Silting Objects, Simple-Minded Collections, t-Structures and Co-t-Structures for Finite-Dimensional Algebras

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Abstract. Bijective correspondences are established between (1) silting objects, (2) simple-minded collections, (3) bounded t-structures with length heart and (4) bounded co-t-structures. These correspondences are shown to commute with mutations and partial orders. The results are valid for finite-dimensional algebras. A concrete example is given to illustrate how these correspondences help to compute the space of Bridgeland's stability conditions.

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Contents

1. Introduction

Let Λ be a finite-dimensional associative algebra. Fundamental objects of study in the representation theory of Λ are the projective modules, the simple modules and the category of all (finite-dimensional) Λ-modules. Various structural concepts have been introduced that include one of these classes of objects as particular instances. In this article, four such concepts are related by explicit bijections. Moreover, these bijections are shown to commute with the basic operation of mutation and to preserve partial orders.

These four concepts may be based on two different general points of view, either considering particular generators of categories ((1) and (2)) or considering structures on categories that identify particular subcategories ((3) and (4)):

(1) Focussing on objects that generate categories, the theory of Morita equivalences has been extended to tilting or derived equivalences. In this way, projective generators are examples of tilting modules, which have been generalised further to *silting objects* (which are allowed to have negative self-extensions).
(2) Another, and different, natural choice of 'generators' of a module category is the set of simple modules (up to isomorphism). In the context of derived or stable equivalences, this set is included in the concept of simple-minded system or *simple-minded collection*.
(3) Starting with a triangulated category and looking for particular subcategories, *t-structures* have been defined so as to provide abelian categories as their hearts. The finite-dimensional Λ-modules form the heart of some t-structure in the bounded derived category $D^b(\mathsf{mod}\,\Lambda)$.
(4) Choosing as triangulated category the homotopy category $K^b(\mathsf{proj}\,\Lambda)$, one considers *co-t-structures*. The additive category $\mathsf{proj}\,\Lambda$ occurs as the co-heart of some co-t-structure in $K^b(\mathsf{proj}\,\Lambda)$.

The first main result of this article is:

Theorem (6.1). *Let Λ be a finite-dimensional algebra over a field K. There are one-to-one correspondences between*

(1) *equivalence classes of silting objects in $K^b(\mathsf{proj}\,\Lambda)$,*
(2) *equivalence classes of simple-minded collections in $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$,*
(3) *bounded t-structures of $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ with length heart,*
(4) *bounded co-t-structures of $K^b(\mathsf{proj}\,\Lambda)$.*

Here two sets of objects in a category are *equivalent* if they additively generate the same subcategory.

A common feature of all four concepts it that they allow for comparisons, often by equivalences. In particular, each of the four structures to be related comes with a basic operation, called mutation, which produces a new such structure from a given one. Moreover, on each of the four structures there is a partial order. All the bijections in Theorem 6.1 enjoy the following naturality properties:

Theorem (7.12). *Each of the bijections between the four structures (1), (2), (3) and (4) commutes with the respective operation of mutation.*

Theorem (7.13). *Each of the bijections between the four structures (1), (2), (3) and (4) preserves the respective partial orders.*

The four concepts are crucial in representation theory, geometry and topology. They are also closely related to fundamental concepts in cluster theory such as clusters ([20]), c-matrices and g-matrices ([21, 40]) and cluster-tilting objects ([7]). We refer to the survey paper [16] for more details. A concrete example to be given at the end of the article demonstrates one practical use of these bijections and their properties.
Finally we give some remarks on the literature. For path algebras of Dynkin quivers, Keller and Vossieck [33] have already given a bijection between bounded t-structures and silting objects. The bijection between silting objects and t-structures with length heart has been established by Keller and Nicolás [32] for homologically smooth non-positive dg algebras, by Assem, Souto Salorio and Trepode [5] and by Vitória [46], who are focussing on piecewise hereditary algebras. An unbounded version of this bijection has been studied by Aihara and Iyama [1]. The bijection between simple-minded collections and bounded t-structures has been established implicitely in Al-Nofayee's work [3] and explicitely for homologically smooth non-positive dg algebras in Keller and Nicolás' work [32] and for finite-dimensional algebras in our preprint [37], which has been partly incorporated into the present article, and partially in the work [44] of Rickard and Rouquier. For hereditary algebras, Buan, Reiten and Thomas [17] studied the bijections between silting objects, simple-minded collections (=$\mathrm{Hom}_{\leq 0}$-configurations in their setting) and bounded t-structures. The correspondence between silting objects and co-t-structures appears implicitly on various levels of generality in the work of Aihara and Iyama [1] and of Bondarko [12] and explicitly in full generality in the work of Mendoza, Sáenz, Santiago and Souto Salorio [39] and of Keller and Nicolás [31]. For homologically smooth non-positive dg algebras, all the bijections are due to Keller and Nicolás [31]. The intersection of our results with those of Keller and Nicolás is the case of finite-dimensional algebras of finite global dimension.

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2. Notations and preliminaries

2.1. Notations. Throughout, K will be a field. All algebras, modules, vector spaces and categories are over the base field K, and $\mathrm{D} = \mathsf{Hom}_K(?, K)$ denotes the K-dual. By abuse of notation, we will denote by Σ the suspension functors of all the triangulated categories.

For a category $\mathcal{C}$, we denote by $\mathsf{Hom}_{\mathcal{C}}(X,Y)$ the morphism space from X to Y, where X and Y are two objects of $\mathcal{C}$. We will omit the subscript and write $\mathsf{Hom}(X,Y)$ when it does not cause confusion. For $\mathcal{S}$ a set of objects or a subcategory of $\mathcal{C}$, call

$$^{\perp}\mathcal{S} = \{X \in \mathcal{C} \mid \mathsf{Hom}(X,S) = 0 \text{ for all } S \in \mathcal{S}\}$$

and

$$\mathcal{S}^{\perp} = \{X \in \mathcal{C} \mid \mathsf{Hom}(S,X) = 0 \text{ for all } S \in \mathcal{S}\}$$

the *left* and *right perpendicular category* of $\mathcal{S}$, respectively.

Let $\mathcal{C}$ be an additive category and $\mathcal{S}$ a set of objects or a subcategory of $\mathcal{C}$. Let $\mathsf{Add}(\mathcal{S})$ and $\mathsf{add}(\mathcal{S})$, respectively, denote the smallest full subcategory of $\mathcal{C}$ containing all objects of $\mathcal{S}$ and stable for taking direct summands and coproducts respectively taking finite coproducts. The category $\mathsf{add}(\mathcal{S})$ will be called the *additive closure* of $\mathcal{S}$. If further $\mathcal{C}$ is abelian or triangulated, the *extension closure* of $\mathcal{S}$ is the smallest subcategory of $\mathcal{C}$ containing $\mathcal{S}$ and stable under taking extensions. Assume that $\mathcal{C}$ is triangulated and let $\mathsf{thick}(\mathcal{S})$ denote the smallest triangulated subcategory of $\mathcal{C}$ containing objects in $\mathcal{S}$ and stable under taking direct summands. We say that $\mathcal{S}$ is a *set of generators* of $\mathcal{C}$, or that $\mathcal{C}$ is *generated by* $\mathcal{S}$, when $\mathcal{C} = \mathsf{thick}(\mathcal{S})$.

2.2. Derived categories. For a finite-dimensional algebra Λ, let $\mathsf{Mod}\,\Lambda$ (respectively, $\mathsf{mod}\,\Lambda$, $\mathsf{proj}\,\Lambda$, $\mathsf{inj}\,\Lambda$) denote the category of right Λ-modules (respectively, finite-dimensional right Λ-modules, finite-dimensional projective, injective right Λ-modules), let $K^b(\mathsf{proj}\,\Lambda)$ (respectively, $K^b(\mathsf{inj}\,\Lambda)$) denote the homotopy category of bounded complexes of $\mathsf{proj}\,\Lambda$ (respectively, $\mathsf{inj}\,\Lambda$) and let $\mathcal{D}(\mathsf{Mod}\,\Lambda)$ (respectively, $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$, $\mathcal{D}^-(\mathsf{mod}\,\Lambda)$) denote the derived category of $\mathsf{Mod}\,\Lambda$ (respectively, bounded derived category of $\mathsf{mod}\,\Lambda$, bounded above derived category of $\mathsf{mod}\,\Lambda$). All these categories are triangulated with suspension functor the shift functor. We view $\mathcal{D}^-(\mathsf{mod}\,\Lambda)$ and $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ as triangulated subcategories of $\mathcal{D}(\mathsf{Mod}\,\Lambda)$.

The categories $\mathsf{mod}\,\Lambda$, $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ and $K^b(\mathsf{proj}\,\Lambda)$ are Krull–Schmidt categories. An object M of $\mathsf{mod}\,\Lambda$ (respectively, $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$, $K^b(\mathsf{proj}\,\Lambda)$) is said to be *basic* if every indecomposable direct summand of M has multiplicity 1. The finite-dimensional algebra Λ is said to be *basic* if the free module of rank 1 is basic in $\mathsf{mod}\,\Lambda$ (equivalently, in $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ or $K^b(\mathsf{proj}\,\Lambda)$).

For a differential graded(=dg) algebra A, let $\mathcal{C}(A)$ denote the category of (right) dg modules over A and $K(A)$ the homotopy category. Let $\mathcal{D}(A)$ denote the derived category of dg A-modules, *i.e.* the triangle quotient of $K(A)$ by acyclic dg A-modules, *cf.* [29, 30], and let $\mathcal{D}_{fd}(A)$ denote its full subcategory of dg A-modules whose total cohomology is finite-dimensional. The category $\mathcal{C}(A)$ is abelian and the other three categories are triangulated with suspension functor the shift functor of complexes. Let $\mathsf{per}(A) = \mathsf{thick}(A_A)$, *i.e.* the triangulated subcategory of $\mathcal{D}(A)$ generated by the free dg A-module of rank 1.

For two dg A-modules M and N, let $\mathcal{H}om_A(M,N)$ denote the complex whose degree n component consists of those A-linear maps from M to N which are homogeneous of degree n, and whose differential takes a homogeneous map f of degree n to $d_N \circ f - (-1)^n f \circ d_M$. Then

$$\mathsf{Hom}_{K(A)}(M,N) = H^0\mathcal{H}om_A(M,N). \tag{2.1}$$

A dg A-module M is said to be *$\mathcal{K}$-projective* if $\mathcal{H}om_A(M,N)$ is acyclic when N is an acyclic dg A-module. For example, A_A, the free dg A-module of rank 1 is $\mathcal{K}$-projective, because $\mathcal{H}om_A(A,N) = N$. Dually, one defines *$\mathcal{K}$-injective* dg modules, and $D({}_AA)$ is $\mathcal{K}$-injective. For two dg A-modules M and N such that M is $\mathcal{K}$-projective or N is $\mathcal{K}$-injective, we have

$$\mathsf{Hom}_{\mathcal{D}(A)}(M,N) = \mathsf{Hom}_{K(A)}(M,N). \tag{2.2}$$

Let A and B be two dg algebras. Then a triangle equivalence between $\mathcal{D}(A)$ and $\mathcal{D}(B)$ restricts to a triangle equivalence between $\mathsf{per}(A)$ and $\mathsf{per}(B)$ and also to a triangle equivalence between $\mathcal{D}_{fd}(A)$ and $\mathcal{D}_{fd}(B)$. If A is a finite-dimensional algebra viewed as a dg algebra concentrated in degree 0, then $\mathcal{D}(A)$ is exactly $\mathcal{D}(\mathsf{Mod}\,A)$, $\mathcal{D}_{fd}(A)$ is $\mathcal{D}^b(\mathsf{mod}\,A)$, $\mathsf{per}(A)$ is triangle equivalent to $K^b(\mathsf{proj}\,A)$, and $\mathsf{thick}(\mathrm{D}({}_AA))$ is triangle equivalent to $K^b(\mathsf{inj}\,A)$.

2.3. The Nakayama functor. Let Λ be a finite-dimensional algebra. The *Nakayama functor* $\nu_{\mathsf{mod}\,\Lambda}$ is defined as $\nu_{\mathsf{mod}\,\Lambda} = ? \otimes_\Lambda \mathrm{D}({}_\Lambda\Lambda)$, and the *inverse Nakayama functor* $\nu^{-1}_{\mathsf{mod}\,\Lambda}$ is its right adjoint $\nu^{-1}_{\mathsf{mod}\,\Lambda} = \mathsf{Hom}_\Lambda(\mathrm{D}({}_\Lambda\Lambda), ?)$. They restrict to quasi-inverse equivalences between $\mathsf{proj}\,\Lambda$ and $\mathsf{inj}\,\Lambda$.

The derived functors of $\nu_{\mathsf{mod}\,\Lambda}$ and $\nu^{-1}_{\mathsf{mod}\,\Lambda}$, denoted by ν and ν^{-1}, restrict to quasi-inverse triangle equivalences between $K^b(\mathsf{proj}\,\Lambda)$ and $K^b(\mathsf{inj}\,\Lambda)$. When Λ is self-injective, they restrict to quasi-inverse triangle auto-equivalences of $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$.

The Auslander–Reiten formula for M in $K^b(\mathsf{proj}\,\Lambda)$ and N in $\mathcal{D}(\mathsf{Mod}\,\Lambda)$ (*cf.* [23, Chapter 1, Section 4.6]) provides an isomorphism

$$\mathrm{D}\,\mathsf{Hom}(M,N) \cong \mathsf{Hom}(N, \nu M),$$

which is natural in M and N. When $K^b(\mathsf{proj}\,\Lambda)$ coincides with $K^b(\mathsf{inj}\,\Lambda)$ (that is, when Λ is Gorenstein), it has Auslander–Reiten triangles and the Auslander–Reiten translation is $\tau = \nu \circ \Sigma^{-1}$.

3. The four concepts

In this section we introduce silting objects, simple-minded collections, t-structures and co-t-structure. Let $\mathcal{C}$ be a triangulated category with suspension functor Σ.

3.1. Silting objects. A subcategory $\mathcal{M}$ of $\mathcal{C}$ is called a *silting subcategory* [33, 1] if it is stable for taking direct summands and generates $\mathcal{C}$ (*i.e.* $\mathcal{C} = \mathsf{thick}(\mathcal{M})$) and if $\mathsf{Hom}(M, \Sigma^m N) = 0$ for $m > 0$ and $M, N \in \mathcal{M}$.

Theorem 3.1. ([1, Theorem 2.27]) *Assume that $\mathcal{C}$ is Krull–Schmidt and has a silting subcategory $\mathcal{M}$. Then the Grothendieck group of $\mathcal{C}$ is free and its rank is equal to the cardinality of the set of isomorphism classes of indecomposable objects of $\mathcal{M}$.*

An object M of $\mathcal{C}$ is called a *silting object* if $\mathsf{add}\, M$ is a silting subcategory of $\mathcal{C}$. This notion was introduced by Keller and Vossieck in [33] to study t-structures on the bounded derived category of representations over a Dynkin quiver. Recently it has also been studied by Wei [47] (who uses the terminology *semi-tilting complexes*) from the perspective of classical tilting theory. A *tilting object* is a silting object M such that $\mathsf{Hom}(M, \Sigma^m M) = 0$ for $m < 0$. For an algebra Λ, a tilting object in $K^b(\mathsf{proj}\,\Lambda)$ is called a *tilting complex* in the literature. For example, the free module of rank 1 is a tilting object in $K^b(\mathsf{proj}\,\Lambda)$. Assume that Λ is finite-dimensional. Theorem 3.1 implies that (a) any silting subcategory of $K^b(\mathsf{proj}\,\Lambda)$ is the additive closure of a silting object, and (b) any two basic silting objects have the same number of indecomposable direct summands. We will rederive (b) as a corollary of the existence of a certain derived equivalence (Corollary 5.1).

