Integration of Vector Fields on Smooth and Holomorphic Supermanifolds

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Abstract. We give a new and self-contained proof of the existence and unicity of the flow for an arbitrary (not necessarily homogeneous) smooth vector field on a real supermanifold, and extend these results to the case of holomorphic vector fields on complex supermanifolds. Furthermore we discuss local actions associated to super vector fields, and give several examples and applications, as, e.g., the construction of an exponential morphism for an arbitrary finite-dimensional Lie supergroup.

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1 Introduction

The natural problem of integrating vector fields to obtain appropriate "flow maps" on supermanifolds is considered in many articles and monographs (compare, e.g., [17], [2], [19], [14] and [3]) but a "general answer" was to our knowledge only given in the work of J. Monterde and co-workers (see [12] and [13]). Let us consider a supermanifold $\mathcal{M} = (M, \mathcal{O}_\mathcal{M})$ together with a vector field X in $\mathcal{T}_\mathcal{M}(M)$, and an initial condition ϕ in $\mathrm{Mor}(\mathcal{S}, \mathcal{M})$, where $\mathcal{S} = (S, \mathcal{O}_\mathcal{S})$ is an arbitrary supermanifold and $\mathrm{Mor}(\mathcal{S}, \mathcal{M})$ denotes the set of morphisms from $\mathcal{S}$ to $\mathcal{M}$. The case of classical, ungraded, manifolds leads one to consider the following question: does there exist a "flow map" F defined on an open sub supermanifold $\mathcal{V} \subset \mathbb{R}^{1|1} \times \mathcal{S}$ and having values in $\mathcal{M}$ and an appropriate derivation on $\mathbb{R}^{1|1}$, $D = \partial_t + \partial_\tau + \tau(a\partial_t + b\partial_\tau)$, where $\partial_t = \frac{\partial}{\partial t}$, $\partial_\tau = \frac{\partial}{\partial \tau}$ and a, b are real numbers, such that the following equations are fulfilled

$$\begin{array}{lcl} D \circ F^* & = & F^* \circ X \\ F \circ \mathrm{inj}^{\mathcal{V}}_{\{0\}\times\mathcal{S}} & = & \phi\,. \end{array} \tag{1}$$

Of course, $\mathcal{V}$ should be a "flow domain", i.e. an open sub supermanifold of $\mathbb{R}^{1|1} \times \mathcal{S}$ such that $\{0\} \times S$ is contained in the body V of $\mathcal{V}$ and for x in S, the set $I_x \subset \mathbb{R}$ defined by $I_x \times \{x\} = (\mathbb{R} \times \{x\}) \cap V$ is an open interval. Furthermore $\mathrm{inj}^{\mathcal{V}}_{\{0\}\times\mathcal{S}}$ denotes the natural injection morphism of the closed sub supermanifold $\{0\} \times \mathcal{S}$ of $\mathcal{V}$ into $\mathcal{V}$. Of course, we could concentrate on the case $\mathcal{S} = \mathcal{M}$ and $\phi = \mathrm{id}_{\mathcal{M}}$, but it will be useful for our later arguments to state all results in this (formally) more general setting.

Though for homogeneous vector fields ($X = X_0$ or $X = X_1$) system (1) does always have a solution, in the general case ($X = X_0 + X_1$ with $X_0 \neq 0$ and $X_1 \neq 0$) the system is overdetermined. A simple example of an inhomogeneous vector field such that (1) is not solvable is given by $X = X_0 + X_1 = \left(\frac{\partial}{\partial x} + \xi\frac{\partial}{\partial \xi}\right) + \left(\frac{\partial}{\partial \xi} + \xi\frac{\partial}{\partial x}\right)$ on $\mathcal{M} = \mathbb{R}^{1|1}$. The crucial novelty of [13] is to consider instead of (1) the following modified, weakened, problem

$$\begin{array}{lcl} (\mathrm{inj}^{\mathbb{R}^{1|1}}_{\mathbb{R}})^* \circ D \circ F^* & = & (\mathrm{inj}^{\mathbb{R}^{1|1}}_{\mathbb{R}})^* \circ F^* \circ X \\ F \circ \mathrm{inj}^{\mathcal{V}}_{\{0\}\times\mathcal{S}} & = & \phi\,, \end{array} \tag{2}$$

where $\mathrm{inj}^{\mathbb{R}^{1|1}}_{\mathbb{R}} = \mathrm{inj}^{\mathbb{R}^{1|1}\times\mathcal{S}}_{\mathbb{R}\times\mathcal{S}}$ is again the natural injection (and where the above more general derivation D could be replaced by $\partial_t + \partial_\tau$ since $(\mathrm{inj}^{\mathbb{R}^{1|1}}_{\mathbb{R}})^*$ annihilates germs of superfunctions of the type $\tau \cdot f$, $f \in \mathcal{O}_{\mathbb{R}\times\mathcal{S}}$).

In [13] (making indispensable use of [12]) it is shown that in the smooth case (2) has a unique maximal solution F, defined on the flow domain $\mathcal{V} = (V, \mathcal{O}_{\mathbb{R}^{1|1}\times\mathcal{S}}|_V)$, where $V \subset \mathbb{R} \times S$ is the maximal flow domain for the flow of the reduced vector field $\widetilde{X} = \widetilde{X_0}$ on M with initial condition $\widetilde{\phi}$. Since the results of [12] are obtained by the use of a Batchelor model for $\mathcal{M}$, i.e. a real vector bundle $E \to M$ such that $\mathcal{M} \cong (M, \Gamma^\infty_{\Lambda E^*})$, and a connection on E, we follow here another road, closer to the classical, ungraded, case and also applicable in the case of complex supermanifolds and holomorphic vector fields.

Our new method of integrating smooth vector fields on a supermanifold in Section 2 consists in first locally solving a finite hierarchy of ordinary differential equations, and is here partly inspired by the approach of [3], where the case of homogeneous super vector fields on compact supermanifolds is treated. We then show existence and unicity of solutions of (2) on smooth supermanifolds and easily deduce the results of [13] from our Lemmata 2.1 and 2.2.

A second beautiful result of [13] (more precisely, Theorem 3.6 of that reference) concerns the question if the flow F solving (2) fulfills "flow equations", as in the ungraded case. Hereby, we mean the existence of a Lie supergroup structure on $\mathbb{R}^{1|1}$ such that F is a local action of $\mathbb{R}^{1|1}$ on $\mathcal{M}$ (in case $\mathcal{S} = \mathcal{M}, \phi = \mathrm{id}_{\mathcal{M}}$). Again, the answer is a little bit unexpected: in general, given X and its flow $F : \mathbb{R}^{1|1} \times \mathcal{M} \supset \mathcal{V} \to \mathcal{M}$, there is no Lie supergroup structure on $\mathbb{R}^{1|1}$ such that F is a local $\mathbb{R}^{1|1}$-action (with regard to this structure). The condition for the existence of such a structure on $\mathbb{R}^{1|1}$ is equivalent to the condition that (2) holds without the post-composition with $(\mathrm{inj}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^*$, i.e. the overdetermined system (1) is solvable. Furthermore, both conditions cited are equivalent to the condition that $\mathbb{R}X_0 \oplus \mathbb{R}X_1$ is a sub Lie superalgebra of $\mathcal{T}_{\mathcal{M}}(M)$, the Lie superalgebra of all vector fields on $\mathcal{M}$.

After discussing Lie supergroup structures and right invariant vector fields on $\mathbb{R}^{1|1}$, as well as local Lie group actions in the category of supermanifolds in general, we show in Section 3 the equivalence of the above three conditions, already given in [13]. We include our proof here notably in order to be able to apply it in the holomorphic case in Section 5 (see below) by simply indicating how to adapt it to this context. Let us nevertheless observe that our result is slightly more general since we do not need to ask for any normalization of the supercommutators between X_1 and X_0 resp. X_1, thus giving the criterion some extra flexibility in applications.

In Section 4, we give several examples of vector fields on supermanifolds, homogeneous and inhomogeneous, and explain their integration to flows. Notably, we construct an exponential morphism for an arbitrary finite-dimensional Lie supergroup, via a canonically defined vector field and its flow. We comment here also on the integration of what are usually called "(infinitesimal) supersymmetries" in physics, i.e., purely odd vector fields having non-vanishing self-commutators.

Finally, in Section 5 we adapt our method to obtain flows of vector fields (compare Section 2 and notably Lemma 2.1) to the case of holomorphic vector fields on holomorphic supermanifolds. To avoid monodromy problems one has, of course, to take care of the topology of the flow domains, and maximal flow domains are -as already in the ungraded holomorphic case- no more unique. Otherwise the analogues of all results in Section 2 and 3 continue to hold in the holomorphic setting.

Throughout the whole article we will work in the ringed space-approach to supermanifolds (see, e.g., [9], [10], [11] and [15] for detailed accounts of this approach). Given two supermanifolds $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ and $\mathcal{N} = (N, \mathcal{O}_{\mathcal{N}})$, a "morphism" $\phi = (\widetilde{\phi}, \phi^*) : \mathcal{M} \to \mathcal{N}$ is thus given by a continuous map $\widetilde{\phi} : M \to N$ between the "bodies" of the two supermanifolds and a sheaf homomophism $\phi^* : \mathcal{O}_{\mathcal{N}} \to \widetilde{\phi}_* \mathcal{O}_{\mathcal{M}}$. The topological space M comes canonically with a sheaf $\mathcal{C}^\infty_M = \mathcal{O}_{\mathcal{M}}/\mathcal{J}$, where $\mathcal{J}$ is the ideal sheaf generated by the germs of odd superfunctions, such that $(M, \mathcal{C}^\infty_M)$ is a smooth real manifold. Then $\widetilde{\phi}$ is a smooth map from $(M, \mathcal{C}^\infty_M)$ to $(N, \mathcal{C}^\infty_N)$. Let us recall that a (super) vector field on $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ is, by definition, an element of the Lie superalgebra $\mathcal{T}_{\mathcal{M}}(M) = (\mathrm{Der}_{\mathbb{R}}(\mathcal{O}_{\mathcal{M}}))(M)$ and that X always induces a smooth vector field $\widetilde{X}$ on $(M, \mathcal{C}^\infty_M)$. For p in M and $f + \mathcal{J}_p \in (\mathcal{C}^\infty_M)_p = (\mathcal{O}_{\mathcal{M}}/\mathcal{J})_p$ one defines $\widetilde{X}_p(f + \mathcal{J}_p) = X_0(f)(p)$, where X_0 is the even part of X and for $g \in (\mathcal{O}_{\mathcal{M}})_p$, $g(p) \in \mathbb{R}$ is the value of g in the point p of M.

2 Flow of a vector field on a real supermanifold

In this section we give our main result on the integration of general (i.e. not necessarily homogeneous) vector fields by a new method, avoiding auxiliary choices of Batchelor models and connections, as in [12]. Our more direct approach is inspired, e.g., by [3], where the case of homogeneous vector fields on compact manifolds is treated, and it can be adapted to the holomorphic case (see Section 4).

For the sake of readability we will often use the following shorthand: if $\mathcal{P}$ is a supermanifold, we write $\mathrm{inj}\,{}^{\mathbb{R}^{1|1}}_{\mathbb{R}}$ for $\mathrm{inj}\,{}^{\mathbb{R}^{1|1}\times\mathcal{P}}_{\mathbb{R}\times\mathcal{P}}$. Furthermore, the canonical coordinates of $\mathbb{R}^{1|1}$ will be denoted by t and τ, with ensueing vector fields $\partial_t = \frac{\partial}{\partial t}$ and $\partial_\tau = \frac{\partial}{\partial \tau}$.

