On Artin Representations and Nearly Ordinary Hecke Algebras over Totally Real Fields

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Abstract. We prove many new cases of the strong Artin conjecture for two-dimensional, totally odd, insoluble (icosahedral) representations $\mathrm{Gal}(\overline{F}/F) \to GL_2(\mathbf{C})$ of the absolute Galois group of a totally real field F.

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1 Introduction

Let K be a number field. Artin conjectured that the L-series of any continuous representation $\rho : \mathrm{Gal}(\overline{K}/K) \to GL_n(\mathbf{C})$ of the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$ of K is holomorphic except a possible pole at $s = 1$ when the trivial representation is a constituent of ρ.

A result of Brauer (See [36]) about finite groups immediately implies that $L(\rho, s)$ has meromorphic continuation and satisfies a certain functional equation relating the values at s and $1 - s$. Any such complex representation is semi-simple, and because Artin showed that $L(\rho_1 + \rho_2) = L(\rho_1, s)L(\rho_2, s)$, the conjecture immediately follows from the case where ρ is irreducible. In the case where ρ is irreducible, the strong form of this conjecture, known as the strong Artin conjecture, asserts that there is a cuspidal automorphic representation π of $GL_n(\mathbf{A}_K)$ such that $L(\pi, s) = L(\rho, s)$, and Artin conjecture follows from the strong Artin conjecture (See [22], Theorem 8.8 with its proof (p.286) attributed to Ramakrishnan).

When $n = 2$ and the image of the projective representation $\mathrm{proj}\,\rho$: $\mathrm{Gal}(\overline{K}/K) \to PGL_2(\mathbf{C}) = GL_2(\mathbf{C})/\mathbf{C}^\times$ is dihedral (D_{2n} for some $n \geq 2$), ρ is induced from a character χ of the absolute Galois group $\mathrm{Gal}(\overline{K}/M)$ of a quadratic extension M of $\mathbf{Q}$, and Artin himself proved the conjecture (the holomorphy of $L(\rho, s) = L(\mathrm{Ind}_{G_M}^{G_K}\chi, s) = L(\chi, s)$ follows from earlier work of Hecke).

When $n = 2$ and the image of $\mathrm{proj}\,\rho$ is tetrahedral (A_4) and when $n = 2$, $K = \mathbf{Q}$, ρ odd, and the projective image of ρ is octahedral (S_4), Langlands [23], using his theory of (cyclic) base change, "deduced" the strong Artin conjecture from the dihedral case. Tunnell, building on work of Langlands, completed the octahedral case $n = 2$ and general K. In the octahedral case, in order to "descend" a cuspidal automorphic representation Π of $GL_2(\mathbf{A}_E)$ such that $L(\Pi, s) = L(\rho|_{\mathrm{Gal}(\overline{K}/E)}, s)$ to a cuspidal automorphic representation π of $GL_2(\mathbf{A}_K)$, where E is the quadratic extension of K corresponding to the unique index 2 subgroup ($\simeq A_4$) of S_4, Langlands uses a theorem of Deligne-Serre (and therefore $K = \mathbf{Q}$ and ρ should be necessarily odd) whilst Tunnell uses cubic base change to match up, for all but finitely many places v of K, the restriction of ρ to the decomposition group at v and the local representation π_v.

The icosahedral (A_5) case had remained largely intractable until Buzzard-Dickinson-Shepherd-Barron-Taylor [4] proved many new cases of the strong Artin conjecture for *odd* $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to GL_2(\mathbf{C})$.

[4] follows the program of Taylor ([37]), which may be succinctly described as an approach to deduce results about weight one forms from results about weight two forms (more specifically the idea of Wiles in [42]), and it is a culmination of a series of work: "$R = T$ theorem for 2-adic ordinary finite flat representations" by Dickinson [10], "modularity of mod 2 icosahedral representations" by Shepherd-Barron and Taylor [33], and "modular lifting theorem for two-dimensional p-adic Artin representations unramifed at p (for any prime p)" by Buzzard and Taylor [5]. Buzzard [3] later extended [5] to treat almost all two-dimensional p-adic Artin representations potentially unramifed at p (the image of the inertia group at p is finite) and subsequently it led to modularity of two-dimensional "5-adic" icosahedral Artin representations by Taylor [39]. The strong Artin conjecture for odd two-dimensional representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ is now completely proved by work of Khare-Wintenberger and Kisin on Serre's conjecture for odd two-dimensional mod p representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$.

In this paper, we push through Taylor's program and generalise them to treat new cases of the strong Artin conjecture for two-dimensional, totally odd, icosahedral Artin representations of the absolute Galois group of a totally real field. More precisely, we prove the following theorems.

Theorem 1 *Let F be a totally real field. Suppose that 5 splits completely in F. Suppose that $\rho : \mathrm{Gal}(\overline{F}/F) \to GL_2(\mathbf{C})$ is a totally odd, irreducible, continuous representation satisfying the following conditions.*

- *The image of the projective representation* $\mathrm{proj}\,\rho$ *of ρ is A_5.*

- *The projective image of the decomposition group at every place of F above 5 has order 2.*

Then ρ arises from a holomorphic cuspidal Hilbert modular eigenform of weight 1 and the Artin L-function $L(\rho, s)$ is entire.

THEOREM 2 *Let F be a totally real field. Suppose that 2 splits completely in F and that $[F(\zeta_5) : F] = 4$. Suppose that $\rho : \mathrm{Gal}(\overline{F}/F) \to GL_2(\mathbf{C})$ is a totally odd, irreducible, continuous representation satisfying the following conditions.*

- *The image of the porjective representation* $\mathrm{proj}\,\rho$ *of ρ is A_5.*
- *At every place $\mathfrak{p}$ of F above 2, the projective representation of ρ is unramified, and the image of* $\mathrm{Frob}_{\mathfrak{p}}$ *has order 3.*

Then ρ arises from a holomorphic cuspidal Hilbert modular eigenform of weight 1 and the Artin L-function $L(\rho, s)$ is entire.

These are corollaries of the following theorems, first of which is about "*if* $\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to GL_2(\overline{\mathbf{F}}_p)$ is modular, *then* $\rho : \mathrm{Gal}(\overline{F}/F) \to GL_2(\overline{\mathbf{Q}}_p) \simeq GL_2(\mathbf{C})$ is modular":

THEOREM 3 *Let p be a rational prime. Let K be a finite extension of $\mathbf{Q}_p$ with ring $\mathcal{O}$ of integers and maximal ideal $\mathfrak{m}$. Let F be a totally real field. Suppose that p splits completely in F. Let $\rho : \mathrm{Gal}(\overline{F}/F) \to GL_2(\mathcal{O})$ be a continuous representation satisfying the following conditions.*

- *ρ ramifies at only finitely many primes.*
- *$\overline{\rho} = (\rho \bmod \mathfrak{m})$ is absolutely irreducible when restricted to* $\mathrm{Gal}(\overline{F}/F(\zeta_p))$*, and has a modular lifting which is potentially ordinary and potentially Barsotti-Tate at every prime of F above p.*
- *For every prime $\mathfrak{p}$ of F above p, the restriction $\rho|_{G_{\mathfrak{p}}}$ to the decomposition group $G_{\mathfrak{p}}$ at $\mathfrak{p}$ is the direct sum of 1-dimensional characters $\chi_{\mathfrak{p},1}$ and $\chi_{\mathfrak{p},2}$ of $G_{\mathfrak{p}}$ such that the images of the inertia subgroup at $\mathfrak{p}$ are finite and* $(\chi_{\mathfrak{p},1} \bmod \mathfrak{m}) \neq (\chi_{\mathfrak{p},2} \bmod \mathfrak{m})$.

If $p = 2$, assume moreover *the following conditions.*

- *The image of the complex conjugation, with respect to every embedding of F into $\mathbf{R}$, is not the identity matrix.*
- *$\overline{\rho}$ has insoluble image.*
- *For every prime $\mathfrak{p}$ of F above 2, ρ is unramified at $\mathfrak{p}$.*

Then there exists an embedding $\iota : K \hookrightarrow \overline{\mathbf{Q}}_p \simeq \mathbf{C}$ and a classical holomorphic cuspidal Hilbert modular eigenform f of weight 1 such that $\iota \circ \rho$ is isomorphic to the representation associated to f by Rogawski-Tunnell [28].

In proving the theorem, we shall firstly establish $R = T$ theorems for Hida p-ordinary families over a finite soluble totally real extension F_Σ of F in which $p \geq 2$ remains split completely–for lack of reference we shall prove them. Since $\overline{\rho}$ has a potentially p-Barsotti-Tate and potentially p-ordinary modular lifting, one can deduce $R = T$ in p-adic families from Kisin's $R = T$ theorems in the p-Barsotti-Tate case. Note that, unfortunately, it is not possible to make appeal to Geraghty's $R = T$ theorems in p-ordinary families as they assume that $p > 2$ and that $\overline{\rho}$ is trivial at every prime of F above p. This is because one can not eliminate the possibility that, upon 'soluble' base-changing to F_Σ to set $\overline{\rho}|_{\mathrm{Gal}(\overline{F}/F_\Sigma)}$ trivial at every prime of F_Σ above p, F_Σ may no longer be split at p, which is crucial in constructing weight one forms in our approach.
In the light of [1], the condition about the existence of a potentially ordinary Barsotti-Tate lifting of $\overline{\rho}$ can be weaker, more precisely, it suffices to assume '$\overline{\rho}$ is modular'. It is not necessary to make appeal to their results however.

The next two theorems are about modularity of $\overline{\rho}$.

Theorem 4 *Let F be a totally real field. Suppose that 5 is unramified in F. Let $\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to GL_2(\overline{\mathbf{F}}_5)$ be a continuous representation of satisfying the following conditions.*

- *$\overline{\rho}$ is totally odd.*
- *$\overline{\rho}$ has projective image A_5.*

The there exists a cuspidal Hilbert modular eigenform of weight 2 such that its associated 5-adic Galois representation is potentially Barsotti-Tate and potentially ordinary at every prime of F above 5, and its associated mod 5 Galois representation is isomorphic to $\overline{\rho}$.

The idea is exactly the same as that of Taylor–to prove modularity of $\overline{\rho}$, one firstly finds an elliptic curve E over a finite *soluble* totally real field extension F_Σ of F such that the action of $\mathrm{Gal}(\overline{F}/F_\Sigma)$ on the 5-torsion points of E is isomorphic to $\overline{\rho}|_{\mathrm{Gal}(\overline{F}/F_\Sigma)}$; secondly one proves E modular, therefore $\overline{\rho}|_{\mathrm{Gal}(\overline{F}/F_\Sigma)}$ modular; and finally it follows from Khare-Wintenberger [18] and Kisin [20] that $\overline{\rho}|_{\mathrm{Gal}(\overline{F}/F_\Sigma)}$ has a characteristic zero lifting which is modular. The 'automorphic descent' works as in [39].
In proving E is modular, we make some technical improvements on a 'naive' generalisation over totally real fields of the main theorem of Taylor in [39] by making appeal to the main result of Kisin [20] rather than the main result of Skinner-Wiles [35]. While Taylor/Skinner-Wiles requires the mod 3 representation $E[3](\overline{F}_\Sigma)$ of $\mathrm{Gal}(\overline{F}/F_\Sigma)$ to be reducible with distinct characters on the diagonal at every prime of F_Σ above 3, we no longer requires this and consequently remove the '3-distinguishedness condition' in the main theorems of [39]. The key observation is that the weight 2 specialisation $F_{\mathrm{H},2}$ of the Hida family F_{H}, whose weight 1 specialisation $F_{\mathrm{H},1}$ renders $E[3](\overline{F}_\Sigma)$ modular by

Langlands-Tunnell, does indeed render the 3-adic Barsotti-Tate representation T_3E 'strongly residually modular' in the sense of Kisin [20] if $E[3](\overline{F}_\Sigma)$ is unramified at every prime above 3.

As is clear from its proof, what we are proving indeed is modularity of general mod 5 representations $\mathrm{Gal}(\overline{F}/F) \to GL_2(\mathbf{F}_5)$, and this allows us to work with the prime 2–proving modularity of $\overline{\rho}_2 : \mathrm{Gal}(\overline{F}/F) \to SL_2(\mathbf{F}_4)$ with $\mathrm{proj}\,\overline{\rho}_2 \simeq A_5$–instead of the prime 5, going back to the original approach of Buzzard-Dickinson-Shepherd-Barron-Taylor; in [4] one firstly finds an abelian surface A over F with real multiplication $\mathbf{Z}[(1+\sqrt{5})/2]$ such that $A(\overline{F})[2] \simeq \overline{\rho}_2$; secondly proves the mod 5 representation $\mathrm{Gal}(\overline{F}/F) \to GL_2(A(\overline{F})[\sqrt{5}]) \simeq GL_2(\mathbf{F}_5)$ is modular; and deduce A is modular by a modular lifting theorem.

Theorem 5 *Let F be a totally real field. Suppose that $[F(\zeta_5) : F] = 4$. Let $\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to SL_2(\mathbf{F}_4)$ be a continuous representation. Then there exists a cuspidal Hilbert modular eigenform of weight 2 such that its associated 2-adic Galois representation is potentially Barsotti-Tate and potentially unramified at every prime of F above 2 and its associated mod 2 Galois representation is isomorphic to $\overline{\rho}$.*

Lastly it might come in useful comparing our work and others. After the first draft of this paper was written in 2010, Kassaei announced a result proving an analogue of the main theorem 3 in the case when p is odd, p is unramified in F, and $\chi_{\mathfrak{p},1}/\chi_{\mathfrak{p},2}$ and $\chi_{\mathfrak{p},2}/\chi_{\mathfrak{p},1}$ are both unramified at every prime $\mathfrak{p}$ of F above p. Pilloni, on the other hand, seems to have proved a slightly stronger analogue in which p is allowed to ramify a little in F. The fundamental ideas in both works and ours are essentially the same and are due to Buzzard, more specifically to Buzzard's Theorem 9.1 in [3]. In forthcoming joint work with Kassaei and Tian, we extend Kassaei's work to the case where $\chi_{\mathfrak{p},1}/\chi_{\mathfrak{p},2}$ and $\chi_{\mathfrak{p},2}/\chi_{\mathfrak{p},1}$ are of conductor $\mathfrak{p}$ for every prime $\mathfrak{p}$ of F above p (unramified in F) *and* prove many new cases of the strong Artin conjecture for $\rho : \mathrm{Gal}(\overline{F}/F) \to GL_2(\mathbf{C})$ in the insoluble case as above.
To prove an analogue of the main theorem 3 in the case where $\chi_{\mathfrak{p},1}/\chi_{\mathfrak{p},2}$ and $\chi_{\mathfrak{p},2}/\chi_{\mathfrak{p},1}$ are of conductor $\mathfrak{p}^r$ with $r > 1$ for every prime $\mathfrak{p}$ of F above p, one needs to know precise geometry of Hilbert modular varieties of level p^r and, unless p splits completely in F which we solve, this may not even be possible. Calculating q-expansions at cusps to glue weight one forms does not seem to depend on the ramification of p in F and, for that, this work is very useful in general. On the other hand, in order to prove the general case (p ramifies arbitrarily in F), the author [30] considers new moduli spaces of Hilbert-Blumenthal abelian varieties; and he expects to make progress in the general case in his forthcoming work.

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2 Modularity of mod 5 icosahedral representations of $\mathrm{Gal}(\overline{F}/F)$

Lemma 6 *Let F be a totally real field. Suppose that 5 is unramified in F. Suppose that $\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to GL_2(\overline{\mathbf{F}}_5)$ is a continuous representation satisfying the following conditions.*

- *$\overline{\rho}$ is totally odd.*
- *$\overline{\rho}$ has projective image A_5.*

Then there is a finite soluble totally real field extension F_Σ of F and an elliptic curve E over F_Σ such that

- *$F(\sqrt{5}) \subset F_\Sigma \subset \overline{F}$, and $\sqrt{5}$ splits completely in F_Σ;*
- *E has good ordinary reduction at every prime of F above 3 and has potentially ordinary reduction at every prime of F above 5;*
- *$\overline{\rho}_{E,5} : \mathrm{Gal}(\overline{F}/F_\Sigma) \to \mathrm{Aut}(E(\overline{F}_\Sigma)[5])$ is equivalent to a twist of $\overline{\rho}|_{\mathrm{Gal}(\overline{F}/F_\Sigma)}$;*
- *$\overline{\rho}_{E,3}|_{\mathrm{Gal}(\overline{F}/F_\Sigma(\zeta_3))} : \mathrm{Gal}(\overline{F}/F_\Sigma(\zeta_3)) \to \mathrm{Aut}(E(\overline{F}_\Sigma)[3])$ is absolutely irreducible.*

Proof. Firstly, as in [39], find a biquadratic totally real extension $K_1 \subset \overline{F}$ of F, which is a quadratic totally real extension of $F(\sqrt{5})$ in which $\sqrt{5}$ splits completely, such that $\mathrm{proj}\,\overline{\rho} : \mathrm{Gal}(\overline{F}/K_1) \to PSL_2(\mathbf{F}_5) \simeq A_5$ lifts to a representation $\overline{\rho}_1 : \mathrm{Gal}(\overline{F}/K_1) \to GL_2(\mathbf{F}_5)$ with determinant the mod 5 cyclotomic character ϵ. Choose, by class field theory, a finite soluble totally real extension $K_2 \subset \overline{F}$ of K_1 such that $\overline{\rho}_1|_{\mathrm{Gal}(\overline{F}/K_2)}$ is trivial when restricted to the decomposition group at every prime of K_2 above 3. Let F_Σ denote the Galois closure of K_2 over F. Let $\overline{\rho}_\Sigma$ denote the restriction of $\overline{\rho}$ to $\mathrm{Gal}(\overline{F}/F_\Sigma)$.
As in section 1 of [33], let $Y_{\overline{\rho}_\Sigma}/F_\Sigma$ (resp. $X_{\overline{\rho}_\Sigma}/F_\Sigma$) denote the twist of the (resp. compactified) modular curve Y_5 (resp. X_5) with full level 5 structure by the cohomology class in $H^1(\mathrm{Gal}(\overline{F}/F_\Sigma), \mathrm{Aut}\,X_5)$ defined by an isomorphism $\overline{\rho}_\Sigma \simeq (\mathbf{Z}/5\mathbf{Z}) \times \mu_5$ of the $\mathbf{F}_5$-vector spaces. As proved in Lemma 1.1 in [33], the 'twist' cohomology class is indeed trivial, and therefore $X_{\overline{\rho}_\Sigma} \simeq X_5$ and $Y_{\overline{\rho}_\Sigma}$ is isomorphic over F_Σ to a Zariski open subset of the projective line $\mathbf{P}^1$. In particular, $Y_{\overline{\rho}_\Sigma}$ has infinitely many rational points.
Let $Y_{\overline{\rho}_\Sigma,0}(3)$ denote the degree 4 cover over $Y_{\overline{\rho}_\Sigma}$ which parameterises isomorphism classes of elliptic curves E equipped with an isomorphism $E[5] \simeq \overline{\rho}_\Sigma$ taking

the Weil pairing on $E[5]$ to $\epsilon : \wedge^2\overline{\rho}_\Sigma \to \mu_5$ and a finite flat subgroup scheme $C \subset E[3]$ of order 3.

Let $Y_{\overline{\rho}_\Sigma,\mathrm{split}}(3)$ denote the étale cover over $Y_{\overline{\rho}_\Sigma}$ which parameterises isomorphism classes of elliptic curves E equipped with an isomorphism $E[5] \simeq \overline{\rho}_\Sigma$ taking the Weil pairing on $E[5]$ to $\epsilon : \wedge^2\overline{\rho}_\Sigma \to \mu_5$ and an unordered pair, fixed by $\mathrm{Gal}(\overline{F}/F_\Sigma)$, of finite flat subgroup schemes $C, D \subset E[3]$ of order 3 which intersect trivially. Then it follows from Lemma 12 in [27] that $Y_{\overline{\rho}_\Sigma,\mathrm{split}}(3)$ and $Y_{\overline{\rho}_\Sigma,0}(3)$ has only finitely many rational points.

For every prime $\mathfrak{p}$ of F_Σ above 3, the elliptic curve $y^2 = x^3 + x^2 - x$ defines an element of $Y_{\overline{\rho}_\Sigma}(F_{\Sigma,\mathfrak{p}})$ with good ordinary reduction, and we let $\mathcal{U}_\mathfrak{p} \subset Y_{\overline{\rho}_\Sigma}(F_{\Sigma,\mathfrak{p}})$ denote a (non-empty) open neighbourhood (for the 3-adic topology) of the point, consisting of elliptic curves with good ordinary reduction at $\mathfrak{p}$.

For every prime $\mathfrak{p}$ of F_Σ above 5, we define a non-empty open subset (for the 5-adic topology) $\mathcal{U}_\mathfrak{p} \subset Y_{\overline{\rho}_\Sigma}(F_{\Sigma,\mathfrak{p}})$ as in the proof of Lemma 2.3 in [39].

By Hilbert irreducibility theorem (Theorem 1.3 in [11]; see also Theorem 3.5.7 in [32]), we may then find a rational point in $Y_{\overline{\rho}_\Sigma}(F_\Sigma)$ which lies in $\mathcal{U}_\mathfrak{p}$ for every $\mathfrak{p}$ of F_Σ above either 3 or 5 and does *not* lie in the images of $Y_{\overline{\rho}_\Sigma,0}(3)(F_\Sigma) \to Y_{\overline{\rho}_\Sigma}(F_\Sigma)$ and $Y_{\overline{\rho}_\Sigma,\mathrm{split}}(3)(F_\Sigma) \to Y_{\overline{\rho}_\Sigma}(F_\Sigma)$. The elliptic curve over F_Σ corresponding to the rational point is what we are looking for. $\square$

Theorem 7 *Let F be a totally real field. Suppose that* 5 *is unramified in F. Let $\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to GL_2(\overline{\mathbf{F}}_5)$ be a continuous representation of satisfying the following conditions.*

- *$\overline{\rho}$ is totally odd.*
- *$\overline{\rho}$ has projective image A_5.*

Then there exists a cuspidal Hilbert modular eigenform of weight 2 such that its associated 5*-adic Galois representation is potentially Barsotti-Tate and potentially ordinary at every prime of F above* 5 *and its associated mod 5 Galois representation is isomorphic to $\overline{\rho}$.*

Proof. Choose an elliptic curve over a finite soluble totally real extension F_Σ of F as in the lemma. Replace F_Σ by its finite soluble totally real extension if necessary to assume that the mod 3 representation $\overline{\rho}_{E,3}$ of $\mathrm{Gal}(\overline{F}/F_\Sigma)$ on $E(\overline{F}_\Sigma)[3]$ is unramified when restricted to the decomposition subgroup at every prime of F_Σ above 3. By the Langlands-Tunnell theorem, there exists a weight 1 holomorphic cuspidal Hilbert modular eigenform f_1 which gives rise to $\overline{\rho}_{E,3}$. By 3-adic Hida theory [14], we may find a holomorphic cuspidal Hilbert modular eigenform f_2 of weight 2 and level prime to 3, ordinary at every prime of F_Σ above 3, which gives rise to $\overline{\rho}_{E,3}$. As E is ordinary at 3, $\rho_{E,3}$ is strongly residually modular in the sense of Kisin [20] (3.5.4), and it follows from Theorem 3.5.5 in [20] that T_3E is modular. By Faltings' isogeny theorem, E is therefore modular. As $\overline{\rho}_{E,5}$ is modular, $\overline{\rho}|_{\mathrm{Gal}(\overline{F}/F_\Sigma)}$ is modular. Since F_Σ is a soluble extension of F, $\overline{\rho}_\Sigma$ remains absolutely irreducible when restricted to $\mathrm{Gal}(\overline{F}/F_\Sigma(\zeta))$. Furthermore, since 5 is unramfied in F, the kernel of proj $\overline{\rho}_\Sigma$ does not fix $F_\Sigma(\zeta_5)$. It then follows from results of Khare-Wintenberger [18] and Kisin [20] that there exists a modular lifting of $\overline{\rho}_\Sigma$. The 'soluble descent' to F is exactly as in [39]. □

Remark. In the forthcoming work with Kassaei and Tian, we remove the assumption that 5 is unramified in F in Lemma 6, and thereby in Theorem 7. Essentially the same argument works.

