The Variety of Polar Simplices

Kristian Ranestad and Frank-Olaf Schreyer[1]

Received: April 25, 2012
Revised: January 29, 2013

Communicated by Gavril Farkas

Abstract. A collection of n distinct hyperplanes $L_i = \{l_i = 0\} \subset \mathbf{P}^{n-1}$, the $(n-1)$-dimensional projective space over an algebraically closed field of characteristic not equal to 2, is a polar simplex of a smooth quadric $Q^{n-2} = \{q = 0\}$, if each L_i is the polar hyperplane of the point $p_i = \bigcap_{j \neq i} L_j$, equivalently, if $q = l_1^2 + \ldots + l_n^2$ for suitable choices of the linear forms l_i. In this paper we study the closure $VPS(Q,n) \subset \mathrm{Hilb}_n(\check{\mathbf{P}}^{n-1})$ of the variety of sums of powers presenting Q from a global viewpoint: $VPS(Q,n)$ is a smooth Fano variety of index 2 and Picard number 1 when $n < 6$, and $VPS(Q,n)$ is singular when $n \geq 6$.

2010 Mathematics Subject Classification: 14J45, 14M
Keywords and Phrases: Fano n-folds, Quadric, polar simplex, syzygies

Contents

[1]The first author was partially supported by an UCM-EEA grant under the NILS program, and both authors were supported by Institut Mittag-Leffler.

1. Introduction

Let $Q = \{q = 0\}$ be a $(n-2)$-dimensional smooth quadric defined over the complex numbers, or any algebraically closed field of characteristic not equal to 2. We denote the projective space containing Q by $\check{\mathbf{P}}^{n-1}$ because its dual space $\mathbf{P}^{n-1}$ plays the major role in this paper. A collection $L_1 = \{l_1 = 0\}, \dots, L_n = \{l_n = 0\}$ of n hyperplanes is a polar simplex iff each L_i is the polar of the point $p_i = \bigcap_{j \neq i} L_j$, equivalently, iff the quadratic equation

$$q = \sum_{i=1}^{n} l_i^2$$

holds for suitable choices of the linear forms l_i defining L_i. In this paper we study the collection of polar simplices, or equivalently, the variety of sums of powers presenting q from a global viewpoint.
We may regard a polar simplex as a point in $\mathrm{Hilb}_n(\mathbf{P}^{n-1})$. Let $VPS(Q,n) \subset \mathrm{Hilb}_n(\mathbf{P}^{n-1})$ be the closure of the variety of sums of n squares presenting Q. The first main result is:

Theorem 1.1. *If $2 \le n \le 5$, then $VPS(Q,n)$ is a smooth rational $\binom{n}{2}$-dimensional Fano variety of index 2 and Picard number 1. If $n \ge 6$, then $VPS(Q,n)$ is a singular rational $\binom{n}{2}$-dimensional variety.*

If $n = 2$, then $VPS(Q,n) = \mathbf{P}^1$, and if $n = 3$, then $VPS(Q,n)$ is a rational Fano threefold of index 2 and degree 5 (cf.[Muk92]).
The quadratic form defines a collineation $q : \check{\mathbf{P}}^{n-1} \to \mathbf{P}^{n-1}$, let $q^{-1} : \mathbf{P}^{n-1} \to \check{\mathbf{P}}^{n-1}$ be the inverse collineation, and $Q^{-1} = \{q^{-1} = 0\} \subset \mathbf{P}^{n-1}$ the corresponding quadric. Consider the double Veronese embedding $Q^{-1} \to \mathbf{P}^{\binom{n+1}{2}-2}$, and let TQ^{-1} be the image by the Gauss map of tangent spaces $Q^{-1} \to \mathbb{G}(n-1, \binom{n+1}{2}-1)$. Our second main result is:

Theorem 1.2. *$VPS(Q,n)$ has a natural embedding in the Grassmannian variety $\mathbb{G}(n-1, \binom{n+1}{2}-1)$ and contains the image TQ^{-1} of the Gauss map of the quadric Q^{-1} in its Veronese embedding. When $n = 4$ or $n = 5$. the restriction of the Plcker line bundle generates the Picard group of $VPS(Q,n)$, and the degree is 310, resp. 395780.*

We denote the coordinate ring of $\mathbf{P}^{n-1}$ by $S = \mathbb{C}[x_1, \dots, x_n]$ and the coordinate ring of the dual $\check{\mathbf{P}}^{n-1}$ by $T = \mathbb{C}[y_1, \dots, y_n]$. In particular $S_1 = (T_1)^*$, so we may set $\mathbf{P}^{n-1} = \mathbf{P}(T_1)$, the projective space of 1-dimensional subspaces of T_1 with coordinate functions in S, and $\check{\mathbf{P}}^{n-1} = \mathbf{P}(S_1)$ with coordinate functions in T. Let $q \in T = \mathbb{C}[y_1, \dots, y_n]$ be a quadratic form defining the smooth $(n-2)$-dimensional quadric $Q \subset \check{\mathbf{P}}^{n-1} = \mathbf{P}(S_1)$. Regard $[q]$ as a point in $\mathbf{P}(T_2)$ and consider the Veronese variety $V_2 \subset \mathbf{P}(T_2)$ of squares,

$$V_2 = \{[l^2] \in \mathbf{P}(T_2) | l \in \mathbf{P}(T_1)\}.$$

Then a polar simplex to Q is simply a collection of n points on V_2 whose linear span contains $[q]$. Any length n subscheme $\Gamma \subset V_2$ whose span in $\mathbf{P}(T_2)$

contains $[q]$ is called an APOLAR SUBSCHEMES OF LENGTH n TO Q. The closure $VPS(Q,n)$ of the polar simplices in $\mathrm{Hilb}_n(\mathbf{P}(T_1))$ consists of apolar subschemes of length n. We denote by $VAPS(Q,n)$ the subset of $\mathrm{Hilb}_n(\mathbf{P}(T_1))$, with reduced scheme structure, parameterizing all apolar subschemes of length n to Q. Our third main result is:

THEOREM 1.3. *The algebraic set $VAPS(Q,n)$ is isomorphic to the complete linear section*

$$VAPS(Q,n) = \langle TQ^{-1}\rangle \cap \mathbb{G}(n-1,T_2/q) \subset \mathbf{P}(\wedge^{n-1}(T_2/q))$$

in the Plücker space. For $n \leq 6$ the two subschemes $VPS(Q,n)$ and $VAPS(Q,n)$ coincide. For $n \geq 24$, the scheme $VAPS(Q,n)$ has more than one component.

Notice that we do not claim that the linear section $\langle TQ^{-1}\rangle \cap \mathbb{G}(n-1,T_2/q)$ is reduced, only that its reduced structure coincides with $VAPS(Q,n)$. The linear span $\langle TQ^{-1}\rangle$ has dimension $\binom{2n-1}{n-1}-\binom{2n-3}{n-2}-1$, while the Grassmannian has dimension $(n-1)\binom{n}{2}$ in $\binom{\binom{n+1}{2}-1}{n-1}$-dimensional Plücker space. So this linear section is far from a proper linear section when $n \geq 4$, i.e. the codimension of $VAPS(Q,n)$ in the Grassmannian is much less than the codimension of its linear span in the Plücker space.

We find a covering of $VAPS(Q,n)$ by affine subschemes $V_h^{aff}(n)$ that are contractible to a point $[\Gamma_p] \in VPS(Q,n)$ (Lemma 5.3). Therefore the apolar subschemes Γ_p play a crucial point. Let us explain what they are: The projection of the Veronese variety $V_2 \subset \mathbf{P}(T_2)$ from $[q] \in \mathbf{P}(T_2)$ is a variety $V_{2,q} \subset \mathbf{P}(T_2/q)$. Since a polar simplex to Q is a collection on n points on V_2 whose span contains $[q]$, the variety $VSP(Q,n)$ is naturally embedded in and in fact coincides with the variety of $(n-2)$-secant spaces of the projected Veronese variety $V_{2,q}$. The double Veronese embedding of Q^{-1} is a linearly normal subvariety in $V_{2,q}$ that spans $\mathbf{P}(T_2/q)$. For each point $p \in Q^{-1}$ consider the tangent space to Q^{-1} in this embedding. This tangent space intersects $V_{2,q}$ along the subscheme Γ_p, and belong to the boundary of variety of $(n-2)$-secant spaces of $V_{2,q}$.

The affine subscheme $V_h^{aff}(n)$ is contractible to Γ_p, but depend only on a hyperplane: It consists of the apolar subschemes that do not intersect a tangent hyperplane h to Q^{-1}. The point p is simply a point on Q^{-1} that does not lie in this hyperplane.

Our computations show that the affine scheme $V_h^{aff}(n)$ and certain natural subschemes has particularly interesting structure: $V_h^{aff}(n)$ is isomorphic to an affine space when $n < 6$ while $V_h^{aff}(6)$ is isomorphic to a 15-dimensional cone over the 10-dimensional spinor variety (Corollary 5.16). Why this spinor variety appears is quite mysterious to us. Recall that Mukai showed that a general canonical curve of genus 7 is a linear section of the spinor variety. Let $V_p^{loc}(n) \subset VAPS(Q,n)$ be the subscheme of apolar subschemes in $VAPS(Q,n)$ with support at a single point $p \in Q^{-1}$. The subscheme $V_p^{loc}(n)$ is naturally contained in $V_p^{sec}(n)$, the variety of apolar subschemes in $V_h^{aff}(n)$ that contains the point p. We compute these subschemes with *Macaulay2* [GS] when $n < 6$ and find that $V_p^{loc}(5)$ is isomorphic to a 3-dimensional cone over the tangent developable of a rational normal sextic curve. This cone is a codimension 3 linear section of the scheme $V_p^{sec}(5)$, which is isomorphic to a 6-dimensional cone over the intersection of the Grassmannian $\mathbb{G}(2,5)$ with a quadric. Mukai showed that a general canonical curve of genus 6 is a linear section of the intersection of $\mathbb{G}(2,5)$ with a quadric. The appearances in the cases $n = 5, 6$ of a natural variety whose curve sections are canonical curves is both surprising and unclear to us. The computational results are summarized in Table 1 in Section 5.