Lemma 2.1. *Let $\mathcal{U} \subset \mathbb{R}^{m|n}$ and $\mathcal{W} \subset \mathbb{R}^{p|q}$ be superdomains, $X \in \mathcal{T}_{\mathcal{W}}(W)$ be a super vector field on $\mathcal{W}$ (not necessarily homogeneous) and $\phi \in Mor(\mathcal{U}, \mathcal{W})$, and $t_0 \in \mathbb{R}$. Let furthermore $H : V \to W$ be the maximal flow of $\widetilde{X} \in \mathcal{X}(W)$, i.e. $\partial_t \circ H^* = H^* \circ \widetilde{X}$, subject to the initial condition $H(t_0, \cdot) = \widetilde{\phi} : U \to W$. Let now $\mathcal{V}$ be $(V, \mathcal{O}_{\mathbb{R}^{1|1}\times\mathcal{U}}|_V)$ and (t, τ) the canonical coordinates on $\mathbb{R}^{1|1}$, then there exists a unique $F : \mathcal{V} \to \mathcal{W}$ such that*

$$(inj\,{}^{\mathbb{R}^{1|1}}_{\mathbb{R}})^* \circ (\partial_t + \partial_\tau) \circ F^* = (inj\,{}^{\mathbb{R}^{1|1}}_{\mathbb{R}})^* \circ F^* \circ X \quad and \tag{3}$$

$$F \circ inj\,{}^{\mathcal{V}}_{\{t_0\}\times\mathcal{U}} = \phi. \tag{4}$$

Moreover, $\widetilde{F} : V \to W$ equals the underlying classical flow map H of the vector field $\widetilde{X}$ with initial condition $\widetilde{\phi}$.

Proof. Let $(u_i) = (x_i, \xi_r)$ and $(w_j) = (y_j, \eta_s)$ denote the canonical coordinates on $\mathbb{R}^{m|n}$ and $\mathbb{R}^{p|q}$, respectively. Then there exist smooth functions $a^j_J \in \mathcal{C}^\infty_{\mathbb{R}^p}(W)$ such that

$$X = \sum_{j=1}^{p+q} \left(\sum_J a^j_J(y) \eta^J \right) \partial_{w_j},$$

where $J=(\beta_1,\ldots,\beta_q)$ runs over the index set $\{0,1\}^q$ and $\eta^J=\prod_{s=1}^{q}\eta_s^{\beta_s}$. We then have, of course,

$$X_0=\sum_j\left(\sum_{|J|=|w_j|}a_J^j(y)\eta^J\right)\partial_{w_j}\text{ resp. }X_1=\sum_j\left(\sum_{|J|=|w_j|+1}a_J^j(y)\eta^J\right)\partial_{w_j}.$$

Here, $|J|$ equals $\beta_1+\cdots+\beta_q$ mod 2 and $|w_j|$ is the parity of the coordinate function w_j. The morphism F determines and is uniquely determined by functions $f_I^j,g_I^j\in\mathcal{C}^\infty_{\mathbb{R}\times\mathbb{R}^m}(V)$ fulfilling for each $j\in\{1,\ldots,p+q\}$

$$F^*(w_j)=\sum_{|I|=|w_j|}f_I^j(t,x)\xi^I+\sum_{|I|=|w_j|+1}g_I^j(t,x)\tau\xi^I$$

(and $f_I^j=0$ if $|I|\neq|w_j|$, $g_I^j=0$ if $|I|\neq|w_j|+1$) as is well-known from the standard theory of supermanifolds (compare, e.g., Thm. 4.3.1 in [20]). Here and in the sequel $I=(\alpha_1,\ldots,\alpha_n)$ is an element of the set $\{0,1\}^n$ and ξ^I stands for the product $\xi_1^{\alpha_1}\cdot\xi_2^{\alpha_2}\cdots\xi_n^{\alpha_n}$. The notation $|I|$ again denotes the parity of I, i.e. $|I|=\alpha_1+\cdots+\alpha_n$ mod 2.

Equation (3) is equivalent to the following equations:

$$\begin{aligned}(\operatorname{inj}{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^*\circ\partial_t\circ F^* &= (\operatorname{inj}{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^*\circ F^*\circ X_0 &(5)\\ (\operatorname{inj}{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^*\circ\partial_\tau\circ F^* &= (\operatorname{inj}{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^*\circ F^*\circ X_1 &(6)\end{aligned}$$

Applying (5) to the canonical coordinate functions on $\mathcal{W}$, we get the following system, which is equivalent to (5):

$$\sum_{|I|=|w_j|}\partial_t f_I^j\cdot\xi^I=\sum_{|J|=|w_j|}\check{F}^*(a_J^j)\check{F}^*(\eta^J)\text{ for all }j\text{ in }\{1,\ldots,p+q\},\tag{7}$$

and (6) is equivalent to

$$\sum_{|I|=|w_j|+1}g_I^j\cdot\xi^I=\sum_{|J|=|w_j|+1}\check{F}^*(a_J^j)\check{F}^*(\eta^J)\text{ for all }j\text{ in }\{1,\ldots,p+q\},\tag{8}$$

where $\check{F}:=F\circ\operatorname{inj}{}_{\mathbb{R}}^{\mathbb{R}^{1|1}}:\check{\mathcal{V}}:=(V,\mathcal{O}_{\mathbb{R}\times\mathcal{U}}|_V)\to\mathcal{W}$. Let us immediately observe that the underlying smooth map of $\check{F}$ equals $\tilde{F}$, the smooth map underlying the morphism F.

Moreover the initial condition (4) is equivalent to

$$\sum_{|I|=|w_j|}f_I^j(t_0,x)\xi^I=\phi^*(w_j)\text{ for all }j\text{ in }\{1,\ldots,p+q\}.\tag{9}$$

We are going to show that (7) and (9) uniquely determine the functions f_I^j on V, i.e. the morphism $\check{F}$. Then the functions g_I^j are unambiguously given by

(8) on V, and the morphism F is fully determined.

Let us develop Equation (7) for a fixed j:

$$\sum_{|I|=|w_j|} \partial_t f_I^j \cdot \xi^I = \sum_{\substack{|J|=|w_j| \\ J=(\beta_1,\ldots,\beta_q)}} \check{F}^*(a_J^j) \prod_{s=1}^{q} \check{F}^*(\eta_s)^{\beta_s} \tag{10}$$

$$\text{and thus} \sum_{|I|=|w_j|} \partial_t f_I^j \cdot \xi^I = \sum_{\substack{|J|=|w_j| \\ J=(\beta_1,\ldots,\beta_q)}} \check{F}^*(a_J^j) \prod_{s=1}^{q} \left(\sum_{|L|=1} f_L^{p+s} \xi^L \right)^{\beta_s} \tag{11}$$

For fixed j this is an equation of Grassmann algebra-valued maps in the variables t and x that can be split in a system of scalar equations as follows. For $K = (\alpha_1, \ldots, \alpha_n) \in \{0,1\}^n$, we will denote the coefficient h_K in front of ξ^K of a superfunction $h = \sum_M h_M(t,x)\xi^M \in \mathcal{O}_{\mathbb{R}^{m+1|n}}$ compactly by $(h|\xi^K)$ in the sequel of this proof.

Let us first describe the coefficients for $\check{F}^*(a_J^j)$ in (11):

$$(\check{F}^*(a_J^j)|\xi^K) = 0 \quad \text{if } |K| = 1 \tag{12}$$

and, if $|K| = 0$,

$$(\check{F}^*(a_J^j)|\xi^K) = a_J^j \circ \widetilde{F} \quad \text{if } K = (0, \ldots, 0),$$

and

$$(\check{F}^*(a_J^j)|\xi^K) = \sum_{\mu=1}^{p} (\partial_{y_\mu} a_J^j)(\widetilde{F}(t,x)) \cdot f_K^\mu + R\left(a_J^j, (f_I^\nu)_{\nu, \deg(I)<\deg(K)}\right) \tag{13}$$

$$\text{if } \deg(K) > 0.$$

Here for $I = (\alpha_1, \ldots, \alpha_n)$, $\deg(I) = \alpha_1 + \cdots + \alpha_n$, and -more importantly- $R = R_{j,J,K}$ is a polynomial function in a_J^j and its derivatives in the y-variables up to order q included, and in the functions $\{f_I^\nu | 1 \le \nu \le p+q, 0 \le \deg(I) < \deg(K)\}$. Equation (12) is obvious since a_J^j is an even function, whereas equation (13) can be deduced from standard analysis on superdomains. More precisely, let a be a smooth function on $\mathbb{R}^p$ and $\psi : \mathbb{R}^{m+1|n} \to \mathbb{R}^{p|q}$ a morphism (of course to be applied to $a = a_J^j, \psi = \check{F}$). Then we can develop $\psi^*(a)$ as follows (compare the proof of Theorem 4.3.1 in [20]):

$$\begin{aligned} \psi^*(a) &= \sum_\gamma \frac{1}{\gamma!} (\partial_\gamma a)(\widetilde{\psi}^*(y_1), \ldots, \widetilde{\psi}^*(y_p)) \cdot \prod_{\mu=1}^{p} (\psi^*(y_\mu) - \widetilde{\psi}^*(y_\mu))^{\gamma_\mu} \\ &= a(\widetilde{\psi}(t,x)) + \sum_{\mu=1}^{p} (\partial_{y_\mu} a)(\widetilde{\psi}(t,x)) \cdot \left(\sum_{M \neq 0} f_M^\mu \cdot \xi^M \right) + \end{aligned}$$

$$\frac{1}{2}\sum_{\mu',\mu''=1}^{p}(\partial_{y_{\mu'}}\partial_{y_{\mu''}}a)(\widetilde{\psi}(t,x))\cdot\left(\sum_{M'\neq 0} f^{\mu'}_{M'}\cdot\xi^{M'}\right)\cdot\left(\sum_{M''\neq 0} f^{\mu''}_{M''}\cdot\xi^{M''}\right)+\dots,$$

where $\sum_{M\neq 0} f^{\mu}_{M}\cdot\xi^{M} = \psi^*(y_\mu)-\widetilde{\psi}^*(y_\mu)$ with f^{μ}_{M} depending on t and x. We observe that the last RHS is a finite sum since we work in the framework of finite-dimensional supermanifolds.
In order to get a contribution to $(\psi^*(a)|\xi^K)$ we can either extract f^{μ}_{K} from the "linear term" or from products coming from the higher order terms in the above development. Thus

$$(\psi^*(a)|\xi^K)=\sum_{\mu=1}^{p}(\partial_{y_\mu}a)(\widetilde{\psi}(t,x))\cdot f^{\mu}_{K}+R(a,(f^{\nu}_{I})_{\nu,\deg(I)<\deg(K)}),$$

where R is a polynomial as described after Equation (13).

Furthermore, for an element $J=(\beta_1,\dots,\beta_q)$ with $|J|=0$ we have for $\deg(K)>0$

$$\left(\prod_{s=1}^{q}\left(\sum_{|L|=1} f^{p+s}_{L}\xi^{L}\right)^{\beta_s}\middle|\,\xi^K\right)=R\left((f^{j}_{I})_{j,\deg(I)<\deg(K)}\right). \tag{14}$$

And for an element $J=(\beta_1,\dots,\beta_q)$ with $|J|=1$ we get for $\deg(K)>0$

$$\left(\prod_{s=1}^{q}\left(\sum_{|L|=1} f^{p+s}_{L}\xi^{L}\right)^{\beta_s}\middle|\,\xi^K\right)=\begin{cases} f^{p+l}_{K}+R\left((f^{j}_{I})_{j,\deg(I)<\deg(K)}\right) \\ \text{if } \deg(J)=1 \text{ and } l\in\{1,\dots,q\} \\ \text{such that } \beta_s=\delta_{s,l}\quad\forall s, \\ R\left((f^{j}_{I})_{j,\deg(I)<\deg(K)}\right) \\ \text{if } \deg(J)>1. \end{cases} \tag{15}$$

Obviously, the coefficient of ξ^K of the LHS of Equation (7) is given by

$$\left(\sum_{|I|=|w_j|}\partial_t f^{j}_{I}\cdot\xi^{I}\middle|\,\xi^K\right)=\begin{cases} \partial_t f^{j}_{K} & \text{if } |K|=|w_j| \\ 0 & \text{if } |K|=|w_j|+1 \end{cases}\qquad \text{for } 1\le j\le p+q.$$

Taking into account the above descriptions of the ξ^K-coefficients, we will show the existence (and uniqueness) of the solution functions $\{f^{j}_{I}|1\le j\le p+q, I\in\{0,1\}^n\}$ for $(t,x)\in V$ by induction on $\deg(I)$ and upon observing that all ordinary differential equations occuring are (inhomogeneous) linear equations for the unknown functions.