3.2. Simple-minded collections.

Definition 3.2. *A collection $X_1, \ldots, X_r$ of objects of $\mathcal{C}$ is said to be* simple-minded *(cohomologically Schurian in* [3]*) if the following conditions hold for $i, j = 1, \ldots, r$*

- $\mathsf{Hom}(X_i, \Sigma^m X_j) = 0, \ \forall\ m < 0,$
- $\mathsf{End}(X_i)$ *is a division algebra and* $\mathsf{Hom}(X_i, X_j)$ *vanishes for $i \neq j$,*
- $X_1, \ldots, X_r$ *generate $\mathcal{C}$ (*i.e. $\mathcal{C} = \mathsf{thick}(X_1, \ldots, X_r)$*).*

Simple-minded collections are variants of simple-minded systems in [36] and were first studied by Rickard [43] in the context of derived equivalences of symmetric algebras. For a finite-dimensional algebra Λ, a complete collection of pairwise non-isomorphic simple modules is a simple-minded collection in $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$. A natural question is: do any two simple-minded collections have the same collection of endomorphism algebras?

3.3. t-structures. A *t-structure* on $\mathcal{C}$ ([8]) is a pair $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ of strict (that is, closed under isomorphisms) and full subcategories of $\mathcal{C}$ such that

- $\Sigma\mathcal{C}^{\leq 0} \subseteq \mathcal{C}^{\leq 0}$ and $\Sigma^{-1}\mathcal{C}^{\geq 0} \subseteq \mathcal{C}^{\geq 0}$;
- $\mathsf{Hom}(M, \Sigma^{-1}N) = 0$ for $M \in \mathcal{C}^{\leq 0}$ and $N \in \mathcal{C}^{\geq 0}$,
- for each $M \in \mathcal{C}$ there is a triangle $M' \to M \to M'' \to \Sigma M'$ in $\mathcal{C}$ with $M' \in \mathcal{C}^{\leq 0}$ and $M'' \in \Sigma^{-1}\mathcal{C}^{\geq 0}$.

The two subcategories $\mathcal{C}^{\leq 0}$ and $\mathcal{C}^{\geq 0}$ are often called the *aisle* and the *co-aisle* of the t-structure respectively. The *heart* $\mathcal{C}^{\leq 0} \cap \mathcal{C}^{\geq 0}$ is always abelian. Moreover, $\mathsf{Hom}(M, \Sigma^m N)$ vanishes for any two objects M and N in the heart and for any $m < 0$. The t-structure $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ is said to be *bounded* if

$$\bigcup_{n\in\mathbb{Z}} \Sigma^n \mathcal{C}^{\leq 0} = \mathcal{C} = \bigcup_{n\in\mathbb{Z}} \Sigma^n \mathcal{C}^{\geq 0}.$$

A bounded t-structure is one of the two ingredients of a Bridgeland stability condition [15]. A typical example of a t-structure is the pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ for the derived category $\mathcal{D}(\mathsf{Mod}\,\Lambda)$ of an (ordinary) algebra Λ, where $\mathcal{D}^{\leq 0}$ consists of complexes with vanishing cohomologies in positive degrees, and $\mathcal{D}^{\geq 0}$ consists of complexes with vanishing cohomologies in negative degrees. This t-structure restricts to a bounded t-structure of $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ whose heart is $\mathsf{mod}\,\Lambda$, which is a *length category*, *i.e.* every object in it has finite length. The following lemma is well-known.

Lemma 3.3. *Let $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ be a bounded t-structure on $\mathcal{C}$ with heart $\mathcal{A}$.*

(a) *The embedding $\mathcal{A} \to \mathcal{C}$ induces an isomorphism $K_0(\mathcal{A}) \to K_0(\mathcal{C})$ of Grothendieck groups.*
(b) *$\mathcal{C}^{\leq 0}$ respectively $\mathcal{C}^{\geq 0}$ is the extension closure of $\Sigma^m\mathcal{A}$ for $m \geq 0$ respectively for $m \leq 0$.*
(c) *$\mathcal{C} = \mathsf{thick}(\mathcal{A})$.*

Assume further $\mathcal{A}$ is a length category with simple objects $\{S_i \mid i \in I\}$.

(d) *$\mathcal{C}^{\leq 0}$ respectively $\mathcal{C}^{\geq 0}$ is the extension closure of $\Sigma^m\{S_i \mid i \in I\}$ for $m \geq 0$ respectively for $m \leq 0$.*
(e) *$\mathcal{C} = \mathsf{thick}(S_i, i \in I)$.*
(f) *If I is finite, then $\{S_i \mid i \in I\}$ is a simple-minded collection.*

3.4. Co-t-structures. According to [41], a *co-t-structure* on $\mathcal{C}$ (or *weight structure* in [12]) is a pair $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ of strict and full subcategories of $\mathcal{C}$ such that

· both $\mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq 0}$ are additive and closed under taking direct summands,
· $\Sigma^{-1}\mathcal{C}_{\geq 0} \subseteq \mathcal{C}_{\geq 0}$ and $\Sigma\mathcal{C}_{\leq 0} \subseteq \mathcal{C}_{\leq 0}$;
· $\mathsf{Hom}(M, \Sigma N) = 0$ for $M \in \mathcal{C}_{\geq 0}$ and $N \in \mathcal{C}_{\leq 0}$,
· for each $M \in \mathcal{C}$ there is a triangle $M' \to M \to M'' \to \Sigma M'$ in $\mathcal{C}$ with $M' \in \mathcal{C}_{\geq 0}$ and $M'' \in \Sigma\mathcal{C}_{\leq 0}$.

The *co-heart* is defined as the intersection $\mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0}$. This is usually not an abelian category. For any two objects M and N in the co-heart, the morphism space $\mathsf{Hom}(M, \Sigma^m N)$ vanishes for any $m > 0$. The co-t-structure $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ is said to be *bounded* [12] if

$$\bigcup_{n\in\mathbb{Z}} \Sigma^n\mathcal{C}_{\leq 0} = \mathcal{C} = \bigcup_{n\in\mathbb{Z}} \Sigma^n\mathcal{C}_{\geq 0}.$$

A bounded co-t-structure is one of the two ingredients of a Jørgensen–Pauksztello costability condition [27]. A typical example of a co-t-structure is the pair $(K_{\geq 0}, K_{\leq 0})$ for the homotopy category $K^b(\mathsf{proj}\,\Lambda)$ of a finite-dimensional algebra Λ, where $K_{\geq 0}$ consists of complexes which are homotopy equivalent to a complex bounded below at 0, and $K_{\leq 0}$ consists of complexes which are homotopy equivalent to a complex bounded above at 0. The co-heart of this co-t-structure is $\mathsf{proj}\,\Lambda$.

Lemma 3.4. ([39, Theorem 4.10 (a)]) *Let* $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ *be a bounded co-t-structure on* $\mathcal{C}$ *with co-heart* $\mathcal{A}$. *Then* $\mathcal{A}$ *is a silting subcategory of* $\mathcal{C}$.

Proof. For the convenience of the reader we give a proof. It suffices to show that $\mathcal{C} = \mathsf{thick}(\mathcal{A})$. Let M be an object of $\mathcal{C}$. Since the co-t-structure is bounded, there are integers $m \geq n$ such that $M \in \Sigma^m \mathcal{C}_{\geq 0} \cap \Sigma^n \mathcal{C}_{\leq 0}$. Up to suspension and cosuspension we may assume that $m = 0$. If $n = 0$, then $M \in \mathcal{A}$. Suppose $n < 0$. There exists a triangle

$$M' \longrightarrow M \longrightarrow M'' \longrightarrow \Sigma M'$$

with $M' \in \Sigma^{-1}\mathcal{C}_{\geq 0}$ and $M'' \in \mathcal{C}_{\leq 0}$. In fact, $M'' \in \mathcal{A}$, see [12, Proposition 1.3.3.6]. Moreover, $\Sigma M' \in \Sigma^{n+1}\mathcal{C}_{\leq 0}$ due to the triangle

$$M'' \longrightarrow \Sigma M' \longrightarrow \Sigma M \longrightarrow \Sigma M''$$

since both M'' and ΣM belong to $\Sigma^{n+1}\mathcal{C}_{\leq 0}$ and $\mathcal{C}_{\leq 0}$ is extension closed (see [12, Proposition 1.3.3.3]). So $\Sigma M' \in \mathcal{C}_{\geq 0} \cup \Sigma^{n+1}\mathcal{C}_{\leq 0}$. We finish the proof by induction on n. $\surd$

Proposition 3.5. ([1, Proposition 2.22], [12, (proof of) Theorem 4.3.2], [39, Theorem 5.5] and [31]) *Let* $\mathcal{A}$ *be a silting subcategory of* $\mathcal{C}$. *Let* $\mathcal{C}_{\leq 0}$ *respectively* $\mathcal{C}_{\geq 0}$ *be the extension closure of* $\Sigma^m \mathcal{A}$ *for* $m \geq 0$ *respectively for* $m \leq 0$. *Then* $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ *is a bounded co-t-structure on* $\mathcal{C}$ *with co-heart* $\mathcal{A}$.

4. Finite-dimensional non-positive dg algebras

In this section we study derived categories of *non-positive dg algebras*, *i.e.* dg algebras $A = \bigoplus_{i \in \mathbb{Z}} A^i$ with $A^i = 0$ for $i > 0$, especially finite-dimensional non-positive dg algebras, *i.e.* , non-positive dg algebras which, as vector spaces, are finite-dimensional. These results will be used in Sections 5.1 and 5.4.

Non-positive dg algebras are closely related to silting objects. A triangulated category is said to be *algebraic* if it is triangle equivalent to the stable category of a Frobenius category.

Lemma 4.1. (a) *Let* A *be a non-positive dg algebra. The free dg* A*-module of rank* 1 *is a silting object of* $\mathsf{per}(A)$.

(b) *Let* $\mathcal{C}$ *be an algebraic triangulated category with split idempotents and let* $M \in \mathcal{C}$ *be a silting object. Then there is a non-positive dg algebra* A *together with a triangle equivalence* $\mathsf{per}(A) \xrightarrow{\sim} \mathcal{C}$ *which takes* A *to* M.

Proof. (a) This is because $\mathsf{Hom}_{\mathsf{per}(A)}(A, \Sigma^i A) = H^i(A)$ vanishes for $i > 0$.
(b) By [30, Theorem 3.8 b)] (which is a 'classically generated' version of [29, Theorem 4.3]), there is a dg algebra A' together with a triangle equivalence $\mathsf{per}(A') \xrightarrow{\sim} \mathcal{C}$. In particular, there are isomorphisms $\mathsf{Hom}_{\mathsf{per}(A')}(A', \Sigma^i A') \cong \mathsf{Hom}_{\mathcal{C}}(M, \Sigma^i M)$ for all $i \in \mathbb{Z}$. Since M is a silting object, A' has vanishing cohomologies in positive degrees. Therefore, if $A = \tau_{\leq 0} A'$ is the standard

truncation at position 0, then the embedding $A \hookrightarrow A'$ is a quasi-isomorphism. It follows that there is a composite triangle equivalence

$$\mathsf{per}(A) \xrightarrow{\sim} \mathsf{per}(A') \xrightarrow{\sim} \mathcal{C}$$

which takes A to M. $\surd$

In the sequel of this section we assume that A is a finite-dimensional non-positive dg algebra. The 0-th cohomology $\bar{A} = H^0(A)$ of A is a finite-dimensional K-algebra. Let $\mathsf{Mod}\,\bar{A}$ and $\mathsf{mod}\,\bar{A}$ denote the category of (right) modules over $\bar{A}$ and its subcategory consisting of those finite-dimensional modules. Let $\pi : A \to \bar{A}$ be the canonical projection. We view $\mathsf{Mod}\,\bar{A}$ as a subcategory of $\mathcal{C}(A)$ via π.
The total cohomology $H^*(A)$ of A is a finite-dimensional graded algebra with multiplication induced from the multiplication of A. Let M be a dg A-module. Then the total cohomology $H^*(M)$ carries a graded $H^*(A)$-module structure, and hence a graded $\bar{A} = H^0(A)$-module structure. In particular, a stalk dg A-module concentrated in degree 0 is an $\bar{A}$-module.

4.1. The standard t-structure. We follow [22, 4, 34], where the dg algebra is not necessarily finite-dimensional.
Let $M = \ldots \to M^{i-1} \stackrel{d^{i-1}}{\to} M^i \stackrel{d^i}{\to} M^{i+1} \to \ldots$ be a dg A-module. Consider the standard truncation functors $\tau_{\leq 0}$ and $\tau_{>0}$:

$$\tau_{\leq 0}M = \cdots \to M^{-2} \xrightarrow{d^{-2}} M^{-1} \xrightarrow{d^{-1}} \mathrm{ker} d^0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

$$\tau_{>0}M = \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow M^0/\mathrm{ker} d^0 \xrightarrow{d^0} M^1 \xrightarrow{d^1} M^2 \xrightarrow{d^2} M^3 \to \cdots$$

Since A is non-positive, $\tau_{\leq 0}M$ is a dg A-submodule of M and $\tau_{>0}M$ is the corresponding quotient dg A-module. Hence there is a distinguished triangle in $\mathcal{D}(A)$

$$\tau_{\leq 0}M \to M \to \tau_{>0}M \to \Sigma\tau_{\leq 0}M.$$

These two functors define a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D}(A)$, where $\mathcal{D}^{\leq 0}$ is the subcategory of $\mathcal{D}(A)$ consisting of dg A-modules with vanishing cohomology in positive degrees, and $\mathcal{D}^{\geq 0}$ is the subcategory of $\mathcal{D}(A)$ consisting of dg A-modules with vanishing cohomology in negative degrees.
By the definition of the t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$, the heart $\mathcal{H} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ consists of those dg A-modules whose cohomology is concentrated in degree 0. Thus the functor H^0 induces an equivalence

$$\begin{array}{rcl} H^0 : \mathcal{H} & \longrightarrow & \mathsf{Mod}\,\bar{A}. \\ M & \mapsto & H^0(M) \end{array}$$

See also [26, Theorem 1.3]. The t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D}(A)$ restricts to a bounded t-structure on $\mathcal{D}_{fd}(A)$ with heart equivalent to $\mathsf{mod}\,\bar{A}$.