By the very construction of polar simplices, it is clear that $VPS(Q,n)$ has dimension $\binom{n}{2}$. On the other hand, the special orthogonal group $SO(n,q)$ that preserves the quadratic form q, acts on the set of polar simplices: If we assume that the symmetric matrix of q with respect to the variables in T is the identity matrix, then regarding $SO(n,q)$ as orthogonal matrices the rows define a polar simplex. Matrix multiplication therefore defines a transitive action of $SO(n,q)$ on the set of polar simplices. By dimension count, this action has a finite stabilizer at a polar simplex. This stabilizer is the subgroup $H \subset SO(n,q)$ of rotational symmetries of the hypercube $[-1,1]^n \subset \mathbb{R}^n$ of order $2^{n-1} \cdot n!$ as suggested by an anonymous referee. We get

Proposition 1.4. *$VPS(Q,n)$ is a compactification of the group $SO(n,q)/H$.*

The linear representation of $SO(n,q)$ on T_2 decomposes

$$T_2 = \langle q \rangle \oplus T_{2,q},$$

where the hyperplane $\mathbf{P}(T_{2,q})$ intersect the Veronese variety V_2 along the Veronese image of Q^{-1}. Therefore we may identify $T_2/q = T_{2,q}$ and the projection from $[q]$: $\mathbf{P}(T_2) \dashrightarrow \mathbf{P}(T_{2,q})$ is an $SO(n,q)$-equivariant projection. $Q^{-1} \subset \mathbf{P}(T_{2,q})$ is a closed orbit, and similarly the image TQ^{-1} of the Gauss map is a closed orbit for the induced representation on the Plücker space of $\mathbb{G}(n-1, T_{2,q})$. The linear span of this image is therefore the projectivization of an irreducible representation of $SO(n,q)$. The set of polar simplices form an orbit for the action of $SO(n,q)$, so the linear span of $VPS(Q,n)$ is also the projectivization of an irreducible representation of $SO(n,q)$. Therefore

$$VPS(Q,n) \subset \langle TQ^{-1} \rangle \cap \mathbb{G}(n-1, T_{2,q}).$$

We show that the intersection $\langle TQ^{-1} \rangle \cap \mathbb{G}(n-1, T_{2,q})$ parameterizes all apolar subschemes of length n, hence Theorem 1.1.

The organization of the paper follows distinct approaches to $VPS(Q,n)$. To start with we introduce the classical notion of apolarity and regard polar simplices as apolar subschemes in $\mathbf{P}(T_1)$ of length n with respect to q. We use syzygies to characterize these subschemes among elements of the Hilbert scheme. In fact, polar simplices are characterized by their smoothness, the Betti numbers of their resolution, and their apolarity with respect to q. Allowing singular subschemes, we consider all apolar subschemes of length n. We show in Section 2 that these subschemes naturally appear in the closure $VPS(Q,n)$ of the

set of polar simplices in the Hilbert scheme. For $n > 6$ there may be apolar subschemes of length n that do not belong to the closure $VPS(Q,n)$ of the smooth ones. In fact, we show in Section 2 that at least for $n \geq 24$, there are nonsmoothable apolar subschemes of length n, i.e. that $VPS(Q,n)$ is not the only component of $VAPS(Q,n)$.
The variety $VPS(Q,n)$, in its embedding in $\mathbb{G}(n-1,T_{2,q})$, has order one, i.e. through a general point in $\mathbf{P}(T_{2,q})$ there is a unique $(n-2)$-dimensional linear space that form the span of an apolar subscheme Γ of length n. This is a generalization of the fact that a general symmetric $n \times n$ matrix has n distinct eigenvalues. In Section 3 we use a geometric approach to characterize the generality assumption.
The fact that $VPS(Q,n)$ has order one, means that it is the image of a rational map

$$\gamma : \mathbf{P}(T_{2,q}) \dashrightarrow \mathbb{G}(n-1,T_{2,q}).$$

In Section 4 we use a trilinear form introduced by Mukai to give equations for the map γ. With respect to the variables in T we may associate a symmetric matrix A to each quadratic form $q' \in T_{2,q}$. The Mukai form associates to q' a space of quadratic forms in S_2 that vanish on all the projectivized eigenspaces of the matrix A. For general q' these quadratic forms generate the ideal of the unique common polar simplex of q and q'. This is Proposition 4.2. The Mukai form therefore defines the universal family of polar simplices, although it does not extend to the whole boundary. Common apolar subschemes to q and q', when q' has rank at most $n-2$, form the exceptional locus of the map γ.
We do not compute the image of γ in $\mathbb{G}(n-1,T_{2,q})$. Instead we compute affine perturbations of $[\Gamma_p]$ in $\mathbb{G}(n-1,T_{2,q})$ that correspond to apolar subschemes to Q. These perturbations form the affine subschemes $V_h^{aff}(n)$ that cover $VAPS(Q,n)$. In Section 5 we make extensive computations of these affine subschemes. Each once of them is contractible to a point $[\Gamma_p]$ on the subvariety $TQ^{-1} \subset VPS(Q,n)$. The question of smoothness of $VPS(Q,n)$ is reduced to a question of smoothness of the affine scheme $V_h^{aff}(n)$ at the point $[\Gamma_p]$. For $n \leq 5$ we show that such a point is smooth, while for $n \geq 6$, it is singular. The main result of Section 5 is however Theorem 1.3, that $VAPS(Q,n)$ is a linear section of the Grassmannian.
In the final Section 6 we return to the geometry of $VPS(Q,n)$ and compute the degree by a combinatorial argument for any n. The Fano-index is computed using the natural $\mathbf{P}^{n-2}$-bundle on $VPS(Q,n)$, obtained by restricting the incidence variety over the Grassmannian, and its birational morphism to $\mathbf{P}(T_{2,q})$.
We thank Tony Iarrobino for sharing his insight on Artinian Gorenstein rings with us, Francesco Zucconi for valuable comments on a previous version of this paper.
Let us briefly summarize the notation:

- $\mathbb{C}$ denotes the field of complex numbers.

- $q \in T_2$ is a non-degenerate quadratic form, and defines a collineation $q : S_1 \to T_1$ and a linear form $q : S_2 \to \mathbb{C}$.
- Q is the quadratic hypersurface $\{q = 0\} \subset \mathbf{P}(S_1)$.
- $q^{-1} \in S_2$ is a quadratic form, that defines the collineation $q^{-1} : T_1 \to S_1$ inverse to q and a linear form $q^{-1} : T_2 \to \mathbb{C}$.
- Q^{-1} is the quadratic hypersurface $\{q^{-1} = 0\} \subset \mathbf{P}(T_1)$.
- $q^\perp \subset S_2$ is the kernel of the linear form $q : S_2 \to \mathbb{C}$.
- $T_{2,q}$ is the kernel $(q^{-1})^\perp$ of the linear form $q^{-1} : T_2 \to \mathbb{C}$
- $\pi_q : \mathbf{P}(T_2) \dashrightarrow \mathbf{P}(T_{2,q})$ is the projection from $[q] \in \mathbf{P}(T_2)$, and $V_{2,q} \subset \mathbf{P}(T_{2,q})$ is the image under this projection of the Veronese variety $V_2 \subset \mathbf{P}(T_2)$.

2. Apolar subschemes of length n

We follow the approach of [RS00]: The apolarity action is defined as the action of $S = \mathbb{C}[x_1, \dots, x_n]$ as polynomial differential forms on $T = \mathbb{C}[y_1, \dots, y_n]$ by setting $x_i = \frac{\partial}{\partial y_i}$. This makes the duality between S_1 and T_1 explicit and, in fact, defines a natural duality between T_i and S_i. The form $q \in T_2$ define the smooth $(n-2)$-dimensional quadric hypersurface

$$Q = \{\, [\sum a_i x_i] \mid (\sum a_i \frac{\partial}{\partial y_i})^2(q) = 0\} \subset \mathbf{P}(S_1).$$

Apolarity defines a graded Artinian Gorenstein algebra associated to Q:

$$A^Q = \mathbb{C}[x_1, \dots, x_n]/(q^\perp)$$

where

$$q^\perp = \{D \in S_2 = \mathbb{C}[x_1, \dots, x_n]_2 | D(q) = 0\}.$$

A subscheme $Y \subset \mathbf{P}(T_1)$ is apolar to Q, or equivalently apolar to q, if the space of quadratic forms in its ideal $I_{Y,2} \subset q^\perp$. The apolarity lemma (cf.[RS00] 1.3) says that any smooth Γ, $[\Gamma] \in \mathrm{Hilb}_n(\mathbf{P}(T_1))$ is a polar simplex with respect to $Q \subset \mathbf{P}(S_1) = \check{\mathbf{P}}^{n-1}$ if and only if $I_{\Gamma,2} \subset q^\perp \subset S_2$, i.e. Γ is apolar to Q. We drop, for the moment, the smoothness criterium and consider any $[\Gamma] \in \mathrm{Hilb}_n(\mathbf{P}(T_1))$, such that Γ is apolar to Q. Notice that since Q is nonsingular, Γ is nondegenerate. But more is known: The following are the graded Betti numbers of A^Q and Γ, given in *Macaulay2* notation [GS].