Let us start with $\deg(I) = 0$ that is $I = (0, \cdots, 0)$. The "0-level" of the equations (11) and (9) is $\partial_t f^j_{(0,\cdots,0)} = a^j_{(0,\cdots,0)} \circ \widetilde{F}$ and $f^j_{(0,\cdots,0)}(t_0, x) = y_j \circ \widetilde{\phi}(x)$ for all j such that $|w_j| = 0$. We remark that $f^j_{(0,\cdots,0)}$ is simply $y_j \circ \widetilde{F}$ and $a^j_{(0,\cdots,0)}$ is $\widetilde{X}(y_j)$. Thus $\widetilde{F}$ is the flow of $\widetilde{X}$ with initial condition $\widetilde{\phi}$ at $t = t_0$, i.e., $\widetilde{F} = H$ on V. Thus the claim is true for $I = (0, \cdots, 0)$.

Suppose $k > 0$ and that the functions f^j_I are uniquely defined on V for all j and all I such that $\deg(I) < k$. Let K be such that $\deg(K) = k$. Let us distinguish the two possible parities of k in order to determine f^j_K for all j. Recall that $f^j_K = 0$ if the parities of K and j are different.

If k is even, i.e., $|K| = 0$, we only have to consider j such that $|w_j| = 0$. Putting (13) and (14) together, we find in this case

$$\begin{aligned}
\partial_t f^j_K &= \left(\sum_{\substack{|J|=0 \\ J=(\beta_1,\ldots,\beta_q)}} \check{F}^*(a^j_J) \prod_{s=1}^{q} \left(\sum_{|L|=1} f^{p+s}_L \xi^L \right)^{\beta_s} \Bigg| \xi^K \right) \\
&= \left(\sum_{\substack{\deg(J)=0 \\ J=(\beta_1,\ldots,\beta_q)}} \check{F}^*(a^j_J) \prod_{s=1}^{q} \left(\sum_{|L|=1} f^{p+s}_L \xi^L \right)^{\beta_s} \right. \\
&\qquad \left. + \sum_{\substack{|J|=0 \\ \deg(J)>0 \\ J=(\beta_1,\ldots,\beta_q)}} \check{F}^*(a^j_J) \prod_{s=1}^{q} \left(\sum_{|L|=1} f^{p+s}_L \xi^L \right)^{\beta_s} \Bigg| \xi^K \right) \\
&= \left(\check{F}^*(a^j_{(0,\cdots,0)}) + \sum_{\substack{|J|=0 \\ \deg(J)>0 \\ J=(\beta_1,\ldots,\beta_q)}} \check{F}^*(a^j_J) \prod_{s=1}^{q} \left(\sum_{|L|=1} f^{p+s}_L \xi^L \right)^{\beta_s} \Bigg| \xi^K \right) \\
&= \sum_{\mu=1}^{p} \left(\partial_{y_\mu} a^j_{(0,\cdots,0)} \circ \widetilde{F} \right) f^\mu_K + R\left((a^j_J)_J, (f^\nu_I)_{\nu, \deg(I)<\deg(K)} \right).
\end{aligned}$$

Moreover, the initial condition gives $f^j_K(t_0, x) = (\phi^*(y_j)|\xi^K)$, for all j in $\{1, \ldots, p\}$. Since the a^j_J are the (given) coefficients of the vector field X and the functions f^ν_I with $\deg(I) < k$ are known by the induction hypothesis, we have a unique local solution function f^j_K. Since the ordinary differential equation for f^j_K is linear its solution is already defined for all $(t, x) \in V$. Thus in the case that k is even f^j_K is unambiguously defined on V for all $j \in \{1, \ldots, p + q\}$ and for all K with $\deg(K) = k$.

Now, if k is odd, i.e., $|K| = 1$, we only have to consider j such that $|w_j| = 1$. Using (13) and (15), we find in this case:

$$\begin{aligned}
\partial_t f_K^j &= \left(\sum_{\substack{|J|=1 \\ J=(\beta_1,\dots,\beta_q)}} \check{F}^*(a_J^j) \prod_{s=1}^q \left(\sum_{|L|=1} f_L^{p+s} \xi^L \right)^{\beta_s} \middle| \xi^K \right) \\
&= \left(\sum_{\substack{\deg(J)=1 \\ J=(\beta_1,\dots,\beta_q)}} \check{F}^*(a_J^j) \prod_{s=1}^q \left(\sum_{|L|=1} f_L^{p+s} \xi^L \right)^{\beta_s} + \right. \\
&\qquad \left. + \sum_{\substack{|J|=1 \\ \deg(J)>1 \\ J=(\beta_1,\dots,\beta_q)}} \check{F}^*(a_J^j) \prod_{s=1}^q \left(\sum_{|L|=1} f_L^{p+s} \xi^L \right)^{\beta_s} \middle| \xi^K \right) \\
&= \left(\sum_{s=1}^q \check{F}^*(a^j_{(\delta_{1s},\cdots,\delta_{qs})}) \left(\sum_{|L|=1} f_L^{p+s} \xi^L \right) \right. \\
&\qquad \left. + \sum_{\substack{|J|=1 \\ \deg(J)>1 \\ J=(\beta_1,\dots,\beta_q)}} \check{F}^*(a_J^j) \prod_{s=1}^q \left(\sum_{|L|=1} f_L^{p+s} \xi^L \right)^{\beta_s} \middle| \xi^K \right) \\
&= \sum_{s=1}^q \left(a^j_{(\delta_{1s},\cdots,\delta_{qs})} \circ \widetilde{F} \right) f_K^{p+s} + R\left((a_J^j)_J, (f_I^\nu)_{\nu,\deg(I)<\deg(K)} \right).
\end{aligned}$$

Moreover, the initial condition gives

$$f_K^j(t_0, x) = (\phi^*(w_j) | \xi^K) \text{ for all } j \text{ in } \{p+1, \dots, p+q\}.$$

It follows as in the case of $|K| = 0$, that f_K^j exists uniquely for all $(t, x) \in V$, for all $j \in \{1, \dots, p+q\}$ and for all K with $\deg(K) = k$.

We conclude that the functions $\{f_I^j | 1 \le j \le p+q, I \in \{0,1\}^n\}$ are uniquely defined on the whole of V. Since the $\{g_I^j | 1 \le j \le p+q, I \in \{0,1\}^n\}$ are determined by Equation (8) from the $\{f_I^j | 1 \le j \le p+q\}$ via comparison of coefficients, the morphism $F : \mathcal{V} \to \mathcal{W}$ is uniquely determined. □

We now consider the global problem of integrating a vector field on a supermanifold. In order to prove that there exists a unique maximal flow of a vector field, the following lemma will be crucial.

Lemma 2.2. *Let $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ and $\mathcal{S} = (S, \mathcal{O}_{\mathcal{S}})$ be supermanifolds, X a vector field in $\mathcal{T}_{\mathcal{M}}(M)$ and ϕ in $Mor(\mathcal{S}, \mathcal{M})$. Then*

(i) there exists an open sub supermanifold $\mathcal{V} = (V, \mathcal{O}_{\mathbb{R}^{1|1}\times\mathcal{S}}|_V)$ of $\mathbb{R}^{1|1}\times\mathcal{S}$ with V open in $\mathbb{R}\times S$ such that $\{0\}\times S \subset V$ and for all x in S, $(\mathbb{R}\times\{x\})\cap V$ is an interval, and a morphism $F : \mathcal{V} \to \mathcal{M}$ satisfying:

$$\begin{aligned}(inj\,{}^{\mathbb{R}^{1|1}}_{\mathbb{R}})^* \circ (\partial_t + \partial_\tau) \circ F^* &= (inj\,{}^{\mathbb{R}^{1|1}}_{\mathbb{R}})^* \circ F^* \circ X \text{ and} \qquad (16)\\ F \circ inj\,{}^{\mathcal{V}}_{\{0\}\times\mathcal{S}} &= \phi\,. \qquad (17)\end{aligned}$$

(ii) Let furthermore $F_1 : \mathcal{V}_1 \to \mathcal{M}$ and $F_2 : \mathcal{V}_2 \to \mathcal{M}$ be morphisms satisfying (16) and (17) where $\mathcal{V}_i = (V_i, \mathcal{O}_{\mathbb{R}^{1|1}\times\mathcal{S}}|_{V_i})$ with V_i open in $\mathbb{R}\times S$ such that $\{0\}\times S \subset V_i$ and for all x in S, $(\mathbb{R}\times\{x\})\cap V_i$ is an interval, for $i = 1, 2$. Then $F_{1|\mathcal{V}_{12}} = F_{2|\mathcal{V}_{12}}$ on $\mathcal{V}_{12} = (V_{12}, \mathcal{O}_{\mathbb{R}^{1|1}\times\mathcal{S}}|_{V_{12}})$, where $V_{12} = V_1 \cap V_2$.

Proof. (i) Let $\widetilde{\phi} : S \to M$ denote the induced map of the underlying classical manifolds. Given now s in S and coordinate domains $\mathcal{U}_s$ of s and $\mathcal{W}_s$ of $\widetilde{\phi}(s)$, isomorphic to superdomains $\check{\mathcal{U}}_s \subset \mathbb{R}^{m|n}$ resp. $\check{\mathcal{W}}_s \subset \mathbb{R}^{p|q}$, by Lemma 2.1 we get solutions of (16) and (17) near s (upon reducing the size of $\mathcal{U}_s$ if necessary): $\mathbb{R}^{1|1}\times\mathcal{S} \supset \mathbb{R}^{1|1}\times\mathcal{U}_s \supset \mathcal{V}_s \xrightarrow{F^s} \mathcal{W}_s \subset \mathcal{M}$. If $\mathcal{V}_{s_1}\cap\mathcal{V}_{s_2} \neq \emptyset$ (compare Figure 1) we know, again by Lemma 2.1, that F^{s_1} and F^{s_2} coincide on this intersection. Thus, by taking the union $\mathcal{V}$ of $\mathcal{V}_s$ for all s in S, we get a morphism $F : \mathbb{R}^{1|1}\times\mathcal{S} \supset \mathcal{V} \to \mathcal{M}$ such that $F_{|\mathcal{V}_s} = F^s$ for all s, and fulfilling (16) and (17).

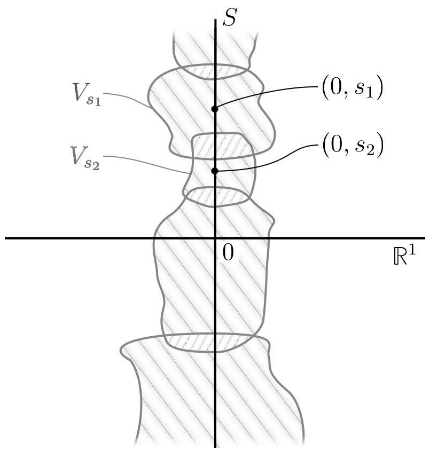

Figure 1

(ii) We define A as the set of points $(t, x) \in V_{12}$ such that there exists $\epsilon = \epsilon_{(t,x)} > 0$ and $\mathcal{U} = \mathcal{U}_{(t,x)}$ an open sub supermanifold of $\mathcal{S}$, such that its body U contains x and for $\mathcal{V} = \mathcal{V}_{(t,x)} = (V_{(t,x)}, \mathcal{O}_{\mathbb{R}^{1|1}\times\mathcal{S}}) = (]-\epsilon, t+\epsilon[\times U, \mathcal{O}_{\mathbb{R}^{1|1}\times\mathcal{S}})$ we have $F_{1|\mathcal{V}} = F_{2|\mathcal{V}}$. Of course, if $t < 0$ the interval will be of the type $]t-\epsilon, \epsilon[$ (See Figure 2). The claim of the Lemma is now equivalent to $A = V_{12}$. The set A is obviously open.