4.2. Morita reduction. Let d be the differential of A. Then $d(A^0)=0$.
Let e be an idempotent of A. For degree reasons, e must belong to A^0, and the graded subspace eA of A is a dg submodule: $d(ea)=d(e)a+ed(a)=ed(a)$. Therefore for each decomposition $1=e_1+\ldots+e_n$ of the unity into a sum of primitive orthogonal idempotents, there is a direct sum decomposition $A=e_1A\oplus\ldots\oplus e_nA$ of A into indecomposable dg A-modules. Moreover, if e and e' are two idempotents of A such that $eA\cong e'A$ as ordinary modules over the ordinary algebra A, then this isomorphism is also an isomorphism of dg modules. Indeed, there are two elements of A such that $fg=e$ and $gf=e'$. Again for degree reasons, f and g belong to A^0. So they induce isomorphisms of dg A-modules: $eA\to e'A$, $a\mapsto ga$ and $e'A\to eA$, $a\mapsto fa$. It follows that the above decomposition of A into a direct sum of indecomposable dg modules is essentially unique. Namely, if $1=e'_1+\ldots+e'_n$ is another decomposition of the unity into a sum of primitive orthogonal idempotents, then $m=n$ and up to reordering, $e_1A\cong e'_1A$, …, $e_nA\cong e'_nA$.

4.3. The perfect derived category. Since A is finite-dimensional (and thus has finite-dimensional total cohomology), $\mathsf{per}(A)$ is a triangulated subcategory of $\mathcal{D}_{fd}(A)$.
We assume, as we may, that A is basic. Let $1=e_1+\ldots+e_n$ be a decomposition of 1 in A into a sum of primitive orthogonal idempotents. Since $d(x)=\lambda_1e_{i_1}+\ldots+\lambda_se_{i_s}$ implies that $d(e_{i_j}x)=\lambda_je_{i_j}$, the intersection of the space spanned by $e_1,\ldots,e_n$ with the image of the differential d has a basis consisting of some e_i's, say $e_{r+1},\ldots,e_n$. So, $e_{r+1}A,\ldots,e_nA$ are homotopic to zero.
We say that a dg A-module M is *strictly perfect* if its underlying graded module is of the form $\bigoplus_{j=1}^N R_j$, where R_j belongs to $\mathsf{add}(\Sigma^{t_j}A)$ for some t_j with $t_1<t_2<\ldots<t_N$, and if its differential is of the form $d_{int}+\delta$, where d_{int} is the direct sum of the differential of the R_j's, and δ, as a degree 1 map from $\bigoplus_{j=1}^N R_j$ to itself, is a strictly upper triangular matrix whose entries are in A. It is *minimal* if in addition no shifted copy of $e_{r+1}A,\ldots,e_nA$ belongs to $\mathsf{add}(R_1,\ldots,R_j)$, and the entries of δ are in the radical of A, *cf.* [42, Section 2.8]. Strictly perfect dg modules are $\mathcal{K}$-projective. If A is an ordinary algebra, then strictly perfect dg modules are precisely bounded complexes of finitely generated projective modules.

Lemma 4.2. *Let M be a dg A-module belonging to $\mathsf{per}(A)$. Then M is quasi-isomorphic to a minimal strictly perfect dg A-module.*

Proof. Bearing in mind that $e_1A,\ldots,e_rA$ have local endomorphism algebras and $e_{r+1}A,\ldots,e_nA$ are homotopic to zero, we prove the assertion as in [42, Lemma 2.14]. $\surd$

4.4. Simple modules. Assume that A is basic. According to the preceding subsection, we may assume that there is a decomposition $1=e_1+\ldots+e_r+e_{r+1}+\ldots+e_n$ of the unity of A into a sum of primitive orthogonal idempotents such that $1=\bar{e}_1+\ldots+\bar{e}_r$ is a decomposition of 1 in $\bar{A}$ into a sum of primitive orthogonal idempotents.

Let $S_1, \ldots, S_r$ be a complete set of pairwise non-isomorphic simple $\bar{A}$-modules and let $R_1, \ldots, R_r$ be their endomorphism algebras. Then

$$\mathcal{H}om_A(e_iA, S_j) = \begin{cases} {}_{R_j}R_j & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, by (2.1) and (2.2),

$$\mathsf{Hom}_{\mathcal{D}(A)}(e_iA, \Sigma^m S_j) = \begin{cases} {}_{R_j}R_j & \text{if } i = j \text{ and } m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, $\{e_1A, \ldots, e_rA\}$ and $\{S_1, \ldots, S_r\}$ characterise each other by this property. On the one hand, if M is a dg A-module such that for some integer $1 \leq j \leq r$

$$\mathsf{Hom}_{\mathcal{D}(A)}(e_iA, \Sigma^m M) = \begin{cases} {}_{R_j}R_j & \text{if } i = j \text{ and } m = 0, \\ 0 & \text{otherwise,} \end{cases}$$

then M is isomorphic in $\mathcal{D}(A)$ to S_j. On the other hand, let M be an object of $\mathsf{per}(A)$ such that for some integer $1 \leq i \leq r$

$$\mathsf{Hom}_{\mathcal{D}(A)}(M, \Sigma^m S_j) = \begin{cases} {}_{R_j}R_j & \text{if } i = j \text{ and } m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then by replacing M by its minimal perfect resolution (Lemma 4.2), we see that M is isomorphic in $\mathcal{D}(A)$ to e_iA.
Further, recall from Section 4.1 that $\mathcal{D}_{fd}(A)$ admits a standard t-structure whose heart is equivalent to $\mathsf{mod}\,\bar{A}$. This implies that the simple modules $S_1, \ldots, S_r$ form a simple-minded collection in $\mathcal{D}_{fd}(A)$.

4.5. The Nakayama functor. For a complex M of K-vector spaces, we define its dual as $\mathrm{D}(M) = \mathcal{H}om_K(M, K)$, where K in the second argument is considered as a complex concentrated in degree 0. One checks that D defines a duality between finite-dimensional dg A-modules and finite-dimensional dg A^{op}-modules.
Let e be an idempotent of A and M a dg A-module. Then there is a canonical isomorphism

$$\mathcal{H}om_A(eA, M) \cong Me.$$

If in addition each component of M is finite-dimensional , there are canonical isomorphisms

$$\mathcal{H}om_A(eA, M) \cong Me \cong \mathrm{D}\mathcal{H}om_A(M, \mathrm{D}(Ae)).$$

Let $\mathcal{C}(A)$ denote the category of dg A-modules. The Nakayama functor ν : $\mathcal{C}(A) \to \mathcal{C}(A)$ is defined by $\nu(M) = \mathrm{D}\mathcal{H}om_A(M, A)$ [29, Section 10]. There are canonical isomorphisms

$$\mathrm{D}\mathcal{H}om_A(M, N) \cong \mathcal{H}om_A(N, \nu M)$$

for any strictly perfect dg A-module M and any dg A-module N. Then $\nu(eA) = \mathrm{D}(Ae)$ for an idempotent e of A, and the functor ν induces a triangle equivalences between the subcategories $\mathsf{per}(A)$ and $\mathsf{thick}(\mathrm{D}(A))$ of $\mathcal{D}(A)$ with quasi-inverse given by $\nu^{-1}(M) = \mathcal{H}om_A(\mathrm{D}(A), M)$. Moreover, we have the Auslander–Reiten formula

$$\mathrm{D}\,\mathsf{Hom}(M,N) \cong \mathsf{Hom}(N, \nu M),$$

which is natural in $M \in \mathsf{per}(A)$ and $N \in \mathcal{D}(A)$.

Let $e_1, \ldots, e_r$, $S_1, \ldots, S_r$ and $R_1, \ldots, R_r$ be as in the preceding subsection. Then

$$\mathcal{H}om_A(S_j, \mathrm{D}(Ae_i)) \cong \mathrm{D}\mathcal{H}om_A(e_iA, S_j) = \begin{cases} (R_j)_{R_j} & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, by (2.1) and (2.2),

$$\mathsf{Hom}_{\mathcal{D}(A)}(S_j, \Sigma^m \mathrm{D}(Ae_i)) = \begin{cases} (R_j)_{R_j} & \text{if } i = j \text{ and } m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, $\{\mathrm{D}(Ae_1), \ldots, \mathrm{D}(Ae_r)\}$ and $\{S_1, \ldots, S_r\}$ characterise each other in $\mathcal{D}(A)$ by this property. This follows from the arguments in the preceding subsection by applying the functors ν and ν^{-1}.

4.6. The standard co-t-structure. Let $\mathcal{P}_{\leq 0}$ (respectively, $\mathcal{P}_{\geq 0}$) be the smallest full subcategory of $\mathsf{per}(A)$ containing $\{\Sigma^m A \mid m \geq 0\}$ (respectively, $\{\Sigma^m A \mid m \leq 0\}$) and closed under taking extensions and direct summands. The following lemma is a special case of Proposition 3.5. For the convenience of the reader we include a proof.

Lemma 4.3. *The pair $(\mathcal{P}_{\geq 0}, \mathcal{P}_{\leq 0})$ is a co-t-structure on $\mathsf{per}(A)$. Moreover, its co-heart is $\mathsf{add}(A_A)$.*

Proof. Since $\mathsf{Hom}(A, \Sigma^m A) = 0$ for $m \geq 0$, it follows that $\mathsf{Hom}(X, \Sigma Y) = 0$ for $M \in \mathcal{P}_{\geq 0}$ and $N \in \mathcal{P}_{\leq 0}$. It remains to show that any object M in $\mathsf{per}(A)$ fits into a triangle whose outer terms belong to $\mathcal{P}_{\geq 0}$ and $\mathcal{P}_{\leq 0}$, respectively. By Lemma 4.2, we may assume that M is minimal perfect. Write $M = (\bigoplus_{j=1}^N R_j, d_{int} + \delta)$ as in Section 4.3. Let $N' \in \{1, \ldots, N\}$ be the unique integer such that $t_{N'} \geq 0$ but $t_{N'+1} < 0$. Let M' be the graded module $\bigoplus_{j=1}^{N'} R_j$ endowed with the differential restricted from $d_{int} + \delta$. Because $d_{int} + \delta$ is upper triangular, M' is a dg submodule of M. Clearly M' belongs to $\mathcal{P}_{\geq 0}$ and the quotient $M'' = M/M'$ belongs to $\Sigma\mathcal{P}_{\leq 0}$. Thus we obtain the desired triangle

$$M' \longrightarrow M \longrightarrow M'' \longrightarrow \Sigma M'$$

with M' in $\mathcal{P}_{\geq 0}$ and M'' in $\Sigma\mathcal{P}_{\leq 0}$. $\surd$

5. The maps

Let Λ be a finite-dimensional basic K-algebra. This section is devoted to defining the maps in the following diagram.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{\text{equivalence classes of silting objects in } K^b(\mathsf{proj}\,\Lambda)} \arrow[r, shift left, "\phi_{41}"] \arrow[d, shift left, "\phi_{21}"] \arrow[dr, "\phi_{31}"] & \boxed{\text{bounded co-}t\text{-structures on } K^b(\mathsf{proj}\,\Lambda)} \arrow[l, shift left, "\phi_{14}"] \arrow[d, "\phi_{34}"] \\
\boxed{\text{equivalence classes of simple-minded collections in } \mathcal{D}^b(\mathsf{mod}\,\Lambda)} \arrow[u, shift left, "\phi_{12}"] \arrow[r, shift left, "\phi_{32}"] & \boxed{\text{bounded } t\text{-structures on } \mathcal{D}^b(\mathsf{mod}\,\Lambda) \text{ with length heart}} \arrow[l, shift left, "\phi_{23}"]
\end{tikzcd}$$

5.1. Silting objects induce derived equivalences. Let M be a basic silting object of the category $K^b(\mathsf{proj}\,\Lambda)$. By definition, M is a bounded complex of finitely generated projective Λ-modules such that $\mathsf{Hom}_{K^b(\mathsf{proj}\,\Lambda)}(M, \Sigma^m M)$ vanishes for all $m > 0$. By Lemma 4.1, there is a non-positive dg algebra whose perfect derived category is triangle equivalent to $K^b(\mathsf{proj}\,\Lambda)$. This equivalence sends the free dg module of rank 1 to M. Below we explicitly construct such a dg algebra.

Consider $\mathcal{H}om_{\Lambda}(M, M)$. Recall that the degree n component of $\mathcal{H}om_{\Lambda}(M, M)$ consists of those Λ-linear maps from M to itself which are homogeneous of degree n. The differential of $\mathcal{H}om_{\Lambda}(M, M)$ takes a homogeneous map f of degree n to $d \circ f - (-1)^n f \circ d$, where d is the differential of M. This differential and the composition of maps makes $\mathcal{H}om_{\Lambda}(M, M)$ into a dg algebra. Therefore $\mathcal{H}om_{\Lambda}(M, M)$ is a finite-dimensional dg algebra. Moreover, $H^m(\mathcal{H}om_{\Lambda}(M, M)) = \mathsf{Hom}_{\mathcal{D}(\Lambda)}(M, \Sigma^m M)$ for any integer m, by (2.1) and (2.2). Because M is a silting object, $\mathcal{H}om_{\Lambda}(M, M)$ has cohomology concentrated in non-positive degrees. Take the truncated dg algebra $\tilde{\Gamma} = \tau_{\leq 0}\mathcal{H}om_{\Lambda}(M, M)$, where $\tau_{\leq 0}$ is the standard truncation at position 0. Then the embedding $\tilde{\Gamma} \to \mathcal{H}om_{\Lambda}(M, M)$ is a quasi-isomorphism of dg algebras, and hence $\tilde{\Gamma}$ is a finite-dimensional non-positive dg algebra. Therefore, the derived category $\mathcal{D}(\tilde{\Gamma})$ carries a natural t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ with heart $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ equivalent to $\mathsf{Mod}\,\Gamma$, where $\Gamma = H^0(\tilde{\Gamma}) = \mathsf{End}_{\mathcal{D}(\Lambda)}(M)$. This t-structure restricts to a t-structure on $\mathcal{D}_{fd}(\tilde{\Gamma})$, denoted by $(\mathcal{D}_{fd}^{\leq 0}, \mathcal{D}_{fd}^{\geq 0})$, whose heart is equivalent to $\mathsf{mod}\,\Gamma$. Moreover, there is a standard co-t-structure $(\mathcal{P}_{\geq 0}, \mathcal{P}_{\leq 0})$ on $\mathsf{per}(\tilde{\Gamma})$, see Section 4.

The object M has a natural dg $\tilde{\Gamma}$-Λ-bimodule structure. Moreover, since it generates $K^b(\mathsf{proj}\,\Lambda)$, it follows from [29, Lemma 6.1 (a)] that there are triangle equivalences

$$F = ?\overset{L}{\otimes}_{\tilde{\Gamma}} M : \quad \begin{tikzcd} \mathcal{D}(\tilde{\Gamma}) \arrow[r, "\sim"] & \mathcal{D}(\Lambda) \arrow[r, equal] & \mathcal{D}(\mathsf{Mod}\,\Lambda) \\ \mathcal{D}_{fd}(\tilde{\Gamma}) \arrow[r, "\sim"] \arrow[u, hook] & \mathcal{D}_{fd}(\Lambda) \arrow[r, equal] \arrow[u, hook] & \mathcal{D}^b(\mathsf{mod}\,\Lambda) \arrow[u, hook] \\ \mathsf{per}(\tilde{\Gamma}) \arrow[r, "\sim"] \arrow[u, hook] & \mathsf{per}(\Lambda) \arrow[r, equal] \arrow[u, hook] & K^b(\mathsf{proj}\,\Lambda) \arrow[u, hook] \end{tikzcd}$$

These equivalences take $\tilde{\Gamma}$ to M. The following special case of Theorem 3.1 is a consequence.