Proposition 2.1. *a) For a smooth quadric $Q \subset \check{\mathbf{P}}^{n-1}$ the syzygies of the apolar Artinian Gorenstein ring A^Q are*

$$\begin{matrix} 1 & - & \dots & - & \dots & - & - \\ - & \frac{n-1}{n+1}\binom{n+2}{2} & \dots & \frac{k(n-k)}{n+1}\binom{n+2}{k+1} & \dots & \frac{n-1}{n+1}\binom{n+2}{n} & - \\ - & - & \dots & - & \dots & - & 1 \end{matrix}$$

b) A zero-dimensional nondegenerate scheme $\Gamma \subset \mathbf{P}^{n-1}$ of length n has syzygies

$$\begin{matrix} 1 & - & \dots & - & \dots & - & - \\ - & \binom{n}{2} & \dots & k\binom{n}{k+1} & \dots & (n-1)\binom{n}{n} & - \end{matrix}$$

Proof. Eg. [Beh81] and [ERS81] $\square$

Corollary 2.2. *The natural morphism*

$$VAPS(Q,n) \to \mathbb{G}(\binom{n}{2}, q^\perp); \quad \Gamma \mapsto I_{\Gamma,2} \subset q^\perp$$

is injective. Equivalently, there is a natural injective morphism

$$VAPS(Q,n) \to \mathbb{G}(n-1, T_{2,q}); \quad \Gamma \mapsto I_{\Gamma,2}^\perp \subset T_{2,q}$$

into the variety of $(n-2)$*-dimensional subspaces of* $\mathbf{P}(T_{2,q})$ *that intersect the projected Veronese variety* $V_{2,q}$ *in a scheme of length* n*. In particular, the Hilbert scheme and Grassmannian compactification in* $\mathbb{G}(n-1, T_{2,q})$ *of the variety of polar simplices coincide.*

Proof. Apolarity defines a natural isomorphism $q^\perp \cong T^*_{2,q}$. Therefore the subspace $I_{\Gamma,2} \subset q^\perp$ defines a $(n-1)$-dimensional subspace $I^\perp_{\Gamma,2} \subset T_{2,q}$. The intersection $\mathbf{P}(I^\perp_{\Gamma,2}) \cap V_{2,q}$ with the projected Veronese variety is precisely $\pi_q(\Gamma)$. The variety $VPS(Q,n) \subset \mathrm{Hilb}_n(\mathbf{P}^{n-1})$ is the closure of the set of polar simplices inside the set of apolar subschemes of length n. The former set is irreducible, while the latter set is a closed variety defined by the condition that the generators of the ideal of the subscheme lie in $q^\perp$. By Proposition 2.1, the map $\Gamma \mapsto I_{\Gamma,2} \subset q^\perp$ extends to all of $VPS(Q,n)$ as an injective morphism. □

We relate apolarity to polarity with respect to a quadric hypersurface. The classical notion of polarity is the composition of the linear map q^{-1} with apolarity: The polar to a point $[l] \in \mathbf{P}(T_1)$ with respect to Q^{-1} is the hyperplane $h_l = \mathbf{P}(q^{-1}(l)^\perp) \subset \mathbf{P}(T_1)$, where

$$(q^{-1}(l))^\perp = \{l' \in T_1 | l'(q^{-1}(l)) = q^{-1}(l \cdot l') = 0\}.$$

In particular, the polar hyperplane to l contains l if and only if $q^{-1}(l^2) = 0$, i.e. the point $[l]$ lies on the hypersurface Q^{-1}.

Let $\Gamma \subset \mathbf{P}(T_1)$ be a length n subscheme that contains $[l]$ and is apolar to Q. The subscheme $\Gamma' \subset \Gamma$ residual to $[l]$ is defined by the quotient $I_{\Gamma'} = I_\Gamma : (l^\perp)$. Since Γ is non degenerate, Γ' spans a unique hyperplane. This hyperplane is defined by a unique linear form $u' \in S_1$, and is characterized by the fact that $u' \cdot u(q) = u'q(u) = 0$ for all $u \in l^\perp$, so it is the hyperplane $\mathbf{P}(q(l^\perp))$. But

$$l' \in q(l^\perp) \Leftrightarrow 0 = q^{-1}(l')l = q^{-1}(l \cdot l') = q^{-1}(l)l' \Leftrightarrow l' \in (q^{-1}(l))^\perp,$$

so $\mathbf{P}(q(l^\perp))$ is the polar hyperplane $\mathbf{P}(q^{-1}(l)^\perp)$ to $[l]$ with respect to Q^{-1}. Thus the subscheme Γ' residual to $[l]$ in Γ spans the polar hyperplane to $[l]$ with respect to Q^{-1}.

Lemma 2.3. *A component of an apolar subscheme has support on* Q^{-1} *if and only if this component is nonreduced.*

Proof. If a component is a reduced point, the residual is contained in the polar hyperplane to this point, so by nondegeneracy the polar hyperplane cannot contain the point. If a component is nonreduced, the residual to the point supporting the component lies in the polar hyperplane to this point, so the point is on Q^{-1}. □

Each component Γ_0 of an apolar subscheme to q is apolar to a quadratic form q_0 defined on the span of Γ_0 and uniquely determined as a summand q. This is the content of the next proposition.

Proposition 2.4. *Let $\Gamma = \Gamma_1 \cup \Gamma_2$ be an apolar subscheme of length n to q that decomposes into two disjoint subschemes Γ_1 and Γ_2 of length n_1 and n_2. Let $U_1 \subset T_1$ and $U_2 \subset T_1$ be subspaces such that Γ_i spans $\mathbf{P}(U_i)$. Then there is a unique decomposition $q = q_1 + q_2$ with $q_i \in (U_i)^2$.*
Furthermore, subschemes $\Gamma_1 \subset \mathbf{P}(U_1)$ and $\Gamma_2 \subset \mathbf{P}(U_2)$ of length n_1 and n_2 are apolar to q_1 and q_2 respectively, if and only if $\Gamma_1 \cup \Gamma_2$ is apolar to q.

Proof. Since Γ is nondegenerate, $T_1 = U_1 \oplus U_2$. Let $U_i^\perp \subset S_1$ be the space of forms vanishing on U_i via apolarity. Then $U_1^\perp$ are natural coordinates on $\mathbf{P}(U_2)$ and likewise, $U_2^\perp$ are natural coordinates on $\mathbf{P}(U_1)$. Let $I_1 \subset (U_2^\perp)^2$ be the quadratic forms generating the ideal of Γ_1 in $\mathbf{P}(U_1)$, and likewise I_2 the quadratic forms generating the ideal of Γ_2 in $\mathbf{P}(U_2)$. Then $I_1 \oplus I_2 \oplus (U_1^\perp)\cdot(U_2^\perp) \subset S_2$ is the space of quadratic forms in the ideal of Γ.
Consider the intersections, $q_2^\perp = q^\perp \cap (U_1^\perp)^2$ and $q_1^\perp = q^\perp \cap (U_2^\perp)^2$. Since q is non degenerate, $q^\perp$ does not contain either of the subspaces $(U_i^\perp)^2$. Therefore $q_2^\perp$ is a codimension one subspace in $(U_1^\perp)^2$ and is apolar to a quadratic form $q_2 \in (U_2)^2$, unique up to scalar. Similarly, $q_1^\perp$ is apolar to a unique quadratic from $q_1 \in (U_1)^2$. The space of quadratic forms $q_1^\perp \oplus q_2^\perp \oplus (U_1^\perp)\cdot(U_2^\perp)$ is contained in $q^\perp$ and is apolar to the subspace $\langle q_1, q_2\rangle \subset T_2$. Therefore, there are unique nonzero coefficients c_1 and c_2 such that $q = c_1 q_1 + c_2 q_2$. Furthermore, each Γ_i is apolar to $q_i, i = 1, 2$.
It remains only to show the last statement. Assume Γ_1 and Γ_2 are apolar to q_1 and q_2 respectively. Then $\Gamma_1 \cup \Gamma_2$ is non degenerate of length n. Let $I_1 \subset (U_2^\perp)^2$ be the generators of the ideal of Γ_1 and $I_2 \subset (U_1^\perp)^2$ be the generators of the ideal of Γ_2. Then the quadratic forms in

$$I_1 \oplus I_2 \oplus (U_1)^\perp \cdot (U_2)^\perp$$

all lie in the ideal of $\Gamma_1 \cup \Gamma_2$. The dimension of this space of quadratic forms is

$$\binom{n_1}{2} + \binom{n_2}{2} + n_1 \cdot n_2 = \binom{n}{2},$$

so they generate the ideal of $\Gamma_1 \cup \Gamma_2$. Since all these forms are apolar to $q = q_1 + q_2$, the subscheme $\Gamma_1 \cup \Gamma_2$ is apolar to q. □

Remark 2.5. *By Proposition 2.4, the orbits of $SO(n, q)$ in $VASP(Q, n)$ are characterized by their components.*

We shall return to the set of local apolar subschemes $V_p^{loc}(n)$ supported at a point $p \in Q^{-1}$ in section 5.
Here we show that apolar subschemes of length n to q are all locally Gorenstein.

Lemma 2.6. *Let B be a local Artinian $\mathbb{C} = B/m_B$-algebra of length n and $\Phi : \mathrm{Spec} B \to \mathbf{A}^{n-1} \subset \mathbf{P}^{n-1}$ the reembedding given by $\mathbb{C}$-basis of m_B. The subscheme* $\mathrm{Im}\,\Phi$ *is apolar to a full rank quadric if and only if B is Gorenstein.*

Proof. Let $\phi : A = \mathbb{C}[x_1, .., x_{n-1}] \to B$ be the ring homomorphism corresponding to Φ. Thus ϕ is defined by a linear k-isomorphism $\phi_1 = A_{\leq 1} = \langle 1, x_1, ..., x_{n-1}\rangle \to B$. Let $\pi : B \to (0 : m_B)$ be the projection onto the socle of B, let $\psi : (0\colon m_B) \to \mathbb{C}$ be a linear form and consider the bilinear form

$$A \times A \xrightarrow{\phi\cdot\phi} B \xrightarrow{\pi} (0 : m_B) \xrightarrow{\psi} \mathbb{C},$$

where the first map is the composition of ϕ with multiplication. This map extends to the tensor product $A \otimes A$, and the restriction then to the symmetric part

$$(A_{\leq 1})^2 \subset A_{\leq 1} \otimes A_{\leq 1}$$

defines a linear form

$$\beta_\psi : (A_{\leq 1})^2 \to \mathbb{C}$$

and an associated quadratic form

$$q_\psi : A_{\leq 1} \to \mathbb{C}.$$

Clearly the kernel of β_ψ generate an ideal in A that is apolar to q_ψ. On the other hand, B is Gorenstein if and only if the socle is 1-dimensional. So for the lemma, it suffices to prove that q_ψ is non degenerate, i.e. has rank n, if and only if the linear form ψ is an isomorphism.

