Ramified Satake Isomorphisms for Strongly Parabolic Characters

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Abstract. For certain characters of the compact maximal torus of a reductive p-adic group, which we call strongly parabolic characters, we prove Satake-type isomorphisms. Our results generalize those of Satake, Howe, Bushnell and Kutzko, and Roche.

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Contents

1. Introduction

1.1. The problem we study. Let F be a local non-Archimedean field with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}_q$. Let G be a connected split reductive group over F with maximal split torus T and Weyl group $W = N_G(T)/T$. Let $\check{T}$ denote the dual torus. Replacing G by an isomorphic group, we may, and henceforth will, assume that G is defined over $\mathbb{Z}$ (see, for instance, [DG93],[Con11, §5] for the definition of a split reductive group over an arbitrary scheme). Then $G(\mathcal{O})$ is a maximal compact (open) subgroup of $G(F)$. Let $\mathscr{H}(G(F), G(\mathcal{O}))$ denote the convolution algebra of compactly supported $G(\mathcal{O})$-bi-invariant complex valued functions on $G(F)$. A celebrated theorem of Satake [Sat63] states that we have a canonical isomorphism of algebras

$$\mathscr{H}(G(F), G(\mathcal{O})) \simeq \mathbb{C}[\check{T}/W]. \tag{1.1}$$

We are interested in generalizing this isomorphism to nontrivial smooth characters $\bar{\mu} : T(\mathcal{O}) \to \mathbb{C}^\times$, as follows. Let $W_{\bar{\mu}} \subseteq W$ denote the stabilizer of $\bar{\mu}$ under the action of the Weyl group. Then it is natural to pose:

Problem 1. *Construct a pair (K, μ) consisting of a compact open subgroup $T(\mathcal{O}) \subseteq K \subseteq G(\mathcal{O})$ and a character $\mu : K \to \mathbb{C}^\times$ extending $\bar{\mu}$, such that we have an isomorphism of algebras*

$$\mathscr{H}(G(F), K, \mu) \simeq \mathbb{C}[\check{T}/W_{\bar{\mu}}], \tag{1.2}$$

where $\mathscr{H}$ is the convolution algebra of (K, μ)-bi-invariant compactly supported functions on $G(F)$.

The Satake isomorphism provides a solution for the above problem for $\bar{\mu} = 1$. In this paper, we solve the above problem for a large class of characters of $T(\mathcal{O})$ which we call "strongly parabolic characters," which are by definition characters such that $W_{\bar{\mu}}$ is the Weyl group of a Levi subgroup $L < G$, and moreover such that $\bar{\mu}$ extends to $L(F)$. This appears to be the proper generality where the problem has a positive solution. Our construction of K is tied to L. We think of the isomorphism $\mathscr{H}(G(F), K, \mu) \simeq \mathbb{C}[\check{T}/W_{\bar{\mu}}]$ as a Satake isomorphism for the (possibly) ramified character $\bar{\mu}$. Therefore, we call these isomorphisms *ramified Satake isomorphisms.* For characters that are not strongly parabolic, we do not have a reason to expect a positive answer to Problem 1.

1.2. History. Following Satake, R. Howe studied Problem 1 for $G = \mathrm{GL}_N$ [How73]. Via an isomorphism which he called the $\bar{\mu}$-spherical Fourier transform, he completely solved the problem for the general linear group. Howe's paper went largely unnoticed; however, several cases of Problem 1 were subsequently solved using other methods.

In [Ber84], [Ber92], Bernstein constructed a decomposition of the category of representations of $G(F)$ using the theory of Bernstein center. Each block admits a projective generator. In particular, for every character $\bar{\mu} : T(\mathcal{O}) \to \mathbb{C}^\times$, one has a block of representations of $G(F)$, which we denote by $\mathscr{R}_{\bar{\mu}}(G)$. Bernstein proved that the center of $\mathscr{R}_{\bar{\mu}}(G)$ is canonically isomorphic to $\mathbb{C}[\check{T}/W_{\bar{\mu}}]$;

see, for instance, [Roc09, Theorem 1.9.1.1]. Moreover, he gave an explicit description of a projective generator for each of these blocks; see the RHS of (1.8). When the character $\bar{\mu}$ is *regular*; i.e., $W_{\bar{\mu}} = \{1\}$, then the center is $\mathbb{C}[\check{T}]$, and it identifies canonically with the endomorphism ring of Bernstein's generator.
In a fundamental paper [BK98], Bushnell and Kutzko organized the study of representations of $G(F)$ via compact open subgroups into the theory of *types*. Namely, they proposed that one should be able to obtain a projective generator for every block of representations of $G(F)$ by inducing a finite dimensional representation from a compact open subgroup. The pair of the compact open subgroup and its finite dimensional representation, up to a certain equivalence, is called the type. In [BK99] and [BK93], they explicitly construct types for every block of representations of GL_N. In particular, they construct projective generators for the principal series blocks $\mathscr{R}_{\bar{\mu}}(\mathrm{GL}_N)$. When the character $\bar{\mu}$ is regular, their construction provides a pair (K, μ) satisfying the requirement of Problem 1. We note, however, that Bushnell and Kutzko's construction of types is technically involved, since they consider all blocks (not merely the principal series blocks); in particular, we were not able to locate exactly where in their papers they construct types for the principal series blocks of GL_N.
Finally, Roche [Roc98] constructed types for principal series representations of arbitrary reductive groups in good characteristics (which excluded in particular those listed in Convention 6). In the case that $\bar{\mu}$ is regular, the type itself is a pair (K, μ) satisfying the conditions of Problem 1.
In this paper, we build on the methods introduced by Bushnell and Kutzko and Roche, and solve the problem for all strongly parabolic characters. We make use of Roche's type in order to construct a pair (K, μ) satisfying the conditions of Problem 1.

1.3. On characters of $T(\mathcal{O})$. A significant part of this paper, which may be of independent interest, is devoted to defining and studying certain smooth characters of $T(\mathcal{O})$. Recall that a subgroup $W' \subseteq W$ is *parabolic* if it is generated by simple reflections. The Levi subgroup L associated to W' is the subgroup generated by T and the the simple root subgroups corresponding to the simple reflections in W' along with their negatives.

Definition 2. Let $\bar{\mu} : T(\mathcal{O}) \to \mathbb{C}^\times$ be a smooth character.

(i) $\bar{\mu}$ is *parabolic* if the stabilizer $\mathrm{Stab}_W(\bar{\mu})$ of $\bar{\mu}$ in W is a parabolic subgroup.
(ii) $\bar{\mu}$ is *strongly parabolic* if it is parabolic with Levi L and extends to a character of $L(F)$.
(iii) $\bar{\mu}$ is *easy* if it is parabolic and it extends to a character of $L(F)$ which is trivial on $[L, L](F)$.

It follows immediately from the definition that the trivial character and all regular characters are easy. Moreover, it is clear that

$$\text{easy} \implies \text{strongly parabolic} \implies \text{parabolic.} \tag{1.3}$$

The reverse implications can all fail; see Examples 19 and 28.

To state our results regarding these characters, we need some notation. Let Φ denote the set of roots of G. Let X, $X^\vee$, Q, $Q^\vee$ denote the character, cocharacter, root and coroot lattices of G, respectively. Below we will frequently impose the conditions that either X/Q is free or $X^\vee/Q^\vee$ is free (or both). We remark that $X^\vee/Q^\vee$ being free is equivalent to $[G(\mathbb{C}), G(\mathbb{C})]$ being simply-connected, while X/Q being free is equivalent to the statement that $G(\mathbb{C})$ has connected center.[1]

In the following, given a coweight $\lambda \in X^\vee$, we view λ as a morphism $\mathbb{G}_m \to T$, and for every ring R (e.g., $R = \mathcal{O}$), we abusively use λ also to denote the morphism $\mathbb{G}_m(R) = R^\times \to T(R)$. Thus, given a character $\bar{\mu}$ of $T(\mathcal{O})$, we obtain a character $\bar{\mu} \circ \lambda$ of $\mathcal{O}^\times$. We will particularly use this when $\lambda = \alpha^\vee$ is a coroot.

Theorem 3. *Let $\bar{\mu} : T(\mathcal{O}) \to \mathbb{C}^\times$ be a smooth character.*

(i) *$\bar{\mu}$ is easy if and only if it is parabolic and can be written as a product $\chi_1 \cdots \chi_l$, where each χ_i is a character $T(\mathcal{O}) \to \mathbb{C}^\times$ which is a composition of a $W_{\bar{\mu}}$-invariant rational character $T(\mathcal{O}) \to \mathcal{O}^\times$ and a smooth character $\mathcal{O}^\times \to \mathbb{C}^\times$.*

(ii) *The following are equivalent:*
   (a) *$\bar{\mu}$ is strongly parabolic;*
   (b) $\bar{\mu} \circ \alpha^\vee|_{\mathcal{O}^\times} = 1, \quad \forall \alpha \in \Phi_L$.
   Moreover, if $q > 2$, then these are also equivalent to:
   (c) *$\bar{\mu}$ extends to a character of $L(\mathcal{O})$.*

(iii) *If X/Q is free or Φ has no factors of type A_1 or C_n, then every parabolic character of $T(\mathcal{O})$ is strongly parabolic.*

(iv) *If $X^\vee/Q^\vee$ is free, then every strongly parabolic character of $T(\mathcal{O})$ is easy.*

(v) *If Φ is simply-laced and X/Q is free, then every character of $T(\mathcal{O})$ is strongly parabolic.*

Section 2 is devoted to the proof of the above theorem.

We now indicate what the above theorem implies for characters of various groups. By G/Z we mean $G/Z(G)$. The letter N denotes a positive integer. We let E_n, $n = 6, 7, 8$ (resp. F_4 and G_2) denote the split reductive group whose associated complex group is the connected, simply-connected, simple group of type E_n (resp. F_4 and G_2).

[1]This follows from the fact that if G is a (connected split) semisimple group, then X/Q equals the dual of $Z(G(\mathbb{C}))$ and $X^\vee/Q^\vee$ equals the dual of $\pi_1(G(\mathbb{C}))$; see, for example [Con12, Example 6.7]. For example, for SL_2, we have $X \cong \mathbb{Z}$, $Q = 2X$, and $Q^\vee = X^\vee \cong \mathbb{Z}$.

