us note that we have short exact sequences

$$0 \to G^n \to F_k^{n+1} \to B^n \to 0,$$

where $G^n = F_k^n/B^n$. Let C^n be the kernel of the composition $B^{n+1} \to F_k^{n+1} \to B^n$.
If $C_n = 0$ then $B_{n+1} \subset B_n$ and hence $\mu(B^{n+1}) \leq \mu(B^n)$. If $C^n \neq 0$ then

$$\mu(C^n) \leq \mu_{\max}(G^n) < \mu(B^n),$$

where the first inequality comes from the fact that $C^n \subset G^n$ and the second one follows from the fact that $B^n \subset F_k^n$ is the maximal destabilizing subsheaf and $G^n = F_k^n/B^n$.
We claim that $\mu(B^{n+1}) < \mu(B^n)$. If $\mu(C^n) \geq \mu(B^{n+1})$ then this inequality follows from the above inequality. If $\mu(C^n) < \mu(B^{n+1})$ then $\mu(B^{n+1}) < \mu(B^{n+1}/C^n)$. But B^{n+1}/C^n is isomorphic to a subsheaf of B^n and B^n is semistable, so in this case we also have $\mu(B^{n+1}) < \mu(B^n)$.
Therefore the sequence $\{\mu(B^n)\}$ is non-increasing. But $\mu(B^{n+1}) < \mu(B^n)$ is possible for only finitely many n since $r!\mu(B^n) \in \mathbb{Z}$ are bounded below by $r!\mu(F_k)$. Therefore for all large n we have $C^n = 0$, i.e., we have inclusions $B^n \supset B^{n+1} \supset \dots$ and $G^n \subset G^{n+1} \subset \dots$. For sufficiently large n these sequences consist of torsion free sheaves with the same slope, so they must stabilize to B and G, respectively. Then $F_k^n = B \oplus G$ for $n \gg 0$. Set $\hat{R} := \varprojlim R/\pi^n R$ and let $\hat{K}$ be the quotient field of $\hat{R}$. Note that F/F^n is R/π^n-flat and as $\mathcal{O}_{X_k}$-module has a filtration with quotients isomorphic to G. Then $\hat{Q} := \varprojlim F/F^n$ is a destabilizing quotient of $F_{\hat{K}}$. But the Harder–Narasimhan filtration is stable under base field extension and therefore F_K is also unstable, contradicting our assumption. □

Our exposition of proof of Langton's theorem is based on [HL] with some small changes (one of the inequalities in proof of [HL, Theorem 2.B.1] is false and we need to give a slightly different argument).

Note that in the above theorem we allow the case when all D_i's are zero. In this case we claim that there exists an L-submodule $E \subset F$ such that $E_K = F_K$ and E_k is torsion free as $\mathcal{O}_{X_k}$-module (by definition slope semistable sheaves are torsion free!).

Let us recall that every slope semistable L-module E has a Jordan–Hölder filtration $E_0 = 0 \subset E_1 \subset ... \subset E_m = E$ by L-submodules such that the associated graded sheaf $\mathrm{Gr}(E) = \oplus E_i/E_{i-1}$ is *slope polystable*, i.e., a direct sum of slope stable (torsion free) L-modules of the same slope.
The following theorem is motivated by theory of moduli spaces and it generalizes [Lt, Theorem 1)].

THEOREM 5.2. *Assume that the collection $(D_0, D_1, \dots, D_{n-1})$ consists of relatively ample divisors. Let F be an R-flat $\mathcal{O}_X$-coherent L-module of relative pure dimension n such that the L_K-module $F_K = F \otimes_R K$ is slope semistable. Let E_1 and E_2 be L-submodules of F such that $(E_1)_K = (E_2)_K = F_K$, $(E_1)_k$ and $(E_2)_k$ are slope semistable. Then the reflexivizations of the associated graded slope polystable sheaves $\mathrm{Gr}((E_1)_k)$ and $\mathrm{Gr}((E_2)_k)$ are isomorphic. Moreover, if at least one of $(E_1)_k$ and $(E_2)_k$ is slope stable then there exists an integer n such that $E_1 = \pi^n E_2$.*

