Burniat Surfaces III: Deformations of Automorphisms and Extended Burniat Surfaces

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Abstract. We continue our investigation of the connected components of the moduli space of surfaces of general type containing the Burniat surfaces, correcting a mistake in part II. We define the family of extended Burniat surfaces with $K_S^2 = 4$, resp. 3, and prove that they are a deformation of the family of nodal Burniat surfaces with $K_S^2 = 4$, resp. 3. We show that the extended Burniat surfaces together with the nodal Burniat surfaces with $K_S^2 = 4$ form a connected component of the moduli space. We prove that the extended Burniat surfaces together with the nodal Burniat surfaces with $K_S^2 = 3$ form an irreducible open set in the moduli space. Finally we point out an interesting pathology of the moduli space of surfaces of general type given together with a group of automorphisms G. In fact, we show that for the minimal model S of a nodal Burniat surface ($G = (\mathbb{Z}/2\mathbb{Z})^2$) we have $\mathrm{Def}(S,G) \neq \mathrm{Def}(S)$, whereas for the canonical model X it holds $\mathrm{Def}(X,G) = \mathrm{Def}(X)$. All deformations of S have a G-action, but there are different deformation types for the pairs (S,G) of the minimal models S together with the G-action, while the pairs (X,G) have a unique deformation type.

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Dedicated to David Mumford.

Introduction

In the present article we continue our investigation, begun in [BC09b] and [BC10], of the connected components of the moduli space (of minimal surfaces S of general type) which contain the Burniat surfaces. We also correct an error in [BC10].

Recall that Burniat surfaces have $K_S^2 = 6 - m$, $m = 0, \dots, 4$, and the case $m = 2$ bifurcates in the subcases: the one of non-nodal Burniat surfaces, and the one of nodal Burniat surfaces. For $m = 3$ Burniat surfaces are three-nodal (this means that their canonical model has three nodes).

The main goals that we achieve in this paper are the following:

(1) We define the family of extended Burniat surfaces for $K_S^2 = 3$, resp. 4, and prove that they are a deformation of the family of nodal Burniat surfaces with $K_S^2 = 3$, resp. 4.

(2) We show that the extended Burniat surfaces with $K_S^2 = 4$, together with the nodal Burniat surfaces with $K_S^2 = 4$, form a set $\mathcal{NEB}_4$ which is a connected component of the moduli space: thereby we correct Theorem 1.1 of [BC10] and simultaneously we answer a question posed on page 562 of [BC10]. [1]

(3) We show that the extended Burniat surfaces with $K_S^2 = 3$, together with the nodal Burniat surfaces with $K_S^2 = 3$ form an irreducible open set $\mathcal{NEB}_3$ of the moduli space, whose closure $\overline{\mathcal{NEB}_3}$ consists of bidouble covers of normal cubic surfaces in $\mathbb{P}^3$ and is shown in Section 7 to be strictly larger than $\mathcal{NEB}_3$.

(4) We point out a truly interesting pathology of the moduli space of varieties with a group G of automorphisms, which is the reason of our mistake mentioned above (Murphy's law applies then, but in a different way than foreseen).

For nodal Burniat surfaces S, we have a group $G \cong (\mathbb{Z}/2\mathbb{Z})^2$ of automorphisms, which is also the group of automorphisms of the canonical model X. But whereas $\mathrm{Def}(X) = \mathrm{Def}(X, G)$, i.e., all deformations of X carry along a deformation of the G-action, $\mathrm{Def}(S) \neq \mathrm{Def}(S, G)$: thus even if all deformations of S have a G-action, the local moduli space $\mathrm{Def}(S, G)$ for the pairs yields a proper subvariety in the smooth germ $\mathrm{Def}(S)$.

We refer to [BC09b] and [BC10] for more details concerning investigation of the connected components of the moduli space containing the Burniat surfaces with $K_S^2 = 6, 5, 4, 2$, which is fully achieved thanks to the results of the present article.

What remains to be done in order to finish the investigation of Burniat surfaces is to decide, in the case $K_S^2 = 3$ of tertiary Burniat surfaces, whether the irreducible component mentioned above is also a connected component, describing in detail all the surfaces which are in the closure and their local deformations.

[1] Namely, the integer $m \geq 2$ in Theorem 1.1 is indeed $= +\infty$, and the local moduli space of nodal Burniat surfaces is smooth.

The description of the closure of the irreducible component of the moduli space given by the extended nodal Burniat surfaces with $K_S^2 = 3$ has been carried out by Y. Chen in his Ph. D. thesis (cf. [Ch12]).
In [BC10] we proved that 3 of the 4 irreducible families of Burniat surfaces with $K_S^2 \geq 4$, i.e., of primary and secondary Burniat surfaces, are a connected component of the moduli space of surfaces of general type.
In this paper we consider only nodal Burniat surfaces with $K_S^2 = 4, 3$, showing that a general deformation of a nodal Burniat surface with $K_S^2 = 4$, resp. with $K_S^2 = 3$, is an extended Burniat surface, still a bidouble cover (through the bicanonical map) of a normal Del Pezzo surface of degree 4 with one ordinary double point, resp. of a cubic surface with three nodes.
The main results of the present paper are the following:

Theorem 0.1. *1) The subset $\mathcal{NEB}_4$ of the moduli space of canonical surfaces of general type $\mathfrak{M}^{can}_{1,4}$ given by the union of the open set corresponding to* extended *Burniat surfaces with $K_S^2 = 4$ with the irreducible closed set parametrizing nodal Burniat surfaces with $K_S^2 = 4$ is a three dimensional irreducible connected component, normal and unirational.*
Moreover the base of the Kuranishi family of deformations of any such a minimal model S is smooth.
2) The subset $\mathcal{NEB}_3$ of the moduli space of canonical surfaces of general type $\mathfrak{M}^{can}_{1,3}$ corresponding to extended *and nodal Burniat surfaces with $K_S^2 = 3$ is an irreducible open set, normal, unirational of dimension 4.*
Moreover the base of the Kuranishi family of deformations of any such a minimal model S is smooth.

A very surprising and new phenomenon occurs for nodal surfaces, confirming Vakil's 'Murphy's law' philosophy ([Va06]).
To explain what happens for the moduli spaces of extended and nodal Burniat surfaces, let us recall again an old result due to Burns and Wahl (cf. [BW74]).
Let S be a minimal surface of general type and let X be its canonical model.
Denote by $\mathrm{Def}(S)$, resp. $\mathrm{Def}(X)$, the base of the Kuranishi family of S, resp. of X.
Their result explains the relation between $\mathrm{Def}(S)$ and $\mathrm{Def}(X)$.

Theorem (Burns-Wahl).
Assume that K_S is not ample and let $p : S \to X$ be the canonical morphism.
Denote by $\mathcal{L}_X$ the space of local deformations of the singularities of X and by $\mathcal{L}_S$ the space of deformations of a neighbourhood of the exceptional curves of p.
Then $\mathrm{Def}(S)$ is realized as the fibre product associated to the Cartesian diagram

$$\begin{tikzcd} \mathrm{Def}(S) \arrow[r] \arrow[d] & \mathcal{L}_S \cong \mathbb{C}^\nu, \arrow[d, "\lambda"] \\ \mathrm{Def}(X) \arrow[r] & \mathcal{L}_X \cong \mathbb{C}^\nu, \end{tikzcd}$$

where ν is the number of rational (-2)-curves in S, and λ is a Galois covering with Galois group $W := \oplus_{i=1}^r W_i$, the direct sum of the Weyl groups W_i of the singular points of X.

An immediate consequence is the following

Corollary (Burns-Wahl).
1) $\psi : \mathrm{Def}(S) \to \mathrm{Def}(X)$ is a finite morphism, in particular, ψ is surjective.
2) If $\mathrm{Def}(X) \to \mathcal{L}_X$ is not surjective (i.e., the singularities of X cannot be smoothened independently by deformations of X), then $\mathrm{Def}(S)$ is singular.

Assume now that we have $1 \neq G \leq \mathrm{Aut}(S) = \mathrm{Aut}(X)$.
Then we can consider the space $\mathrm{Def}(S, G)$ of local deformations of S together with the G-action (by [Cat88] this is the space of G-invariant local deformations of S), and similarly consider the space $\mathrm{Def}(X, G)$ of local deformations of X and its G-action; we have then a natural map $\mathrm{Def}(S, G) \to \mathrm{Def}(X, G)$.
We indeed show here that, unlike the case for the corresponding morphism of local deformation spaces of the surfaces, this map needs not to be surjective; and, as far as we know, the following result gives the first global example of such a phenomenon.

Theorem 0.2. *The deformations of nodal Burniat surfaces with $K_S^2 = 4, 3$ to extended Burniat surfaces with $K_S^2 = 4, 3$ yield examples where $\mathrm{Def}(S, (\mathbb{Z}/2\mathbb{Z})^2) \to \mathrm{Def}(X, (\mathbb{Z}/2\mathbb{Z})^2)$ is not surjective.*
Moreover, $\mathrm{Def}(S, (\mathbb{Z}/2\mathbb{Z})^2) \subsetneq \mathrm{Def}(S)$, whereas for the canonical model we have: $\mathrm{Def}(X, (\mathbb{Z}/2\mathbb{Z})^2) = \mathrm{Def}(X)$.
The moduli space of pairs, of an extended (or nodal) Burniat surface with $K_S^2 = 4, 3$ and a $(\mathbb{Z}/2\mathbb{Z})^2$-action, is disconnected; but its image in the moduli space is a connected open set.

The above phenomenon can already be seen locally around a node, as it will be explained in Section 2. Our results show that the local pathology does indeed globalize.

Our paper is organized as follows: in Section 1 we give the definition of *extended* Burniat surfaces and describe the respective branch loci of the bidouble covers yielding nodal Burniat surfaces, respectively extended Burniat surfaces.
In the second chapter we analyse bidouble covers of a nodal surface singularity, explaining the phenomenon of Theorem 0.2 locally.
In the third section we show that nodal Burniat surfaces with $K_S^2 = 4, 3$ deform to extended Burniat surfaces with $K_S^2 = 4, 3$.
Section 4 is instead devoted to the calculation of $H^1(S, \Theta_S)$ for nodal and extended Burniat surfaces, and its eigenspaces for the $G = (\mathbb{Z}/2\mathbb{Z})^2$ action.
In the course of doing this we need to amend a small mistake in [BC10] Lemma 2.10; this is done in an appendix, where we actually generalize this lemma substantially in order to make it appropriate for our present purposes and also applicable in other situations.

In the end we succeed to prove that the subset $\mathcal{NEB}_4$ of the moduli space of canonical surfaces of general type $\mathfrak{M}^{can}_{1,4}$ corresponding to nodal and extended Burniat surfaces with $K_S^2 = 4$ is an irreducible open set, normal, unirational of dimension 3 (similarly we show that the subset $\mathcal{NEB}_3$ of the moduli space of canonical surfaces of general type $\mathfrak{M}^{can}_{1,3}$ corresponding to nodal and extended Burniat surfaces with $K_S^2 = 3$ is an irreducible open set, normal, unirational of dimension 4).
Section 5 is dedicated to the study of one-parameter limits of extended Burniat surfaces with $K_S^2 = 4$, showing that the subset of the moduli space of canonical surfaces of general type $\mathfrak{M}^{can}_{1,4}$ corresponding to nodal and extended Burniat surfaces with $K_S^2 = 4$ is closed.
In Section 6 we give examples of other surfaces which lie in the closure of the family of extended Burniat surfaces with $K_S^2 = 3$.
In another appendix we give an alternative proof of three of the four assertions of Proposition 4.1, by other methods which could be of independent interest.

1. Definition of extended and nodal Burniat surfaces

Burniat surfaces are minimal surfaces of general type with $K^2 = 6, 5, 4, 3, 2$ and $p_g = 0$, which were constructed in [Bu66] as minimal resolutions of singular bidouble covers (that is, Galois covers with group $(\mathbb{Z}/2\mathbb{Z})^2$) of the projective plane branched on 9 lines.
We refer the reader to [BC10] for their construction, and we shall adhere to the notation introduced there.

Let $P_1, P_2, P_3 \in \mathbb{P}^2$ be three non-collinear points (which we assume to be the points $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$), and let $P_4, \dots, P_{3+m}$, $\quad m = 2, 3$, be further (distinct) points not lying on the sides of the triangle with vertices P_1, P_2, P_3.
We make the further assumptions:

- for $m = 2$, the points P_1, P_4, P_5 are collinear, while,
- for $m = 3$, we assume moreover that also P_2, P_4, P_6 and P_3, P_5, P_6 are collinear (in particular, no four points are collinear); we may also use the notation $P'_1 := P_6, P'_2 := P_5, P'_3 := P_4$, so that P_i, P'_{i+1}, P'_{i+2} are collinear, where we use the convention $i \in \{1, 2, 3\} \mod 3$.

Let's denote by $\tilde{Y} := \hat{\mathbb{P}}^2(P_1, P_2, \dots, P_{3+m})$ the weak Del Pezzo surface of degree $6 - m$, obtained blowing up $\mathbb{P}^2$ in the points $P_1, P_2, \dots, P_{3+m}$.
Saying that $\tilde{Y}$ is a weak Del Pezzo surface means that the anticanonical divisor $-K_{\tilde{Y}}$ is nef and big; in our case it is not ample, because of the existence of (-2)-curves, i.e. curves $N_i \cong \mathbb{P}^1$, with $N_i \cdot K_{\tilde{Y}} = 0$: for $m = 3$ N_i is the strict transform of the line passing through P_i, P'_{i+1}, P'_{i+2}.
Contracting the (-2)-curves N_i we obtain a normal singular Del Pezzo surface Y' with $-K_{Y'}$ very ample.
We denote by L the divisor on $\tilde{Y}$ which is the total transform of a general line in $\mathbb{P}^2$, by E_i the exceptional curve lying over P_i, by E'_i the exceptional curve

lying over P_i' (hence $E_i' = E_{7-i}$) and by $D_{i,1}$ the strict transform of the line $y_{i-1} = 0$, side of the triangle joining the points $P_i, P_{[i+1]}$; that is, the unique effective divisor in $|L - E_i - E_{[i+1]}|$, where $[i+1]$ represents the residue class of $i+1 \mod 3$, an element of $\{1,2,3\}$.
For $m = 2$ we have only one (-2)-curve N_1, such that $\{N_1\} = |L - E_1 - E_4 - E_5|$, while for $m = 3$ we also have the curves N_2, N_3 such that $\{N_2\} = |L - E_2 - E_4 - E_6|$, $\{N_3\} = |L - E_3 - E_5 - E_6|$.
Therefore the anticanonical image of $\tilde{Y}$ is a normal surface $Y' \subset \mathbb{P}^{6-m}$ of degree $6 - m$, whose singularities are one node ν_1 (an A_1 singularity) in the case $m = 2$, and three nodes ν_1, ν_2, ν_3 in the case $m = 3$ (the (-2)-curve N_i is the total transform of the point ν_i).
In order to improve readability we separate the definitions for $m = 2$ and $m = 3$.

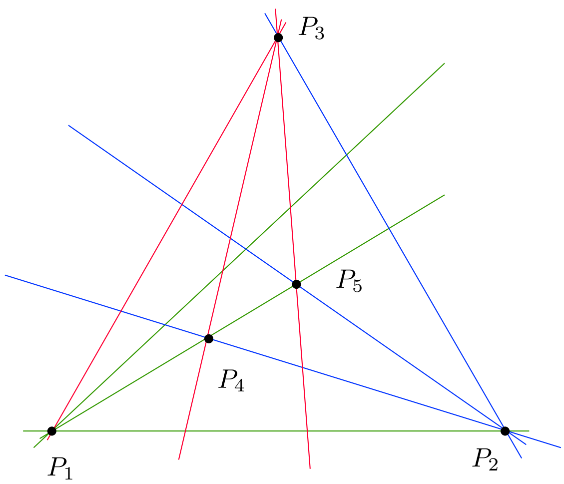
1.1. Nodal and extended Burniat surfaces with $K_S^2 = 4$.

Definition 1.1. *1) Define the* Burniat divisors for $m = 2$ *as follows:*

$$D_1 \in |L - E_1| + |L - E_1 - E_2| + |L - E_1 - E_4 - E_5| + E_3,$$

i.e., $D_1 = D_{1,1} + N_1 + C_1 + E_3$*, where* $C_1 \in |L - E_1|$ *is assumed to be irreducible, whereas* D_2*,* D_3 *are divisors such that*

$$\{D_2\} = |L - E_2 - E_3| + |L - E_2 - E_4| + |L - E_2 - E_5| + E_1,$$
$$\{D_3\} = |L - E_3 - E_1| + |L - E_3 - E_4| + |L - E_3 - E_5| + E_2.$$



2) The extended Burniat divisors for $m = 2$ *are given as follows:*

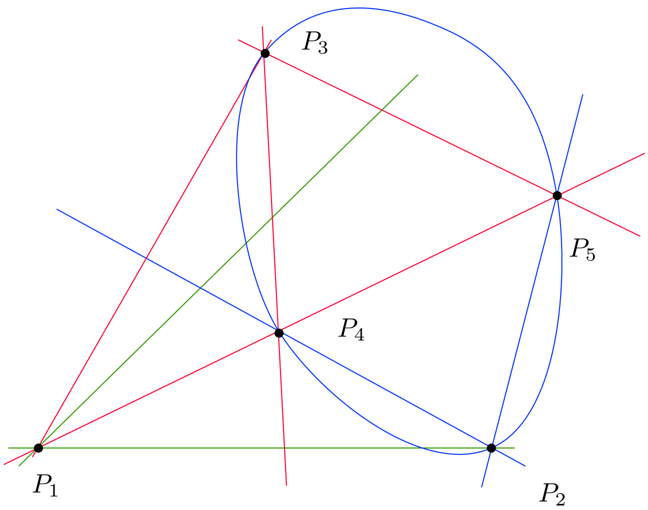
$$\Delta_1 \in |L - E_1| + |L - E_1 - E_2| + E_3,$$

i.e., $\Delta_1 = D_{1,1} + C_1 + E_3$*, where* $C_1 \in |L - E_1|$ *is assumed to be irreducible, and*

$$\Delta_2 \in |L - E_2 - E_4| + |L - E_2 - E_5| + |2L - E_2 - E_3 - E_4 - E_5|,$$

where we assume the divisor $\Gamma_2 \in |2L - E_2 - E_3 - E_4 - E_5|$ *to be irreducible; and* Δ_3 *is the divisor such that*

$$\{\Delta_3\} = (|L - E_3 - E_4| + |L - E_3 - E_5|) + (|L - E_3 - E_1| + E_2) + |L - E_1 - E_4 - E_5|$$
$$=: \Delta_{3,1} + \Delta_{3,2} + N_1.$$



3) Define $L_i := \frac{1}{2}(D_{[i-1]} + D_{[i+1]})$ *and observe that it is an integral divisor; define also* $\Lambda_1 := L_1 + N_1$ *and* $\Lambda_j := L_j$ *for* $j = 2, 3$.