(1.4)

Reductive group	Properties	Characters
GL_N, E_8	simply-laced, X/Q and $X^\vee/Q^\vee$ free	all characters are easy
PGL_N, GO_{2N}, SO_{2N}/Z, E_6/Z, E_7/Z	simply-laced and X/Q free	all characters are strongly parabolic
SL_N $(N \geq 3)$, GSp_{2N}, Spin_N, E_N $(N \geq 6)$, $\mathrm{F}_4, \mathrm{G}_2$	$X^\vee/Q^\vee$ free, and hypothesis of (iii)	all parabolic characters are easy
Sp_{2N}/Z, GO_N, SO_N	hypothesis of (iii)	all parabolic characters are strongly parabolic

Remark 4. Let G be a (connected) algebraic group over a field k. Let $\bar{k}$ denote an algebraic closure of k. Then G is said to be *easy* if every $g \in G(\bar{k})$ is in the neutral connected component of its centralizer in $G \otimes_k \bar{k}$. This definition is due to V. Drinfeld. Based on the discussion in, e.g., [Boy10, §2.2], there appears to be a relationship between Drinfeld's notion of easy and ours, when k has characteristic zero. Namely, here we show that, if $[G, G]$ is simply connected and $Z(G)$ is connected, then every parabolic character is easy (and the parabolic assumption is not needed in the simply-laced case); in [Boy10, §2.2] it is asserted, without proof, that these two assumptions are equivalent (over a field of characteristic zero) to G being easy in Drinfeld's sense.

Remark 5. To every character $\bar{\mu} : T(\mathcal{O}) \to \mathbb{C}^\times$, Roche [Roc98, §8] associated a possibly disconnected split reductive group $\tilde{H} = \tilde{H}_{\bar{\mu}}$ over F. The connected component of $\tilde{H}$ is an endoscopy group for G. It follows from Theorem 3.(ii) that strongly parabolic characters are exactly those characters for which $\tilde{H}$ is the Levi of a parabolic of G (and in particular connected). In more detail, by [Roc98, Definition 6.1], the coroots $\alpha^\vee$ of the connected component H of the identity of $\tilde{H}$ (as a complex reductive group) are exactly those for which $\bar{\mu} \circ \alpha^\vee|_{\mathcal{O}^\times} = 1$, and by [Roc98, Lemma 8.1.(i)], the stabilizer of $\bar{\mu}$ equals the Weyl group of H (and is not bigger) if and only if $\tilde{H} = H$. Then, we conclude because the Weyl group of H is a parabolic subgroup of the Weyl group of G if and only if H is a Levi subgroup of G (i.e., its roots form a closed root subsystem of those of G).

1.4. Satake isomorphisms. As before, G denotes a connected split reductive group over $\mathbb{Z}$. Let F be a local non-Archimidean field with ring of integers $\mathcal{O}$. We impose the following restrictions on the residue characteristic of F.

Convention 6. For every irreducible direct factor of the root system of G, we assume that the residue characteristic of F is not one of the following primes:

(1.5)

Root system	Excluded primes
B_n, C_n, D_n	$\{2\}$
F_4, G_2, E_6, E_7	$\{2,3\}$
E_8	$\{2,3,5\}$

Theorem 7. *For every strongly parabolic character $\bar{\mu} : T(\mathcal{O}) \to \mathbb{C}^\times$, there exists a compact open subgroup $K < G(\mathcal{O})$ and an extension $\mu : K \to \mathbb{C}^\times$ such that*

$$\mathscr{H}(G(F), K, \mu) \simeq \mathbb{C}[\check{T}/W_{\bar{\mu}}] \tag{1.6}$$

As mentioned above, in the case of $G = \mathrm{GL}_N$, the above theorem is due to Howe [How73], and if $\bar{\mu}$ is regular, then the above theorem follows by combining results of Bernstein [Ber84], [Ber92], Bushnell-Kutzko [BK98], [BK99] and Roche [Roc98]. As far as we know, the generalization to strongly parabolic characters is new.

Example 8. Let $G = \mathrm{GL}_3$ and let $T(\mathcal{O}) \simeq (\mathcal{O}^\times)^3$ denote the group of diagonal matrices. Write $\bar{\mu} = (\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3)$ where each $\bar{\mu}_i$ is a smooth character $\mathcal{O}^\times \to \mathbb{C}^\times$. Suppose $\bar{\mu}_1 = \bar{\mu}_2$ and that the conductor $\mathrm{cond}(\bar{\mu}_1/\bar{\mu}_3)$ equals $n \geq 2$. (The conductor of a character $\chi : \mathcal{O}^\times \to \mathbb{C}^\times$ is the smallest positive integer c for which $\chi(1 + \mathfrak{p}^c) = \{1\}$.) If we follow Howe's approach, we would take

$$K = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathcal{O} \end{pmatrix} \cap G(\mathcal{O}).$$

On the other hand, in the present article, following more closely the types of [Roc98], we take instead

$$K = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathfrak{p}^{[\frac{n}{2}]} \\ \mathcal{O} & \mathcal{O} & \mathfrak{p}^{[\frac{n}{2}]} \\ \mathfrak{p}^{[\frac{n+1}{2}]} & \mathfrak{p}^{[\frac{n+1}{2}]} & \mathcal{O} \end{pmatrix} \cap G(\mathcal{O}).$$

In both cases, $\bar{\mu}$ extends to a character $\mu : K \to \mathbb{C}^\times$ and one has an isomorphism $\mathscr{H}(G(F), K, \mu) \simeq \mathbb{C}[\check{T}]$. This example shows that the subgroup K of Theorem 7 is not necessarily unique.

To prove Theorem 7, we use Roche's result on types for principal series representations. Given an arbitrary smooth character $\bar{\mu} : T(\mathcal{O}) \to \mathbb{C}^\times$, Roche [Roc98] constructed a compact open subgroup $J \subset G(F)$ (which depends on the choice of a Borel B) and an extension $\mu^J : J \to \mathbb{C}^\times$ such that the compactly induced representation

$$\mathscr{W} := \mathrm{ind}_J^{G(F)} \mu^J \tag{1.7}$$

is a progenerator for the principal series Bernstein block of G defined by $\bar{\mu}$. More precisely, a combination of results of Bushnell and Kutzko, Dat, and

Roche implies that in this situation, one has an explicit isomorphism of $G(F)$-modules

$$\Psi : \mathscr{W} \xrightarrow{\simeq} \Pi := \iota_{B(F)}^{G(F)} \left(\mathrm{ind}_{T(\mathcal{O})}^{T(F)} \bar{\mu} \right). \tag{1.8}$$

Here, ι denotes the functor of parabolic induction.[2] See §3.4 for the explicit description of Ψ. Note that the endomorphism algebra of $\mathscr{W}$ is canonically isomorphic with $\mathscr{H}(G(F), J, \mu^J)$.

Now suppose the character $\bar{\mu}$ is strongly parabolic. Let L denote the corresponding Levi and let $\mu^{L(F)} : L(F) \to \mathbb{C}^\times$ denote an extension of $\bar{\mu}$ to $L(F)$. Let $\mu^L = \mu^{L(\mathcal{O})} := \mu^{L(F)}|_{L(\mathcal{O})}$ denote its restriction to $L(\mathcal{O})$. We prove that $K = JL(\mathcal{O})$ is a subgroup of $G(F)$. Moreover, we show that there exists a canonical character $\mu : K \to \mathbb{C}^\times$ which extends μ^J and μ^L. Theorem 7 states that the Hecke algebra $\mathscr{H}(G(F), K, \mu)$, consisting of compactly supported (K, μ)-bi-invariant functions on $G(F)$, is isomorphic to $\mathbb{C}[\check{T}/W_{\bar{\mu}}]$. To prove this result, we realize $\mathscr{H}(G(F), K, \mu)$ as an endomorphism ring of a family of principal series representations, which we call a *central family*.

1.5. Central families. In order to prove Theorem 7, we will introduce a certain representation attached to strongly parabolic characters.

Definition 9. Let $\bar{\mu}$ be a strongly parabolic character with the corresponding Levi L. Let $K = JL(\mathcal{O})$ denote the corresponding compact open subgroup. The *central family* of principal series representations of G attached to $\bar{\mu}$ is defined by

$$\mathscr{V} := \mathrm{ind}_K^{G(F)} \mu. \tag{1.9}$$

We will now give an alternative description of $\mathscr{V}$. Let $P \supseteq B$ be a parabolic subgroup whose Levi is isomorphic to L. Let

$$\Theta := \iota_{P(F)}^{G(F)} \left(\mathrm{ind}_{L(\mathcal{O})}^{L(F)} \mu^L \right). \tag{1.10}$$

Theorem 10. *Under the assumptions of Theorem 7, we have a canonical isomorphism of $G(F)$-modules $\mathscr{V} \xrightarrow{\simeq} \Theta$.*

We prove the above theorem by identifying $\mathscr{V}$ and Θ with submodules of $\mathscr{W}$ and Π, respectively. Then, using the explicit description of Ψ in (1.8), we show that $\Psi|_{\mathscr{V}} : \mathscr{V} \to \Pi$ defines an isomorphism onto Θ. On the other hand, the endomorphism ring of $\mathscr{V}$ identifies with $\mathscr{H}(G(F), K, \mu)$. Thus, to prove Theorem 7, we need to compute the endomorphism algebra of Θ. To this end, we will use a theorem of Roche [Roc02] on parabolic induction of Bernstein blocks.

[2] Note that here and in (1.10), it does not matter if we use normalized or unnormalized parabolic induction since the representation being induced is isomorphic to its twist by any unramified character.

Remark 11. (i) As mentioned above, in this paper, we construct the pair (K, μ) satisfying requirement of Problem 1 by using Roche's pair (J, μ^J). In this case, the subgroup K depends only on the kernel of $\bar{\mu}$; that is, if $\ker(\bar{\mu}) = \ker(\bar{\mu}')$ then $K_{\bar{\mu}} = K_{\bar{\mu}'}$. In fact, it only depends on the conductors of the restrictions of $\bar{\mu}$ to the coroot subgroups (i.e., the minimal $c_\alpha \geq 1$ such that $\bar{\mu}|_{\alpha^\vee(1+\mathfrak{p}^{c_\alpha})}$ is trivial) together with the collection of roots α such that the entire restriction $\bar{\mu}|_{\alpha^\vee(\mathcal{O}^\times)}$ is trivial. This follows immediately from the construction of J; see §3.2.

(ii) The pair (J, μ^J) is a *type* for the Bernstein block $\mathscr{R}_{\bar{\mu}}(G)$. Types for Bernstein blocks are not, however, necessarily unique. Therefore, it is natural to wonder if our construction could work using a different type $(J', \mu^{J'})$. In the case $G = \mathrm{GL}_N$, this is true in view of the results of [How73], as we observed (for $N = 3$) in Example 8. We do not, however, pursue this question in the current text.

(iii) Note that $\mathscr{V}$ is a submodule of $\mathscr{W}$ and the latter is a progenerator for the principal series block corresponding to $\bar{\mu}$. According to Theorem 7, the endomorphism ring $\mathscr{H}$ of this family identifies with the center of the corresponding Bernstein block (which is isomorphic to the center of $\mathscr{H}(G(F), J, \mu^J)$, and hence isomorphic to $\mathbb{C}[\check{T}/W_{\bar{\mu}}]$; cf. §1.2). Moreover, one can show that, for generic maximal ideals $\mathfrak{m} \subset \mathscr{H}$, the $G(F)$-module $\mathscr{V}/\mathfrak{m}\mathscr{V}$ is an irreducible principal series representation. (We will neither prove nor use the last statement.)