3 Modularity of mod 2 icosahedral representations of $\mathrm{Gal}(\overline{F}/F)$

Theorem 8 *Let F be a totally real field. Suppose that $[F(\zeta_5) : F] = 4$. Let $\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to SL_2(\mathbf{F}_4)$ be a continuous representation. Then there exists a cuspidal Hilbert modular eigenform of weight 2 such that its associated 2-adic Galois representation is potentially Barsotti-Tate and potentially unramified at every prime of F above 2 and its associated mod 2 Galois representation is isomorphic to $\overline{\rho}$.*

Proof. By Theorem 3.4 in [33], there exists a principally polarised abelian surface A over F with real multiplication by $\mathbf{Z}[(1+\sqrt{5})/2]$ compatible with the polarisation such that the action of $\mathrm{Gal}(\overline{F}/F)$ on $A(\overline{F})[2] \simeq \mathbf{F}_4^2$ is equivalent to $\overline{\rho}$; and the action of $\mathrm{Gal}(\overline{F}/F)$ on $A(\overline{F})[\sqrt{5}] \simeq \mathbf{F}_5^2$ is given via a homomorphism

$$\overline{\rho}_{A,\sqrt{5}} : \mathrm{Gal}(\overline{F}/F) \to GL_2(\mathbf{F}_5)$$

which is surjective and whose image contains $SL_2(\mathbf{F}_5)$. It suffices to prove that A is modular.

Firstly, the Weil pairing on $A(\overline{F})[\sqrt{5}]$ shows that $\det\overline{\rho}_{A,\sqrt{5}}$ is the mod 5 cyclotomic character. Since $[F(\zeta_5) : F] = 4$, the determinant is indeed surjective, and therefore $\overline{\rho}_{A,\sqrt{5}}$ is absolutely irreducible.

If $\overline{\rho}_{A,\sqrt{5}}$ is irreducible at some place of F above 5, the absolute irreducibility of $\overline{\rho}_{A,\sqrt{5}}$ implies the absolute irreducibility of its restriction to $\mathrm{Gal}(\overline{F}/F(\zeta_5))$. Otherwise, $\overline{\rho}_{A,\sqrt{5}}$ is reducible at every place of F above 5; in which case, it is also equally easy to check that its restriction to $\mathrm{Gal}(\overline{F}/F(\sqrt{5}))$ of $\overline{\rho}_{A,\sqrt{5}}$ is absolutely irreducible (See Proposition 7 in [27], for example). It follows from results of Khare-Wintenberger [16] and Kisin [20] that it is possible to construct a modular lifting of $\overline{\rho}_{A,\sqrt{5}}$; more precisely, $\overline{\rho}_{A,\sqrt{5}}$ is strongly residually modular. The modularity of $\rho_{A,\sqrt{5}}$ follows from Theorem 3.5.5 in [20] and [12]. □

4 Holomorphic Hilbert modular forms and Hida theory of modular Galois representations

Let F be a totally real field. We let $\mathcal{O}_F$ denote the ring of integers, $\mathfrak{d}_F$ the different of F, $\mathbf{A}_F = \mathbf{A}_F^\infty \times F_\infty$, and $\mathcal{O}_F^\wedge$ denote $\mathcal{O}_F \otimes_{\mathbf{Z}} \mathbf{Z}^\wedge \subset \mathbf{A}_F^\infty$. Let S_∞

denote the set of infinite places of F. For an ideal $\mathfrak{n}$ of $\mathcal{O}_F$, let $F_{\mathfrak{n}}$ denote the strict ray class field of conductor $\mathfrak{n}S_\infty$.
For an ideal $\mathfrak{n}$, let $U^1(\mathfrak{n})$ (resp. $U_1(\mathfrak{n})$) denote the open compact subgroup of $GL_2(\mathcal{O}_F^\wedge)$ consisting of matrices which are congruent modulo $\mathfrak{n}\mathcal{O}_F^\wedge$ to matrices with first column $(1,0)$ (resp. the second row $(0,1)$). Let $I_{\mathfrak{n}}$ denote $\mathbf{A}_F^\times/(F^\times(\mathbf{A}_F^{\infty\times}\cap U^1(\mathfrak{n}))F_\infty^{+\times})$.
For $k\in\mathbf{Z}$ and an open compact subgroup U of $GL_2(\mathcal{O}_F^\wedge)$, let $S_k(U)$ denote the space, $S_{k,k/2}(U)$ in the sense of Hida [14], of cuspidal holomorphic Hilbert modular forms f of parallel weight k and level U with the Fourier coefficient $c(\mathfrak{n},f)\in\mathbf{Z}$ for all ideals $\mathfrak{n}$ of $\mathcal{O}_F$. The spaces $S_k(U^1(\mathfrak{n}))$ and $S_k(U_1(\mathfrak{n}))$ for an ideal $\mathfrak{n}$ of $\mathcal{O}_F$ come equipped with an action of $I_{\mathfrak{n}}$ via the diamond operator $\langle\ \rangle$, and Hecke operators $T_{\mathfrak{q}}$ for every prime $\mathfrak{q}$ of $\mathcal{O}_F$ not dividing $\mathfrak{n}$ and $U_{\mathfrak{q}}$ for every prime of $\mathfrak{q}$ dividing $\mathfrak{n}$.
Let $h_k(\mathfrak{n})$ denote the sub $\mathbf{Z}$-algebra of $\mathrm{End}(S_k(U^1(\mathfrak{n}))$ generated over $\mathbf{Z}$ by all these operators (See Proposition 2.3, Theorem 4.10, and Theorem 4.11 of [14]). For every prime $\mathfrak{q}$ not dividing $\mathfrak{n}$, let $S_{\mathfrak{q}}=(\mathbf{N}_{F/\mathbf{Q}}\mathfrak{q})^{k-2}\langle\mathfrak{q}\rangle\in h_k(\mathfrak{n})$; this corresponds to the action of the scalar matrix with a uniformiser of $\mathcal{O}_F$ at $\mathfrak{q}$ on the diagonal. Following [14], for *every* ideal $\mathfrak{m}$ of $\mathcal{O}_F$, we may define $T_{\mathfrak{m}}\in h_k(\mathfrak{n})$.

Let p be a rational prime and let S_{P} denote the set of prime ideals of $\mathcal{O}_F$ dividing p. Fix an algebraic closure $\overline{\mathbf{Q}}_p$, an isomorphism $\overline{\mathbf{Q}}_p\simeq\mathbf{C}$, and an embedding $\overline{\mathbf{Q}}\to\overline{\mathbf{Q}}_p$.

For a ring $R\subset\overline{\mathbf{Q}}_p$, we shall let $S_k(U_1(\mathfrak{n}))_R$ denote $S_k(U_1(\mathfrak{n}))\otimes_{\mathbf{Z}}R$ and $h_k(\mathfrak{n})_R$ denote $h_k(\mathfrak{n})\otimes_{\mathbf{Z}}R$; there is a pairing $(\ ,\ ):h_k(\mathfrak{n})_R\times S_k(U_1(\mathfrak{n}))_R\to R$ defined by $(T,f)=c(\mathcal{O}_F,Tf)$.

For a ray class character $\psi:I_{\mathfrak{n}}\to\overline{\mathbf{Q}}_p^\times$ mod $\mathfrak{n}S_\infty$, let $S_{k,\psi}(U_1(\mathfrak{n}))_{\mathbf{Z}_p[\psi]}$ denote the submodule of $S_k(U_1(\mathfrak{n}_1))_{\mathbf{Z}_p[\psi]}$ consisting of cuspidal Hilbert modular forms of parallel weight k and level $U_1(\mathfrak{n})$ with central character ψ–$S_{\mathfrak{q}}$ acts via ψ at $\mathfrak{q}$; the forms in $S_{k,\psi}(U_1(\mathfrak{n}))_{\mathbf{Z}_p[\psi]}$ may be thought of as $|I_{\mathfrak{n}}|$-tuple of classical Hilbert modular forms of 'level $\Gamma_1(\mathfrak{n})$' on the $|I_{\mathfrak{n}}|$-copies of $(GL_2(\mathbf{R})/(\mathbf{R}^\times SO_2(\mathbf{R}))^{\mathrm{Hom}(F,\mathbf{R})}$ with 'Dirichlet character mod $\mathfrak{n}$'.

Fix an ideal $\mathfrak{n}$ of $\mathcal{O}_F$ coprime to p. For a finite extension K of $\mathbf{Q}_p$ with ring $\mathcal{O}$ of integers, Hida [14] defines the idempotent e and we set $h^0_{\mathcal{O}}(\mathfrak{n})$ to be the inverse limit with respect to $r\in\mathbf{Z}_{\geq1}$ of $h_2(\mathfrak{n}p^r)_{\mathbf{Z}_p}\otimes_{\mathbf{Z}_p}\mathcal{O}$. Let $I_{\mathfrak{n}p^\infty}$ denote the inverse limit of the $I_{\mathfrak{n}p^r}$ and the diamond operators $\langle\ \rangle:I_{\mathfrak{n}p^r}\to eh_2(\mathfrak{n}p^r)_{\mathcal{O}}$ induce

$$\langle\ \rangle:I_{\mathfrak{n}p^\infty}\to h^0_{\mathcal{O}}(\mathfrak{n})^\times.$$

One can also see $\langle\ \rangle$ as the action of $(\mathcal{O}_F/\mathfrak{n})^\times\times(\mathcal{O}_F\otimes_{\mathbf{Z}}\mathbf{Z}_p)^\times$ by the composite:

$$(\mathcal{O}_F/\mathfrak{n})^\times\times(\mathcal{O}_F\otimes_{\mathbf{Z}}\mathbf{Z}_p)^\times\to I_{\mathfrak{n}p^\infty}\xrightarrow{\langle\ \rangle}h^0_{\mathcal{O}}(\mathfrak{n}).$$

We let $\mathrm{Tor}_{\mathfrak{n}p^\infty}$ (resp. $\mathrm{Fr}_{\mathfrak{n}p^\infty}$) denote the torsion subgroup (resp. the maximal $\mathbf{Z}_p$ free subgroup of rank $1+\delta$ with $\delta = 0$ if the Leopoldt conjecture holds) of $I_{\mathfrak{n}p^\infty}$; let Λ denote the completed group algebra over $\mathbf{Z}_p$ of $\mathrm{Fr}_{\mathfrak{n}p^\infty}$ and $\Lambda_K = \Lambda \otimes_{\mathbf{Z}_p} \mathcal{O}$. Then $h^0_{\mathcal{O}}(\mathfrak{n})$ is a Λ_K-module by $\langle\ \rangle$. We will let

$$\mathrm{Art} : \mathbf{A}_F^\times / \overline{F^\times F_\infty^{+\times}} \simeq \mathrm{Gal}(\overline{F}/F)^{\mathrm{ab}}$$

denote the (global) Artin map, normalised compatibly with the local Artin maps normalised to take uniformisers to arithmetic Frobenius elements. By abuse of notation, we shall let Art also denote the induced homomorphism $I_{\mathfrak{n}p^\infty} \to \mathrm{Gal}(F_{\mathfrak{n}}(\mu_{p^\infty})/F)$ and let ϵ denote the cyclotomic character $\epsilon : \mathrm{Gal}(F_{\mathfrak{n}}(\mu_{p^\infty})/F) \to \mathbf{Z}_p^\times$.

Hida [14] proves that $h^0_{\mathcal{O}}(\mathfrak{n})$ is a torsion free Λ_K-module and, for a character $\psi : I_{\mathfrak{n}p^\infty} \to K$ which factors through $I_{\mathfrak{n}p^r}$ for $r \in \mathbf{Z}_{\geq 1}$, if $k \geq 2$, then $h^0_{\mathcal{O}}(\mathfrak{n})_{\ker((\epsilon\circ\mathrm{Art})^{k-2}\psi)}$ is isomorphic to the subspace of $eS_k(U_1(\mathfrak{n}p^r))_{\mathcal{O}}$ where $\langle\ \rangle = \psi$ on $\mathrm{Fr}_{\mathfrak{n}p^\infty}$.
We will let $\epsilon^{\mathrm{cyclo}}$ denote the character

$$\mathrm{Gal}(\overline{F}/F) \twoheadrightarrow \mathrm{Gal}(\overline{F}/F)^{\mathrm{ab}} \twoheadrightarrow I_{\mathfrak{n}p^\infty} \hookrightarrow \mathcal{O}[[I_{\mathfrak{n}p^\infty}]]^\times = \Lambda_K[\mathrm{Tor}_{\mathfrak{n}p^\infty}]^\times.$$

Note that $\mathfrak{q} \mapsto \mathbf{N}\mathfrak{q}S_{\mathfrak{q}}$ extends to $\mathbf{N}S : (\mathcal{O}_F/\mathfrak{n})^\times \times (\mathcal{O}_F \otimes_{\mathbf{Z}} \mathbf{Z}_p)^\times \to I_{\mathfrak{n}p^\infty} \to h^0_{\mathcal{O}}(\mathfrak{n})^\times$. Let $\mathbf{N}S_\Sigma$ (resp. $\mathbf{N}S^{\mathrm{P}}$) denote the Σ (resp. the prime to S_{P}) part $\prod_{\mathfrak{p}\in\Sigma} \mathcal{O}_{F_{\mathfrak{p}}}^\times \to h^0_{\mathcal{O}}(\mathfrak{n})$ (resp. $(\mathcal{O}_F/\mathfrak{n})^\times \to h^0_{\mathcal{O}}(\mathfrak{n})$) for s subset Σ of S_{P}.

If $\mathfrak{m}$ is a maximal ideal of $h^0_{\mathcal{O}}(\mathfrak{n})$ with residue field $k_{\mathfrak{m}}$, there is a continuous representation

$$\overline{\rho}_{\mathfrak{m}} : G_F = \mathrm{Gal}(\overline{F}/F) \to GL_2(k_{\mathfrak{m}})$$

such that, for every prime ideal $\mathfrak{q}$ of $\mathcal{O}_F$ not dividing $\mathfrak{n}p$, $\overline{\rho}_{\mathfrak{m}}$ is unramified at $\mathfrak{q}$ and $\mathrm{tr}\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_{\mathfrak{q}}) = T_{\mathfrak{q}}$. Set $S^0_{\mathcal{O}}(\mathfrak{n}) = \mathrm{Hom}_{\Lambda_K}(h^0_{\mathcal{O}}(\mathfrak{n}), \Lambda_K)$. For a finite field extension L of the field $\mathrm{Frac}\,\Lambda_K$ of fractions of Λ_K with its integral closure $\mathcal{O}_L$ of Λ_K in L, Buzzard-Taylor [5] calls a Λ_K-algebra homomorphism $F_{\mathrm{H}} \in S^0_{\mathcal{O}}(\mathfrak{n}) \otimes_{\Lambda_K} L = \mathrm{Hom}_{\Lambda_K}(h^0_{\mathcal{O}}, L)$ a Λ-adic *eigenform* (of level $\mathfrak{n}$).
If the unique maximal ideal $\mathfrak{m}$ above $\ker F_{\mathrm{H}} \subset h^0_{\mathcal{O}}(\mathfrak{n})$ is non-Eisenstein, i.e., $\overline{\rho}_{\mathfrak{m}}$ as above is absolutely irreducible, then there is a continuous representation

$$\rho_{F_{\mathrm{H}}} : G_F \to GL_2(h^0_{\mathcal{O}}(\mathfrak{n})_{\mathfrak{m}}) \xrightarrow{F_{\mathrm{H}}} GL_2(\mathcal{O}_L)$$

which is unramified at every prime ideal $\mathfrak{q}$ of $\mathcal{O}_F$ not dividing $\mathfrak{n}p$ and satisfies $\mathrm{tr}\,\rho_{F_{\mathrm{H}}}(\mathrm{Frob}_{\mathfrak{q}}) = T_{\mathfrak{q}}$ and $\det \rho_{F_{\mathrm{H}}} = (\mathbf{N}S)\circ\epsilon^{\mathrm{cyclo}}$. Moreover, it is a result of Wiles [43] that, for every place $\mathfrak{p}$ of F above p, the restriction to the decomposition group $G_{\mathfrak{p}}$ at $\mathfrak{p}$ of $\rho_{F_{\mathrm{H}}}$ is of the form

$$\begin{pmatrix} \chi_{F_{\mathrm{H}},\mathfrak{p},2} & * \\ 0 & \chi_{F_{\mathrm{H}},\mathfrak{p},1} \end{pmatrix}$$

where $\chi_{F_{\mathrm{H}},\mathfrak{p},1}$ is an unramified character of $G_{\mathfrak{p}}$ such that $\chi_{F_{\mathrm{H}},\mathfrak{p},1}(\mathrm{Frob}_{\mathfrak{p}}) = U_{\mathfrak{p}}$ and $\chi_{F_{\mathrm{H}},\mathfrak{p},1}\chi_{F_{\mathrm{H}},\mathfrak{p},2} = (F_{\mathrm{H}} \circ \mathbf{N}S) \circ \epsilon^{\mathrm{cyclo}}|_{G_{\mathfrak{p}}}$.

DEFINITION. Following [5], we call two Λ-adic eigenforms $F_{\mathrm{H},1}$ and $F_{\mathrm{H},2}$: $h^0_{\mathcal{O}}(\mathfrak{n}) \to \mathcal{O}_L$ of level $\mathfrak{n}$ Λ-adic companion form with respect to primes $\wp_1$ and $\wp_2$ of $\mathcal{O}_L$ which do not divide p, if there are embeddings $\iota_1 : \mathcal{O}_L/\wp_1 \hookrightarrow \overline{\mathbf{Q}}_p$ and $\iota_2 : \mathcal{O}_L/\wp_2 \hookrightarrow \overline{\mathbf{Q}}_p$ such that, for every ideal $\mathfrak{m}$ of $\mathcal{O}_F$ not dividing p, there exists a subset Σ of S_{P} such that $(F_{\mathrm{H},2}(T_{\mathfrak{m}}) \bmod \wp_2) = (F_{\mathrm{H},1}(T_{\mathfrak{m}}(\mathbf{N}S_\Sigma)(\mathfrak{m})^{-1}) \bmod \wp_1)$; and such that, for every place $\mathfrak{p}$ in Σ, $(F_{\mathrm{H},2}(U_{\mathfrak{p}}) \bmod \wp_2) = (F_{\mathrm{H},1}(U_{\mathfrak{p}}^{-1}(\mathbf{N}S^P)(\mathfrak{p})) \bmod \wp_1)$ while, for every $\mathfrak{p}$ in $S_{\mathrm{P}} - \Sigma$, $(F_{\mathrm{H},2}(U_{\mathfrak{p}}) \bmod \wp_2) = (F_{\mathrm{H},1}(U_{\mathfrak{p}}) \bmod \wp_1)$.

5 DEFORMATION RINGS AND HECKE ALGEBRAS

Let F be a totally real field of even degree in which p is unramified; if $p = 2$ assume furthermore that 2 splits completely in F. If p is odd, suppose $p \geq 5$. Let D be the quaternion algebra over F which ramifies exactly at a finite set Σ of finite places of F not dividing p and the infinite places S_∞ of F. Let $\mathcal{O}_D$ denote a maximal order and fix an isomorphism $\mathcal{O}_{D\mathfrak{q}} \simeq M_2(\mathcal{O}_{F_{\mathfrak{q}}})$ for $\mathfrak{q}$ not in Σ. Let S denote the disjoint union of Σ, the set S_{P} of places of F above p, and the infinite places of F.
For a topological $\mathbf{Z}_p$-algebra R, let $\psi : \mathbf{A}_F^{\infty,\times}/F \to R^\times$ be a continuous character such that $\psi|_{\mathcal{O}^\times_{F_{\mathfrak{p}}}}$ is trivial for every place $\mathfrak{p}$ of F above p, and, for an open compact subgroup $U = \prod_{\mathfrak{q}} U_{\mathfrak{q}} \subset \prod_{\mathfrak{q}} \mathcal{O}^\times_{D\mathfrak{q}}$, let $S^D_{2,\psi}(U)_R$ denote the R-module of R-valued modular forms on $D^\times \backslash (D \otimes_F \mathbf{A}_F^\infty)^\times$ of weight 2 and of level U in the sense of Taylor [40].
Let $\mathfrak{n}_\Sigma$ denote the square-free product of the primes in Σ and define $U_\Sigma \subset (D \otimes_F \mathbf{A}_F^\infty)^\times$ by $U_{\Sigma,\mathfrak{q}} = GL_2(\mathcal{O}_{F\mathfrak{q}})$ for $\mathfrak{q}$ not in Σ; and $U_{\Sigma,\mathfrak{q}} = \mathcal{O}^\times_{D\mathfrak{q}}$ for $\mathfrak{q} \in \Sigma$.

We shall write $S^D_{2,\psi}(\mathfrak{n}_\Sigma)$ for $S^D_{2,\psi}(U_\Sigma)$ and $h^D_{2,\psi}(\mathfrak{n}_\Sigma)_R$ for the R-subalgebra of $\mathrm{End}_R(S^D_{2,\psi}(\mathfrak{n}_\Sigma)_R)$ generated by $T_{\mathfrak{q}}$ and $S_{\mathfrak{q}}$ for all $\mathfrak{q}$ not in S; and $T_{\mathfrak{p}}$ and $S_{\mathfrak{p}}$ for all $\mathfrak{p}$ in S_{P}.

Let K be a finite extension of $\mathbf{Q}_p$ and $\mathcal{O}$ be the ring of integers with maximal ideal $\mathfrak{m}$ ad residue field k. Let

$$\rho : \mathrm{Gal}(\overline{F}/F) \to GL_2(\mathcal{O})$$

be a continuous representation such that

- $\overline{\rho} = (\rho \bmod \mathfrak{m})$ is unramified outside S_{P},
- $\overline{\rho}$ is not scalar at every place $\mathfrak{p}$ above p,
- if p is odd, the restriction of $\overline{\rho}$ to $\mathrm{Gal}(\overline{F}/F(\zeta_p))$ is absolutely irreducible; if $p = 2$, $\overline{\rho}$ has insoluble image,

- there exists a holomorphic automorphic representation π of $(D \otimes_F \mathbf{A}_F)^\times$ generated by a cusp form in $S^D_{2,\psi}(\mathfrak{n}_\Sigma)_{\mathcal{O}}$ such that $\pi_\mathfrak{q}$ is unramified for every $\mathfrak{q}$ not in $\Sigma \cup S_{\mathrm{P}}$, $\pi_\mathfrak{p}$ is ordinary at every $\mathfrak{p}$ in S_{P}, for every $\mathfrak{q} \in \Sigma$, $\pi_\mathfrak{q}$ corresponds by the local Jacquet-Langlands correspondence to a special representation of conductor $\mathfrak{q}$, and such that $\overline{\rho}_\pi \simeq \overline{\rho}$,

- ρ ramifies at Σ and possibly at S_{P}; for every $\mathfrak{p}$ in S_{P}

$$\rho|_{G_\mathfrak{p}} \sim \begin{pmatrix} * & * \\ 0 & \chi_\mathfrak{p} \end{pmatrix}$$

  with $\chi_\mathfrak{p}$ unramified; and for $\mathfrak{q} \in \Sigma$

$$\rho|_{G_\mathfrak{q}} \sim \begin{pmatrix} \epsilon\chi_\mathfrak{q} & * \\ 0 & \chi_\mathfrak{q} \end{pmatrix}$$

  with $\chi_\mathfrak{q}$ unramified such that $\chi_\mathfrak{q}^2 = (\psi \circ \mathrm{Art})|_{G_\mathfrak{q}}$.