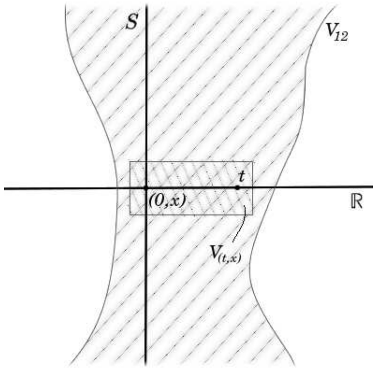

Figure 2

By an easy application of Lemma 2.1, A contains $\{0\} \times S$. The assumptions imply that for all $x \in S$, the set $I_x \subset \mathbb{R}$, defined by $(\mathbb{R} \times \{x\}) \cap V_{12} = I_x \times \{x\}$, is an open interval containing 0. The definition of A implies that the set $J_x \subset I_x$ such that $(\mathbb{R} \times \{x\}) \cap A = J_x \times \{x\}$ is an open interval containing 0 as well. Assuming now that $A \neq V_{12}$, then there exists a point $(t, x_0) \in V_{12} \backslash A$ such that $J_{x_0} \neq I_{x_0}$. Without loss of generality we can assume that $t > 0$ and that for $0 \leq t' < t$, $(t', x_0) \in A$. Let U_0 be an open coordinate neighborhood of x_0 in S and $\delta > 0$ such that, with $V_0 :=]t-\delta, t+\delta[\times U_0 \subset V_{12}$, $H(V_0) \subset W$, where $\mathcal{W} = (W, \mathcal{O}_{\mathcal{M}}|_W)$ is a coordinate patch of $\mathcal{M}$ and H is the maximal flow of $\widetilde{X}$ as in Lemma 2.1. Choose $t_0 \in]t-\delta, t[$. Then $(t_0, x_0) \in A$ and thus there exists $\epsilon > 0$ and $\mathcal{U}$ an open sub supermanifold of $\mathcal{U}_0 = (U_0, \mathcal{O}_{\mathcal{S}}|_{U_0})$ containing x_0 such that

$$F_{1|\mathcal{V}} = F_{2|\mathcal{V}}, \quad \text{where } \mathcal{V} =]-\epsilon, t_0 + \epsilon[\times \mathbb{R}^{0|1} \times \mathcal{U} \subset \mathcal{V}_{12}. \tag{18}$$

On $\mathcal{V}' =]t-\delta, t+\delta[\times \mathbb{R}^{0|1} \times \mathcal{U} \subset \mathcal{V}_{12}$, F_1 and F_2 are defined and for $i = 1, 2$ the maps $F_i \circ \mathrm{inj}^{\mathcal{V}}_{\{t_0\} \times \mathcal{U}}$ coincide by (18) (Compare Figure 3 for the relative positions of the underlying topological spaces of these open sub supermanifolds of $\mathbb{R}^{1|1} \times \mathcal{S}$).

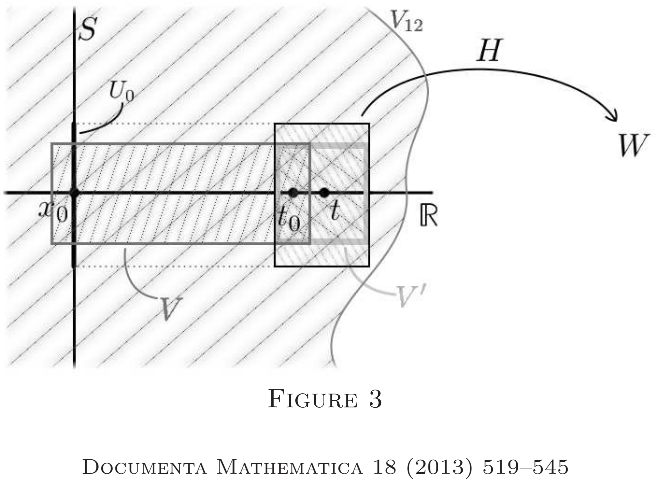

Figure 3

By Lemma 2.1 we have $F_{1|\mathcal{V}'} = F_{2|\mathcal{V}'}$. Thus $F_1 = F_2$ on $\mathcal{V} \cup \mathcal{V}'$, and we conclude that $(t, x_0) \in A$. This contradiction shows that $V_{12} = A$. □

Remarks. (1) Obviously, Lemma 2.2 holds true for an arbitrary $t_0 \in \mathbb{R}$ replacing $t_0 = 0$.
(2) Let us call a "flow domain for X with initial condition $\phi \in \mathrm{Mor}(\mathcal{S}, \mathcal{M})$ (with respect to $t_0 \in \mathbb{R}$)" a domain $\mathcal{V} \subset \mathbb{R}^{1|1} \times \mathcal{S}$ such that $\{t_0\} \times S \subset V$ and for all s in S, $(\mathbb{R} \times \{s\}) \cap V$ is connected, i.e. an interval (times $\{s\}$) and such that a solution F (a "flow") of (16) and (17) exists on $\mathcal{V}$. By the preceding lemma there exists such "flow domains".

Theorem 2.3. *Let $\mathcal{M}$ and $\mathcal{S}$ be supermanifolds, X be a vector field in $\mathcal{T}_{\mathcal{M}}(M)$, ϕ in Mor$(\mathcal{S}, \mathcal{M})$ and t_0 in $\mathbb{R}$. Then there exists a unique map $F : \mathcal{V} \to \mathcal{M}$ such that*

$$\begin{aligned} (inj\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ (\partial_t + \partial_\tau) \circ F^* &= (inj\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ F^* \circ X \ \textit{and} \\ F \circ inj\,{}_{\{t_0\}\times\mathcal{S}}^{\mathcal{V}} &= \phi\,, \end{aligned}$$

where $\mathcal{V} = (V, \mathcal{O}_{\mathbb{R}^{1|1}\times\mathcal{U}}|_V)$ is the maximal flow domain for X with the given initial condition.
Moreover, $\widetilde{F} : V \to M$ is the maximal flow of $\widetilde{X} \in \mathcal{X}(M)$ subject to the initial condition $\widetilde{\phi}$ at $t = t_0$.

Proof. The proof of the theorem follows immediately from the Lemmata 2.1 and 2.2, upon taking the union of all flow domains and flows for X as defined in the preceding remark. □

3 Supervector fields and local $\mathbb{R}^{1|1}$-actions

Given a vector field on a classical, ungraded, manifold, the flow map $\widetilde{F}$ (for $S = M$, $\widetilde{\phi} = \mathrm{id}_M$) is always a local action of $\mathbb{R}$ with its usual (and unique up to isomorphism) Lie group structure, the standard addition. The flow maps for vector fields described in the preceding section (taking here $\mathcal{S} = \mathcal{M}$, $\phi = \mathrm{id}_{\mathcal{M}}$), do not always have the analogous property of being local actions of $\mathbb{R}^{1|1}$ with an appropriate Lie supergroup structure. Two characterizations of those vector fields $X = X_0 + X_1$ that generate a local $\mathbb{R}^{1|1}$-action were found by J. Monterde and O. A. Sánchez-Valenzuela. We will give in this section a short proof of a slightly more general result, whose condition (iii) seems to be more easily verified in practice than those given in [13] (compare Thm. 3.6 and its proof there).

Let us begin by giving a useful two-parameter family of Lie supergroup structures on the supermanifold $\mathbb{R}^{1|1}$ and their right invariant vector fields.

Lemma 3.1. *Let a and b be real numbers such that $a \cdot b = 0$ and $\mu_{a,b} = \mu$:*

$\mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}$ *be defined by*

$$\begin{array}{rcl} \widetilde{\mu}(t_1, t_2) & = & t_1 + t_2, \\ \mu^*(t) & = & t_1 + t_2 + a\tau_1\tau_2, \\ \mu^*(\tau) & = & \tau_1 + e^{bt_1}\tau_2 \qquad . \end{array}$$

Then

(i) there exists a unique Lie supergroup structure on $\mathbb{R}^{1|1}$ *such that the multiplication morphism is given by* $\mu_{a,b}$*,*

(ii) the right invariant vector fields on $(\mathbb{R}^{1|1}, \mu_{a,b})$ *are given by the graded vector space* $\mathbb{R}D_0 \oplus \mathbb{R}D_1$*, where*

$$D_0 := \partial_t + b \cdot \tau\partial_\tau \text{ and } D_1 := \partial_\tau + a \cdot \tau\partial_t,$$

and they obey $[D_0, D_0] = 0$*,* $[D_0, D_1] = -bD_1$ *and* $[D_1, D_1] = 2aD_0$*.*

Proof. Both assertions follow by straightforward verifications. □

Remarks. (1) It can easily be checked that the above family yields only three non-isomorphic Lie supergroup structures on $\mathbb{R}^{1|1}$, since $(\mathbb{R}^{1|1}, \mu_{a,0})$ with $a \neq 0$ is isomorphic to $(\mathbb{R}^{1|1}, \mu_{1,0})$ and $(\mathbb{R}^{1|1}, \mu_{0,b})$ with $b \neq 0$ is isomorphic to $(\mathbb{R}^{1|1}, \mu_{0,1})$ and the three multiplications $\mu_{0,0}$, $\mu_{1,0}$ and $\mu_{0,1}$ correspond to non-isomorphic Lie supergroup structures on $\mathbb{R}^{1|1}$. Nevertheless it is very convenient to work here with the more flexible two-parameter family of multiplications.
(2) In fact, all Lie supergroup structures on $\mathbb{R}^{1|1}$ are equivalent to $\mu_{0,0}$, $\mu_{1,0}$ or $\mu_{0,1}$. See, e.g., [4] for a direct approach to the classification of all Lie supergroup structures on $\mathbb{R}^{1|1}$.

Definition 3.2. *Let* $\mathcal{G} = (G, \mathcal{O}_\mathcal{G})$ *resp.* $\mathcal{M} = (M, \mathcal{O}_\mathcal{M})$ *be a Lie supergroup with multiplication morphism* μ *and unit element* e *resp. a supermanifold. A "local action of* $\mathcal{G}$ *on* $\mathcal{M}$*" is given by the following data:*
a collection Π *of pairs of open subsets* $\pi = (U_\pi, W_\pi)$ *of* M*, where* U_π *is relatively compact in* W_π*, with associated open sub supermanifolds* $\mathcal{U}_\pi \subset \mathcal{W}_\pi \subset \mathcal{M}$ *such that* $\{U_\pi | \pi \in \Pi\}$ *is an open covering of* M*, and for all* π *in* Π *an open sub supermanifold* $\mathcal{G}_\pi \subset \mathcal{G}$*, containing the neutral element* e *and a morphism*

$$\Phi_\pi : \mathcal{G}_\pi \times \mathcal{U}_\pi \to \mathcal{W}_\pi$$

fulfilling

(1) $\Phi_\pi \circ (e \times id_{\mathcal{U}_\pi}) = id_{\mathcal{U}_\pi}$*, where* $e : \{pt\} \to \mathcal{G}$ *is viewed as a morphism,*

(2) $\Phi_\pi \circ (\mu \times id_\mathcal{M}) = \Phi_\pi \circ (id_\mathcal{G} \times \Phi_\pi)$*, where both sides are defined,*

(3) if $U_\pi \cap U_{\pi'} \neq \emptyset$*,* $\Phi_\pi = \Phi_{\pi'}$ *on* $(\mathcal{G}_\pi \cap \mathcal{G}_{\pi'}) \times (\mathcal{U}_\pi \cap \mathcal{U}_{\pi'})$*.*

Proposition 3.3. *Let $\mathcal{G} = (G, \mathcal{O}_\mathcal{G})$ resp. $\mathcal{M} = (M, \mathcal{O}_\mathcal{M})$ be a Lie supergroup resp. a supermanifold. Then*

(i) a local $\mathcal{G}$-action on $\mathcal{M}$, specified by a set Π and morphisms $\{(\mathcal{U}_\pi, \mathcal{W}_\pi, \mathcal{G}_\pi, \Phi_\pi) | \pi \in \Pi\}$, gives rise to an open sub supermanifold $\mathcal{V} \subset \mathcal{G} \times \mathcal{M}$ containing $\{e\} \times \mathcal{M}$ and a morphism $\Phi_\mathcal{V} : \mathcal{V} \to \mathcal{M}$ such that

$$\Phi_\mathcal{V} \circ (\mu \times id_\mathcal{M}) = \Phi_\mathcal{V} \circ (id_\mathcal{G} \times \Phi_\mathcal{V}), \tag{$*$}$$

where both sides are defined and such that

$$\Phi_\pi = \Phi_\mathcal{V} \text{ on } (\mathcal{G}_\pi \times \mathcal{U}_\pi) \cap \mathcal{V}, \ \forall \pi \in \Pi, \tag{$**$}$$

(ii) an open sub supermanifold $\mathcal{V} \subset \mathcal{G} \times \mathcal{M}$ containing $\{e\} \times \mathcal{M}$ and a morphism $\Phi : \mathcal{V} \to \mathcal{M}$ such that $(*)$ is fulfilled, where it makes sense, yields a local $\mathcal{G}$-action on $\mathcal{M}$ such that $(**)$ holds.