Remark 1.2. 1) Observe that $(D_1 + D_2 + D_3) \in |-3K_{\tilde{Y}}|$ is a reduced normal crossing divisor.

2) Similarly, $(\Delta_1 + \Delta_2 + \Delta_3) \in |-3K_{\tilde{Y}} + N_1|$ is a reduced normal crossing divisor.

3) On the normal Del Pezzo surface Y'

- D_1 yields a conic plus two lines, and Δ_1 does the same (indeed $D_1 = \Delta_1 + N_1$ and N_1 is the 'nodal' exceptional curve)
- D_2 yields four lines, Δ_2 yields a conic plus two lines (indeed $\Delta_2 \equiv D_2 + N_1$)
- D_3 yields four lines, the same does Δ_3 (indeed $\Delta_3 = D_3 + N_1$).

In particular, if the conic corresponding to Δ_2 specializes to contain the line corresponding to E_1, we obtain then D_2 subtracting the divisor $N_1 \equiv L - E_1 - E_4 - E_5$.

Finally, the four lines of Δ_3 divide into two groups, i.e., we can write $\Delta_3 = \Delta_{3,1} + \Delta_{3,2} + N_1$ so that, setting $\Gamma_1 := C_1$ and writing $\Delta_i = \Gamma_i + \Delta'_i$, for $i = 1, 2$, then

$$(*) : \Delta'_i + \Delta_{3,i} \equiv -K_{\tilde{Y}}$$

$$(**) : \Gamma_1 + \Gamma_2 \equiv -K_{\tilde{Y}}.$$

4) The divisors D_i enjoy the property (cf. [BC10]) that there are divisor classes L_i such that $D_{[i-1]} + D_{[i+1]} \equiv 2L_i$.
Hence, in particular, $\Delta_{[i-1]} + \Delta_{[i+1]} \equiv 2\Lambda_i$, where, $\Lambda_1 := L_1 + N_1$ and $\Lambda_j := L_j$ for $j = 2, 3$.
5) Assume now that the conic corresponding to Γ_2 becomes reducible: if the conic passes through P_1, then necessarily Γ_2 splits as $N_1 + E_1 + |L - E_2 - E_3|$, hence the conic is the union of two lines. If the conic is the union of two lines in another fashion, then necessarily either $|L - E_2 - E_5|$ or $|L - E_2 - E_4|$ is a component of Γ_2, hence Δ_2 is not reduced.

We can now consider (cf. [Cat84b], [Cat99]) the associated bidouble covers $S \to \tilde{Y}$ with branching divisors the Burniat divisors, respectively the extended Burniat divisors.

Definition 1.3. *A* secondary nodal Burniat surface *is a bidouble cover $S \to \tilde{Y}$ with branch divisors three Burniat divisors for $m = 2$.*
S is then a minimal surface of general type with $p_g(S) = q(S) = 0$, $K_S^2 = 6 - m = 4$ (cf. [BC10]*).*
If we let the three branch divisors be extended Burniat divisors, then we obtain a non minimal surface $\tilde{S}$ whose minimal model S is called a secondary extended Burniat surface*.*

1.2. Nodal and extended Burniat surfaces with $K_S^2 = 3$.

Definition 1.4. *1) The* Burniat divisors for $m = 3$ *are defined to be the divisors D_1, D_2, D_3 such that*

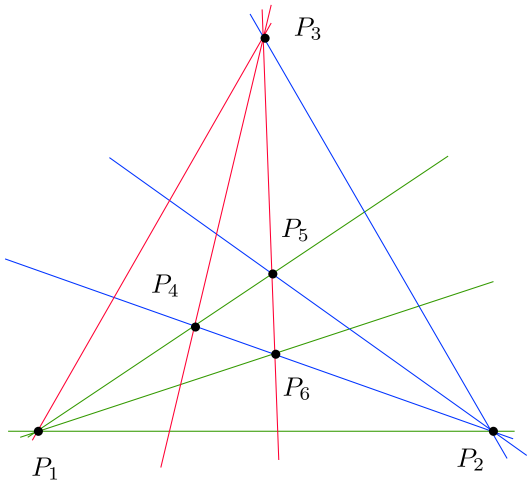
$$D_i := D_{i,1} + N_i + E_{[i-1]} + |L - E_i - E'_i|.$$

More in detail,

$$\{D_1\} = |L - E_1 - E_2| + |L - E_1 - E_4 - E_5| + |L - E_1 - E_6| + E_3,$$

$$\{D_2\} = |L - E_2 - E_3| + |L - E_2 - E_4 - E_6| + |L - E_2 - E_5| + E_1,$$

$$\{D_3\} = |L - E_3 - E_1| + |L - E_3 - E_5 - E_6| + |L - E_3 - E_4| + E_2.$$



2) The strictly extended Burniat divisors for $m = 3$ *are defined as follows:*

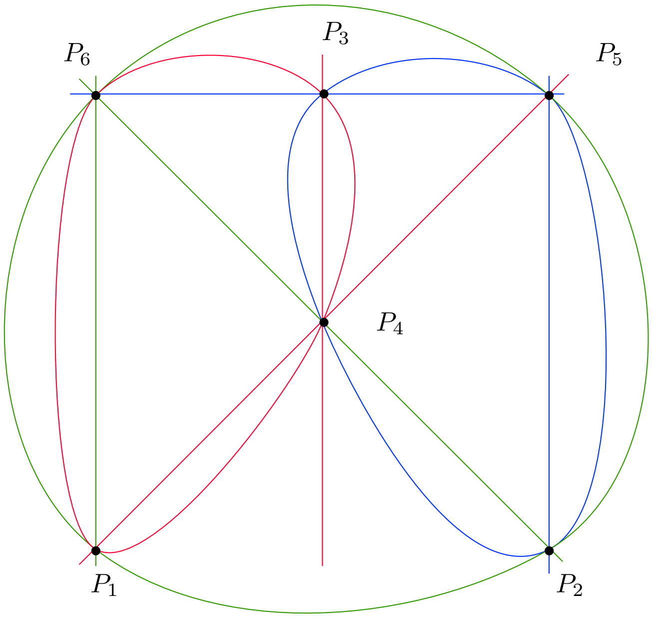
$$\Delta_i := |L - E_i - E_i'| + N_{[i+1]} + \Gamma_i,$$

where we assume $\Gamma_i \in |2L - E_i - E_{[i+1]} - E_i' - E_{[i+1]}'|$ *to be the strict transform of an irreducible conic.*
More in detail,

$$\Delta_1 \in |L - E_1 - E_6| + |2L - E_1 - E_2 - E_5 - E_6| + |L - E_2 - E_4 - E_6|,$$

$$\Delta_2 \in |L - E_2 - E_5| + |2L - E_2 - E_3 - E_4 - E_5| + |L - E_3 - E_5 - E_6|,$$

$$\Delta_3 \in |L - E_3 - E_4| + |2L - E_1 - E_3 - E_4 - E_6| + |L - E_1 - E_4 - E_5|.$$



3) Define $L_i := \frac{1}{2}(D_{[i-1]} + D_{[i+1]})$ *and observe that it is an integral divisor; define also* $\Lambda_i := L_i + N_i$.

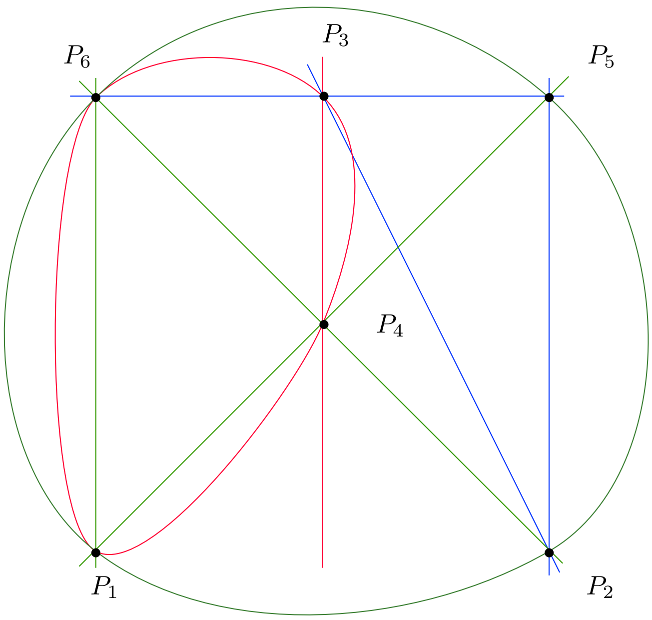
Remark 1.5. 1) Observe that $(D_1 + D_2 + D_3) \in |-3K_{\tilde{Y}}|$ is a reduced normal crossing divisor.
2) Similarly, $(\Delta_1 + \Delta_2 + \Delta_3) \in |-3K_{\tilde{Y}} + \sum N_i|$ is a reduced normal crossing divisor.
3) On the normal Del Pezzo surface Y', for $m = 3$,
Δ_j yields a conic and one line, D_j yields three lines (indeed $\Delta_j \equiv D_j - N_j + N_{j-1} + N_{j+1}$).
In particular, if the conic corresponding to Δ_j specializes to contain the line corresponding to $E_{[j-1]}$ (here as before $[j-1] \in \mathbb{Z}/3\mathbb{Z}$), we obtain D_2 subtracting the divisor $N_{j-1} + N_{j+1}$ and adding the divisor N_j.
4) The divisors D_i enjoy the property (cf. [BC10]) that there are divisor classes L_i such that $D_{[i-1]} + D_{[i+1]} \equiv 2L_i$.
Hence, in particular, $\Delta_{[i-1]} + \Delta_{[i+1]} \equiv 2\Lambda_i$, recalling that, for $m = 3$, $\Lambda_i := L_i + N_i$.
5) Assume that one or more of these conics become reducible. E.g., assume that the conic corresponding to Γ_2 becomes reducible, and observe that this will be the case if the conic passes through P_1 or P_6. We disregard this degeneration if the corresponding divisor Δ_2 will be non reduced. The only possibility left

over is that Γ_2 splits as before, $N_1 + E_1 + |L - E_2 - E_3|$. This degeneration will be considered admissible. This motivates the following definition:

Definition 1.6. *Assume $m = 3$ and that one or two of these conics Γ_j become reducible in the admissible way $\Gamma_j = N_{j-1} + E_{j-1} + |L - E_j - E_{j+1}|$ (here, as usual, $j \in \mathbb{Z}/3\mathbb{Z}$).*
In this case we define the (not strictly) extended Burniat divisors *by subtracting to Γ_j the nodal divisor N_{j-1} it contains. Moreover, we replace*

- *Δ_{j+1} by $(\Delta_{j+1} - N_{j-1})$, and*
- *Δ_{j-1} by $(\Delta_{j-1} + N_{j-1})$.*

For the convenience of the reader we have drawn the non strictly extended Burniat divisors in the case, where only Γ_2 degenerates.



Remark 1.7. If all three conics Γ_j become reducible in the admissible way and we define in the same way as in the previous definition the three divisors by subtracting to Γ_j the nodal divisor N_{j-1} it contains, by subtracting again the nodal divisor N_{j-1} from Δ_{j+1} and adding it to Δ_{j-1}, we get the Burniat divisors from Definition 1.4.

We can now consider (cf. [Cat84b], [Cat99]) the associated bidouble covers $S \to \tilde{Y}$ with branching divisors the Burniat divisors, respectively the extended Burniat divisors.

Definition 1.8. *A* tertiary (three-)nodal Burniat surface *S is a bidouble cover $S \to \tilde{Y}$ with branch divisors three Burniat divisors for $m = 3$.*
S is then a minimal surface of general type with $p_g(S) = q(S) = 0$, $K_S^2 = 6 - m = 3$ (cf. [BC10]*).*
If we let the three branch divisors be extended Burniat divisors (i.e., either strictly extended or non strictly extended!), then we obtain a non minimal surface $\tilde{S}$ whose minimal model S is called a tertiary extended Burniat surface*.*

Remark 1.9. 1) In the nodal Burniat case the surface S does not have an ample canonical divisor K_S, due to the existence of (-2)-curves, which are exactly the inverse images of the (-2)-curves $N_i \subset \tilde{Y}$.
For this reason we call the above Burniat surfaces of *nodal type*. We denote their canonical model by X, and observe that X is a finite bidouble cover of the normal Del Pezzo surface Y'.
If $m = 2$, then X has precisely one node (an A_1-singularity, corresponding to the contraction of the (-2)-curve) as singularity. While, for $m = 3$, the canonical model X has exactly three nodes as singularities.
2) In the extended Burniat case $\tilde{S}$ is not minimal. In the strictly extended Burniat case the inverse image of each N_i splits as the union of two disjoint (-1)-curves. S has ample canonical divisor (hence $S = X$) exactly in the strictly extended case.
3) In all cases, the morphism $X \to Y'$ is exactly the bicanonical map of X (see [BC10]).
4) Nodal Burniat surfaces are parametrized by a family with smooth base of dimension 2 for $m = 2$, of dimension 1 for $m = 3$.
Strictly extended Burniat surfaces are parametrized by a family with smooth base of dimension 3 for $m = 2$, of dimension 4 for $m = 3$.

The key feature is that, both for nodal Burniat surfaces, and for extended Burniat surfaces, the canonical model X is a finite bidouble cover of a singular Del Pezzo surface Y', which has one node in the case $m = 2$, and three nodes for $m = 3$ (in this case Y' is a cubic surface in $\mathbb{P}^3$).
In this case the direct image $p_*(\mathcal{O}_X)$ splits as a direct sum of four reflexive character sheaves of generic rank 1.
In the next section we shall describe how the covering behaves in the neighbourhood of a node in the two respective cases, and how these local coverings deform to each other (the Burniat case deforms to the extended Burniat case).

2. Local calculations around the nodes

In this section we consider finite bidouble covers of a node which are of Du Val type, i.e., yielding singularities which are at worst RDP 's (rational double points).
We obtain a classification which is a subset of the one made in [Cat87], classifying quotients of RDP's by actions of $\mathbb{Z}/2\mathbb{Z}$ or of $G = (\mathbb{Z}/2\mathbb{Z})^2$.

We only need to look at Table 2, page 90, and Table 3, page 93, ibidem, to see which quotients of a rational double point by an involution, or by a pair of commuting involutions, yield an A_1 singularity, i.e., a node.
There are six cases for such coverings of Du Val type of a node Y, which in local holomorphic coordinates is given by

$$xy - z^2 = 0.$$

In order to be more informative in our description, we denote by $\tilde{Y}$ the resolution of Y, which is the total space of a line bundle on $N \cong \mathbb{P}^1$ of degree -2 (hence $N^2 = -2$). Denoting the bidouble cover of Y by X, we shall obtain, through the normalization of the fibre product, a finite bidouble cover of $\tilde{Y}$, for which we shall give the three corresponding branch divisors.
In the case where X is not irreducible, we shall describe a connected component X' of X.

(1) $X' = Y$ (the covering is étale).

(2) $X' = \mathbb{C}^2$, X has two components and the covering morphism is given by

$$(u, v) \mapsto (x = u^2, y = v^2, z = uv).$$

The branch divisor on $\tilde{Y}$ is just the (-2)-curve N.

(3) $X' = \{w^4 = xy\}$, X has two components and the covering morphism is given by

$$(x, y, w) \mapsto (x, y, z = w^2).$$

The branch divisor on $\tilde{Y}$ consists of the (-2)-curve N plus two fibres; the double cover of $\tilde{Y}$ has two nodes and resolving them we get the minimal resolution of the A_3 singularity X'.

(4) $X = \{w^2 = uv\}$ and the covering morphism is given by

$$(u, v) \mapsto (x = u^2, y = v^2, z = w^2).$$

The three intermediate $\mathbb{Z}/2\mathbb{Z}$ covers are the two double covers (2), (3) described above, plus the intermediate cover (here $a := uw, b := vw$)

$$\{(x, y, z, a, b)| \text{ Rank} \begin{pmatrix} x, a, z, b \\ a, z, b, y \end{pmatrix} = 1\},$$

which is the cone over a rational normal quartic (set $x = t_0^4, a = t_0^3 t_1, z = t_0^2 t_1^2, z = t_0 t_1^3, z = t_1^4$).

The branch divisors on $\tilde{Y}$ are two: the (-2)-curve N and the divisor D formed by two fibres. The three intermediate double covers depend on the choice of the branch locus: N, respectively $N + D$, respectively D.

(5) $X' = \{z^2 = (w^2 + y^{k+1}) \cdot y\}$, X has two components having a singularity of type D_{k+3}, and the covering morphism is given by

$$(y, z, w) \mapsto (x = w^2 + y^{k+1}, y, z).$$

The branch divisor on $\tilde{Y}$ is the total transform of the divisor $C := \{x = y^{k+1}, z^2 = y^{k+2}\}$ which is irreducible with a cusp for k odd, else it is reducible with a $\frac{k}{2}$-tacnode for k even.

In particular, N is part of the branch locus.

(6) $X = \{w^2 = (u - v^{k+1})(u + v^{k+1})\} = \{w^2 = u^2 - v^{2k+2}\}$ and the covering morphism is given by

$$(u, v, w) \mapsto (x = u^2, y = v^2, z = uv).$$

X is a singularity of type A_{2k+1} and, in order to treat a new case, we make the assumption $k \geq 1$.

The three intermediate $\mathbb{Z}/2\mathbb{Z}$ covers are the smooth double cover (2), the double cover (5) $\{w^2 = x - y^{k+1}\}$, and a third singularity which we omit to describe.

The branch divisors on $\tilde{Y}$ are two: the (-2)-curve N and the the total transform of the divisor C above.

The three intermediate covers depend on the choice of the branch locus: N, or $N + C'$, or C', where C' is the strict transform of C.

Letting $p : X \to Y$ be the finite bidouble cover, the direct image sheaf $p_*\mathcal{O}_X$ splits as

$$\mathcal{O}_Y \oplus (\bigoplus_{i=1,2,3} \mathcal{L}_i),$$

where in the first case the reflexive sheaves $\mathcal{L}_i$ are locally free.
To describe the other cases we use the reflexive sheaf $\mathcal{F}$ generated by u, v as $\mathcal{O}_Y$-module, with relations

$$yu - zv = 0, zu - xv = 0.$$

We get

(2) $X' = \mathbb{C}^2$, $(u, v) \mapsto (x = u^2, y = v^2, z = uv)$,

$$p_*\mathcal{O}_X = (\mathcal{O}_Y \oplus \mathcal{F})^{\oplus 2}$$

(3) $X' = \{w^4 = xy\}$, $(x, y, w) \mapsto (x, y, z = w^2)$

$$p_*\mathcal{O}_X = (\mathcal{O}_Y \oplus \mathcal{O}_Y)^{\oplus 2}$$

(4) $X = \{w^2 = uv\}$, $(u, v) \mapsto (x = u^2, y = v^2, z = w^2)$

$$p_*\mathcal{O}_X = (\mathcal{O}_Y \oplus \mathcal{F})^{\oplus 2},$$

with generators $1, \{u, v\}, w, \{a = wu, b = vw\}$.