Let $\mathbf{A}_F^\times = \mathbf{A}_F^{\times S} \times \mathbf{A}_{FS}^\times$ for a finite subset S of the places of F. Let ψ be a character of $\mathbf{A}_F^{\times S_{\mathrm{P}}}$. For $p = 2$ let $\psi_{\mathrm{P},\pm}$ denote the $\mathbf{Z}_p$-linear extension of the norm $\mathbf{N} : (\mathcal{O}_F \otimes_\mathbf{Z} \mathbf{Z}_2)^\times \to \mathbf{Z}_2^\times$ followed by the character $\mathbf{Z}_2^\times \to \mathbf{Z}_p^\times$ whose restriction to $(\mathbf{Z}/4)^\times = \{\pm 1\}$ sends -1 to ∓ 1 and whose restriction to $(1 + 4\mathbf{Z}_2)^\times$ is trivial. For p odd, let ψ_{P} denote the norm followed by the trivial character on $\mathbf{Z}_p^\times$.

5.1 (Framed) deformation rings R

$\underline{\mathfrak{p}|p}$: if p is odd, let $R_\mathfrak{p}^{\square,\mathrm{ord}}$ (resp. $R_\mathfrak{p}^{\square,\mathrm{BT},\mathrm{ord}}$) denote the $\mathcal{O}$-algebra which represents the $\mathfrak{p}$-ordinary (resp. Barsotti-Tate $\mathfrak{p}$-ordinary) framed deformations of $\overline{\rho}|_{G_\mathfrak{p}}$ of the form

$$\begin{pmatrix} * & * \\ 0 & \chi_\mathfrak{p}^{\mathrm{ur}} \end{pmatrix}$$

with an unramified lifting $\overline{\chi}_\mathfrak{p}^{\mathrm{ur}}$ of $\chi_\mathfrak{p}$ (resp. *and* its determinant is $\epsilon\psi_P$); if $p = 2$, we shall write '$\pm$' in shorthand to mean two independent cases– '$+$' corresponds to the 2-old case while '$-$' corresponds to the 2-new case in the sense to be made precise below, and let $R_{\mathfrak{p},\pm}^{\square,\mathrm{ord}}$ (resp. $R_{\mathfrak{p},\pm}^{\square,\mathrm{BT},\mathrm{ord}}$) denote the complete local noetherian $\mathcal{O}$-algebra which represents the $\mathfrak{p}$-ordinary (resp. Barsotti-Tate $\mathfrak{p}$-ordinary) liftings of $\overline{\rho}|_{G_\mathfrak{p}}$ of the form

$$\begin{pmatrix} * & * \\ 0 & \chi_\mathfrak{p}^{\mathrm{ur}} \end{pmatrix}$$

with an unramified lifting $\chi_\mathfrak{p}^{\mathrm{ur}}$ of $\chi_\mathfrak{p}$, and with its determinant corresponding, by the local class field theory, to the norm $\mathcal{O}_{F_\mathfrak{p}}^\times \hookrightarrow (\mathcal{O}_F \otimes_\mathbf{Z} \mathbf{Z}_2)^\times \xrightarrow{\mathbf{N}} \mathbf{Z}_2^\times$ followed by the character $\mathbf{Z}_2^\times \to \mathbf{Z}_2^\times$ whose restriction to $(\mathbf{Z}/4)^\times = \{\pm 1\}$ sends -1 to

∓ 1 (resp. with its determinant $\epsilon\psi_{P,\pm}$).

Let $R_{\mathrm{P}}^{\square,\mathrm{ord}} = \bigotimes^{\wedge}_{\mathfrak{p}\in S_{\mathrm{P}}} R_{\mathfrak{p}}^{\square,\mathrm{ord}}$ (resp. $R_{\mathrm{P}}^{\square,\mathrm{BT},\mathrm{ord}} = \bigotimes^{\wedge}_{\mathfrak{p}\in S_{\mathrm{P}}} R_{\mathfrak{p}}^{\square,\mathrm{BT},\mathrm{ord}}$) if p is odd; and $R_{\mathrm{P},\pm}^{\square,\mathrm{ord}} = \bigotimes^{\wedge}_{\mathfrak{p}\in S_{\mathrm{P}}} R_{\mathfrak{p},\pm}^{\square,\mathrm{ord}}$ (resp. $R_{\mathrm{P}}^{\square,\mathrm{BT},\mathrm{ord}} = \bigotimes^{\wedge}_{\mathfrak{p}\in S_{\mathrm{P}}} R_{\mathfrak{p},\pm}^{\square,\mathrm{BT},\mathrm{ord}}$) if $p = 2$.

$\underline{\mathfrak{q}\in\Sigma}$: let $R_{\mathfrak{q}}^{\square,\psi}$ denote the domain (see 2.6 in [20], or Proposition 2.12 and 3.3.4 in [18]) parameterising liftings of $\overline{\rho}|_{G_{\mathfrak{q}}}$ of the form

$$\begin{pmatrix} \epsilon\chi_{\mathfrak{q}}^{\mathrm{ur}} & * \\ 0 & \chi_{\mathfrak{q}}^{\mathrm{ur}} \end{pmatrix}$$

with $\chi_{\mathfrak{q}}^{\mathrm{ur}}$ an unramified lifting of $\overline{\chi}_{\mathfrak{q}}$ such that $(\chi_{\mathfrak{q}}^{\mathrm{ur}})^2 = (\psi\circ\mathrm{Art}^{-1})|_{G_{\mathfrak{q}}}$.

Let $R_{\Sigma}^{\square,\psi}$ denote the completed tensor product $\bigotimes^{\wedge}_{\mathfrak{q}\in\Sigma} R_{\mathfrak{q}}^{\square,\psi}$.

$\underline{\tau|\infty}$: let $R_{\tau}^{\square,\mathrm{odd}}$ denote the formally smooth ring which represents the liftings of $\overline{\rho}|_{G_\tau}$ which, if p is odd, are odd ; and, if $p = 2$, the image of complex conjugation in $G_\tau \simeq \mathrm{Gal}(\mathbf{C}/\mathbf{R})$ is not the identity matrix.

Let $R_{\infty}^{\square,\mathrm{odd}}$ denote the completed tensor product $\bigotimes^{\wedge}_{\tau|\infty} R_{\tau}^{\square,\mathrm{odd}}$

Fix a k-basis of $\overline{\rho}$ and let

$$\rho^{\square S} : G_F \to GL_2(R^{\square S})$$

denote the S-framed universal deformation ring. Let $R_S^{\square}$ denote the completed tensor product of the local framed deformation rings at places in S.

Let

$$\begin{aligned} R_S^{\square,\mathrm{ord},\psi} &= R^{\square S} \otimes^{\wedge}_{R_S^{\square}} (R_{\mathrm{P}}^{\square,\mathrm{ord}} \otimes^{\wedge} R_{\Sigma}^{\square,\psi} \otimes^{\wedge} R_{\infty}^{\square,\mathrm{odd}}) \\ R_S^{\square,\mathrm{BT},\mathrm{ord},\psi} &= R^{\square S} \otimes^{\wedge}_{R_S^{\square}} (R_{\mathrm{P}}^{\square,\mathrm{BT},\mathrm{ord}} \otimes^{\wedge} R_{\Sigma}^{\square,\psi} \otimes^{\wedge} R_{\infty}^{\square,\mathrm{odd}}) \end{aligned}$$

if p is odd; and

$$\begin{aligned} R_{S,\pm}^{\square,\mathrm{ord},\psi} &= R^{\square S} \otimes^{\wedge}_{R_S^{\square}} (R_{\mathrm{P},\pm}^{\square,\mathrm{ord}} \otimes^{\wedge} R_{\Sigma}^{\square,\psi} \otimes^{\wedge} R_{\infty}^{\square,\mathrm{odd}}) \\ R_{S,\pm}^{\square,\mathrm{BT},\mathrm{ord},\psi} &= R^{\square S} \otimes^{\wedge}_{R_S^{\square}} (R_{\mathrm{P},\pm}^{\square,\mathrm{BT},\mathrm{ord}} \otimes^{\wedge} R_{\Sigma}^{\square,\psi} \otimes^{\wedge} R_{\infty}^{\square,\mathrm{odd}}) \end{aligned}$$

if $p = 2$.

Let $R_S^{\mathrm{ord},\psi}$ (resp. $R_S^{\mathrm{ord},\mathrm{BT},\psi}$) denote the subring of $R_S^{\square,\mathrm{ord},\psi}$ (resp. $R_S^{\square,\mathrm{BT},\mathrm{ord},\psi}$) generated by the traces of $\rho^{\square S}$. Similarly define $R_{S,\pm}^{\mathrm{ord},\psi}$ and $R_{S,\pm}^{\square,\mathrm{BT},\mathrm{ord},\psi}$.

5.2 Hecke algebras

Since $\overline{\rho}$ arises from a holomorphic cusp form in $S_2^D(\mathfrak{n}_\Sigma,\psi)_{\mathcal{O}}$ on the quaternion algebra D over F_Σ by assumption, there exists a maximal ideal $\mathfrak{m}^D \subset h_2^D(\mathfrak{n}_\Sigma,\psi\psi_{\mathrm{P}})_{\mathcal{O}}$ if p odd (resp. $\mathfrak{m}^D \subset h_2^D(\mathfrak{n}_\Sigma,\psi\psi_{\mathrm{P},+})_{\mathcal{O}}$ if $p=2$). It then follows that there exists a maximal ideal $\mathfrak{m} \subset h_2(\mathfrak{n}_\Sigma p,\psi\psi_{\mathrm{P}})_{\mathcal{O}}$ such that

$$h_2(\mathfrak{n}_\Sigma p,\psi)_{\mathfrak{m}} \simeq h_2^D(\mathfrak{n}_\Sigma,\psi\psi_{\mathrm{P}})_{\mathfrak{m}^D}$$

if p odd (resp. $\mathfrak{m}_+ \subset h_2(\mathfrak{n}_\Sigma 2,\psi\psi_{\mathrm{P},+})_{\mathcal{O}}$ such that

$$h_2(\mathfrak{n}_\Sigma 2,\psi\psi_{\mathrm{P},+})_{\mathfrak{m}_+} \simeq h_2^D(\mathfrak{n}_\Sigma,\psi\psi_{\mathrm{P},+})_{\mathfrak{m}^D}$$

if $p=2$). When $p=2$, there also exists $\mathfrak{m}_- \subset h_2(\mathfrak{n}_\Sigma 4,\psi\psi_{\mathrm{P},-})$ such that

$$h_2(\mathfrak{n}_\Sigma 4,\psi\psi_{\mathrm{P},-})_{\mathfrak{m}_-}/(2) \simeq h_2(\mathfrak{n}_\Sigma 2,\psi\psi_{\mathrm{P},+})_{\mathfrak{m}_+}/(2)$$

This can be proved exactly as in the proof of Lemma 3.2 in [4]; instead use the 0-dimensional Shimura variety corresponding to D over F_Σ.
For $p=2$ define $e_{\mathrm{BDST},\pm}$ and let $h^0(\mathfrak{n})_\pm = e_{\mathrm{BDST},\pm}h^0(\mathfrak{n})$. Let

$$\begin{aligned} h_2^\square(\mathfrak{n}_\Sigma p,\psi\psi_P)_{\mathfrak{m}} &= h_2(\mathfrak{n}_\Sigma p,\psi\psi_{\mathrm{P}})_{\mathfrak{m}} \otimes_{R_S^{\mathrm{BT,ord},\psi}} R_S^{\square,\mathrm{BT,ord},\psi} \\ h^{0\square}(\mathfrak{n}_\Sigma,\psi)_{\mathfrak{m}} &= h^0(\mathfrak{n}_S,\psi\psi_{\mathrm{P}})_{\mathfrak{m}} \otimes_{R_S^{\mathrm{ord},\psi}} R_S^{\square,\mathrm{ord},\psi} \end{aligned}$$

if p is odd; and let

$$\begin{aligned} h_2^\square(\mathfrak{n}_\Sigma 4,\psi\psi_{\mathrm{P},-})_{\mathfrak{m}_-} &= h_2(\mathfrak{n}_\Sigma 4,\psi\psi_{\mathrm{P},-})_{\mathfrak{m}_-} \otimes_{R_{S,-}^{\mathrm{BT,ord},\psi}} R_{S,-}^{\square,\mathrm{BT,ord},\psi} \\ h^{0\square}(\mathfrak{n}_S,\psi)_{-,\mathfrak{m}_-} &= h^0(\mathfrak{n}_\Sigma,\psi\psi_{\mathrm{P}})_{-,\mathfrak{m}} \otimes_{R_{S,-}^{\mathrm{ord},\psi}} R_{S,-}^{\square,\mathrm{ord},\psi} \end{aligned}$$

if $p=2$. It then follow from results of Kisin and Khare-Wintenberger that there is a natural surjection

$$R_S^{\square,\mathrm{BT,ord},\psi} \to h_2^\square(\mathfrak{n}_\Sigma p,\psi\psi_{\mathrm{P}})_{\mathfrak{m}}$$

if p odd and

$$R_{S,-}^{\square,\mathrm{BT,ord},\psi} \to h_2^\square(\mathfrak{n}_\Sigma 4,\psi\psi_{\mathrm{P},-})_{\mathfrak{m}_-}$$

if $p=2$, which induce isomorphisms

$$R_S^{\square,\mathrm{BT,ord},\psi}[1/p] \simeq h_2^\square(\mathfrak{n}_\Sigma p,\psi\psi_{\mathrm{P}})_{\mathfrak{m}}[1/p]$$

if p odd and

$$R_{S,-}^{\square,\mathrm{BT,ord},\psi}[1/2] \simeq h_2^\square(\mathfrak{n}_\Sigma 4,\psi\psi_{\mathrm{P},-})_{\mathfrak{m}_-}[1/2].$$

The determinant of $\rho^{\square S}$ defines

$$\mathbf{N}S : I_{\mathfrak{n}_\Sigma p^\infty} \to R_S^{\mathrm{ord},\psi}$$

and $R_S^{\mathrm{ord},\psi}/\ker(S-(\psi\psi_{\mathrm{P}}\circ\epsilon^{\mathrm{cyclo}}))\simeq R_S^{\mathrm{BT,ord},\psi}$ if p odd, and

$$\mathbf{N}S : I_{\mathfrak{n}_\Sigma p^\infty}\to R_{S,-}^{\mathrm{ord},\psi}$$

induces $R_{S,-}^{\mathrm{ord},\psi}/\ker(S-(\psi\psi_{\mathrm{P},-}\circ\epsilon^{\mathrm{cyclo}}))\simeq R_{S,-}^{\mathrm{BT,ord},\psi}$ if $p=2$. On the other hand, $h^0(\mathfrak{n}_\Sigma,\psi)_{\mathfrak{m}}/\ker(S-(\psi\psi_{\mathrm{P}}\circ\epsilon^{\mathrm{cyclo}}))\simeq h_2(\mathfrak{n}_\Sigma p,\psi\psi_{\mathrm{P}})_{\mathfrak{m}}$ and $h^0(\mathfrak{n}_\Sigma,\psi)_{-,\mathfrak{m}_-}/\ker(S-(\psi\psi_{\mathrm{P},-}\circ\epsilon^{\mathrm{cyclo}}))\simeq h_2(\mathfrak{n}_\Sigma 4,\psi\psi_{\mathrm{P},-})_{\mathfrak{m}_-}$. Then the surjective Λ-algebra homomorphisms

$$R_S^{\Box,\mathrm{ord},\psi}\to h^{0\Box}(\mathfrak{n}_\Sigma,\psi)_{\mathfrak{m}}$$

if p odd and

$$R_{S,-}^{\Box,\mathrm{ord},\psi}\to h^{0\Box}(\mathfrak{n}_\Sigma,\psi)_{-,\mathfrak{m}_-}$$

if $p=2$ induce the isomorphisms

$$R_S^{\Box,\mathrm{ord},\psi}[1/p]\simeq h^{0\Box}(\mathfrak{n}_\Sigma,\psi)_{\mathfrak{m}}[1/p]$$

and

$$R_{S,-}^{\Box,\mathrm{ord},\psi}[1/2]\simeq h^{0\Box}(\mathfrak{n}_\Sigma,\psi)_{-,\mathfrak{m}_-}[1/2].$$

6 Companion forms mod p

Let F be a totally real field and p be a rational prime. Suppose that $[F(\zeta_p):F]>3$ if $p>3$ and that 2 splits completely in F if $p=2$. Let f_2 be a holomorphic cuspidal Hilbert eigenform of weight $2\le k_2\le p$ and of level prime to p. Assume that the associated p-adic representation ρ_2 of $\mathrm{Gal}(\overline{F}/F)$ is crystalline and ordinary at every prime $\mathfrak{p}$ of F above p. It is a well-known theorem of Wiles (Theorem 2.1.4 in [43]) that, for every prime $\mathfrak{p}$ of F above p, the restriction $\rho|_{G_\mathfrak{p}}$ to the decomposition group $G_\mathfrak{p}$ at $\mathfrak{p}$ is of the form

$$\rho|_{G_\mathfrak{p}}\simeq\begin{pmatrix}\epsilon^{k_2-1}\chi_{\mathfrak{p},1} & *\\ 0 & \chi_{\mathfrak{p},2}\end{pmatrix}$$

where $\chi_{\mathfrak{p},1}$ and $\chi_{\mathfrak{p},2}$ are unramified characters of $G_\mathfrak{p}$, and $\chi_{\mathfrak{p},2}(\mathrm{Frob}_\mathfrak{p})$ is a unit $U_\mathfrak{p}$-eigenvalue of the p-stabilised newform of f_2.

Theorem 9 *Let f_2 be a holomorphic cuspidal Hilbert eigenform of weight $2\le k_2\le p$ and of level prime to p as above. Let $k_1\stackrel{\mathrm{def}}{=}p$ if $k_2=p$ and $k_1\stackrel{\mathrm{def}}{=}p+1-k_2$ if $2\le k_2<p$. Suppose that*

- *if $p>2$, the associated mod p representation $\overline{\rho}_2:\mathrm{Gal}(\overline{F}/F)\to GL_2(\overline{\mathbf{F}}_p)$ is absolutely irreducible when restricted to $\mathrm{Gal}(\overline{F}/F(\zeta_p))$, and if $p=2$, $\overline{\rho}_2:\mathrm{Gal}(\overline{F}/F)\to GL_2(\overline{\mathbf{F}}_p)$ has insoluble image;*
- *if $p>2$ and if $\overline{\epsilon}^{k_1-2}\overline{\chi}_{\mathfrak{p},2}\neq\overline{\chi}_{\mathfrak{p},1}$, the ramification index of $F_\mathfrak{p}$ is strictly less than $p-1$ for every prime $\mathfrak{p}$ of F above p, and if $p=2$, $\overline{\rho}_2$ is unramified at every prime of F above 2;*

- *if* $p > 2$, $\overline{\epsilon}^{k_2-1}\overline{\chi}_{\mathfrak{p},1} \neq \overline{\chi}_{\mathfrak{p},2}$ *and if* $p = 2$, $\overline{\chi}_{\mathfrak{p},2} \neq \overline{\chi}_{\mathfrak{p},1}$

- $\overline{\rho}_2$ *is the direct sum of the characters* $\overline{\epsilon}^{k_2-1}\overline{\chi}_{\mathfrak{p},1}$ *and* $\overline{\chi}_{\mathfrak{p},2}$ *at every prime* $\mathfrak{p}$ *of* F *above* p.

Then there exists a holomorphic cuspidal Hilbert eigenform of weight $2 \leq k_1 \leq p$ *and of level prime to* p *with its associated mod* p *representation* $\rho_1 : \mathrm{Gal}(\overline{F}/F) \to GL_2(\overline{\mathbf{F}}_p)$ *satisfying* $\overline{\rho}_1 \simeq \overline{\rho}_2 \otimes \overline{\epsilon}^{k_1-1}$ *if* $p > 2$ *and* $\overline{\rho}_1 \simeq \overline{\rho}_2$ *if* $p = 2$, *and the* $U_{\mathfrak{p}}$*-eigenvalue of the* p*-stabilised new form is a lifting of* $\overline{\chi}_{\mathfrak{p},1}(\mathrm{Frob}_{\mathfrak{p}})$.

Proof. For $p > 2$, this is a result of Gee (Theorem 2.1 [13]). Let $p = 2$; thus $k_1 = k_2 = 2$. For clarity, let $\overline{\rho}$ denote $\overline{\rho}_2 \otimes \overline{\epsilon}$ where $\overline{\epsilon}$ is the mod 4 cyclotomic character. Clearly the twist of ρ_2 by the Teichmuller lift of $\overline{\epsilon}$ defines a modular lifting of $\overline{\rho}$ potentially ordinary and potentially Barsotti-Tate at p. By class field theory, find a finite totally real soluble extension $F_\Sigma \subset \overline{F}$ of F of even degree in which 2 remains split completely, and satisfies the following conditions:

- there exists a quaternion algebra D over F_Σ ramified exactly at a finite set Σ of finite primes of F_Σ not dividing 2;

- $\overline{\rho}|_{\mathrm{Gal}(\overline{F}/F_\Sigma)}$ is ramified exactly at Σ and the infinite places, and , in particular, for every prime $\mathfrak{q} \in \Sigma$, $\overline{\rho}|_{\mathrm{Gal}(\overline{F}/F_\Sigma)}$ at $\mathfrak{q}$ is an extension of an unramified character by the twist of the character by ϵ at $\mathfrak{q}$;

- there exists a maximal open compact subgroup $U \subset (D \otimes_{F_\Sigma} \mathbf{A}_{F_\Sigma}^\infty)^\times$ such that $U_{\mathfrak{q}} = GL_2(\mathcal{O}_{F_\Sigma \mathfrak{q}})$ for $\mathfrak{q} \notin S^D$ and $U_{\mathfrak{q}} = \mathcal{O}_{D\mathfrak{q}}^\times$ for $\mathfrak{q} \in S^D$, and a holomorphic cuspidal automorphic representation π_2 of $(D \otimes_{F_\Sigma} \mathbf{A}_{F_\Sigma})^\times$ with central character ψ such that $\overline{\rho}|_{\mathrm{Gal}(\overline{F}/F_\Sigma)} \simeq \overline{\rho}_{\pi_2} : \mathrm{Gal}(\overline{F}/F_\Sigma) \to GL_2(\overline{\mathbf{F}}_p)$ and $\det\overline{\rho}|_{\mathrm{Gal}(\overline{F}/F_\Sigma)} = \overline{\psi}\overline{\epsilon}$ and such that π is unramified at every prime of F_Σ above 2.