1.6. FURTHER DIRECTIONS. The proof of Theorem 7 given in this paper is rather indirect; moreover, it relies on nontrivial results of Bernstein, Bushnell and Kutzko, Roche, and Dat. In a forthcoming paper [KS], we hope to give a direct proof of this theorem by writing an explicit support preserving isomorphism $\mathscr{H}(L(F), L(\mathcal{O}), \mu^L) \xrightarrow{\simeq} \mathscr{H}(G(F), K, \mu)$. In other words, we hope to prove Theorem 7 using combinatorics and the classical Satake isomorphism. This proof should also make clear the geometric nature of the group K and some of its double cosets in $G(F)$; in particular, we expect that it will help with the geometrization program (see below).

In Definition 9, for strongly parabolic characters of the compact maximal torus, we constructed "central families". The endomorphism ring of the central family identifies canonically with the center of the block defined by the character; moreover, generic irreducible representations in the block appear with multiplicity one in the central family. It would be interesting to find analogous central families for other Bernstein blocks.

It is well-known that the Satake isomorphism allows one to realize the local unramified Langlands correspondence. In more detail, let $\check{G}$ denote the complex reductive group which is the Langlands dual of G. Using the classical Satake isomorphism (1.1), one can show that we have a bijection

$$\boxed{\text{unramified irreducible representations of } G(F)} \leftrightarrow \boxed{\text{characters of } \mathscr{H}}$$

Combining this with the bijections

$$\boxed{\text{characters of } \mathscr{H}} \leftrightarrow \boxed{\text{points of } \check{T}/W} \leftrightarrow \boxed{\text{semisimple conjugacy classes in } \check{G}}$$

we obtain a bijection between unramified representations of $G(F)$ and semisimple conjugacy classes in $\check{G}$. It would be interesting to study the role of the ramified Satake isomorphisms (i.e., the ones given by Theorem 7) in the local Langlands program.
In [HR10], a version of the Satake isomorphism for non-split groups is proved. On the other hand, there is also now a Satake isomorphism in characteristic p; see [Her11]. We expect that there is also a version of Theorem 7 for non-split groups and one in characteristic p.
Finally, we expect that there is a geometric version of Theorem 7. The geometric version of the usual Satake isomorphism is proved by Mirkovic and Vilonen [MV07], completing a project initiated by Lusztig, Beilinson and Drinfeld, and Ginzburg. In the case of regular characters; i.e., in the case that the stabilizer of the character in the Weyl group is trivial, a geometric version of Theorem 7 is proved in [KS11]. In [KS11, §1.4], we conjectured the theorems proved in this article; moreover, we formulated precise conjectures for geometrizing these results. We hope to return to this theme in future work.

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2. PARABOLIC, STRONGLY PARABOLIC, AND EASY CHARACTERS

2.1. CONVENTIONS. Let F be a local field with ring of integers $\mathcal{O}$, unique maximal ideal $\mathfrak{p}$, residue field $\mathbb{F}_q$, and uniformizer t. Let G be a connected split reductive group over $\mathbb{Z}$ with split maximal torus T. Let $W = N_G(T)/T$ denote the Weyl group.
Let $\Phi = \Phi_G$ denote the roots of G (with respect to T). For α a root in Φ, we write $\alpha^\vee$ for the corresponding coroot. Let $X = \mathrm{Hom}(T, \mathbb{G}_m)$ and $X^\vee = \mathrm{Hom}(\mathbb{G}_m, T)$ denote the the character and cocharacter lattices, respectively. Let $Q \subseteq X$ be the root lattice, and let $Q^\vee \subseteq X^\vee$ denote the coroot lattice. Let $(Q^\vee)_{\mathrm{sat}}$ be the saturation of $Q^\vee$ in $X^\vee$, i.e.,

$$(Q^\vee)_{\mathrm{sat}} = \{\lambda \in X^\vee \mid m \cdot \lambda \in Q^\vee, \text{some } m \in \mathbb{Z}\}.$$

By definition $X^\vee/(Q^\vee)_{\mathrm{sat}}$ is a torsion free abelian group. To an element $\lambda \in X^\vee$, we associate $t^\lambda = \lambda(t) \in T(F)$.

For every $\alpha \in \Phi$, let $u_\alpha : \mathbb{G}_a \to G$ be the one-parameter root subgroup, where $\mathbb{G}_a$ is the additive group. We assume these root subgroups satisfy the conditions specified in [Roc98, §2]. Let $U_\alpha < G$ be the image of u_α. For all $i \in \mathbb{Z}$, let $U_{\alpha,i} = u_\alpha(\mathfrak{p}^i) < G(F)$. In particular, $U_{\alpha,0} = u_\alpha(\mathcal{O})$.
Let H and K be topological groups and suppose $H < K$. Let $\chi : H \to \mathbb{C}^\times$ be a character of H. We write $\operatorname{ind}_H^K \chi$ for the space of left (H,χ)-invariant relatively compactly supported functions on K; that is, those functions $f : K \to \mathbb{C}$ whose support has compact image in K/H and satisfy $f(hk) = \chi(h)f(k)$ for all $h \in H$ and $k \in K$. The group K acts on this space by right translation.

2.2. W-invariant rational characters. We start this section with a general lemma which we will repeatedly use below.

Lemma 12. *Let H be a group and $K < H$ a subgroup. Then a character $\chi : K \to \mathbb{C}^\times$ extends to a character of H if and only if $\chi|_{K\cap[H,H]}$ is trivial. The same is true if H is an l-group (i.e., a locally compact totally disconnected Hausdorff topological group), K is a closed subgroup, and χ is smooth.*[3] *Finally, the same is true if H is a connected split reductive algebraic group, K is a closed subgroup, and $\chi : K \to \mathbb{G}_m$ is a rational character.*

Proof. It is clear that the assumption that χ be trivial on $K \cap [H,H]$ is necessary. Conversely, if this is true, extending the character is the same as extending the induced character of $K/(K\cap[H,H])$ to $H/[H,H]$. Therefore, all the statements of the lemma reduce to the case that H is commutative.
Then, the statement that any character of a subgroup of an abstract (discrete) abelian group extends to the entire group follows from the fact that $\mathbb{C}^\times$ is divisible, and hence injective.
For the locally compact analogue, write $\mathbb{C}^\times \cong S^1 \times \mathbb{R}_{>0}$. For characters to S^1, the statement follows from Pontryagin duality. For $\mathbb{R}_{>0}$, note first that, if H is compact, then there are no nontrivial continuous characters to $\mathbb{R}_{>0}$. As an l-group always contains a compact open subgroup, this reduces the problem to the case H is discrete, where it follows as in the previous paragraph, since $\mathbb{R}_{>0}$ is divisible, and hence injective, as a discrete abelian group.
For the algebraic analogue, i.e., where H and K are connected split tori, the statement follows because applying $\operatorname{Hom}(-,\mathbb{G}_m)$ to a short exact sequence $1 \to K \to H \to H/K \to 1$ of split tori is well-known to be an equivalence of short exact sequences of split tori with that of their weight lattices. Hence, the restriction map from characters of H to characters of K is surjective. □

Lemma 13. *Let G be a connected split reductive algebraic group over $\mathbb{Z}$ with split maximal torus T. Let $\chi : T \to \mathbb{G}_m$ be a rational character. The following are equivalent:*

(1) *χ is trivial on $T \cap [G,G]$.*
(2) *χ extends to a character $G \to \mathbb{G}_m$;*

[3] We don't need to assume that H is totally disconnected, if we use the fact [HM06, Corollary 7.54] that every locally compact Hausdorff topological group contains a compact subgroup H' such that the quotient H/H' is isomorphic to $\mathbb{R}^n \times D$ for a discrete group D.

(3) χ *is* W*-invariant;*
(4) $\chi \circ \alpha^\vee : \mathbb{G}_m \to \mathbb{G}_m$ *is trivial, for every* $\alpha \in \Phi$.

Proof. Lemma 12 implies immediately that (1) $\Longrightarrow$ (2). Next, it is clear that $[N_G(T), T] \subseteq [G,G] \cap T$; therefore, if we restrict a character of G to T, we obtain a character which is invariant under the conjugation action of $N_T(G)$. This proves (2) $\Longrightarrow$ (3). Next, suppose χ is W-invariant. Then

$$\chi \circ \alpha^\vee = (s_\alpha . \chi) \circ \alpha^\vee = \chi \circ (-\alpha^\vee) = (\chi \circ \alpha)^{-1} \tag{2.1}$$

It follows that $(\chi \circ \alpha^\vee)^2 = 1$. Since $\mathbb{G}_m$ has no nontrivial character of order 2, it follows that $\chi \circ \alpha^\vee = 1$. Hence, (3) $\Longrightarrow$ (4). For the final implication, we use the canonical identification

$$T \cap [G,G] = \mathbb{G}_m \otimes_{\mathbb{Z}} (Q^\vee)_{\mathrm{sat}}. \tag{2.2}$$

By the notation on the RHS we mean the group subscheme of T whose R points equals $R^\times \otimes_{\mathbb{Z}} (Q^\vee)_{\mathrm{sat}}$, where R is a ring over k.[4] Now if $\chi \circ \alpha^\vee$ is trivial for every $\alpha \in \Phi$, then χ is trivial on $T \cap [G,G]$. This proves (4) $\Longrightarrow$ (1). $\square$

Remark 14. It follows from the above lemma that the group of characters of T which satisfy the above equivalent conditions is canonically isomorphic to

(2.3)
$$\mathrm{Hom}(T/(T \cap [G,G]), \mathbb{G}_m) \simeq X^W \simeq \mathrm{Hom}(X^\vee/Q^\vee, \mathbb{Z}) \simeq \mathrm{Hom}(X^\vee/(Q^\vee)_{\mathrm{sat}}, \mathbb{Z}).$$

The last isomorphism follows from the following: the quotient $X^\vee/Q^\vee \twoheadrightarrow X^\vee/(Q^\vee)_{\mathrm{sat}}$ splits, since $X^\vee/(Q^\vee)_{\mathrm{sat}}$ is free, and the resulting pullback maps $\mathrm{Hom}(X^\vee/Q^\vee, \mathbb{Z}) \leftrightarrow \mathrm{Hom}(X^\vee/(Q^\vee)_{\mathrm{sat}}, \mathbb{Z})$ are inverse to each other since the quotient $X^\vee/Q^\vee \twoheadrightarrow X^\vee/(Q^\vee)_{\mathrm{sat}}$ has finite kernel and $\mathbb{Z}$ is torsion-free. (More generally, for any finite-kernel quotient of finitely-generated abelian groups, the pullback map on $\mathrm{Hom}(-, \mathbb{Z})$ is an isomorphism.)