Proof. As in the classical case of ungraded manifolds and Lie groups. □

Theorem 3.4. *Let $\mathcal{M}$ be a supermanifold, X a vector field on $\mathcal{M}$ and $\mathcal{V} \subset \mathbb{R}^{1|1} \times \mathcal{M}$ the domain of the maximal flow $F : \mathcal{V} \to \mathcal{M}$ satisfying*

$$\begin{aligned} (inj\,_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ (\partial_t + \partial_\tau) \circ F^* &= (inj\,_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ F^* \circ X \text{ and} \\ F \circ inj\,_{\{0\}\times\mathcal{M}}^{\mathcal{V}} &= id_\mathcal{M}. \end{aligned}$$

Let a and b be real numbers such that $a \cdot b = 0$. Then the following assertions are equivalent:

(i) the map F fulfills

$$(\partial_t + \partial_\tau + \tau(a\partial_t + b\partial_\tau)) \circ F^* = F^* \circ X,$$

(ii) the map F is a local $(\mathbb{R}^{1|1}, \mu_{a,b})$-action on $\mathcal{M}$,

(iii) $\mathbb{R}X_0 \oplus \mathbb{R}X_1$ is a sub Lie superalgebra of $\mathcal{T}_\mathcal{M}(M)$ with commutators $[X_0, X_1] = -bX_1$ and $[X_1, X_1] = 2aX_0$.

Proof. Recall that F fulfills

$$\begin{aligned} (\mathrm{inj}\,_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ \partial_t \circ F^* &= (\mathrm{inj}\,_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ F^* \circ X_0 \quad \text{and} \\ (\mathrm{inj}\,_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ \partial_\tau \circ F^* &= (\mathrm{inj}\,_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ F^* \circ X_1. \end{aligned}$$

Denoting the projection from $\mathbb{R}^{1|1}$ to $\mathbb{R}$ by p, we have

$$\mathrm{id}^*_{\mathbb{R}^{1|1}} \quad = \quad p^* \circ (\mathrm{inj}\,_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* + \tau \cdot p^* \circ (\mathrm{inj}\,_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ \partial_\tau$$

which we will write more succinctly as

$$\mathrm{id}^*_{\mathbb{R}^{1|1}} \quad = \quad (\mathrm{inj}\,_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* + \tau \cdot (\mathrm{inj}\,_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ \partial_\tau. \tag{19}$$

Using relation (19) and the equations fulfilled by F^* we get

$$\begin{aligned} F^* \circ X &= (\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ F^* \circ X + \tau \cdot (\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ \partial_\tau \circ F^* \circ X \\ &= (\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ (\partial_t + \partial_\tau) \circ F^* + \tau \cdot (\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ F^* \circ X_1 \circ X \\ &= (\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ (\partial_t + \partial_\tau) \circ F^* \\ &+ \tau \cdot (\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ F^* \circ \left([X_1, X_0] + X_0 \circ X_1 + \frac{1}{2}[X_1, X_1] \right). \end{aligned}$$

Since

$$\begin{aligned} (\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ F^* \circ X_0 \circ X_1 &= (\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ \partial_t \circ F^* \circ X_1 \\ &= \partial_t \circ (\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ F^* \circ X_1 \\ &= \partial_t \circ (\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ \partial_\tau \circ F^* \\ &= \partial_t \circ \partial_\tau \circ F^* \\ &= \partial_\tau \circ \partial_t \circ F^*, \end{aligned}$$

we arrive at

$$\begin{aligned} F^* \circ X = {} & (\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ (\partial_t + \partial_\tau) \circ F^* + \\ & \tau \cdot F^* \circ \left([X_1, X_0] + \frac{1}{2}[X_1, X_1] \right) + \tau \cdot \partial_\tau \circ \partial_t \circ F^*. \end{aligned} \tag{20}$$

On the other hand, if a and b are real numbers, we have, again using (19)

$$\begin{aligned} & (\partial_t + \partial_\tau + \tau(a\partial_t + b\partial_\tau)) \circ F^* \\ &= \left((\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* + \tau \cdot (\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ \partial_\tau \right) \circ (\partial_t + \partial_\tau) \circ F^* \\ &\qquad + \tau \cdot \left(a \cdot (\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ \partial_t + b \cdot (\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ \partial_\tau \right) \circ F^* \\ &= (\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ (\partial_t + \partial_\tau) \circ F^* + \tau \cdot \partial_\tau \circ \partial_t \circ F^* \\ &\qquad + \tau \cdot F^* \circ (aX_0 + bX_1). \end{aligned}$$

Thus we have

$$\begin{aligned} & (\partial_t + \partial_\tau + \tau(a\partial_t + b\partial_\tau)) \circ F^* - F^* \circ X \\ &\qquad = \tau \cdot F^* \circ \left(aX_0 - \frac{1}{2}[X_1, X_1] + bX_1 - [X_1, X_0] \right). \end{aligned} \tag{21}$$

Since F satisfies the initial condition $(\mathrm{inj}\,{}_{\{0\}\times\mathcal{M}}^{\mathcal{V}})^* \circ F^* = \mathrm{id}_{\mathcal{M}}$, $\tau \cdot F^*$ is injective and thus Equation (21) easily implies the equivalence of (i) and (iii).

We remark that, in this case, we automatically have $a \cdot b = 0$ since the Jacobi identity implies that $[X_1, [X_1, X_1]] = [[X_1, X_1], X_1] + (-1)^{1\cdot 1}[X_1, [X_1, X_1]]$, i.e., $2a \cdot b \cdot X_1 = [X_1, [X_1, X_1]] = 0$.

Assume now that a and b are real numbers such that (i) satisfied, and let $\mu = \mu_{a,b}$ be as in Lemma 3.1. We have to show that F is a local action of $(\mathbb{R}^{1|1}, \mu)$.
Let us define

$$G := F \circ (\mathrm{id}_{\mathbb{R}^{1|1}} \times F) : \mathbb{R}^{1|1} \times (\mathbb{R}^{1|1} \times \mathcal{M}) \to \mathcal{M}$$

and

$$H := F \circ (\mu \times \mathrm{id}_{\mathcal{M}}) : (\mathbb{R}^{1|1} \times \mathbb{R}^{1|1}) \times \mathcal{M} \cong \mathbb{R}^{1|1} \times (\mathbb{R}^{1|1} \times \mathcal{M}) \to \mathcal{M}.$$

In order to prove that F is a $\mathbb{R}^{1|1}$-action on, we have to show that $G = H$. We observe that G is the integral curve of X subject to the initial condition $F \in \mathrm{Mor}(\mathbb{R}^{1|1} \times \mathcal{M}, \mathcal{M})$.

Let us prove that the morphism H satisfies the following conditions:

$$\begin{aligned}
(\mathrm{inj}\, {}^{\mathbb{R}^{1|1}\times(\mathbb{R}^{1|1}\times\mathcal{M})}_{\mathbb{R}\times(\mathbb{R}^{1|1}\times\mathcal{M})})^* \circ (\partial_{t_1} + \partial_{\tau_1}) \circ H^* &= (\mathrm{inj}\, {}^{\mathbb{R}^{1|1}\times(\mathbb{R}^{1|1}\times\mathcal{M})}_{\mathbb{R}\times(\mathbb{R}^{1|1}\times\mathcal{M})})^* \circ H^* \circ X &(22)\\
H \circ \mathrm{inj}\, {}^{\mathbb{R}^{1|1}\times(\mathbb{R}^{1|1}\times\mathcal{M})}_{\{0\}\times(\mathbb{R}^{1|1}\times\mathcal{M})} &= F. &(23)
\end{aligned}$$

Then by the unicity of integral curves we have $H = G$.

Equation (23) holds true since $\mu \circ \mathrm{inj}\, {}^{\mathbb{R}^{1|1}\times\mathbb{R}^{1|1}}_{\{0\}\times\mathbb{R}^{1|1}} = \mathrm{id}_{\mathbb{R}^{1|1}}$.

Defining $D := D_0 + D_1 = \partial_{t_1} + \partial_{\tau_1} + \tau_1(a\partial_{t_1} + b\partial_{\tau_1})$ and writing $\mathrm{inj}_{\,|t_1}$ for $\mathrm{inj}\, {}^{\mathbb{R}^{1|1}\times(\mathbb{R}^{1|1}\times\mathcal{M})}_{\mathbb{R}\times(\mathbb{R}^{1|1}\times\mathcal{M})}$ and using right invariance of D, we arrive at equation (22) as follows

$$\begin{aligned}
&(\mathrm{inj}\, {}^{\mathbb{R}^{1|1}\times(\mathbb{R}^{1|1}\times\mathcal{M})}_{\mathbb{R}\times(\mathbb{R}^{1|1}\times\mathcal{M})})^* \circ (\partial_{t_1} + \partial_{\tau_1}) \circ H^* \\
&\qquad = (\mathrm{inj}_{\,|t_1})^* \circ D \circ H^* \\
&\qquad = (\mathrm{inj}_{\,|t_1})^* \circ (((D \otimes \mathrm{id}^*_{\mathbb{R}^{1|1}}) \circ \mu^*) \times \mathrm{id}^*_{\mathcal{M}}) \circ F^* \\
&\qquad = (\mathrm{inj}_{\,|t_1})^* \circ ((\mu^* \circ D) \times \mathrm{id}^*_{\mathcal{M}}) \circ F^* \\
&\qquad = (\mathrm{inj}_{\,|t_1})^* \circ (\mu^* \times \mathrm{id}^*_{\mathcal{M}}) \circ F^* \circ X \\
&\qquad = (\mathrm{inj}_{\,|t_1})^* \circ H^* \circ X.
\end{aligned}$$

Thus we obtain that (i) implies (ii).

Assume now that (ii) is satisfied, i.e., there exists a Lie supergroup structure on $\mathbb{R}^{1|1}$ with multiplication μ such that

$$F \circ (\mathrm{id}_{\mathbb{R}^{1|1}} \times F) = F \circ (\mu \times \mathrm{id}_{\mathcal{M}}). \tag{24}$$

Since F is a flow for X, with initial condition $\phi = \mathrm{id}_{\mathcal{M}}$, the LHS of the preceding equality is a flow for X with initial condition $\phi = F$, (24) implies

$$(\mathrm{inj}_{\,|t_1})^* \circ (\partial_{t_1} + \partial_{\tau_1}) \circ (\mu^* \times \mathrm{id}^*_{\mathcal{M}}) \circ F^* = (\mathrm{inj}_{\,|t_1})^* \circ (\mu^* \times \mathrm{id}^*_{\mathcal{M}}) \circ F^* \circ X. \tag{25}$$

By Equation (20), the RHS gives for $t_1 = 0$:

$$\begin{aligned}(\mathrm{inj}_{\,|t_1=0})^* \circ (\mu^* \times \mathrm{id}^*_{\mathcal{M}}) \circ F^* \circ X &= F^* \circ X \\ &= (\mathrm{inj}\,{}^{\mathbb{R}^{1|1}}_{\mathbb{R}})^* \circ (\partial_t + \partial_\tau) \circ F^* \\ +\tau \cdot \Bigg[F^* \circ \bigg([X_1, X_0] + \frac{1}{2}[X_1, X_1]\bigg) &+ \partial_\tau \circ \partial_t \circ F^*\Bigg].\end{aligned}$$

Moreover, we have by direct comparison

$$\begin{aligned}(\partial_{t_1} + \partial_{\tau_1}) \circ \mu^* &= (\partial_{t_1} + \partial_{\tau_1})(\mu^*(t)) \cdot (\mu^* \circ \partial_t) \\ &\quad +(\partial_{t_1} + \partial_{\tau_1})(\mu^*(\tau)) \cdot (\mu^* \circ \partial_\tau).\end{aligned}$$