Proof. We prove only the second part, leaving proof of the first one to the reader. Assume that $(E_1)_k$ is slope stable. Consider the discrete valuation ring $\mathcal{O}_{X,\eta}$, where η is the generic point of X_k. Multiplying E_1 by some power of π, we can assume that $E_1 \otimes_{\mathcal{O}_X} \mathcal{O}_{X,\eta} \subset E_2 \otimes_{\mathcal{O}_X} \mathcal{O}_{X,\eta}$ and the induced map $E_1 \otimes k(\eta) \to E_2 \otimes k(\eta)$ is non-zero. But E_1 and E_2 are torsion free so $E_1 \subset E_2$. Since $(E_1)_k$ is slope stable the non-zero map $(E_1)_k \to (E_2)_k$ between slope semistable sheaves of the same slope must be an inclusion. Since the Hilbert polynomials of $(E_1)_k$ and $(E_2)_k$ coincide (from flatness of E_1 and E_2), it must be an isomorphism. ☐

Let Y be a projective scheme over a field k and let L_Y be a k-Lie algebroid on Y. Let us fix an ample line bundle $\mathcal{O}_Y(1)$ on Y. Let $\mathrm{Coh}_d^L(Y)$ be the full subcategory of the category of L-modules which are coherent as $\mathcal{O}_Y$-modules and whose objects are sheaves supported in dimension $\leq d$. Then we can consider the quotient category $\mathrm{Coh}_{d,d'}^L(Y) := \mathrm{Coh}_d^L(Y)/\mathrm{Coh}_{d'-1}^L(Y)$. For any object of $\mathrm{Coh}_{d,d'}^L(Y)$ one can define its Hilbert polynomial which can be used to define notion of (semi)stability in this category.

We can generalize Langton's theorem to singular schemes at the cost of dealing with only one ample polarization. In this case compatibility of intersection product with specialization follows from computation of the Hilbert polynomial. One can also generalize Theorem 5.1 so that it works for other kinds of stability as defined above.
Let $X \to \operatorname{Spec} R$ be a projective morphism with relatively ample line bundle $\mathcal{O}_X(1)$ and let L be a smooth $\mathcal{O}_S$-Lie algebroid on X. The following Langton's type theorem generalizes [Si4, Theorem 10.1] and [HL, Theorem 2.B.1]:

THEOREM 5.3. *Let F be an R-flat $\mathcal{O}_X$-coherent L-module of relative dimension d. Assume that the L_K-module $F_K = F \otimes_R K$ is pure of dimension d and semistable in $\mathrm{Coh}_{d,d'}^L(X_K)$ for some $d' < d$. Then there exists an L-submodule $E \subset F$ such that $E_K = F_K$ and E_k is semistable in $\mathrm{Coh}_{d,d'}^L(X_k)$.*

Proof. The proof is almost the same as the proof of [HL, Theorem 2.B.1]. However, there are a few small problems that we meet in the proof. The first one is that we need to define reflexive hulls of sheaves on the special fiber X_k. This can be done by embedding X into a fixed smooth R-scheme (e.g., use some multiple of the polarization $\mathscr{O}_X(1)$ to embedd X into some projective space over R).

The second problem is the same as before: one of the inequalities in proof of [HL, Theorem 2.B.1] is false and we need to use a slightly different argument similar to the one used in proof of Theorem 5.1. We sketch the necessary changes using the notation of proof of [HL, Theorem 2.B.1]. If $C^n \neq 0$ then we only have

$$p(C^n) \le p_{\max}(G^n) < p(B^n) \mod \mathbb{Q}[T]_{\delta-1}.$$

Hence if $p_{d,\delta}(C^n) \ge p_{d,\delta}(B^{n+1})$ then $p_{d,\delta}(B^{n+1}) < p_{d,\delta}(B^n)$. If $p_{d,\delta}(C^n) < p_{d,\delta}(B^{n+1})$ then we have $p_{d,\delta}(B^{n+1}) < p_{d,\delta}(B^{n+1}/C^n) \le p_{d,\delta}(B^n)$. This proves that if $C^n \neq 0$ then we always have $p_{d,\delta}(B^{n+1}) < p_{d,\delta}(B^n)$ as needed in the argument.