(5) $X' = \{w^2 = x - y^{k+1}\}$, $(y, z, w) \mapsto (x = w^2 + y^{k+1}, y, z)$,

$$p_*\mathcal{O}_X = (\mathcal{O}_Y \oplus \mathcal{O}_Y)^{\oplus 2}.$$

(6) $X = \{w^2 = u^2 - v^{2k+2}\}$, $(u, v, w) \mapsto (x = u^2, y = v^2, z = uv)$

$$p_*\mathcal{O}_X = (\mathcal{O}_Y \oplus \mathcal{F})^{\oplus 2}.$$

Remark 2.1. Cases 1, 3 and 5 are the case where we have a flat bidouble cover, i.e., $p_*\mathcal{O}_X$ is locally free. In cases 2, 4 and 6 we have non-flat bidouble covers, but with the same character sheaves. We shall soon show how case 4 deforms to case 2.

Proposition 2.2. *In case 2) $X = Spec((\mathcal{O}_Y \oplus \mathcal{F}) \oplus (\mathcal{O}_Y \oplus \mathcal{F}))$, where the two addenda are orthogonal, and the algebra structure is determined by the nondegenerate pairing $\mathcal{F} \times \mathcal{F} \to \mathcal{O}_Y$.*
In case 4) $X = Spec((\mathcal{O}_Y \oplus \mathcal{F}) \oplus w(\mathcal{O}_Y \oplus \mathcal{F}))$, and the algebra structure is determined by the nondegenerate pairing $\mathcal{F} \times \mathcal{F} \to \mathcal{O}_Y$, together with the assignment $w^2 = z$.

We omit the simple proof.
Case 4) deforms now to case 2) by changing the assignment $w^2 = z$ to $w^2 = z+t$, $t \neq 0$, so that w becomes then a local unit at the origin.
We can relate the resulting picture with the local semiuniversal deformation of a node.

Proposition 2.3. *Let $t \in \mathbb{C}$, and consider the action of $G := (\mathbb{Z}/2\mathbb{Z})^2$ on $\mathbb{C}^3$ generated by $\sigma_1(u,v,w) = (u,v,-w)$, $\sigma_2(u,v,w) = (-u,-v,w)$. Then the hypersurfaces $X_t = \{(u,v,w) | w^2 = uv + t\}$ are G-invariant, and the quotient X_t/G is the hypersurface*

$$Y_t = Y_0 = \{(x,y,z) | z^2 = xy\},$$

which has a nodal singularity at the point $x = y = z = 0$.
$X_t \to Y_t$ is a bidouble covering of type 2 for $t \neq 0$, and of type 4 for $t = 0$. We get in this way a flat family of (non flat) bidouble covers.

Proof. The invariants for the action of G on $\mathbb{C}^3 \times \mathbb{C}$ are:

$$x := u^2, y := v^2, z := uv, s := w^2, t.$$

Hence the family $\mathfrak{X}$ of the hypersurfaces X_t is the inverse image of the family of hypersurfaces $s = z + t$ on the product

$$Y' \times \mathbb{C}^2 = \{x,y,z,s,t) | xy = z^2\}.$$

Hence the quotient of X_t is isomorphic to Y'.
The rest was already explained before. □

Remark 2.4. i) The simplest way to view X_t is to see $\mathbb{C}^2$ as a double cover of Y' branched only at the origin, and then X_t as a family of double covers of $\mathbb{C}^2$ branched on the curve $uv + t = 0$, which acquires a double point for $t = 0$.
ii) The involution $\sigma_3(u,v,w) = (-u,-v,-w)$ has only the origin as fixed point, which lies on X_0. Whereas σ_3 acts freely on X_t, for $t \neq 0$.
$Fix(\sigma_1) = \{w = 0\}$, and $\{w = 0\} \cap X_t = \{uv + t = w = 0\}$.
Finally, $Fix(\sigma_2) = \{u = v = 0\}$, and $\{u = v = 0\} \cap X_t = \{u = v = 0, w^2 = t\}$, which consists of two points for $t \neq 0$, one for $t = 0$.
The corresponding branch loci are the origin, for $t = 0$, the divisor $z = 0$, and the point $x = y = z - t = 0$.

iii) If we pull back the bidouble cover X_t to $\tilde{Y}$, and we normalize it, we can see that

- D_3 is, for $t = 0$, the nodal curve N, and is the empty divisor for $t \neq 0$;
- D_1 is, for $t \neq 0$, the inverse image of the curve $z+t = 0$; while, for $t = 0$, it is only its strict transform, i.e. the divisor D considered previously, made up of two fibres;
- D_2 is an empty divisor for $t = 0$, and the nodal curve N for $t \neq 0$.

Remark 2.5. Part iii) of the previous remark shows that, as $t \to 0$, one subtracts the nodal divisor N to D_2, and adds it to D_3; while for D_1, it specializes to $D + N$, and then we subtract N.
This is precisely the algorithm which applies when passing from extended Burniat to Burniat divisors.

The really interesting part of the story comes now: the family X_t admits a simultaneous resolution only after that we perform a base change $t = \tau^2$ and the equation of X_t becomes

$$X_\tau = \{w^2 - \tau^2 = uv\}.$$

Definition 2.6. *Let $\mathfrak{X} \to T'$ be the family where*

$$\mathfrak{X} = \{(u, v, w, \tau) | w^2 - \tau^2 = uv\}$$

and T' is the affine line with coordinate τ.
Define $\mathcal{S} \subset \mathfrak{X} \times \mathbb{P}^1$ to be one of the small resolutions of $\mathfrak{X}$, and $\mathcal{S}'$ to be the other one, namely:

$$\mathcal{S} : \{(u, v, w, \tau)(\xi) \in \mathfrak{X} \times \mathbb{P}^1 | \ \frac{w - \tau}{u} = \frac{v}{w + \tau} = \xi\}$$
$$\mathcal{S}' : \{(u, v, w, \tau)(\eta) \in \mathfrak{X} \times \mathbb{P}^1 | \ \frac{w + \tau}{u} = \frac{v}{w - \tau} = \eta\}.$$

Let G be the group $G \cong (\mathbb{Z}/2\mathbb{Z})^2$ acting on $\mathfrak{X}$ trivially on the variable τ, and else as in Proposition 2.3. Let further σ_4 act by $\sigma_4(u, v, w, \tau) = (u, v, w, -\tau)$, let $G' \cong (\mathbb{Z}/2\mathbb{Z})^3$ be the group generated by G and σ_4, and let $H \cong (\mathbb{Z}/2\mathbb{Z})^2$ be the subgroup $\{Id, \sigma_2, \sigma_1\sigma_4, \sigma_3\sigma_4\}$.

The following is a rephrasing and a generalization of a discovery of Atiyah in our context: we omit the simple proof. For more details and a discussion of how these examples fit into the general theory of moduli spaces, see the 'working guide' written by the second author in [Cat11].

Lemma 2.7. *The biregular action of G' on $\mathfrak{X}$ lifts only to a birational action on $\mathcal{S}$, respectively $\mathcal{S}'$. The subgroup H acts on $\mathcal{S}$, respectively $\mathcal{S}'$, as a group of biregular automorphisms.*
The elements of $G' \setminus H = \{\sigma_1, \sigma_3, \sigma_4, \sigma_2\sigma_4\}$ yield isomorphisms between $\mathcal{S}$ and $\mathcal{S}'$.
The group G acts on the punctured family $\mathcal{S} \setminus \mathcal{S}_0$, in particular it acts on each fibre $\mathcal{S}_\tau$.

Since σ_4 acts trivially on $\mathcal{S}_0$, the group G' acts on $\mathcal{S}_0$ through its direct summand G.
The biregular actions of G on $\mathcal{S} \setminus \mathcal{S}_0$ and on $\mathcal{S}_0$ do not patch together to a biregular action on $\mathcal{S}$, in particular σ_1 and σ_3 yield birational maps which are not biregular: they are called Atiyah flops (cf. [At58]*).*

3. Nodal Burniat surfaces deform to extended Burniat surfaces

In this section we will show that

- the canonical models X of nodal Burniat surfaces with $K_X^2 = 4$, together with the extended Burniat surfaces with $K_X^2 = 4$ are parametrized by a family with smooth connected base of dimension 3, which maps to the moduli space via a finite morphism;
- the canonical models X of nodal Burniat surfaces with $K_X^2 = 3$, together with the extended Burniat surfaces with $K_X^2 = 3$ are parametrized by a family with smooth connected base of respective dimension 4, which maps to the moduli space via a finite morphism.

We shall treat first the easier case

3.1. (Extended) Burniat surfaces with $K^2 = 4$.

Proposition 3.1. *There exists a family, with connected smooth 3-dimensional base*

$$B \subset \{(P_5, C_1, \Gamma_2) | C_1 \in |L - E_1|, \Gamma_2 \in |2L - E_2 - E_3 - E_4 - E_5|\}$$

parametrizing a flat family of canonical models, including exactly all the nodal Burniat surfaces and the extended Burniat surfaces with $K_X^2 = 4$. The family maps to the moduli space via a quasi-finite morphism.
Here, P_1, P_2, P_3, P_4 are the standard projective basis in $\mathbb{P}^2$, the point P_5 belongs to the line $\overline{P_1 P_4}$ and, in the blow up of the plane in the given five points $P_j, j = 1, \dots, 5$, C_1, Γ_2 are as in Definition 1.1 (C_1 is irreducible and either Γ_2 is irreducible, or splits as $N_1 + E_1 + |L - E_2 - E_3|$).

Proof. Recall that in this case $D_1 + D_3 = \Delta_1 + \Delta_3$, and that N_1 is a connected component of the above divisor $D_1 + D_3 = \Delta_1 + \Delta_3$.
We can therefore construct a family of double covers

$$\tilde{W}_b \to \tilde{Y}$$

such that the inverse image of N_1 is a (-1)-curve. Blowing down this (-1)-curve we get a family of finite double covers $W'_b \to Y'$, which are nodal and equisingular.
Consider the pull back of the divisors Δ_2 in the case where Γ_2 is irreducible, and of the divisors D_2 in the case where Γ_2 is reducible.
Since $\Delta_2 \equiv D_2 + N_1$, and the divisor N_1 becomes trivial on W'_b, since it contracts to a smooth point, it follows that all these divisors are linearly equivalent, and we have a family of divisors on the family W'_b

We consider then the family of double covers $X_b \to W'_b$ branched on these divisors, and on the nodes of W'_b.
Finally, assume that two surfaces S_1, S_2 in the above family are isomorphic, equivalently that their canonical models X_1, X_2 are isomorphic. Then this isomorphism would yield an isomorphism of the bicanonical morphisms of each X_j, hence we would have isomorphisms of the image normal Del Pezzo surfaces Y'_1, Y'_2, sending the respective triple of branch curves for X_1 to the ones for X_2.
Y'_j has exactly one node and contains exactly 12 lines (cf. Proposition 3.7 of [BC10]. Y'_j is obtained blowing down the (-2) curve N_1 and conversely, blowing up the singular point of Y'_j and five disjoint (-1)-curves we obtain an isomorphism with the plane. Since these (-1)-curves correspond exactly to the lines in Y'_j, we have only a finite number of such birational contractions to the plane, and each of them determines a triple of branch curves, and five points in the plane.
Therefore the number of surfaces S_2 in the family which are isomorphic to S_1 is finite.

□

3.2. (Extended) Burniat surfaces with $K^2 = 3$. We argue similarly, but it may be useful to make right away a simple geometrical observation.
Let P_1, P_2, P_3, P_4 be the standard projective basis in $\mathbb{P}^2$, and consider a line L' with $P_3 \in L'$, different from te coordinate lines: then the line configuration of a ternary Burniat surface is completely determined by the line L', since then

$$P_5 := L' \cap \overline{P_1P_4},\ P_6 := L' \cap \overline{P_2P_4}.$$

Proposition 3.2. *There exists a family, with connected smooth 4-dimensional base*

$$T \subset \{(L', \Gamma_1, \Gamma_2, \Gamma_3)\}$$

where L' is as above and $\Gamma_1, \Gamma_2, \Gamma_3$ are as in Definitions 1.4 and 1.6, parametrizing a flat family of canonical models, including exactly all the nodal Burniat surfaces and the extended Burniat surfaces with $K_X^2 = 3$. The family maps to the moduli space via a quasi-finite morphism.

Proof. Given a triple $(\Gamma_1, \Gamma_2, \Gamma_3)$, according to the reducibility of each Γ_i, there corresponds either a Burniat divisor, or an extended Burniat divisor. We take the corresponding bidouble cover of $\tilde{Y}$, hence we construct four families of smooth surfaces, which are not necessarily minimal. We take now the corresponding canonical models, which are finite bidouble covers of the normal Del Pezzo surface Y'.
Observe that, given $p' : \tilde{S} \to \tilde{Y}$, and $\pi : \tilde{Y} \to Y'$,

$$X = Spec(\pi_*(p')_*\mathcal{O}_{\tilde{S}}) = Spec(\mathcal{O}_{Y'} \bigoplus (\oplus_{i=1}^3 \mathcal{F}_i)).$$

Now the reflexive sheaves $\mathcal{F}_i$ correspond to Weil divisors on Y', and they are independent of $t \in T$ by virtue of 4) of remark 1.5.

The multiplication maps correspond to a family of Weil divisors on Y': whence we get a flat family on $Y' \setminus \mathrm{Sing}(Y')$. Locally around the nodes the structure of the deformation is as described in the previous section, therefore the family is flat everywhere.
The assertion that for a given surface S_1 in the family parametrized by T the number of surfaces S_2 in the family which isomorphic to S_1 is finite follows as in the case $K^2 = 4$, using that the three nodal cubic surface Y' contains only a finite number of lines.

□

Observe that proof given for the case $K^2 = 3$ works also in the case $K^2 = 4$.

4. Local deformations of the extended Burniat surfaces

The aim of this section is to calculate the dimension of $H^1(S, \Theta_S)$ for

- nodal Burniat surfaces with $K_S^2 = 4$,
- extended Burniat surfaces with $K_S^2 = 4$,
- nodal Burniat surfaces with $K_S^2 = 3$,
- extended Burniat surfaces with $K_S^2 = 3$.

The main results of this paragraph are the following (note that we use the notation introduced in Section 1:

Proposition 4.1. *1) Assume that S is a nodal Burniat surface with $K_S^2 = 4$ ($m = 2$). Then the dimension of the vector space*

$$H^0(\Omega^1_{\tilde{Y}}(\log(D_i))(K_{\tilde{Y}} + L_i)) = H^0(\Omega^1_{\tilde{Y}}(\log(D_i))(E_i - E_{i+2}))$$

is 1 for $i = 3$, else it is 0.
2) Consider instead extended Burniat divisors for $m = 2$, and the corresponding vector spaces

$$H^0(\Omega^1_{\tilde{Y}}(\log(\Delta_i))(K_{\tilde{Y}} + \Lambda_i)).$$

Then their dimensions are the same as in the Burniat case, namely, 1 for $i = 3$, else 0.
3) Assume that S is a Burniat surface with $K_S^2 = 3$ (m=3).
Then each vector space

$$H^0(\Omega^1_{\tilde{Y}}(\log(D_i))(K_{\tilde{Y}} + L_i)) = H^0(\Omega^1_{\tilde{Y}}(\log(D_i))(E_i - E_{i+2}))$$

is equal to 0.
4) In the case of (strictly or not strictly) extended Burniat divisors for $m = 3$ we have $\forall i$:

$$H^0(\Omega^1_{\tilde{Y}}(\log(\Delta_i))(K_{\tilde{Y}} + \Lambda_i)) = 0.$$

The proof of Proposition 4.1 is to be found in the second appendix.
Using Proposition 4.1 we can explicitly determine the several G-character spaces of $H^i(S, \Theta_S)$ and their dimensions (here $G = (\mathbb{Z}/2\mathbb{Z})^2 = \{1, g_1, g_2, g_3\}$) . In the following, given a G-space V, we denote by V^i, for $i \in 1, 2, 3$, the eigenspace corresponding to the character whose kernel consists of $\{1, g_i\}$.

Proposition 4.2. *1) Let S be the minimal model of a Burniat surface. Then the dimensions of the eigenspaces of the cohomology groups of the tangent sheaf Θ_S (for the natural $(\mathbb{Z}/2\mathbb{Z})^2$-action) are as follows.*

(1) $\underline{K^2 = 4}$ *of nodal type:*
$h^1(S, \Theta_S)^{\mathrm{inv}} = 2$, $h^2(S, \Theta_S)^{\mathrm{inv}} = 0$,
$h^1(S, \Theta_S)^3 = 1 = h^2(S, \Theta_S)^3$,
$h^j(S, \Theta_S)^i = 0$, *for* $i \in \{1, 2\}$, $\forall j$;

(2) $\underline{K^2 = 3}$:
$h^1(S, \Theta_S)^{\mathrm{inv}} = 1$, $h^2(S, \Theta_S)^{\mathrm{inv}} = 0$,
$h^1(S, \Theta_S)^i = 1$, $h^2(S, \Theta_S)^i = 0$, *for* $i \in \{1, 2, 3\}$.

2) Let S be a minimal model of an extended Burniat surface with $K_S^2 = 4$. Then the dimensions of the eigenspaces of the cohomology groups of the tangent sheaf Θ_S (for the natural $(\mathbb{Z}/2\mathbb{Z})^2$-action) are as follows.

- $h^1(S, \Theta_S)^{\mathrm{inv}} = 3$, $h^2(S, \Theta_S)^{\mathrm{inv}} = 0$,
- $h^1(S, \Theta_S)^i = 0 = h^2(S, \Theta_S)^i$, *for* $i \in \{1, 2\}$,
- $h^1(S, \Theta_S)^3 = 1 = h^2(S, \Theta_S)^3$.

3) Let S be the minimal model of an extended Burniat surface with $K_S^2 = 3$. Then the dimensions of the eigenspaces of the cohomology groups of the tangent sheaf Θ_S (for the natural $(\mathbb{Z}/2\mathbb{Z})^2$-action) are as follows:

(1) *strictly extended case:*
$h^1(S, \Theta_S)^{\mathrm{inv}} = 4$, $h^2(S, \Theta_S)^{\mathrm{inv}} = 0$,
$h^j(S, \Theta_S)^i = 0$, *for* $i \in \{1, 2, 3\}$;

(2) *case where exactly one conic, w.l.o.g. the conic Γ_1,because of symmetry, degenerates to two lines:*
$h^1(S, \Theta_S)^{\mathrm{inv}} = 3$, $h^2(S, \Theta_S)^{\mathrm{inv}} = 0$,
$h^1(S, \Theta_S)^i = 0 = h^2(S, \Theta_S)^i$, *for* $i \in \{1, 3\}$,
$h^1(S, \Theta_S)^2 = 1$, $h^2(S, \Theta_S)^2 = 0$;

(3) *case where exactly two conics, w.l.o.g. Γ_1, Γ_2, degenerate to two lines each:*
$h^1(S, \Theta_S)^{\mathrm{inv}} = 2$, $h^2(S, \Theta_S)^{\mathrm{inv}} = 0$,
$h^1(S, \Theta_S)^1 = 0 = h^2(S, \Theta_S)^1$,
$h^1(S, \Theta_S)^i = 1$, $h^2(S, \Theta_S)^i = 0$, *for* $i \in \{1, 3\}$.