But q_ψ is degenerate if and only if the kernel of β_ψ contains $x \cdot A_{\leq 1}$ for some nonzero element $x \in A_{\leq 1}$. Now, $\beta_\psi(x \cdot A_{\leq 1}) = 0$ if and only if $\phi(x) \cdot B \cap (0 : m_B) \subset \ker \psi$. Since B is Artinian, $\phi(x) \cdot B \cap (0 : m_B)$ is a nonzero subspace of $(0 : m_B)$, so it suffices to consider elements x, which map to the socle. But then the kernel of β_ψ contains $x \cdot A_{\leq 1}$ precisely when x is in the kernel of ψ and the lemma follows. □

COROLLARY 2.7. *$VAPS(Q, n)$ is reducible for $n \geq 24$*

Proof. Consider a general graded Artinian Gorenstein algebra B of embedding dimension e and socle in degree 3. The length of B is $2e + 2$. By the Macaulay correspondence [Mac16], such algebras are in bijection with homogeneous forms, up to scalars, of degree 3 in e variables, hence depends on $\binom{e-1+3}{3} = (e + 2)(e + 1)e/6 - 1$ variables. The family of smoothable algebras have dimension at most $e(2e + 2) - 1$ So for $e + 2 > 2 \cdot 6$ a general algebra B cannot be smoothable, for trivial reason. In particular, $e = 11$ hence $n = 24$ is enough. □

We do not believe the bound $n \geq 24$ is sharp.

3. A RATIONAL PARAMETERIZATION

In this section we show that through a general point in $\mathbf{P}(T_{2,q})$ there is a unique n-secant $(n-2)$-space to the projected Veronese variety $V_{2,q}$. Furthermore, we give a characterization of the points for which there are more than one, i.e. infinitely many n-secant $(n - 2)$-spaces to $V_{2,q}$.

If we choose basis a for T_1 such that the symmetric matrix associated to q is the identity matrix, then the eigenvectors of the symmetric matrix associated to a general quadric q' are distinct. Thus, the symmetric matrices associated to q and q' have a unique set of n common 1-dimensional eigenspaces. Hodge & Pedoe [HP52, XIII.8,Theorem II] and Gantmacher [Gan59, Chapter XII, Theorem 3] found canonical forms for any pair of quadratic forms q, q' as soon as one of them is nonsingular. Here we are concerned with the simplex formed by the set of common eigenspaces and give a geometric formulation and proof.

PROPOSITION 3.1. *Let $q, q' \in T_2$ be two general quadrics. Then there exists a unique $n-$simplex $\{L_1, \ldots, L_n\}$ polar to both q and q'.*

Proof. By the above, it suffices to show the relation between the collection of common eigenspaces of the associated symmetric matrices and the common simplex. So we assume that q, q' are quadrics of rank n and that

$$q = \sum_{i=1}^{n} l_i^2, \quad \text{and} \quad q' = \sum_{i=1}^{n} \lambda_i l_i^2,$$

where the λ_i are pairwise distinct coefficients and $L_i = \{l_i = 0\}, \quad i = 1, \ldots, n$. Let

$$q_i = \lambda_i q - q', \quad i = 1, \ldots, n.$$

Then the q_i are precisely the quadratic forms of the pencil generated by q and q' that have rank less than n. The rank of q_i is exactly $n-1$ since $\lambda_i \neq \lambda_j$ for $i \neq j$, so $q_i \in (U_i)^2$ for a unique rank $n-1$ subspace $U_i \subset T_1$. The intersection $\cap_{i \neq j} U_i$ is the 1-dimensional subspace generated by the nonzero linear form l_j. These forms are therefore determined uniquely by the pencil generated by q and q'. □

A precise condition for generality in the proposition is given by rank:

LEMMA 3.2. *A pencil of quadratic forms in n variables have a unique common apolar subscheme of length n if and only if every quadric in the pencil have rank at least $n-1$ and some, hence the general quadric has rank n. Furthermore the unique apolar subscheme is curvilinear, i.e. embeddable in a smooth curve.*

Proof. Let

$$\langle q' + \lambda q \rangle_{\lambda \in \mathbf{A}^1_{\mathbb{C}}}$$

be a pencil with discriminant $\Delta \subset \mathbf{A}^1_{\mathbb{C}}$, a scheme of length n. Consider the incidence

$$\{(D, \lambda) | D(q' + \lambda q) = 0\} \subset \mathbf{P}(T_1) \times \mathbf{A}^1_{\mathbb{C}}$$

with projections p_T and $p_{\mathbb{C}}$. Clearly the fibers of each projection are all linear. Now as in the proof of the proposition, a general length n subscheme of $p_T(p_{\mathbb{C}}^{-1}(\Delta)$ is a common apolar subscheme to the pencil of quadratic forms. Therefore, the common apolar subscheme is unique if and only if $p_T(p_{\mathbb{C}}^{-1}(\Delta)$ is finite, i.e. the corank of any quadric in L is at most 1. In this case both projections restricted to the incidence are isomorphisms onto their images. In particular the apolar subscheme is isomorphic to Δ, so it is curvilinear. □

REMARK 3.3. *The ideal of the curvilinear image* Γ *of the map*

$$\operatorname{Spec}\mathbb{C}[t]/(t^n) \to \mathbf{P}^{n-1}$$
$$t \mapsto (1 : t : t^2 : \ldots : t^{n-1}),$$

is generated by the 2×2 *minors of*

$$\begin{pmatrix} x_1 & x_2 & \ldots & x_{n-1} & x_n \\ x_2 & x_3 & \ldots & x_n & 0 \end{pmatrix},$$

so the Γ *is apolar to the maximal rank quadric*

$$\sum_{k=1}^{n} y_k y_{n+1-k}.$$

This remark generalizes to a partial converse of Lemma 3.2.

LEMMA 3.4. *Any curvilinear nondegenerate zero-dimensional subscheme* $\Gamma \subset \mathbf{P}^{n-1}$ *of length* n *is apolar to a quadric* $Q \subset \check{\mathbf{P}}^{n-1}$ *of maximal rank.*

Proof. Let Γ be a nondegenerated curvilinear subscheme with r components of length $n_1, \ldots, n_r$ such that $n_1 + \ldots + n_r = n$. Then Γ is projectively equivalent to $\Gamma' = \Gamma_1 \cup \ldots \cup \Gamma_r$, where Γ_i is the image of

$$\operatorname{Spec}\mathbb{C}[t]/(t^{n_i}) \to \mathbf{P}(\mathbb{C}^{n_i}) \subset \mathbf{P}(\mathbb{C}^{n_1} \oplus \ldots \oplus \mathbb{C}^{n_r})$$
$$t \mapsto (1 : t : t^2 : \ldots : t^{n_i - 1}),$$

where the nonzero coordinates in the image are $x_{i,1}, \ldots, x_{i,n_i}$. The ideal of Γ' is generated by the 2×2 minors of the r matrices

$$\begin{pmatrix} x_{(1,1)} & \ldots & x_{(1,n_1-1)} & x_{(1,n_1)} \\ x_{(1,2)} & \ldots & x_{(1,n_1)} & 0 \end{pmatrix} \quad \ldots \quad \begin{pmatrix} x_{(r,1)} & \ldots & x_{(r,n_r-1)} & x_{(r,n_r)} \\ x_{(r,2)} & \ldots & x_{(r,n_r)} & 0 \end{pmatrix},$$

and the products

$$x_{(i,k_i)} x_{(j,k_j)} \quad \text{for} \quad 1 \le i < j \le r, 1 \le k_i \le n_i, 1 \le k_j \le n_j.$$

So Γ' is apolar to the maximal rank quadric

$$\sum_{i=1}^{r} \sum_{k=1}^{n_i} y_{(i,k)} y_{(i,n_i+1-k)}.$$

$\square$

More important to us will be that rank n quadrics have apolar subschemes of length n that are not curvilinear (when $n > 3$).

REMARK 3.5. *Consider the rank* n *quadric*

$$q = 2y_1 y_n + y_2^2 + \ldots + y_{n-1}^2.$$

The subscheme $\Gamma_p \subset \mathbf{P}^{n-1}$ *defined by*

$$(x_1^2, x_1x_2, x_2^2 - x_1x_n, x_1x_3, \ldots, x_{n-1}^2 - x_1x_n),$$

has degree n *and is apolar to* q*, but it is clearly not curvilinear when* $n > 3$*. It contains the tangency locus of the quadric* $\{q^{-1} = \frac{1}{2}x_1x_n + \frac{1}{4}x_2^2 + \ldots + \frac{1}{4}x_{n-1}^2 =$

$0\}$ *at the point* $[0:0:...:1]$. *The tangency locus has length* $n-1$ *and is defined by*

$$(x_1,(x_2^2-x_1x_n,x_2x_3,...,x_{n-1}^2-x_1x_n)).$$

The subscheme Γ_p *is itself not contained in the tangent hyperplane* $\{x_1=0\}$, *but it is the unique apolar subscheme to* q *that contains the first order neighborhood of* $[0:0:...:1]$ *on* $\{q^{-1}=0\}$. *It will be the focus of our attention in Section 5.*

It follows immediately from Proposition 3.1 that there is a rational and dominant map

$$\gamma:\mathbf{P}(T_{2,q})\dashrightarrow VPS(Q,n)\subset\mathbb{G}(n-1,T_{2,q})$$

whose general fiber is a n-secant $(n-2)$-space to the projected Veronese variety $V_{2,q}$. In the next section we find equations for this map.