It then follows from work of Khare-Wintenberger (See Corollary 4.7 and Theorem 10.1 in [18]) that there is a lifting $\rho : \mathrm{Gal}(\overline{F}/F_\Sigma) \to GL_2(\overline{\mathbf{Q}}_p)$ of $\overline{\rho}|_{\mathrm{Gal}(\overline{F}/F_\Sigma)}$, unramified outside $S_{\Sigma,\mathrm{P}} \coprod \Sigma \coprod S_{\Sigma,\infty}$ with $\det\rho = \psi\epsilon$ such that, for every prime $\mathfrak{p}$ of F_Σ above 2, ρ is ordinary at $\mathfrak{p}$ and Barsotti-Tate and is of the form

$$\begin{pmatrix} \epsilon\widetilde{\chi}_{\mathfrak{p},2} & * \\ 0 & \widetilde{\chi}_{\mathfrak{p},1} \end{pmatrix}$$

where $\widetilde{\chi}_{\mathfrak{p},1}$ and $\widetilde{\chi}_{\mathfrak{p},2}$ are unramified liftings of $\overline{\chi}_{\mathfrak{p},1}|_{\mathrm{Gal}(\overline{F}/F_\Sigma)}$ and $\overline{\chi}_{\mathfrak{p},2}|_{\mathrm{Gal}(\overline{F}/F_\Sigma)}$ respectively. It then follows from the main theorem of Kisin [19] and Khare-Wintenberegr [18], and by soluble descent that there exists a holomorphic cuspidal Hilbert eigenform f_1 of weight $k_1 = 2$ and of level prime to 2 such that $\rho_{f_1}|_{\mathrm{Gal}(\overline{F}/F_\Sigma)} \simeq \rho$. $\square$

7 Λ-adic companion forms

Theorem 10 *Let p be a rational prime. Let F be a totally real field. Suppose that p splits completely in F. Let K be a finite extension of $\mathbf{Q}_p$ with ring of integers $\mathcal{O}$ and residue field $k = \mathcal{O}/\mathfrak{m}$. Suppose that*

$$\rho : \mathrm{Gal}(\overline{F}/F) \to GL_2(\mathcal{O})$$

is a continuous representation satisfying

- *ρ ramifies at only finite many primes;*
- *$\overline{\rho} = (\rho \bmod \mathfrak{m})$ is absolutely irreducible when restricted to $\mathrm{Gal}(\overline{F}/F(\zeta_p))$, and has a modular lifting which is potentially ordinary and potentially Barsotti-Tate at every prime of F above p;*
- *for every prime ideal $\mathfrak{p}$ of F above p, the restriction $\rho|_{G_\mathfrak{p}}$ to the decomposition group $G_\mathfrak{p}$ at $\mathfrak{p}$ is the direct sum of characters $\chi_{\mathfrak{p},1}$ and $\chi_{\mathfrak{p},2} : G_\mathfrak{p} \to \mathcal{O}^\times$ such that the images of the inertia subgroup at $\mathfrak{p}$ are finite and $(\chi_{\mathfrak{p},1} \bmod \mathfrak{m}) \neq (\chi_{\mathfrak{p},2} \bmod \mathfrak{m})$;*

If $p = 2$, assume furthermore that

- *the image of the complex conjugation with respect to every embedding of F into $\mathbf{R}$ is not the identity matrix;*
- *$\overline{\rho}$ has insoluble image;*
- *for every $\mathfrak{p}$ of F above p, ρ is unramified at $\mathfrak{p}$.*

Then there is a finite totally real soluble extension $F_\Sigma \subset \overline{F}$ of F in which p splits completely; a finite set Σ of finite places of F_Σ (at which $\rho|_{G_\Sigma}$, where $G_\Sigma \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{F}/F_\Sigma)$ is ramified of conductor $\mathfrak{n}_\Sigma$); an ideal $\mathfrak{n}$ of $\mathcal{O}_{F_\Sigma}$ coprime to p which $\mathfrak{n}_\Sigma$ divides; and, for every subset P of the set $S_{\Sigma,\mathrm{P}}$ of places of F_Σ above p,

1. *a character*
$$\chi_P : G_\Sigma \to \mathcal{O}^\times$$
of finite order, unramified outside a finite set of places containing $S_{\Sigma,\mathrm{P}}$, such that the restriction to the inertia subgroup of G_Σ at $\mathfrak{p}$ of χ_P equals that of $\chi_{\mathfrak{p},1}$ (resp. $\chi_{\mathfrak{p},2}$) for all $\mathfrak{p}$ in P (resp. $S_{\Sigma,\mathrm{P}} - P$);
2. *a finite extension L of $\mathrm{Frac}\,\Lambda_K$ and a Λ-adic form*
$$F_{\mathrm{Hida},P} : h^0_{\mathcal{O}}(\mathfrak{n}_\Sigma) \to L;$$
3. *a homomorphism $f_P : h^0_{\mathcal{O}}(\mathfrak{n}) \to \mathcal{O}$ if $p > 2$ while $f_P : h^0_{\mathcal{O}}(\mathfrak{n})_- \to \mathcal{O}$ if $p = 2$ satisfying*

- $f_P(T_{\mathfrak{q}}) = \mathrm{tr}\ \rho(\mathrm{Frob}_{\mathfrak{q}})/\chi_P(\mathrm{Frob}_{\mathfrak{q}})$ *for all* $\mathfrak{q}$ *not dividing* $\mathfrak{n}p$*;*
- $f_P(\mathbf{N}\mathfrak{q}S_{\mathfrak{q}}) = \det\ \rho(\mathrm{Frob}_{\mathfrak{q}})/\chi_P^2(\mathrm{Frob}_{\mathfrak{q}})$ *for all* $\mathfrak{q}$ *not dividing* $\mathfrak{n}p$*;*
- $f_P(U_{\mathfrak{q}}) = 0$ *for* $\mathfrak{q}$ *dividing* $\mathfrak{n}$ *but* not *dividing* p*;*
- $f_P(U_{\mathfrak{p}}) = (\chi_{\mathfrak{p},1}/\chi_P)(\mathrm{Frob}_{\mathfrak{p}})$ *for every* $\mathfrak{p}$ *in* P *and* $f_P(U_{\mathfrak{p}}) = (\chi_{\mathfrak{p},2}/\chi_P)(\mathrm{Frob}_{\mathfrak{p}})$ *for every* $\mathfrak{p}$ *in* $S_{\Sigma,\mathrm{P}} - P$*.*

Proof. Choose a finite soluble totally real extension F_Σ of F in which p splits completely such that the restriction of ρ is absolutely irreducible when restricted to $\mathrm{Gal}(\overline{F}/F_\Sigma(\zeta_p))$, unramified outside a finite set $\Sigma \coprod S_{\Sigma,\mathrm{P}} \coprod S_{\Sigma,\infty}$ of finite places $\mathfrak{q}$ of F_Σ such that $\rho|_{G_\Sigma}$ is of of conductor 1 or $\mathfrak{q}$ at $\mathfrak{q}$, and arises from–by Jacquet-Langlands–a cuspidal automorphic representation, *nearly* ordinary at every $\mathfrak{p} \in S_{\Sigma,\mathrm{P}}$ and special at $\mathfrak{q} \in \Sigma$, of the quaternion algebra D_Σ over F_Σ as in the previous section.

For every $P \subseteq S_{\Sigma,\mathrm{P}}$, it follows from class field theory that one can choose χ_P, of conductor 1 away from a finite set of places containing the set of places above p, as asserted in the theorem.

Let ρ_P denote $\rho \otimes_{\mathrm{Gal}(\overline{F}/F_\Sigma)} \chi_P^{-1}$ and $\overline{\rho}_P$ denote (ρ_P mod $\mathfrak{m}$). If we let ρ_Σ denote the modular lifting of $\overline{\rho}$, then $\rho_\Sigma \otimes \chi_P^{-1}$ *is* a modular lifting of $\overline{\rho}_P$; in fact it is ordinary at every $\mathfrak{p} \in S_{\Sigma,\mathrm{P}}$ by Jarvis' level lowering results [15]–by which one shows $\rho_\Sigma \otimes \chi_P^{-1}$ is crystalline at $\mathfrak{p}$– followed by Fontaine-Laffaille theory. Let $\mathfrak{m}_P$ denote the corresponding maximal ideal of $eh_2(\mathfrak{n}_\Sigma p)_{\mathcal{O}}$ if $p > 2$ and $eh_2(\mathfrak{n}_\Sigma 4)_{\mathcal{O},-}$ if $p = 2$. It then follows from Hida theory [14] and results from preceding sections that there exists a finite extension L in an algebraic closure of $\mathrm{Frac}\,\Lambda_K$ which we fix; and, for every $P \subseteq S_{\Sigma,\mathrm{P}}$, a Λ-adic eigenform $F_{\mathrm{H},P} : h^0_{\mathcal{O}}(\mathfrak{n}_\Sigma) \to h^0_{\mathcal{O}}(\mathfrak{n}_\Sigma)_{\mathfrak{m}_P} \to \mathcal{O}_L$, and a height one prime $\wp_P$ of $\mathcal{O}_L$ such that

- $(\rho_{F_{\mathrm{H}}} \bmod \wp_P) \sim \rho_P$
- for every distinct subsets P and Q, $(F_{\mathrm{H},P}, \wp_P)$ and $(F_{\mathrm{H},Q}, \wp_Q)$ are in companion; more precisely, for every $\mathfrak{q}$ not dividing $\mathfrak{n}_\Sigma p$, $(F_{\mathrm{H},Q}(T_{\mathfrak{q}}) \bmod \wp_Q) = (F_{\mathrm{H},P}(T_{\mathfrak{q}} S_{(P-(P\cap Q))\cup(Q-(P\cap Q))}(\mathfrak{q})^{-1}) \bmod \wp_P)$; for $\mathfrak{p}$ in $(P \cap Q) \cup ((S_{\Sigma,\mathrm{P}} - P) \cap (S_{\Sigma,\mathrm{P}} - Q))$, $(F_{\mathrm{H},Q}(U_{\mathfrak{p}}) \bmod \wp_Q) = (F_{\mathrm{H},P}(U_{\mathfrak{p}}) \bmod \wp_P)$, while for $\mathfrak{p}$ in $(P \cap (S_{\Sigma,\mathrm{P}} - Q)) \cup ((S_{\Sigma,\mathrm{P}} - P) \cap Q)$, $(F_{\mathrm{H},Q}(U_{\mathfrak{p}}) \bmod \wp_Q) = (F_{\mathrm{H},P}(U_{\mathfrak{p}}^{-1} S^P(\mathfrak{p})) \bmod \wp_P)$.

Let f_P be the composite $h^0_{\mathcal{O}}(\mathfrak{n}_\Sigma) \to \mathcal{O}_L \to \mathcal{O}_L/\wp_P \simeq \mathcal{O}$; if the image of $U_{\mathfrak{q}}$ for $\mathfrak{q}$ dividing $\mathfrak{n}_\Sigma$ but not dividing p is not zero, we may increase the level at $\mathfrak{q}$ if necessary to assume the image of indeed zero (See [34] for example) . □

8 Models of Hilbert modular varieties

Let F be a totally real field–F_Σ in the preceding section–of degree $d = [F : \mathbf{Q}]$ with ring of integers $\mathcal{O}_F$. Fix a rational prime p and an ideal $\mathfrak{n}$ of $\mathcal{O}_F$ prime to p. For every integer $r \geq 1$, fix a p^r-th primitive root ζ_{p^r} of unity. For a prime

$\mathfrak{p}$ of F above p, let $F_{\mathfrak{p}}$ denote the completion of F with respect to the absolute value corresponding to $\mathfrak{p}$, $k_{\mathfrak{p}}$ the residue field of $F_{\mathfrak{p}}$, $f_{\mathfrak{p}}$ the residue class degree, and $e_{\mathfrak{p}}$ the ramification index.
Fix embeddings $\mathbf{Q} \to \overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_p$. Let K denote a finite extension of $\mathbf{Q}_p$ which contains the image of F by every embedding of F into $\overline{\mathbf{Q}}_p$; and let $\mathcal{O}$ denote its ring of integers and k denote the residue field.
For a fractional ideal I of F canonically ordered, let I^+ denote the totally positive elements. Fix a set T of representatives in $\mathbf{A}_F^\times$ of the strict ideal class group $\mathbf{A}_F^\times/(F^\times(\mathcal{O}_F \otimes \mathbf{Z}^\wedge)^\times F_\infty^{+\times})$, and we shall let t also mean the fractional ideal $t\mathfrak{d}$ corresponding to a representative t in T.

Definition. A t-polarised Hilbert-Blumenthal abelian variety (henceforth abbreviated as HBAV) with level $\Gamma_1(\mathfrak{n})$-structure over a $\mathcal{O}$-scheme S is an abelian variety A over S of relative dimension d together with

- $i : \mathcal{O}_F \to \mathrm{End}(A/S)$;
- a homomorphism $\lambda : (t, t^+) \to (\mathrm{Sym}(A/S), \mathrm{Pol}(A/S))$ of ordered invertible $\mathcal{O}_F$-modules, where $\mathrm{Sym}(A/S)$ (resp. $\mathrm{Pol}(A/S)$) denotes the invertible $\mathcal{O}_F$-module (via i) of symmetric homomorphisms (resp. polarisations), such that $A \otimes_{\mathcal{O}_F} t \to A^\vee$, induced by λ, is an isomorphism of HBAVs–it is shown in [41] that this is equivalent to the condition that there exists a prime-to-p polarisation $A \to A^\vee$; and to the 'determinant condition' on $\mathrm{Lie}(A)$ in the sense of Kottwitz;
- an $\mathcal{O}_F/\mathfrak{n}$-module morphism $\eta : (\mathcal{O}_F/\mathfrak{n})^\vee = (GL_1 \otimes \mathfrak{d}_F^{-1})[\mathfrak{n}] \to A[\mathfrak{n}]$.

Definition. Let $Y_{\Gamma_1(\mathfrak{n},t)}$ (resp. $Y_{\Gamma_1(\mathfrak{n},t)\cap \mathrm{Iw}}$) denote the $\mathcal{O}$-scheme representing the functor which sends an $\mathcal{O}$-scheme S to the set of isomorphism classes (A, i, λ, η) (resp. (A, i, λ, η, C)) of t-polarised HBAVs with level $\Gamma_1(\mathfrak{n})$-level structure (resp. and a finite flat subgroup scheme C of $A[p]$ with compatible $\mathcal{O}_F$-action locally free of rank $\sum_{\mathfrak{p}} |\mathcal{O}_F/\mathfrak{p}|$).
It follows from [27] and [8] that if $\mathfrak{n}$ does not divide 2, nor 3, $Y_{\Gamma_1(\mathfrak{n},t)}$ is a smooth scheme over $\mathcal{O}$ of relative dimension $[F : \mathbf{Q}]$. If $\mathfrak{n}$ does divide 2, or 3, we let $Y_{\Gamma_1(\mathfrak{n},t)}$ denote the $\mathcal{O}$-scheme

$$(\Gamma_1(\mathfrak{n}, t)/\Gamma_1(\mathfrak{m}, t))\backslash Y_{\Gamma_1(\mathfrak{m},t)}$$

for an auxiliary ideal $\mathfrak{m}$ of $\mathcal{O}_F$ such that $\mathfrak{n}|\mathfrak{m}$ and $\Gamma_1(\mathfrak{m})$ small enough, i.e., torsion-free.

Let $\overline{Y}_{\Gamma_1(\mathfrak{n},t)}$ denote the fibre over k of $Y_{\Gamma_1(\mathfrak{n},t)}$; and let Let $\overline{Y}_{\Gamma_1(\mathfrak{n},t)\cap \mathrm{Iw}}$ denote the fibre over k of $Y_{\Gamma_1(\mathfrak{n},t)\cap \mathrm{Iw}}$.
It is a well-known result of Deligne-Ribet that the fibre $\overline{Y}_{\Gamma_1(\mathfrak{n},t)}$ is irreducible (Corollary 4.6 in [9]). It is a result of local model theory by Pappas that $\overline{Y}_{\Gamma_1(\mathfrak{n},t)\cap \mathrm{Iw}}$ is normal (Corollary 2.2.3 in [25]).

Suppose that p splits completely in F. In which case, the p-divisible group of a HBAV over the ring of integers of a finite extension of $\mathbf{Q}_p$ decomposes as the product of $[F:\mathbf{Q}]$ one-dimensional p-divisible groups, one for each prime $\mathfrak{p}$ of F above p, and this allows us to define 'Katz-Mazur-Drinfeld' higher level structures at p by defining level structures at $\mathfrak{p}$ on the '$\mathfrak{p}$-divisible group' for each $\mathfrak{p}$.

DEFINITION. Let r be an integer ≥ 1. Define $Y_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p^r)}$ to be the $\mathcal{O}$-scheme representing the functor which sends an $\mathcal{O}$-scheme S to the set of isomorphism classes of the sextuples $(A,i,\lambda,\eta,C,\eta_{\mathrm{KM}})$ over S where (A,i,λ,η) is a t-polarised HBAV over S with $\Gamma_1(\mathfrak{n})$-level structure, C is a finite flat subgroup scheme of $A[p^r]$ locally free of finite rank $|\mathcal{O}_F/p^r| = \sum_{\mathfrak{p}} |\mathcal{O}_F/\mathfrak{p}^r|$ with compatible action of $\mathcal{O}_F$, and an $\mathcal{O}_F$-linear group homomorphism $\eta_{\mathrm{KM}}: \mathcal{O}_F/p^r \to \mathrm{Mor}(S,C) \subset \mathrm{Mor}(S,A[p^r])$ such that the image of η_{KM} defines a 'full set of sections' in the sense of Katz-Mazur [17] (See 1.10.5 and 1.10.10 in [17]).

DEFINITION. For every prime $\mathfrak{p}$ of F above p, let $Y_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p^r),\mathrm{Iw}_{\mathfrak{p}},K}$ denote the fine moduli space over K of the septuples $(A,i,\lambda,\eta,C,\eta_{\mathrm{KM}},D_{\mathfrak{p}})$ where the sextuple $(A,i,\lambda,\eta,C,\eta_{\mathrm{KM}})$ defines a point of $Y_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p^r)} \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathrm{Spec}\,K$, and $D_{\mathfrak{p}}$ is finite flat subgroup scheme of $A[\mathfrak{p}]$ of rank $|\mathcal{O}_F/\mathfrak{p}|$ which has trivial intersection with C.

9 COMPACTIFICATION

By an *unramified* cusp C of $Y_{\Gamma_1(\mathfrak{n},t)}$ over R, we shall mean a pair of fractional ideals M_1, M_2 of F such that $M_1M_2^{-1} \simeq t$ which comes equipped with

- an $\mathcal{O}_F \otimes_{\mathbf{Z}} R$-linear isomorphism $\lambda: M_1^{-1} \otimes_{\mathbf{Z}} R \simeq \mathcal{O}_F \otimes_{\mathbf{Z}} R$;
- an $\mathcal{O}_F$-linear embedding $\eta: \mathcal{O}_F/\mathfrak{n} \to \mathfrak{n}^{-1}M_2^{-1}/M_2^{-1}$.

For brevity, let $M = M_1M_2$, $M^\vee = \mathrm{Hom}_{\mathbf{Z}}(M,\mathbf{Z}) = \mathrm{Hom}_{\mathcal{O}_F}(M,\mathfrak{d}_F^{-1}) \simeq \mathfrak{c}M_2^{-1}\mathfrak{d}_F^{-1}$, and $M^{\vee+} \subset M^\vee$ of the totally positive elements in $(\mathfrak{c}M_2^{-1}\mathfrak{d}_F^{-1})^+$. Choose a rational polyhedral cone decomposition Σ_C of $(M^{\vee+}\otimes_{\mathbf{Z}}\mathbf{R}) \cup \{0\}$. For a cone $\sigma \subset (M^{\vee+}\otimes_{\mathbf{Z}}\mathbf{R})$, we let $\sigma^\vee \subset M\otimes_{\mathbf{Z}}\mathbf{R}$ denote the dual cone.

Let $S_{\mathfrak{n}} = \mathrm{Spec}\,\mathbf{Z}[q^{\mathfrak{n}^{-1}M}]$ and $S_{\mathfrak{n}} \hookrightarrow S_{\mathfrak{n},\sigma}$ denote the affine torus embedding (see Theorem 2.5 in [6]) corresponding to the cone σ and let $S_{\mathfrak{n},\sigma}^\wedge = \mathrm{Spf}\,\mathbf{Z}[[q^{\mathfrak{n}^{-1}M\cap\sigma^\vee}]]$ denote the formal completion of $S_{\mathfrak{n},\sigma}$ along the boundary $S_{\mathfrak{n},\sigma}^\infty \stackrel{\mathrm{def}}{=} S_{\mathfrak{n},\sigma} - S_{\mathfrak{n}}$.

Let $T_{\mathfrak{n},\sigma} = \mathrm{Spec}\,\mathbf{Z}[[q^{\mathfrak{n}^{-1}M\cap\sigma^\vee}]]$ and $T_{\mathfrak{n},\sigma}^0 = T_{\mathfrak{n},\sigma} - S_{\mathfrak{n},\sigma}^\infty = \mathrm{Spec}\,\mathbf{Z}[[q^{\mathfrak{n}^{-1}M\cap\sigma^\vee}, q^{-\mathfrak{n}^{-1}M\cap\sigma^\vee}]]$. The henselisation of $(S_{\mathfrak{n},\sigma}, S_{\mathfrak{n},\sigma}^\infty)$ projects onto an affine étale scheme $U_{\mathfrak{n},\sigma}$ over $S_{\mathfrak{n},\sigma}$ which approximates $S_{\mathfrak{n},\sigma}^\wedge$ in the sense of Artin, and let $U_{\mathfrak{n},\sigma}^0 = U_{\mathfrak{n},\sigma} \times_{T_{\mathfrak{n},\sigma}} T_{\mathfrak{n},\sigma}^0$.

The Mumford construction applied to the $\mathcal{O}_F$-linear 'period' map $q : M_2 \to GL_1(U_{\mathfrak{n},\sigma}) \otimes_{\mathbf{Z}} \mathfrak{d}_F^{-1} M_1^{-1}$ gives rise to a semi-abelian scheme

$$\mathrm{Tate}_{M_1,M_2}(q) \stackrel{\mathrm{def}}{=} (GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1} M_1^{-1})/q^{M_2}$$

over the complete ring $U_{\mathfrak{n},\sigma}$ with action of $\mathcal{O}_F$, whose pull-back, which we shall denote by $\mathrm{Tate}^0_{M_1,M_2}(q)$ to $U^0_{\mathfrak{n},\sigma}$, is naturally a HBAV, t-polarised

$$\begin{array}{ccc} \mathrm{Tate}_{M_1,M_2}(q) \otimes_{\mathcal{O}_F} M_1 M_2^{-1} \simeq & (GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1} M_2^{-1})/q^{M_1} & \\ & \| & \\ & \mathrm{Tate}_{M_2,M_1}(q) & \simeq \mathrm{Tate}_{M_1,M_2}(q)^{\vee}, \end{array}$$

with level $\Gamma_1(\mathfrak{n})$-structure, and which gives rise to a map

$$U^0_{\mathfrak{n},\sigma} \times_{\mathrm{Spec}\,\mathbf{Z}} \mathrm{Spec}\,\mathcal{O} \to Y_{\Gamma_1(\mathfrak{n},t)}.$$

We glue $\coprod_{T/\simeq} \coprod_{\sigma \in \Sigma_C} U_{\mathfrak{n},\sigma} \times_{\mathrm{Spec}\,\mathbf{Z}} \mathrm{Spec}\,\mathcal{O}$ along the map to get a toroidal compactification $X_{\Gamma_1(\mathfrak{n},t)}$ over $\mathcal{O}$ of $Y_{\Gamma_1(\mathfrak{n},t)}$ ([26]). Similarly, one can define a compactification $X_{\Gamma_1(\mathfrak{n},t)\cap \mathrm{Iw}}$ over $\mathcal{O}$ of $Y_{\Gamma_1(\mathfrak{n},t)\cap \mathrm{Iw}}$ with its choice of a rational cone decomposition compatible with that of $X_{\Gamma_1(\mathfrak{n},t)}$.

Let

$$\mathrm{Tate}^0_{M_1,M_2,S}(q) \stackrel{\mathrm{def}}{=} \mathrm{Tate}_{M_1,M_2}(q) \times_{\mathrm{Spec}\,\mathbf{Z}[[q^M,q^{-M}]]} S$$

for a $\mathbf{Z}[[q^M, q^{-M}]]$-scheme S; it is t-polarised. Let S be a $\mathcal{O} \otimes_{\mathbf{Z}} \mathbf{Z}[[q^M, q^{-M}]]$-scheme. Then there is a 'connected-étale' exact sequence

$$0 \to (GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1} M_1^{-1})[p^r] \to \mathrm{Tate}_{M_1,M_2,S}(q)[p^r] \to (1/p^r) M_2/M_2 \to 0$$

of $(\mathcal{O}_F/p^r)$-modules schemes over S.