2.3. EASY CHARACTERS. Let G be a connected split reductive group defined over $\mathbb{Z}$. Let $\mathrm{Hom}_{\mathrm{sm}}(\mathcal{O}^\times, \mathbb{C}^\times)$ denote the group of smooth characters $\mathcal{O}^\times \to \mathbb{C}^\times$.

PROPOSITION 15. *The following conditions are equivalent for a smooth character* $\bar{\mu} : T(\mathcal{O}) \to \mathbb{C}^\times$*:*

(i) *The restriction* $\bar{\mu}|_{([G,G]\cap T)(\mathcal{O})}$ *is trivial;*
(ii) *The character* $\bar{\mu}$ *is a product of compositions of* W*-invariant rational characters* $T(\mathcal{O}) \to \mathcal{O}^\times$ *with smooth characters* $\mathcal{O}^\times \to \mathbb{C}^\times$.

Remark 16. The same statement and proof holds when $\mathcal{O}$ is replaced by any (topological) ring.

DEFINITION 17. A smooth character $\bar{\mu} : T(\mathcal{O}) \to \mathbb{C}^\times$ is *easy* with respect to G if the equivalent conditions of Proposition 15 are satisfied.

Note that, in particular, any easy character with respect to G must be W-invariant.

[4] For a proof of this statement over an algebraically closed field see, for instance, [DM91], §0.20. Note that over an algebraically closed field, one does not need to use $(Q^\vee)_{\mathrm{sat}}$; more precisely, we have $\bar{k}^\times \otimes_{\mathbb{Z}} Q^\vee = T(\bar{k}) \cap [G,G](\bar{k}) = (T \cap [G,G])(\bar{k})$.

Remark 18. By Lemma 13 and Proposition 15, the group of easy characters of $T(\mathcal{O})$ identifies canonically with

$$\mathrm{Hom}_{\mathrm{sm}}((T/(T\cap[G,G]))(\mathcal{O}),\mathbb{C}^\times)\simeq$$
$$\mathrm{Hom}(X^\vee/(Q^\vee)_{\mathrm{sat}},\mathbb{Z})\otimes_{\mathbb{Z}}\mathrm{Hom}_{\mathrm{sm}}(\mathcal{O}^\times,\mathbb{C}^\times)\simeq\mathrm{Hom}_{\mathrm{sm}}(X^\vee/(Q^\vee)_{\mathrm{sat}}\otimes_{\mathbb{Z}}\mathcal{O}^\times,\mathbb{C}^\times).$$

The last isomorphism follows from the fact that $X^\vee/(Q^\vee)_{\mathrm{sat}}$ is free.

Proof of Proposition 15. The implication (ii) $\Rightarrow$ (i) is immediate from Lemma 13. For the reverse implication, note that by assumption $\bar{\mu}$ is a character of $T(\mathcal{O})=(\mathbb{G}_m\otimes_{\mathbb{Z}}X^\vee)(\mathcal{O})$ which is trivial on $([G,G]\cap T)(\mathcal{O})=(\mathbb{G}_m\otimes_{\mathbb{Z}}(Q^\vee)_{\mathrm{sat}})(\mathcal{O})$. Therefore $\bar{\mu}$ is canonically a character of

$$(\mathbb{G}_m\otimes_{\mathbb{Z}}X^\vee)(\mathcal{O})/(\mathbb{G}_m\otimes_{\mathbb{Z}}(Q^\vee)_{\mathrm{sat}})(\mathcal{O})=$$
$$((\mathbb{G}_m\otimes_{\mathbb{Z}}X^\vee)/(\mathbb{G}_m\otimes_{\mathbb{Z}}(Q^\vee)_{\mathrm{sat}}))(\mathcal{O})=$$
$$(\mathbb{G}_m\otimes_{\mathbb{Z}}X^\vee/(Q^\vee)_{\mathrm{sat}})(\mathcal{O})=(X^\vee/(Q^\vee)_{\mathrm{sat}})\otimes_{\mathbb{Z}}\mathcal{O}^\times.$$

We conclude that $\bar{\mu}$ is a product of compositions of (smooth) characters of $\mathcal{O}^\times$ with rational characters $\mathrm{Hom}(X^\vee/(Q^\vee)_{\mathrm{sat}},\mathbb{Z})$. By Remark 14, the group of such rational characters is canonically isomorphic to the sublattice X^W of W-invariant rational characters. Therefore, $\bar{\mu}$ has the form claimed in Part (ii). $\square$

Note that, if $\bar{\mu}$ is easy, then Lemma 13 implies that $\bar{\mu}$ extends to a character of $G(F)$, and hence of $G(\mathcal{O})$. As the following example illustrates, the converse is not, in general, true.

Example 19. Let $G=PGL_2$. Then the determinant map $\mathrm{GL}_2(F)\to F^\times$ descends to a map $G(F)\to F^\times/(F^\times)^2\cong\{\pm1\}\times t^{X^\vee}/t^{2X^\vee}$. Take the composition and the further quotient by the second factor, and view it as a character $G(F)\to\mathbb{C}^\times$ (which is trivial on $t^{X^\vee}$). The restriction of this character to $T(\mathcal{O})$ is non-trivial, even though there are no nonzero W-invariant rational characters (and hence no nontrivial easy characters).

Nonetheless, in the next subsection, we give a combinatorial description of all characters of $T(\mathcal{O})$ which extend to characters of $G(F)$, similar to the description of easy characters above.

2.4. Extendable characters.

Proposition 20. *The following conditions are equivalent for a smooth character $\bar{\mu}:T(\mathcal{O})\to\mathbb{C}^\times$:*

(a) *$\bar{\mu}$ extends to a character of $G(F)$;*
(b) *For all $\alpha\in\Phi$, we have*

$$\bar{\mu}\circ\alpha^\vee|_{\mathcal{O}^\times}=1. \tag{2.4}$$

If in addition $q>2$, then these are also equivalent to

(c) *$\bar{\mu}$ extends to a character of $G(\mathcal{O})$.*

Definition 21. A smooth character $\bar{\mu} : T(\mathcal{O}) \to \mathbb{C}^\times$ is said to be *extendable* (to $G(F)$) if it satisfies the equivalent conditions of the above proposition.

Note, in particular, that any extendable character to $G(F)$ must be W-invariant (as is easy to verify from either of the conditions above: for (a), see the proof of Corollary 27, and for (b), see (i') in the proof of Lemma 30). We will not need this fact in this subsection.

Remark 22. (i) Let R be an arbitrary ring. Since the span $\langle \alpha^\vee \rangle_{\alpha \in \Phi}$ is, by definition, the coroot lattice $Q^\vee \subseteq X^\vee$, we have $\langle \alpha^\vee(R^\times) \rangle = Q^\vee \otimes_{\mathbb{Z}} R$. Also note that $T = X^\vee \otimes_{\mathbb{Z}} \mathbb{G}_m$; therefore, $T(R) = X^\vee \otimes_{\mathbb{Z}} R^\times$.

(ii) It follows from the previous remark and the above proposition that the group of extendable (to $G(F)$) characters of $T(\mathcal{O})$ identifies with

$$\begin{aligned}\mathrm{Hom}_{\mathrm{sm}}(T(\mathcal{O})/\langle \alpha^\vee(\mathcal{O}^\times)\rangle_{\alpha\in\Phi}, \mathbb{C}^\times) \\ = \mathrm{Hom}_{\mathrm{sm}}((X^\vee \otimes_{\mathbb{Z}} \mathcal{O}^\times)/(Q^\vee \otimes_{\mathbb{Z}} \mathcal{O}^\times), \mathbb{C}^\times) \\ \simeq \mathrm{Hom}_{\mathrm{sm}}(X^\vee/Q^\vee \otimes_{\mathbb{Z}} \mathcal{O}^\times, \mathbb{C}^\times).\end{aligned}$$

The last isomorphism holds because $(X^\vee \otimes_{\mathbb{Z}} \mathcal{O}^\times)/(Q^\vee \otimes_{\mathbb{Z}} \mathcal{O}^\times) = X^\vee/Q^\vee \otimes_{\mathbb{Z}} \mathcal{O}^\times$ (by definition of $\otimes$, or by its right-exactness). Note that $X^\vee/Q^\vee \otimes_{\mathbb{Z}} \mathcal{O}^\times$ has a topology which is induced by the topology on $\mathcal{O}^\times$. Therefore, we can speak of its smooth characters.

To prove Proposition 20, we will prove the following:

Proposition 23. (i) *Let R be an arbitrary ring. Then, we have an inclusion of groups*

$$[G(R), G(R)] \cap T(R) \subseteq \langle \alpha^\vee(R^\times) \rangle_{\alpha\in\Phi}. \tag{2.5}$$

(ii) *If $R = \mathcal{O}$ for $q > 2$ or $R = F$ for arbitrary q, then*

$$[G(R), G(R)] \cap T(R) \supseteq \langle \alpha^\vee(R^\times) \rangle_{\alpha\in\Phi}. \tag{2.6}$$

Proof of Proposition 20. In view of Lemma 12, part (i) of Proposition 23 is equivalent to the statement that all characters of $T(R)$ trivial on $\langle \alpha^\vee(R) \rangle_{\alpha\in\Phi}$ extend to characters of $G(R)$; the same holds in the context R is an l-group and we restrict to smooth characters (because, in both contexts, any nontrivial abelian group admits a nontrivial character). Thus, condition (b) of Proposition 20 implies condition (c). Moreover, note that every character of $T(\mathcal{O})$ trivial on $\langle \alpha^\vee(\mathcal{O}) \rangle_{\alpha\in\Phi}$ extends to a character of $T(F)$ trivial on $\langle \alpha^\vee(F) \rangle_{\alpha\in\Phi}$; a unique such extension is given by requiring the character to be trivial on $t^{X^\vee}$. Thus, condition (b) of Proposition 20 implies condition (a).
Similarly, part (ii) of Proposition 23 is equivalent to the statement that every character of $G(R)$ in the cases mentioned must be trivial on $\langle \alpha^\vee(R) \rangle_{\alpha\in\Phi}$. Thus, condition (a) of Proposition 20 implies condition (b), and also (c) implies (b). Thus, Proposition 23 implies Proposition 20. □

The rest of this subsection is devoted to proving Proposition 23. To this end, we need to recall some facts about the universal group cover of the commutator subgroup.