4. The Mukai form

Mukai introduced in [Muk92] a trilinear form in his approach to varieties of sums of powers of conics in particular, and to forms of even degree in general (see also [Dol12, Sections 1.4 and 2.1.3]). In this section we show how this form naturally gives equations for the map γ and for the universal family of polar simplices. The main result of this section, Proposition 4.2, gives the equations for the common apolar subscheme of length n of a pencil of quadrics in n variables, whenever this subscheme is unique, cf. Lemma 3.2.
Both the quadratic form $q\in T_2$ and the inverse $q^{-1}\in S_2$ play a crucial role in the definition of the Mukai form. Recall that the form q defines an invertible linear map $q:S_1\to T_1$, and q^{-1} defines the inverse map: $q^{-1}:T_1\to S_1$. In coordinates, if $q=(\alpha_1y_1^2+...+\alpha_ny_n^2)$, then $q^{-1}=(\frac{1}{4\alpha_1}x_1^2+...+\frac{1}{4\alpha_n}x_n^2)$.
We will arrive at Mukai's form from

$$\tau\in\mathrm{Hom}(\wedge^2S_1\otimes T_2\otimes T_2\otimes S_2,\mathbb{C})$$

defined by

$$z_1\wedge z_2\otimes q_1\otimes q_2\otimes\alpha\mapsto(z_1(q_1)z_2(q_2)-z_2(q_1)z_1(q_2))(\alpha)$$

where $f(g)$, as above, means f viewed as differential operator applied to g. Interpreting $\omega=z_1\wedge z_2\in\wedge^2S_1\subset\mathrm{Hom}(T_1,S_1), q_j\in T_2\subset\mathrm{Hom}(S_1,T_1)$ and $\alpha\in S_2\subset\mathrm{Hom}(T_1,S_1)$ the expression

$$\begin{aligned}(\omega\otimes q_1\otimes q_2\otimes\alpha)\mapsto&\frac{1}{2}\mathrm{trace}\,(\alpha\circ q_2\circ\omega\circ q_1-\alpha\circ q_1\circ\omega\circ q_2)\\ &=\frac{1}{2}\mathrm{trace}\,(\omega\circ q_1\circ\alpha\circ q_2-\alpha\circ q_1\circ\omega\circ q_2)\end{aligned}$$

gives an alternative description of τ. In fact $f(g)=\frac{1}{2}(\mathrm{trace}\ f\circ g)$ holds for $f\otimes g\in S_2\otimes T_2\subset\mathrm{Hom}(T_1,S_1)\otimes\mathrm{Hom}(S_1,T_1)$ and trace $((\alpha\circ q_2)\circ(\omega\circ q_1))=$ trace $((\omega\circ q_1)\circ(\alpha\circ q_2))$. We now substitute $\alpha=q^{-1}$. Then $\frac{1}{2}$trace $(\omega\circ q_1\circ q^{-1}\circ q_2-q^{-1}\circ q_1\circ\omega\circ q_2)=0$ for $q_1=q$, and, since the first expression for τ is alternating on $T_2\otimes T_2$, we have $\frac{1}{2}\mathrm{trace}\,(\omega\circ q_1\circ q^{-1}\circ q_2-q^{-1}\circ q_1\circ\omega\circ q_2)=0$ for $q_2=q$ as well. Thus τ induces a well defined trilinear form

$$\tau_q\in \mathrm{Hom}(\wedge^2 S_1\otimes T_{2,q}\otimes T_{2,q},\mathbb{C})$$

on the quotient space $T_{2,q}=T_2/\langle q\rangle$.
Since $T^*_{2,q}=q^\perp\subset S_2=T_2^*$ and

$$\mathrm{Hom}(\wedge^2 S_1\otimes T_{2,q}\otimes T_{2,q},\mathbb{C})\cong \mathrm{Hom}(T_{2,q},\mathrm{Hom}(\wedge^2 S_1,q^\perp))$$

we have a second interpretation of τ_q. With this interpretation, the image of $\tau_q(q_1)\in \mathrm{Hom}(\wedge^2 S_1,q^\perp)\subset \mathrm{Hom}(\wedge^2 S_1,S_2)$ is defined by

$$\omega\mapsto[\omega\circ q_1\circ q^{-1}-q^{-1}\circ q_1\circ\omega]\in q^\perp\subset S_2\subset \mathrm{Hom}(T_1,S_1).$$

The form τ is alternating on $T_2\otimes T_2$, so $\tau(\omega,q',q',q^{-1})=0$ for every $\omega\in\wedge^2 S_1$. Therefore

$$\tau_q(q')(\wedge^2 S_1)\subset (q')^\perp.$$

If Q' is the quadric $\{q'=0\}\subset \mathbf{P}(S_1)$, we may therefore conclude:

Lemma 4.1. *Any quadratic form in $\tau_q(q')(\wedge^2 S_1)$ is apolar to both Q and Q':*

$$\tau_q(q')(\wedge^2 S_1)\subset q^\perp\cap(q')^\perp.$$

□

Notice that the linear space of quadratic forms $\tau_q(q')(\wedge^2 S_1)$ is not all of $q^\perp\cap(q')^\perp$. It is a special subspace of the intersection. Since $\tau_q(q)=0$, we have $\tau_q(q')=\tau_q(q'+\lambda q)$ for any λ, so the space $\tau_q(q')(\wedge^2 S_1)$ of quadratic forms depends only on the pencil $\langle q,q'\rangle$.
If the pencil of quadratic forms $\langle q,q'\rangle\subset T_2$ contains no forms of corank at least 2, then, by Lemma 3.2, there is a unique common apolar subscheme $\Gamma_{q'}$ of length n to q and q'. The significance of the form τ_q is

Proposition 4.2. *Let $q'\in T_{2,q}$. Then the linear map*

$$\tau_q(q'):\wedge^2 S\to q^\perp$$

is injective if and only if q and q' have a unique common apolar subscheme of length n. Furthermore, in this case the image generates the ideal in S of this subscheme.

Proof. Our argument depends on several lemmas, in which we study $\mathrm{Im}\;\tau_q(q')\subset S_2$ by considering the symmetric matrices associated to these quadratic forms with respect to a suitable basis. Thus, we choose coordinates such that $q=\frac{1}{2}(y_1^2+y_2^2+...+y_n^2)$ and hence $q^{-1}=\frac{1}{2}(x_1^2+x_2^2+...+x_n^2)$. The symmetric matrices of these quadratic forms with respect to the coordinate basis of T_1 and S_1 are both the identity matrix. We denote by A the symmetric matrix of q', i.e. $q'=\frac{1}{2}(y_1,...,y_n)A(y_1,...,y_n)^t$. For a form $\omega\in\wedge^2 S_1$ there is similarly an associated skew symmetric matrix Λ_ω. For a form $l\in T_1$ we denote by v_l the column vector of its coordinates. The quadratic forms in the image $\tau_q(q')$ are the forms associated to the symmetric bilinear forms

$$\{\omega\circ q'\circ q^{-1}-q^{-1}\circ q'\circ\omega|\omega\in\wedge^2 S_1\},$$

so their associated symmetric matrices are

$$\{\Lambda_\omega A - A\Lambda_\omega | \omega \in \wedge^2 S_1\}.$$

Lemma 4.3. *Let $[l] \in \mathbf{P}(T_1)$, then every quadric in $\tau_q(q')(\wedge^2 S) \subset q^\perp$ vanishes at the point $[l]$ if and only if there is a quadric $q_\lambda = q' + \lambda q$ for some $\lambda \in \mathbb{C}$, such that l lies in the kernel of the linear transformation $q_\lambda \circ q^{-1} : T_1 \to T_1$. Equivalently, in terms of matrices: If v_l is the column coordinate vector of l, then $v_l^t(\Lambda_\omega A - A\Lambda_\omega)v_l = 0$ for every $\omega \in \wedge^2 S_1$ if and only if v_l is an eigenvector for the matrix A.*

Proof. Note first that the matrix of the linear transformation $q_\lambda \circ q^{-1}$, with respect to the coordinate basis of T_1, is simply $A + \lambda I$. Hence, the two parts of the lemma are equivalent.
In the matrix notation, if v_l is an eigenvector for A with eigenvalue λ, then

$$\begin{aligned} v_l^t(\Lambda_\omega A - A\Lambda_\omega)v_l &= v_l^t \Lambda_\omega A v_l - v_l^t A \Lambda_\omega v_l \\ &= v_l^t \Lambda_\omega \lambda v_l - \lambda v_l^t \Lambda_\omega v_l = 0, \end{aligned}$$

so the if part follows.
Conversely, assume that

$$v_l^t(\Lambda_\omega A - A\Lambda_\omega)v_l = 0$$

for every skew symmetric $n \times n$ matrix Λ_ω. Again

$$v_l^t(\Lambda_\omega A - A\Lambda_\omega)v_l = v_l^t \Lambda_\omega A v_l - v_l^t \Lambda_\omega A v_l,$$

and since A is symmetric and Λ_ω is skewsymmetric, $(v_l^t \Lambda_\omega A v_l)^t = -v_l^t A \Lambda_\omega v_l$, so we deduce that

$$v_l^t \Lambda_\omega A v_l = 0.$$

But $v_l^t \Lambda_\omega u = 0$ for every skew symmetric matrix Λ_ω only if u is proportional to v_l, so we conclude that $A(v_l) = \lambda v_l$ for some λ. □