Thus, if $\mu : \mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}$ is given by

$$\begin{aligned}\mu^*(t) &= \widetilde{\mu}(t_1, t_2) + \alpha(t_1, t_2)\tau_1\tau_2 \\ \mu^*(\tau) &= \beta(t_1, t_2)\tau_1 + \gamma(t_1, t_2)\tau_2,\end{aligned}$$

and upon using $(\mathrm{inj}\,{}^{\mathbb{R}^{1|1}\times\mathbb{R}^{1|1}}_{\{0\}\times\mathbb{R}^{1|1}})^* \circ \mu^* = \mathrm{id}^*_{\mathbb{R}^{1|1}}$, we have

$$\begin{aligned}(\mathrm{inj}\,{}^{\mathbb{R}^{1|1}\times\mathbb{R}^{1|1}}_{\{0\}\times\mathbb{R}^{1|1}})^*(\partial_{t_1} + \partial_{\tau_1}) \circ \mu^* &= ((\partial_{t_1}\widetilde{\mu})(0, t) + \alpha(0, t)\tau) \cdot \partial_t \\ &\quad + (\beta(0, t) + (\partial_{t_1}\gamma)(0, t)\tau) \cdot \partial_\tau.\end{aligned}$$

Using again (19), we have

$$\begin{aligned}\partial_t \circ F^* &= (\mathrm{inj}\,{}^{\mathbb{R}^{1|1}}_{\mathbb{R}})^* \circ F^* \circ X_0 + \tau \cdot \partial_\tau \circ \partial_t \circ F^* \quad \text{and} \\ \partial_\tau \circ F^* &= (\mathrm{inj}\,{}^{\mathbb{R}^{1|1}}_{\mathbb{R}})^* \circ F^* \circ X_1.\end{aligned}$$

Then the LHS of (25) at $t_1 = 0$ is

$$\begin{aligned}&(\mathrm{inj}_{\,|t_1=0})^* \circ (\partial_{t_1} + \partial_{\tau_1}) \circ (\mu^* \times \mathrm{id}^*_{\mathcal{M}}) \circ F^* \\ &\quad = (\partial_{t_1}\widetilde{\mu})(0, t) \cdot (\mathrm{inj}\,{}^{\mathbb{R}^{1|1}}_{\mathbb{R}})^* \circ F^* \circ X_0 \\ &\qquad +\beta(0, t) \cdot (\mathrm{inj}\,{}^{\mathbb{R}^{1|1}}_{\mathbb{R}})^* \circ F^* \circ X_1 \\ &\qquad\quad +\tau \cdot \big((\partial_{t_1}\widetilde{\mu})(0, t) \cdot \partial_\tau \circ \partial_t \circ F^* \\ &\qquad\qquad +\alpha(0, t) \cdot F^* \circ X_0 \\ &\qquad\qquad\quad +(\partial_{t_1}\gamma)(0, t) \cdot F^* \circ X_1\big).\end{aligned}$$

Using the obtained identities for its LHS and RHS, the "τ-part" of Equation (25) at $t_1 = 0$ gives us:

$$\begin{aligned}&\tau \cdot \Bigg[F^* \circ \bigg([X_1, X_0] + \frac{1}{2}[X_1, X_1]\bigg) + \partial_\tau \circ \partial_t \circ F^*\Bigg] \\ &\quad = \tau \cdot \big[(\partial_{t_1}\widetilde{\mu})(0, t) \cdot \partial_\tau \circ \partial_t \circ F^* + \alpha(0, t) \cdot F^* \circ X_0 + (\partial_{t_1}\gamma)(0, t) \cdot F^* \circ X_1\big].\end{aligned}$$

Since $\widetilde{\mu}(t_1, 0) = t_1$, we have $(\partial_{t_1}\widetilde{\mu})(0, 0) = 1$ and therefore the preceding equation evaluated at $t = 0$ yields

$$[X_1, X_0] + \frac{1}{2}[X_1, X_1] = (\partial_{t_1}\gamma)(0, 0) \cdot X_1 + \alpha(0, 0) \cdot X_0$$

finishing the proof that (ii) implies (iii). □

4 Examples and applications

(4.1) If $X = X_0$ is an even vector field, the fact that it integrates to a (local) action of $\mathbb{R} = \mathbb{R}^{1|0}$ is almost folkloristic. The relatively recent proof of [3] - in the case of compact supermanifolds - is close to our approach. A non-trivial (local) action of $\mathbb{R}^{1|0}$ can obviously be extended to a (local) action of $(\mathbb{R}^{1|1}, \mu_{a,b})$ if and only if $a = 0$. Of course, the ensueing action of $\mathbb{R}^{1|1}$ will not even be almost-effective, since the positive-dimensional sub Lie supergroup $\mathbb{R}^{0|1}$ acts trivially.

(4.2) Our preferred example of an even vector field gives rise to the exponential map on Lie supergroups.

Let us first recall that an even vector field X on a supermanifold $\mathcal{M}$ corresponds to a section σ_X of the tangent bundle $T\mathcal{M} \to \mathcal{M}$ (see, e.g., Sections 7 and 8 of [15] for a construction of $T\mathcal{M}$ and a proof of this statement, and compare also the remark after Thm. 2.19 in [5]). Given an auxiliary supermanifold $\mathcal{S}$ and a morphism $\psi : \mathcal{S} \to \mathcal{M}$, one calls for $i \in \{0, 1\}$

$$\begin{aligned}
&\mathrm{Der}_\psi(\mathcal{O}_\mathcal{M}(M), \mathcal{O}_\mathcal{S}(S))_i := \\
&\quad \{D : \mathcal{O}_\mathcal{M}(M) \to \mathcal{O}_\mathcal{S}(S) | D \text{ is } \mathbb{R}\text{-linear and } \forall f, g \in \mathcal{O}_\mathcal{M}(M) \text{ homogeneous,} \\
&\quad D(f \cdot g) = D(f) \cdot \psi^*(g) + (-1)^{i \cdot |f|}\psi^*(f) \cdot D(g)\},
\end{aligned}$$

the "space of derivations of parity i along ψ". In category-theoretical terms the tangent bundle $T\mathcal{M}$ represents then the functor from supermanifolds to sets given by $\mathcal{S} \mapsto \{(\psi, D)|\psi \in \mathrm{Mor}(\mathcal{S}, \mathcal{M}) \text{ and } D \in \mathrm{Der}_\psi(\mathcal{O}_\mathcal{M}(M), \mathcal{O}_\mathcal{S}(S))_0\}$ (compare, e.g., Section 3 of [6]).

Let now $\mathcal{G} = (G, \mathcal{O}_\mathcal{G})$ be a Lie supergroup with multiplication $\mu{=}\mu^\mathcal{G}$ and neutral element e. We define X in $\mathrm{Der}\,(\mathcal{O}_{\mathcal{G}\times T_e\mathcal{G}}(G \times T_eG))$ to be the even vector field on $\mathcal{G} \times T_e\mathcal{G}$ corresponding to the following section σ_X of $T\mathcal{G} \times T(T_e\mathcal{G}) \cong T(\mathcal{G} \times T_e\mathcal{G}) \to \mathcal{G} \times T_e\mathcal{G}$. We denote the zero-section of $T\mathcal{G} \to \mathcal{G}$ by σ_0 and the canonical inclusion $T_e\mathcal{G} \to T\mathcal{G}$ by i_e. Then $\sigma_X := (T\mu \circ (\sigma_0 \times i_e), 0)$, where $T\mu : T\mathcal{G} \times T\mathcal{G} \cong T(\mathcal{G} \times \mathcal{G}) \to T\mathcal{G}$ is the tangential morphism associated to the multiplication morphism. (For simplicity, we write 0 for the zero-section of $T(T_e\mathcal{G}) \to T_e\mathcal{G}$ here and in the sequel.)

Let us recall that for $\mathcal{S}$ an arbitrary supermanifold, and $\phi : \mathcal{M} \to \mathcal{N}$ a morphism between supermanifolds, we have an induced map $\phi(\mathcal{S}) : \mathcal{M}(\mathcal{S}) = \mathrm{Mor}(\mathcal{S}, \mathcal{M}) \to \mathrm{Mor}(\mathcal{S}, \mathcal{N}) = \mathcal{N}(\mathcal{S})$, $\phi(\mathcal{S})(\psi) := \phi \circ \psi$. Given a finite-dimensional Lie superalgebra $\mathfrak{g}$ or a Lie supergroup $\mathcal{G}$, one easily checks that for all $k \geq 0$, $\mathfrak{g}(\mathbb{R}^{0|k})$ resp. $\mathcal{G}(\mathbb{R}^{0|k})$ is a finite-dimensional classical (i.e. even) Lie algebra resp. Lie group. Furthermore, $T_e(\mathcal{G}(\mathbb{R}^{0|k}))$ is canonically isomorphic to $(T_e\mathcal{G})(\mathbb{R}^{0|k})$, where the first e is the obvious constant morphism from $\mathbb{R}^{0|k}$ to $\mathcal{G}$ and the second e denotes the neutral element of $\mathcal{G}$. (Compare, e.g., [16] for more information on the superpoint approach to Lie supergroups.)

Lemma 4.1. *Let $\mathcal{G}$ be a Lie supergroup with multiplication $\mu^{\mathcal{G}}$, and the vector field X as above. Then*

(i) the induced vector field $\widetilde{X}$ on the underlying manifold $G \times T_eG$ is given as

$$\widetilde{X}_{(g,\xi)} = (\xi^L(g), 0) \quad \forall (g, \xi) \in G \times T_eG,$$

(ii) the (even) vector fields $\widetilde{X}$ and X are complete.

Proof. (i) For $k \geq 0$, let $\sigma_X(\mathbb{R}^{0|k})$ be the section of $T(\mathcal{G}(\mathbb{R}^{0|k})) \times T(T_e\mathcal{G}(\mathbb{R}^{0|k})) \to \mathcal{G}(\mathbb{R}^{0|k}) \times T_e\mathcal{G}(\mathbb{R}^{0|k})$ induced by σ_X, and let X^k be the corresponding derivation on $\mathcal{G}(\mathbb{R}^{0|k}) \times T_e\mathcal{G}(\mathbb{R}^{0|k})$. Since $\mathcal{G}(\mathbb{R}^{0|k}) \times (T_e\mathcal{G})(\mathbb{R}^{0|k})$ is an ungraded manifold,

$$\begin{aligned}
\sigma_X(\mathbb{R}^{0|k})(g, \xi) &= (T\mu^{\mathcal{G}} \circ (\sigma_0 \times i_e) \circ (g \times \xi), 0) \\
&= (T\mu^{\mathcal{G}} \circ (0_g \times \xi), 0) \\
&= \Big((T\mu^{\mathcal{G}(\mathbb{R}^{0|k})})_{(g,e)}(0, \xi), 0\Big) \\
&= \Big((Tl_g^{\mathcal{G}(\mathbb{R}^{0|k})})_e(\xi), 0\Big),
\end{aligned}$$

where $\mu^{\mathcal{G}(\mathbb{R}^{0|k})}$ is the multiplication on $\mathcal{G}(\mathbb{R}^{0|k})$ and $l_g^{\mathcal{G}(\mathbb{R}^{0|k})}$ is the left-multiplication by the element g of the group $\mathcal{G}(\mathbb{R}^{0|k})$. We conclude that $\sigma_X(\mathbb{R}^{0|k})(g, \xi)$ (or equivalently $X^k_{(g,\xi)}$) corresponds to $(\xi^L(g), 0)$, where ξ^L is the unique left-invariant vector field on $\mathcal{G}(\mathbb{R}^{0|k})$ such that its value in e is ξ. (Observe that $\mathcal{G}(\mathbb{R}^{0|k})$ is a classical Lie group and not only a group object in the category of supermanifolds, allowing us to argue “point-wise”.)
(ii) The flows of X^k are simply given by $F^{X^k} : \mathbb{R} \times \mathcal{G}(\mathbb{R}^{0|k}) \times (T_e\mathcal{G})(\mathbb{R}^{0|k}) \to \mathcal{G}(\mathbb{R}^{0|k}) \times (T_e\mathcal{G})(\mathbb{R}^{0|k})$, $(t, g, \xi) \mapsto (g \cdot \exp^{\mathcal{G}(\mathbb{R}^{0|k})}(t\xi), \xi)$. All fields X^k are thus complete, in particular this holds for $\widetilde{X} = X^0$, the induced vector field on $G = \mathcal{G}(\mathbb{R}^{0|0})$. By Theorem 2.3 the flow $F^X : \mathbb{R} \times \mathcal{G} \times T_e\mathcal{G} \to \mathcal{G} \times T_e\mathcal{G}$ is then global as well, i.e. X is complete. □