2.4.1. *Universal cover of* $[G,G]$. Let $\tilde{G}$ be the connected reductive algebraic group over $\mathbb{Z}$ such that $\tilde{G}(\mathbb{C})$ is the universal cover of $[G,G](\mathbb{C})$. More precisely, this is the group corresponding to the root datum $(\mathrm{Hom}_{\mathbb{Z}}(Q^\vee,\mathbb{Z}),\Phi,Q^\vee,\Phi^\vee)$. By [DG93], cf. [Con11, §6], the category of split reductive groups over an arbitrary scheme is equivalent to the category of root data, so one can define $\tilde{G}$ as a split reductive group over $\mathbb{Z}$. Let π denote the canonical morphism $\tilde{G}\to G$ of group schemes over $\mathbb{Z}$ (i.e., the composition of the canonical isogeny $\tilde{G}\to[G,G]$, cf. [Con11, Theorem 6.1.6(1)], and the inclusion $[G,G]\subseteq G$).
We claim that the commutator morphism $G\times G\to G$ factors through the map $\tilde{G}\to G$. Let $G^{\mathrm{ad}}:=G/Z(G)$, where $Z(G)$ is the center of G. This coincides with $(\tilde{G})^{\mathrm{ad}}:=\tilde{G}/Z(\tilde{G})$, i.e., the natural map $(\tilde{G})^{\mathrm{ad}}\to G^{\mathrm{ad}}$ is an isomorphism. The commutator map $\tilde{G}\times\tilde{G}\to\tilde{G}$ descends to $(\tilde{G})^{\mathrm{ad}}\times(\tilde{G})^{\mathrm{ad}}=G^{\mathrm{ad}}\times G^{\mathrm{ad}}$. Composing with the quotient $G\times G\to G^{\mathrm{ad}}\times G^{\mathrm{ad}}$, we obtain a morphism $G\times G\to\tilde{G}$ whose composition with $\tilde{G}\to G$ is evidently the commutator morphism, as desired. We note that the above argument that the commutator map lifts to $\tilde{G}$ also appears in [Del, §2.0.2]. On the other hand, the first author of this paper also studied the group $\tilde{G}$ under the name of "true commutator" of G; see [Kam09, §B.2 and Corollary 4.4].
Next, let $\tilde{T}=\pi^{-1}(T)$. Then $\tilde{T}$ is a commutative subgroup of $\tilde{G}$ containing a maximal split torus. Therefore, $\tilde{T}$ is itself a maximal split torus.

Lemma 24. *For every ring R, $\pi(\tilde{T}(R))=\langle\alpha^\vee(R^\times)\rangle_{\alpha\in\Phi}$.*

Proof. The cocharacter lattice of $\tilde{G}$ equals its coroot lattice. Therefore, by Remark 22.(i), we have

$$\tilde{T}(R)=X^\vee_{\tilde{G}}\otimes_{\mathbb{Z}}R^\times=Q^\vee_{\tilde{G}}\otimes_{\mathbb{Z}}R^\times=\langle\alpha^\vee(R^\times)\rangle_{\alpha\in\Phi}$$

Now the set of roots for $\tilde{G}$ and G coincide. If, abusively, by $\alpha^\vee$ we also denote the corresponding subgroup of T, then we see that $\pi(\tilde{T}(R))=\langle\alpha^\vee(R^\times)\rangle_{\alpha\in\Phi}$, as required. □

2.4.2. *Proof of Proposition 23.* Part (i) of the proposition follows easily from Lemma 24: since the commutator map factors through π, we have $[G(R),G(R)]\subseteq\pi(\tilde{G}(R))$; hence, $[G(R),G(R)]\cap T(R)\subseteq\pi(\tilde{T}(R))$. By Lemma 24, $\pi(\tilde{T}(R))$ equals the RHS of (2.5).
To prove part (ii) of the proposition, we first reduce to the case $G=\mathrm{SL}_2$:

Lemma 25. *For a fixed ring R, the inclusion* (2.6) *holds for all G if and only if it holds for $G=\mathrm{SL}_2$.*

Proof. For each $\alpha\in\Phi$, we have $[G(R),G(R)]\supseteq\alpha^\vee(R^\times)$ if this fact holds when G is replaced by the centralizer of the root $\alpha:T\to\mathbb{G}_m$, which is split connected reductive of semisimple rank one (in fact, it is the subgroup generated by T and the root subgroups $U_{\pm\alpha}$). So we can assume that G has semisimple rank one. Then, the group $\tilde{G}$ is isomorphic to SL_2. In view of the morphism $\tilde{G}(R)\to G(R)$, it suffices to prove (2.6) for $\tilde{G}\cong\mathrm{SL}_2$. So once we establish (2.6) for SL_2, it follows for all split reductive groups G. □

Thus part (ii) reduces to:

Lemma 26. *The inclusion* (2.6) *holds for* $G = \mathrm{SL}_2$ *in the case that either* $R = \mathcal{O}$ *for* $q > 2$ *or* R *is a field.*

Proof. Let $f \in R^\times$ and let α be the positive simple root. Then one can verify that

(2.7)
$$\begin{pmatrix} 1 & 0 \\ -(f-1)^2/f & 1 \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ f-1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} 1 & (1-f)/f \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} f & 0 \\ 0 & f^{-1} \end{pmatrix} = \alpha^\vee(f).$$

We now consider the question of when the first and last matrices on the LHS are in $[G(R), G(R)]$. Generally, for $g \in R^\times$,

$$\left[\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \right] = \begin{pmatrix} 1 & 0 \\ x(1-g^2) & 1 \end{pmatrix}. \tag{2.8}$$

Let $I_R \coloneqq (1-g^2 \mid g \in R^\times) = \{x(1-g^2) \mid x \in R,\, g \in R^\times\}$ be the ideal of elements appearing in the lower-left entry of the final matrix. If this ideal is the unit ideal, then the LHS of (2.7) is in $[G(R), G(R)]$, as desired. This is clearly true if R is a field such that $|R| > 3$. It therefore also true if R is a discrete valuation ring whose residue field has cardinality greater than 3 (then let g be a unit of R whose image $\bar{g}$ in the residue field has the property that $1 - \bar{g}^2$ is invertible, so that $1 - g^2$ itself is invertible).
Therefore, we only need to show that, for $R = \mathcal{O}$ for $q = 3$, or $R = \mathbb{F}_q$ for $q \le 3$, then (2.6) holds.
First, if $R = \mathbb{F}_q$ and $q = 2$, there is nothing to show because now $T(\mathbb{F}_q)$ is trivial. If $R = \mathbb{F}_q$ for $q = 3$, then it is well known that $[G(\mathbb{F}_q), G(\mathbb{F}_q)]$ has index three in $G(\mathbb{F}_q)$; since $T(\mathbb{F}_q)$ has order two, it follows that $T(\mathbb{F}_q)$ must be in the kernel of the abelianization map $G(\mathbb{F}_q) \to G(\mathbb{F}_q)/[G(\mathbb{F}_q), G(\mathbb{F}_q)]$, i.e., that $[G(\mathbb{F}_q), G(\mathbb{F}_q)] \supseteq T(\mathbb{F}_q)$. This completes the proof of the lemma for R equal to a field.
Finally, suppose that $R = \mathcal{O}$ and $q = 3$. Since we already showed that $[G(\mathbb{F}_q), G(\mathbb{F}_q)] \supseteq \alpha^\vee(\mathbb{F}_q^\times)$, (2.6) will follow if we show that $[G(\mathcal{O}), G(\mathcal{O})] \supseteq \alpha^\vee(1+\mathfrak{p})$.
To see this, more generally when q is odd, we claim that $I_{\mathcal{O}} \supseteq \mathfrak{p}$. Indeed, the squaring operation is bijective on $1+\mathfrak{p}$, for all $z \in \mathfrak{p}$. So, we can take $g \in \mathcal{O}$ such that $g^2 = 1 + z$, and hence $z \in I_R$.
Hence, we can apply (2.8) to the case $f \in 1+\mathfrak{p}$, and we conclude that $\alpha^\vee(1+\mathfrak{p}) \subseteq [G(\mathcal{O}), G(\mathcal{O})]$, as desired. □

2.5. Comparison between easy and extendable.

Corollary 27. *Let* $\bar{\mu} : T(\mathcal{O}) \to \mathbb{C}^\times$ *be a smooth character of* G. *Then*

$$\bar{\mu} \text{ is easy for } G \implies \bar{\mu} \text{ is extendable to } G(F) \implies \bar{\mu} \text{ is } W\text{-invariant} \implies (\bar{\mu} \circ \alpha^\vee|_{\mathcal{O}^\times})^2 = 1,\ \forall \alpha \in \Phi.$$

Proof. The first implication was already discussed before Example 19. The second implication follows from the facts $W \simeq N_{G(\mathcal{O})}(T(\mathcal{O}))/T(\mathcal{O})$ and $[N_{G(\mathcal{O})}(T(\mathcal{O})), T(\mathcal{O})] \subseteq [G(\mathcal{O}), G(\mathcal{O})] \cap T(\mathcal{O})$. For the last implication, note that $(\bar{\mu} \circ \alpha^\vee)^2(x) = \bar{\mu}([\alpha^\vee(x), s_\alpha])$, where s_α is any lift to $N_{G(\mathcal{O})}(T(\mathcal{O}))$ of the simple reflection s_α. □

The reverse implications can all fail. For the first implication, see Example 19. For the remaining two, we have the following:

Example 28. (i) Let $G = \mathrm{SL}_2$. Let $\bar{\mu} : T(\mathcal{O}) \to \mathbb{C}^\times$ denote the composition

$$T(\mathcal{O}) \simeq \mathcal{O}^\times \to \mathcal{O}^\times/(\mathcal{O}^\times)^2 \xrightarrow{\theta} \mathbb{C}^\times,$$

where θ is a nontrivial character. Then $\bar{\mu} : T(\mathcal{O}) = \mathcal{O}^\times \to \mathbb{C}^\times$ is W-invariant; however, it does not extend to $G(F)$ by Proposition 20.

(ii) [Roc98, Example 8.4] Let $G = \mathrm{Sp}_{2n}$, $n \geq 2$. Identify $T(\mathcal{O})$ with $(\mathcal{O}^\times)^n$, and let $\bar{\mu} = (\theta, \cdots, \theta)$. Then $\bar{\mu}$ is W-invariant; however, it does not extend to $G(F)$. This is because, as observed in [Roc98, Example 8.4], the composition $\bar{\mu} \circ \alpha^\vee$ is not trivial for all α (and in fact, the root subsystem whose coroots have trivial composition produces an endoscopic group SO_{2n}, which is not a subgroup of G).

Example 29. Let $G = \mathrm{SL}_3$. Define

$$\bar{\mu}(\mathrm{diag}(a, b, a^{-1}b^{-1})) = \theta(a)\theta(b), \quad a, b \in \mathcal{O}^\times,$$

where θ is a nontrivial quadratic character of $\mathcal{O}^\times$. By assumption, $(\bar{\mu} \circ \alpha^\vee)^2 = 1$ for both coroots of G; however, $\bar{\mu}$ is not invariant under the transformation $(a, b, a^{-1}b^{-1}) \mapsto (a^{-1}b^{-1}, b, a)$; in particular, it is not W-invariant.

In certain situations, either (or both) of the first two implications in the above corollary become biconditionals.