Corollary 5.1. *The number of indecomposable direct summands of M equals the rank of the Grothendieck group of $K^b(\mathsf{proj}\,\Lambda)$. In particular, any two basic silting objects of $K^b(\mathsf{proj}\,\Lambda)$ have the same number of indecomposable direct summands.*

Proof. The number of indecomposable direct summands of M equals the rank of the Grothendieck group of $\mathsf{mod}\,\Gamma$, which equals the rank of the Grothendieck group of $\mathcal{D}_{fd}(\tilde{\Gamma}) \cong \mathcal{D}^b(\mathsf{mod}\,\Lambda)$ since $\mathsf{mod}\,\Gamma$ is the heart of a bounded t-structure (Lemma 3.3). $\surd$

Write $M = M_1 \oplus \ldots \oplus M_r$ with M_i indecomposable. Suppose that $X_1, \ldots, X_r$ are objects in $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ such that their endomorphism algebras $R_1, \ldots, R_r$ are division algebras and that the following formula holds for $i, j = 1, \ldots, r$ and $m \in \mathbb{Z}$

$$\mathsf{Hom}(M_i, \Sigma^m X_j) = \begin{cases} {}_{R_j}R_j & \text{if } i = j \text{ and } m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then up to isomorphism, the objects $X_1, \ldots, X_r$ are sent by the derived equivalence $?\overset{L}{\otimes}_{\tilde{\Gamma}} M$ to a complete set of pairwise non-isomorphic simple Γ-modules, see Section 4.4.

Lemma 5.2. (a) *Let $X'_1, \ldots, X'_r$ be objects of $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ such that the following formula holds for $1 \leq i, j \leq r$ and $m \in \mathbb{Z}$*

$$\mathsf{Hom}(M_i, \Sigma^m X'_j) = \begin{cases} {}_{R_j}R_j & \textit{if } i = j \textit{ and } m = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Then $X_i \cong X'_i$ for any $i = 1, \ldots, r$.

(b) *Let $M'_1, \ldots, M'_r$ be objects of $K^b(\mathsf{proj}\,\Lambda)$ such that the following formula holds for $1 \leq i, j \leq r$ and $m \in \mathbb{Z}$*

$$\mathsf{Hom}(M'_i, \Sigma^m X_j) = \begin{cases} {}_{R_j}R_j & \textit{if } i = j \textit{ and } m = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Then $M_i \cong M_i'$ for any $i = 1, \ldots, r$.

Proof. This follows from the corresponding result in $\mathcal{D}(\tilde{\Gamma})$, see Section 4.4. $\surd$

5.2. From co-t-structures to silting objects. Let $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ be a bounded co-t-structure of $K^b(\mathsf{proj}\,\Lambda)$. By Lemma 3.4, the co-heart $\mathcal{A} = \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0}$ is a silting subcategory of $K^b(\mathsf{proj}\,\Lambda)$. Since Λ is a silting object of $K^b(\mathsf{proj}\,\Lambda)$, it follows from Theorem 3.1 that $\mathcal{A}$ has an additive generator, say M, *i.e.* $\mathcal{A} = \mathsf{add}(M)$. Then M is a silting object in $K^b(\mathsf{proj}\,\Lambda)$. Define

$$\phi_{14}(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}) \quad = \quad M.$$

5.3. From t-structures to simple-minded collections. Let $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ be a bounded t-structure of $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ with length heart $\mathcal{A}$. Boundedness implies that the Grothendieck group of $\mathcal{A}$ is isomorphic to the Grothendieck group of $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$, which is free, say, of rank r. Therefore, $\mathcal{A}$ has precisely r isomorphism classes of simple objects, say $X_1, \ldots, X_r$. By Lemma 3.3 (f), $X_1, \ldots, X_r$ is a simple-minded collection in $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$. Define

$$\phi_{23}(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0}) \quad = \quad \{X_1, \ldots, X_r\}.$$

5.4. From silting objects to simple-minded collections, t-structures and co-t-structures. Let M be a silting object of $K^b(\mathsf{proj}\,\Lambda)$. Define full subcategories of $\mathcal{C}$

$$\begin{aligned}
\mathcal{C}^{\leq 0} &= \{N \in \mathcal{D}^b(\mathsf{mod}\,\Lambda) \mid \mathsf{Hom}(M, \Sigma^m N) = 0,\ \forall\ m > 0\},\\
\mathcal{C}^{\geq 0} &= \{N \in \mathcal{D}^b(\mathsf{mod}\,\Lambda) \mid \mathsf{Hom}(M, \Sigma^m N) = 0,\ \forall\ m < 0\},\\
\mathcal{C}_{\leq 0} &= \text{the additive closure of the extension closure}\\
&\ \text{of } \Sigma^m M,\ m \geq 0 \text{ in } K^b(\mathsf{proj}\,\Lambda),\\
\mathcal{C}_{\geq 0} &= \text{the additive closure of the extension closure}\\
&\ \text{of } \Sigma^m M,\ m \leq 0 \text{ in } K^b(\mathsf{proj}\,\Lambda).
\end{aligned}$$

Lemma 5.3. (a) *The pair $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ is a bounded t-structure on $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ whose heart is equivalent to $\mathsf{mod}\,\Gamma$ for $\Gamma = \mathsf{End}(M)$. Write $M = M_1 \oplus \ldots \oplus M_r$ and let $X_1, \ldots, X_r$ be the corresponding simple objects of the heart with endomorphism algebras $R_1, \ldots, R_r$ respectively. Then the following formula holds for $1 \leq i, j \leq r$ and $m \in \mathbb{Z}$*

$$\mathsf{Hom}(M_i, \Sigma^m X_j) = \begin{cases} {}_{R_j}R_j & \text{if } i = j \text{ and } m = 0,\\ 0 & \text{otherwise.} \end{cases}$$

(b) *The pair $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ is a bounded co-t-structure on $K^b(\mathsf{proj}\,\Lambda)$ whose co-heart is $\mathsf{add}(M)$.*

The first statement of part (a) is proved by Keller and Vossieck [33] in the case when Λ is the path algebra of a Dynkin quiver and by Assem, Souto and Trepode [5] in the case when Λ is hereditary.

Proof. Let $\tilde{\Gamma}$ be the truncated dg endomorphism algebra of M, see Section 5.1. Then $\mathsf{per}(\tilde{\Gamma})$ has a standard bounded co-t-structure $(\mathcal{P}_{\geq 0}, \mathcal{P}_{\leq 0})$ and $\mathcal{D}_{fd}(\tilde{\Gamma})$ has a standard bounded t-structure $(\mathcal{D}_{fd}^{\leq 0}, \mathcal{D}_{fd}^{\geq 0})$ with heart equivalent to $\mathsf{mod}\,\Gamma$. One checks that the triangle equivalence $?\overset{L}{\otimes}_{\tilde{\Gamma}} M$ takes $(\mathcal{P}_{\geq 0}, \mathcal{P}_{\leq 0})$ to $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ and it takes $(\mathcal{D}_{fd}^{\leq 0}, \mathcal{D}_{fd}^{\geq 0})$ to $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$. $\surd$

Define

$$\begin{aligned} \phi_{31}(M) &= (\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0}), \\ \phi_{41}(M) &= (\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}), \\ \phi_{21}(M) &= \{X_1, \ldots, X_r\}. \end{aligned}$$

5.5. FROM SIMPLE-MINDED COLLECTIONS TO t-STRUCTURES. Let $X_1, \ldots, X_r$ be a simple-minded collection of $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$. Let $\mathcal{C}^{\leq 0}$ (respectively, $\mathcal{C}^{\geq 0}$) be the extension closure of $\{\Sigma^m X_i \mid i = 1, \ldots, r, m \geq 0\}$ (respectively, $\{\Sigma^m X_i \mid i = 1, \ldots, r, m \leq 0\}$) in $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$.

PROPOSITION 5.4. *The pair $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ is a bounded t-structure on $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$. Moreover, the heart of this t-structure is a length category with simple objects $X_1, \ldots, X_r$. The same results hold true with $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ replaced by a Hom-finite Krull–Schmidt triangulated category $\mathcal{C}$.*

Proof. The first two statements are [3, Corollary 3 and Proposition 4]. The proof there still works if we replace $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ by $\mathcal{C}$. $\surd$

Define

$$\phi_{32}(X_1, \ldots, X_r) = (\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0}).$$

Later we will show that the heart of this t-structure always is equivalent to the category of finite-dimensional modules over a finite-dimensional algebra (Corollary 6.2). This was proved by Al-Nofayee for self-injective algebras Λ, see [3, Theorem 7].

COROLLARY 5.5. *Any two simple-minded collections in $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ have the same cardinality.*

Proof. By Proposition 5.4, the cardinality of a simple-minded collection equals the rank of the Grothendieck group of $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$. The assertion follows. $\surd$

5.6. FROM SIMPLE-MINDED COLLECTIONS TO SILTING OBJECTS. Let $X_1, \ldots, X_r$ be a simple-minded collection in $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$. We will construct a silting object $\nu^{-1}T$ of $K^b(\mathsf{proj}\,\Lambda)$ following a method of Rickard [43]. Then we define

$$\phi_{12}(X_1, \ldots, X_r) = \nu^{-1}T.$$

The same construction is studied by Keller and Nicolás [32] in the context of positive dg algebras. In the case of Λ being hereditary, Buan, Reiten and Thomas [17] give an elegant construction of $\nu^{-1}(T)$ using the Braid group action on exceptional sequences. Unfortunately, their construction cannot be generalised.

Let $R_1, \ldots, R_r$ be the endomorphism algebras of $X_1, \ldots, X_r$, respectively.
Set $X_i^{(0)} = X_i$. Suppose $X_i^{(n-1)}$ is constructed. For $i, j = 1, \ldots, r$ and $m < 0$, let $B(j, m, i)$ be a basis of $\mathsf{Hom}(\Sigma^m X_j, X_i^{(n-1)})$ over R_j. Put

$$Z_i^{(n-1)} = \bigoplus_{m<0} \bigoplus_{j} \bigoplus_{B(j,m,i)} \Sigma^m X_j$$

and let $\alpha_i^{(n-1)} : Z_i^{(n-1)} \to X_i^{(n-1)}$ be the map whose component corresponding to $f \in B(j, m, i)$ is exactly f.
Let $X_i^{(n)}$ be a cone of $\alpha_i^{(n-1)}$ and form the corresponding triangle

$$Z_i^{(n-1)} \xrightarrow{\alpha_i^{(n-1)}} X_i^{(n-1)} \xrightarrow{\beta_i^{(n-1)}} X_i^{(n)} \longrightarrow \Sigma Z_i^{(n-1)}.$$

Inductively, a sequence of morphisms in $\mathcal{D}(\mathsf{Mod}\,\Lambda)$ is constructed:

$$X_i^{(0)} \xrightarrow{\beta_i^{(0)}} X_i^{(1)} \longrightarrow \cdots \longrightarrow X_i^{(n-1)} \xrightarrow{\beta_i^{(n-1)}} X_i^{(n)} \longrightarrow \cdots.$$

Let T_i be the homotopy colimit of this sequence. That is, up to isomorphism, T_i is defined by the following triangle

$$\bigoplus_{n\geq 0} X_i^{(n)} \xrightarrow{id-\beta} \bigoplus_{n\geq 0} X_i^{(n)} \longrightarrow T_i \longrightarrow \Sigma \bigoplus_{n\geq 0} X_i^{(n)} .$$

Here $\beta = (\beta_{mn})$ is the square matrix with rows and columns labeled by non-negative integers and with entries $\beta_{mn} = \beta_i^{(n)}$ if $n + 1 = m$ and 0 otherwise.
These properties of T_i's were proved by Rickard in [43] for symmetric algebras Λ over algebraically closed fields. Rickard remarked that they hold for arbitrary fields, see [43, Section 8]. In fact, his proofs verbatim carry over to general finite-dimensional algebras.

LEMMA 5.6. (a) ([43, Lemma 5.4]) *For $1 \leq i, j \leq r$, and $m \in \mathbb{Z}$,*

$$\mathsf{Hom}(X_j, \Sigma^m T_i) = \begin{cases} (R_j)_{R_j} & \text{if } i = j \text{ and } m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

(b) ([43, Lemma 5.5]) *For each $1 \leq i \leq r$, T_i is quasi-isomorphic to a bounded complex of finitely generated injective Λ-modules.*

(c) ([43, Lemma 5.8]) *Let C be an object of $\mathcal{D}^-(\mathsf{mod}\,\Lambda)$. If $\mathsf{Hom}(C, \Sigma^m T_i) = 0$ for all $m \in \mathbb{Z}$ and all $1 \leq i \leq r$, then $C = 0$.*

From now on we assume that T_i is a bounded complex of finitely generated injective Λ-modules. Recall from Section 2.3 that the Nakayama functor ν and the inverse Nakayama functor ν^{-1} are quasi-inverse triangle equivalences between $K^b(\mathsf{proj}\,\Lambda)$ and $K^b(\mathsf{inj}\,\Lambda)$ The following is a consequence of Lemma 5.6 and the Auslander–Reiten formula.

Lemma 5.7. (a) *For $1 \leq i, j \leq r$, and $m \in \mathbb{Z}$,*

$$\mathsf{Hom}(\nu^{-1}T_i, \Sigma^m X_j) = \begin{cases} {}_{R_j}R_j & \text{if } i = j \text{ and } m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

(b) *For each $1 \leq i \leq r$, $\nu^{-1}T_i$ is a bounded complex of finitely generated projective Λ-modules.*

(c) *Let C be an object of $\mathcal{D}^-(\mathsf{mod}\,\Lambda)$. If $\mathsf{Hom}(\nu^{-1}T_i, \Sigma^m C) = 0$ for all $m \in \mathbb{Z}$ and all $1 \leq i \leq r$, then $C = 0$.*

Put $T = \bigoplus_{i=1}^r T_i$ and $\nu^{-1}T = \bigoplus_{i=1}^r \nu^{-1}T_i$.

Lemma 5.8. *We have $\mathsf{Hom}(\nu^{-1}T, \Sigma^m T) = 0$ for $m < 0$. Equivalently, $\mathsf{Hom}(\nu^{-1}T, \Sigma^m \nu^{-1}T) = \mathsf{Hom}(T, \Sigma^m T) = 0$ for $m > 0$.*

Proof. Same as the proof of [43, Lemma 5.7], with the T_i in the first entry of Hom there replaced by $\nu^{-1}T_i$. $\surd$

It follows from Lemma 5.7 (c) that $\nu^{-1}T$ generates $K^b(\mathsf{proj}\,\Lambda)$. Combining this with Lemma 5.8 implies

Proposition 5.9. *$\nu^{-1}T$ is a silting object of $K^b(\mathsf{proj}\,\Lambda)$.*

Rickard's construction was originally motivated by constructing tilting complexes over symmetric algebras which yield certain derived equivalences, see [43, Theorem 5.1]. His work was later generalised by Al-Nofayee to self-injective algebras, see [2, Theorem 4].