Definition 4.2. *Let $\mathcal{G} = (G, \mathcal{O}_{\mathcal{G}})$ be a Lie supergroup with multiplication μ and neutral element e, and with the even vector field X and its flow morphism*

$F = F^X$ *as above. Then the "exponential morphism of $\mathcal{G}$" is given by* $exp^{\mathcal{G}} = proj_1 \circ F \circ inj^{\mathbb{R}\times\mathcal{G}\times T_e\mathcal{G}}_{\{1\}\times\{e\}\times T_e\mathcal{G}} : T_e\mathcal{G} \to \mathcal{G}$, *where* $proj_1 : \mathcal{G}\times T_e\mathcal{G} \to \mathcal{G}$ *is the projection on the first factor. Diagrammatically, one has*

$$\begin{tikzcd} \mathbb{R}\times\mathcal{G}\times T_e\mathcal{G} \supset \{1\}\times\{e\}\times T_e\mathcal{G} \arrow[r, "F"] & \mathcal{G}\times T_e\mathcal{G} \arrow[d, "proj_1"] \\ T_e\mathcal{G} \arrow[u, "\cong"] \arrow[r, "exp^{\mathcal{G}}"'] & \mathcal{G}. \end{tikzcd}$$

THEOREM 4.3. *The exponential morphism* $exp^{\mathcal{G}} : T_e\mathcal{G} \to \mathcal{G}$ *for a Lie supergroup* $\mathcal{G}$ *fulfills and is uniquely determined by the following condition: for all* $k \geq 0$, $exp^{\mathcal{G}}(\mathbb{R}^{0|k}) : T_e\mathcal{G}(\mathbb{R}^{0|k}) \to \mathcal{G}(\mathbb{R}^{0|k})$ *is the exponential map* $exp^{\mathcal{G}(\mathbb{R}^{0|k})}$ *of the finite-dimensional, ungraded Lie group* $\mathcal{G}(\mathbb{R}^{0|k})$.

PROOF. Using the notations of Lemma 4.1, a straightforward calculation shows that the flow F^{X^k} of X^k on $(\mathcal{G}\times T_e\mathcal{G})(\mathbb{R}^{0|k})$ is given as follows $(t,(g,\xi)) \mapsto F^X \circ \mathrm{inj}_t \circ (g\times\xi)$ and, notably, we have $F^{X^k} \circ \mathrm{inj}_{(1,e)}(\xi) = F^X \circ \mathrm{inj}_1 \circ (e\times\xi) = F^X \circ \mathrm{inj}_{(1,e)} \circ \xi$. Hence

$$\begin{aligned} exp^{\mathcal{G}(\mathbb{R}^{0|k})}(\xi) &= \mathrm{proj}_1 \circ F^{X^k} \circ \mathrm{inj}_{(1,e)}(\xi) \\ &= \mathrm{proj}_1 \circ F^X \circ \mathrm{inj}_{(1,e)} \circ \xi \\ &= \exp^{\mathcal{G}} \circ \xi \\ &= (\exp^{\mathcal{G}})(\mathbb{R}^{0|k})(\xi). \end{aligned}$$

On the other hand, it is clear that the subcategory of superpoints with objects $\{\mathbb{R}^{0|k} | k \geq 0\}$ generates the category of supermanifolds in the following sense: given two different morphisms $\phi_1, \phi_2 : \mathcal{M} \to \mathcal{N}$ between supermanifolds, there exists a $k \geq 0$ and a morphism $\psi : \mathbb{R}^{0|k} \to \mathcal{M}$ such that $\phi_1 \circ \psi \neq \phi_2 \circ \psi$. Thus it follows that the family $\{exp^{\mathcal{G}}(\mathbb{R}^{0|k}) | k \geq 0\}$ uniquely fixes $exp^{\mathcal{G}}$. □

(4.3) Part of our interest in the integration of supervector fields stemmed from the construction of a geodesic flow in [5] . Given a homogeneous (i.e., even or odd) Riemannian metric on a supermanifold $\mathcal{M}$, the associated geodesic flow is, in fact, defined as the flow of an appropriate Hamiltonian vector field on its (co-)tangent bundle.

(4.4) An odd vector field X_1 on a supermanifold $\mathcal{M}$ is called "homological" if $X_1 \circ X_1 = \frac{1}{2}[X_1, X_1] = 0$. Its flow is given by the following $\mathbb{R}^{0|1}$-action $\Phi : \mathbb{R}^{0|1} \times \mathcal{M} \to \mathcal{M}$, $\Phi^*(f) = f + \tau \cdot X_1(f), \forall f \in \mathcal{O}_{\mathcal{M}}(M)$. This action can, of course, be extended to a (not almost-effective) $(\mathbb{R}^{1|1}, \mu_{a,b})$-action if and only if $b = 0$.

Typical examples arise as follows: let $E \to M$ be a vector bundle over a classical manifold and T be an "appropriate" $\mathbb{R}$-linear operator on sections of $\wedge E^*$, then T yields a vector field on $\Pi E := (M, \Gamma^\infty_{\wedge E^*})$, the supermanifold associated to $E \to M$ by the Batchelor construction. If $E = TM \to M$, we have $\Gamma^\infty_{\wedge E^*} = \Omega^\bullet_M(M)$, the sheaf of differential forms on M, with its natural $\mathbb{Z}/2\mathbb{Z}$-grading. Taking $T = d$, we get an odd vector field that is obviously homological. Taking $T = \iota_\xi$, the contraction of differential forms with a vector field ξ on the (here classical) base manifold M, we again get a homological vector field on ΠTM. Since $\iota_\xi \circ \iota_\eta + \iota_\eta \circ \iota_\xi = 0$, the vector space of all vector fields on M is realized as a commutative, purely odd sub Lie superalgebra of all vector fields on ΠTM. More generally, a section s of $E \to M$ always gives rise to a contraction $\iota_s : \Gamma^\infty_{\wedge E^*}(M) \to \Gamma^\infty_{\wedge E^*}(M)$ that is an odd derivation (i.e. an anti-derivation of degree -1 in more classical language). Furthermore, given two sections s and t of E, the associated odd vector fields commute. In the article [1] this construction is studied in the special case that E is the spinor bundle over a classical spin manifold M.

(4.5) If G is a Lie group acting on a classical manifold M, then the action can of course be lifted to an action on the total space of the tangent and the cotangent bundle of M. The induced vector fields on ΠTM are even and ξ in $\mathfrak{g} = \mathrm{Lie}(G)$, the Lie algebra of G, acts on $\mathcal{O}_{\Pi TM}$ by $\mathcal{L}_\xi$, the Lie derivative with respect to the fundamental vector field on M associated to ξ. Putting together these fields and the contractions constructed in Example (4.4), we get a Lie superalgebra with underlying vector space $\mathfrak{g} \oplus \mathfrak{g}$, the first resp. second summand being the even resp. odd part. The commutators in $\mathfrak{g} \oplus \mathfrak{g}$ are given as follows: $[\mathcal{L}_\xi, \mathcal{L}_\eta] = \mathcal{L}_{[\xi,\eta]}$, $[\iota_\xi, \iota_\eta] = 0$ and $[\mathcal{L}_\xi, \iota_\eta] = \iota_{[\xi,\eta]}$ for all $\xi, \eta \in \mathfrak{g}$. In fact, the above can be interpreted as an action of the Lie supergroup ΠTG on ΠTM.
The Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ can be extended by a one-dimensional odd direction generated by the exterior derivative d. The extended algebra $\mathfrak{g} \oplus (\mathfrak{g} \oplus \mathbb{R} \cdot d)$, still a sub Lie superalgebra of $\mathcal{T}_{\Pi TM}(M)$, has the following additional commutators:

$$[\mathcal{L}_\xi, d] = 0 \text{ and } [d, \iota_\xi] = \mathcal{L}_\xi \text{ for all } \xi \in \mathfrak{g}.$$

(4.6) If $\mathcal{M} = \mathbb{R}^{1|1}$ with coordinates (x, ξ), the vector field $X_1 = \partial_\xi + \xi\partial_x$ is obviously odd and non-homological since $X_1 \circ X_1 = \partial_x$. Direct inspection shows that the map $\Phi : \mathbb{R}^{0|1} \times \mathcal{M} \to \mathcal{M}, \Phi^*(f) = f + \tau \cdot X_1(f)$ (compare Example (4.1)) does not fulfill $\partial_\tau \circ \Phi^* = \Phi^* \circ X_1$. Nevertheless, the trivial extension of Φ to a morphism $F : \mathbb{R}^{1|1} \times \mathcal{M} \to \mathcal{M}$ is the flow of X_1 in the sense of Theorem 2.3, fulfilling the initial condition $\phi = \mathrm{id}_\mathcal{M}$. We underline that this map is not an action of $\mathbb{R}^{1|1}$. Upon extending X_1 to $X := X_0 + X_1$ with $X_0 := \frac{1}{2}[X_1, X_1] = X_1 \circ X_1$, we obtain by Theorem 3.4 an action of $(\mathbb{R}^{1|1}, \mu_{1,0})$ as the flow map of X. Let us observe that the above vector field X_1 (and not X) is the prototype of what is called a "supersymmetry" in the physics literature (compare, e.g., [21] and the other relevant texts in these volumes.). More recently, the associated Lie supergroup structure on $\mathbb{R}^{1|1}$ (and an analogous structure on $\mathbb{R}^{2|1}$) were introduced by S. Stolz and P. Teichner into their program to geometrize the cocycles of elliptic cohomology (compare

[7] and [18]).

Obviously, one can generalize this construction to $\mathbb{R}^{m|n}$ $(m, n \geq 1)$ with coordinates $(x_1, \ldots, x_m, \xi_1, \ldots, \xi_n)$ by setting for $1 \leq k \leq m$, $1 \leq \alpha \leq n$

$$D_{\alpha,k} := \partial_{\xi_\alpha} + \xi_\alpha \cdot \partial_{x_k}.$$

We then have $[D_{\alpha,k}, D_{\beta,l}] = \delta_{\alpha,\beta} \cdot (\partial_{x_k} + \partial_{x_l})$ and $[D_{\alpha,k}, \partial_{x_l}] = 0$. Taking $X_1 = D_{\alpha,k}$ and $X_0 = \partial_{x_k}$ we reproduce a copy of the preceding situation.

(4.7) The vector field $X = X_0 + X_1$ on $\mathcal{M} = \mathbb{R}^{1|1}$ with $X_0 = \partial_x + \xi \cdot \partial_\xi$ and $X_1 = \partial_\xi + \xi \cdot \partial_x$, already mentioned in the introduction, is a very simple example of an inhomogeneous vector field not generating any local $\mathbb{R}^{1|1}$-action, since, e.g., condition (iii) in Theorem 3.4 is violated. Thus integration of X is only possible in the sense of Theorem 2.3, i.e. upon using the evaluation map. Let us observe that the sub Lie superalgebra $\mathfrak{g}$ of $\mathcal{T}_\mathcal{M}(M)$ generated by X, i.e. by X_0 and X_1 since sub Lie superalgebras are by definition graded sub vector spaces, is four-dimensional with two even generators Z, W and two odd generators D, Q such that: Z is central, $[W, D] = Q$, $[W, Q] = D$, $D^2 = -Q^2 = Z$ and $[D, Q] = 0$. (This amounts in physical interpretation to the presence of two commuting supersymmetries D and Q, generating the same supersymmetric Hamiltonian Z plus an even symmetry commuting with the Hamiltonian and exchanging the supersymmetries D and Q.)

5 Flow of a holomorphic vector field on a holomorphic supermanifold

In this section we extend our results to the holomorphic case. We will always denote the canonical coordinates on $\mathbb{C}^{1|1}$ by z and ζ and write ∂_z resp. ∂_ζ for $\frac{\partial}{\partial z}$ resp. $\frac{\partial}{\partial \zeta}$. All "auxiliary" supermanifolds $\mathcal{S}$ and morphisms having these as sources will be assumed to be holomorphic in this section.