Lemma 30. (i) *Suppose that $Q^\vee = \langle \lambda - w(\lambda) \mid \lambda \in X^\vee, w \in W \rangle$. Then every W-invariant character of $T(\mathcal{O})$ is extendable to $G(F)$.*

(i') *The hypothesis of (i) is equivalent to the statement that, for some choice of simple roots α_i (or, equivalently, for any choice of simple roots), there exist cocharacters $\lambda_i \in X^\vee$ such that $\langle \lambda_i, \alpha_i \rangle = 1$. Moreover, this condition is implied by either of the following:*

  (a) *X/Q is free*

  (b) *The root system of G has no factors of type A_1 or C_n.*

(ii) *Suppose $X^\vee/Q^\vee$ is torsion-free. Then every extendable character of $T(\mathcal{O})$ (to $G(F)$) is easy.*

Proof. (i) If the coroot lattice equals the span of the elements $\lambda - w(\lambda)$ for $w \in W$ and $\lambda \in X^\vee$, then (2.4) is satisfied. This is because W-invariance implies $\bar{\mu}(\lambda(x)) = \bar{\mu}(w(\lambda)(x))$ for all $x \in \mathbb{G}_m(\mathcal{O})$, and hence $\bar{\mu}((\lambda - w(\lambda))(x)) = 1$ for all $x \in \mathbb{G}_m(\mathcal{O})$.

(i') First, we claim that $Q^\vee \supseteq \langle \lambda - w(\lambda) \mid \lambda \in X^\vee, w \in W\rangle$. Let $\alpha_i, i \in I$ be a choice of simple roots. Since W is generated by the s_{α_i},

$$\langle \lambda - w(\lambda) \mid \lambda \in X^\vee, w \in W\rangle = \langle \lambda - s_{\alpha_i}\lambda \mid \lambda \in X^\vee, i \in I\rangle = \langle \langle \lambda, \alpha_i\rangle \alpha_i^\vee \mid \lambda \in X^\vee, i \in I\rangle.$$

This proves the desired containment. So, we need to show that the opposite inclusion is equivalent to the condition stated in (i').
Given λ_i such that $\langle \lambda_i, \alpha_i\rangle = 1$, we obviously get $\alpha_i^\vee$ in the RHS of the above equation. Conversely, if $\alpha_i^\vee \in \langle \langle \lambda, \alpha_j\rangle \alpha_j^\vee \mid \lambda \in X^\vee, j \in I\rangle$, then there must exist $\lambda_i \in X^\vee$ such that $\langle \lambda_i, \alpha_i\rangle = 1$. Applying this to all i yields the desired equivalence (since $Q^\vee$ is spanned by the $\alpha_i^\vee$).
(a) If X/Q is torsion-free, then Q must be saturated in X, so the condition (i') is satisfied.
(b) For the root system A_2, with simple roots α_1 and α_2, one has $s_{\alpha_1}(\alpha_2^\vee) - \alpha_2^\vee = \alpha_1^\vee$, and similarly with indices 1 and 2 swapped, so that one concludes that $\alpha_1^\vee, \alpha_2^\vee \in \langle \lambda - w(\lambda)\rangle$ and hence $Q^\vee = \langle \lambda - w(\lambda)\rangle$. The same argument shows that, for every root system in which every simple root is contained in a root subsystem of type A_2, then every coroot is contained in $\langle \lambda - w(\lambda)\rangle$ and hence (i) is also satisfied.
This takes care of all root systems except for types A_1, B_n, C_n, and G_2. For type B_n with $n \geq 3$, the above argument shows that, for the standard choice of simple roots $\alpha_1, \dots, \alpha_n$ where α_n is the short simple root, then $\alpha_i^\vee \in \langle \lambda - w(\lambda)\rangle$ for $i < n$, since these are incident to a subdiagram of type A_2; for $\alpha_n^\vee$, it is still true that $s_{\alpha_n}(\alpha_{n-1}^\vee) - \alpha_{n-1}^\vee = \alpha_n^\vee$, so also $\alpha_n^\vee \in \langle \lambda - w(\lambda)\rangle$. For type G_2, if the simple roots are α_1 and α_2, we see that $s_{\alpha_1}(\alpha_1^\vee + \alpha_2^\vee) - (\alpha_1^\vee + \alpha_2^\vee) = \pm\alpha_1^\vee$, so $\alpha_1^\vee \in \langle \lambda - w(\lambda)\rangle$, and the same fact holds (with opposite sign) when indices 1 and 2 are swapped. Note also that $B_2 = C_2$, so we do not need to separately exclude B_2.
(ii) The hypothesis is equivalent to the condition that $Q^\vee$ is saturated in $X^\vee$; i.e., $Q^\vee = (Q^\vee)_{\mathrm{sat}}$. The result then follows from Remarks 18 and 22. □

2.6. On parabolic characters. Recall that a smooth character $\bar{\mu} : T(\mathcal{O}) \to \mathbb{C}^\times$ is parabolic if its stabilizer in W is a parabolic subgroup. Here is an example of a character which is not parabolic.

Example 31. (cf. [Roc98, Example 8.3], due to Sanje-Mpacko) Let $N \geq 3$ and $G = \mathrm{SL}_N$. Define

$$\bar{\mu}(\mathrm{diag}(a_1, a_2, \dots, a_{N-1}, a_1^{-1} \cdots a_{N-1}^{-1})) = \chi(a_1)\chi^2(a_2)\cdots\chi^{N-1}(a_{N-1}),$$

where $\chi : \mathcal{O}^\times \to \mathbb{C}^\times$ is a character of order N. Then the stabilizer of $\bar{\mu}$ in W is the subgroup $\mathbb{Z}/N$ of cyclic permutations, which is not parabolic.

On the other hand, as the following proposition illustrates, in certain situations all characters are parabolic.

Proposition 32. *Let G be a connected simply laced split reductive group over $\mathbb{Z}$. If X/Q is free, then every smooth character of $T(\mathcal{O})$ is strongly parabolic. If, moreover, $X^\vee/Q^\vee$ is free, then every smooth character of $T(\mathcal{O})$ is easy.*

Proof. Let $\Phi_{\bar{\mu}}$ denote the collection of roots $\alpha \in \Phi$ such that $\bar{\mu} \circ \alpha^\vee = 1$. We claim that $\Phi_{\bar{\mu}}$ is a closed root subsystem. Indeed, if $\alpha, \beta \in \Phi_{\bar{\mu}}$ and $\langle \alpha, \beta \rangle = -1$, then $(\alpha + \beta)^\vee = \alpha^\vee + \beta^\vee$, and so $\alpha + \beta \in \Phi_{\bar{\mu}}$ as well. Let L denote the Levi subgroup corresponding to $\Phi_{\bar{\mu}}$. It follows from Proposition 20 along with [Roc98], Lemma 8.1.(i) and the comment at the end of p. 395, that $\bar{\mu}$ is strongly parabolic with Levi L (cf. Remark 5). Alternatively, if we use only from [Roc98] that $\bar{\mu}$ is parabolic, then we can apply Lemma 30 to deduce strong parabolicity. For the final statement, we again apply Lemma 30. □

2.7. PROOF OF THEOREM 3. Parts (i) and (ii) follow from Propositions 15 and 20, respectively. Next, we need a basic fact from the theory of reductive groups.

LEMMA 33. *Let G be a connected split reductive group over $\mathbb{Z}$ with split maximal torus T. Let $L < G$ be a Levi containing T.*

(i) *If $X^\vee/Q^\vee$ is torsion free, so is $X^\vee/Q_L^\vee$.*
(ii) *If X/Q is torsion free, then so is X/Q_L.*
(iii) *If the equivalent conditions (i) or (i') of Lemma 30 are satisfied for G (i.e., $Q^\vee = \langle \lambda - w(\lambda) \mid \lambda \in X^\vee \rangle$ or, for some choice of simple roots α_i, there exist cocharacters $\lambda_i \in X^\vee$ such that $\langle \lambda_i, \alpha_i \rangle = 1$), then they are also satisfied when G is replaced by L.*

Proof. Parts (i) and (ii) follow from the fact that Q/Q_L and $Q^\vee/Q_L^\vee$ are always torsion free. For part (iii), we consider the condition (i'), i.e., the second condition. But, by definition, one can choose simple roots of L which form a subset of a choice of simple roots of G (and note that the (co)weight lattices are the same for L as for G). Hence condition (i') is satisfied for L. □

Then, parts (iii) and (iv) both follow from Lemmas 33 and 30. Finally, part (v) follows from Proposition 32 and Lemma 33.

3. CENTRAL FAMILIES AND SATAKE ISOMORPHISMS

3.1. RECOLLECTIONS ON DECOMPOSED SUBGROUPS. We begin this section with some general remarks on compact open subgroups of $G(F)$. Let P be a parabolic subgroup of G with Levi decomposition LU_P^+. Let $P^- = LU_P^-$ denote the opposite of P relative to L. (According to [Bor91, Proposition 14.21], the opposite parabolic is unique up to conjugation by a unique element of U_P^+.) In what follows, by a Levi subgroup of G we always mean a Levi for a parabolic subgroup containing T.

Let $J \subset G$ be a compact open subgroup. Let

$$J_P^+ = J \cap U_P^+(F), \quad J_P^0 = J \cap L(F), \quad J_P^- = J \cap U_P^-(F).$$

For a parabolic $P = LU_P^+$, we let Φ_P^+ denote the set of roots of U_P^+. Similarly, we let Φ_P^- denote the set of roots of U_P^-. Note that $\Phi = \Phi_L \sqcup \Phi_P^+ \sqcup \Phi_P^-$.

DEFINITION 34. (1) The subgroup J is *decomposed* with respect to P if the product

$$J_P^+ \times J_P^0 \times J_P^- \to J$$

is surjective (and hence bijective).

(2) The group J is *totally decomposed* with respect to P if it is decomposed, and in addition, the product maps

$$\prod_{\alpha\in\Phi_P^{\pm}} U_\alpha(F)\cap J \to J_P^{\pm}$$

are surjective (and hence bijective) for any ordering of the factors on the left hand side.

(3) We say that J is *absolutely totally decomposed* if it is totally decomposed with respect to all parabolic subgroups P.

The above definitions are closely related to the ones given in [BK98, §6] and [Bus01, §1.1]. (Note, however, that similar decompositions appear in [BT72, §6].) The following result, which is immediate from the definitions, is similar to a statement in [Bus01, §1.1].

Lemma 35. *Let J be totally decomposed in G with respect to a Borel subgroup B. Then J is totally decomposed with respect to every parabolic P containing B.*

In particular, if J is totally decomposed with respect to all Borels, then it is absolutely totally decomposed. The following lemma is also immediate from the definitions.

Lemma 36. *Let J be a compact open subgroup of G which is decomposed with respect to a parabolic P. Suppose $L(\mathcal{O})$ normalizes J_P^+ and J_P^- and $J_P^0\subseteq L(\mathcal{O})$. Then the subset $K = JL(\mathcal{O})$ is a subgroup of $G(F)$; moreover, it is decomposed with respect to P; that is, $K = K_P^+K_P^0K_P^-$ where $K_P^{\pm} = J_P^{\pm}$ and $K_P^0 = L(\mathcal{O})$.*

3.2. The subgroup K. Let $f:\Phi\to\mathbb{Z}$ be a function satisfying the properties

(a) $f(\alpha)+f(\beta)\geq f(\alpha+\beta)$, whenever $\alpha,\beta,\alpha+\beta\in\Phi$;

(b) $f(\alpha)+f(-\alpha)\geq 1$.