Remark 4.4. *A point $l \in T_1$ lies in the kernel of $q_\lambda \circ q^{-1}$ if and only if $q^{-1}(l)$ lies in the kernel of $q_\lambda : S_1 \to T_1$. Equivalently, $\{q_\lambda = 0\} \subset \mathbf{P}(S_1)$ is a singular quadric and $[q^{-1}(l)] \in \mathbf{P}(S_1)$ lies in its singular locus.*

Corollary 4.5. *$\tau_q(q')$ is injective only if $\langle q, q' \rangle$ contains no quadratic form of rank less than $n - 1$.*

Proof. If the quadratic form $q_\lambda = q' + \lambda q$ has rank less than $n - 1$, then there are independent forms $l, l' \in T_1$ such that $\langle q^{-1}(l), q^{-1}(l') \rangle$ is contained in the kernel of $q_\lambda : S_1 \to T_1$. In particular, viewed as differential operators applied to q_λ,

$$q^{-1}(l)(q_\lambda) = q^{-1}(l')(q_\lambda) = 0 \in T_1.$$

Let

$$\omega = q^{-1}(l) \wedge q^{-1}(l') \in \wedge^2 S_1 \subset \mathrm{Hom}(T_1, S_1).$$

Then

$$\omega \otimes q_\lambda \otimes q_2 \otimes q^{-1} \mapsto [q^{-1}(l)(q_\lambda) \cdot q^{-1}(l')(q_2) - q^{-1}(l')(q_\lambda) \cdot q^{-1}(l)(q_2)](q^{-1}) = 0$$

for every $q_2 \in T_2$, so $\tau_q(q')(\omega) = 0$ and $\tau_q(q')$ is not injective. □

To complete the proof of Proposition 4.2, we assume that q and q' have a unique common apolar subscheme Γ of length n, i.e. by Lemma 3.2, no quadratic form in $\langle q, q'\rangle$ has rank less than $n-1$. We want to show that $\tau_q(q')$ is injective and that the image generates the ideal of Γ.
Let $\Gamma = \Gamma_1 \cup ... \cup \Gamma_r$ be a decomposition of Γ into its connected components. Then each Γ_i is a finite local curvilinear scheme. Let n_i be the length of Γ_i. By Proposition 2.4, there is a decomposition $T_1 = \oplus_i U_i$ such that each $\Gamma_i \subset \mathbf{P}(U_i)$. Furthermore U_i has dimension n_i and q and q' have unique decompositions $q = q_1 + ... + q_r$ and $q' = q'_1 + ... + q'_r$ with $q_i, q'_i \in (U_i)^2 \subset T_2$. Denote by $U'_i = \oplus_{j\neq i} U_j$, and let $(U'_i)^\perp$ be the orthogonal subspace of linear forms in S_1. Then

$$\sum_i U_i^\perp \cdot (U'_i)^\perp \subset S_2$$

generate the ideal of $\cup_i \mathbf{P}(U_i) \subset \mathbf{P}(T_1)$.
The linear forms $q^{-1}(U_i) \subset S_1$ are natural coordinates on $\mathbf{P}(U_i)$. Denote by $I_{\Gamma_i,2}$ the quadratic forms in these coordinates in the ideal of Γ_i. Then $I_{\Gamma_i,2} \subset (q^{-1}(U_i))^2 \subset S_2$ and the space of quadratic forms in the ideal of Γ is

$$I_{\Gamma,2} = \sum_i I_{\Gamma_i,2} + \sum_i U_i^\perp \cdot (U'_i)^\perp \subset S_2.$$

We

CLAIM 4.6.

$$\text{Im } \tau_q(q') \supset \sum_i I_{\Gamma_i,2} + \sum_i U_i^\perp \cdot (U'_i)^\perp.$$

If the claim holds, $\tau_q(q')$ is injective, since $\dim \wedge^2 S_1 = \dim I_{\Gamma,2} \quad (= \binom{n}{2})$, so the equality $\text{Im } \tau_q(q') = I_{\Gamma,2}$ holds and the proof of Proposition 4.2 is complete.
We use matrices to prove the claim. To interpret the decomposition of q and q' in terms of matrices, we choose a basis for each U_i such that the symmetric matrix associated to each q_i is the $n_i \times n_i$ identity matrix. Let A_i be the symmetric $n_i \times n_i$ matrix associated to q'_i. The union of the bases for the U_i form a basis for T_1 with respect to which the symmetric matrix A of q' has r diagonal blocks A_i and zeros elsewhere.
The matrices A_i each have a unique eigenvalue λ_i, and these eigenvalues are pairwise distinct. Furthermore, each A_i has a 1-dimensional eigenspace, so their Jordan form has a unique Jordan block, and we may write $A_i = \lambda_i I_{n_i} + B_i$ with B_i a nilpotent symmetric matrix. (See ([DZ04, Theorem 2.3]) for a nice normal form for the matrices B_i.)
By extending each A_i with zeros to $n \times n$ matrices, we may write $A = \sum A_i$. The decomposition $T_1 = \oplus_i U_i$ is then defined by $U_i = \ker(\lambda_i I - A)^{n_i} \subset T_1$.
Denote by $U'_i = \oplus_{j\neq i} U_j$. Then $\mathbf{P}(U_i)$ and $\mathbf{P}(U'_i)$ have complementary dimension in $\mathbf{P}(T_1)$. We shall use the techniques applied by Gantmacher in the analysis of commuting matrices ([Gan59, Chapter VIII]) to show

LEMMA 4.7. *Let A be the symmetric matrix of the quadratic form $q' \in T_{2,q}$ as above. Let $T_1 = U_i \oplus U'_i$ be the decomposition associated to the eigenvalue λ_i.*

Then

$$U_i^{\perp} \cdot (U_i')^{\perp} \subset \operatorname{Im} \tau_q(q') \subset S_2.$$

Proof. Set $d = n_i$ and $\lambda = \lambda_i$ and choose coordinates such that $U_\lambda = \langle y_1, ..., y_d \rangle$ and $U'_\lambda = \langle y_{d+1}, ..., y_n \rangle$. Then $(U'_\lambda)^{\perp} = \langle x_1, ..., x_d \rangle$ and $(U_\lambda)^{\perp} = \langle x_{d+1}, ..., x_n \rangle$. Consider the matrix B of the quadratic form $\tau_q(q')(x_i \wedge x_j)$ with $i \leq d$ and $j > d$. The skew symmetric matrix $\Lambda_{(ij)}$ of $x_i \wedge x_j$ has (ij)-th entry 1, consequently (ji)-th entry -1, and 0 elsewhere, and

$$B = \Lambda_{(ij)} A - A \Lambda_{(ij)}.$$

The nonzero entries in $\Lambda_{(ij)} A$ are in positions (i, k) with $k > d$ and (j, k) with $k \leq d$, while the nonzero entries in $A\Lambda_{(ij)}$ are in positions (k, i) with $k > d$ and (k, j) with $k \leq d$. Therefore the quadratic form $\tau_q(q')(x_i \wedge x_j)$ lies in the space

$$\langle x_a x_b \rangle_{a \leq d < b} = (U'_\lambda)^{\perp} \cdot (U_\lambda)^{\perp}.$$

A linear relation between these quadratic forms would correspond to a skew symmetric matrix Λ with nonzero entries only in the rectangular block $(ij), i \leq d, j > d$, such that $\Lambda A - A\Lambda = 0$. Write A as a sum $A = A_\lambda + A_{\mu_1} + ... + A_{\mu_s}$ where the μ_i are the eigenvalues of A distinct from λ. Let Λ be a skew symmetric matrix and let Λ_{λ,μ_i} be the rectangular submatrix with rows equal to the nonzero rows of A_λ and columns equal to the nonzero columns of A_{μ_i}. Then the corresponding submatrix

$$(\Lambda A - A\Lambda)_{\lambda,\mu_i} = \Lambda_{\lambda,\mu_i} A_{\mu_i} - A_\lambda \Lambda_{\lambda,\mu_i}.$$

So $\Lambda A - A\Lambda = 0$ only if $\Lambda_{\lambda,\mu_i} A_{\mu_i} - A_\lambda \Lambda_{\lambda,\mu_i} = 0$ for each μ_i.
Let μ be one of the μ_i, and assume for simplicity $U_\mu = \langle y_{d+1}, ..., y_{d+e} \rangle$. Let I_d be the diagonal matrix with 1 in the d first entries and 0 elsewhere, and let I_e be the diagonal matrix with 1 in the entries $d+1, ..., d+e$ and 0 elsewhere. Then the special summand A_λ of A can be written as a sum $A_\lambda = \lambda I_d + B_d$ where B_d is nilpotent of order d. Likewise, $A_\mu = \mu I_e + B_e$ where B_e is nilpotent of order e. So we may write $A = \lambda I_d + B_d + \mu I_e + B_e + A'$, where $A' = A - A_\lambda - A_\mu$. But then $(\Lambda A - A\Lambda)_{\lambda,\mu} = 0$ only if

$$\Lambda_{\lambda,\mu} A_\mu - A_\lambda \Lambda_{\lambda,\mu} = 0,$$

i.e. when

$$\Lambda_{\lambda,\mu}(\mu I_e + B_e) - (\lambda I_d + B_d)\Lambda_{\lambda,\mu} = 0.$$

This is equivalent to

$$(\lambda - \mu)\Lambda_{\lambda,\mu} = \Lambda_{\lambda,\mu} B_e - B_d \Lambda_{\lambda,\mu}.$$