Lemma 11 *Fix an integer $r \geq 1$. Let S be a connected $\mathcal{O} \otimes_{\mathbf{Z}} \mathbf{Z}[[q^M, q^{-M}]]$-scheme. Suppose that C is an $\mathcal{O}_F$-stable finite flat subgroup scheme of $\mathrm{Tate}^0_{M_1,M_2,S}(q)[p^r]$ of order $|\mathcal{O}_F/p^r|$. Then for every $\tau = \tau_{\mathfrak{p}}$, there exists a unique pair of non-negative integers $\rho_{\tau,1}, \rho_{\tau,2}$ such that $\rho_{\tau,1} + \rho_{\tau,2} = r$ and such that*

$$C_{\mathfrak{p}} \cap (GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1} M_1^{-1})[p^r] \simeq (GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1} M_1^{-1})[\mathfrak{p}^{\rho_{\tau,1}}]$$

and the image of $C_{\mathfrak{p}}$ in $(1/p^r)M_2/M_2$ is isomorphic to $\mathfrak{p}^{-\rho_{\tau,2}} M_2/M_2$.

Proof. This is essentially Proposition 13.6.2 in [17]. □

By a cusp of C of $Y_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p^r)}$ over R, we shall mean a pair of fractional ideals M_1, M_2 of F such that $M_1 M_2^{-1} \simeq t$ which comes equipped with

- an $\mathcal{O}_F \otimes_{\mathbf{Z}} R$-linear isomorphism $\lambda : M_1^{-1} \otimes_{\mathbf{Z}} R \simeq \mathcal{O}_F \otimes_{\mathbf{Z}} R$;
- an $\mathcal{O}_F$-linear embedding $\eta : \mathcal{O}_F/\mathfrak{n} \to \mathfrak{n}^{-1} M_2^{-1}/M_2^{-1}$.

- an $\mathcal{O}_F$-linear isomorphism $\eta_{\mathrm{KM}} : \mathcal{O}_F/p^r \simeq p^{-r}M_2/M_2$.

Let $M = M_1M_2$ as above. Fix an integer $r \geq 1$. Suppose that S is an $\mathcal{O} \otimes_{\mathbf{Z}} \mathbf{Z}[\zeta_{p^r}][[q^{(1/p^r)M,-(1/p^r)M}]]$-scheme.
Definition Let ζ_r denote the image of 1 by

$$\zeta_{\mathrm{KM},r} : (\mathcal{O}_F/p^r) \simeq \mathfrak{d}^{-1}/p^r\mathfrak{d}^{-1} \simeq GL_1[p^r] \otimes_{\mathbf{Z}} \mathfrak{d}^{-1} \simeq (GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1})[p^r]$$

and $\zeta_{r,\tau}$ denote its $\tau = \tau_{\mathfrak{p}}$ component. We often allow ζ_r and $\zeta_{r,\tau}$ to mean their images in $(GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1})(S)$ and $\mathrm{Tate}_{M_1,M_2,S}(q)(S)$.
Let $\eta_r^{\text{ét}}$ denote the image of 1 by

$$\eta^{\text{ét}}_{\mathrm{KM},r} : (\mathcal{O}_F/p^r) \overset{\eta_{\mathrm{KM}}}{\simeq} p^{-r}M_2/M_2 \xrightarrow{q} q^{p^{-r}M_2/M_2}$$

defining a point of $\mathrm{Tate}_{M_1,M_2}(q)(S)$ of exact order $|\mathcal{O}_F/p^r|$. Let $\eta^{\text{ét}}_{r,\tau}$ denote its $\tau = \tau_{\mathfrak{p}}$ component.

Lemma 12 *Fix an integer $r \geq 1$. Let S be a connected $\mathcal{O} \otimes_{\mathbf{Z}} \mathbf{Z}[[q^M, q^{-M}]]$-scheme. Suppose that C is an $\mathcal{O}_F$-stable finite flat subgroup scheme of $\mathrm{Tate}^0_{M_1,M_2,S}(q)[p^r]$ of order $|\mathcal{O}_F/p^r|$. Suppose that C is of type $\rho = (\rho_{\tau,1}, \rho_{\tau,2})_\tau$. Let $P_{\mathrm{KM}} \in C(S)$ denote a point of exact order $|\mathcal{O}_F/p^r|$. Then for every $\tau = \tau_{\mathfrak{p}}$, $P_{\mathrm{KM},\tau}$ is of the form $\zeta_{r,\tau}^{\sigma_{\tau,1}}\eta_{r,\tau}^{\acute{e}t,\sigma_{\tau,2}}$ for a pair of integers $0 \leq \sigma_{\tau,1} \leq \rho_{\tau,1}$ and $0 \leq \sigma_{\tau,2} \leq \rho_{\tau,2}$ such that both $\sigma_{\tau,1}$ and $\sigma_{\tau,2}$ are coprime to p.*

proof. This is essentially 13.6.3 in [17]. □

10 Generic fibres

With $\mathfrak{n}$ fixed, for every integer $r \geq 1$, let $\overline{\mathbf{U}}_r$ denote the quotient group of the totally positive units of F by the subgroup of elements which are squares of elements in $\mathcal{O}_F$ which are congruent to 1 mod $\mathfrak{n}p^r$. If $r = 0$, we simply write $\overline{\mathbf{U}}$.

Let $Y_{\Gamma_1(\mathfrak{n})}, X_{\Gamma_1(\mathfrak{n})}, Y_{\Gamma_1(\mathfrak{n})\cap\mathrm{Iw}}, X_{\Gamma_1(\mathfrak{n})\cap\mathrm{Iw}}, Y_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r)}, Y_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),\mathrm{Iw}_{\mathfrak{p}},K}$ respectively denote the disjoint unions, t ranging over T, of $Y_{\Gamma_1(\mathfrak{n},t)}, X_{\Gamma_1(\mathfrak{n},t)}, Y_{\Gamma_1(\mathfrak{n},t)\cap\mathrm{Iw}}, X_{\Gamma_1(\mathfrak{n},t)\cap\mathrm{Iw}}, Y_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p^r)}, Y_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p^r),\mathrm{Iw}_{\mathfrak{p}},K}$.

Let $X_{\Gamma_1(\mathfrak{n}),K}, X_{\Gamma_1(\mathfrak{n})\cap\mathrm{Iw},K}$ respectively denote the generic fibres over K of the $\mathcal{O}_K$-schemes $X_{\Gamma_1(\mathfrak{n})}, X_{\Gamma_1(\mathfrak{n})\cap\mathrm{Iw}}$.

Let $X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),K}, X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),\mathrm{Iw}_{\mathfrak{p}},K}$ respectively denote the toroidal compactifications of the K-schemes $Y_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),K}, Y_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),\mathrm{Iw}_{\mathfrak{p}},K}$.

Let $Y^\wedge_{\Gamma_1(\mathfrak{n},t)}, X^\wedge_{\Gamma_1(\mathfrak{n},t)}, Y^\wedge_{\Gamma_1(\mathfrak{n},t)\cap\mathrm{Iw}}, X^\wedge_{\Gamma_1(\mathfrak{n},t)\cap\mathrm{Iw}}$ respectively denote the formal completions of $Y_{\Gamma_1(\mathfrak{n},t)}, X_{\Gamma_1(\mathfrak{n},t)}, Y_{\Gamma_1(\mathfrak{n},t)\cap\mathrm{Iw}}, X_{\Gamma_1(\mathfrak{n},t)\cap\mathrm{Iw}}$ along their closed k-fibres $\overline{Y}_{\Gamma_1(\mathfrak{n},t)}, \overline{X}_{\Gamma_1(\mathfrak{n},t)}, \overline{Y}_{\Gamma_1(\mathfrak{n},t)\cap\mathrm{Iw}}, \overline{X}_{\Gamma_1(\mathfrak{n},t)\cap\mathrm{Iw}}$. Let

$Y^{\wedge}_{\Gamma_1(\mathfrak{n})}, X^{\wedge}_{\Gamma_1(\mathfrak{n})}, Y^{\wedge}_{\Gamma_1(\mathfrak{n})\cap \mathrm{Iw}}, X^{\wedge}_{\Gamma_1(\mathfrak{n})\cap \mathrm{Iw}}$ denote their disjoint unions over T.

Finally, let $Y^{\mathrm{rig}}_{\Gamma_1(\mathfrak{n})}, X^{\mathrm{rig}}_{\Gamma_1(\mathfrak{n})}, Y^{\mathrm{rig}}_{\Gamma_1(\mathfrak{n})\cap \mathrm{Iw}}, X^{\mathrm{rig}}_{\Gamma_1(\mathfrak{n})\cap \mathrm{Iw}}$ respectively denote the Raynaud rigid generic fibres of $Y^{\wedge}_{\Gamma_1(\mathfrak{n})}, X^{\wedge}_{\Gamma_1(\mathfrak{n})}, Y^{\wedge}_{\Gamma_1(\mathfrak{n})\cap \mathrm{Iw}}, X^{\wedge}_{\Gamma_1(\mathfrak{n})\cap \mathrm{Iw}}$.

11 p-adic classical Hilbert modular forms

Suppose that $(k = \sum_{\tau\in \mathrm{Hom}(F,K)} k_\tau \tau, w = \sum_{\tau\in\mathrm{Hom}(F,K)} w_\tau\tau) \in \mathbf{Z}^{\mathrm{Hom}(F,K)} \times \mathbf{Z}^{\mathrm{Hom}(F,K)}$ is such that $w = 2w_\tau - k_\tau$ is independent of τ (this is Taylor's μ in [38]).

For $S \in \{Y_{\Gamma_1(\mathfrak{n}),K}, Y_{\Gamma_1(\mathfrak{n})\cap\mathrm{Iw},K}, Y_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),K}\}$, let $\mathrm{Lie}^\vee(A/S)$ (resp. $H^1_{\mathrm{dR}}(A/S)$) denote the pull-back by the identity section of the sheaf of relative differentials of the universal HBAV A over S (resp. the higher direct image of the relative de Rham complex). By the decomposition,

$$\mathcal{O}_F \otimes_{\mathbf{Z}} \mathcal{O} \simeq \prod_{\tau\in\mathrm{Hom}(F,K)} \mathcal{O}_\tau$$

where $\mathcal{O}_\tau$ is $\mathcal{O}$ into which F embeds by τ, we have

$$\mathrm{Lie}^\vee(A/S) = \bigoplus_{\tau\in\mathrm{Hom}(F,K)} \mathrm{Lie}^\vee(A/S)_\tau,\ H^1_{\mathrm{dR}}(A/S) = \bigoplus_{\tau\in\mathrm{Hom}(F,K)} H^1_{\mathrm{dR}}(A/S)_\tau$$

where $\mathrm{Lie}^\vee(A/S)$ and $H^1_{\mathrm{dR}}(A/S)$ are locally free sheaves of $\mathcal{O}_S$-modules of rank 1 and 2 respectively. Following Hida [14], let

$$L_{(k,w)} = \bigotimes_{\tau\in\mathrm{Hom}(F,K)} (\bigwedge H^1_{\mathrm{dR}}(A/S)_\tau)^{\otimes w/2} \otimes_{\mathcal{O}_S} (\mathrm{Lie}^\vee(A/S))_\tau^{\otimes k_\tau}$$

If k is parallel, more precisely, if $(k,w) = ((k,\dots,k),(k/2,\dots,k/2))$, we will often write L_k for $L_{(k,w)}$. We shall also let $L_{(k,w)}$ denote its extension to the compactification.

Let π_1 (resp. $\pi_{2,\mathfrak{p}}$) denote the degeneracy map

$$X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),\mathrm{Iw}_{\mathfrak{p}},K} \longrightarrow X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),K}$$

defined, on the non-cuspdail points, by

$$(A,i,\lambda,\eta,C,\eta_{\mathrm{KM}},D_{\mathfrak{p}}) \mapsto (A,i,\lambda,\eta,C,\eta_{\mathrm{KM}})$$

$$\text{(resp. } (A/D_{\mathfrak{p}}, (i \bmod D_{\mathfrak{p}}), (\lambda \bmod D_{\mathfrak{p}}), (\eta \bmod D_{\mathfrak{p}}), (\eta_{\mathrm{KM}} \bmod D_{\mathfrak{p}})).$$

12 Canonical subgroups for one-dimensional formal groups

Let L be a finite extension of K, and let val_L be a valuation on L normalised so that $\mathrm{val}_L(p) = 1$. Let G be a one-dimensional principally polarised p-divisible/Barsotti-Tate group over $\mathcal{O}_L$.

Definition. The identity component $G^\wedge$ of G is a one-dimensional formal group, and define $\mathrm{Ha}(G)$ to be $\mathrm{val}_L(a)$ for a as defined in Proposition 3.6.6, [16] (see also [29]).

By definition, G is ordinary if and only if $\mathrm{Ha}(G) = 0$.

Let C be a finite flat subgroup scheme of $G[p]$ of order p.

Definition. Define $\deg(G, C)$ to be $1 - \mathrm{val}_L(\mathrm{Ann}(\mathrm{coker}(\mathrm{Lie}^\vee(G/C) \to \mathrm{Lie}^\vee(G))))$.

It follows immediately from the definition that $\deg(G, C) + \deg(G/C, G[p]/C) = 1$.

Suppose that $\deg(G, C) < p/(p+1)$. Then there exists a canonical subgroup $H(G)$ of G. If $C = H(G)$, then $\deg(G, C) = \mathrm{Ha}(G)$. To see this, note that $H(G)(L)$ consists of 0 and $p-1$ points P of the formal group $G^\wedge$ of valuation $(1 - \mathrm{Ha}(G))/(p-1)$ (Theorem 3.10.7, [16]). Since $\deg(G, C) = 1 - \prod_P \mathrm{val}(P)$ (Lemma 1.3 [24]), $\deg(G, C) = \mathrm{Ha}(G)$.

Lemma 13 *Let r be a rational number $< p/(p+1)$. Suppose that G is not ordinary. Then*
$$\{(G, C) \mid \mathrm{Ha}(G) \le r\}$$
divides into two disjoint subsets, namely
$$\{(G, C) \mid C = H(G) \text{ and } \deg(G, C) \in (0, r]\}$$
and
$$\{(G, C) \mid C \neq H(G) \text{ and } \deg(G, C) \in [1 - r/p, 1)\}.$$
On the other hand,
$$\{(G, C) \mid \mathrm{Ha}(G/C) \le r\}$$
divides into two disjoint subsets, namely
$$\begin{aligned} &\{(G, C) \mid \deg(G, C) \in (0, r/p],\ C = H(G), \text{ and } \mathrm{Ha}(G) < 1/(p+1)\ \} \\ \cup\ &\{(G, C) \mid \deg(G, C) \in (0, r/p],\ C \neq H(G), \text{ and } \mathrm{Ha}(G) < p/(p+1)\ \} \end{aligned}$$
and
$$\{(G, C) \mid \deg(G, C) \in [1 - r, 1),\ C = H(G),\ 1/(p+1) < \mathrm{Ha}(G) < p/(p+1)\}$$
$$\cup\ \{(G, C) \mid \deg(G, C) \in [1 - r, 1),\ C \neq H(G),\ \mathrm{Ha}(G) \ge p/(p+1)\}.$$

Proof. This follows from canonical subgroup theorem in [29]. $\square$

Fix an integer $n \geq 1$ and suppose furthermore that $\deg(G, C) \leq p^{1-n}/(p+1) < p/(p+1)$. Then define subgroup $H_n = H_n(G)$ of G order p^n inductively as follows: If $n = 1$, set $H_1 = D$. If $n > 1$, then let H_n to be the pre-image by the map $G \to G/H(G)$ of $H_{n-1}(G/H(G)) \subset G/H(G)$.

Proposition 14 *Suppose that one-dimensional principally polarised p-divisible group G over $\mathcal{O}_L$ has a subgroup $H_n(G)$ as defined above. Suppse that $m \geq 1$ is an integer. Suppose that C_m is a subgroup of G of order p^m such that $H_n(G) \cap C_m = \{0\}$, and suppose that D_{m+n} is a cyclic subgroup of G of order p^{m+n} such that $H_n(G) \subseteq D_{m+n}$. Then $\deg(G/C_m) < p^{1-(m+n)}/(p+1)$ and G/C_m has the subgroup $H_{m+n}(G/C_m)$. Indeed, $H_{m+n}(G/C_m) = (D_{m+n} + C_m)/C_m$.*

Proof. This can be proved as in Proposition 3.5 in [3]. $\square$

13 p-adic overconvergent Hilbert modular forms

Let $X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n}),K}, X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n})\cap \mathrm{Iw},K}, X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),K}$ respectively denote the rigid analytic spaces in the sense of Tate ([2]) associated to the K-schemes $X_{\Gamma_1(\mathfrak{n}),K}, X_{\Gamma_1(\mathfrak{n})\cap \mathrm{Iw},K}, X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),K}$.

Given a closed point of $Y^{\mathrm{rig}}_{\Gamma_1(\mathfrak{n})}$, it corresponds to a point (A, λ, η) defined over the integer $\mathcal{O}_L$ of a finite extension L of K. We then define $\deg_\tau(A)$, for $\tau = \tau_{\mathfrak{p}}$ for a place $\mathfrak{p}$ of F above p, to be 'deg' as in the previous section with the (one-dimension) Barsotti-Tate group of $\mathfrak{p}$-power torsions of A in place of 'G'.

The $\mathcal{O}$-scheme $X_{\Gamma_1(\mathfrak{n})}$ is of finite type, hence $X^{\mathrm{rig}}_{\Gamma_1(\mathfrak{n})}$ is quasi-compact. There exists a finitely many sufficiently small affine formal schemes $U^\wedge$ such that their generic fibres U^{rig} form an admissible covering of $X^{\mathrm{rig}}_{\Gamma_1(\mathfrak{n})}$. Let $U^\wedge_{\mathrm{good}}$ denote the smooth formal scheme $U^\wedge \cap Y^\wedge_{\Gamma_1(\mathfrak{n})}$ and let $i : U^\wedge_{\mathrm{good}} \hookrightarrow U^\wedge$. On each $U^\wedge_{\mathrm{good}}$, there is a function whose corresponding rigid function has its valuation $\deg_{\mathfrak{p}}$; indeed, apply the construction to the formal completion of the 'universal' semi-abelian scheme over $X_{\Gamma_1(\mathfrak{n})}$ along the underlying scheme of $U^\wedge_{\mathrm{good}}$. We may think of the function on $U^\wedge_{\mathrm{good}}$ as a lift of the Hasse invariant at $\mathfrak{p}$, and it follows from Kocher's principle that $i_*\mathcal{O}_{U^\wedge_{\mathrm{good}}} = \mathcal{O}_{U^\wedge}$, i.e., the function extends to $U^\wedge$. The valuation of its induced function on the generic fibre U^{rig} extends the function on $U^{\mathrm{rig}}_{\mathrm{good}}$. Glue these functions on U^{rig}'s, there is a rigid function on $X^{\mathrm{rig}}_{\Gamma_1(\mathfrak{n})} \simeq X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n})}$ that defines deg.

Definition. If $I \subset [0, 1)$ is a closed, open, or half open interval with endpoint in $\mathbf{Q}$, define the rigid space $X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n}),K} I = \coprod_t X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n},t),K} I$ to be the admissible open set of points whose degrees are all in the range I.

For every t, $X^{\rm an}_{\Gamma_1(\mathfrak{n},t),K}I$ is connected; this follows from the fact that $X^{\rm rig}_{\Gamma_1(\mathfrak{n},t),K}$ is connected (since $\overline{X}_{\Gamma_1(\mathfrak{n},t)}$ is irreducible) and its ordinary locus is open, dense, and connected.

Similarly, given a closed point of $Y^{\rm rig}_{\Gamma_1(\mathfrak{n})\cap{\rm Iw}}$, it corresponds to a point (A,λ,η,C) defined over the integer $\mathcal{O}_L$ of a finite extension L of K. Let $B=A/C$ and $S=\operatorname{Spec}\mathcal{O}_L$; let ${\rm val}_S$ denote the valuation on L normalised such that ${\rm val}_S(p)=1$. Then the $\mathcal{O}_F$-equivariant map of $\mathcal{O}_S$-modules

$$\mathrm{Lie}^\vee(B/S)\longrightarrow\mathrm{Lie}^\vee(A/S)$$

decomposes into

$$\mathrm{Lie}^\vee(B/S)_\tau\longrightarrow\mathrm{Lie}^\vee(A/S)_\tau$$

for every $\tau\in\mathrm{Hom}(F,K)$, and, for the unique prime $\mathfrak{p}$ of F above p corresponding to τ, let $\deg_{\mathfrak{p}}((A,C))$ denote $1-{\rm val}_S(\mathrm{Ann}(\mathrm{Coker}(\mathrm{Lie}^\vee(B/S)_\tau\to\mathrm{Lie}^\vee(A/S)_\tau)))$. Applying the construction to the universal HBAV over $Y^{\rm rig}_{\Gamma_1(\mathfrak{n})\cap{\rm Iw}}$, we locally have functions on $Y^{\rm rig}_{\Gamma_1(\mathfrak{n})\cap{\rm Iw}}$ whose valuations define the degrees. As for $\deg(A)$, Kocher's principle allows us to extend the function to $X^{\rm rig}_{\Gamma_1(\mathfrak{n})\cap{\rm Iw}}\simeq X^{\rm an}_{\Gamma_1(\mathfrak{n})\cap{\rm Iw}}$

Definition. If S_1 and S_2 are disjoint subsets of $\mathrm{Hom}(F,K)$ and if $I,I_1,I_2\subseteq[0,1]$ are closed, open, or half open intervals with endpoints in $\mathbf{Q}$, define the rigid space $(X^{\rm an}_{\Gamma_1(\mathfrak{n})\cap{\rm Iw},K}I),I_{1S_1}I_{2S_2}$ to be the admissible open set of points whose degree at $\tau\in\mathrm{Hom}(F,K)-S_1-S_2$ (resp. S_1, resp. S_2) is in the range I (resp. I_1, resp. I_2).

Definition. Let π_1 (resp. $\pi_{2,\mathfrak{p}}$) denote the degeneracy map

$$X^{\rm an}_{\Gamma_1(\mathfrak{n})\cap{\rm Iw},K}\longrightarrow X^{\rm an}_{\Gamma_1(\mathfrak{n}),K}$$

which, on the non-cuspidal points, is defined by

$$(A,i,\lambda,\eta,C)\mapsto(A,i,\lambda,\eta)$$

$$(\text{resp. } (A,i,\lambda,\eta,C)\mapsto(A/C_{\mathfrak{p}},(i\bmod C_{\mathfrak{p}}),(\lambda\bmod C_{\mathfrak{p}}),(\eta\bmod C_{\mathfrak{p}}))$$

Definition. Let π denote the degeneracy map

$$X^{\rm an}_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),K}\longrightarrow X^{\rm an}_{\Gamma_1(\mathfrak{n})\cap{\rm Iw},K}$$

which, on the non-cuspidal points, is defined by

$$(A,i,\lambda,\eta,\eta_{\rm KM})\mapsto$$
$$(A/\langle pP_{\eta_{\rm KM}}\rangle,(i\bmod\langle pP_{\eta_{\rm KM}}\rangle),(\lambda\bmod\langle pP_{\eta_{\rm KM}}\rangle),(\eta\bmod\langle pP_{\eta_{\rm KM}}\rangle)).$$

where by $P_{\eta_{\mathrm{KM}}}$, we mean the image of 1 by η_{KM}.

DEFINITION. Define $(X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),K}I)I_{1S_1}I_{2S_2}$ to be the preimage by π of $(X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n})\cap\mathrm{Iw},K}I,)I_{1S_1}I_{2S_2}$.