The last problem is the use of Quot-schemes in [HL], which do not exist as projective schemes in our situation. This can be solved as in proof of Theorem 5.1. ☐

THEOREM 5.4. *Let F be an R-flat $\mathscr{O}_X$-coherent L-module of relative pure dimension d such that the L_K-module $F_K = F \otimes_R K$ is semistable in $\mathrm{Coh}^L_{d,d'}(X_K)$ for some $d' < d$. Let E_1 and E_2 be L-submodules of F such that $(E_1)_K = (E_2)_K = F_K$, $(E_1)_k$ and $(E_2)_k$ are semistable in $\mathrm{Coh}^L_{d,d'}(X_k)$ and at least one of them is stable. Then there exists an integer n such that $E_1 = \pi^n E_2$ in $\mathrm{Coh}^L_{d,d'}(X_K)$.*

5.2 SEMISTABLE FILTRATIONS ON SHEAVES WITH CONNECTION

Let L be a smooth Lie algebroid on a normal projective variety X defined over an algebraically closed field k. Let us consider a torsion free coherent $\mathscr{O}_X$-module E with an integrable d_{Ω_L}-connection ∇ (i.e., an L-module whose underlying sheaf is coherent and torsion free as an $\mathscr{O}_X$-module). We say that a filtration $E = N^0 \supset N^1 \supset \ldots \supset N^m = 0$ satisfies *Griffiths transversality* if $\nabla(N^i) \subset N^{i-1} \otimes_{\mathscr{O}_X} \Omega_L$ and the quotients N^i/N^{i+1} are torsion free. For every such filtration the associated graded object $\mathrm{Gr}_N(E) := \bigoplus_i N^i/N^{i+1}$ carries a canonical Ω_L-Higgs field θ defined by ∇. Note that $(\mathrm{Gr}_N(E), \theta)$ is a system of L-Hodge sheaves. A convenient way of looking at this is by means of the Rees construction. More precisely, if $N^\bullet$ is a Griffiths transverse filtration on (E, ∇) then we can consider the subsheaf

$$\xi(E, N^\bullet) := \sum t^{-i} N^i \otimes \mathscr{O}_{X \times \mathbb{A}^1} \subset p_X^* E$$

on $X \times \mathbb{A}^1$. By Griffiths transversality of the filtration $N^\bullet$ the connection $t\nabla$ on $\xi(E, N^\bullet)|_{X \times \mathbb{G}_m}$ extends to a t-d_{Ω_L}-connection on $X \times \mathbb{A}^1$ (i.e., we get an L^R-module on $X \times \mathbb{A}^1$). In the limit as $t \to 0$ we get exactly the above described system of L-Hodge sheaves $(\mathrm{Gr}_N(E), \theta)$.

In the remainder of this section to define semistability we use a fixed collection $(D_0, D_1, \ldots, D_{n-1})$ of nef divisors such that $D_0 = D_1$.

After Simpson [Si4] we say that a Griffiths transverse filtration $N^\bullet$ on (E, ∇) is *slope gr-semistable* if the associated Ω_L-Higgs sheaf $(\mathrm{Gr}_N(E), \theta)$ is slope semistable. A *partial L-oper* is a triple $(E,\nabla,N^\bullet)$ consisting of a torsion free coherent $\mathscr{O}_X$-module E with an integrable d_{Ω_L}-connection ∇ and a Griffiths transverse filtration $N^\bullet$ which is slope gr-semistable.

Theorem 5.5. *If (E,∇) is slope semistable then there exists a canonically defined slope gr-semistable Griffiths transverse filtration $N^\bullet$ on (E,∇) providing it with a partial L-oper structure. This filtration is preserved by the automorphisms of (E,∇).*