For the reader's convenience the above dimensions can also be found in Table 1. We begin with an easy but useful observation

Lemma 4.3. *Assume that N is a connected component of a smooth divisor $D \subset Y$, where Y is a smooth projective surface. Moreover, let M be a divisor on Y. Then*

$$H^0(\Omega^1_Y(\log(D - N))(N + M)) = H^0(\Omega^1_Y(\log(D))(M))$$

provided $(K_Y + 2N + M) \cdot N < 0$.

Proof. The cokernel of $\Omega^1_Y(\log(D))(M) \to \Omega^1_Y(\log(D - N))(N + M)$ is supported on N and equal to $\Omega^1_N(N + M) = \mathcal{O}_N(K_Y + 2N + M)$.

□

The lemma will be applied several times in the case where $N \cong \mathbb{P}^1$ and $N^2 < 0$.
Another useful lemma which will be crucial in some calculation is the following

Lemma 4.4. *Assume that we have three linearly independent linear forms on $\mathbb{P}^2$, $l_1 := x_1, l_2 := x_2, l_3 := x_3$. Then*

(1) $H^0(\Omega^1_{\mathbb{P}^2}(2))$ *has as basis the three 1-forms, for* $j < i$,

$$\eta_{ji} := x_j dx_i - x_i dx_j = -\eta_{ij}.$$

(2) $H^0(\Omega^1_{\mathbb{P}^2}(\operatorname{dlog} l_1, \operatorname{dlog} l_2, \operatorname{dlog} l_3)(1))$ *has as basis the six 1-forms*

$$\omega_{ij} := \frac{x_j dx_i - x_i dx_j}{x_i}.$$

(3) $H^0(\Omega^1_{\mathbb{P}^2}(\operatorname{dlog} l_1, \operatorname{dlog} l_2, \operatorname{dlog} l_3)(2))$ *has as basis the three 1-forms* η_{ji}, *for* $j < i$, *plus the six 1-forms* $x_j\omega_{ij}$ *and the three 1-forms* $x_1\omega_{23}, x_2\omega_{31}, x_3\omega_{12}$.

Proof. 1) is well known and follows from the Euler sequence.
2) Take the chart $x_i \neq 0 \Leftrightarrow x_i = 1$: then in this chart $\omega_{ij} := -dx_j$ is a regular 1-form.
In the chart $x_j = 1$ we have $\omega_{ij} := \frac{dx_i}{x_i}$, while in the chart $x_h = 1$ we have $\omega_{ij} := x_j \frac{dx_i}{x_i} - dx_j$.
Hence ω_{ij} has logarithmic poles on $x_i = 0$, and the coefficient of the logarithmic term vanishes for $x_i = x_j = 0$, and is equal to 1 in $x_i = x_h = 0$.
The above observation shows the linear independence of the above 6 forms.
Moreover, ω_{ij} is an eigenvector with character λ for the $\mathbb{C}^*$-action $x_i \mapsto \lambda x_i$, hence $\omega_{ij} \in H^0(\Omega^1_{\mathbb{P}^2}(\operatorname{dlog} l_1, \operatorname{dlog} l_2, \operatorname{dlog} l_3)(1))$.
It suffices to show that this space has vector dimension equal to 6.
This follows however from the exact sequence

$$0 \to \Omega^1_{\mathbb{P}^2}(1) \to \Omega^1_{\mathbb{P}^2}(\operatorname{dlog} l_1, \operatorname{dlog} l_2, \operatorname{dlog} l_3)(1) \to \oplus_{i=1}^3 \mathcal{O}_{l_i}(1) \to 0$$

and the vanishing of $H^j(\Omega^1_{\mathbb{P}^2}(1))$ for $j = 0, 1$.
3) Observe that $\omega_{ij} = \frac{1}{x_i}\eta_{ji}$, so that $x_i\omega_{ij} = \eta_{ji} = -\eta_{ij} = x_j\omega_{ji}$.
Moreover, if $h \neq i, j$, $x_h\omega_{ij} - x_j\omega_{ih} = \eta_{jh}$, so that the products $x_r\omega_{ij}$ generate a subspace of dimension at most 12.
By the exact sequence

$$0 \to \Omega^1_{\mathbb{P}^2}(2) \to \Omega^1_{\mathbb{P}^2}(\operatorname{dlog} l_1, \operatorname{dlog} l_2, \operatorname{dlog} l_3)(2) \to \oplus_{i=1}^3 \mathcal{O}_{l_i}(2) \to 0$$

and since $H^1(\Omega^1_{\mathbb{P}^2}(2)) = 0$, $h^0(\mathcal{O}_{l_i}(2)) = 3$ we infer that the dimension is indeed 12.
Since $H^0(\mathcal{O}_{l_i}(2))$ is generated by $H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \otimes_{\mathbb{C}} H^0(\mathcal{O}_{l_i}(1))$ we conclude that the twelve 1-forms are a basis. □

Lemma 4.5. *Assume that we have two linearly independent linear forms on $\mathbb{P}^2$, $l_1 := x_1, l_2 := x_2$.*

(1) $H^0(\Omega^1_{\mathbb{P}^2}(\operatorname{dlog} l_1, \operatorname{dlog} l_2)(1))$ *has as basis the 4 forms*

$$\omega_{ij} := \frac{x_j dx_i - x_i dx_j}{x_i}, \ 1 \leq i, j \leq 3, \ i \neq 3.$$

(2) $H^0(\Omega^1_{\mathbb{P}^2}(\operatorname{dlog} l_1, \operatorname{dlog} l_2)(2))$ *has as basis the 3 forms* η_{ji}*, for* $j < i$*, plus the 6 forms* $x_2\omega_{12}, x_1\omega_{21}, x_3\omega_{13}, x_3\omega_{23}, x_2\omega_{12}, x_1\omega_{23}$.

Proof. Follows from Lemma 4.4 observing that $H^0(\Omega^1_{\mathbb{P}^2}(\operatorname{dlog} l_1, \operatorname{dlog} l_2)(i))$ is a subspace of $H^0(\Omega^1_{\mathbb{P}^2}(\operatorname{dlog} l_1, \operatorname{dlog} l_2, \operatorname{dlog} l_3)(i))$. The above two sets of vectors are linearly independent and the dimensions are 4, resp. 9. □

Corollary 4.6. *1) Let* $\omega \in H^0(\Omega^1_{\mathbb{P}^2}(\operatorname{dlog} l_1, \operatorname{dlog} l_2)(1))$.
Then there are complex numbers a_{ij} *such that:*

$$\omega = a_{12}\omega_{12} + a_{21}\omega_{21} + a_{13}\omega_{13} + a_{23}\omega_{23} = \frac{dx_1}{x_1}(a_{12}x_2 - a_{21}x_1 + a_{13}x_3) + \\ + \frac{dx_2}{x_2}(-a_{12}x_2 + a_{21}x_1 + a_{23}x_3) + dx_3(-a_{13} - a_{23}).$$

2) Let $\omega \in H^0(\Omega^1_{\mathbb{P}^2}(\operatorname{dlog} l_1, \operatorname{dlog} l_2)(2))$*: then we can write*

$$\omega = a_{12}\eta_{12} + a_{13}\eta_{13} + a_{23}\eta_{23} + a_{212}x_2\omega_{12} + a_{121}x_1\omega_{21} + a_{313}x_3\omega_{13} + \\ + a_{323}x_3\omega_{23} + a_{213}x_2\omega_{13} + a_{123}x_1\omega_{23} =$$

$$= \frac{dx_1}{x_1}(-a_{12}x_1x_2 - a_{13}x_3x_1 + a_{212}x_2^2 - a_{121}x_1^2 + a_{313}x_3^2 + a_{213}x_2x_3) + \\ + \frac{dx_2}{x_2}(a_{12}x_1x_2 - a_{23}x_3x_2 - a_{212}x_2^2 + a_{121}x_1^2 + a_{323}x_3^2 + a_{123}x_1x_3) + \\ + dx_3(a_{13}x_1 + a_{23}x_2 - a_{313}x_3 - a_{323}x_3 - a_{213}x_2 - a_{123}x_1).$$

3) Any $\omega \in H^0(\Omega^1_{\mathbb{P}^2}(\operatorname{dlog} l_1, \operatorname{dlog} l_2, \operatorname{dlog} l_3)(1))$ *can be written as:*

$$\omega = a_{12}\omega_{12} + a_{13}\omega_{13} + a_{23}\omega_{23} + a_{21}\omega_{21} + a_{31}\omega_{31} + a_{32}\omega_{32} =$$

$$= \frac{dx_1}{x_1}(a_{12}x_2 - a_{21}x_1 + a_{13}x_3 - a_{31}x_1) + \\ + \frac{dx_2}{x_2}(-a_{12}x_2 + a_{21}x_1 + a_{23}x_3 - a_{32}x_2) + \\ + \frac{dx_3}{x_3}(-a_{13}x_3 + a_{31}x_1 + a_{32}x_2 - a_{23}x_3).$$

4) Any $\omega \in H^0(\Omega^1_{\mathbb{P}^2}(\operatorname{dlog} l_1, \operatorname{dlog} l_2, \operatorname{dlog} l_3)(2))$ *can be written as:*

$$\omega = a_{12}\eta_{12} + a_{13}\eta_{13} + a_{23}\eta_{23} + a_{212}x_2\omega_{12} + a_{313}x_3\omega_{13} + a_{323}x_3\omega_{23} + \\ + a_{121}x_1\omega_{21} + a_{131}x_1\omega_{31} + a_{232}x_2\omega_{32} + a_{123}x_1\omega_{23} + a_{231}x_2\omega_{31} + a_{312}x_3\omega_{12} =$$

$$
\begin{aligned}
= \frac{dx_1}{x_1}(-a_{12}x_1x_2 - a_{13}x_3x_1 + a_{212}x_2^2 + a_{313}x_3^2 - \\
- a_{121}x_1^2 - a_{131}x_1^2 - a_{231}x_2x_1 + a_{312}x_3x_2) + \\
+ \frac{dx_2}{x_2}(a_{12}x_1x_2 - a_{23}x_3x_2 - a_{212}x_2^2 + a_{121}x_1^2 + a_{323}x_3^2 + \\
+ a_{123}x_1x_3 - a_{232}x_2^2 - a_{312}x_3x_2) + \\
+ \frac{dx_3}{x_3}(a_{13}x_1x_3 + a_{23}x_2x_3 - a_{313}x_3^2 - a_{323}x_3^2 + a_{131}x_1^2 + \\
+ a_{232}x_2^2 - a_{123}x_1x_3 + a_{231}x_1x_2).
\end{aligned}
$$

Proof. This is an easy verification. □

Now we have prepared everything for the proof of Proposition 4.1. Since the proof is long and technical we prefer to put it in a separate section (cf. Appendix 8).

Proof of Proposition 4.2. For the invariant part, the calculation goes exactly as the proof of Lemma 2.9. of [BC10], using that $h^i(\Theta_{\tilde{S}})^{inv} = h^i(\Theta_S)^{inv}$.
For the other character spaces, we use the same argument as in Lemma 2.12. in [BC10] to calculate $\chi(\Omega^1_{\tilde{Y}}(\log D_i)(K_{\tilde{Y}} + \mathcal{L}_i))$ (resp. $\chi(\Omega^1_{\tilde{Y}}(\log \Delta_i)(K_{\tilde{Y}} + \Lambda_i))$ for extended Burniat surfaces).
We first observe that

$$\chi(\Omega^1_{\tilde{Y}}(\log D_i)(K_{\tilde{Y}} + \mathcal{L}_i)) = \chi(\Omega^1_{\tilde{Y}}(K_{\tilde{Y}} + \mathcal{L}_i)) + \chi(\mathcal{O}_{D_i}(K_{\tilde{Y}} + \mathcal{L}_i)),$$

(and analogously for $\chi(\Omega^1_{\tilde{Y}}(\log \Delta_i)(K_{\tilde{Y}} + \Lambda_i))$ for extended Burniat surfaces). Moreover, note that with the same calculation as in Lemma 2.12. of [BC10], we see that $\chi(\Omega^1_{\tilde{Y}}(K_{\tilde{Y}} + \mathcal{L}_i)) = K^2_{\tilde{Y}} - 12$.
Each D_i (resp. Δ_i) consists of k_i irreducible connected components, each of them being a smooth rational curve. Write $D_i = D_{i,1} + \ldots + D_{i,k_i}$ as disjoint union of smooth rational curves and let $n_j := D_{i,j} \cdot (K_{\tilde{Y}} + L_i)$. Then

$$\chi(\mathcal{O}_{D_i}(\log D_i)(K_{\tilde{Y}} + \mathcal{L}_i)) = \sum_{j=1}^{k_i} max(0, n_j + 1).$$

Therefore

$$\chi(\Omega^1_{\tilde{Y}}(\log D_i)(K_{\tilde{Y}} + \mathcal{L}_i)) = K^2_{\tilde{Y}} - 12 + \sum_{j=1}^{k_i} max(0, n_j + 1).$$

We summarize the calculations in the following table (note that we write χ_i for $\chi(\Omega^1_{\tilde{Y}}(\log D_i)(K_{\tilde{Y}} + \mathcal{L}_i))$). The values for $h^2(\Theta_{\tilde{S}})^i$ have been calculated in prop. 4.1. The notation: extended case (2), resp. (3), refers to Proposition 4.2.
Moreover, we use Lemma 9.22 of [Cat88] to compare $h^1(\Theta_{\tilde{S}})$ and $h^1(\Theta_S)$: it asserts that for a single blow up of a point P

$$\pi_* \Theta_{\tilde{S}} = \mathfrak{m}_P \Theta_S, \quad \mathcal{R}^1 \pi_* \Theta_{\tilde{S}} = 0.$$

Table 1.

K_S^2	i	$(n_1, \dots, n_{k_i})$	χ_i	$h^2(\Theta_{\tilde{S}})^i$	$h^1(\Theta_{\tilde{S}})^i$	$h^2(\Theta_S)^i$	$h^1(\Theta_S)^i$
	0		-2	0	2	0	2
4	1	$(1,1,1,1)$	0	0	0	0	0
n.B.	2	$(1,1,1,1)$	0	0	0	0	0
	3	$(1,1,1,1)$	0	1	1	1	1
	0		-3	0	3	0	3
4	1	$(1,1,1)$	-2	0	2	0	0
ext.	2	$(1,1,1)$	-2	0	2	0	0
	3	$(1,1,1,0,1)$	1	1	0	1	0
	0		-1	0	1	0	1
3	1	$(1,1,1,1)$	-1	0	1	0	1
B.	2	$(1,1,1,1)$	-1	0	1	0	1
	3	$(1,1,1,1)$	-1	0	1	0	1
	0		-4	0	4	0	4
3	$1,2,3$	$(1,1,0)$	-4	0	4	0	0
str.ext.							
	0		-3	0	3	0	3
3	1	$(1,1,0,1)$	-2	0	2	0	0
ext. (2)	2	$(1,1)$	-5	0	5	0	1
	3	$(1,1,0,1)$	-2	0	2	0	0
	0		-2	0	2	0	2
3	1	$(1,1,1,1,0)$	0	0	0	0	0
ext. (3).	2	$(1,1,1)$	-3	0	3	0	1
	3	$(1,1,1)$	-3	0	3	0	1

$\square$

From the above calculations and from Propositions 3.1, 3.2 follow all the statements of our first main theorem, with the exception of the statement that $\mathcal{NEB}_4$ is a connected component. It follows that $\mathcal{NEB}_4$ is open, while the statement that $\mathcal{NEB}_4$ is closed will be shown in the forthcoming section.

Theorem 0.1 *1) The subset $\mathcal{NEB}_4$ of the moduli space of canonical surfaces of general type $\mathfrak{M}^{can}_{1,4}$ given by the union of the open set corresponding to* extended *Burniat surfaces with $K_S^2 = 4$ with the irreducible closed set parametrizing nodal Burniat surfaces with $K_S^2 = 4$ is an irreducible connected component, normal, unirational of dimension 3.*
Moreover the base of the Kuranishi family of deformations of such a minimal model S is smooth.
2) The subset $\mathcal{NEB}_3$ of the moduli space of canonical surfaces of general type $\mathfrak{M}^{can}_{1,3}$ corresponding to extended *and nodal Burniat surfaces with $K_S^2 = 3$ is an irreducible open set, normal, unirational of dimension 4.*
Moreover the base of the Kuranishi family of S is smooth.

Proof. We show here the smoothness of the base $\mathfrak{B}$ of the Kuranishi family. This follows, in the case $K^2 = 4$, from the fact that we have shown (Proposition 3.1) the existence of a family with smooth three dimensional base B mapping to the moduli space, hence also to the base $\mathfrak{B}$ of the Kuranishi family, in a quasi-finite way. Since the tangent dimension of $\mathfrak{B}$ is also equal to three, it follows that the dimension of $\mathfrak{B}$ is exactly three, and that $\mathfrak{B}$ is smooth.
The argument in the case $K^2 = 3$ is identical, using Proposition 3.2.