For $0 \leq r \leq p/(p+1)$, it follows from the previous section that

$$\pi_1^{-1}(X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n}),K}[0,r]) \simeq X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n})\cap\mathrm{Iw},K}[0,r]\coprod X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n})\cap\mathrm{Iw},K}[1-r/p,1];$$

and for $\tau = \tau_{\mathfrak{p}}$

$$\begin{array}{cl} & \pi_{2,\mathfrak{p}}^{-1}(X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n}),K}[0,r]) \\ \simeq & (X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n})\cap\mathrm{Iw},K}[0,r])_\tau[0,r/p]\coprod(X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n})\cap\mathrm{Iw},K}[0,r])_\tau[1-r,1]. \end{array}$$

The theory of canonical subgroups provides rigid sections:

$$\pi_1 : X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n})\cap\mathrm{Iw},K}[0,r] \xrightarrow{\simeq} X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n}),K}[0,r]$$

and

$$\pi_{2,\mathfrak{p}} : X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n})\cap\mathrm{Iw},K}[1-r,1] \xrightarrow{\simeq} X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n}),K}[0,r].$$

On the other hand,

$$\pi_1 : X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n})\cap\mathrm{Iw},K}[1-r/p,1]\longrightarrow X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n}),K}[0,r]$$

is finite flat of degree $|\mathcal{O}_F/p|$, and

$$\pi_{2,\mathfrak{p}} : X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n})\cap\mathrm{Iw},K}[0,r/p]\longrightarrow X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n}),K}[0,r]$$

is finite flat of degree $|\mathcal{O}_F/\mathfrak{p}|$.

Hida [14] proves (Theorem 5.6 in [14]) that, for a character $\psi : \mathrm{Fr}_{\mathfrak{n}p^\infty} \to K^\times$ which factors through $I_{\mathfrak{n}p^r}$ and $k \geq 2$, an element $F_{\mathrm{H}} : h^0_{\mathcal{O}}(\mathfrak{n}) \to L$ of $\mathcal{S}^0_{\mathcal{O}}(\mathfrak{n}) \otimes L$ defines, modulo$(\ker(\epsilon\circ\mathrm{Art})^{k-2}\psi)$, a cusp eigenform of weight k and level $\Gamma_1(\mathfrak{n}p^r)$ which is an eigenform with its $T_{\mathfrak{m}}$-eigenvalue $F_{\mathrm{H}}(T_{\mathfrak{m}})$ mod $(\ker(\epsilon\circ\mathrm{Art})^{k-2}\psi)$ and S acting by $(\epsilon\circ\mathrm{Art})^{k-2}\psi$. Indeed $I_{\mathfrak{n}p^\infty}$-action defines the character of F_{H} mod $(\ker(\epsilon\circ\mathrm{Art})^{k-2}\psi)$, i.e,

$$\begin{array}{cl} & (F_{\mathrm{H}} \bmod (\ker(\epsilon\circ\mathrm{Art})^{k-2}\psi))(\langle\ \rangle) \\ = & \psi(\psi_{F_{\mathrm{H}}} \bmod (\ker(\epsilon\circ\mathrm{Art})^{k-2}\psi))(\psi_{\mathrm{T}}\circ\epsilon^{2-k}) \end{array}$$

where $\psi_{F_{\mathrm{H}}}$ is the composite $\mathrm{Tor}_{\mathfrak{n}p^\infty} \hookrightarrow I_{\mathfrak{n}p^\infty} \xrightarrow{\langle\,\rangle} h^0_{\mathcal{O}}(\mathfrak{n})$ followed by $F_{\mathrm{H}} : h^0_{\mathcal{O}}(\mathfrak{n}) \to L$; and ψ_{T} is the 'Teichmuller character', the projection from $\mathbf{Z}_p^\times$ to its torsion subgroup of finite order. We shall prove that the specialisation F_{H} mod $\ker(\epsilon\circ\mathrm{Art})^{k-2}\psi$ defines a p-ordinary overconvergent eigenform of weight k and of level $\Gamma_1(\mathfrak{n}p^r)$ for any $k=1$.

For ϵ such that $0 \leq \epsilon < 1/(p^{r-2}(p+1))$, the theory of canonical subgroups in [29] (see also Proposition 2.3.1 and 2.4.1 in [21]) shows that $U_p \overset{\text{def}}{=} \prod_{\mathfrak{p}} U_{\mathfrak{p}}$ defines a completely continuous endomorphism on $H^0(X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),K}[0,\epsilon], L_k(\mathrm{cusps}))^{\overline{\mathbf{U}}}$, where $X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),K}[0,\epsilon]$ is the pre-image by the forgetful morphism of $X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n}),K}[0,\epsilon]$. We remark that, when $F = \mathbf{Q}$, this is proved in [4] Lemma 2.3 as a result of calculations with q-expansions.

By Serre's theory [31], there is an idempotent $\mathbf{e}$ commuting with U_p by which we may write

$$\begin{aligned}
&H^0(X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),K}[0,\epsilon], L_k(\mathrm{cusps}))^{\overline{\mathbf{U}}}\\
&= \mathbf{e}H^0(X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),K}[0,\epsilon], L_k(\mathrm{cusps}))^{\overline{\mathbf{U}}}\\
&+(1-\mathbf{e})H^0(X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),K}[0,\epsilon], L_k(\mathrm{cusps}))^{\overline{\mathbf{U}}}
\end{aligned}$$

where $\mathbf{e}H^0(X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),K}[0,\epsilon], L_k(\mathrm{cusps}))^{\overline{\mathbf{U}}}$ is finite-dimensional K-vector space and all the generalised eigenvalues of U_p are units, while U_p is topologically nilpotent on the complement. It is well-known that $\mathbf{e} = e|H^0(X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),K}[0,\epsilon], L_k(\mathrm{cusps}))^{\overline{\mathbf{U}}}$.

Lemma 15 *For any integer k, the p-adic eigenform F_{H} mod $(\ker(\epsilon\circ\mathrm{Art})^{k-2}\psi)$ as above is overconvergent of weight k and of level $\Gamma_1(\mathfrak{n}p^r)$.*

Proof. This can be proved as in Lemma 1 in [5]; replace the Eisenstein series 'E' of weight $(p-1)$ therein by the pull-back to $X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),K}$ of a characteristic zero lifting of a sufficiently large power of the Hasse invariant. □

It follows from the theorem in the previous section that, given a p-adic representation

$$\rho : \mathrm{Gal}(\overline{F}/F) \to GL_2(\mathcal{O})$$

as in the main theorem, there are

1. a finite soluble totally real field extension $F_\Sigma \subset \overline{F}$ of F in which p splits completely,

2. a finite set $S = \Sigma \coprod S_{\Sigma,\mathrm{P}} \coprod S_{\Sigma,\infty}$ of places in F_Σ, where $S_{\Sigma,\mathrm{P}}$ denotes the set of places of F above p and $S_{\Sigma,\infty}$ denotes the set of infinite places of F_Σ,

3. an ideal $\mathfrak{n}$ of $\mathcal{O}_F$ divisible by $\mathfrak{n}_\Sigma = \prod_{\mathfrak{q}\in\Sigma} \mathfrak{q}$,

4. $2^{|S_{\Sigma,\mathrm{P}}|}$ characters $\chi_P : \mathrm{Gal}(\overline{F}/F_\Sigma) \to \mathcal{O}^\times$ of finite order and $2^{|S_{\Sigma,\mathrm{P}}|}$ weight one p-ordinary overconvergent cuspidal Hilbert modular eigenforms f_P of 'tame level' $\mathfrak{n}$, one for every subset P of $S_{\Sigma,\mathrm{P}}$, such that:

- f_P is the weight one specialisation of the Λ-adic companion form $F_{\mathrm{Hida},P} : h^0_{\mathcal{O}}(\mathfrak{n}) \to K$, with character $\psi_P = \psi_P^{S_{\Sigma,\mathrm{P}}}\psi_{P,S_{\Sigma,\mathrm{P}}}$ of $(\mathcal{O}_{F_\Sigma}/\mathfrak{n})^\times \times (\mathcal{O}_{F_\Sigma}/p)^\times$
- the Galois representation ρ_P associated to f_P is $\rho|_{\mathrm{Gal}(\overline{F}/F_\Sigma)} \otimes {\chi_P}^{-1}$,
- ρ_P is unramified outside S and ordinary at every place in $S_{\Sigma,\mathrm{P}}$,

and the f_P's are 'in companion' in the sense that

- $c(\mathcal{O}_{F_\Sigma}, f_P) = 1$, and $c(\mathfrak{m}, f_P) = 0$ if $\mathfrak{m}$ is not coprime to $\mathfrak{n}$;
- $c(\mathfrak{q}, f_P) = \mathrm{tr}\,\rho(\mathrm{Frob}_{\mathfrak{q}})/\chi_P(\mathrm{Frob}_{\mathfrak{q}})$ for every prime ideal $\mathfrak{q}$ not dividing $\mathfrak{n}p$;
- for $\mathfrak{p}$ in P, $c(\mathfrak{m}, f_P)(\chi_P \circ \mathrm{Art})(\mathfrak{m}) = c(\mathfrak{m}, f_{P-\{\mathfrak{p}\}})(\chi_{P-\{\mathfrak{p}\}} \circ \mathrm{Art})(\mathfrak{m})$ for every ideal $\mathfrak{m}$ coprime to $\mathfrak{n}p$;
- for $\mathfrak{p}$ in P, the character of f_P at $\mathfrak{p}$ is $\chi_{P-\{\mathfrak{p}\}}\chi_P^{-1}$ while for $\mathfrak{p} \in S_{\Sigma,\mathrm{P}} - P$, the character of f_P at $\mathfrak{p}$ is $\chi_{P\cup\{\mathfrak{p}\}}\chi_P^{-1}$;
- for $\mathfrak{p}$ in P, $(\chi_P^{S_{\Sigma,\mathrm{P}}} \circ \mathrm{Art})(\mathfrak{p}) = (\psi_{P-\{\mathfrak{p}\}}^{S_{\Sigma,\mathrm{P}}} \circ \mathrm{Art})(\mathfrak{p})$;
- for a place $\mathfrak{p}$ of P, the $U_{\mathfrak{p}}$-eigenvalue of f_P is $(\chi_{\mathfrak{p},1}\chi_P^{-1})(\mathrm{Frob}_{\mathfrak{p}})$ while for $\mathfrak{p}$ in $S_{\Sigma,\mathrm{P}} - P$, the $U_{\mathfrak{p}}$-eigenvalue of f_P is $(\chi_{\mathfrak{p},2}\chi_P^{-1})(\mathrm{Frob}_{\mathfrak{p}})$.

14 Analytic continuation of overconvergent eigenforms

Fix $\tau = \tau_{\mathfrak{p}}$ throughout the section (except the last two assertions).

Definition. Fix t. For brevity, let $X_{\Gamma_1(\mathfrak{n},t)\cap\mathrm{Iw},\tau}$ denote $(X_{\Gamma_1(\mathfrak{n},t)\cap\mathrm{Iw},K}[0,r])[0,1)_\tau$; and for an integer $n \geq 0$, let $X_{\Gamma_1(\mathfrak{n},t)\cap\mathrm{Iw},\tau,n}$ denote $(X_{\Gamma_1(\mathfrak{n},t)\cap\mathrm{Iw},K}[0,r])[0,1-1/p^n(p+1)]_\tau$.
Let $X_{\Gamma_1(\mathfrak{n})\cap\mathrm{Iw},\tau}$ (resp. $X_{\Gamma_1(\mathfrak{n})\cap\mathrm{Iw},\tau,n}$) denote the disjoint union over T of $X_{\Gamma_1(\mathfrak{n},t)\cap\mathrm{Iw},\tau}$ (resp. $X_{\Gamma_1(\mathfrak{n},t)\cap\mathrm{Iw},\tau,n}$).

Proposition 16 *For every integer $n \geq 0$, $X_{\Gamma_1(\mathfrak{n})\cap\mathrm{Iw},\tau,n}$ is an admissible open subset of $X_{\Gamma_1(\mathfrak{n})\cap\mathrm{Iw},\tau}$, and the $X_{\Gamma_1(\mathfrak{n})\cap\mathrm{Iw},\tau,n}$ form an admissible covering of $X_{\Gamma_1(\mathfrak{n})\cap\mathrm{Iw},\tau}$. For every t and every $n \in \mathbf{Z}_{\geq 0}$, $X_{\Gamma_1(\mathfrak{n},t)\cap\mathrm{Iw},\tau,n}$ is connected.*

Proof. Clear. $\square$

Definition. Let $X_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p^r),\tau}$ (resp. $X_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p^r),\tau,n}$) denote the preimage by the degeneracy morphism

$$\pi : X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p^r),K} \to X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n},t)\cap\mathrm{Iw},K}$$

of $X_{\Gamma_1(\mathfrak{n},t)\cap\mathrm{Iw},\tau}$ (resp. $X_{\Gamma_1(\mathfrak{n},t)\cap\mathrm{Iw},\tau,n}$).
Let $X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),\tau}$ (resp. $X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),\tau,n}$) denote the disjoint union over T of $X_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p^r),\tau}$ (resp. $X_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p^r),\tau,n}$).

Proposition 17 *For every integer $n \geq 0$, $X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),\tau,n}$ is an admissible open subset of $X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),\tau}$, and the $X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),\tau,n}$ form an admissible covering of $X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),\tau}$. For every t and an integer $n \geq 0$, $X_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p^r),\tau,n}$ is connected.*

Proof. Analogous to the proposition above. □

Corollary 18 *We have $\pi_1^{-1}(X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),\tau,n+1}) \subset \pi_{2,\mathfrak{p}}^{-1}(X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),\tau,n})$.*

Proof. This follows from [29]. □

Let $(\mathrm{Tate}_{M_1,M_2}(q) = (GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1}M_1^{-1})/q^{M_2}, i, \lambda, \eta, \eta_{\mathrm{KM}} : 1 \mapsto \zeta_r)$ over $\mathcal{O} \otimes \mathbf{Z}((q^{M_1M_2^{-1}}))$ for the pair M_1, M_2 of the fractional ideals such that $M_1M_2^{-1} \simeq t$ be a family of HBAVs *around* a cusp of $X_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p^r),n}$. Choose (non-canonically) once for all a basis of the pull-back by $\mathrm{Max}\,(\mathcal{O} \otimes_{\mathbf{Z}} \mathbf{Z}((q^M))) \to X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),K}$ of the line bundle L_k, since a subgroup of $\mathrm{Tate}_{M_1,M_2}(q)[\mathfrak{p}]$ of order $|\mathcal{O}_F/\mathfrak{p}|$, disjoint from η_r, is of the form $\zeta\eta + q^{M_2}$ where ζ ranges over the $|\mathcal{O}_F/\mathfrak{p}|$ points of $(GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1}M_1^{-1})(S)[\mathfrak{p}]$ and $\eta = \eta^{\text{ét}}_{1,\mathfrak{p}} \in q^{\mathfrak{p}^{-1}M_2/M_2}$, $U_{\mathfrak{p}}(f)(\mathrm{Tate}_{M_1,M_2}(q), i, \lambda, \eta, \eta_{\mathrm{KM}})$ is:

$$\begin{aligned}
& |\mathcal{O}_F/\mathfrak{p}|^{-1} \sum_{\zeta} f(((GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1}M_1^{-1})/q^{M_2})/(\zeta\eta)) \\
= \; & |\mathcal{O}_F/\mathfrak{p}|^{-1} \sum_{\zeta} f((GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1}M_1^{-1})/(\zeta q_\eta)^{\mathfrak{p}^{-1}M_2}) \\
= \; & |\mathcal{O}_F/\mathfrak{p}|^{-1} \sum_{\zeta} |\mathcal{O}_F/t_1|^{-1} \sum_{\nu \in (\mathfrak{p}^{-1}M)^+} c(\mathfrak{p}M^{-1}\nu, f)(\zeta q_\eta)^\nu \\
= \; & |\mathcal{O}_F/\mathfrak{p}|^{-1} |\mathcal{O}_F/t_{\mathfrak{p}}|^{-1} \sum_{\nu \in (\mathfrak{p}^{-1}M)^+} (\sum_{\zeta} \zeta^\nu) c(\mathfrak{p}M^{-1}\nu, f) q_\eta^\nu \\
= \; & |\mathcal{O}_F/\mathfrak{p}|^{-1} |\mathcal{O}_F/t_{\mathfrak{p}}|^{-1} \sum_{\nu \in M^+} |\mathcal{O}_F/\mathfrak{p}| c(\mathfrak{p}M^{-1}\nu, f) q^\nu
\end{aligned}$$

where q_η denotes a representative in $q^{\mathfrak{p}^{-1}M_2}$ of the class $\eta \in q^{\mathfrak{p}^{-1}M_2/M_2} = q^{\mathfrak{p}^{-1}M_2}/q^{M_2}$ defined earlier; and $t_{\mathfrak{p}}$ represents the class of $\mathfrak{p}t \simeq \mathfrak{p}M_1M_2^{-1}$.

Theorem 19 *Suppose that $f \in H^0((X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),K})_\tau[0,\epsilon], L_k)$ is an eigenform for $U_{\mathfrak{p}}$ with non-zero eigenvalue, then f extends to $X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),\tau} = (X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r)})_\tau[0,1)$.*

Definition. Let

$$\begin{array}{ccccc}
X^{[0]}_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p^r),\tau} & \subset & X^{[1]}_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p^r),\tau} & \subset \cdots \subset & X^{[r-1]}_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p^r),\tau} \\
\| & & & & \\
X_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p^r),\tau} & & & &
\end{array}$$

denote the admissible open subsets of $X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p^r)}$ defined in such a way that the non-cuspidal S-points of $X^{[s]}_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p^r),\tau}$ parameterises $(A/S, i, \lambda, \eta)$

equipped with a point $P_{\eta_{\mathrm{KM}}}$ of exact of order $\sum_{\mathfrak{p}} |\mathcal{O}_F/\mathfrak{p}^r|$ where A/S is either $\mathfrak{p}$-non-ordinary, or it is $\mathfrak{p}$-ordinary *and* $H_{r-s}(A[\mathfrak{p}])$ equals the subgroup generated by $|\mathcal{O}_F/\mathfrak{p}|^s P_{\eta_{\mathrm{KM}},\mathfrak{p}}$.

For every $0 \leq s \leq r-1$, $X^{[s]}_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p^r),\tau}$ is connected since it is the pre-image of a closed subset of the union of irreducible components intersecting precisely at the $\mathfrak{p}$-non-ordinary locus of $X_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p^r),\tau}$.

Theorem 20 *If r is an integer ≥ 2 and suppose that $f \in H^0(X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),\tau}, L_k)$ is an eigenform for $U_{\mathfrak{p}}$ with non-zero eigenvalue. Then f extends to $X^{[r-1]}_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p^r),\tau}$.*

Proof. This can be proved as Lemma 6.1 in [3] □

Corollary 21 *If $f \in H^0(X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n})\cap\mathrm{Iw},K}[0,\epsilon], L_k)$ for some $0 < \epsilon < 1$ is an eigenform for every $U_{\mathfrak{p}}$, $\mathfrak{p}|p$, with non-zero eigenvalue, then f extends to $H^0(X_{\Gamma_1(\mathfrak{n})\cap\mathrm{Iw},K}[0,1), L_k)$. Similarly, if $f \in H^0(X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),K}[0,\epsilon], L_k)$, for some $0 < \epsilon < 1$ is an eigenform for every $U_{\mathfrak{p}}$, $\mathfrak{p}|p$, with non-zero eigenvalue, then g extends to $H^0(X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),K}[0,1), L_k)$.*

15 Gluing eigenforms

15.1 The Iwahori case

Definition. For every subset P of the set of places of F above p, let w_P denote the automorphism of $X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n})\cap\mathrm{Iw},K}$ defined by a composite (independent of ordering) of the $w_{\mathfrak{p}}$ for all $\mathfrak{p}$ in P.

Theorem 22 *For every subset P of the set $S = S_{\mathrm{P}}$ of places of F above p, suppose $f_P \in H^0(X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n}),K}, L_k)$ is an overconvergent modular form of parallel weight $k = \sum_{\tau\in\mathrm{Hom}(F,K)} k\tau \in \mathbf{Z}$ and of level $\Gamma_1(\mathfrak{n})$. Assume furthermore that*

- *the Fourier coefficient $c(f_P, \mathcal{O}_F) = 1$;*
- *for every place $\mathfrak{p}$ of F above p,there exist $\alpha_{\mathfrak{p}}, \beta_{\mathfrak{p}} \in K$ such that $\alpha_{\mathfrak{p}} \neq \beta_{\mathfrak{p}}$ and such that, for every P, f_P is an eigenform for $U_{\mathfrak{p}}$ with eigenvalue $\alpha_{\mathfrak{p}}$ if $\mathfrak{p} \in P$ whilst with eigenvalue $\beta_{\mathfrak{p}}$ if $\mathfrak{p} \notin P$;*
- *for all ideal $\mathfrak{m}$ of $\mathcal{O}_F$ coprime to p, $c(\mathfrak{m}, f_P)$ are equal for every P.*

Then every f_P is a classical Hilbert modular eigenform of weight k and of level $\Gamma_1(\mathfrak{n}) \cap \mathrm{Iw}$.

Proof. By the isomorphism

$$\pi_1 : X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n})\cap\mathrm{Iw},K}[0,r] \xrightarrow{\simeq} X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n}),K}[0,r]$$

for $r < p/(p+1)$ given by the canonical subgroups theorem [29], we may think of f_P as an element of $H^0(X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n})\cap\mathrm{Iw},K}[0,r], L_k)$. It follows from results in [29] that $\pi_1^* f_P$ extends to a section over $X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n})\cap \mathrm{Iw},K}[0,1)$. For brevity, we shall only show that f_P, with P the (full) set S of places of F above p, is classical; the general case follows by changing the roles of $\alpha_{\mathfrak{p}}$ and $\beta_{\mathfrak{p}}$.

Choose a rational number $r \in \mathbf{Q}$ with $1/2 < r < p/(p+1)$. Suppose that f_S extends to a section of L_k over $(X_{\Gamma_1(\mathfrak{n})\cap \mathrm{Iw},K}[0,r])[0,1]_{S-P}$ for some $P \subseteq S$. Fix a prime $\mathfrak{p} \in P$. It suffices to show that f_S extends to $(X_{\Gamma_1(\mathfrak{n})\cap \mathrm{Iw},K}[0,r])[0,1]_{S-(P-\{\mathfrak{p}\})}$.

For $f \in H^0(X_{\Gamma_1(\mathfrak{n}),K}[0,r], L_k)$ and for every subset $Q \subseteq S-P$, let f^Q denote the restriction of f to $(X_{\Gamma_1(\mathfrak{n})\cap \mathrm{Iw},K}[0,r])[1-r,1]_Q[0,r]_{(S-P)-Q}$ by the map $\pi_1 \circ w_Q$ which defines an isomorphism

$$\begin{array}{ccc} (X_{\Gamma_1(\mathfrak{n})\cap \mathrm{Iw},K}[0,r])[1-r,1]_Q,[0,r]_{(S-P)-Q} & \simeq & X_{\Gamma_1(\mathfrak{n}),K}[0,r] \\ || & & \\ (X_{\Gamma_1(\mathfrak{n})\cap \mathrm{Iw},K}[0,r])[1-r,1]_Q & & . \end{array}$$

The pre-image by $\pi_{2,\mathfrak{p}} \circ w_Q$ of $(X_{\Gamma_1(\mathfrak{n}),K}[0,r])(0,r)_{\mathfrak{p}}$ is the union of two components

$$(X_{\Gamma_1(\mathfrak{n})\cap \mathrm{Iw},K}[0,r])[1-r,1]_Q(1-r,1)_{\mathfrak{p}} \coprod (X_{\Gamma_1(\mathfrak{n})\cap \mathrm{Iw},K}[0,r])[1-r,1]_Q(0,r/p)_{\mathfrak{p}}$$

and it induces an isomorphism

$$(X_{\Gamma_1(\mathfrak{n})\cap \mathrm{Iw},K}[0,r])[1-r,1]_Q(1-r,1)_{\mathfrak{p}} \simeq (X_{\Gamma_1(\mathfrak{n}),K}[0,r])(0,r)_{\mathfrak{p}}$$

on the one component and a finite flat morphism of degree $|\mathcal{O}_F/\mathfrak{p}|$

$$(X_{\Gamma_1(\mathfrak{n})\cap \mathrm{Iw},K}[0,r])[1-r,1]_Q(0,r/p)_{\mathfrak{p}} \longrightarrow (X_{\Gamma_1(\mathfrak{n}),K}[0,r])(0,r)_{\mathfrak{p}}$$

on the other.