In particular, f is concave in the sense of Bruhat and Tits (see [BT72], §6.4.3 and §6.4.5). Let

$$J = J_f := \langle U_{\alpha,f(\alpha)}, T(\mathcal{O}) \mid \alpha\in\Phi\rangle. \tag{3.1}$$

Using the results of Bruhat and Tits, specifically [BT72, Proposition 6.4.9], Roche proved the following lemma.

Lemma 37. [Roc98, Lemma 3.2] *The group J is absolutely totally decomposed in G. Moreover, $J\cap U_\alpha(F) = U_{\alpha,f(\alpha)}$ for all $\alpha\in\Phi$.*

We are interested in $K = JL(\mathcal{O})$, in the case that it is a group. In view of Lemma 36, to check that it is a group, it is enough to require that $J_P^0\subseteq L(\mathcal{O})$ and to check that $L(\mathcal{O})$ normalizes $J_P^{\pm}$ for some choice of parabolic P with Levi component L.

Lemma 38. *Let P be a parabolic with Levi component L. Suppose that*

$$f(\beta) = f(\alpha+\beta),\quad \forall\alpha\in\Phi_L, \beta\in\Phi\smallsetminus\Phi_L \text{ such that } \alpha+\beta\in\Phi. \tag{3.2}$$

Then $L(\mathcal{O})$ normalizes $J_P^{\pm}$.

To prove the above, we will make use of the following lemma, which will also be useful later:

Lemma 39. *Assume that* $g : \Phi \smallsetminus \Phi_L \to \mathbb{Z}$ *satisfies* (3.2)*, in addition to conditions (a) and (b) (restricting* α, β*, and* $\alpha + \beta$ *to lie in* $\Phi \smallsetminus \Phi_L$*). Let* $\Phi_L^+ \subseteq \Phi_L$ *be any choice of positive roots, and let* $f : \Phi \to \mathbb{Z}$ *be the function defined by*

$$f|_{\Phi \smallsetminus \Phi_L} = g, \quad f|_{\Phi_L^+} = 0, \quad f|_{\Phi_L^-} = 1. \tag{3.3}$$

Then f *satisfies conditions (a) and (b).*

Note that $J_{f|_{\Phi_L}} = I_L$ is the Iwahori subgroup of $L(\mathcal{O})$ corresponding to $\Phi_L^+ \subseteq \Phi_L$.

Proof. It is clear (and standard) that $f|_{\Phi_L}$ satisfies conditions (a) and (b) (where we require in (a) that α, β, and $\alpha + \beta$ lie in Φ_L). By hypothesis, $f|_{\Phi \smallsetminus \Phi_L}$ satisfies conditions (a) and (b) (requiring α, β, and $\alpha + \beta$ to be in $\Phi \smallsetminus \Phi_L$ in (a)). So we only need to check that, if $\alpha \in \Phi_L$ and $\beta \in \Phi \smallsetminus \Phi_L$, then condition (a) is satisfied in the case that $\alpha + \beta \in \Phi$. This is immediate from (3.2). □

Proof of Lemma 38. Choose a subset $\Phi_L^+ \subseteq \Phi_L$ of positive roots for L. Let $g = f|_{\Phi \smallsetminus \Phi_L}$, and let $f' : \Phi \to \mathbb{Z}$ be as in Lemma 39 (i.e., $f'|_{\Phi \smallsetminus \Phi_L} = f|_{\Phi \smallsetminus \Phi_L}$, $f'|_{\Phi_L^+} = 0$ and $f'|_{\Phi_L^-} = 1$). Let $I_L = J_{f'|_{\Phi_L}} < L(\mathcal{O})$ be the corresponding Iwahori subgroup containing $T(\mathcal{O})$. Then $I_L \leq J_{f'}$, and hence I_L normalizes $J_{f'}$. It also normalizes $U_P^{\pm}$ (since L normalizes the unipotent radical U_P^+), so I_L normalizes $J_{f'} \cap U_P^{\pm} = J_f \cap U_P^{\pm} = J_P^{\pm}$. On the other hand, $L(\mathcal{O})$ is generated by all its Iwahori subgroups, so $L(\mathcal{O})$ also normalizes $J_P^{\pm}$. (Note that we could have also used the decomposition $L(\mathcal{O}) = I_L W_L I_L$, for W_L the Weyl group of L, and the fact that W_L normalizes J_P^+ under hypothesis (3.2).) □

Proposition 40. *Let* L *be a Levi subgroup of* G*. Assume that the function* $f : \Phi \to \mathbb{Z}$ *satisfies conditions (a) and (b) as well as* (3.2)*, and that* $f(\alpha) \geq 0$ *for all* $\alpha \in \Phi_L$*. Set* $J = J_f$*. Then* $K = JL(\mathcal{O})$ *is a group; moreover,* K *is decomposed with respect to every parabolic* P *with Levi* L*.*

Proof. By Lemma 38, $L(\mathcal{O})$ normalizes J_P^+ and J_P^-. Since $f(\alpha) \geq 0$ for all $\alpha \in \Phi_L$, it follows also that $J_P^0 \subseteq L(\mathcal{O})$. The result then follows from Lemma 36. □

The following corollary gives an alternative definition of K.

Corollary 41. *Let* L *be a Levi subgroup of* G*. Let* $g : \Phi \to \mathbb{Z}$ *be a function satisfying the following properties:*

(i) $g(\alpha) = 0$ *for all* $\alpha \in \Phi_L$*;*
(ii) $g(\alpha) + g(-\alpha) \geq 1$ *for all* $\alpha \in \Phi \smallsetminus \Phi_L$*;*
(iii) $g(\alpha) + g(\beta) \geq g(\alpha + \beta)$*, whenever* $\alpha, \beta, \alpha + \beta \in \Phi$*.*

Then $K = \langle U_{\alpha, g(\alpha)}, T(\mathcal{O}) \rangle$ *is a compact open subgroup of* G*. Moreover,* $K \cap U_\alpha = U_{\alpha, g(\alpha)}$*. Finally,* $K = L(\mathcal{O}) J_f$*, where* $f : \Phi \to \mathbb{C}^\times$ *is defined from* $g|_{\Phi \smallsetminus \Phi_L}$ *by* (3.2) *(for any choice of positive roots* $\Phi_L^+ \subseteq \Phi_L$*).*

Proof. The inclusion $K \subseteq J_f L(\mathcal{O})$ is clear. For the reverse inclusion, note that it is obvious that $J_f^{\pm} \subset K$, so we only need to show that $L(\mathcal{O}) \subset K$. This follows from the fact that L is generated by T and the root subgroups. Thus, $K = J_f L(\mathcal{O})$. In particular, by the above proposition, we have a direct product decomposition $K = K_P^+ K_P^0 K_P^-$ for every parabolic with Levi L. This implies that for $\alpha \in \Phi_P^{\pm}$, $K \cap U_\alpha = U_{\alpha, g(\alpha)}$. On the other hand it is clear that for $\alpha \in \Phi_L$, we have $K \cap U_\alpha = U_{\alpha,0}$ since $U_{\alpha,0} \subset L(\mathcal{O})$. □

3.3. Extension of $\bar{\mu}$. Let $\bar{\mu} : T(\mathcal{O}) \to \mathbb{C}^\times$ be a smooth character. Following Roche [Roc98], we define a compact open subgroup J associated to $\bar{\mu}$. To this end, we have to choose a partition $\Phi = \Phi^+ \sqcup \Phi^-$. (Note that this amounts to choosing a Borel $B \subset G$.) For every $\alpha \in \Phi$, let

$$c_\alpha := \mathrm{cond}(\bar{\mu} \circ \alpha^\vee) \tag{3.4}$$

denote the conductor of $\bar{\mu} \circ \alpha^\vee$; that is, the smallest positive integer c for which $\bar{\mu}(\alpha^\vee(1 + \mathfrak{p}^c)) = \{1\}$. Let

$$f_{\bar{\mu}}(\alpha) = \begin{cases} [c_\alpha/2], & \text{if } \alpha > 0, \\ [(c_\alpha + 1)/2], & \text{if } \alpha < 0. \end{cases} \tag{3.5}$$

Lemma 42. [Roc98, §3] *Suppose that characteristic of $\mathbb{F}_q$ satisfies the conditions in* (1.5)*. Then $f_{\bar{\mu}}$ satisfies conditions (a) and (b) of* (3.2)*.*

In particular, in view of Lemma 37, we obtain an associated compact open subgroup $J = J_{\bar{\mu}} = J_{f_{\bar{\mu}}}$. Note that the function $f_{\bar{\mu}}$ and the corresponding group $J_{\bar{\mu}}$ depend on the partition of Φ into positive and negative roots (or equivalently, on the chosen Borel B). While we ignore this in the notation, the reader should keep in mind that the Borel B is present. In particular, we have a decomposition $J = J^+ J^0 J^-$, where $J^{\pm} = J_B^{\pm}$. Let $J^\bullet = \langle J^+, J^- \rangle$.

Lemma 43. [Roc98, §3] *There exists a unique character $\mu^J : J \to \mathbb{C}^\times$ whose restriction to $J^0 = T(\mathcal{O})$ equals $\bar{\mu}$ and whose restriction to $J^\bullet$ is trivial.*

Let $\bar{\mu}$ be a strongly parabolic character of $T(\mathcal{O})$ and let L denote the corresponding Levi. Let P be a parabolic for L, and B the Borel subgroup of P. In terms of B, let $f = f_{\bar{\mu}}$ denote the function associated by Roche, and let $J = J_{\bar{\mu}}$ denote the corresponding compact open subgroup of $G(F)$.

Lemma 44. *The set $K = JL(\mathcal{O})$ is a compact open subgroup of $G(F)$. Moreover, for every parabolic subgroup P containing L, we have a decomposition $K = K_P^+ K_P^0 K_P^-$ where $K_P^{\pm} = J_P^{\pm}$ and $K_P^0 = L(\mathcal{O})$.*

Proof. If $\alpha \in \Phi_L$, then $\bar{\mu} \circ \alpha^\vee$ is trivial by (2.4). Now, if $\beta \in \Phi$ is such that $\alpha + \beta \in \Phi$, then $(\alpha + \beta)^\vee = a\alpha^\vee + b\beta^\vee$ where a and b are relatively prime to q (by our assumption on the characteristic of $\mathbb{F}_q$; see Conventions 6). Therefore, for every $\beta \in \Phi$ such that $\alpha + \beta \in \Phi$, the conductor of $\bar{\mu} \circ (\alpha + \beta)^\vee$ equals the conductor of $\bar{\mu} \circ (\beta^\vee)$; i.e.,

$$f_{\bar{\mu}}(\alpha + \beta) = f_{\bar{\mu}}(\beta). \tag{3.6}$$

The result then follows from Proposition 40. □

Let $B_L = B \cap L$ denote the corresponding Borel subgroup of L. Let I_L denote the corresponding Iwahori subgroup of L. Note that by (3.5), we have $J_P^0 = J \cap L(\mathcal{O}) = I_L$. Let $\mu^{L(F)}$ denote an extension of $\bar{\mu}$ to $L(F)$. Let $\mu^L = \mu^{L(\mathcal{O})} := \mu^{L(F)}|_{L(\mathcal{O})}$ denote its restriction to $L(\mathcal{O})$. Note that μ^L is automatically trivial on I_L^+ and I_L^-, since these groups are in $[L(F), L(F)]$. Set $K_P^\bullet = \langle K_P^+, K_P^- \rangle$.