5.7. From co-t-structures to t-structures. Let $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ be a bounded co-t-structure of $K^b(\mathsf{proj}\,\Lambda)$. Let

$$\begin{aligned} \mathcal{C}^{\leq 0} &= \{N \in \mathcal{D}^b(\mathsf{mod}\,\Lambda) \mid \mathsf{Hom}(M, N) = 0,\ \forall\ M \in \Sigma^{-1}\mathcal{C}_{\geq 0}\} \\ \mathcal{C}^{\geq 0} &= \{N \in \mathcal{D}^b(\mathsf{mod}\,\Lambda) \mid \mathsf{Hom}(M, N) = 0,\ \forall\ M \in \Sigma\mathcal{C}_{\leq 0}\}. \end{aligned}$$

Lemma 5.10. *The pair $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ is a bounded t-structure on $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ with length heart.*

Proof. Because $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0}) = \phi_{31} \circ \phi_{14}(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. $\surd$

By definition $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ is right orthogonal to the given co-t-structure in the sense of Bondarko [11, Definition 2.5.1]. Define

$$\phi_{34}(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}) \quad = \quad (\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0}).$$

If Λ has finite global dimension, then $K^b(\mathsf{proj}\,\Lambda)$ is identified with $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$. As a consequence, $\mathcal{C}_{\leq 0} = \mathcal{C}^{\leq 0}$ and $\mathcal{C}^{\geq 0} = \nu\mathcal{C}_{\geq 0}$. Thus the t-structure $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ is right adjacent to the given co-t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ in the sense of Bondarko [12, Definition 4.4.1].

5.8. Some remarks. Some of the maps ϕ_{ij} are defined in more general setups:

- ϕ_{14} and ϕ_{41} are defined for all triangulated categories, with silting objects replaced by silting subcategories, by Proposition 3.5 and Lemma 3.4, see also [12, 31, 39].
- ϕ_{23} is defined for all triangulated categories, with simple-minded collections allowed to contain infinitely many objects (Lemma 3.3).
- ϕ_{32} is defined for all algebraic triangulated categories (see [32]) and for Hom-finite Krull–Schmidt triangulated categories (see Proposition 5.4).
- ϕ_{21} and ϕ_{31} are defined for all algebraic triangulated categories (replacing $K^b(\mathsf{proj}\,\Lambda)$), with $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ replaced by a suitable triangulated category; then we may follow the arguments in Sections 4.1 and 5.4.
- ϕ_{34} is defined for all algebraic triangulated categories (replacing $K^b(\mathsf{proj}\,\Lambda)$), with $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ replaced by a suitable triangulated category. Then we may follow the argument in Section 5.7.
- ϕ_{12} is defined for finite-dimensional non-positive dg algebras, since these dg algebras behave like finite-dimensional algebras from the perspective of derived categories. Similarly, ϕ_{12} is defined for homologically smooth non-positive dg algebras, see [31].

6. The correspondences are bijections

Let Λ be a finite-dimensional K-algebra. In the preceding section we defined the maps ϕ_{ij}. In this section we will show that they are bijections. See [5, 46] for related work, focussing on piecewise hereditary algebras.

Theorem 6.1. *The ϕ_{ij}'s defined in Section 5 are bijective. In particular, there are one-to-one correspondences between*

(1) *equivalence classes of silting objects in $K^b(\mathsf{proj}\,\Lambda)$,*
(2) *equivalence classes of simple-minded collections in $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$,*
(3) *bounded t-structures on $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ with length heart,*
(4) *bounded co-t-structures on $K^b(\mathsf{proj}\,\Lambda)$.*

There is an immediate consequence:

Corollary 6.2. *Let $\mathcal{A}$ be the heart of a bounded t-structure on $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$. If $\mathcal{A}$ is a length category, then $\mathcal{A}$ is equivalent to $\mathsf{mod}\,\Gamma$ for some finite-dimensional algebra Γ.*

Proof. By Theorem 6.1, such a t-structure is of the form $\phi_{31}(M)$ for some silting object M of $K^b(\mathsf{proj}\,\Lambda)$. The result then follows from Lemma 5.3 (a). $\surd$

The proof of the theorem is divided into several lemmas, which are consequences of the material collected in the previous sections.

Lemma 6.3. *The maps ϕ_{14} and ϕ_{41} are inverse to each other.*

Proof. Let M be a basic silting object. The definitions of ϕ_{14} and ϕ_{41} and Lemma 5.3 (b) imply that $\phi_{14} \circ \phi_{41}(M) \cong M$.
Let $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ be a bounded co-t-structure on $K^b(\mathsf{proj}\,\Lambda)$. It follows from Lemma 3.4 that $\phi_{41} \circ \phi_{14}(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}) = (\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. $\surd$

Recall from Section 5.8 that ϕ_{14} and ϕ_{41} are defined in full generality. Lemma 6.3 holds in full generality as well, see [39, Corollary 5.8] and [31].

Lemma 6.4. *The maps ϕ_{21} and ϕ_{12} are inverse to each other.*

Proof. This follows from the Hom-duality: Lemma 5.7 (a), Lemma 5.3 (a) and Lemma 5.2. $\surd$

Lemma 6.5. *The maps ϕ_{23} and ϕ_{32} are inverse to each other.*

Proof. Let $X_1, \ldots, X_r$ be a simple-minded collection in $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$. It follows from Proposition 5.4 that $\phi_{23} \circ \phi_{32}(X_1, \ldots, X_r) = \{X_1, \ldots, X_r\}$.
Let $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ be a bounded t-structure on $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ with length heart. It follows from Lemma 3.3 that $\phi_{32} \circ \phi_{23}(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0}) = (\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$. $\surd$

Lemma 6.6. *For a triple i, j, k such that ϕ_{ij}, ϕ_{jk} and ϕ_{ik} are defined, there is the equality $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$. In particular, ϕ_{31} and ϕ_{34} are bijective.*

Proof. In view of the preceding three lemmas, it suffices to prove $\phi_{23} \circ \phi_{31} = \phi_{21}$ and $\phi_{31} \circ \phi_{14} = \phi_{34}$, which is clear from the definitions. $\surd$

7. Mutations and partial orders

In this section we introduce mutations and partial orders on the four concepts in Section 3, and we show that the maps defined in Section 5 commute with mutations and preserve the partial orders.
Let $\mathcal{C}$ be a Hom-finite Krull–Schmidt triangulated category with suspension functor Σ.

7.1. Silting objects. We follow [1, 18] to define silting mutation. Let M be a silting object in $\mathcal{C}$. We assume that M is basic and $M = M_1 \oplus \ldots \oplus M_r$ is a decomposition into indecomposable objects. Let $i = 1, \ldots, r$. The *left mutation* of M at the direct summand M_i is the object $\mu_i^+(M) = M_i' \oplus \bigoplus_{j \neq i} M_j$ where M_i' is the cone of the minimal left $\mathsf{add}(\bigoplus_{j \neq i} M_j)$-approximation of M_i

$$M_i \longrightarrow E.$$

Similarly one can define the *right mutation* $\mu_i^-(M)$.

Theorem 7.1. ([1, Theorem 2.31 and Proposition 2.33]) *The objects $\mu_i^+(M)$ and $\mu_i^-(M)$ are silting objects. Moreover, $\mu_i^+ \circ \mu_i^-(M) \cong M \cong \mu_i^- \circ \mu_i^+(M)$.*

Let $\mathsf{silt}\,\mathcal{C}$ be the set of isomorphism classes of basic tilting objects of $\mathcal{C}$. The *silting quiver* of $\mathcal{C}$ has the elements in $\mathsf{silt}\,\mathcal{C}$ as vertices. For $P, P' \in \mathsf{silt}\,\mathcal{C}$, there are arrows from P to P' if and only if P' is obtained from P by a left mutation, in which case there is precisely one arrow. See [1, Section 2.6].
For $P, P' \in \mathsf{silt}\,\mathcal{C}$, define $P \geq P'$ if $\mathsf{Hom}(P, \Sigma^m P') = 0$ for any $m > 0$. According to [1, Theorem 2.11], $\geq$ is a partial order on $\mathsf{silt}\,\mathcal{C}$.

Theorem 7.2. ([1, Theorem 2.35]) *The Hasse diagram of $(\mathsf{silt}\,\mathcal{C}, \geq)$ is the silting quiver of $\mathcal{C}$.*

Next we define (a generalisation of) the Brenner–Butler tilting module for a finite-dimensional algebra, and show that it is a left mutation of the free module of rank 1. The corresponding right mutation is the Okuyama–Rickard complex, see [1, Section 2.7]. Let Λ be a finite-dimensional basic algebra and $1 = e_1 + \ldots + e_n$ be a decomposition of the unity into the sum of primitive idempotents and $\Lambda = P_1 \oplus \ldots \oplus P_n$ the corresponding decomposition of the free module of rank 1. Fix $i = 1, \ldots, n$ and let S_i be the corresponding simple module and let $S_i^+ = \mathrm{D}(\Lambda/\Lambda(1-e_i)\Lambda)$. Assume that

- S_i^+ is not injective,
- the projective dimension of $\tau^{-1}_{\mathsf{mod}\,\Lambda} S_i^+$ is at most 1.

Definition 7.3. *Define the* BB tilting module *with respect to i by*

$$T = \tau^{-1}_{\mathsf{mod}\,\Lambda} S_i^+ \oplus \bigoplus_{j \neq i} P_j.$$

We call it the APR tilting module *if $\Lambda/\Lambda(1-e_i)\Lambda$ is projective as a Λ-module.*

When $\Lambda/\Lambda(1-e_i)\Lambda$ is a division algebra (*i.e.* there are no loops in the quiver of Λ at the vertex i), this specialises to the 'classical' BB tilting module [13] and APR tilting module [6]. The following proposition generalises [1, Theorem 2.53].

Proposition 7.4. (a) *T is isomorphic to the left mutation $\mu_i^+(\Lambda)$ of Λ.*
(b) *T is a tilting Λ-module of projective dimension at most 1.*

Proof. We modify the proof in [1]. Take a minimal injective copresentation of S_i^+:

$$0 \longrightarrow S_i^+ \longrightarrow \mathrm{D}(e_i\Lambda) \xrightarrow{f} I.$$

Since $\mathsf{Ext}^1_\Lambda(S_i, S_i^+) = \mathsf{Ext}^1_{\Lambda/\Lambda(1-e_i)\Lambda}(S_i, S_i^+) = 0$, it follows that the injective module I belongs to $\mathsf{add}\,\mathrm{D}((1-e_i)\Lambda)$. Applying the inverse Nakayama functor $\nu^{-1}_{\mathsf{mod}\,\Lambda}$ yields an exact sequence

$$P_i \xrightarrow{\nu^{-1}_{\mathsf{mod}\,\Lambda} f} \nu^{-1}_{\mathsf{mod}\,\Lambda} I \longrightarrow \tau^{-1}_{\mathsf{mod}\,\Lambda} S_i^+ \longrightarrow 0.$$

Moreover, $\nu^{-1}_{\mathsf{mod}\,\Lambda} f$ is a minimal left approximation of P_i in $\mathsf{add}(P_j, j \neq i)$. Since the projective dimension of $\tau^{-1}_{\mathsf{mod}\,\Lambda} S_i^+$ is at most 1, it follows that $\nu^{-1}_{\mathsf{mod}\,\Lambda} f$ is injective. This completes the proof for (a).
(b) follows from [1, Theorem 2.32]. $\surd$

7.2. Simple-minded collections. Let $X_1, \ldots, X_r$ be a simple-minded collection in $\mathcal{C}$ and fix $i = 1, \ldots, r$. Let $\mathcal{X}_i$ denote the extension closure of X_i in $\mathcal{C}$. Assume that for any j the object $\Sigma^{-1} X_j$ admits a minimal left approximation $g_j : \Sigma^{-1} X_j \to X_{ij}$ in $\mathcal{X}_i$.

Definition 7.5. *The* left mutation $\mu_i^+(X_1, \ldots, X_r)$ *of $X_1, \ldots, X_r$ at X_i is a new collection $X'_1, \ldots, X'_r$ such that $X'_i = \Sigma X_i$ and X'_j $(j \neq i)$ is the cone of* *the above left approximation*

$$\Sigma^{-1}X_j \xrightarrow{g_j} X_{ij}.$$

Similarly one defines the right mutation $\mu_i^-(X_1,\ldots,X_r)$.

This generalises Kontsevich–Soibelman's mutation of spherical collections [38, Section 8.1] and appeared in [35] in the case of derived categories of acyclic quivers.

PROPOSITION 7.6. (a) $\mu_i^+\circ\mu_i^-(X_1,\ldots,X_r)\cong(X_1,\ldots,X_r)\cong\mu_i^-\circ\mu_i^+(X_1,\ldots,X_r)$.

(b) *Assume that*

- *for any $j\neq i$ the object $\Sigma^{-1}X_j$ admits a minimal left approximation $g_j:\Sigma^{-1}X_j\to X_{ij}$ in $\mathcal{X}_i$,*
- *the induced map $\mathsf{Hom}(g_j,X_i):\mathsf{Hom}(X_{ij},X_i)\to\mathsf{Hom}(\Sigma^{-1}X_j,X_i)$ is injective,*
- *the induced map $\mathsf{Hom}(g_j,\Sigma X_i):\mathsf{Hom}(X_{ij},\Sigma X_i)\to\mathsf{Hom}(\Sigma^{-1}X_j,\Sigma X_i)$ is injective.*

Then the collection $\mu_i^+(X_1,\ldots,X_r)$ is simple-minded.

(c) *Assume that*

- *for any $j\neq i$ the object X_j admits a minimal right approximation $g_j^-:\Sigma^{-1}X_{ij}^-\to X_j$ in $\Sigma^{-1}\mathcal{X}_i$,*
- *the induced map $\mathsf{Hom}(X_i,\Sigma g_j^-):\mathsf{Hom}(X_i,X_{ij}^-)\to\mathsf{Hom}(X_i,\Sigma X_j)$ is injective,*
- *the induced map $\mathsf{Hom}(X_i,\Sigma^2 g_j^-,):\mathsf{Hom}(X_i,\Sigma X_{ij}^-)\to\mathsf{Hom}(X_i,\Sigma^2X_j)$ is injective.*

Then the collection $\mu_i^-(X_1,\ldots,X_r)$ is simple-minded.

Proof. (a) Because in the triangle

$$\Sigma^{-1}X_j \xrightarrow{g_j} X_{ij} \xrightarrow{g_j^-} X_j' \longrightarrow X_j$$

g_j is a minimal left approximation of $\Sigma^{-1}X_j$ in $\mathcal{X}_i$ if and only if g_j^- is a minimal right approximation of X_j in $\mathcal{X}_i=\Sigma^{-1}(\Sigma\mathcal{X}_i)$.
(b) and (c) The proof uses long exact Hom sequences induced from the defining triangles of the X_j'. We leave it to the reader. $\surd$

REMARK 7.7. *In the course of the proof of Proposition 7.6 (b) and (c), one notices that the collection of endomorphism algebras of the mutated simple-minded collection is the same as that of the given simple-minded collection.*

If $\mathsf{Hom}(X_i,\Sigma X_i)=0$, then $\mathcal{X}_i=\mathsf{add}(X_i)$. In this case, all six assumptions in Proposition 7.6 (b) and (c) are satisfied.