Proof. Let R be a localization of $\mathbb{A}^1$ at 0 and let $\tilde{L}$ be the smooth Lie algebroid on $X_R = X\times_k \operatorname{Spec} R$ obtained by restricting of L^R from $X\times_k \mathbb{A}^1$. Consider the trivial filtration $E=N^0\supset N^1=0$. It satisfies Griffiths tranversality so we can associate to it via the Rees construction and restricting to X_R, an R-flat $\mathscr{O}_{X_R}$-coherent $\tilde{L}$-module $F^0=F$ (in fact $F=(p_X^*E,\pi\nabla)$).
Now suppose that we have defined an $\tilde{L}$-submodule $F^n\subset F$ coming by restriction from the Rees construction associated to a Griffiths transverse filtration $N_n^\bullet$ of E. If the associated Ω_L-Higgs sheaf $F_k^n=(\mathrm{Gr}_{N_n}(E),\theta_n)$ is semistable then we get the required filtration. Otherwise, we consider its maximal destabilizing Ω_L-Higgs subsheaf B^n. But $(\mathrm{Gr}_{N_n}(E),\theta_n)$ is a system of Ω_L-Hodge sheaves, so by Corollary 3.5 B^n is also a system of Ω_L-Hodge sheaves. Let us write $B^n=\bigoplus B^n_m$, where $B^n_m\subset \mathrm{Gr}^m_{N_n}(E)=N_n^m/N_n^{m+1}$. Then we can define a new Griffiths transverse filtration $N^\bullet_{n+1}$ on E by setting
$$N^m_{n+1}:=\ker\left(E\to \frac{E/N^m_n}{B^n_{m-1}}\right).$$
Let F^{n+1} denote the restriction to X_R of the L^R-module associated by the Rees construction to $N^\bullet_{n+1}$. We need to prove that this procedure cannot continue indefinitely. To show it, it is sufficient to check that we follow the same procedure as the one described in the proof of Theorem 5.3.
By construction $\pi F^n\subset F^{n+1}\subset F^n$ and in particular $F_K^{n+1}=F_K^n$. On the other hand, on the special fiber of $X_R\to \operatorname{Spec} R$ we have a short exact sequence
$$0\to F_k^n/B^n\to F_k^{n+1}=(\mathrm{Gr}_{N_{n+1}}(E),\theta_{n+1})\to B^n\to 0$$
coming from the definition of the filtration $N^\bullet_{n+1}$. This shows that πF^n is the kernel of the composition $F^{n+1}\to F_k^{n+1}\to B^n$. But then F^{n+1} is the kernel of the composition $F^n\to F^n_k\to F^n_k/B^n$. Now the proof of Theorem 5.3 shows that this procedure must finish.
Since the Harder–Narasimhan filtration is canonically defined, the above described procedure is also canonical and the obtained filtration is preserved by the automorphisms of (E,∇). ☐

In the following the canonical filtration $N^\bullet$ from Theorem 5.5 will be called *Simpson's filtration* of (E,∇) and denoted by $N_S^\bullet$. The reason is that apart from many spectacular results due to Simpson in non-abelian Hodge theory, the construction of the filtration described in the proof of the above theorem was done by Simpson in [Si4, Section 3] for the usual Higgs bundles on complex projective curves. However, our proof of the fact that the procedure stops is different.

Theorem 5.5 generalizes [Si4, Theorem 2.5] to higher dimensions as asked for at the end of [Si4, Section 3]. Indeed, in the characteristic zero case every vector bundle with an integrable connection has vanishing Chern classes. In particular, any saturated subsheaf of such a vector bundle which is preserved by the connection (is locally free and) has vanishing Chern classes. So any vector bundle with an integrable connection is slope semistable (with respect to an arbitrary polarization). This argument fails in the logarithmic case which shows that the above theorem is a correct analogue in this case.

Note that there can be many slope gr-semistable filtrations providing (E,∇) with a partial L-oper structure. This depends on the choice of the Griffiths transverse filtration at the beginning of our procedure (in the proof of Theorem 5.5 we used the canonical choice). In general, all the obtained filtrations are related as described by the following corollary which follows from Theorem 5.2:

COROLLARY 5.6. *If $N^\bullet$ and $M^\bullet$ are two slope gr-semistable Griffiths transverse filtrations on (E,∇) then the reflexivizations of the associated-graded slope polystable Ω_L-Higgs sheaves obtained from their Jordan–Hölder filtrations are isomorphic. In particular, if the associated Ω_L-Higgs sheaf is slope stable then (E,∇) carries a unique gr-semistable Griffiths transverse filtration.*

The above corollary generalizes [Si4, Corollary 4.2]. Note that Simpson's proof does not work so easily in our situation as in higher dimensions we do not have appropriate moduli spaces at our disposal.
Let us also note that any slope gr-semistable filtration can be refined so that the associated graded Ω_L-Higgs sheaf is slope polystable (in which case its reflexivization is uniquely determined by (E,∇) up to an isomorphism).

As an immediate application of Theorem 5.5 we also get the following interesting corollary:

COROLLARY 5.7. *Let L be a smooth trivial Lie algebroid. Let (E,θ) be a torsion free, slope semistable Ω_L-Higgs sheaf on X. Then we can deform it to a slope semistable system of Ω_L-Hodge sheaves.*

5.3 HIGGS-DE RHAM SEQUENCES

Let k be an algebraically closed field of characteristic $p>0$. Let X be a smooth projective k-variety of dimension n that can be lifted to a smooth scheme $\mathscr{X}$ over $W_2(k)$.
Let $\mathrm{MIC}_{p-1}(X/k)$ be the category of $\mathscr{O}_X$-modules with an integrable connection whose p-curvature is nilpotent of level less or equal to $p-1$. Similarly, let $\mathrm{HIG}_{p-1}(X/k)$ denote the category of Higgs $\mathscr{O}_{X'}$-modules with a nilpotent Higgs sheaf of level less or equal to $p-1$. In this case one of the main results of Ogus and Vologodsky (see [OV, Theorem 2.8]) says that:

THEOREM 5.8. *The Cartier operator*

$$C_{\mathscr{X}/\mathscr{S}}:\mathrm{MIC}_{p-1}(X/k)\to \mathrm{HIG}_{p-1}(X'/k)$$

defines an equivalence of categories with quasi-inverse

$$C^{-1}_{\mathscr{X}/\mathscr{S}} : \mathrm{HIG}_{p-1}(X/k) \to \mathrm{MIC}_{p-1}(X'/k).$$

A small variant of the following lemma can be found in proof of [OV, Theorem 4.17]:

LEMMA 5.9. *Let* $(E,\theta) \in \mathrm{HIG}_{p-1}(X'/S)$. *Then*

$$[C^{-1}_{\mathscr{X}/\mathscr{S}}(E)] = F^*_{X/S}[E],$$

where $[\cdot]$ *denotes the class of a coherent* $\mathscr{O}_X$*-module in Grothendieck's* K*-group* $K_0(X)$.

As a corollary to Theorem 5.8 and Lemma 5.9 we get the following:

COROLLARY 5.10. *Let* (E,θ) *be a torsion free Higgs sheaf with nilpotent Higgs field of level less than p. Then it is slope semistable if and only if the corresponding sheaf with integrable connection* $(V,\nabla) := C^{-1}_{\mathscr{X}/\mathscr{S}}(E,\theta)$ *is slope semistable.*

Now let (E,θ) be a rank r torsion free Higgs sheaf with nilpotent Higgs field. Let us assume that $r \le p$ so that level of nilpotence of (E,θ) is less than p. Let us recall the following definition taken from [LSZ].

DEFINITION 5.11. A *Higgs–de Rham sequence* of (E,θ) is an infinite sequence

$$\begin{tikzcd}[column sep=small] & (V_0,\nabla_0) \arrow[dr, "\mathrm{Gr}_{N_0}"] & & (V_1,\nabla_1) \arrow[dr, "\mathrm{Gr}_{N_1}"] & \\ (E_0,\theta_0)=(E,\theta) \arrow[ur, "C^{-1}"] & & (E_1,\theta_1) \arrow[ur, "C^{-1}"] & & \dots \end{tikzcd}$$

in which $C^{-1} = C^{-1}_{\mathscr{X}/S}$ is the inverse Cartier transform, $N_i^\bullet$ is a Griffiths transverse filtration of (V_i,∇_i) and $(E_{i+1} := \mathrm{Gr}_{N_i}(V_i), \theta_{i+1})$ is the associated Higgs sheaf.

The following theorem proves the conjecture of Lan-Sheng-Zuo [LSZ, Conjecture 2.8]:

THEOREM 5.12. *If* (E,θ) *is slope semistable then there exists a canonically defined Higgs–de Rham sequence*

$$\begin{tikzcd}[column sep=small] & (V_0,\nabla_0) \arrow[dr, "\mathrm{Gr}_{N_S}"] & & (V_1,\nabla_1) \arrow[dr, "\mathrm{Gr}_{N_S}"] & \\ (E_0,\theta_0)=(E,\theta) \arrow[ur, "C^{-1}"] & & (E_1,\theta_1) \arrow[ur, "C^{-1}"] & & \dots \end{tikzcd}$$

in which each (V_i,∇_i) *is slope semistable and* (E_{i+1},θ_{i+1}) *is the slope semistable Higgs sheaf associated to* (V_i,∇_i) *via Simpson's filtration.*

Proof. The proof is by induction on index i. Once we defined slope semistable (E_i, θ_i), we can construct (V_i, ∇_i), which is slope semistable by Corollary 5.10. So by Theorem 5.5 there exists Simpson's filtration on (V_i, ∇_i) and hence we can construct a slope semistable Higgs sheaf (E_{i+1}, θ_{i+1}). Since (E_{i+1}, θ_{i+1}) is a system of Hodge sheaves and $r \leq p$, it satisfies the nilpotence condition required to define C^{-1}. $\square$

In the above theorem slope semistability is defined with respect to an arbitrary fixed collection $(D_1, \dots, D_{n-1})$ of nef divisors on X.

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