□

We are also almost done with the proof of our second main theorem

Theorem 0.2 *The deformations of nodal Burniat surfaces with* $K_S^2 = 4, 3$ *to extended Burniat surfaces with* $K_S^2 = 4, 3$ *yield examples where* $\mathrm{Def}(S, (\mathbb{Z}/2\mathbb{Z})^2) \to \mathrm{Def}(X, (\mathbb{Z}/2\mathbb{Z})^2)$ *is not surjective.*
Moreover, $\mathrm{Def}(S, (\mathbb{Z}/2\mathbb{Z})^2) \subsetneq \mathrm{Def}(S)$*, whereas for the canonical model we have:* $\mathrm{Def}(X, (\mathbb{Z}/2\mathbb{Z})^2) = \mathrm{Def}(X)$.
The moduli space of pairs, of an extended (or nodal) Burniat surface with $K_S^2 = 4, 3$ *and a* $(\mathbb{Z}/2\mathbb{Z})^2$*-action, is disconnected; but its image in the moduli space is a connected open set.*

Proof. By Propositions 3.1 and 3.2 we have two families with smooth connected rational base of dimension 3, resp. 4, parametrizing all the canonical models X of the surfaces in $\mathcal{NEB}_4$, resp. $\mathcal{NEB}_3$.
In the previous Theorem 0.1 we showed that the base of the Kuranishi family of S is smooth, hence base change of these families yield the Kuranishi family of S.
The above families of canonical models X yield the Kuranishi family of X, e.g., by the Theorem of Burns and Wahl.
Propositions 3.1 and 3.2, exhibiting all the canonical models as bidouble covers of normal Del Pezzo surfaces, immediately show that $\mathrm{Def}(X, (\mathbb{Z}/2\mathbb{Z})^2) = \mathrm{Def}(X)$.
Let now S be a nodal Burniat surface.
Since, by (7.1), page 23, of [Cat88] $\mathrm{Def}(S, (\mathbb{Z}/2\mathbb{Z})^2) \subsetneq \mathrm{Def}(S)$ is the intersection with $H^1(\Theta_S)^0$, which is the smooth subvariety corresponding to the nodal Burniat surfaces, we obtain that $\mathrm{Def}(S, (\mathbb{Z}/2\mathbb{Z})^2) \subsetneq \mathrm{Def}(S)$.
On the other hand, for instance in the case $K_S^2 = 4$, we explicitly see that $\mathcal{NEB}_4$ is the union of two families of bidouble covers, the family of nodal Burniat surfaces, respectively the family of extended Burniat surfaces: hence the moduli space of pairs $(S, (\mathbb{Z}/2\mathbb{Z})^2)$ has exactly two connected components.

□

5. One parameter limits of extended Burniat surfaces with $K_S^2 = 4$

In this section we shall prove the following:

Theorem 5.1. *The family of extended Burniat surfaces with $K_S^2 = 4$ yields, together with the family of nodal Burniat surfaces with $K_S^2 = 4$, a closed subset $\mathcal{NEB}_4$ of the moduli space.*

This will be accomplished through the study of limits of one parameter families of such extended Burniat surfaces: we shall indeed show that only nodal Burniat surfaces (or extended Burniat surfaces) occur.
Let Y' be a normal $\mathbb{Q}$- Gorenstein surface and denote the dualizing sheaf of Y' by $\omega_{Y'}$.
Then there is a minimal natural number m such that $\omega_{Y'}^{\otimes m}$ is an invertible sheaf and it makes sense to define $\omega_{Y'}$ to be ample, respectively anti-ample; Y' is Gorenstein iff $m = 1$.
We recall the following result which was shown in [BC10].

Proposition 5.2. *Let T be a smooth affine curve, $t_0 \in T$, and let $f\colon \mathcal{X} \to T$ be a flat family of canonical surfaces. Suppose that $\mathcal{X}_t$ is the canonical model of a Burniat surface with $4 \leq K^2_{\mathcal{X}_t}$ for $t \neq t_0 \in T$. Then there is a biregular action of $G := (\mathbb{Z}/2\mathbb{Z})^2$ on $\mathcal{X}$ yielding a one parameter family of finite $(\mathbb{Z}/2\mathbb{Z})^2$-covers*

$$\begin{tikzcd} \mathcal{X} \arrow[rr] \arrow[dr, "f"'] & & \mathcal{Y} \arrow[dl] \\ & T & \end{tikzcd},$$

(i.e., $\mathcal{X}_t \to \mathcal{Y}_t$ is a finite $(\mathbb{Z}/2\mathbb{Z})^2$-cover), such that $\mathcal{Y}_t$ is a Gorenstein Del Pezzo surface for each $t \in T$.

Observe that the above result remains true if we replace "Burniat surface" by "extended Burniat surface".
This implies immediately the following:

Corollary 5.3. *Consider a one parameter family of bidouble covers $\mathcal{X} \to \mathcal{Y}$ as in prop. 5.2. Then the branch locus of $\mathcal{X}_{t_0} \to \mathcal{Y}_{t_0}$ is the limit of the branch locus of $\mathcal{X}_t \to \mathcal{Y}_t$, and it is reduced.*

Note that the limit of a line on the del Pezzo surfaces $\mathcal{Y}_t$ is a line on the del Pezzo surface $\mathcal{Y}_{t_0}$, and, as a consequence of the above assertion,
two lines in the branch locus in $\mathcal{Y}_t$ cannot tend to the same line in $\mathcal{Y}_{t_0}$.

Remark 5.4. Let X be the canonical model of an extended or nodal Burniat surface with $K_X^2 = 4$. Recall that X is smooth for a general member of the family of extended Burniat surfaces, whereas X has one ordinary node if X is the canonical model of a nodal Burniat surface with $K^2 = 4$.
In the extended case the branch locus consists of the union of 3 hyperplane sections, containing 8 lines, 2 conics and the node. In the nodal Burniat case one of the conics degenerates to two lines, hence the branch locus consists instead of 10 lines and one conic.

The first step towards proving Theorem 5.1 is the following:

Proposition 5.5. *Consider a one parameter family of bidouble covers $\mathcal{X} \to \mathcal{Y}$ as in prop. 5.2 except that $\mathcal{X}_t$ is an extended Burniat surface with $K^2_{X_t} = 4$ for $t \neq 0$.*
Then $\mathcal{Y}_0$ is a normal Del Pezzo surface with exactly one node as singularity.

Lemma 5.6. *A normal singular Del Pezzo surface with $K^2_{\mathcal{Y}_0} = 4$ containing at least 8 lines has as singularities either*

1. *one node, and then it contains 12 lines, or*
2. *two nodes, and then it contains 9 lines, or*
3. *an A_2 singularity, and then it contains 8 lines, 4 of which pass through the singular point.*

Proof. The assertion is a generalization of Proposition 3.6 of [BC10], page 581, see especially the proof in the appendix ibidem, pages 585-587.
We blow up $r = 5$ points in the plane.
By [BC10], p.586, we have that the loss of number of lines when one has a chain of k infinitely near points is bounded by

$$(k-1)(r-(k-1)) + \frac{1}{2}(k+1)(k-2) = (k-1)(5-(k-1)) + \frac{1}{2}(k+1)(k-2).$$

Since a smooth Del Pezzo surface of degree four has 16 lines, $k \geq 4$ implies that the number of lines is less than or equal to $16 - 11 = 5$.
If there is a chain with $k = 3$, the same estimate gives a loss of 8, and we cannot then have other (-2)-curves, else the number would be strictly smaller than $16 - 8 = 8$.
In this case we get an A_2 singularity and 8 lines.
In fact, in the chosen plane model we have 5 points lying on an irreducible conic C, of which P_2 infinitely near to P_1, and P_3 infinitely near to P_2. The lines are given by

$$E_3, E_4, E_5, |L - E_1 - E_4|, |L - E_1 - E_5|, |L - E_1 - E_2|, |L - E_4 - E_5|, C',$$

where C' is the strict transform of C.
In this case the 4 lines passing through the singular point are

$$E_3, |L - E_1 - E_4|, |L - E_1 - E_5|, |L - E_1 - E_2|.$$

In the case where there is no chain of three infinitely near points by a standard Cremona transformation as in [BC10], ibidem, we may reduce to the case where there are no infinitely near points and then we have
that the weak Del Pezzo surface is $\hat{Y}_0 := \hat{\mathbb{P}}^2(P_1, \dots, P_5)$, where P_1, P_2, P_3 and P_1, P_4, P_5 are collinear.
Then $\hat{Y}_0$ contains nine lines. In fact, the set of lines of $\hat{Y}_0$ is:

$$\mathcal{L} := \{E_1, \dots, E_5, L - E_2 - E_4, L - E_2 - E_5, L - E_3 - E_4, L - E_3 - E_5\}.$$

□

Proof of prop. 5.5. Since the branch locus of $\mathcal{X}_t \to \mathcal{Y}_t$ contains eight lines for $t \neq 0$, also the branch locus of $\mathcal{X}_0 \to \mathcal{Y}_0$ contains eight lines.
We want to show that cases (2) and (3) of the previous lemma cannot occur.

We start by eliminating case (3).
Here, the A_2 singularity must be a limit of the node of $\mathcal{Y}_t$, hence the bidouble cover is branched at the singular point.
The bidouble cover is a RDP, hence, looking at table 2, page 90 of [Cat87], and table 3, page 93 ibidem, we see that the branch locus is analytically isomorphic to

- an ordinary cusp $\{y = 0 = z^2 + x^3 = 0\}$ for $E_6 = \{z^2 + x^3 + t^4 = 0\} \to A_2 = \{z^2 + x^3 + y^2 = 0\}$,
- two lines $\{x = 0 = z^2 + y^2 = 0\}$ for $A_5 = \{z^2 + w^6 + y^2 = 0\} \to A_2 = \{z^2 + x^3 + y^2 = 0\}$,
- two lines $\{x = 0 = z^2 + y^2 = 0\}$ for the composition of $A_2 \to A_5$ (ramified only at the singular point) with the previous $A_5 = \{z^2 + w^6 + y^2 = 0\} \to A_2 = \{z^2 + x^3 + y^2 = 0\}$.

We observe however that by our previous arguments the branch locus contains the 8 lines, 4 of which pass through the A_2 singularity, contradicting the above local description of the branch locus.
Assume now by contradiction that we have case (2), i.e., $\mathcal{Y}_0$ has two nodes. Then

Claim 5.7. *E_1 is not a component of the total branch locus Δ of $\hat{X}_0 \to \hat{Y}_0$, i.e.,*

$$E_2, \dots, E_5, L - E_2 - E_4, L - E_2 - E_5, L - E_3 - E_4, L - E_3 - E_5$$

are exactly the 8 lines contained in Δ.

Proof of the claim. Assume that E_1 is contained in the total branch locus Δ of the bidouble cover $\hat{X}_0 \to \hat{Y}_0$. Then Δ contains three lines intersecting one of the two (-2) curves. But a bidouble cover of a node branched in at least three lines does not give a rational double point, as shown by the classification recalled in Section 2. A contradiction. □

Since for each node ν_1, ν_2 there are two lines in the total branch divisor passing through ν_i, it follows by the classification given in Section 2, that $N_1, N_2 \leq \Delta$ and that $(\Delta - N_i)N_i = 2$.
Denote by $\pi \colon \hat{Y}_0 \to Y'$ be the desingularization map.
Then $\pi_*(\Delta) \equiv -3K_{Y'}$, whence

$$\Delta \equiv -3K_{\hat{Y}_0} + n_1 N_1 + n_2 N_2.$$

Then $2 = (\Delta - N_i)N_i = (n_i - 1)N_i^2 = 2(1 - n_i) \iff n_i = 0$.
We conclude that

$$\Delta \equiv -3K_{\hat{Y}_0}.$$

Observe that

$$-3K_{\hat{Y}_0} - \sum_{l \in \mathcal{L} \setminus \{E_1\}} l - N_1 - N_2 \equiv 3L - E_1 - E_2 - \ldots - E_5.$$

Since no other component of Δ can intersect the (-2)-curves, we see immediately that the remaining two components of Δ are:

$$L - E_1, 2L - E_2 - E_3 - E_4 - E_5.$$

We write now

$$\Delta_1 = \lambda_1 L - E_1 - a_2E_2 - a_3E_3 - a_4E_4 - a_5E_5,$$
$$\Delta_2 = \lambda_2 L - E_1 - b_2E_2 - b_3E_3 - b_4E_4 - b_5E_5,$$
$$\Delta_3 = \lambda_3 L - E_1 - c_2E_2 - c_3E_3 - c_4E_4 - c_5E_5.$$

Here we have used that, since E_1 is not a component of Δ and since $\Delta_i + \Delta_j$ has to be divisible by two, the only possibility is

$$E_1 \cdot (\Delta_1, \Delta_2, \Delta_3) = (1, 1, 1).$$

Note that, since $\lambda_1 + \lambda_2 + \lambda_3 = 9$ (and again since $\Delta_i + \Delta_j$ is divisible by two) we have:

$$(\lambda_1, \lambda_2, \lambda_3) \in \{(3, 3, 3), (1, 3, 5), (1, 1, 7)\}.$$

Moreover, since the branch divisor is reduced, for each i it happens that, among the three numbers a_i, b_i, c_i, there cannot be two which are negative, and if one such a number is negative, then it is $= -1$; hence the only possibilities are: $\{a_i, b_i, c_i\} = \{1, 1, 1\}$ or $\{-1, 1, 3\}$, for $i \in \{2, \ldots, 5\}$.

$\underline{(\lambda_1, \lambda_2, \lambda_3) = (3, 3, 3)}$: then we get for the character sheaves:

$$\mathcal{L}_1 = \mathcal{O}(3L - E_1 - \frac{b_2 + c_2}{2}E_2 - \frac{b_3 + c_3}{2}E_3 - \frac{b_4 + c_4}{2}E_4 - \frac{b_5 + c_5}{2}E_5),$$
$$\mathcal{L}_2 = \mathcal{O}(3L - E_1 - \frac{a_2 + c_2}{2}E_2 - \frac{a_3 + c_3}{2}E_3 - \frac{a_4 + c_4}{2}E_4 - \frac{a_5 + c_5}{2}E_5),$$
$$\mathcal{L}_3 = \mathcal{O}(3L - E_1 - \frac{a_2 + b_2}{2}E_2 - \frac{a_3 + b_3}{2}E_3 - \frac{a_4 + b_4}{2}E_4 - \frac{a_5 + b_5}{2}E_5).$$

Note that $(a_i, b_i, c_i) = (1, 1, 1)$ for all $i \in \{2, \ldots, 5\}$ implies that $p_g(\mathcal{X}_0) \neq 0$, whence w.l.o.g.

$$(a_2, b_2, c_2) = (-1, 1, 3).$$

Then $E_2 \leq \Delta_1$, and by the local calculations in Section 2 this implies that also $E_3 \leq \Delta_1$ (since the two lines of the branch locus intersecting a (-2)-curve belong to the same Δ_i). Therefore

$$(a_3, b_3, c_3) \in \{(-1, *, *), (1, 1, 1)\}.$$

Again using $p_g(\mathcal{X}_0) = 0$, we conclude (looking at $\mathcal{L}_3$) that (up to exchanging P_4 with P_5)

$$(a_4, b_4, c_4) \in \{(3, 1, -1), (1, 3, -1)\},$$

and again this implies that

$$(a_5, b_5, c_5) \in \{(*, *, -1), (1, 1, 1)\}.$$

But in all of these cases we have

$$\frac{a_i + c_i}{2} \in \{0, 1\} \, \forall \in \{2, \ldots, 5\},$$

contradicting $p_g = 0$.

$\underline{(\lambda_1, \lambda_2, \lambda_3) = (1, 3, 5):}$ here we have

$$\mathcal{L}_1 = \mathcal{O}(4L - E_1 - \frac{b_2 + c_2}{2}E_2 - \frac{b_3 + c_3}{2}E_3 - \frac{b_4 + c_4}{2}E_4 - \frac{b_5 + c_5}{2}E_5),$$
$$\mathcal{L}_2 = \mathcal{O}(3L - E_1 - \frac{a_2 + c_2}{2}E_2 - \frac{a_3 + c_3}{2}E_3 - \frac{a_4 + c_4}{2}E_4 - \frac{a_5 + c_5}{2}E_5),$$
$$\mathcal{L}_3 = \mathcal{O}(2L - E_1 - \frac{a_2 + b_2}{2}E_2 - \frac{a_3 + b_3}{2}E_3 - \frac{a_4 + b_4}{2}E_4 - \frac{a_5 + b_5}{2}E_5).$$

Again, $p_g = 0$ implies that there is an $i \in \{2, \dots, 5\}$ such that $\frac{a_i+c_i}{2} = 2$. W.l.o.g. we can assume that $\frac{a_2+c_2}{2} = 2$. Therefore

$$(a_2, b_2, c_2) \in \{(3, -1, 1), (1, -1, 3)\},$$

whence

$$(a_3, b_3, c_3) \in \{(3, -1, 1), (1, -1, 3), (1, 1, 1)\}.$$

Then $\frac{b_2+c_2}{2}, \frac{b_3+c_3}{2} \leq 1$ and $\frac{b_4+c_4}{2}, \frac{b_5+c_5}{2} \leq 2$, which implies that $\mathcal{O}(L - E_2 - E_4) \subset \mathcal{O}(K_{\hat{Y}_0}) \otimes \mathcal{L}_1$, contradicting $p_g(\mathcal{X}_0) = 0$.

$\underline{(\lambda_1, \lambda_2, \lambda_3) = (1, 1, 7):}$ this case can be excluded since

$$4 = \Delta_3 \cdot (-K_{\hat{Y}_0}) = 3\lambda_3 - 1 - \sum_{i=2}^{5} c_i \Rightarrow 12 \geq \sum_{i=2}^{5} c_i = 16,$$

a contradiction.

This proves the proposition. □

Consider a one parameter family of bidouble covers $\mathcal{X} \to \mathcal{Y}$ as in prop. 5.5. Then $Y' := \mathcal{Y}_0$ is a normal Del Pezzo surface with exactly one node.
Let $\tilde{Y}$ be the blow up of Y' in the node and denote the exceptional (-2)-curve of $\tilde{Y}$ over the node by A.
The following result concludes the proof of Theorem 5.1.

Proposition 5.8. *For the limit of a one parameter family of extended Burniat surfaces with $K_S^2 = 4$ we have:*

(1) *if A does not intersect $\Delta - A$, then $\mathcal{X}_0$ is an extended Burniat surface with $K_S^2 = 4$;*
(2) *if A intersects $\Delta - A$, then $\mathcal{X}_0$ is a nodal Burniat surface with $K_S^2 = 4$.*

Proof. We can assume that $\tilde{Y} = \hat{\mathbb{P}}^2(P_1, \dots, P_5)$, and w.l.o.g. P_1, P_4, P_5 collinear, i.e., $A \equiv L - E_1 - E_4 - E_5$.
Recall that we have shown that in both cases A is contained in the branch locus, hence the two alternatives are that A is a connected component of the branch locus, or not.
1) In the first case, argueing as in Proposition 5.5, we get that the total branch locus is $\Delta \equiv -3K_{\tilde{Y}} + A$.