Multiplying both sides by $(\lambda - \mu)$ and substituting on the right hand side $(\lambda - \mu)\Lambda_{\lambda,\mu}$ with $\Lambda_{\lambda,\mu} B_e - B_d \Lambda_{\lambda,\mu}$ we get

$$\begin{aligned}(\lambda - \mu)^2 \Lambda_{\lambda,\mu} &= (\Lambda_{\lambda,\mu} B_e - B_d \Lambda_{\lambda,\mu}) B_e - B_d(\Lambda_{\lambda,\mu} B_e - B_d \Lambda_{\lambda,\mu}) \\ &= (\Lambda_{\lambda,\mu}(B_e)^2 - 2B_d \Lambda_{\lambda,\mu} B_e + (B_d)^2 \Lambda_{\lambda,\mu}).\end{aligned}$$

Iterating $m = d + e - 1$ times we get

$$(\lambda-\mu)^m\Lambda_{\lambda,\mu}=\sum_{s+t=m}(-1)^s\binom{s+t}{t}(B_d)^s\Lambda_{\lambda,\mu}(B_e)^t.$$

But on the right hand side either $(B_d)^s=0$ or $(B_e)^t=0$ when $s+t=d+e$, so $\Lambda_{\lambda,\mu}=0$.
Thus $\Lambda_{\lambda,\mu_i}=0$ for all i, and the symmetric matrices $A\Lambda_{ij}-\Lambda_{ij}A$ with $i\leq d, j>d$ are linearly independent. The corresponding quadratic forms therefore are linearly independent in the space $\langle x_1,...,x_d\rangle\times\langle x_{d+1},...,x_n\rangle$. Since the dimensions coincides, the quadratic forms span this space, and the lemma follows. □

Next, we consider the case when the symmetric matrix A only has one eigenvalue. Thus we assume that Γ has only one component, the symmetric matrix A of q' has only one eigenvalue and up to scalars only one nonzero eigenvector. Hence $\langle q,q'\rangle$ contains exactly one quadratic form of rank $n-1$. In particular, by Lemma 3.2, Γ is curvilinear. Without loss of generality we may assume that q' has rank $n-1$ i.e. that the eigenvalue is 0. Then A is nilpotent, and since A is a one-dimensional eigenvector space, $A^n=0$ and $A^i\neq 0$ for any $i<n$.

Lemma 4.8. *Let $q'\in T_{2,q}$ be a quadratic form whose associated $n\times n$ matrix A is symmetric, nilpotent and has rank $n-1$. Then the ideal generated by the quadratic forms $\tau_q(q')\subset q^\perp$ is the ideal of the unique common apolar subscheme Γ of length n of q and q'. Moreover Γ is a local curvilinear subscheme.*

Proof. Let Λ be a skew symmetric $n\times n$ matrix and think of A and Λ as the matrices of linear endomorphisms of a n-dimensional vector space V. Then we may choose a basis $v_1,\dots,v_n\in V$ such that $Av_1=0$ and $Av_i=v_{i-1}$ for $i=2,\dots,n$. Let

$$\rho:\mathrm{Spec}(\mathbb{C}[t]/t^n)\to\mathbf{P}(V):t\mapsto[v_1+tv_2+\ldots+t^{n-1}v_n]$$

and set $\Gamma=\mathrm{Im}\,\rho$. Then I_Γ is generated by $\binom{n}{2}$ quadratic forms. We shall show that the symmetric matrices of these forms coincide with the matrices $\Lambda A-A\Lambda$ as Λ varies. We evaluate the quadratic form associated to $\Lambda A-A\Lambda$ on the vector $v=v_1+tv_2+\ldots+t^{n-1}v_n$:

$$v^t(\Lambda A-A\Lambda)v=v^t\Lambda Av-v^tA\Lambda v.$$

But

$$\begin{aligned}v^t\Lambda Av&=(v_1+tv_2+\ldots+t^{n-1}v_n)^t\Lambda A(v_1+tv_2+\ldots+t^{n-1}v_n)\\&=(v_1+tv_2+\ldots+t^{n-1}v_n)^t\Lambda(tv_1+\ldots+t^{n-1}v_{n-1})\\&=(v_1+\ldots+t^{n-2}v_{n-1})^t\Lambda(v_1+\ldots+t^{n-2}v_{n-1})t\\&\quad+t^nv_n^t\Lambda(v_1+\ldots+t^{n-2}v_{n-1})\\&=0\end{aligned}$$

since Λ is skew symmetric and $t^n=0$. Therefore the quadratic forms with matrices $\Lambda A-A\Lambda$ are in the ideal of Γ. They are independent and therefore generate the ideal unless $\Lambda A - A\Lambda = 0$ for some nontrivial Λ. But then Λ and A commute, hence have common eigenvectors. Λ is nontrivial and skew symmetric so it has at least 2 independent eigenvectors, while A has only one, so this is impossible. Clearly, Γ is curvilinear, and any non degenerate local curvilinear subscheme of length n in $\mathbf{P}(V)$ is projectively equivalent to it, so the lemma follows. $\square$

To complete the proof of the claim 4.6 and the proof of Proposition 4.2, we consider the common apolar subscheme $\Gamma = \Gamma_1 \cup ... \cup \Gamma_r$ to q and q', and the corresponding decompositions $q = \sum q_i$ and $q' = \sum q'_i$ as above. By Lemma 4.7,

$$\sum_i U_i^\perp \cdot (U'_i)^\perp \subset \mathrm{Im}\ \tau_q(q').$$

Furthermore, applying Lemma 4.8 to each component q_i and q'_i, the image of $\tau_{q_i}(q'_i)$ in $(q^{-1}(U_i))^2$ is $I_{\Gamma_i,2}$. But $\tau_{q_i}(q'_i)$ is the restriction of $\tau_q(q')$ to $\wedge^2(q^{-1}(U_i))$, so

$$I_{\Gamma_i,2} \subset \mathrm{Im}\ \tau_q(q') \qquad i = 1, ..., r$$

and the claim and Proposition 4.2 follows. $\square$

By Lemma 4.3, the quadratic forms in Im $\tau_q(q')$ vanish in every point on any common apolar subscheme of length n to q and q'. Combined with Proposition 4.2 it may be reasonable to guess that Im $\tau_q(q')$ is precisely the quadratic forms in the intersection of the ideals of these common apolar subschemes. We do not have a clear answer and leave this as an open question.

We are now ready to analyze our main object $VPS(Q, n)$ in its embedding in $\mathbb{G}(n-1, T_{2,q})$, i.e. as the image of the rational map

$$\gamma : \mathbf{P}(T_{2,q}) \dashrightarrow \mathbb{G}(n-1, T_{2,q}).$$

We identify the restriction of the Plücker divisor to $VPS(Q, n)$.

Let $h \subset \mathbf{P}(T_1)$ be a hyperplane, and denote by $H_h \subset VSQ(Q, n)$ the set

$$H_h = \{[\Gamma] \in VSQ(Q, n) | \Gamma \cap h \neq \emptyset\}.$$

Lemma 4.9. *H_h is the restriction to $VPS(Q, n)$ of a Plücker divisor on $\mathbb{G}(n-1, T_{2,q})$.*

Proof. The hyperplane $h \subset \mathbf{P}(T_1)$ is defined by some $l \in S_1$. Let $V(l) = \{q' \in T_2 | l(q') = 0\}$, then $V(l)^\perp = l \cdot S_1 = \{l \cdot l' | l' \in S_1\} \subset S_2$.

For any nondegenerate subscheme $\Gamma \subset \mathbf{P}(T_1)$ of length n, the ideal $I_\Gamma \subset S$ contains a reducible quadric $l_1 \cdot l_2$ only if Γ intersects both hyperplanes $\{l_1 = 0\}$ and $\{l_2 = 0\}$. On the other hand the subspace of quadrics $I_{\Gamma,2} \subset S_2$ has codimension n, which coincides with the dimension of $l \cdot S_1$. Therefore

$$I_{\Gamma,2} \cap l \cdot S_1 \neq \{0\} \subset S_2 \quad \text{if and only if} \quad (I_{\Gamma,2})^\perp \cap V(l) \neq \{0\} \subset T_2.$$

Notice that $\mathbf{P}((I_{\Gamma,2})^\perp)$ equals the span $\langle \Gamma \rangle \subset \mathbf{P}(T_2)$ of Γ in the Veronese embedding.

For the lemma we now consider apolar subschemes to q and the projection from $\mathbf{P}(T_2)$ to $\mathbf{P}(T_{2,q})$. Since q has maximal rank, $l(q) \neq 0$, i.e. $q \notin V(l)$. Thus

$\mathbf{P}(V(l))$ is projected isomorphically to its image $\mathbf{P}(V_q(l)) \subset \mathbf{P}(T_{2,q})$. For an apolar subscheme Γ of length n the quadratic form q lies in the linear span of $\Gamma \subset \mathbf{P}(T_2)$, so this subspace is mapped to the $(n-2)$-dimensional linear span of Γ in $\mathbf{P}(T_{2,q})$. We therefore deduce from the above equivalence: If Γ is apolar to q, then the linear span of Γ in $\mathbf{P}(T_{2,q})$ intersects the codimension n linear space $\mathbf{P}(V_q(l))$ if and only if Γ intersects the hyperplane $h \subset \mathbf{P}(T_1)$.
But the set of $(n-2)$-dimensional subspaces in $\mathbf{P}(T_{2,q})$ that intersect a linear space of codimension n form a Plücker divisor, so the lemma follows. □

In the next section we use the special Plücker divisors H_h of this lemma to give a local affine description of $VPS(Q,n)$, or better, the variety $VAPS(Q,n)$ of all apolar subschemes of length n.