Proposition 45. *There exists a unique character $\mu = \mu^K : K \to \mathbb{C}^\times$ whose restrictions to $K_P^\bullet$, J and $L(\mathcal{O})$ equal 1, μ^J, and μ^L, respectively.*

Proof. We need the following elementary fact: let H^+, H^0, H^- be subgroups of a group H which generate the group. Suppose that H^0 normalizes $H^\pm$. Let χ be a character of H^0 which is trivial on $\langle H^+, H^- \rangle \cap H^0$. Then the map $\tilde{\chi} : H \to \mathbb{C}^\times$ defined by $\tilde{\chi}(h^+ h^0 h^-) = \chi(h^0)$ is a well-defined extension of χ to H.

By assumption the characters μ^J and μ^L agree on $J \cap L(\mathcal{O}) = I_L$; in particular, μ^L is trivial on $K_P^\bullet \cap L(\mathcal{O})$ (since μ is trivial on $J^\bullet$). Applying the above fact, we conclude that there exists a character $\mu : K \to \mathbb{C}^\times$ whose restriction to $K_P^\pm$ is trivial and whose restriction to $L(\mathcal{O})$ equals μ^L. The latter statement implies that the restriction of μ to $I_L^\pm$ is trivial; hence, the restriction of μ to $J^\pm = K_P^\pm I_L^\pm$ is also trivial. Moreover, the restriction of μ to $T(\mathcal{O})$ equals $\bar{\mu}$. By Lemma 43, the restriction of μ to J equals μ^J. ☐

3.4. Proof of Theorem 10. Let $\bar{\mu} : T(\mathcal{O}) \to \mathbb{C}^\times$ be a strongly parabolic character of $T(\mathcal{O})$ with Levi L, and extensions $\mu^{L(F)}$ and $\mu^{L(\mathcal{O})} = \mu^L$ as above. Pick a parabolic P containing L and a Borel $B < P$. Let $J = J_{\bar{\mu}}$ denote the compact open subgroup associated by Roche to B and $\bar{\mu}$. Let $\mu^J : J \to \mathbb{C}^\times$ denote the canonical extension of $\bar{\mu}$ to J. Let

$$\mathscr{W} := \operatorname{ind}_J^{G(F)} \mu^J. \tag{3.7}$$

By definition, $\mathscr{W}$ is realized on the space of left (J, μ^J)-invariant compactly supported functions on $G(F)$. The group $G(F)$ acts on this space by right translation.

Note that $J \cap L(\mathcal{O}) = I_L$ is the Iwahori subgroup of J. Let μ^I denote the restriction of μ^J to I_L. Let $P_I^\circ := I_L U_P^+(F)$. The character μ^I extends uniquely to a character of P_I° which is trivial on $U_P^+(F)$. By an abuse of notation, we denote this character of P_I° by μ^I as well. Let

$$\Pi := \operatorname{ind}_{P_I^\circ}^{G(F)} \mu^I.\text{[5]} \tag{3.8}$$

Then Π is realized as the space of left (P_I°, μ^I)-invariant smooth functions on $G(F)$ which are compactly supported modulo P_I°. The group $G(F)$ acts by right translation.

Proposition 46. *The map $\Psi(f)(x) := \frac{1}{|J_P^+|} \int_N f(nx) dn$ is an isomorphism $\mathscr{W} \xrightarrow{\simeq} \Pi$.*

[5] It is easy to check that Π, thus defined, is isomorphic to the Π defined in (1.8).

Proof. According to [Roc98, Theorem 7.7], (J, μ^J) is a cover of $(T(\mathcal{O}), \bar{\mu})$, in the sense of [BK98, Definition 8.1] (in [Roc98], the residue characteristic is further restricted so as to obtain a nondegenerate bilinear form on the Lie algebra, but this restriction can be relaxed using the dual Lie algebra as in [Yu01]: see [KS11, §3.1.2, §A.2]). It follows from [BK98, Proposition 8.5] that (J, μ^J) is also a cover of (I_L, μ^L). The explicit isomorphism above is given by [Blo05, Theorem 2]. □

Recall from (1.9) that $\mathscr{V} := \operatorname{ind}_K^{G(F)} \mu$. By definition, this is a submodule of $\mathscr{W}$. On the other hand, let $P^\circ = L(\mathcal{O})U_P^+(F)$. The character μ^L extends uniquely to a character of P° which is trivial on $U_P^+(F)$. By an abuse of notation, we denote this character by μ^L as well. Then, recalling the definition of Θ in (1.10), we have an isomorphism

$$\Theta := \iota_{P(F)}^{G(F)} \operatorname{ind}_{L(\mathcal{O})}^{L(F)} \mu^L \simeq \operatorname{ind}_{P^\circ}^{G(F)} \mu^L.$$

We identify Θ with the $G(F)$-module on the RHS of the above isomorphism. With this convention, it is clear that we have an inclusion $\Theta \hookrightarrow \Pi$. To establish Theorem 10, we prove that the restriction of Ψ to $\mathscr{V}$ defines an isomorphism $G(F)$-modules $\mathscr{V} \xrightarrow{\simeq} \Theta$. To this end, we define averaging (or symmetrization) maps $\mathscr{W} \to \mathscr{V}$ and $\Pi \to \Theta$ and show that they are compatible with Ψ.
Recall that $\mathscr{W}$ is realized on the space of left (J, μ^J)-invariant smooth functions on $G(F)$. Under this identification, the subspace $\mathscr{V} \subset \mathscr{W}$ is identified with the space of left $(L(\mathcal{O}), \mu^L)$-invariant functions in $\mathscr{W}$. On the other hand, Θ is the subspace of left $(L(\mathcal{O}), \mu^L)$-invariant functions in Π.
Choose a Haar measure on L such that the volume of $L(\mathcal{O})$ equals 1. For every function $f : G(F) \to \mathbb{C}$, define f^c by

$$f^c(x) = \int_{L(\mathcal{O})} \mu^L(l) f(l^{-1}x) dl \tag{3.9}$$

Then $f \mapsto f^c$ defines a splitting of the natural inclusion of left $(L(\mathcal{O}), \mu^L)$-invariant functions on $G(F)$ into the space of all functions on $G(F)$. Note that this splitting obviously commutes with the action of $G(F)$ on the space of all smooth functions by right translation. Therefore, the map $f \mapsto f^c$ defines a splitting of the natural inclusions of $G(F)$-modules $\mathscr{V} \hookrightarrow \mathscr{W}$ and $\Theta \hookrightarrow \Pi$. The definition of Ψ, given in Proposition 46, implies that the diagram

$$\begin{tikzcd} \mathscr{W} \arrow[r, "\Psi"] \arrow[d, dotted, bend left] & \Pi \arrow[d, dotted, bend left] \\ \mathscr{V} \arrow[u, hook] \arrow[r] & \Theta \arrow[u, hook] \end{tikzcd} \tag{3.10}$$

commutes. Since the dotted arrows are surjective and the top horizontal arrow is an isomorphism, $\Psi|_{\mathscr{V}} : \mathscr{V} \to \Theta$ is surjective. Since it is the restriction of Ψ, which is an isomorphism, it is also injective. Thus it is an isomorphism. □

3.5. Proof of Theorem 7. We will continue with the notation of the previous subsection.

Proposition 47. *We have a canonical isomorphism* $\mathrm{End}_{G(F)}(\Theta) \simeq \mathscr{H}(L(F), L(\mathcal{O}), \mu^L)$.

Proof. The fact that $\bar{\mu}$ is parabolic with Levi L means that $W_{\bar{\mu}} = N_G(\bar{\mu})/T = N_L(T)/T$. In particular, $N_G(\bar{\mu}) \subset L(F)$. By [Roc02, Theorem 3.1], parabolic induction with respect to P defines an equivalence of categories between Bernstein block of L corresponding to the pair $(T(\mathcal{O}), \bar{\mu})$ and that of G. Under this equivalence, the $L(F)$-module $\mathrm{ind}_{L(\mathcal{O})}^{L(F)} \mu^L$ is mapped to Θ. Thus, we obtain a canonical isomorphism $\mathrm{End}_{G(F)}(\Theta) \simeq \mathrm{End}_{L(F)}(\mathrm{ind}_{L(\mathcal{O})}^{L(F)} \mu^L) \simeq \mathscr{H}(L(F), L(\mathcal{O}), \mu^L)$. □

Note that the algebra $\mathscr{H}(G(F), K, \mu)$ acts by convolution on the module $\mathscr{V} = \mathrm{ind}_K^{G(F)} \mu$. It is a standard fact that $\mathscr{H}(G(F), K, \mu) \simeq \mathrm{End}_{G(F)}(\mathscr{V})$. By Theorem 10, $\mathscr{V}$ is canonically isomorphic to $\Theta = \iota_P^G \left(\mathrm{ind}_{L(\mathcal{O})}^{L(F)} \mu^L\right)$. By the preceding paragraph, the endomorphism ring of Θ is canonically isomorphic to the endomorphism ring of the $L(F)$-module $\mathrm{ind}_{L(\mathcal{O})}^{L(F)} \mu^L$. Therefore, we obtain a canonical isomorphism $\mathscr{H}(G(F), K, \mu) \simeq \mathscr{H}(L(F), L(\mathcal{O}), \mu^L)$.

Finally, recall that $\mu^L = \mu^{L(F)}|_{L(\mathcal{O})}$, where $\mu^{L(F)} : L(F) \to \mathbb{C}^\times$ is a character of $L(F)$. Then multiplication by $\mu^{L(F)}$ defines a canonical isomorphism of algebras $\mathscr{H}(L(F), L(\mathcal{O})) \simeq \mathscr{H}(L(F), L(\mathcal{O}), \mu^L)$. Moreover, by the Satake isomorphism, we have a canonical isomorphism $\mathscr{H}(L(F), L(\mathcal{O})) \simeq \mathbb{C}[\check{T}/W_L] = \mathbb{C}[\check{T}/W_{\bar{\mu}}]$. Theorem 7 is established. □

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