It is easy to see that $\tilde{Y}$ contains exactly 8 lines $l_1, \dots, l_8$ which do not intersect A. Then these 8 lines have to be contained in Δ.
Then $\Delta - A - \sum_{i=1}^{8} l_i \equiv 3L - \sum E_i$, which has to split into two Del Pezzo conics, which then have to be $L - E_1$ and $2L - \sum_{i=2}^{5} E_i$. Hence we get an extended Burniat surface.
2) Here $L - E_1 - E_4 - E_5 \equiv A \leq \Delta \equiv -K_{\tilde{Y}}$.
Observe that $\tilde{Y}$ contains exactly 4 lines intersecting A: $E_1, E_4, E_5, L - E_2 - E_3$. By our local calculations in Section 2 two of these four lines are components of the total branch divisor and the two other not.
W.l.o.g. we can assume $E_1, L - E_2 - E_3 \leq \Delta$. Since E_4 and E_5 are not contained in the branch divisor, we see (writing Δ_i as in the proof of Proposition 5.5) that$(a_4, b_4, c_4) = (a_5, b_5, c_5) = (1, 1, 1)$.
Now it is straightforward that $(\lambda_1, \lambda_2, \lambda_3) = (3, 3, 3)$ (use the same argument as in the proof of prop. 5.5 to exclude the cases $(1, 3, 5)$ and $(1, 1, 7)$).
Since $p_g = 0$, we have (up to a permutation of {1,2,3})

$$\frac{b_1 + c_1}{2} = \frac{a_2 + c_2}{2} = \frac{a_3 + b_3}{2} = 2.$$

W.l.o.g. we can assume $(a_1, b_1, c_1) = (-1, 1, 3)$; then $E_1, L - E_2 - E_3 \leq \Delta_1$. Therefore

$$(a_2, b_2, c_2) \in \{(3, -1, 1), (1, -1, 3)\}$$

and

$$(a_3, b_3, c_3) \in \{(3, 1, -1), (1, 3, -1)\}.$$

But only $(a_2, b_2, c_2) = (3, -1, 1)$ and $(a_3, b_3, c_3) = (1, 3, -1)$ is possible (since a cubic cannot have two triple points, i.e., this would contradict the effectivity of Δ_i for some i).
Therefore we get a nodal Burniat surface. □

6. Nodal and extended Burniat surfaces do not form a closed set for $K_S^2 = 3$

We are going to exhibit surfaces which are in the closure of the family of nodal and extended Burniat surfaces, but for which the image of the bicanonical map is a normal cubic with other singularities than 3 nodes.
In our first example we exhibit a 3 -dimensional family with a 4-nodal cubic as image.
Consider a specialization of the 6 points $P_1, \dots, P_6$ in $\mathbb{P}^2$ so that P_1, P_2, P_3 become collinear, and, more precisely, the point P_2 moves in the line joining P_4 and P_6 till it reaches the line joining P_1 and P_3.
Then $P_1, \dots, P_6$ are the vertices of a complete quadrilateral with sides N_1, N_2, N_3, N_4: here we identify N_4 to the (-2) curve $N_4 \equiv L - E_1 - E_2 - E_3$ on the weak Del Pezzo $\tilde{Y}$ of degree 3 obtained blowing up the 6 points. Our notation for N_1, N_2, N_3 remains the same, and $\tilde{Y}$ is the minimal resolution of the 4-nodal cubic surface $Y' := \Sigma$.

We consider exactly the same divisors as the strictly extended Burniat divisors in 4) of Definition 1.4. We obtain a three dimensional family of bidouble covers X of Σ, with total branch locus consisting of 9 connected components, namely:

$$N_1, N_2, N_3;\ \Gamma_1, \Gamma_2, \Gamma_3;\ G_1, G_2, G_3.$$

G_1, G_2, G_3 correspond to the three diagonals of the quadrilateral, and are the 3 lines of Σ not passing through the nodes, whereas $\Gamma_1, \Gamma_2, \Gamma_3$ are conics as in Definition 1.4. The canonical models X have therefore 4 nodes lying over the node of Σ corresponding to N_4.
We have therefore proven:

Proposition 6.1. *The closure of the (4-dimensional) open set corresponding to nodal and extended Burniat surfaces with $K_X^2 = 3$ contains a 3-dimensional family of canonical models which are bidouble covers of a 4-nodal cubic surface Σ.*
Each such surface X has 4 nodes, lying over one fixed node of Σ, and where the bicanonical map $\Phi_2 \colon X \to \Sigma$ is unramified.

In our second example we obtain a 3-dimensional family of bidouble covers of a cubic surface Y' with a singularity of type D_4.
We give this example using the different planar realization which was indeed the way we found our first description of the deformation of nodal Burniat surfaces with $K_S^2 = 3$ to extended Burniat surfaces.
To do this, we relabel the 6 points in the plane as follows:

$$P_3' := P_4,\ P_2' := P_5,\ P_1' := P_6.$$

We have therefore irreducible rational curves

$$D_{i,1} := L - E_i - E_{i+1},\ D_{i,2} := N_i = L - E_i - E'_{i+1} - E'_{i+2},$$
$$D_{i,3} := G_i = L - E_i - E'_i$$

on the weak Del Pezzo $\hat{Y}$ of degree 3.
Blowing down the 3 (-1) curves $D_{i,1}$ $(i = 1,2,3)$ first, and then the strict transform of the 3 (-2) curves $D_{i,2}$ ($i = 1,2,3$) we obtain another copy of the projective plane where one has blown up three points Q_i $(i = 1,2,3)$ and three points Q'_i $(i = 1,2,3)$, where Q'_i is infinitely near to Q_i.
We denote by slight abuse of notation by Q_i the full transform of the point Q_i, namely, the divisor $D_{i,1} + D_{i-1,2}$, and by Q'_i the full transform of the point Q'_i,namely, the divisor $D_{i,1}$.
The pull back of the system of lines in the new $\mathbb{P}^2$ is, by the Hurwitz formula, the linear system

$$\mathcal{L} := 4L - 2\sum_i E_i - \sum_i E'_i.$$

And the curve $D_{i,3}$ is linearly equivalent to

$$D_{i,3} \equiv \mathcal{L} - 2D_{i+1,1} - D_{i,2} = \mathcal{L} - Q_{i+1} - Q'_{i+1}.$$

Hence $D_{i-1,2} = Q_i - Q'_i$, and we can write the branch loci for the extended Burniat surfaces as:

$$\Delta_i \in D_{i,3} + D_{i+1,2} + |D_{i,3} + D_{i+1,3}| = D_{i,3} + |Q_{i-1} - Q'_{i-1}| + |D_{i,3} + D_{i+1,3}| =$$
$$= |\mathcal{L} - Q_{i+1} - Q'_{i+1}| + |Q_{i-1} - Q'_{i-1}| + |2\mathcal{L} - Q_{i+1} - Q'_{i+1} - Q_{i-1} - Q'_{i-1}| =$$
$$= |\mathcal{L} - Q_{i+1} - Q'_{i+1}| + N_{i-1} + |2\mathcal{L} - Q_{i+1} - Q'_{i+1} - Q_{i-1} - Q'_{i-1}|.$$

Now, we simply let the three points Q_1, Q_2, Q_3 become collinear, but we let the tangent directions Q'_i remain general.
The blow up of the plane in the 6 points possesses now 4 (-2) curves, the three curves N_1, N_2, N_3 and the strict transform N of the line through Q_1, Q_2, Q_3. Since N intersects each N_i and these are disjoint, the corresponding normal Del Pezzo surface Y' has a singularity of type D_4.
Letting the branch divisor be as before (namely, take pull backs of general conics in $|2\mathcal{L} - Q_{i+1} - Q'_{i+1} - Q_{i-1} - Q'_{i-1}|$), we obtain

PROPOSITION 6.2. *The closure of the (4-dimensional) open set corresponding to nodal and extended Burniat surfaces with $K_X^2 = 3$ contains a 3-dimensional family of canonical models which are bidouble covers of a normal cubic surface Y' with a singularity of type D_4.*
The branch locus on Y' has the singular point as an isolated point, and the local covering is determined by the epimorphism $D_4 \to (\mathbb{Z}/2\mathbb{Z})^2 = (D_4)^{ab}$ of the local fundamental group of the singularity to its abelianization.

Proof. The inverse image of the (-2) curves in the bidouble cover are: the inverse image N' of N, which is a (-8) curve, and, for each N_i, there is a pair of (-1) curves meeting N'. After contracting the 6 (-1) curves we obtain a (-2) curves.

□

7. APPENDIX 1: A CORRIGENDUM TO BURNIAT SURFACES II

Parts 1), 2) and 3) of the following lemma were contained in Lemma 2.10 of [BC10], while 4) corrects a wrongly stated assertion of 2) of loc. cit.
We also amend the proof for the correct assertions.

LEMMA 7.1. *Consider a finite set of distinct linear forms*

$$l_\alpha := y - c_\alpha x, \alpha \in A$$

vanishing at the origin in $\mathbb{C}^2$.
Let $p\colon Z \to \mathbb{C}^2$ be the blow up of the origin, let D_α be the strict transform of the line $L_\alpha := \{l_\alpha = 0\}$, and let E be the exceptional divisor.
Let $\Omega^1_{\mathbb{C}^2}((\mathrm{dlog}\, l_\alpha)_{\alpha\in A})$ be the sheaf of rational 1-forms η generated by $\Omega^1_{\mathbb{C}^2}$ and by the differential forms $d\log l_\alpha$ as an $\mathcal{O}_{\mathbb{C}^2}$-module and define similarly $\Omega^1_Z((\log D_\alpha)_{\alpha\in A})$. Then:
1) $p_*\Omega^1_Z(\log E)(-E) = \Omega^1_{\mathbb{C}^2}$,
2) $p_*\Omega^1_Z(\log E, (\log D_\alpha)_{\alpha\in A}) = \Omega^1_{\mathbb{C}^2}((\mathrm{dlog}\, l_\alpha)_{\alpha\in A})$,

3) $p_*\Omega^1_Z((\log D_\alpha)_{\alpha\in A}) =$
$\{\eta \in \Omega^1_{\mathbb{C}^2}((\mathrm{dlog}\, l_\alpha)_{\alpha\in A}) | \eta = \sum_\alpha g_\alpha \,\mathrm{dlog}\, l_\alpha + \omega, \omega \in \Omega^1_{\mathbb{C}^2}, \sum_\alpha g_\alpha(0) = 0\}$.
4) $p_*\Omega^1_Z((\log D_\alpha)_{\alpha\in A})(E) \supset \Omega^1_{\mathbb{C}^2}((\mathrm{dlog}\, l_\alpha)_{\alpha\in A})$ *and*

$$dim_{\mathbb{C}}[p_*\Omega^1_Z((\log D_\alpha)_{\alpha\in A})(E)/\Omega^1_{\mathbb{C}^2}((\mathrm{dlog}\, l_\alpha)_{\alpha\in A})] = d-2$$

is supported at the origin, where $d := |A|$. *More precisely, we have an exact sequence*

$$0 \to \Omega^1_{\mathbb{C}^2} \to p_*\Omega^1_Z((\log D_\alpha)_{\alpha\in A})(E) \to \bigoplus_{\alpha=1}^{d} \mathcal{O}_{D_\alpha}(0) \to \mathbb{C}^2_0 \to 0.$$

5)
Assume w.l.o.g. $c_1 = 0$ *in the following formulae: then*
$p_*\Omega^1_Z(\log D_1)(-E) \subset \Omega^1_{\mathbb{C}^2}(\mathrm{dlog}\, l_1)$ *is the subsheaf of forms*

$$\{\omega = \alpha dx + \beta \frac{dy}{y} | \beta(0) = 0, \frac{\partial \beta}{\partial y}(0) = 0, \frac{\partial \beta}{\partial x}(0) + \alpha(0) = 0\}.$$

6) $p_*\Omega^1_Z(-E) = \mathfrak{m}_0\Omega^1_{\mathbb{C}^2}$, *where* $\mathfrak{m}_0$ *is the maximal ideal of the origin* 0 *in the sheaf* $\mathcal{O}_{\mathbb{C}^2}$.
7) $p_*\Omega^1_Z(\log D_1, \log D_2)(-E) \subset \Omega^1_{\mathbb{C}^2}(\mathrm{dlog}\, l_1, \mathrm{dlog}\, l_2)$ *is the subsheaf of forms*

$$\{\omega = \alpha\frac{dx}{x} + \beta\frac{dy}{y} | \alpha(0) = 0, \beta(0) = 0, \frac{\partial(\alpha+\beta)}{\partial x}(0) = 0, \frac{\partial(\alpha+\beta)}{\partial y}(0) = 0\}.$$

Proof. We show 2), 3), 4), 5) and 7).
Observe that

$$p_*\Omega^1_Z((\log D_\alpha)_{\alpha\in A})(mE)$$

consists of rational differential 1-forms ω which, when restricted to $\mathbb{C}^2 \setminus \{0\}$, yield sections of $\Omega^1_{\mathbb{C}^2}((\mathrm{dlog}\, l_\alpha)_{\alpha\in A})$.
Therefore, in particular, $\omega \prod_{\alpha\in A} l_\alpha$ is a regular holomorphic 1-form on $\mathbb{C}^2$.
Hence ω, modulo holomorphic 1-forms, can be written as

$$\omega = \frac{f}{\prod_{\alpha\in A} l_\alpha} dx + \frac{g}{\prod_{\alpha\in A} l_\alpha} dy,$$

where f, g are Weierstrass pseudopolynomials of degree in y strictly less than $d := \mathrm{card}(A)$.
Since $dy = dl_\alpha + c_\alpha dx$, the condition that ω restricted to $\mathbb{C}^2 \setminus \{0\}$ yields a section of $\Omega^1_{\mathbb{C}^2}((\mathrm{dlog}\, l_\alpha)_{\alpha\in A})$ implies that $l_\alpha | (f + c_\alpha g)$.
Whence l_α divides $fx + yg$, and we conclude, since $\prod_{\alpha\in A} l_\alpha$ is a pseudo polynomial of degree d, that

$$fx + yg = c(x) \prod_{\alpha\in A} l_\alpha.$$

This allows us to write, modulo holomorphic 1-forms,

$$\omega = \frac{g(dy - \frac{y}{x}dx)}{\prod_{\alpha\in A} l_\alpha} + \frac{c}{x}dx,$$

where now $c \in \mathbb{C}$.
Let us pull back ω to Z, using local coordinates (x,t) such that $y = xt$, and where we make the assumption $c_\alpha \neq 0, \forall \alpha$.

$$p^*\omega = \frac{x^{-d} g(x,xt)(xdt)}{\prod_{\alpha \in A}(t - c_\alpha)} + \frac{c}{x} dx.$$

The pull back form has logarithmic poles along $E = \{x = 0\}$ iff $g(x,y)$ has multiplicity at least $d-1$ at the origin, and poles of order at most one along E iff $g(x,y)$ has multiplicity at least $d-2$ at the origin.
Observe that the d polynomials $P_\beta := \prod_{\alpha \in A, \alpha \neq \beta} l_\alpha$ are linearly independent and homogeneous of degree $d-1$, hence they generate the vector space of homogeneous polynomials of degree $d-1$, hence they generate the ideal of holomorphic functions vanishing at the origin of multiplicity at least $d-1$.
Hence $g(x,y)$ has multiplicity at least $d-1$ iff we can write

$$g = \sum_{\alpha \in A} g_\alpha P_\alpha.$$

And since g is a pseudo polynomial of degree $\leq d-1$, the g_α's are just functions of x.
In this case we can write

$$\omega = \frac{c}{x} dx + \sum_{\alpha \in A} \frac{g_\alpha}{l_\alpha}(dy - \frac{y}{x} dx) = \frac{1}{x}\left(cdx + \sum_{\alpha \in A} \frac{g_\alpha}{l_\alpha}(xdy - ydx)\right).$$

$$= \frac{1}{x}\left(cdx + \sum_{\alpha \in A} \frac{g_\alpha}{l_\alpha}(xdl_\alpha + xc_\alpha dx - ydx)\right) = \sum_{\alpha \in A} \frac{g_\alpha}{l_\alpha} dl_\alpha + \frac{1}{x} dx \left(c - \sum_{\alpha \in A} g_\alpha\right).$$

The above form ω does not have poles on the line $x = 0$ if and only if $c = (\sum_{\alpha \in A} g_\alpha(0))$.
Observing that the strict transform of the line $x = 0$ is not among the divisors D_α, we establish claim (2), while (3) follows since $c = 0$ iff there are no poles along E.
The very first assertion of (4) follows by (2), so let's proceed to verify the other assertions.
Assume now that $p^*\omega$ has poles of order 1 along E; equivalently, assume that $g(x,y)$ has multiplicity at least $d-2$ at the origin. Since we already dealt with the case where this multiplicity is at least $d-1$, we may assume that $g(x,y)$ is homogeneous of degree $d-2$, and that $c = 0$.
Argueing as before, the space of homogeneous polynomials of degree $d-2$ has as basis the $d-1$ polynomials $(\beta = 1, \ldots, d-1)$

$$Q_\beta := \prod_{\alpha \in A, \alpha \neq \beta, \alpha \neq d} l_\alpha.$$

Whence $g = \sum_{\alpha \in A, \alpha \neq d} g_\alpha Q_\alpha$, where $g_\alpha \in \mathbb{C}$, and we may write:

$$\omega = \sum_{\alpha=1}^{d-1} \frac{g_\alpha}{l_\alpha l_d}(dy - \frac{y}{x}dx).$$

Since we want no poles on the line $x = 0$, we must have

$$\sum_{\alpha=1}^{d-1} \frac{g_\alpha y}{y^2} = 0 \Leftrightarrow \sum_{\alpha=1}^{d-1} g_\alpha = 0.$$

Under this condition we may then write

$$\omega = \sum_{\alpha=1}^{d-1} \frac{g_\alpha}{l_\alpha l_d}(dl_\alpha),$$

which has logarithmic poles along $l_\alpha = 0$.
Logarithmic poles along $l_d = 0$ follow by writing

$$\omega = \sum_{\alpha=1}^{d-1} \frac{g_\alpha}{l_\alpha l_d}(dl_d) + \sum_{\alpha=1}^{d-1} \frac{g_\alpha(c_d - c_\alpha)}{l_\alpha l_d}dx,$$

and observing that $\sum_{\alpha=1}^{d-1} \frac{g_\alpha(c_d - c_\alpha)}{y - c_\alpha x}$ vanishes for $l_d = 0$ since on $\{l_d = 0\}$ we have $y = c_d x$.
Applying the residue sequence, we see that each such form ω has as residue on D_α a function with a single pole at most at the origin O, and with coefficient of $\frac{1}{x}$ respectively equal to $r_d := \sum_{\alpha=1}^{d-1} \frac{g_\alpha}{(c_d - c_\alpha)}$ in the case of D_d, and $r_\alpha := -\frac{g_\alpha}{(c_d - c_\alpha)}$ in the case of D_α.
In other words, the sum of the 'double' residues is 0, and the other condition $\sum_{\alpha=1}^{d-1} g_\alpha = 0$ can be also written down as $\sum_{\alpha=1}^{d} c_\alpha r_\alpha = 0$.
To show 5), observe that

$$p_* \Omega^1_Z(\log D_1)(-E) \subset p_* \Omega^1_Z(\log D_1) \subset \Omega^1_{\mathbb{C}^2}(\operatorname{dlog} l_1).$$

Take coordinates x, y such that $l_1 = y$, and write $\omega = \alpha dx + \beta \frac{dy}{y}$.
We just pull back ω on the blow up Z in the chart where we have $y = tx$, and impose that it lies in the span of

$$x\frac{dt}{t}, xdx.$$

We have

$$\omega = \alpha(x, tx)dx + \beta(x, tx)(\frac{dt}{t} + \frac{dx}{x})$$

and we must clearly have $\beta(0) = 0$.
Then $\beta(x, tx)\frac{dt}{t}$ is a multiple of $x\frac{dt}{t}$, and it suffices to require that $\alpha(x, tx) + \frac{1}{x}\beta(x, tx)$ be divisible by x.
Writing $\beta(x, y) = \beta_1 x + \beta_2 y + \dots$, our condition boils down to the divisibility by x of

$$\alpha(0) + \beta_1 + \beta_2 t \Leftrightarrow \beta_2 = 0, \ \alpha(0) + \beta_1 = 0.$$

Finally, let us show 7). Write

$$\omega = \alpha \frac{dx}{x} + \beta \frac{dy}{y}$$

and pull back to the blow up in the chart where $y = tx$: we get

$$(\alpha + \beta)\frac{dx}{x} + \beta \frac{dt}{t},$$

which must be divisible by x, hence in particular $\beta(0) = 0$. Looking at the other chart we get symmetrically $\alpha(0) = 0$.
Now, $\alpha + \beta$ must vanish of order two, in order that its pull back be divisible by x^2.