We are going to glue f_S and $f_{S-\{\mathfrak{p}\}}$; more precisely glue f_S^Q and $f^Q_{S-\{\mathfrak{p}\}}$.

Let F denote the section

$$(\alpha_{\mathfrak{p}} f_S^Q - \beta_{\mathfrak{p}} f^Q_{S-\{\mathfrak{p}\}})/(\alpha_{\mathfrak{p}} - \beta_{\mathfrak{p}}) \in H^0((X_{\Gamma_1(\mathfrak{n}),K}[0,r])(0,r)_{\mathfrak{p}}, L_k)$$

and G denote the section

$$|\mathcal{O}_F/\mathfrak{p}|\langle\mathfrak{p}\rangle((f_S^Q - f^Q_{S-\{\mathfrak{p}\}})/(\alpha_{\mathfrak{p}} - \beta_{\mathfrak{p}}) \in H^0((X_{\Gamma_1(\mathfrak{n})\cap \mathrm{Iw},K}[0,r])[1-r,1]_Q(0,r)_{\mathfrak{p}}, L_k)$$

Since one can show readily the q-expansions of $\pi^*_{2,\mathfrak{p}}F$ and G are equal at around $C = (\mathrm{Tate}_{M_1,M_2}(q),\ldots,\langle\zeta_1\rangle)$, we shall glue $\pi^*_{2,\mathfrak{p}}F$ and G at $(X_{\Gamma_1(\mathfrak{n})\cap \mathrm{Iw},K}[0,r])[1-r]_Q(0,r/p)_{\mathfrak{p}}$ to construct an extension F' of F to a section over $(X_{\Gamma_1(\mathfrak{n}),K}[0,r])[0,1)_{\mathfrak{p}}$; this extension constructs an extension of f_S^Q (and $f^Q_{S-\{\mathfrak{p}\}}$) to $(X_{\Gamma_1(\mathfrak{n})\cap \mathrm{Iw},K}[0,r])[1-r,1]_Q[0,r]_{(S-P)-Q}[0,1]_{\mathfrak{p}}$ and therefore to $(X_{\Gamma_1(\mathfrak{n})\cap \mathrm{Iw},K}[0,r])[0,1]_{S-(P-\{\mathfrak{p}\})}$ (by assumption, there is an extension 'over $[0,1]$' at $S-P$)

Gluing of $\pi_{2,\mathfrak{p}}^*F$ and G is analogous to [3] since we have a commutative diagram

$$\begin{array}{ccc}(X_{\Gamma_1(\mathfrak{n})\cap\mathrm{Iw},K}[0,r])[1-r]_Q(0,r/p)_{\mathfrak{p}} & \longrightarrow & (X_{\Gamma_1(\mathfrak{n})\cap\mathrm{Iw},K}[0,r])[1-r]_Q(0,1)_{\mathfrak{p}} \\ \downarrow & & \downarrow \\ (X_{\Gamma_1(\mathfrak{n}),K}[0,r])(0,r)_{\mathfrak{p}} & \longrightarrow & (X_{\Gamma_1(\mathfrak{n}),K}[0,r])(0,1)_{\mathfrak{p}}\end{array}$$

where the vertical arrows are $\pi_{2,\mathfrak{p}}\circ w_Q$ but of degree $|\mathcal{O}_F/\mathfrak{p}|$ on the left and $1+|\mathcal{O}_F/\mathfrak{p}|$ on the right.$\square$

15.2 The $\Gamma_1(p)$ case

For evert t in T and for every subset P of the set $S=S_{\mathrm{P}}$ of places of F above p, we well let

$$\begin{aligned}&(X_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p),\mathrm{Iw}_{\mathfrak{p}},K}[0,1))(0,1]_{S-P}\\ \stackrel{\mathrm{def}}{=}\;&\pi_1^{-1}(X_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p),K}[0,1)(0,1]_{S-P})\subset X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p),\mathrm{Iw}_{\mathfrak{p}},K}.\end{aligned}$$

Let w_P denote the composite of the $w_{\zeta_{\mathfrak{p}}}$ for all $\mathfrak{p}\in P$. Note that

$$(X_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p),\mathrm{Iw}_{\mathfrak{p}},K}[0,1))(0,1]_{S-P}\stackrel{\simeq}{\to}w_{S-P}^{-1}X_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p),\mathrm{Iw}_{\mathfrak{p}},K}[0,1)$$

and each $(X_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p),\mathrm{Iw}_{\mathfrak{p}},K}[0,1))(0,1]_{S-P}$ is connected since it is isomorphic to $X_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p),\mathcal{O}_K}[0,1)$ and the latter is connected since it is the pre-image of a connected component in the Zariski topology of the closed fibre. Let $(X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p),\mathrm{Iw}_{\mathfrak{p}},K}[0,1))(0,1]_{S-P}$ denote the disjoint union over T of $(X_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p),\mathrm{Iw}_{\mathfrak{p}},K}[0,1))(0,1]_{S-P}$.

Definition. If f is a section of L_k over $X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p),K}[0,1)$, then w_{S-P}^*f is a section of $w_{S-P}^*L_k$ over $w_{S-P}^{-1}(X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p),K},[0,1)) = (X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p),K}[0,1))(0,1]_{S-P}$. Because p is inverted, the natural morphism of invertible sheaves

$$L_k|(X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p),K}[0,1))(0,1]_{S-P}\stackrel{\simeq}{\to}w_{S-P}^*(L_k|X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p),K}[0,1))$$

is an isomorphism and we let $f|w_P$ denote the section of L_k over $(X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p),K}[0,1))(0,1]_{P-S}$ corresponding to w_{S-P}^*f by the isomorphism.

Theorem 23 *For every subset P of $S=S_{\mathrm{P}}$, let $f_P\in H^0(X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p),K}[0,1),L_k)^{\overline{\mathbf{U}}_1}$ be an overconvergent Hilbert modular form of parallel weight $k=\sum_{\tau\in\mathrm{Hom}(F,K)}k\tau\in\mathbf{Z}$ and of level $\Gamma_1(\mathfrak{n}p)$. For every subset P of S, suppose that f_P has a Hecke character*

$$\psi_P\stackrel{\mathrm{def}}{=}\psi_P^S\psi_{S,P}:(\mathcal{O}_F/\mathfrak{n}p)^\times\simeq(\mathcal{O}_F/\mathfrak{n})^\times\times(\mathcal{O}_F/p)^\times\longrightarrow\mathcal{O}^\times;$$

and that $\psi_P^S(\mathfrak{p})=\psi_{P-\{\mathfrak{p}\}}^S(\mathfrak{p})$ for every $\mathfrak{p}$ in P. Suppose that

- *the Fourier coefficients* $c(\mathcal{O}_F, f_P) = 1$ *and* $c(\mathfrak{m}, f_P) = 0$ *if* $\mathfrak{m}$ *and* $\mathfrak{n}$ *are not coprime,*
- *for every* $\mathfrak{p} \in P$, $c(\mathfrak{m}, f_P) = \psi_{P,\mathfrak{p}}(\mathfrak{m})c(\mathfrak{m}, f_{P-\{\mathfrak{p}\}})$ *for every ideal* $\mathfrak{m}$ *coprime to* $\mathfrak{n}p$*, where by* $\psi_{P,\mathfrak{p}}$ *we mean the* $\mathfrak{p}$*-component of* $\psi_{P,S}$ *which we assume non-trivial,*
- *for every* $\mathfrak{p}$ *in* S*,* f_P *is an* $U_\mathfrak{p}$*-eigenform with non-zero eigenvalue* $\alpha(\mathfrak{p}, f_P)$*, and for every* $\mathfrak{p} \in P$*,* $\alpha(\mathfrak{p}, f_P)\alpha(\mathfrak{p}, f_{P-\{\mathfrak{p}\}}) = \psi_P^S(\mathfrak{p})|\mathcal{O}_F/\mathfrak{p}|^{k-1} = \psi_{P-\{\mathfrak{p}\}}^S(\mathfrak{p})|\mathcal{O}_F/\mathfrak{p}|^{k-1}$.

Then f_P *is a section of* L_k *over* $X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p),\mathcal{O}_K}$.

Proof. For every subset P of S, let g_P denote $f_P|w_P \in H^0((X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p),K}[0,1))(0,1]_{S-P}, L_k)$. Clearly $g_S = f_S$. We shall prove that f_S is classical.
Fix an integer $0 \leq n \leq |S|$ and suppose that the g_P with $P \subseteq S$ such that $|P| \geq n$ glue together to define sections, which will again be denoted by g_P, over

$$\bigcup_{P\subset S, |P|\geq n} (X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p),K}[0,1))(0,1]_{P-S}.$$

Fix a subset $P \subseteq S$ with $\#P = n$ and fix $\mathfrak{p} \in P$. It suffice to show that $g_P(\simeq w^*_{S-P}f_P)$ and (a constant multiple of) $g_{P-\{\mathfrak{p}\}}(\simeq w^*_{S-(P-\{\mathfrak{p}\})}f_{P-\{\mathfrak{p}\}} = w^*_{\{\mathfrak{p}\}}w^*_{S-P}f_{P-\{\mathfrak{p}\}})$ glue.
Let $\alpha_\mathfrak{p}$ (resp. $\beta_\mathfrak{p}$) denote the $U_\mathfrak{p}$-eigenvalue $\alpha(\mathfrak{p}, f_P)$ (resp. $\alpha(\mathfrak{p}, f_{P-\{\mathfrak{p}\}})$).
Fix a p-th root ζ_1 of unity. Let $(\mathrm{Tate}_{M_1,M_2}(q), \ldots, \eta_{\mathrm{KM}} : 1 \mapsto \zeta_1)$. be a point around a cusp C. By abuse of notation, we call it C.
There is a morphism

$$\pi_1 : X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p),\mathrm{Iw}_\mathfrak{p},K} \longrightarrow X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p),K}$$

defined, on the non-cuspidal points, by

$$(A, i, \lambda, \eta, \eta_{\mathrm{KM}}, D_\mathfrak{p}) \mapsto (A, i, \lambda, \eta, \eta_{\mathrm{KM}})$$

and, for $\gamma \in (\mathcal{O}_F/\mathfrak{p})^\times$,

$$\pi_{2,\gamma} : X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p),\mathrm{Iw}_\mathfrak{p},K} \longrightarrow X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p),K}$$

defined, on the non-cuspidal points, by

$$(A, i, \lambda, \eta, \eta_{\mathrm{KM}}, D_\mathfrak{p}) \mapsto (A/(\gamma\eta_{\mathrm{KM}}(1)_\mathfrak{p} + D_\mathfrak{p}), \ldots, (\eta_{\mathrm{KM}} \bmod (\gamma\eta_{\mathrm{KM}}(1)_\mathfrak{p} + D_\mathfrak{p}))).$$

To single out, let

$$\pi_{2,\mathfrak{p}} : X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p),\mathrm{Iw}_\mathfrak{p},K} \longrightarrow X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p),K}$$

denote the morphism '$\gamma = 0$ in $\mathcal{O}_F/\mathfrak{p}$' which takes $(A, i, \lambda, \eta, \eta_{\mathrm{KM}}, D_{\mathfrak{p}})$ to $(A/D_{\mathfrak{p}}, \ldots, (\eta_{\mathrm{KM}} \bmod D_{\mathfrak{p}}))$.

By abuse of notation, let C also denote the pre-image

$$(\mathrm{Tate}_{M_1,M_2}(q), \ldots, \eta_{\mathrm{KM}} : 1 \mapsto \zeta_1, \langle \eta_1^{\mathrm{et}} \rangle) \in X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p),\mathrm{Iw}_{\mathfrak{p}},K}$$

by π_1 above of $C = (\mathrm{Tate}_{M_1,M_2}(q) \ldots, \zeta_1)$ for $(M_1, M_2) = (\mathcal{O}_F, t^{-1})$ and $M = M_1M_2 = t^{-1}$; and let $C_P \in (X_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p),\mathrm{Iw}_{\mathfrak{p}},K}[0,1))(0,1]_{S-P}$ denote the cups $w^*_{S-P}C$. Then

$$(g_P|\pi_1)(C_P) = pr^*f(\mathrm{Tate}_{M_1,M_2}(q), \ldots, \zeta_1) = |\mathcal{O}_F/t|^{-1} \sum_{\nu\in M^+} c(\nu M^{-1}, f_P)q^\nu$$

On the other hand, for $\gamma \in (\mathcal{O}_F/\mathfrak{p})^\times$,

$$\begin{array}{cl} & (g_P|\pi_{2,\gamma})(C_P) \\ = & (f_P|\pi_{2,\gamma})(\mathrm{Tate}_{M_1,M_2}(q), \ldots, \zeta_1) \\ = & pr^*f_P(\mathrm{Tate}_{M_1,M_2}(q)/(\zeta^\gamma\eta), \ldots) \text{ where } \eta := \eta^{\mathrm{et}}_{1,\mathfrak{p}} \text{ and } \zeta := \zeta_{1,\mathfrak{p}} \\ = & pr^*f_P((GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1}M_1^{-1})/(\zeta^\gamma q_\eta)^{\mathfrak{p}^{-1}M_2}, \ldots) \\ = & |\mathcal{O}_F/t_{\mathfrak{p}}|^{-1} \sum_{\nu\in(\mathfrak{p}^{-1}M)^+} c(\nu\mathfrak{p}M^{-1}, f_P)\zeta^{\gamma\nu}q_\eta^\nu \end{array}$$

where $t_{\mathfrak{p}}$ is one of the (fixed) representatives of the narrow class group of F representing the class of $t\mathfrak{p}$, and where q_η denote a representative in $q^{\mathfrak{p}^{-1}M_2}$ of the class $\eta \in q^{\mathfrak{p}^{-1}M_2}/q^{M_2}$. Finally

$$\begin{array}{rcl} (g_P|\pi_{2,\mathfrak{p}})(C_P) & = & pr^*(f_P|\pi_{2,\mathfrak{p}})(\mathrm{Tate}_{M_1,M_2}(q), \ldots, \zeta_p) \\ & = & |\mathcal{O}_F/t_{\mathfrak{p}}|^{-1} \sum_{\nu\in(\mathfrak{p}^{-1}M)^+} c(\nu\mathfrak{p}M^{-1}, f_P)q_\eta^\nu \end{array}$$

For brevity, let S denote the 'Gauss sum'

$$S \stackrel{\mathrm{def}}{=} \sum_{\gamma\in(\mathcal{O}_F/\mathfrak{p})^\times} \zeta^\gamma\psi_{P,\mathfrak{p}}(\gamma)$$

for $(M_1, M_2) = (\mathcal{O}_F, t^{-1})$. Then for $\nu \in (\mathfrak{p}^{-1}M)^+$ such that $\nu\mathfrak{p}M^{-1} \subset \mathcal{O}_F^+$ is not divisible by $\mathfrak{p}$,

$$S = \sum_{\gamma\in(\mathcal{O}_F/\mathfrak{p})^\times} \zeta^{\gamma\nu\mathfrak{p}M^{-1}}\psi_{P,\mathfrak{p}}(\gamma\nu\mathfrak{p}M^{-1}) = \psi_{P,\mathfrak{p}}(\nu\mathfrak{p}M^{-1}) \sum_{\gamma\in(\mathcal{O}_F/\mathfrak{p})^\times} \zeta^{\gamma\nu}\psi_{P,\mathfrak{p}}(\gamma).$$

It then follows that

$$\begin{array}{rl} & \sum_{\gamma\in(\mathcal{O}_F/\mathfrak{p})^\times}\psi_{P,\mathfrak{p}}(\gamma)(g_P|\pi_{2,\gamma})(C_P) \\ = & \sum_{\gamma\in(\mathcal{O}_F/\mathfrak{p})^\times}\psi_{P,\mathfrak{p}}(\gamma)|\mathcal{O}_F/t_\mathfrak{p}|^{-1}\sum_{\nu\in(\mathfrak{p}^{-1}M)^+}c(\nu\mathfrak{p}M^{-1},f_P)\zeta^{\gamma\nu}q_\eta^\nu \\ = & |\mathcal{O}_F/t_\mathfrak{p}|^{-1}\sum_{\nu\in(\mathfrak{p}^{-1}M)^+}c(\nu\mathfrak{p}M^{-1},f_P)q_\eta^\nu\sum_{\gamma\in(\mathcal{O}_F/\mathfrak{p})}\psi_{P,\mathfrak{p}}(\gamma)\zeta^{\gamma\nu} \\ = & S|\mathcal{O}_F/t_\mathfrak{p}|^{-1}\sum_{\nu\in(\mathfrak{p}^{-1}M)^+,\mathfrak{p}\nmid\nu\mathfrak{p}M^{-1}}c(\nu\mathfrak{p}M^{-1},f_P)\psi_{P,\mathfrak{p}}^{-1}(\nu\mathfrak{p}M^{-1})q_\eta^\nu \\ = & S|\mathcal{O}_F/t_\mathfrak{p}|^{-1}\sum_{\nu\in(\mathfrak{p}^{-1}M)^+,\mathfrak{p}\nmid\nu\mathfrak{p}M^{-1}}c(\nu\mathfrak{p}M^{-1},f_{P-\{\mathfrak{p}\}})q_\eta^\nu \\ = & S|\mathcal{O}_F/t_\mathfrak{p}|^{-1}\sum_{\nu\in(\mathfrak{p}^{-1}M)^+}c(\nu\mathfrak{p}M^{-1},f_{P-\{\mathfrak{p}\}})q_\eta^\nu \\ & -S|\mathcal{O}_F/t_\mathfrak{p}|^{-1}\sum_{\nu\in(\mathfrak{p}^{-1}M)^+,\mathfrak{p}|\nu\mathfrak{p}M^{-1}}c(\nu\mathfrak{p}M^{-1},f_{P-\{\mathfrak{p}\}})q_\eta^\nu \\ = & S|\mathcal{O}_F/t_\mathfrak{p}|^{-1}\sum_{\nu\in(\mathfrak{p}^{-1}M)^+}c(\nu\mathfrak{p}M^{-1},f_{P-\{\mathfrak{p}\}})q_\eta^\nu \\ & -S|\mathcal{O}_F/t_\mathfrak{p}|^{-1}\sum_{\nu\in M^+}c(\nu M,f_{P-\{\mathfrak{p}\}})q_\eta^\nu \\ = & S(|\mathcal{O}_F/t_\mathfrak{p}|^{-1}\sum_{\nu\in(\mathfrak{p}^{-1}M)^+}c(\nu\mathfrak{p}M^{-1},f_{P-\{\mathfrak{p}\}})q_\eta^\nu-(U_\mathfrak{p}g_{P-\{\mathfrak{p}\}}|\pi_1)(C_P)) \\ = & S(g_{P-\{\mathfrak{p}\}}|\pi_{2,\mathfrak{p}}-\beta_\mathfrak{p}g_{P-\{\mathfrak{p}\}}|\pi_1)(C_P) \end{array}$$

By the connectedness of $(X_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p),K}[0,1))(0,1]_{S-P}$,

$$\sum_{\gamma\in(\mathcal{O}_F/\mathfrak{p})^\times}\psi_{P,\mathfrak{p}}(\gamma)(g_P|\pi_{2,\gamma})=S(g_{P-\{\mathfrak{p}\}}|\pi_{2,\mathfrak{p}}-\beta_\mathfrak{p}g_{P-\{\mathfrak{p}\}}|\pi_1)$$

on $(X_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p),K}[0,1))(0,1]_{S-P}$.
Let $(A,i,\lambda,\eta,\eta_{\mathrm{KM}},D_\mathfrak{p}=\langle Q_\mathfrak{p}\rangle)$ be a non-cuspidal point of $X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p),\mathrm{Iw}_\mathfrak{p},K}$ and let $P=\eta_{\mathrm{KM}}(1)=P^\mathfrak{p}\times P_\mathfrak{p}$ and $Q=P^\mathfrak{p}\times Q_\mathfrak{p}$. Then

$$|\mathcal{O}_F/\mathfrak{p}|\alpha_\mathfrak{p}g_P(A,i,\lambda,\eta,Q)-pr^*g_P(A/\langle P_\mathfrak{p}\rangle,\ldots,\overline{\eta},\overline{Q}))$$

where $\overline{\eta}:=\eta \bmod \langle P_\mathfrak{p}\rangle$, and $\overline{Q}:=Q \bmod \langle P_\mathfrak{p}\rangle$ is:

$$\begin{array}{rl} = & |\mathcal{O}_F/\mathfrak{p}|U_\mathfrak{p}g_P(A,i,\lambda,\eta,Q)-pr^*g_P(A/\langle P_\mathfrak{p}\rangle,\ldots,\overline{\eta},\overline{Q}) \\ = & \sum_{C_\mathfrak{p}\subset A[\mathfrak{p}],C_\mathfrak{p}\neq\langle P_\mathfrak{p}\rangle,\langle Q_\mathfrak{p}\rangle}pr^*g_P(A/C_\mathfrak{p},\ldots,(Q \bmod C_\mathfrak{p})) \\ = & \sum_{C_\mathfrak{p}=\langle\gamma P_\mathfrak{p}+Q_\mathfrak{p}\rangle,\gamma\in(\mathcal{O}_F/\mathfrak{p})^\times}pr^*g_P(A/\langle\gamma P_\mathfrak{p}+Q_\mathfrak{p}\rangle,\ldots,(Q \bmod \langle\gamma P_\mathfrak{p}+Q_\mathfrak{p}\rangle)) \\ = & \sum_{\gamma\in(\mathcal{O}_F/\mathfrak{p})^\times}\psi_{P,\mathfrak{p}}(-\gamma)pr^*g_P(A/\langle\gamma P_\mathfrak{p}+Q_\mathfrak{p}\rangle,\ldots,(P^\mathfrak{p}\times P_\mathfrak{p} \bmod \langle\gamma P_\mathfrak{p}+Q_\mathfrak{p}\rangle) \\ = & \psi_{P,\mathfrak{p}}(-1)\sum_{\gamma\in(\mathcal{O}_F/\mathfrak{p})^\times}\psi_{\mathfrak{p},P}(\gamma)(g_P|\pi_{2,\gamma})(A,\ldots,\eta,P) \\ = & \psi_{P,\mathfrak{p}}(-1)S(g_{P-\{\mathfrak{p}\}}|\pi_{2,\mathfrak{p}}-\beta_\mathfrak{p}g_{P-\{\mathfrak{p}\}}|\pi_1)(A,\ldots,\eta,P) \\ = & \psi_{P,\mathfrak{p}}(-1)S(pr^*g_{P-\{\mathfrak{p}\}}(A/\langle Q_\mathfrak{p}\rangle,\ldots,P \bmod \langle Q_\mathfrak{p}\rangle)-\beta_\mathfrak{p}g_{P-\{\mathfrak{p}\}}(A,\ldots,P) \\ = & \psi_{P,\mathfrak{p}}(-1)S((g_{P-\{\mathfrak{p}\}}|w_{\zeta_\mathfrak{p}})(A,\ldots,P^\mathfrak{p}\times(-Q_\mathfrak{p})) \\ & -\alpha_\mathfrak{p}^{-1}\psi_P^S(\mathfrak{p})|\mathcal{O}_F/\mathfrak{p}|pr^*(g_{P-\{\mathfrak{p}\}}|w_{\zeta_\mathfrak{p}})(A/\langle P_\mathfrak{p}\rangle,\ldots,\mathfrak{p}^{-1}\overline{\eta},(P^\mathfrak{p}\times(-Q_\mathfrak{p}) \bmod \langle P_\mathfrak{p}\rangle)) \\ = & S((g_{P-\{\mathfrak{p}\}}|w_{\zeta_\mathfrak{p}})(A,\ldots,Q)-|\mathcal{O}_F/\mathfrak{p}|^{-1}\alpha_\mathfrak{p}^{-1}pr^*(g_{P-\{\mathfrak{p}\}}|w_{\zeta_\mathfrak{p}})(A/\langle P_\mathfrak{p}\rangle,\ldots,\overline{Q}). \end{array}$$