Definition 5.1. *Let $\mathcal{M} = (M, \mathcal{O}_\mathcal{M})$ be a holomorphic supermanifold and X a holomorphic vector field on $\mathcal{M}$ and $\mathcal{S}$ a supermanifold with a morphism $\phi \in Mor(\mathcal{S}, \mathcal{M})$ and z_0 in $\mathbb{C}$.*

(1) A "flow for X (with initial condition ϕ and with respect to z_0)" is an open sub supermanifold $\mathcal{V} \subset \mathbb{C}^{1|1} \times \mathcal{S}$, such that $\{z_0\} \times S \subset V$ (S and V the bodies of $\mathcal{S}$ and $\mathcal{V}$) and such that for all s in S, $(\mathbb{C} \times \{s\}) \cap V$ is connected, together with a morphism of holomorphic supermanifolds $F : \mathcal{V} \to \mathcal{M}$ such that

$$\begin{aligned} (inj\,{}_{\mathbb{C}}^{\mathbb{C}^{1|1}})^* \circ (\partial_z + \partial_\zeta) \circ F^* &= (inj\,{}_{\mathbb{C}}^{\mathbb{C}^{1|1}})^* \circ F^* \circ X \text{ and} \\ F \circ inj\,{}_{\{z_0\}\times\mathcal{S}}^{\mathcal{V}} &= \phi. \end{aligned}$$

Sometimes we call the supermanifold $\mathcal{V}$ (or abusively its body V) the "flow domain" of X.

(2) A flow domain $\mathcal{V}$ of a flow $(\mathcal{V}, F)$ for X is called "fibrewise 1-connected (relative to the projection $\mathcal{V} \to \mathcal{S}$)" (or "fibrewise 1-connected over $\mathcal{S}$") if for all s in S, $(\mathbb{C} \times \{s\}) \cap V$ is connected and simply connected.

Remark. We avoid the term "complex supermanifold" here, since it is often used to describe supermanifolds that are, as ringed spaces, locally isomorphic to open sets $D \subset \mathbb{R}^k$ with structure sheaf $\mathcal{C}^\infty_D \otimes_{\mathbb{R}} \wedge \mathbb{C}^l$. "Holomorphic supermanifolds" are of course locally isomorphic to open sets $D \subset \mathbb{C}^k$ with structure sheaf $\mathcal{O}_D \otimes_{\mathbb{C}} \wedge \mathbb{C}^l$, where $\mathcal{O}_D$ denotes the sheaf of holomorphic functions on D.

Let us first give the holomorphic analogue of Lemma 2.1.

Lemma 5.2. *Let $\mathcal{U} \subset \mathbb{C}^{m|n}$ and $\mathcal{W} \subset \mathbb{C}^{p|q}$ be superdomains, X a holomorphic vector field on $\mathcal{W}$ (not necessarily homogeneous), ϕ in Mor$(\mathcal{U}, \mathcal{W})$ and z_0 in $\mathbb{C}$. Then*

(i) it exists a holomorphic flow $(V, \widetilde{F})$ for the reduced holomorphic vector field $\widetilde{X}$ on U with initial condition $\widetilde{\phi}$ and with respect to z_0 such that the flow domain $V \subset \mathbb{C} \times U$ is fibrewise 1-connected over U. Furthermore on every flow domain in the sense of Definition 5.1 the holomorphic flow is unique.

(ii) Let now $(V, \widetilde{F})$ be a fibrewise 1-connected flow domain for $\widetilde{X}$ over U. Then there exists a unique holomorphic flow $F : \mathcal{V} \to \mathcal{W}$ for X, with $\mathcal{V}$ the open sub supermanifold of $\mathbb{C}^{1|1} \times \mathcal{U}$ with body equal to V.

Remark. The example of the holomorphic vector field $X = (w^2 + w^3\xi_1\xi_2)\frac{\partial}{\partial w}$ on $\mathcal{W} = \mathbb{C}^{1|2}$ with coordinates (w, ξ_1, ξ_2) shows that the condition of fibrewise 1-connectivity of V is not only a technical assumption to our proof. The underlying vector field $\widetilde{X} = w^2\frac{\partial}{\partial w}$ on $\mathbb{C}$, with initial condition $\widetilde{\phi} = \mathrm{id} : \mathbb{C} \to \mathbb{C}$ with respect to $z_0 = 0$, can be integrated to the flow $\widetilde{F} : V = \mathbb{C}^2 \backslash \{z \cdot w = 1\} \to \mathbb{C}$, $\widetilde{F}(z, w) = \frac{1}{1/w - z}$ for $w \neq 0$ and $\widetilde{F}(z, 0) = 0$. Obviously, for $w \neq 0$, $(\mathbb{C} \times \{w\}) \cap V$ is connected, but not simply connected. Direct inspection now shows that the flow F of X with initial condition $\phi = \mathrm{id}$ and with respect to $z_0 = 0$ cannot be defined on the whole of $\mathcal{V} = (V, \mathcal{O}_{\mathbb{C}^{1|1} \times \mathbb{C}^{1|2}}|_V)$.

Proof of Lemma 5.2. (i) The existence (and the stated unicity property) of a flow $(\check{V}, \widetilde{F})$ for $\widetilde{X}$, with $\{z_0\} \times U \subset \check{V} \subset \mathbb{C} \times U$ fulfilling the initial condition $\widetilde{\phi}$ with respect to $z_0 \in \mathbb{C}$ is of course a classical application of the existence of solutions of holomorphic ordinary differential equations (see, e.g., [8]). Upon reducing the size of $\check{V}$ we always find flow domains that are fibrewise 1-connected.

(ii) The induction procedure of the proof of Lemma 2.1 can be applied here upon recalling the following standard facts from the theory of holomorphic linear ordinary differential equations (compare, e.g., [8]):

Fact 1. *Let $\Omega \subset \mathbb{C}$ be open and 1-connected (i.e. connected and simply connected), and $z_0 \in \Omega$. If $A : \Omega \to Mat(N \times N, \mathbb{C})$ and $b : \Omega \to \mathbb{C}^N$ are holomorphic and $\psi_0 \in \mathbb{C}^N$, then there exists a unique holomorphic map $\psi : \Omega \to \mathbb{C}^N$ fulfilling*

$$\frac{\partial}{\partial z}\psi(z) = A(z)\psi(z) + b(z)$$

such that $\psi(z_0) = \psi_0$.

Fact 2. *Let Ω and z_0 be as in Fact 1, and let P be a holomorphic manifold ("a parameter space"), and let $A : \Omega \times P \to Mat(N \times N, \mathbb{C})$, $b : \Omega \times P \to \mathbb{C}^N$, as well as $\psi_0 : P \to \mathbb{C}^N$ be holomorphic maps. Then there exists a unique holomorphic map $\psi : \Omega \times P \to \mathbb{C}^N$ fulfilling*

$$\frac{\partial}{\partial z}\psi(z, x) = A(z, x)\psi(z, x) + b(z, x)$$

such that $\psi(z_0, x) = \psi_0(x)$, $\forall x \in P$.

Obviously, to apply these facts in our context, we need the fibrewise 1-connectivity of the "underlying flow domain" V for $\widetilde{X}$. $\square$

Before stating and proving our central result in the holomorphic case, we give the following useful shorthand.

Definition 5.3. *Let $\mathcal{S}$ be a supermanifold, z_0 in $\mathbb{C}$ and $\mathcal{N} \subset \mathbb{C}^{1|1} \times \mathcal{S}$ be an open sub supermanifold containing $\{z_0\} \times \mathcal{S}$. Then $\mathcal{N}^{z_0}$ is defined as the open sub supermanifold of $\mathcal{N} = (N, \mathcal{O}_{\mathcal{N}})$ whose body equals $\coprod_{s \in S} ((\mathbb{C} \times \{s\}) \cap N)^{(z_0,s)}$, where $((\mathbb{C} \times \{s\}) \cap N)^{(z_0,s)}$ is the connected component of $(\mathbb{C} \times \{s\}) \cap N$ containing (z_0, s).*

Remark. A flow domain $\mathcal{V}$ in the sense of Definition 5.1(1) is always open and contains $\{z_0\} \times \mathcal{S}$. Furthermore for all s in S the section $(\mathbb{C} \times \{s\}) \cap V$ is connected. The preceding definition will in fact be useful for discussing intersections of flow domains in the next theorem.

Theorem 5.4. *Let $\mathcal{M}$ be a holomorphic supermanifold and X a holomorphic vector field on $\mathcal{M}$, and let $\mathcal{S}$ be a holomorphic supermanifold with a holomorphic morphism $\psi : \mathcal{S} \to \mathcal{M}$, and $z_0 \in \mathbb{C}$. Then*

(i) there exists a flow $(V, \widetilde{F})$ for the reduced vector field $\widetilde{X}$ with initial condition $\widetilde{\phi}$ with respect to z_0 such that the flow domain $V \subset \mathbb{C} \times S$ is fibrewise 1-connected over S,

(ii) if $(V, \widetilde{F})$ is as in (i), then there exists a unique flow for X with initial condition ϕ with respect to z_0, $F : \mathcal{V} \to \mathcal{M}$, where $\mathcal{V} \subset \mathbb{C}^{1|1} \times \mathcal{S}$ is the open sub supermanifold with body V,

(iii) if $(\mathcal{V}_1, F_1)$ *and* $(\mathcal{V}_2, F_2)$ *are two flows for* X*, both with initial condition* ϕ *with respect to* z_0*, then* $F_1 = F_2$ *on the flow domain* $(\mathcal{V}_1 \cap \mathcal{V}_2)^{z_0}$*,*

(iv) there exists maximal flow domains for X *and the germs of their flows coincide on* $\{z_0\} \times \mathcal{S}$*.*

Proof. (i) and (ii). It easily follows from Lemma 5.2 that $\mathcal{S}$ can be covered by open sub supermanifolds $\{\mathcal{U}^\alpha | \alpha \in A\}$ such that $X_{|\mathcal{U}^\alpha}$ has a holomorphic flow with initial condition $\phi_{|\mathcal{U}_\alpha}$ with respect to z_0, $F^\alpha : \mathcal{V}^\alpha = \Delta_{r_\alpha}(z_0) \times \mathbb{C}^{0|1} \times \mathcal{U}^\alpha \to \mathcal{M}$, where $r_\alpha > 0$, and for $r > 0$, $\Delta_r(z_0)$ is the open disc of radius r centred in z_0. Since $F^\alpha = F^\beta$ on $\mathcal{V}^\alpha \cap \mathcal{V}^\beta$ by the unicity part of Lemma 5.2, we can glue these flows to obtain $\mathbb{C}^{1|1} \times \mathcal{S} \supset \mathcal{V} := \bigcup_{\alpha \in A} \mathcal{V}^\alpha \xrightarrow{F} \mathcal{M}$, a flow for X on $\mathcal{M}$ with initial condition ϕ with respect to z_0. Obviously, the "fibres" $(\mathbb{C} \times \{s\}) \cap V$ are 1-connected for all s in S, i.e., the flow domain $\mathcal{V}$ is fibrewise 1-connected over $\mathcal{S}$.
Note that if we have a flow $\widetilde{F}$ for the reduced vector field $\widetilde{X}$ on a flow domain V that is fibrewise 1-connected over S, then part (ii) of Lemma 5.2 yields a flow for X defined on $\mathcal{V} = (V, \mathcal{O}_{\mathbb{C}^{1|1} \times \mathcal{S}}|_V)$.

(iii) The body of $(\mathcal{V}_1 \cap \mathcal{V}_2)^{z_0}$ has as a strong deformation retract the body of $\{z_0\} \times \mathcal{S}$. Without loss of generality we can assume that $\mathcal{S}$ and thus $(\mathcal{V}_1 \cap \mathcal{V}_2)^{z_0}$ are connected. The local unicity in Lemma 5.2 together with the identity principle for holomorphic morphisms of holomorphic supermanifolds imply that $F_1 = F_2$ on $(\mathcal{V}_1 \cap \mathcal{V}_2)^{z_0}$.

(iv) By Zorn's lemma we get maximal flow domains and by part (iii) the corresponding flows coincide near $\{z_0\} \times \mathcal{S}$. $\square$

Remarks. (1) The non-unicity of maximal flow domains for holomorphic vector fields is a well-known phenomenon already in the ungraded case. A simple example for this is the vector field X on $\mathbb{C}^*$ such that $X(w) = \frac{1}{w}\frac{\partial}{\partial w}$ for all w in $\mathbb{C}^*$.
(2) Given the above theorem, the analogues of Lemma 3.1, Proposition 3.3 and Theorem 3.4 can now without difficulty be proven to hold for holomorphic supermanifolds.

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