□

8. Appendix 2: Proof of Proposition 4.1

Proof of Proposition 4.1. We can prove 1) and 2) simultaneously for $i = 1$. Observe that $D_1 = \Delta_1 + N_1$, that $\Lambda_1 = L_1 + N_1$, and apply Lemma 4.3 (observing that $(K_{\tilde{Y}} + 2N_1 + (E_1 - E_3))N_1 = -4 + 1 < 0$) in order to conclude that

$$H^0(\Omega^1_{\tilde{Y}}(\log(\Delta_1))(E_1 - E_3 + N_1)) \cong H^0(\Omega^1_{\tilde{Y}}(\log(D_1))(E_1 - E_3)).$$

Moreover we observe that, again by Lemma 4.3,

$$H^0(\Omega^1_{\tilde{Y}}(\log(D_1))(E_1 - E_3)) = H^0(\Omega^1_{\tilde{Y}}(\log(D_1 - (L - E_1)))(L - E_3)).$$

Let $f\colon \tilde{Y} \to \mathbb{P}^2$ be the blow down of $E_1, \ldots, E_5$. Then $f_*(D_1 - (L - E_1))$ splits as the sum of two lines l_1, l_2 in $\mathbb{P}^2$ intersecting in P_1.
W.l.o.g. we can assume that $P_1 = (0:0:1)$, $P_2 = (0:1:0)$, $P_4 = (1:0:0)$ and $P_5 = (1:0:\lambda)$, with $\lambda \neq 0$.
Applying Proposition 7.1 several times for each blow down we get that

$$H^0(\Omega^1_{\tilde{Y}}(\log(D_1 - (L - E_1)))(L - E_3)) = H^0(f_*\Omega^1_{\tilde{Y}}(\log(D_1 - (L - E_1)))(L - E_3))$$

is the subspace V_1 of $H^0(\Omega^1_{\mathbb{P}^2}(\operatorname{dlog} l_1, \operatorname{dlog} l_2)(1))$ consisting of sections satisfying several linear conditions.
We write these conditions using the basis provided by Lemma 4.5 and its corollary, in order to show that $V_1 = 0$. By prop. 7.1, 3) we get for P_1:

$$a_{13} + a_{23} = 0;$$

for P_2, P_4 and P_5 the three equations

$$a_{12} = a_{21} = a_{21} + \lambda a_{23} = 0.$$

This shows that $V_1 = 0$.
We continue with the proof of 1).
For $i = 2$, again by Lemma 4.3 we have to calculate

$$V_2 = H^0(\Omega^1_{\tilde{Y}}(\log(L - E_2 - E_5), \log(L - E_2 - E_4))(L - E_3)),$$

which after blowing down $E_1, \ldots, E_5$ corresponds to a subspace of $H^0(\Omega^1_{\mathbb{P}^2}(\operatorname{dlog} l_1, \operatorname{dlog} l_2)(1))$.

W.l.o.g. we can assume that $P_2 = (0:0:1)$, $P_5 = (0:1:0)$, $P_4 = (1:0:0)$ and $P_3 = (1:1:1)$.
By prop. 7.1, 3), we get for P_2,P_4, P_5 the three linear equations:

$$a_{13} + a_{23} = 0, \ a_{21} = 0, \ a_{12} = 0.$$

We evaluate ω in P_3, and get (using the above equalities)

$$\omega(P_3) = a_{13} dx_1 + a_{23} dx_2,$$

whence by Proposition 7.1, 6) $a_{13} = a_{23} = 0$ and therefore we have verified that $V_2 = 0$.
For $i = 3$, using Lemma 4.3, we have to calculate:

$$V_3 := H^0(\Omega^1_{\tilde{Y}}(\log(L - E_3 - E_4), \log(L - E_3 - E_5))(L - E_1)),$$

which, after blowing down $E_1, \ldots, E_5$, becomes a linear subspace of $H^0(\Omega^1_{\mathbb{P}^2}(\mathrm{dlog}\, l_1, \mathrm{dlog}\, l_2)(1))$.
W.l.o.g. we can assume that $P_3 = (0:0:1)$, $P_4 = (0:1:0)$, $P_5 = (1:0:0)$, $P_1 = (1:1:0)$.
By prop. 7.1, 3), we get for P_3,P_4, P_5 the three linear equations:

$$a_{13} + a_{23} = 0, \ a_{12} = 0, \ a_{21} = 0.$$

Setting the evaluation of ω in P_1 equal to zero is easily seen to give no new conditions, hence $V_3 \cong \mathbb{C}$.
Let's proceed to prove 2) for $i = 2, 3$.
For $i = 2, 3$, by 4) of remark 1.5,

$$H^0(\Omega^1_{\tilde{Y}}(\log \Delta_i)(K_{\tilde{Y}} + \Lambda_i)) = H^0(\Omega^1_{\tilde{Y}}(\log \Delta_i)(E_i - E_{i+2})).$$

For $i = 2$, using again Lemma 4.3, observing that

$$(K_{\tilde{Y}} + 2\Gamma_2 + (E_2 - E_1))\Gamma_2 < 0,$$

we see that we have to calculate

$$V_2 := H^0(\Omega^1_{\tilde{Y}}(\log(L - E_2 - E_4), \log(L - E_2 - E_5))(2L - E_1 - E_3 - E_4 - E_5)),$$

which, after blowing down $E_1, \ldots, E_5$, becomes a linear subspace of $H^0(\Omega^1_{\mathbb{P}^2}(\mathrm{dlog}\, l_1, \mathrm{dlog}\, l_2)(2))$.
W.l.o.g. we can assume that $P_2 = (0:0:1)$, $P_4 = (0:1:0)$, $P_5 = (1:0:0)$, $P_3 = (1:1:1)$, and then $P_1 = (1:\lambda:0)$, where $\lambda \neq 0, 1$.
Using cor. 4.6, we get by prop. 7.1, 3) for P_2 the linear equation

$$a_{313} + a_{323} = 0.$$

By prop. 7.1, 5) the conditions for P_4 are

$$a_{212} = 0, \ a_{12} = 0, \ a_{23} = 0;$$

whereas the conditions for P_5 are

$$a_{121} = 0, \ a_{12} = 0, \ a_{13} = 0.$$

Imposing that ω vanishes in P_3, we get

$$\omega(P_3) = dx_1(a_{313} + 2a_{213} + a_{123}) + dx_2(a_{323} + 2a_{123} + a_{213}) = 0.$$

The above conditions yield:

$$a_{123} = a_{313} = -a_{213} = -a_{323}.$$

Finally, imposing that ω vanishes in P_1 we obtain:

$$\omega(P_1) = -dx_3(\lambda a_{213} + a_{123}) = 0,$$

whence $(\lambda - 1)a_{213} = 0$. Since $\lambda \neq 0, 1$ this implies $a_{213} = 0$, and we have shown that $V_2 = 0$.
We are left with the case $i = 3$. Using repeatedly Lemma 4.3 and Proposition 7.1, we see that we have to calculate

$$\begin{aligned} &V_3 := H^0(\Omega^1_{\tilde{Y}}(\log \Delta_3)(E_3 - E_2)) = \\ &= H^0(\Omega^1_{\tilde{Y}}(\log(L - E_1 - E_3), \log(L - E_1 - E_4 - E_5))(2L - E_3 - E_4 - E_5)). \end{aligned}$$

After blowing down $E_1, \dots, E_5$, we can assume w.l.o.g. that $P_1 = (0 : 0 : 1)$, $P_4 = (0 : 1 : 0)$, $P_3 = (1 : 0 : 0)$, $P_5 = (0 : 1 : 1)$, and V_3 becomes a linear subspace of $H^0(\Omega^1_{\mathbb{P}^2}(\operatorname{dlog} l_1, \operatorname{dlog} l_2)(2))$.
Using cor. 4.6, we get by prop. 7.1, 3) for P_1 the linear equation

$$a_{313} + a_{323} = 0.$$

By prop. 7.1, 5) the conditions for P_4 are

$$a_{212} = 0,\ a_{12} = 0,\ a_{23} = 0;$$

whereas the conditions for P_3 are

$$a_{121} = 0,\ a_{12} = 0,\ a_{13} = 0.$$

For P_5, we get instead (again by prop. 7.1, 5)), the two linear equations (the third is trivial):

$$a_{213} + a_{313} = 0,\ 2a_{313} = 0.$$

This implies that $a_{313} = a_{213} = a_{323} = 0$, but a_{123} is arbitrary. This shows that $V_3 \cong \mathbb{C}$.
Thus 2) is proven.
To prove 3), by symmetry, we may assume without loss of generality that $i = 1$. We have to calculate $V_1 := H^0(\Omega^1_{\tilde{Y}}(\log(D_1)(E_1 - E_3))$, which by Lemma 4.3 is equal to

$$H^0(\Omega^1_{\tilde{Y}}(\log(L - E_1 - E_2), \log(L - E_1 - E_4 - E_5))(L - E_6)),$$

which, after blowing down $E_1, \dots, E_5$, becomes a linear subspace of $H^0(\Omega^1_{\mathbb{P}^2}(\operatorname{dlog} l_1, \operatorname{dlog} l_2)(1))$.
W.l.o.g. we can assume that $P_1 = (0 : 0 : 1)$, $P_5 = (0 : 1 : 0)$, $P_2 = (1 : 0 : 0)$, $P_4 = (0 : 1 : 1)$. Since P_2, P_4, P_6 are collinear, $P_6 = (1 : \mu : \mu)$, where $\mu \neq 0$.
By prop. 7.1, 3), we get for P_1,P_2, P_4 and P_5 the linear equations:

$$a_{13} + a_{23} = 0,\ a_{21} = 0,\ a_{12} + a_{13} = 0,\ a_{12} = 0.$$

This already shows that $V_1 = 0$.
Thus 3) is proven.

Let us treat the subcase of 4) where we have strictly extended Burniat divisors: the situation is here symmetric in the indices i, hence it suffices to show the vanishing of

$$H^0(\Omega^1_{\tilde{Y}}(\log(\Delta_1))(K_{\tilde{Y}} + \Lambda_1)).$$

Recall that we have the decomposition in irreducible connected components $\Delta_1 = G_1 + \Gamma_1 + N_2$, where G_1 is the del Pezzo line $G_1 \equiv L - E_1 - E_6$.
By Lemma 4.3 we get:

$$H^0(\Omega^1_{\tilde{Y}}(\log(\Delta_1))(K_{\tilde{Y}} + \Lambda_1)) = H^0(\Omega^1_{\tilde{Y}}(\log(\Delta_1 + N_1))(E_1 - E_3)),$$

since $(K_{\tilde{Y}} + 2N_1 + (E_1 - E_3))N_1 < 0$. Using again Lemma 4.3 we see that

$$\begin{aligned} H^0(\Omega^1_{\tilde{Y}}(\log(\Delta_1 + N_1))(E_1 - E_3)) = \\ = H^0(\Omega^1_{\tilde{Y}}(\log(\Delta_1 + N_1 - \Gamma_1))((E_1 - E_3) + \Gamma_1)) = \\ = H^0(\Omega^1_{\tilde{Y}}(\log(G_1 + N_1 + N_2))(2L - E_2 - E_3 - E_5 - E_6)), \end{aligned}$$

because $(K_{\tilde{Y}} + 2\Gamma_1 + (E_1 - E_3))\Gamma_1 < 0$.
Let $f \colon \tilde{Y} \to \mathbb{P}^2$ be the blow down of $E_1, \ldots, E_6$. Then $f_*(G_1+N_1+N_2)$ splits as the sum of three lines l_1, l_2, l_3 in $\mathbb{P}^2$ forming a triangle. W.l.o.g. we can assume that $P_6 = (1:0:0)$, $P_1 = (0:1:0)$, $P_4 = (0:0:1)$ and $P_3 = (1:1:1)$. Then $P_5 = (0:1:1)$, whereas P_2 is collinear with P_6, P_4, whence $P_2 = (1:0:\lambda)$, with $\lambda \neq 0, 1$.
Then

$$\begin{aligned} H^0(\Omega^1_{\tilde{Y}}(\log(G_1 + N_1 + N_2))(2L - E_2 - E_3 - E_5 - E_6)) = \\ H^0(f_*\Omega^1_{\tilde{Y}}(\log(G_1 + N_1 + N_2))(2L - E_2 - E_3 - E_5 - E_6)) \end{aligned}$$

is a subspace of

$$H^0(\Omega^1_{\mathbb{P}^2}(\operatorname{dlog} l_1, \operatorname{dlog} l_2, \operatorname{dlog} l_3)(2)),$$

where $l_i = x_i$, whence $P_1, P_4, P_5 \in \{l_1 = 0\}$, $P_6, P_4, P_2 \in \{l_2 = 0\}$, $P_1, P_6 \in \{l_3 = 0\}$, consisting of sections satisfying fourteen linear conditions described in Proposition 7.1.
We explicitly write these conditions using Lemma 4.4 and Lemma 4.6 in order to show that this subspace must be trivial.
Let $\omega \in H^0(\Omega^1_{\mathbb{P}^2}(\operatorname{dlog} l_1, \operatorname{dlog} l_2, \operatorname{dlog} l_3)(2))$ and we write ω in the basis of Lemma 4.4:

$$\begin{aligned} \omega = a_{12}\eta_{12} + a_{13}\eta_{13} + a_{23}\eta_{23} + a_{212}x_2\omega_{12} + a_{313}x_3\omega_{13} + a_{323}x_3\omega_{23} + \\ + a_{121}x_1\omega_{21} + a_{131}x_1\omega_{31} + a_{232}x_2\omega_{32} + a_{123}x_1\omega_{23} + a_{231}x_2\omega_{31} + a_{312}x_3\omega_{12}. \end{aligned}$$

Then by prop. 7.1, 3) the condition for $P_1 = (0:1:0)$ is

$$a_{212} + a_{232} = 0. \tag{1}$$

The same argument shows that the linear condition for $P_4 = (0:0:1)$ is

$$a_{313} + a_{323} = 0. \tag{2}$$

Next we work out the conditions for P_5, P_2 using prop. 7.1, 5). For $P_5 :=$ $(0 : 1 : 1)$ we work in the chart $x_3 = 1$ and write ω locally around $(0, 1)$ as $\alpha(x_1, x_2)dx_2 + \beta(x_1, x_2)\frac{dx_1}{x_1}$. Then we get (using Lemma 4.6):

$$\beta(0,1) = a_{212} + a_{313} + a_{312} = 0; \tag{3}$$

$$\frac{\partial\beta}{\partial x_1}(0,1) = -a_{12} - a_{13} - a_{231} = 0; \tag{4}$$

$$\frac{\partial\beta}{\partial x_2}(0,1) + \alpha(0,1) = -a_{23} + 2a_{212} + a_{323} = 0. \tag{5}$$

The same argument for $P_2 = (1 : 0 : \lambda)$ (working in the chart $x_1 = 1$ and writing ω locally around $(0, \lambda)$ as $\alpha(x_2, x_3)dx_3 + \beta(x_2, x_3)\frac{dx_2}{x_2}$) gives the following three linear equations ($\lambda \neq 0, 1$):

$$\beta(0,\lambda) = a_{323}\lambda^2 + a_{121} + a_{123}\lambda = 0; \tag{6}$$

$$\frac{\partial\beta}{\partial x_2}(0,\lambda) = a_{12} - \lambda a_{23} - \lambda a_{312} = 0; \tag{7}$$

$$\begin{aligned} \frac{\partial\beta}{\partial x_3}(0,\lambda) + \alpha(0,\lambda) &= a_{13} + \frac{1}{\lambda}a_{131} + \lambda a_{323} - \lambda a_{313} = \\ &= a_{13} + \frac{1}{\lambda}a_{131} + 2\lambda a_{323} = 0. \end{aligned} \tag{8}$$

There are four linear conditions coming from $P_6 = (1 : 0 : 0)$, given in prop. 7.1, 7). We work in the chart $x_1 = 1$ and write $\omega = \alpha(x_2, x_3)\frac{dx_2}{x_2} + \beta(x_2, x_3)\frac{dx_3}{x_3}$. Then we get:

$$\alpha(0,0) = a_{121} = 0; \tag{9}$$

$$\beta(0,0) = a_{131} = 0; \tag{10}$$

$$\frac{\partial(\alpha+\beta)}{\partial x_2}(0,0) = a_{12} + a_{231} = 0; \tag{11}$$

$$\frac{\partial(\alpha+\beta)}{\partial x_3}(0,0) = a_{13} = 0. \tag{12}$$

From equation (11) we get: $a_{12} = -a_{231}$.
Since $a_{13} = a_{131} = 0$, equation (8) implies $a_{323} = 0$, whence by (2) also $a_{313} = 0$. Moreover, by (6), we get $a_{123} = 0$.
We write finally the conditions coming from $P_3 = (1 : 1 : 1)$ (using again that certain coefficients are zero).
We evaluate ω in P_3 and work in the affine chart $x_2 = 1$ to obtain

$$\begin{aligned} \omega(P_3) = (-a_{12} + a_{212} - a_{231} + a_{312})dx_1 + \\ (a_{23} + a_{232} + a_{231})dx_3 = 0. \end{aligned} \tag{13}$$