LEMMA 7.8. *Let Λ be a finite-dimensional algebra and let $X_1,\ldots,X_r$ be a simple-minded collection in $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$. Let $i=1,\ldots,r$. Then the left mutation $\mu_i^+(X_1,\ldots,X_r)$ and the right mutation $\mu_i^-(X_1,\ldots,X_r)$ are again simple-minded collections.*

Proof. We will show that the three assumptions in Proposition 7.6 (b) are satisfied, so the left-mutated collection $\mu_i^+(X_1,\dots,X_r)$ is a simple-minded collection. The case for $\mu_i^-(X_1,\dots,X_r)$ is similar.
By Proposition 5.4, $X_1,\dots,X_r$ are the simple objects in the heart of a bounded t-structure on $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$. Moreover, by Corollary 6.2, the heart is equivalent to $\mathsf{mod}\,\Gamma$ for some finite-dimensional algebra Γ. We identify $\mathsf{mod}\,\Gamma$ with the heart via this equivalence. In this way we consider $X_1,\dots,X_r$ as simple Γ-modules. By [8, Section 3.1], there is a triangle functor

$$\mathsf{real}:\mathcal{D}^b(\mathsf{mod}\,\Gamma)\to\mathcal{D}^b(\mathsf{mod}\,\Lambda)$$

such that

- restricted to $\mathsf{mod}\,\Gamma$, real is the identity;
- for $M,N\in\mathsf{mod}\,\Gamma$, the induced map
  $$\mathsf{Ext}^1_\Gamma(M,N)=\mathsf{Hom}_{\mathcal{D}^b(\mathsf{mod}\,\Gamma)}(M,\Sigma N)\to\mathsf{Hom}_{\mathcal{D}^b(\mathsf{mod}\,\Lambda)}(M,\Sigma N)$$
  is bijective;
- for $M,N\in\mathsf{mod}\,\Gamma$, the induced map
  $$\mathsf{Ext}^2_\Gamma(M,N)=\mathsf{Hom}_{\mathcal{D}^b(\mathsf{mod}\,\Gamma)}(M,\Sigma^2 N)\to\mathsf{Hom}_{\mathcal{D}^b(\mathsf{mod}\,\Lambda)}(M,\Sigma^2 N)$$
  is injective.

For $j=1,\dots,r$, there is a short exact sequence

$$0\longrightarrow\Omega X_j\longrightarrow P_j\longrightarrow X_j\longrightarrow 0,$$

where P_j is the projective cover of X_j and ΩX_j is the first syzygy of X_j. Let $\mathcal{X}_i$ be the extension closure of X_i in $\mathsf{mod}\,\Gamma$ (by the second property of real listed in the preceding paragraph, this is the same as the extension closure of X_i in $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$) and let X_{ij} denote the maximal quotient of ΩX_j belonging to $\mathcal{X}_i$. There is the following push-out diagram

$$\begin{tikzcd}
 & 0 \arrow[r] & \Omega X_j \arrow[r] \arrow[d] & P_j \arrow[r] \arrow[d] & X_j \arrow[r] \arrow[d, equal] & 0 \\
\xi: & 0 \arrow[r] & X_{ij} \arrow[r] & X_j' \arrow[r] & X_j \arrow[r] & 0
\end{tikzcd}$$

(a) Suppose we are given an object Y of $\mathcal{X}_i$ and a short exact sequence

$$\eta:\quad 0\longrightarrow Y\longrightarrow Z\longrightarrow X_j\longrightarrow 0.$$

Then there is a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[r] & \Omega X_j \arrow[r] \arrow[d] & P_j \arrow[r] \arrow[d] & X_j \arrow[r] \arrow[d, equal] & 0 \\
\eta: & 0 \arrow[r] & Y \arrow[r] & Z \arrow[r] & X_j \arrow[r] & 0.
\end{tikzcd}$$

Because X_{ij} is the maximal quotient of ΩX_j belonging to $\mathcal{X}_i$, this morphism of short exact sequences factors through ξ. In other words, the morphism $g_j:X_j\to\Sigma X_{ij}$ corresponding to ξ is a minimal left $\Sigma\mathcal{X}_i$-approximation.

(b) The dimension of the space $\mathsf{Hom}_{\mathcal{D}^b(\mathsf{mod}\,\Lambda)}(\Sigma X_{ij}, \Sigma X_i) \cong \mathsf{Hom}_\Gamma(X_{ij}, X_i)$ over $\mathsf{End}(X_i)$ equals the number of indecomposable direct summands of $\mathrm{top}(X_{ij})$, which clearly equals the dimension of $\mathsf{Ext}^1_\Gamma(X_j, X_i) \cong \mathsf{Hom}_{\mathcal{D}^b(\mathsf{mod}\,\Lambda)}(X_j, \Sigma X_i)$ over $\mathsf{End}(X_i)$. Therefore the induced map

$$\mathsf{Hom}(g_j, \Sigma X_i) : \mathsf{Hom}_{\mathcal{D}^b(\mathsf{mod}\,\Lambda)}(\Sigma X_{ij}, \Sigma X_i) \longrightarrow \mathsf{Hom}_{\mathcal{D}^b(\mathsf{mod}\,\Lambda)}(X_j, \Sigma X_i)$$

is injective since by (a) it is surjective.
(c) First observe that the following diagram is commutative

$$\begin{tikzcd}
\mathsf{Hom}_{\mathcal{D}^b(\mathsf{mod}\,\Lambda)}(\Sigma X_{ij}, \Sigma^2 X_i) \arrow[r, "\mathsf{Hom}_\Lambda(g_j, \Sigma^2 X_i)"] & \mathsf{Hom}_{\mathcal{D}^b(\mathsf{mod}\,\Lambda)}(X_j, \Sigma^2 X_i) \\
\mathsf{Hom}_{\mathcal{D}^b(\mathsf{mod}\,\Gamma)}(\Sigma X_{ij}, \Sigma^2 X_i) \arrow[r, "\mathsf{Hom}_\Gamma(g_j, \Sigma^2 X_i)"] \arrow[u, "\mathsf{real}"] & \mathsf{Hom}_{\mathcal{D}^b(\mathsf{mod}\,\Gamma)}(X_j, \Sigma^2 X_i) \arrow[u, "\mathsf{real}"']
\end{tikzcd}$$

The left vertical map is a bijection and the right vertical map is injective, so to prove the injectivity of $\mathsf{Hom}_\Lambda(g_j, \Sigma^2 X_i)$ it suffices to prove the injectivity of $\mathsf{Hom}_\Gamma(g_j, \Sigma^2 X_i)$. Writing

$$\mathsf{Hom}_{\mathcal{D}^b(\mathsf{mod}\,\Gamma)}(\Sigma X_{ij}, \Sigma^2 X_i) = \mathsf{Ext}^1_\Gamma(X_{ij}, X_i) = \mathsf{Hom}_\Gamma(\Omega X_{ij}, X_i)$$

and

$$\mathsf{Hom}_{\mathcal{D}^b(\mathsf{mod}\,\Gamma)}(X_j, \Sigma^2 X_i) = \mathsf{Ext}^2_\Gamma(X_j, X_i) = \mathsf{Ext}^1_\Gamma(\Omega X_j, X_i) = \mathsf{Hom}_\Gamma(\Omega^2 X_j, X_i),$$

we see that $\mathsf{Hom}_\Gamma(g_j, \Sigma^2 X_i)$ is $\mathsf{Hom}_\Gamma(\alpha, X_i)$, where α is defined by the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega^2 X_j \arrow[r] \arrow[d, "\alpha"] & P^0 \arrow[r] \arrow[d, "\beta"] & \Omega X_j \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & \Omega X_{ij} \arrow[r] & Q^0 \arrow[r] & X_{ij} \arrow[r] & 0,
\end{tikzcd}$$

Here, P^0 and Q^0 are projective covers of ΩX_j and X_{ij}, respectively, and γ is the canonical quotient map. As the map γ is surjective, the map β is a split epimorphism. By the snake lemma, there is an exact sequence

$$\mathsf{ker}(\gamma) \longrightarrow \mathsf{cok}(\alpha) \longrightarrow 0.$$

Since X_{ij} is the maximal quotient of ΩX_j in $\mathcal{X}_i$, it follows that $\mathsf{Hom}_\Gamma(\mathsf{ker}(\gamma), X_i) = 0$, and hence $\mathsf{Hom}_\Gamma(\mathsf{cok}(\alpha), X_i) = 0$. Therefore $\mathsf{Hom}_\Gamma(\alpha, X_i)$ is injective. $\surd$

For two simple-minded collections $\{X_1, \ldots, X_r\}$ and $\{X'_1, \ldots, X'_r\}$ of $\mathcal{C}$, define

$$\{X_1, \ldots, X_r\} \geq \{X'_1, \ldots, X'_r\}$$

if $\mathsf{Hom}(X'_i, \Sigma^m X_j) = 0$ for any $m < 0$ and any $i, j = 1, \ldots, r$.

Proposition 7.9. *The relation $\geq$ defined above is a partial order on the set of equivalence classes of simple-minded collections of $\mathcal{C}$.*

Proof. The reflexivity is clear by the definition of a simple-minded collection. Next we show the antisymmetry and transitivity. Let $\{X_1, \dots, X_r\}$ and $\{X'_1, \dots, X'_r\}$ be two simple-minded collections of $\mathcal{C}$ and let $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ and $(\mathcal{C}'^{\leq 0}, \mathcal{C}'^{\geq 0})$ be the corresponding t-structures given in Proposition 5.4 (the general case). Then

$$\begin{aligned} \{X_1, \dots, X_r\} &\geq \{X'_1, \dots, X'_r\} \\ &\Leftrightarrow \mathsf{Hom}(X'_i, \Sigma^m X_j) = 0 \text{ for any } m < 0 \text{ and } i, j = 1, \dots, r \\ &\Leftrightarrow \mathsf{Hom}(\Sigma^{m'} X'_i, \Sigma^m X_j) = 0 \text{ for any } m < 0,\ m' \geq 0 \\ &\qquad \text{and } i, j = 1, \dots, r \\ &\Leftrightarrow \mathcal{C}'^{\leq 0} \perp \Sigma^{-1}\mathcal{C}^{\leq 0} \\ &\Leftrightarrow \mathcal{C}'^{\leq 0} \subseteq \mathcal{C}^{\leq 0}. \end{aligned}$$

(a) If $\{X_1, \dots, X_r\} \geq \{X'_1, \dots, X'_r\}$ and $\{X'_1, \dots, X'_r\} \geq \{X_1, \dots, X_r\}$, then $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0}) = (\mathcal{C}'^{\leq 0}, \mathcal{C}'^{\geq 0})$. In particular, the two t-structures have the same heart. Therefore, both $\{X_1, \dots, X_r\}$ and $\{X'_1, \dots, X'_r\}$ are complete sets of pairwise non-isomorphic simple objects of the same abelian category, and hence they are equivalent.
(b) Let $\{X''_1, \dots, X''_r\}$ be a third simple-minded collection of $\mathcal{C}$, with corresponding t-structure $(\mathcal{C}''^{\leq 0}, \mathcal{C}''^{\geq 0})$. Suppose $\{X_1, \dots, X_r\} \geq \{X'_1, \dots, X'_r\}$ and $\{X'_1, \dots, X'_r\} \geq \{X''_1, \dots, X''_r\}$. Then $\mathcal{C}''^{\leq 0} \subseteq \mathcal{C}'^{\leq 0} \subseteq \mathcal{C}^{\leq 0}$. Consequently, $\{X_1, \dots, X_r\} \geq \{X''_1, \dots, X''_r\}$. $\surd$

7.3. T-STRUCTURES. Let $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ be a bounded t-structure of $\mathcal{C}$ such that the heart $\mathcal{A}$ is a length category which has only finitely many simple objects $S_1, \dots, S_r$ up to isomorphism. Then $\{S_1, \dots, S_r\}$ is a simple-minded collection. Let $\mathcal{F} = \mathcal{S}_i$ be the extension closure of S_i in $\mathcal{A}$ and let $\mathcal{T} = {}^{\perp}\mathcal{S}_i$ be the left perpendicular category of $\mathcal{S}_i$ in $\mathcal{A}$. It is easy to show that $(\mathcal{T}, \mathcal{F})$ is a torsion pair of $\mathcal{A}$. Define the *left mutation* $\mu_i^+(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0}) = (\mathcal{C}'^{\leq 0}, \mathcal{C}'^{\geq 0})$ by

$$\begin{aligned} \mathcal{C}'^{\leq 0} &= \{M \in \mathcal{C} \mid H^m(M) = 0 \text{ for } m > 0 \text{ and } H^0(M) \in \mathcal{T}\}, \\ \mathcal{C}'^{\geq 0} &= \{M \in \mathcal{C} \mid H^m(M) = 0 \text{ for } m < -1 \text{ and } H^{-1}(M) \in \mathcal{F}\}. \end{aligned}$$

Similarly one defines the *right mutation* $\mu_i^-(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$. These mutations provide an effective method to compute the space of Bridgeland's stability conditions on $\mathcal{C}$ by gluing different charts, see [14, 48].

PROPOSITION 7.10. *The pairs $\mu_i^+(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ and $\mu_i^-(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ are bounded t-structures of $\mathcal{C}$. The heart of $\mu_i^+(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ has a torsion pair $(\Sigma\mathcal{F}, \mathcal{T})$ and the heart of $\mu_i^-(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ has a torsion pair $(\mathcal{S}_i^{\perp}, \Sigma^{-1}\mathcal{S}_i)$. Moreover, $\mu_i^+ \circ \mu_i^-(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0}) = (\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0}) = \mu_i^- \circ \mu_i^+(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$.*

Proof. This follows from [24, Proposition 2.1, Corollary 2.2] and [14, Proposition 2.5]. $\surd$

In general the heart of the mutation of a bounded t-structure with length heart is not necessarily a length category. For an example, let Q be the quiver

$$\begin{tikzcd} 1 \arrow[loop left] & 2 \arrow[l] \end{tikzcd}$$

and consider the bounded derived category $\mathcal{C} = \mathcal{D}^b(\mathsf{nil.rep}\,Q)$ of finite-dimensional nilpotent representations of Q. Let S_1 and S_2 be the one-dimensional nilpotent representations associated to the two vertices. Let $\mathcal{F} = \mathcal{S}_1$ be the extension closure of S_1 and $\mathcal{T} = {}^\perp\mathcal{F} = \{M \in \mathsf{nil.rep}\,Q \mid \mathrm{top}(M) \in \mathsf{add}(S_2)\}$. Then the heart $\mathcal{A}'$ of the left mutation at 1 of the standard t-structure has a torsion pair $(\Sigma\mathcal{F}, \mathcal{T})$. Due to $\mathsf{nil.rep}\,Q$ being hereditary, there are no extensions of $\Sigma\mathcal{F}$ by $\mathcal{T}$, and hence any indecomposable object of $\mathcal{A}'$ belongs to either $\mathcal{T}$ or $\Sigma\mathcal{F}$. Suppose that $\mathcal{A}'$ is a length category. Then $\mathcal{A}'$ has two isomorphism classes of simple modules, which respectively belong to $\mathcal{T}$ and $\Sigma\mathcal{F}$, say $S_2' \in \mathcal{T}$ and $S_1' \in \Sigma\mathcal{F}$. For $n \in \mathbb{N}$ define an indecomposable object M_n in $\mathcal{T}$ as

$$\begin{tikzcd} k \arrow[loop left, "J_n(0)"] & k, \arrow[l, "(0,\ldots,0,1)^{tr}"'] \end{tikzcd}$$

where $J_n(0)$ is the (upper triangular) Jordan block of size n and with eigenvalue 0. There are no morphisms from S_1' to M_n for any n. Suppose that the Loewy length of S_2' in $\mathcal{A}$ is l. Then for $n > l$, any morphism from S_2' to M_n factors through $\mathrm{rad}^{n-l}M_n$ which lies in $\mathcal{F}$, and hence the morphism has to be zero. Therefore M_n $(n > l)$, considered as an object in $\mathcal{A}'$, does not have finite length, a contradiction.