Therefore

$$\begin{array}{rl} & (|\mathcal{O}_F/\mathfrak{p}|\alpha_\mathfrak{p}g_P-S(g_{P-\{\mathfrak{p}\}}|w_{\zeta_\mathfrak{p}}))(A,\ldots,Q) \\ = & (|\mathcal{O}_F/\mathfrak{p}|\alpha_\mathfrak{p})^{-1}pr^*(|\mathcal{O}_F/\mathfrak{p}|\alpha_\mathfrak{p}g_P-S(g_{P-\{\mathfrak{p}\}}|w_{\zeta_\mathfrak{p}}))(A/\langle P_\mathfrak{p}\rangle,\ldots,(Q \bmod \langle P_\mathfrak{p}\rangle)) \end{array}$$

It suffices to show that $|\mathcal{O}_F/\mathfrak{p}|\alpha_\mathfrak{p}g_P-S(g_{P-\{\mathfrak{p}\}}|w_{\zeta_\mathfrak{p}})$ is identically zero; in which case, one can glue g_P and $(|\mathcal{O}_F/\mathfrak{p}|\alpha_\mathfrak{p})^{-1}S(g_{P-\{\mathfrak{p}\}}|w_{\zeta_\mathfrak{p}})$ as desired. Showing that it is identically zero is exactly as in [3]. $\square$

15.3 THE $\Gamma_1(p^r)$, $r \geq 2$, CASE

THEOREM 24 *Let S denote the set S_{P} of places of F above p. For any set $P \subseteq S$, let $f_P \in H^0(X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),K}[0,1), L_k)^{\overline{\mathbf{U}}_r}$ be an overconvergent modular form of weight $k = \sum_{\tau\in\mathrm{Hom}(F,K)} k\tau \in \mathbf{Z}$ and of level $\Gamma_1(\mathfrak{n}p^r)$.*
Suppose that, for every $P \subseteq S$, f_P has a character $\psi_P^S\psi_{S,P}$ of $(\mathcal{O}_F/\mathfrak{n}p^r)^\times \simeq (\mathcal{O}_F/\mathfrak{n})^\times\times(\mathcal{O}_F/p^r)^\times$. Suppose furthermore that f_P is an eigenform for $U_\mathfrak{p}$ with non-zero eigenvalue for every $\mathfrak{p} \in S$. Suppose finally that, for every $P \subseteq S$,

- *$c(\mathcal{O}_F, f_P) = 1$;*
- *$c(\mathfrak{m}, f_P) = 0$ if $\mathfrak{m}$ and $\mathfrak{n}$ are not coprime;*
- *for every $\mathfrak{p} \in P$, $c(\mathfrak{m}, f_P) = \psi_{\mathfrak{p},P}(\mathfrak{m})c(\mathfrak{m}, f_{P-\{\mathfrak{p}\}})$ for every ideal $\mathfrak{m}$ coprime to $\mathfrak{n}p$, where $\psi_{\mathfrak{p},P}$ is the $\mathfrak{p}$-component of $\psi_{S,P}$.*

Then the f_P are classical Hilbert modular forms in $H^0(X^{\mathrm{an}}_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),K}, L_k)^{\overline{\mathbf{U}}_r}$.

Proof. As in the previous subsection, we shall prove the theorem by induction. For every subset P of S, let g_P denote $f_P|w_P \in H^0(X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),K}[0,1))(0,1]_{S-P}, L_k)$. We shall prove that f_S is classical. Fix an integer $0 \leq n \leq |S|$ and suppose that the g_P with $P \subseteq S$ such that $|P| \geq n$ glue together to define sections, which will again be denoted by g_P, over

$$\bigcup_{P\subset S, |P|\geq n} X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),K}[0,1)(0,1]_{S-P}.$$

Fix a subset $P \subseteq S$ with $\#P = n$, and fix $\mathfrak{p} \in P$. It suffice to show that g_P and (a constant multiple of) $g_{P-\{\mathfrak{p}\}}$ glue.
Let C denote a point $(\mathrm{Tate}_{M_1,M_2}(q), i, \lambda, \eta, P)$ around a cusp $(M_1, M_2) = (\mathcal{O}_F, t^{-1})$ where

$$P = \eta_{\mathrm{KM}}(1) = P^\mathfrak{p} \times P_\mathfrak{p} \in \mathrm{Tate}_{M_1,M_2}(q)(\mathcal{O}\otimes_{\mathbf{Z}}\mathbf{Z}((q^M)))$$

where $P^\mathfrak{p} \stackrel{\mathrm{def}}{=} \prod_{\mathfrak{q}|p,\mathfrak{q}\neq\mathfrak{p}} \zeta_{1,\mathfrak{q}}$ and $P_\mathfrak{p} \stackrel{\mathrm{def}}{=} \zeta_{1,\mathfrak{p}}\eta^{\mathrm{et}}_{1,\mathfrak{p}}$.
For brevity, let μ denote $\zeta_r^{p^{r-1}}$, and $\mu_\mathfrak{p}$ its $\mathfrak{p}$-component.
We shall compute q-expansions of g_P and $g_{P-\{\mathfrak{p}\}}$ at the cusp $C_P \stackrel{\mathrm{def}}{=} w^*_{S-P}C$.
Let $\alpha_\mathfrak{p}$ denote the $U_\mathfrak{p}$-eigenvalue of f_P.

$$\begin{array}{cl}
 & |\mathcal{O}_F/\mathfrak{p}|\alpha_\mathfrak{p} g_P(C_P) \\
= & |\mathcal{O}_F/\mathfrak{p}|U_\mathfrak{p} g_P(C_P) \\
= & \sum_{C_\mathfrak{p}\subset\mathrm{Tate}(q)[\mathfrak{p}], C_\mathfrak{p}\neq\langle\mu_\mathfrak{p}\rangle} pr^* f_P(\mathrm{Tate}_{M_1,M_2}(q)/C_\mathfrak{p}, \ldots, (P \bmod C_\mathfrak{p})) \\
= & \sum_{\gamma\in(\mathcal{O}_F/\mathfrak{p})^\times} pr^* f_P((GL_1\otimes_{\mathbf{Z}}\mathfrak{d}^{-1}M_1^{-1})/(\mu_\mathfrak{p}^\gamma q_\eta)^{\mathfrak{p}^{-1}M_2}, \ldots, P) \\
= & \sum_\gamma pr^* f_P((GL_1\otimes_{\mathbf{Z}}\mathfrak{d}^{-1}M_1^{-1})/(\mu_\mathfrak{p}^\gamma q_\eta)^{\mathfrak{p}^{-1}M_2}, \ldots, P^\mathfrak{p}\times\{\zeta_{r,\mathfrak{p}}\mu_\mathfrak{p}^{-\gamma}\}) \\
= & \sum_\gamma pr^* f_P((GL_1\otimes_{\mathbf{Z}}\mathfrak{d}^{-1}M_1^{-1})/(\mu_\mathfrak{p}^\gamma q_\eta)^{\mathfrak{p}^{-1}M_2}, \ldots, P^\mathfrak{p}\times\{\zeta_{r,\mathfrak{p}}^{1-p^{r-1}\gamma}\}) \\
= & \sum_\gamma \psi_{\mathfrak{p},P}((1-p^{r-1}\gamma)\mathfrak{p}M^{-1}) pr^* f_P((GL_1\otimes_{\mathbf{Z}}\mathfrak{d}^{-1}M_1^{-1})/(\mu_\mathfrak{p}^\gamma q_\eta)^{\mathfrak{p}^{-1}M_2}, \ldots, \zeta_r) \\
= & \sum_\gamma \psi_{\mathfrak{p},P}((1-p^{r-1}\gamma)\mathfrak{p}M^{-1}) \sum_{\nu\in(\mathfrak{p}^{-1}M)^+} c(\nu\mathfrak{p}M^{-1}, f_P)(\mu_\mathfrak{p}^\gamma q_\eta)^\nu \\
= & \sum_{\nu\in(\mathfrak{p}^{-1}M)^+} c(\nu\mathfrak{p}M^{-1}, f_P) q_\eta^\nu \sum_\gamma \psi_{\mathfrak{p},P}((1-p^{r-1}\gamma)\mathfrak{p}M^{-1})\mu_\mathfrak{p}^{\gamma\nu}.
\end{array}$$

We know that $\psi_{S,P}$ has a conductor p^r, and hence $\psi_{S,P}((1+p^{r-1})\mathfrak{p}M^{-1}) = \mu^{\nu_1}$ for some integer $0 < \nu_1 < p$ (not that $1+p^{r-1}$ is thought of as an element of $\mathfrak{p}^{-1}M_2 \overset{\mathfrak{p}M_2^{-1}M_1}{\twoheadrightarrow} M_1 \twoheadrightarrow M_1/\mathfrak{p}^r M_1 \simeq \mathcal{O}_F/\mathfrak{p}^r$); it therefore follows that $\psi_{\mathfrak{p},P}((1+p^{r-1})\mathfrak{p}M^{-1}) = \mu_{\mathfrak{p}}^{\nu_1}$. In particular, $\psi_{\mathfrak{p},P}((1-p^{r-1}\gamma)\mathfrak{p}M^{-1}) = \mu_{\mathfrak{p}}^{-\gamma\nu_1}$. Hence

$$\sum_{\gamma} \psi_{\mathfrak{p},P}((1-p^{r-1}\gamma)\mathfrak{p}M^{-1})\mu_{\mathfrak{p}}^{\gamma\nu} = \sum_{\gamma} \mu_{\mathfrak{p}}^{\gamma(\nu-\nu_1)} = \begin{cases} |\mathcal{O}_F/\mathfrak{p}| \text{ if } \mathfrak{p}|(\nu-\nu_1) \\ 0 \text{ otherwise} \end{cases}$$

Therefore,

$$g_P(C_P) = (\alpha_{\mathfrak{p}}|\mathcal{O}_F/\mathfrak{p}|)^{-1}|\mathcal{O}_F/\mathfrak{p}| \sum_{\nu\in(\mathfrak{p}^{-1}M)^+,\mathfrak{p}|(\nu-\nu_1)} c(\nu\mathfrak{p}M^{-1}, f_P)q_\eta^\nu.$$

We now calculate the q-expansion of $g_{P-\{\mathfrak{p}\}}|w_{\zeta_{\mathfrak{p}}}$ at C_P. Firstly, note that

$$(g_{P-\{\mathfrak{p}\}}|w_{\zeta_{\mathfrak{p}}})(C_P) = pr^* g_{P-\{\mathfrak{p}\}}(\mathrm{Tate}_{M_1,M_2}(q)/(\zeta_{r,\mathfrak{p}}\eta_{1,\mathfrak{p}}^{\mathrm{et}}), \ldots, P^{\mathfrak{p}} \times Q_{\mathfrak{p}})$$

where $Q_{\mathfrak{p}}$ is defined by $\langle \zeta_{r,\mathfrak{p}}\eta_{1,\mathfrak{p}}^{\mathrm{et}}, Q_{\mathfrak{p}}\rangle = \zeta_{\mathfrak{p}}$. Tensoring over $\mathcal{O}_F$ with $\mathfrak{p}^{r-1}$ on $GL_1 \otimes \mathfrak{d}^{-1}M_1^{-1}$ induces an isomorphism

$$\begin{array}{ll} & (GL_1 \otimes \mathfrak{d}^{-1}M_1^{-1})/\langle q^{M_2}, \zeta_{r,\mathfrak{p}}q\rangle \\ \simeq & (GL_1 \otimes \mathfrak{d}^{-1}M_1^{-1})/\langle q^{\mathfrak{p}^{r-1}M_2}, \zeta_{r-(r-1),\mathfrak{p}}q^{\mathfrak{p}^{r-2}M_2}\rangle \\ \simeq & (GL_1 \otimes \mathfrak{d}^{-1}M_1^{-1})/(\zeta_{1,\mathfrak{p}}q)^{\mathfrak{p}^{r-2}M_2} \end{array}$$

The HBAV

$$\mathrm{Tate}_{M_1,\mathfrak{p}^{r-2}M_2}(\zeta_{1,\mathfrak{p}}q) \overset{\mathrm{def}}{=} (GL_1 \otimes \mathfrak{d}^{-1}M_1^{-1})/(\zeta_{1,\mathfrak{p}}q)^{\mathfrak{p}^{r-2}M_2}$$

is naturally $t\mathfrak{p}^{2-r}(\simeq (\mathfrak{p}^{r-2}M_2)^{-1}M_1)$-polarised, and comes equipped with the level structure $\langle\mathfrak{p}\rangle^{r-1}\eta$ and the point $P^{\mathfrak{p}} \times \{\eta_{1,\mathfrak{p}}^{\mathrm{et}}\}$ of order $|\mathcal{O}_F/p^r|$; it defines a point of $X^{[r-1]}_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),\tau}$. For a point (A,i,λ,η,P) of $X^{[r-1]}_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p^r),\tau}$,

$$\begin{array}{ll} & (|\mathcal{O}_F/\mathfrak{p}|\beta_{\mathfrak{p}})^{r-1} g_{P-\{\mathfrak{p}\}}(A,i,\lambda,\eta,P) \\ = & (|\mathcal{O}_F/\mathfrak{p}|U_{\mathfrak{p}})^{r-1} g_{P-\{\mathfrak{p}\}}(A,i,\lambda,\eta,P) \\ = & \sum_{C_{\mathfrak{p}}\subset A[\mathfrak{p}],|C_{\mathfrak{p}}|=|\mathcal{O}_F/\mathfrak{p}^{r-1}|,C_{\mathfrak{p}}\cap P=\{1\}} pr^* g_{P-\{\mathfrak{p}\}}(A/C_{\mathfrak{p}},\ldots,(P \bmod C_{\mathfrak{p}})) \end{array}$$

In which case, observe that $(A/C_{\mathfrak{p}},\ldots,(P \bmod C_{\mathfrak{p}})) \in X^{[0]}_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p^r),\tau}$ and that it allows one to extend finite slope $U_{\mathfrak{p}}$-eigenforms over $X^{[0]}_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p^r),\tau}$ to $U_{\mathfrak{p}}$-eigenforms over $X^{[r-1]}_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p^r),\tau}$ by 'analytic continuation'. If we let

$$(A,i,\lambda,\eta,P) = (\mathrm{Tate}_{M_1,\mathfrak{p}^{r-2}M_2}(\zeta_{1,\mathfrak{p}}q),\ldots,\langle\mathfrak{p}\rangle^{r-1}\eta, P^{\mathfrak{p}} \times \{\eta_{1,\mathfrak{p}}^{\mathrm{et}}\})$$

then the cyclic subgroups $C_{\mathfrak{p}}$ of order $|\mathcal{O}_F/\mathfrak{p}^{r-1}|$, disjoint from the subgroup of order $|\mathcal{O}_F/\mathfrak{p}|$ generated by $\eta_{1,\mathfrak{p}}^{\mathrm{et}}$, are of the form

$(\zeta_{r,\mathfrak{p}}\zeta_{r-1,\mathfrak{p}}^{\nu_{r-1}}\eta_{1,\mathfrak{p}}^{\mathrm{et},\mathfrak{p}^{-1}M_2})/(\zeta_{1,\mathfrak{p}}q^{\mathfrak{p}^{r-2}M_2})$ for $\nu_{r-1} \in M_1/\mathfrak{p}^{r-1}M_1 \simeq \mathcal{O}_F/\mathfrak{p}^{r-1}$. Then

$$\begin{array}{rl} & ((GL_1 \otimes \mathfrak{d}^{-1}M_1^{-1})/(\zeta_{1,\mathfrak{p}}q^{\mathfrak{p}^{r-2}M_2}))/((\zeta_{r,\mathfrak{p}}\zeta_{r-1,\mathfrak{p}}^{\nu_{r-1}}\eta_{1,\mathfrak{p}}^{\mathrm{et},\mathfrak{p}^{-1}M_2})/(\zeta_{1,\mathfrak{p}}q^{\mathfrak{p}^{r-2}M_2})) \\ \simeq & (GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1}M_1)/(\zeta_{r,\mathfrak{p}}\zeta_{r-1,\mathfrak{p}}^{\nu_{r-1}}\eta_{1,\mathfrak{p}}^{\mathrm{et},\mathfrak{p}^{-1}M_2}) \\ \simeq & (GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1}M_1)/(\zeta_{r,\mathfrak{p}}^{1+p\nu_{r-1}}q_\eta)^{\mathfrak{p}^{-1}M_2}, \end{array}$$

where q_η is a representative in $q^{\mathfrak{p}^{-1}M_2}$ of $\eta_{1,\mathfrak{p}}^{\mathrm{et}} \in q^{\mathfrak{p}^{-1}M_2}/q^{M_2}$, is naturally $(\mathfrak{p}^{-1}M_2)^{-1}M_1 \simeq \mathfrak{p}t$-polarised. Then there exists a non-zero constant κ_1 such that

$$\begin{array}{l} (g_{P-\{\mathfrak{p}\}}|w_{\zeta_\mathfrak{p}})(C_P) \\ = \sum_{\nu_{r-1}} pr^* g_{P-\{\mathfrak{p}\}}((GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1}M_1)/\zeta_{r,\mathfrak{p}}\zeta_{r-1,\mathfrak{p}}^{\nu_{r-1}}\eta_{1,\mathfrak{p}}^{\mathrm{et},\mathfrak{p}^{-1}M_2}), \ldots, \langle\mathfrak{p}\rangle^{r-1}\eta, P^{\mathfrak{p}} \times \{\eta_{1,\mathfrak{p}}^{\mathrm{et}}\})) \\ = \psi^S(\mathfrak{p}^{r-1}) \sum_{\nu_{r-1}} pr^* g_{P-\{\mathfrak{p}\}}((GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1}M_1)/(\zeta_{r,\mathfrak{p}}^{1+p\nu_{r-1}}\eta_{1,\mathfrak{p}}^{\mathrm{et},\mathfrak{p}^{-1}M_2}), \ldots, \eta, P^{\mathfrak{p}} \times \{q_{1,\mathfrak{p}}^{\mathrm{et},-1}\})) \\ = \psi^S(-\mathfrak{p}^{r-1}) \sum_{\nu_{r-1}} pr^* g_{P-\{\mathfrak{p}\}}((GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1}M_1)/(\zeta_{r,\mathfrak{p}}^{1+p\nu_{r-1}}\eta_{1,\mathfrak{p}}^{\mathrm{et},\mathfrak{p}^{-1}M_2}), \ldots, \eta, P^{\mathfrak{p}} \times \{\zeta_{r,\mathfrak{p}}^{1+p\nu_{r-1}}\})) \\ = \kappa_1 \sum_{\nu_{r-1}} \psi_{\mathfrak{p},P}((1+p\nu_{r-1})\mathfrak{p}M^{-1})^{-1} pr^* g_{P-\{\mathfrak{p}\}}((GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1}M_1)/(\zeta_{r,\mathfrak{p}}^{1+p\nu_{r-1}}\eta_{1,\mathfrak{p}}^{\mathrm{et},\mathfrak{p}^{-1}M_2}), \ldots, \zeta_r)) \\ = \kappa_1 \sum_{\nu_{r-1}} \psi_{\mathfrak{p},P}((1+p\nu_{r-1})\mathfrak{p}M^{-1})^{-1} \sum_{\nu\in(\mathfrak{p}^{-1}M)^+} c(\nu\mathfrak{p}M^{-1}, f_{P-\{\mathfrak{p}\}})(\zeta_{r,\mathfrak{p}}^{1+p\nu_{r-1}}q_\eta)^\nu \\ = \kappa_1 \sum_\nu c(\nu\mathfrak{p}M^{-1}, f_{P-\{\mathfrak{p}\}})q_\eta^\nu \sum_{\nu_{r-1}} \psi_{\mathfrak{p},P}((1+p\nu_{r-1})\mathfrak{p}M^{-1})^{-1}\zeta_{r,\mathfrak{p}}^{(1+p\nu_{r-1})\nu} \end{array}$$

where ν_{r-1} ranges over $\in (\mathcal{O}_F/\mathfrak{p}^{r-1})$. For brevity, let $S_\nu \stackrel{\mathrm{def}}{=} \sum_{\nu_{r-1}\in(\mathcal{O}_F/\mathfrak{p}^{r-1})} \psi_{\mathfrak{p},P}((1+p\nu_{r-1})\mathfrak{p}M^{-1})^{-1}\zeta_{r,\mathfrak{p}}^{(1+p\nu_{r-1})\nu}$. As in the proof of Theorem 11.1 in [3], one can deduce that

$$S_\nu = \mu_\mathfrak{p}^{\nu-\nu_1}S_\nu$$

where ν_1 is defined by $\psi_{\mathfrak{p},P}((1+p^{r-1})\mathfrak{p}M^{-1}) = \mu_\mathfrak{p}^{\nu_1}$, and therefore $S_\nu = 0$ unless $\mathfrak{p}|(\nu-\nu_1)$; one also deduces that, for $\nu \in (\mathfrak{p}^{-1}M)^+$ such that $\mathfrak{p}|(\nu-\nu_1)$,

$$S_{\nu\nu'} = \psi_{\mathfrak{p},P}(\nu'\mathfrak{p}M^{-1})S_\nu$$

for $\nu' \in (\mathfrak{p}^{-1}M)^+$ such that $\nu'\mathfrak{p}M^{-1} \equiv 1 \bmod \mathfrak{p}$ and therefore $S_{\nu\nu'}/\psi_{\mathfrak{p},P}((\nu\mathfrak{p}M^{-1})(\nu'\mathfrak{p}M^{-1})) = S_\nu/\psi_{\mathfrak{p},P}(\nu\mathfrak{p}M^{-1})$. Consequently, there is a non-zero constant κ_2 such that

$$\begin{array}{rl} (g_{P-\{\mathfrak{p}\}}|w_{\zeta_\mathfrak{p}})(C_P) = & \kappa_1 \sum_{\nu\in(\mathfrak{p}^{-1}M)^+,\mathfrak{p}|(\nu-\nu_2)} c(\nu\mathfrak{p}^{-1}M, f_{P-\{\mathfrak{p}\}})S_\nu q_\eta^\nu \\ = & \kappa_2 \sum_{\nu,\mathfrak{p}|(\nu-\nu_2)} c(\nu\mathfrak{p}^{-1}M, f_{P-\{\mathfrak{p}\}})\psi_{\mathfrak{p},P}(\nu\mathfrak{p}M^{-1})q_\eta^\nu \\ = & \kappa_2 \sum_{\nu,\mathfrak{p}|(\nu-\nu_2)} c(\nu\mathfrak{p}^{-1}M, f_P)q_\eta^\nu. \end{array}$$

Therefore $\alpha_\mathfrak{p}g_P$ and $\kappa_2^{-1}g_{P-\{\mathfrak{p}\}}$ agree at C_P and hence f_P and $(\alpha_\mathfrak{p}\kappa_2)^{-1}g_{P-\{\mathfrak{p}\}}$ glue together. □

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