Since:

$$\omega(P_3) = (a_{212} + a_{312})dx_1 + ((a_{23} + a_{232} + a_{231})dx_3$$

we get the last two linear equations:

$$a_{231} + a_{23} + a_{232} = 0; \tag{14}$$

$$a_{212} + a_{312} = 0. \tag{15}$$

These immediately imply that

$$a_{312} = a_{232} = -a_{212}.$$

By (14) $a_{23} = -a_{232} - a_{231}$, and using (5), we see that $a_{23} = 2a_{212}$.
Again by (14) we get then that $a_{212} = -a_{231}$. Therefore, we have:

$$a_{212} = -a_{231} = -a_{312} = -a_{232} = \frac{a_{23}}{2}.$$

By (7):

$$0 = a_{12} - \lambda a_{23} - \lambda a_{231} = a_{12} + \lambda a_{231},$$

whence by (4) $\lambda = 1$, which gives a contradiction, or $a_{12} = a_{231} = 0$.
Hence the claim for strictly extended Burniat surfaces with $K_S^2 = 3$ is established.
Next we come to the case of (non strictly) extended Burniat surfaces. Here we have to consider two cases:

a) only one of the three conics Γ_i degenerates to two lines;
b) exactly two of the three conics Γ_i degenerate to two lines.

a) W.l.o.g. and by remark 1.5, (5) we may assume that Γ_1 splits as

$$\Gamma_1 \equiv (L - E_1 - E_2) + N_3 + E_3.$$

Then we get the extended Burniat divisors:

$$\{D_1'\} = |L - E_1 - E_6| + |L - E_1 - E_2| + E_3 + N_2,$$
$$D_2' \in |L - E_2 - E_5| + |2L - E_2 - E_3 - E_4 - E_5|,$$
$$D_3' \in |L - E_3 - E_4| + |2L - E_1 - E_3 - E_4 - E_6| + N_1 + N_3.$$

We make the assumption, for each D_i', $i = 2, 3$ that the strict transform of the conic Γ_i is irreducible.
Then we have

$$K_{\tilde{Y}} + \mathcal{L}_1' \equiv L - E_3 - E_4 - E_5 \equiv K_{\tilde{Y}} + \Lambda_1,$$
$$K_{\tilde{Y}} + \mathcal{L}_2' \equiv L - E_1 - E_4 - E_6 \equiv K_{\tilde{Y}} + \Lambda_2,$$
$$K_{\tilde{Y}} + \mathcal{L}_3' \equiv E_3 - E_2 \equiv K_{\tilde{Y}} + \Lambda_3 - N_3,$$

where the Λ_i are as for the strictly extended Burniat divisors.
Observe that $D_2' + N_3 = \Delta_2$, whence

$$H^0(\Omega^1_{\tilde{Y}}(\log D_2')(K_{\tilde{Y}} + \mathcal{L}_2')) \subset H^0(\Omega^1_{\tilde{Y}}(\log D_2')(K_{\tilde{Y}} + \mathcal{L}_2' + N_3)) =$$
$$H^0(\Omega^1_{\tilde{Y}}(\log(D_2' + N_3))(K_{\tilde{Y}} + \mathcal{L}_2')) = H^0(\Omega^1_{\tilde{Y}}(\log \Delta_2)(K_{\tilde{Y}} + \Lambda_2)) = 0,$$

where the first equality holds by Lemma 4.3 and the last holds by our previous computations for strictly extended Burniat surfaces.
Moreover, $D'_3 = \Delta_3 + N_3$, $\mathcal{L}'_3 = \Lambda_3 - N_3$, whence the vanishing of $H^0(\Omega^1_{\tilde{Y}}(\log D'_3)(K_{\tilde{Y}} + \mathcal{L}'_3))$ follows again using Lemma 4.3 from the analogous vanishing for strictly extended Burniat surfaces. It remains to prove the following

CLAIM 8.1. $H^0(\Omega^1_{\tilde{Y}}(\log D'_1)(K_{\tilde{Y}} + \mathcal{L}'_1)) = 0$

Proof of the claim. By Lemma 4.3 we see that

$$H^0(\Omega^1_{\tilde{Y}}(\log D'_1)(K_{\tilde{Y}} + \mathcal{L}'_1)) = H^0(\Omega^1_{\tilde{Y}}(\log(D'_1 - E_3))(L - E_4 - E_5)).$$

Let $f \colon \tilde{Y} \to \mathbb{P}^2$ be the blow down of $E_1, \dots, E_6$.
Then $f_*(D'_1 - E_3)$ splits as the sum of three lines l_1, l_2, l_3 in $\mathbb{P}^2$ forming a triangle. W.l.o.g. we can assume that $P_1 = (1:0:0)$, $P_2 = (0:1:0)$, $P_6 = (0:0:1)$ and $P_5 = (1:1:1)$. Then $P_4 = (0:1:1)$.
We conclude that $H^0(\Omega^1_{\tilde{Y}}(\log(D'_1 - E_3))(L - E_4 - E_5)) = H^0(f_*\Omega^1_{\tilde{Y}}(\log(D'_1 - E_3))(L - E_4 - E_5))$ is the subspace of $H^0(\Omega^1_{\mathbb{P}^2}(\mathrm{dlog}\, l_1, \mathrm{dlog}\, l_2, \mathrm{dlog}\, l_3)(1))$ consisting of sections satisfying one linear condition for P_1, P_2, P_6 each, two linear conditions for P_5 and three linear conditions for P_4, described in Proposition 7.1.
We write these conditions using Lemma 4.4 in order to show that this subspace must be trivial.
By Lemma 4.4 we write $\omega \in H^0(\Omega^1_{\mathbb{P}^2}(\mathrm{dlog}\, l_1, \mathrm{dlog}\, l_2, \mathrm{dlog}\, l_3)(1)$ as

$$\omega = \sum_{i \neq j} a_{ij}\omega_{ij}.$$

Then the three equations for P_1, P_2, P_6 (cf. prop. 7.1, 3)) are

$$a_{21} + a_{31} = 0,\ a_{12} + a_{32} = 0,\ a_{13} + a_{23} = 0.$$

By prop. 7.1, 5), we get for P_4 the linear equations:

$$a_{12} + a_{13} = 0,\ -a_{21} - a_{31} = 0,\ a_{23} - a_{32} = 0.$$

The above conditions already imply:

$$a_{13} = a_{12} = a_{23} = a_{32} = 0,\ a_{21} = -a_{31}.$$

We impose the vanishing of ω in $P_5 = (1:1:1)$ working in the affine chart $x_3 = 1$ and obtain

$$\omega(1:1:1) = (-a_{21} - a_{31})dx_1 + (a_{21})dx_2 = 0,$$

whence $a_{21} = a_{31} = 0$. □

b) W.l.o.g. we can assume that each of the two conics Γ_1 and Γ_2 degenerate to two lines. Then we get the extended Burniat divisors:

$$\{D''_1\} = |L - E_1 - E_6| + |L - E_1 - E_2| + E_3 + N_1 + N_2,$$
$$\{D''_2\} = |L - E_2 - E_5| + |L - E_2 - E_3| + E_1,$$
$$D''_3 \in |L - E_3 - E_4| + |2L - E_1 - E_3 - E_4 - E_6| + N_3.$$

We make the assumption for D_3'' that the strict transform of the conic Γ_3 passing through P_1, P_3, P_4, P_6 is irreducible.
Then we have

$$K_{\tilde{Y}} + \mathcal{L}_1'' \equiv E_1 - E_3 = K_{\tilde{Y}} + \mathcal{L}_1' - N_1,\ D_1'' = D_1' + N_1;$$

$$K_{\tilde{Y}} + \mathcal{L}_2'' \equiv L - E_1 - E_4 - E_6 \equiv K_{\tilde{Y}} + \mathcal{L}_2 + N_2,\ D_2'' = D_2 - N_2;$$

$$K_{\tilde{Y}} + \mathcal{L}_3'' \equiv E_3 - E_2 \equiv K_{\tilde{Y}} + \Lambda_3',\ D_3'' = D_3' - N_3.$$

Therefore for $i = 1, 2, 3$ the vanishing of $H^0(\Omega^1_{\tilde{Y}}(\log D_i'')(K_{\tilde{Y}} + \mathcal{L}_i''))$ can be reduced via Lemma 4.3 to the analogous vanishing for extended Burniat surfaces of case a) ($i = 1, 3$) and to the analogous vanishing for Burniat divisors (for $i = 2$), which was already proved in part 3).

$\square$

9. Appendix 3: an alternative proof of statements 1), 2), 3) of Proposition 4.1

In this appendix we present other methods to calculate the space of sections of twisted logarithmic sheaves, in particular a fibration method.
Assume that we have d smooth rational curves $C_\alpha \subset Y$ contained in a smooth algebraic surface Y, meeting with distinct tangents in a point O, a divisor B_α on C_α of degree 0, 1 or 2, and disjoint from O, and let Z be the blow up of Y in the point O. Denote by D_α the strict transform of C_α, and denote by $\Omega^1_Y((\log C_\alpha(-B_\alpha))_{\alpha\in A})$ the sheaf which is the inverse image, under the residue sequence, of $\oplus_{\alpha\in A}\mathcal{O}_{C_\alpha}(-B_\alpha)$.
Then by 4) of Proposition 7.1 we have an exact sequence

$$0 \to \Omega^1_Y((\log C_\alpha(-B_\alpha))_{\alpha\in A}) \to$$
$$\to p_*\Omega^1_Z((\log D_\alpha(-B_\alpha))_{\alpha\in A})(E) \to \mathbb{C}_O^{d-2} \to 0$$

which is exact on global sections if

$$h := dim_{\mathbb{C}} H^1(\Omega^1_Y((\log C_\alpha(-B_\alpha))_{\alpha\in A})) = 0.$$

Or, more generally, iff $h = h'$, where

$$h' := dim_{\mathbb{C}} H^1(\Omega^1_Z((\log D_\alpha(-B_\alpha))_{\alpha\in A})).$$

Consider the exact sequence

$$0 \to \Omega^1_Y \to \Omega^1_Y((\log C_\alpha(-B_\alpha))_{\alpha\in A}) \to \bigoplus_{\alpha=1}^{d} \mathcal{O}_{C_\alpha}(-B_\alpha) \to 0,$$

and assume that $H^2(\Omega^1_Y) = 0$.
Then $h = a + b$, where a is the number of α's such that B_α has degree 2, while b is the difference of the dimensions between $H^1(\Omega^1_Y)$ and the subspace generated by the Chern classes of the C_α's such that B_α has degree 0. If we choose $Y = \mathbb{P}^2$ then $h = 0$ as soon as no B_α has degree 2, and some B_α has degree 0.

Otherwise, one can calculate h' in a similar way. We assume for simplicity that $Y = \mathbb{P}^2$. We have a similar exact sequence

$$0 \to \Omega^1_Z(E) \to \Omega^1_Z((\log D_\alpha(-B_\alpha))_{\alpha \in A})(E) \to \bigoplus_{\alpha=1}^{d} \mathcal{O}_{C_\alpha}(O - B_\alpha) \to 0,$$

and since $H^1(\mathcal{O}_{C_\alpha}(O - B_\alpha)) = 0$ by our assumption, we get that h' is the dimension of the cokernel of

$$\bigoplus_{\alpha=1}^{d} H^0(\mathcal{O}_{C_\alpha}(O - B_\alpha)) \to H^1(\Omega^1_Z(E)).$$

To calculate the last space, observe that

$$\Omega^1_Z \otimes \mathcal{O}_E = \mathcal{O}_E(-2) \oplus \mathcal{O}_E(1)$$

whence $h^1(\Omega^1_Z(E)) = h^1(\Omega^1_Z) + 1 = 3$.
These criteria can now be used in order to prove statements 1), 2), 3) of Proposition 4.1.
We can prove 1) and 2) simultaneously for $i = 1$.
Observe that $D_1 = \Delta_1 + N_1$, that $\Lambda_1 = L_1 + N_1$, and apply Lemma 4.3 in order to conclude that

$$H^0(\Omega^1_{\tilde{Y}}(\log(\Delta_1))(E_1 - E_3 + N_1)) \cong H^0(\Omega^1_{\tilde{Y}}(\log(D_1))(E_1 - E_3)).$$

By Lemma 7.1 we can blow down E_3 and obtain $H^0(\Omega^1_{\tilde{Y}'}(\log(D'_1))(E_1))$. In this case the respective degrees of the divisors B_α are $0, 1, 2$ hence $h = 1$. We have to decide whether h' is 0 or 1. We contract E_3, E_4, E_5, and we let Z be the blow up of the plane in P_1. We must calculate $h^0(\Omega^1_Z(\log(C_\alpha(-B_\alpha)))(E_1))$. Here the curves C_α are fibres of the ruling of Z, $f\colon Z \to \mathbb{P}^1$. Using the exact sequence

$$(**)\ 0 \to f^*\Omega^1_{\mathbb{P}^1} \to \Omega^1_Z \to \omega_{Z|\mathbb{P}^1} = \mathcal{O}_Z(-F - 2E_1) \to 0$$

we obtain the analogous sequence

$$0 \to f^*\mathcal{O}_{\mathbb{P}^1}(1)(E_1) \to \Omega^1_Z(\log(C_\alpha)(E_1) \to \mathcal{O}_Z(-F - E_1) \to 0$$

to infer that

$$H^0(f^*\mathcal{O}_{\mathbb{P}^1}(1)(E_1)) = H^0(\Omega^1_Z(\log(C_\alpha))(E_1)).$$

We are imposing some vanishing on three points lying in two fibres, hence we get the sections of $H^0(\mathcal{O}_Z(F + E_1)) = p^*H^0(\mathcal{O}_{\mathbb{P}^2}(1))$ vanishing in the three points P_4, P_5, P_2, whence we conclude that this space has dimension $= 0$.
This argument shows 1) also for $i = 2, 3$.
For a nodal Burniat with $m = 2$ the space $H^0(\Omega^1_{\tilde{Y}}(\log(D_i))(E_i - E_{i+2}))$, vanishes for $i = 2$, but it has dimension equal to 1 for $i = 3$, since then the three points P_4, P_5, P_1 are collinear.
Let's proceed with 2).
For $i = 2, 3$

$$H^0(\Omega^1_{\tilde{Y}}(\log \Delta_i)(K_{\tilde{Y}} + \Lambda_i)) = H^0(\Omega^1_{\tilde{Y}}(\log \Delta_i)(E_i - E_{i+2})).$$

By applying again Lemma 7.1 for $i = 3$ we can blow down the curve E_2 and the curve E_3 and apply the residue sequence to the sheaf $\Omega^1_{\tilde{Y}'}(\log \Delta'_3)$. Since each component is smooth and rational, we find that $H^0(\Omega^1_{\tilde{Y}'}(\log \Delta'_3)) = ker(\mathbb{C}^4 \to H^1(\Omega^1_{\tilde{Y}'}))$, while $H^1(\Omega^1_{\tilde{Y}'}(\log \Delta'_3)) = Coker(\mathbb{C}^4 \to H^1(\Omega^1_{\tilde{Y}'}))$.
The map is given by the Chern classes of $L - E_1, L - E_4, L - E_5, L - E_1 - E_4 - E_5$. These generate a rank 4 subspace of the 4-dimensional space $H^1(\Omega^1_{\tilde{Y}'})$ (we are blowing up 3 points in the plane), whence $h^0(\Omega^1_{\tilde{Y}'}(\log \Delta'_3)) = 0, h^1(\Omega^1_{\tilde{Y}'}(\log \Delta'_3)) = 0$.
We conclude, by the exact cohomology sequence associated to

$$0 \to \Omega^1_{\tilde{Y}'}(\log \Delta'_3) \to f_*(\Omega^1_{\tilde{Y}}(\log \Delta_3)(K_{\tilde{Y}} + \Lambda_3)) \to \mathbb{C}_{P_3} \to 0,$$

that $h^0(\Omega^1_{\tilde{Y}}(\log \Delta_3)(K_{\tilde{Y}} + \Lambda_3)) = 1 + h^0(\Omega^1_{\tilde{Y}'}(\log \Delta'_3)) = 1$.
For the case $i = 2$ recall that $\Delta_2 \in |L - E_2 - E_4| + |L - E_2 - E_5| + |2L - E_2 - E_3 - E_4 - E_5|$ consists of three smooth connected components.
Blow down E_1, E_3, E_4, E_5 and obtain the ruled surface Z equal to the blow up of the plane in P_2. Denote by $f : Z \to \mathbb{P}^1$ the standard fibration.
The direct image $\Delta'_2 := f_*\Delta_2$ decomposes as the union of two fibres F_4 and F_5 and a section C with $C \cdot E_2 = 1$.
We have to calculate the space of global sections of

$$\mathcal{F} := \mathfrak{m}_{P_1}\Omega^1_Z(\log F_4, \log F_5, \log C(-P_3))(E_2)$$

satisfying two linear conditions imposed by the points P_4, P_5.
Using the exact sequence (**) we get the exact sequence

$$0 \to \mathfrak{m}_{P_1}\mathcal{O}_Z(E_2) \to \mathcal{F} \to \mathfrak{M}_{P_3}\mathfrak{m}_{P_1}\mathcal{O}_Z(-F - E_2 + C) \to 0.$$

Observe that $\mathcal{O}_Z(-F - E_2 + C)$ has degree 0 on each fibre and degree 1 on E_2. If $D \equiv -F - E_2 + C \equiv L - E_2$ is effective, then D is a fibre. Since no fibre contains both P_1, P_3, we obtain

$$H^0(\mathfrak{m}_{P_3}\mathfrak{m}_{P_1}\mathcal{O}_Z(-F - E_2 + C)) = 0.$$

Since $|E_2|$ consists of the curve E_2, which does not contain P_1, we conclude that $H^0(M_{P_1}\mathcal{O}_Z(E_2)) = H^0(\mathcal{F}) = 0$.
To prove 3), by symmetry, we may assume without loss of generality that $i = 1$. Blow down all the curves E_j excet E_1, so that , as usual, we have the blow up Z of the plane in a point (P_1) and the standard fibration $f : Z \to \mathbb{P}^1$.
By Lemma 7.1 and since E_3 is a connected component of D_1, the direct image $\mathcal{F}$ of $\Omega^1_{\tilde{Y}}(\log(D_1)(E_1 - E_3)$ is contained in $\Omega^1_Z(\log(F_2 + F_6 + F_{4,5}))(E_1)$ where F_j denotes the unique fibre of f passing through the point P_j.
More precisely, we have an exact sequence

$$0 \to \mathfrak{m}_{P_2}\mathfrak{m}_{P_4}\mathfrak{m}_{P_5}\mathfrak{m}_{P_6}\mathcal{O}_Z(F + E_1) \to \mathcal{F} \to \mathcal{O}_Z(-F - E_1) \to 0.$$

Clearly $H^0(\mathcal{O}_Z(-F - E_1)) = 0$ since $F \cdot (F + E_1) = 1$. On the other hand $H^0(\mathcal{O}_Z(F + E_1)) = H^0(\mathcal{O}_{\mathbb{P}^2}(1))$, hence the fact that the points P_2, P_4, P_5, P_6

are not collinear implies the desired vanishing

$$H^0(\mathfrak{m}_{P_2}\mathfrak{m}_{P_4}\mathfrak{m}_{P_5}\mathfrak{m}_{P_6}\mathcal{O}_Z(F+E_1)) = 0.$$

Thus 3) is proven.

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