For two bounded t-structures $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ and $(\mathcal{C}'^{\leq 0}, \mathcal{C}'^{\geq 0})$ on $\mathcal{C}$, define

$$(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0}) \geq (\mathcal{C}'^{\leq 0}, \mathcal{C}'^{\geq 0})$$

if $\mathcal{C}^{\leq 0} \supseteq \mathcal{C}'^{\leq 0}$. This defines a partial order on the set of bounded t-structures on $\mathcal{C}$.

7.4. Co-t-structures. Let $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ be a bounded co-t-structure of $\mathcal{C}$. Assume that the co-heart admits a basic additive generator $M = M_1 \oplus \ldots \oplus M_r$ with M_i indecomposable. Then M is a silting object of $\mathcal{C}$. Let $i = 1, \ldots, r$. Define $\mathcal{C}'_{\leq 0}$ as the additive closure of the extension closure of $\Sigma^m M_j$, $j \neq i$, and $\Sigma^{m+1} M_i$ for $m \geq 0$ and define $\mathcal{C}'_{\geq 0}$ as the left perpendicular category of $\Sigma\mathcal{C}'_{\leq 0}$. The *left mutation* $\mu_i^+(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ is defined as the pair $(\mathcal{C}'_{\geq 0}, \mathcal{C}'_{\leq 0})$. Similarly one defines the *right mutation* $\mu_i^-(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$.

Proposition 7.11. *The pairs $\mu_i^+(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ and $\mu_i^-(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ are bounded co-t-structures on $\mathcal{C}$. Moreover, $\mu_i^+ \circ \mu_i^-(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}) = (\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}) = \mu_i^- \circ \mu_i^+(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$.*

Proof. This can be proved directly. Here we alternatively make use of the results in Sections 3.1 and 7.1. Recall from Theorem 7.1 that there is a mutated silting object $\mu_i^+(M)$. It is straightforward to check, using the defining triangle for $\mu_i^+(M)$, that $\mu_i^+(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ is the bounded co-t-structure associated to $\mu_i^+(M)$ as defined in Proposition 3.5, and similarly for μ_i^-. The second statement follows from Theorem 7.1. $\surd$

For two bounded co-t-structures $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ and $(\mathcal{C}'_{\geq 0}, \mathcal{C}'_{\leq 0})$ on $\mathcal{C}$, define

$$(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}) \geq (\mathcal{C}'_{\geq 0}, \mathcal{C}'_{\leq 0})$$

if $\mathcal{C}_{\leq 0} \supseteq \mathcal{C}'_{\leq 0}$. This defines a partial order on the set of bounded co-t-structures on $\mathcal{C}$.

7.5. The bijections commute with mutations. Let Λ a finite-dimensional algebra over K.

Theorem 7.12. *The ϕ_{ij}'s defined in Section 5 commute with the left and right mutations defined in previous subsections.*

A priori it it not known that the heart of the mutation of a bounded t-structure with length heart is again a length category. So the theorem becomes well-stated only when the proof has been finished.

Proof. In view of Lemma 6.6, Theorem 7.1, and Propositions 7.6, 7.10 and 7.11, it suffices to prove that ϕ_{41}, ϕ_{31} and ϕ_{23} commute with the corresponding left mutations.
(a) ϕ_{41} commutes with μ_i^+: this was already shown in the proof of Proposition 7.11.
(b) ϕ_{31} commutes with μ_i^+: Let $M = M_1 \oplus \ldots \oplus M_r$ be a silting object with M_i indecomposable and $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0}) = \phi_{31}(M)$. We want to show $\mu_i^+(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0}) = \phi_{31}(\mu_i^+(M))$.
Let $\tilde{\Gamma}$ be the truncated dg endomorphism algebra of M as in Section 5.1. Then there is a triangle equivalence $F = ? \overset{L}{\otimes}_{\tilde{\Gamma}} M : \mathcal{D}_{fd}(\tilde{\Gamma}) \to \mathcal{D}^b(\mathsf{mod}\,\Lambda)$, which takes $\tilde{\Gamma}$ to M and takes the standard t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D}_{fd}(\tilde{\Gamma})$ to $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$. There is a decomposition $1 = e_1 + \ldots + e_r$, where $e_1, \ldots, e_r$ are (not necessarily primitive) idempotents of $\tilde{\Gamma}$ such that F takes $e_j\tilde{\Gamma}$ to M_j for $1 \leq j \leq r$.
Let $\Gamma = H^0(\tilde{\Gamma})$ and $\pi : \tilde{\Gamma} \to \Gamma$ be the canonical projection. By abuse of notation, write $e_1 = \pi(e_1), \ldots, e_r = \pi(e_r)$. Then $e_1\Gamma, \ldots, e_r\Gamma$ are indecomposable projective Γ-modules. Let $S_1, \ldots, S_r$ be the corresponding simple modules. Recall that the heart of the t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is $\mathsf{mod}\,\Gamma$. Let $\mathcal{F} = \mathsf{add}(S_i) \subseteq \mathsf{mod}\,\Gamma$ and $\mathcal{T} = {}^{\perp}S_i$. Define $\mathcal{D}'^{\leq 0}$ (respectively, $\mathcal{D}'^{\geq 0}$) to be the extension closure of $\Sigma\mathcal{D}^{\leq 0}$ and $\mathcal{T}$ (respectively, of $\Sigma\mathcal{F}$ and $\mathcal{D}^{\geq 0}$). Then $F(\mathcal{D}'^{\leq 0}, \mathcal{D}'^{\geq 0}) = \mu_i^+(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$.
The left mutation of $\tilde{\Gamma}$ at $e_i\tilde{\Gamma}$ is $\mu_i^+(\tilde{\Gamma}) = Q_i \oplus \bigoplus_{j \neq i} e_j\tilde{\Gamma}$, where Q_i is defined by the triangle

$$e_i\tilde{\Gamma} \xrightarrow{f} E \longrightarrow P'_i \longrightarrow \Sigma e_i\tilde{\Gamma}\ , \tag{7.1}$$

where f is a minimal left $\mathsf{add}(\bigoplus_{j\neq i} e_j\tilde{\Gamma})$-approximation. Then $F(\mu_i^+(\tilde{\Gamma})) = \mu_i^+(M)$. Define

$$\begin{aligned} \mathcal{D}''^{\leq 0} &= \{N \in \mathcal{D}_{fd}(\tilde{\Gamma}) \mid \mathsf{Hom}(\mu_i^+(\tilde{\Gamma}), \Sigma^m N) = 0, \forall m > 0\}, \\ \mathcal{D}''^{\geq 0} &= \{N \in \mathcal{D}_{fd}(\tilde{\Gamma}) \mid \mathsf{Hom}(\mu_i^+(\tilde{\Gamma}), \Sigma^m N) = 0, \forall m < 0\}. \end{aligned}$$

Thus showing $\mu_i^+(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0}) = \phi_{31}(\mu_i^+(M))$ is equivalent to showing the equality $(\mathcal{D}'^{\leq 0}, \mathcal{D}'^{\geq 0}) = (\mathcal{D}''^{\leq 0}, \mathcal{D}''^{\geq 0})$, equivalently, the inclusions $\mathcal{D}'^{\leq 0} \subseteq \mathcal{D}''^{\leq 0}$ and $\mathcal{D}'^{\geq 0} \subseteq \mathcal{D}''^{\geq 0}$. It suffices to prove $\mathcal{T} \subseteq \mathcal{D}''^{\leq 0}$, $\Sigma\mathcal{D}^{\leq 0} \subseteq \mathcal{D}''^{\leq 0}$, $\Sigma\mathcal{F} \subseteq \mathcal{D}''^{\geq 0}$ and $\mathcal{D}^{\geq 0} \subseteq \mathcal{D}''^{\geq 0}$. We only show the first inclusion, the other three are easy.
Let $T \in \mathcal{T}$. To show $T \in \mathcal{D}''^{\leq 0}$, it suffices to show $\mathsf{Hom}(Q_i, \Sigma T) = 0$. Applying $\mathsf{Hom}(?, T)$ to the triangle (7.1), we obtain a long exact sequence

$$\mathsf{Hom}(E, T) \xrightarrow{f^*} \mathsf{Hom}(e_i\tilde{\Gamma}, \Sigma T) \longrightarrow \mathsf{Hom}(Q_i, \Sigma T) \longrightarrow \mathsf{Hom}(E, \Sigma T) = 0\ .$$

We claim that f^* is surjective. Then the desired result follows. Consider the commutative diagram

$$\begin{tikzcd} \mathsf{Hom}(e_i\Gamma, T) \arrow[r, "\pi_i^*"] & \mathsf{Hom}(e_i\tilde{\Gamma}, T) \\ \mathsf{Hom}(H^0(E), T) \arrow[u, "H^0(f)^*"] \arrow[r, "\pi_E^*"] & \mathsf{Hom}(E, T), \arrow[u, "f^*"'] \end{tikzcd} \tag{7.2}$$

where $\pi_i : e_i\tilde{\Gamma} \to e_i\Gamma$ and $\pi_E : E \to H^0(E)$ are the canonical projections. Let $C = \mathsf{ker}(\pi_i)$. Then there is a triangle

$$C \longrightarrow e_i\tilde{\Gamma} \xrightarrow{\pi_i} e_i\Gamma \longrightarrow \Sigma C\ .$$

Note that C belongs to $\Sigma\mathcal{D}^{\leq 0}$, which implies that $\mathsf{Hom}(C, T) = 0 = \mathsf{Hom}(\Sigma C, T)$. It follows that the map π_i^* is bijective. Similarly, the map π_E^* is also bijective. Thus it suffices to show the surjectivity of $H^0(f)^*$. Now let P_T be a projective cover of T in $\mathsf{mod}\,\Gamma$. Then P_T belongs to $\mathsf{add}(\bigoplus_{j\neq i} e_j\Gamma)$ because $T \in \mathcal{T} = {}^\perp S_i$. It follows that any morphism $e_i\Gamma \to T$ factors through P_T, and hence factors through $H^0(f) : e_i\Gamma \to H^0(E)$, since $H^0(f)$ is a minimal left $\mathsf{add}(\bigoplus_{j\neq i} e_j\Gamma)$-approximation (for $H^0|_{\mathsf{add}(\tilde{\Gamma})} : \mathsf{add}(\tilde{\Gamma}) \to \mathsf{add}(\Gamma)$ is an equivalence). This shows that $H^0(f)^*$ is surjective, completing the proof of the claim.
(c) ϕ_{23} commutes with μ_i^+: Let $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ be a bounded t-structure on $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ with length heart. Let $\{X_1, \ldots, X_r\} = \phi_{23}(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$. The mutated simple-minded collection $\mu_i^+(X_1, \ldots, X_r)$ is contained in the heart of the mutated t-structure $\mu_i^+(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$. Consequently, the aisle and co-aisle of $\phi_{32} \circ \mu_i^+(X_1, \ldots, X_r)$ are respectively contained in the aisle and co-aisle of $\mu_i^+(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$, and hence $\phi_{32} \circ \mu_i^+(X_1, \ldots, X_r) = \mu_i^+(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$, *i.e.* $\phi_{23} \circ \mu_i^+(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0}) = \mu_i^+ \circ \phi_{23}(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$. $\surd$

7.6. The bijections are isomorphisms of partially ordered sets. Let Λ be a finite-dimensional algebra over K.

Theorem 7.13. *The ϕ_{ij}'s defined in Section 5 are isomorphisms of partially ordered sets with respect to the partial orders defined in previous subsections.*

Proof. In view of Theorem 6.1 and Lemma 6.6, it suffices to show that $f(x) \geq f(y)$ if and only if $x \geq y$ for $f = \phi_{41}, \phi_{32}$ and ϕ_{34}.
(a) For ϕ_{41} the desired result follows from [1, Proposition 2.14].
(b) For ϕ_{32} the desired result is included in the proof of Proposition 7.9.
(c) Let $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ and $(\mathcal{C}'_{\geq 0}, \mathcal{C}'_{\leq 0})$ be two bounded co-t-structures on $\mathcal{C}$ and let $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ and $(\mathcal{C}'^{\leq 0}, \mathcal{C}'^{\geq 0})$ be their respective images under ϕ_{34}. Then by definition

$$\begin{aligned}\mathcal{C}^{\leq 0} &= \{M \in \mathcal{D}^b(\mathsf{mod}\,\Lambda) \mid \mathsf{Hom}(N, M) = 0\ \forall N \in \Sigma^{-1}\mathcal{C}_{\geq 0}\},\\ \mathcal{C}'^{\leq 0} &= \{M \in \mathcal{D}^b(\mathsf{mod}\,\Lambda) \mid \mathsf{Hom}(N, M) = 0\ \forall N \in \Sigma^{-1}\mathcal{C}'_{\geq 0}\}.\end{aligned}$$

Here, $\mathcal{C}^{\leq 0} \supseteq \mathcal{C}'^{\leq 0}$ if and only if $\mathcal{C}_{\geq 0} \supseteq \mathcal{C}'_{\geq 0}$, and hence by definition $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0}) \geq (\mathcal{C}'^{\leq 0}, \mathcal{C}'^{\geq 0})$ if and only if $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}) \geq (\mathcal{C}'_{\geq 0}, \mathcal{C}'_{\leq 0})$. $\surd$

8. A concrete example

Let Λ be the finite-dimensional K-algebra given by the quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$$

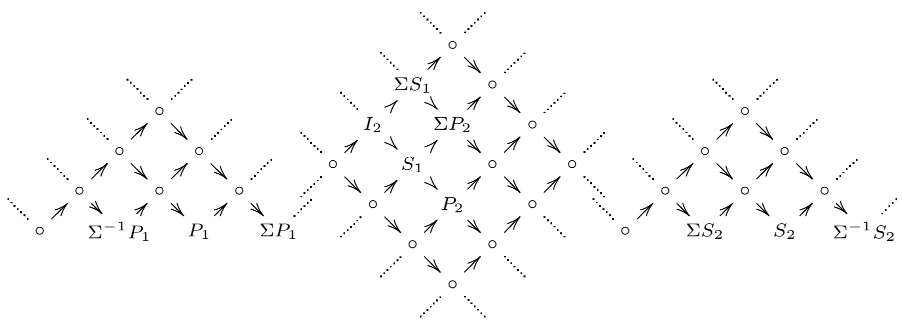
with relation $\alpha\beta = 0$. This algebra has many manifestations: It is, possibly up to Morita equivalence, the Auslander algebra of $k[x]/x^2$, the Schur algebra $S(2,2)$ ($\mathrm{char} K = 2$) and the principal block of the category $\mathcal{O}$ for $\mathfrak{sl}_2(\mathbb{C})$ ($K = \mathbb{C}$). In this section we will compute the derived Picard group for Λ and classify all silting objects/simple-minded collections in $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$. As a consequence of this classification and a result of Woolf [48, Theorem 3.1], the space of stability conditions on $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ is exactly $\mathbb{C}^2$.

8.1. Indecomposable objects. Let P_1 and P_2 be the indecomposable projective Λ-modules corresponding to the vertices 1 and 2. Then up to isomorphism and up to shift an indecomposable object in $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ belongs to one of the following four families (see for example [19, 9])

- $P_1(n) = P_1 \to P_1 \to \ldots \to P_1 \to P_1$, $n \geq 1$,
- $R(n) = P_1 \to P_1 \to \ldots \to P_1 \to P_1 \to P_2$, $n \geq 0$,
- $L(n) = P_2 \to P_1 \to P_1 \to \ldots \to P_1 \to P_1$, $n \geq 0$,
- $B(n) = P_2 \to P_1 \to P_1 \to \ldots \to P_1 \to P_1 \to P_2$, $n \geq 1$,

where the homomorphisms are the unique non-isomorphisms, n is the number of occurrences of P_1 and the rightmost components have been put in degree 0.

8.2. The Auslander–Reiten quiver. The Auslander–Reiten quiver of $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ consists of three components: two $\mathbb{Z}A_\infty$ components and one $\mathbb{Z}A_\infty^\infty$ component (see [10, 28])



The abelian category $\mathsf{mod}\,\Lambda$ has five indecomposable objects up to isomorphism: the two simple modules S_1 and S_2, their projective covers P_1 and P_2 and their injective envelopes $I_1 = P_1$ and I_2. They are marked on the above Auslander–Reiten quiver.

The left $\mathbb{Z}A_\infty$ component consists of shifts of $P_1(n)$, $n \geq 1$. The Auslander–Reiten translation τ takes $P_1(n)$ to $\Sigma^{-1}P_1(n)$. It is straightforward to check that P_1 is a 0-spherical object of $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ in the sense of Seidel and Thomas [45]. The additive closure of this component is the triangulated subcategory generated by P_1. This component will be referred to as the 0-spherical component.

The $\mathbb{Z}A_\infty^\infty$ component consists of shifts of $R(n)$ and $L(n)$, $n \geq 0$. Note that $S_1 = L(1)$, $P_2 = R(0) = L(0)$ and $I_2 = L(2)$. The Auslander–Reiten translation τ takes $R(n)$ $(n \geq 2)$ to $\Sigma R(n-2)$, takes $R(1)$ to $L(1)$ and takes $L(n)$ to $\Sigma^{-1}L(n+2)$.

The right $\mathbb{Z}A_\infty$ component consists of shifts of $B(n)$, $n \geq 1$. The Auslander–Reiten translation takes $B(n)$ to $\Sigma B(n)$. The simple module $S_2 = B(1)$ is a 2-spherical object of $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ and the additive closure of this component is the triangulated subcategory generated by S_2. This component will be referred to as the 2-spherical component.

8.3. The derived Picard group. Let E be a spherical object of a triangulated category $\mathcal{C}$ in the sense of Seidel and Thomas [45]. Then the twist functor Φ_E defined by

$$\Phi_E(M) = \mathrm{Cone}(\bigoplus_{m\in\mathbb{Z}} \mathsf{Hom}(\Sigma^m E, M) \otimes \Sigma^m E \xrightarrow{ev} M),$$

where ev is the evaluation map, is an auto-equivalence of $\mathcal{C}$ by [45, Proposition 2.10].

Recall from the preceding subsection that P_1 is a 0-spherical object and S_2 is a 2-spherical object of $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$. Thus the associated twist functors Φ_{P_1} and Φ_{S_2} are two auto-equivalences of $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$.

Lemma 8.1. *For M in $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ there are isomorphisms $\Phi_{S_2}(M) \cong \Phi_{P_1} \circ \Sigma^{-1}(M)$ and $\Phi^2_{P_1}(M) \cong \nu^{-1} \circ \Sigma^2(M)$. Moreover, if M is indecomposable and belongs to the $\mathbb{Z}A^\infty_\infty$ component, there exists a unique pair of integers (n, n') such that $M \cong \Phi^n_{P_1} \circ \Phi^{n'}_{S_2}(P_2)$.*

Proof. Observe that $\Phi_{P_1}(S_1) \cong \Sigma P_2$, $\Phi_{P_1}(P_1) \cong \Sigma P_1$, $\Phi_{P_1}(S_2) \cong S_2$ and $\Phi_{S_2}(S_1) \cong P_2$, $\Phi_{S_2}(P_1) \cong P_1$, $\Phi_{S_2}(S_2) \cong \Sigma^{-1}S_2$. Since auto-equivalences preserve the shape of the Auslander–Reiten quiver, the statements follow. $\surd$

Remark 8.2. *Inspecting the action of Φ_{P_1} and Φ_{S_2} on maps shows that the isomorphism $\Phi^2_{P_1}(M) \cong \nu^{-1} \circ \Sigma^2(M)$ is functorial, while $\Phi_{S_2}(M) \cong \Phi_{P_1} \circ \Sigma^{-1}(M)$ is not.*

Let $\mathsf{Aut}\,\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ denote the group of algebraic auto-equivalences of $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$, *i.e.* those which admits a dg lift. By [29, Lemma 6.4], such an auto-equivalence is naturally isomorphic to the derived tensor functor of a complex of bimodules.

Lemma 8.3. $\mathsf{Aut}\,\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ *is isomorphic to $\mathbb{Z}^2 \times K^\times$.*

Proof. Let $F \in \mathsf{Aut}\,\mathcal{D}^b(\mathsf{mod}\,\Lambda)$. Since F preserves the Auslander–Reiten quiver, the object $F(P_2)$ is in the $\mathbb{Z}A^\infty_\infty$ component. Thus there is a pair of integers (n_F, n'_F) such that $F(P_2) \cong \Phi^{n_F}_{P_2} \circ \Phi^{n'_F}_{S_2}(P_2)$. This allows us to define a map

$$f: \qquad \begin{aligned} \mathsf{Aut}\,\mathcal{D}^b(\mathsf{mod}\,\Lambda) &\longrightarrow \mathbb{Z}^2 \\ F &\longmapsto (n_F, n'_F). \end{aligned}$$

This map is clearly a surjective group homomorphism. Moreover, the group homomorphism

$$\begin{aligned} \mathbb{Z}^2 &\longrightarrow \mathsf{Aut}\,\mathcal{D}^b(\mathsf{mod}\,\Lambda) \\ (n, n') &\longmapsto \Phi^n_{P_2} \circ \Phi^{n'}_{S_2} \end{aligned}$$

is a retraction of f. Therefore $\mathsf{Aut}\,\mathcal{D}^b(\mathsf{mod}\,\Lambda) \cong \mathbb{Z}^2 \times \mathsf{ker}(f)$.
Let $F \in \mathsf{ker}(f)$. Then $F(P_2) \cong P_2$. This forces $F(P_1) \cong P_1$, and hence F is induced from an outer automorphism of Λ which fixes the two primitive idempotents e_1 and e_2. Thus $\mathsf{ker}(f) \cong K^\times$, finishing the proof. $\surd$

8.4. Morphism spaces. We first compute the morphism spaces between the two $\mathbb{Z}A_\infty$ components.

Lemma 8.4. (a) *For $n \geq 2$, $\mathsf{Hom}(P_1(n), \Sigma^m P_1(n))$ does not vanish for some $m > 0$ and for some $m < 0$. For $n = 1$, $\mathsf{Hom}(P_1, \Sigma^m P_1)$ is isomorphic to $K[x]/x^2$ for $m = 0$ and vanishes for $m \neq 0$.*

(b) *For $n \geq 2$, $\mathsf{Hom}(B(n), \Sigma^m B(n))$ does not vanish for some $m > 0$ and for some $m < 0$. For $n = 1$, $\mathsf{Hom}(S_2, \Sigma^m S_2)$ is K for $m = 0, 2$ and vanishes for $m \neq 0, 2$.*

Proof. Direct computation, or apply some general result (e.g. [25, Section 2]) to the triangulated categories generated by P_1 and S_2. $\surd$

Next we compute the morphism spaces between P_2 and the objects on the $\mathbb{Z}A_\infty^\infty$ component.

Lemma 8.5. *Let $n \geq 0$.*

(a) *$\mathsf{Hom}(P_2, \Sigma^m R(n))$ is K if $-n \leq m \leq 0$ and is 0 otherwise.*
(a') *$\mathsf{Hom}(R(n), \Sigma^m P_2)$ is K if $2 \leq m \leq n$ or if $n = 0, m = 0$ and is 0 otherwise.*
(b) *$\mathsf{Hom}(P_2, \Sigma^m L(n))$ is K if $2 - n \leq m \leq 0$ or if $n = 0, m = 0$ and is 0 otherwise.*
(b') *$\mathsf{Hom}(L(n), \Sigma^m P_2)$ is K if $0 \leq m \leq n$ and is 0 otherwise.*

Proof. (a) and (b) Because $\mathsf{Hom}(P_2, M) = H^0(M)e_2$.
(a') and (b') are obtained from (a) and (b) by applying the Auslander–Reiten formula $\mathrm{D}\,\mathsf{Hom}(M, N) \cong \mathsf{Hom}(N, \tau\Sigma M)$. $\surd$

8.5. Silting objects and simple-minded collections. Now we are ready to classify the silting objects and simple-minded collections in $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$.

Proposition 8.6. *Up to isomorphism, any basic silting object of $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ belongs to one of the following two families*

- *$\Phi_{P_1}^n \circ \Phi_{S_2}^{n'}(P_1 \oplus P_2)$, $n, n' \in \mathbb{Z}$, the corresponding simple-minded collection is $\Phi_{P_1}^n \circ \Phi_{S_2}^{n'}\{S_1, S_2\}$,*
- *$\Phi_{P_1}^n \circ \Phi_{S_2}^{n'}(\Sigma^m S_1 \oplus P_2)$, $n, n' \in \mathbb{Z}$ and $m \leq -1$, the corresponding simple-minded collection is $\Phi_{P_1}^n \circ \Phi_{S_2}^{n'}\{\Sigma^m S_1, I_2\}$.*

Proof. Let N be an indecomposable direct summand of a silting object. By Lemma 8.4, N does not belong to the 2-spherical component, and N belongs to the 0-spherical component if and only if N is a shift of P_1. Moreover, a basic silting object can have at most one shift of P_1 as a direct summand. It follows that a silting object has at least one indecomposable direct summand from the $\mathbb{Z}A_\infty^\infty$ component.
Let $M = M_1 \oplus M_2$ be a silting object with M_1 and M_2 indecomposable. Assume that M_1 belongs to the $\mathbb{Z}A_\infty^\infty$ component. Up to an auto-equivalence of the form $\Phi_{P_1}^n \circ \Phi_{S_2}^{n'}$, we may assume that $M_1 = P_2$. Then, if M_2 belongs to the 0-spherical component it has to be P_1. Thus we assume that M_2 also belongs to the $\mathbb{Z}A_\infty^\infty$ component. Then it follows from Lemma 8.5 that M_2 is isomorphic to $\Sigma^m S_1$ for some $m \leq -1$ or to $\Sigma^m R(1)$ for some $m \geq 0$. Observing $P_2 \oplus \Sigma^m R(1) = \Phi_{P_1}^{-m-1} \circ \Phi_{S_2}^m(P_2 \oplus \Sigma^{-m-1} S_1)$ for $m \geq 0$ finishes the proof for the silting-object part.
That the simple-minded collection corresponding to a silting object is the desired one follows from the Hom-duality they satisfy. $\surd$

8.6. The silting quiver. Recall from [1] that the *silting quiver* has as vertices the isomorphism classes of basic silting objects and there is an arrow from M to M' if M' can be obtained from M by a left mutation.
The vertex set of the silting quiver of $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ is $\{(n,n',m) \mid n,n' \in \mathbb{Z}, m \in \mathbb{Z}_{\leq 0}\}$, where $(n,n',0)$ represents the silting object $\Phi^n_{P_1} \circ \Phi^{n'}_{S_2}(P_1 \oplus P_2)$ and (n,n',m) $(m \leq -1)$ represents the silting object $\Phi^n_{P_1} \circ \Phi^{n'}_{S_2}(\Sigma^m S_1 \oplus P_2)$. It is straightforward to show that from each vertex (n,n',m) there are precisely two outgoing arrows whose targets are respectively

- $(n,n'-1,m)$ and $(n+1,n',m-1)$ if $m=0$,
- $(n+1,n'-1,m-1)$ and $(n,n',m+1)$ if $m \leq -1$.

8.7. Hearts and the space of stability conditions.

Lemma 8.7. *The heart of any t-structure on $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ is a length category.*

Proof. Let $\mathcal{A}$ be the heart of a t-structure on $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$. We will show that $\mathcal{A}$ has only finitely many isomorphism classes of indecomposable objects. Such an abelian category must be a length category.
Due to vanishing of negative extensions, it follows from Lemma 8.4 that $\mathcal{A}$ contains at most one indecomposable object from the 0-spherical component respectively the 2-spherical component.
Suppose that $\mathcal{A}$ contains an indecomposable object from the $\mathbb{Z}A^\infty_\infty$ component. Without loss of generality we may assume that it is P_2. It follows from Lemma 8.5 that for $n \geq 3$ and $m \in \mathbb{Z}$ either $\mathsf{Hom}(P_2, \Sigma^{m'}\Sigma^m R(n)) \neq 0$ for some $m' < 0$ or $\mathsf{Hom}(\Sigma^m R(n), \Sigma^{m'} P_2) \neq 0$ for some $m' < 0$. Similarly for $L(n)$. Therefore an indecomposable object M belongs to the heart only if it is isomorphic to one of $\Sigma^m P_2$, $\Sigma^m R(1)$, $\Sigma^m R(2)$, $\Sigma^m L(1)$ and $\Sigma^m L(2)$, $m \in \mathbb{Z}$. But at most one shift of a nonzero object can belong to a heart. So $\mathcal{A}$ contains at most 7 indecomposable objects up to isomorphism. $\surd$

In view of Lemma 8.7, the result in the preceding subsection shows that all bounded t-structures on $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ are related to each other by a sequence of left or/and right mutations. In particular, this implies that the t-structures Woolf considered in [48, Section 3.1] are already all bounded t-structures on $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$. Therefore we have

Corollary 8.8. (a) *The Bridgeland space of stability conditions on $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ is $\mathbb{C}^2$.*
(b) *An abelian category is the heart of some bounded t-structure on $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ if and only if it is equivalent to $\mathsf{mod}\,\Gamma$ for $\Gamma = \Lambda$ or $\Gamma = K(\cdot \to \cdot)$ or $\Gamma = K \oplus K$.*

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