5. An open affine subvariety

We use a standard basis approach to compute an open affine subvariety of $VAPS(Q,n)$, the variety of all apolar subschemes of length n to Q. Of course this will include our primary object of interest, namely $VPS(Q,n)$. For small n there will be no difference, but for larger n we have already seen that they do not coincide. The distinction between the two will eventually be the main concern in our analysis. The computations in this section extensively use *Macaulay2* [GS]. In particular when we show, by direct computation, that $VAPS(Q,6)$ is irreducible and therefore coincides with $VPS(Q,6)$ (Corollary 5.16).
We choose coordinates such that

$$Q = \{q = 2y_1y_n + y_2^2 + \ldots + y_{n-1}^2 = 0\},$$

and consider the apolar subscheme Γ_p to q defined by

$$x_1^2, x_1x_2, x_2^2 - x_1x_n, x_1x_3, x_2x_3, \ldots x_{n-1}^2 - x_1x_n.$$

It is of length n and corresponds in the setting of the previous section to the intersection of the projected Veronese variety $V_{2,q}$ with the tangent space T_p to $v_2(Q^{-1}) \subset \mathbf{P}(T_{2,q})$ at the point $v_2(p) = [y_n^2] \in \mathbf{P}(T_{2,q})$, where $p = [y_n] = [0 : \ldots : 1] \in \mathbf{P}(T_1)$. The tangent space to the Veronese variety $V_2 \subset \mathbf{P}(T_2)$ at $[y_n^2]$ is spanned by

$$\langle y_1y_n, y_2y_n, \ldots, , y_{n-1}y_n, y_n^2\rangle.$$

The quadric Q^{-1} is defined by $\frac{1}{2}x_1x_n + \frac{1}{4}(x_2^2 + \ldots + x_{n-1}^2)$. Its tangent space in $P(T_1)$ at $[y_n]$ is defined by x_1, so its tangent space in $P(T_2)$ at $[y_n^2]$ is defined by x_1 inside the tangent space to the Veronese variety. Therefore, the tangent space T_p to $v_2(Q^{-1})$ is spanned by

$$\langle y_2y_n, \ldots, y_{n-1}y_n, y_n^2\rangle.$$

The orthogonal space of quadratic forms in S_2 is spanned by

$$\langle x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3, \ldots x_{n-1}^2, x_1x_n\rangle$$

and intersect $q^\perp$ precisely in the ideal of Γ_p given above.

With reverse lexicographically order on the coordinates $x_1, ..., x_n$, the initial ideal of Γ_p is generated by the monomials

$$x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3, \dots x_{n-1}^2.$$

In this monomial order, these monomials have the highest order in the ideal of any apolar scheme Γ that does not intersect the hyperplane $\{x_n = 0\}$. In fact, if the initial ideal of Γ contains x_ix_n, then x_n divides a quadratic form in the ideal of Γ. But if Γ does not intersect $\{x_n = 0\}$, then Γ would be degenerate. We therefore consider the open subvariety $V_h^{aff}(n)$ containing $[\Gamma_p]$ in $VAPS(Q, n)$, parametrizing apolar subschemes Γ of length n with support in $D(x_n)$. This is the complement of the divisor H_h defined by $h = \{x_n = 0\}$, the tangent hyperplane to Q^{-1} at $[y_1] = [1 : 0 : ... : 0] \in \mathbf{P}(T_1)$.
For $\Gamma \in V_h^{aff}(n)$ the initial terms of the generators of the ideal I_Γ coincide with those of I_{Γ_p}. More precisely, the generators of I_Γ may be obtained by adding suitable multiples of the monomials $x_ix_n, i \geq 1$ to these initial terms. We may therefore write these generators in the form

$$\begin{aligned}
&x_1^2 - a_{(11,1)}x_1x_n - a_{(11,2)}x_2x_n - a_{(11,2)}x_2x_n - ... - a_{(11,n)}x_n^2,\\
&x_ix_j - a_{(ij,1)}x_1x_n - a_{(ij,2)}x_2x_n - ... - a_{(ij,n)}x_n^2, \quad 1 \leq i < j \leq n-1,\\
&x_i^2 - x_1x_n - a_{(ii,2)}x_2x_n - ... - a_{(ii,n)}x_n^2, \quad 2 \leq i \leq n-1.
\end{aligned}$$

Analyzing these equations of Γ further, we see that the apolarity condition, i.e. that $I_{\Gamma,2} \subset q^\perp$, means that $a_{(11,1)} = 0$ and that $a_{(ij,1)} = 0$ when $i \neq j$. Therefore they take the form

$$\begin{aligned}
f_{11} =& x_1^2 - a_{(11,2)}x_2x_n - \dots - a_{(11,n)}x_n^2,\\
f_{12} =& x_1x_2 - a_{(12,2)}x_2x_n - \dots - a_{(12,n)}x_n^2,\\
f_{22} =& (x_2^2 - x_1x_n) - a_{(22,2)}x_2x_n - \dots - a_{(22,n)}x_n^2,\\
f_{13} =& x_1x_3 - a_{(13,2)}x_2x_n - \dots - a_{(13,n)}x_n^2,\\
f_{23} =& x_2x_3 - a_{(23,2)}x_2x_n - \dots - a_{(23,n)}x_n^2,\\
f_{33} =& (x_3^2 - x_1x_n) - a_{(33,2)}x_2x_n - \dots - a_{(33,n)}x_n^2,\\
&\vdots\\
f_{(n-1)(n-1)} =& (x_{n-1}^2 - x_1x_n) - a_{((n-1)(n-1),2)}x_2x_n - \dots - a_{((n-1)(n-1),n)}x_n^2.
\end{aligned} \tag{5.1}$$

To insure that these perturbed equations actually define length n subschemes, we ask that the first order relations or syzygies among the generators of I_{Γ_p} lift to the entire family. This is in fact precisely the requirement for the perturbation to define a flat family [Art76, Proposition 3.1], and will be pursued below when we find equations for $V_h^{aff}(n)$.
Here, we introduce weights and a torus action on this family: We give

- x_n and $a_{(ij,k)}$, where $2 \leq i, j, k \leq n-1$, weight 1,
- x_i, where $2 \leq i \leq n-1$, and $a_{(ij,n)}$, where $2 \leq i, j \leq n-1$. weight 2,
- x_1 and $a_{(1i,n)}$ and $a_{(11,i)}$, where $2 \leq i \leq n-1$, weight 3,

- $a_{(11,n)}$ weight 4.

Notice that with these weights each generator f_{ij} is homogeneous. A $\mathbb{C}^*$-action defined by multiplying each parameter with a constant λ^w to the power of its weight, acts on each generator by a scalar multiplication, i.e. on the total family in $\mathbf{P}(T_1) \times V_h^{aff}(n)$. This $\mathbb{C}^*$- action induces an action on the family $V_h^{aff}(n)$. In particular, if $[a] = [a_{(ij,k)}] \in V_h^{aff}(n)$ defines a subscheme $\Gamma_{[a]}$, then $[\lambda^w(a)] = [\lambda^w(a_{ij,k})] \in V_h^{aff}(n)$ and defines a subscheme $\Gamma_{[\lambda^w a]}$, such that $p' \in \Gamma_{[a]}$ if and only if $\lambda^w(p') \in \Gamma_{[\lambda^w a]}$.
Since $\lim_{\lambda\to 0} \lambda^w(a_{ij,k}) = 0$, the limit when $\lambda \to 0$ of the $\mathbb{C}^*$- action is the point in $V_h^{aff}(n)$ representing Γ_p. Thus we have shown

Lemma 5.1. *The affine algebraic set $V_h^{aff}(n)$ of apolar subschemes of length n contained in $D(x_n)$ coincides with the apolar schemes of length n whose equations are affine perturbations of the equations of Γ_p.*
Furthermore, the family $V_h^{aff}(n)$ is contractible to the point $[\Gamma_p]$.

An immediate consequence is the

Corollary 5.2. *The apolar subscheme Γ_p belongs to $VPS(Q,n)$. In particular, the variety of tangent spaces $TQ^{-1} \subset \mathbb{G}(n-1, T_{q,2})$ to the Veronese embedding of the quadric $Q^{-1} \subset \mathbf{P}(T_{q,2})$ is a subvariety of $VPS(Q,n)$.*

Notice that $V_h^{aff}(n)$ depends only on h, and not on p. Only the coordinates on $V_h^{aff}(n)$ depend on p. On the other hand, the contractible varieties $V_h^{aff}(n)$ form a covering of $VAPS(Q,n)$:

Lemma 5.3. *If $h_j = \{l_j = 0\}, j = 1, ..., n^2$ is a collection of tangent hyperplanes to Q^{-1}, so that no subset of n of them have a common point, then the open subvarieties $V_{h_j}^{aff}(l_j)$ parametrizing apolar subschemes Z of length n with support in $D(l_j)$ form a covering of $VAPS(Q,n)$ of isomorphic varieties.*

Proof. If an apolar subscheme Γ has $k \leq n$ components, then the collection of hyperplanes among the $\{l_j = 0\}$ that intersect Γ is at most $k(n-1) < n^2$, so the $V_{h_j}^{aff}(l_j)$ form a covering. The last part follows from the homogeneity. $\square$

To find equations for the family $V_h^{aff}(n)$, we use the parameters for the generators in (5.1), i.e.

$$a_{(ij,k)} \quad i,j \in \{1,\dots,n-1\},\ 2 \leq k \leq n,$$

where we read the first index (ij) as an unordered pair.

It will be useful to write the generators with matrices:

$$\begin{pmatrix} f_{11} \\ f_{12} \\ f_{22} \\ \vdots \\ f_{(n-1)(n-1)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 & -a_{(11,2)} & \cdots & -a_{(11,n)} \\ 0 & 1 & \dots & 0 & -a_{(12,2)} & \cdots & -a_{(12,n)} \\ 0 & 0 & \dots & 0 & -a_{(22,2)} & \cdots & -a_{(22,n)} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 & \vdots & \dots & -a_{((n-1)(n-1),n)} \end{pmatrix} \cdot \begin{pmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 - x_1x_n \\ \vdots \\ x_{n-1}^2 - x_1x_n \\ x_2x_n \\ \vdots \\ x_n^2 \end{pmatrix}.$$

We denote by A_F the $\binom{n}{2} \times (\binom{n}{2} + n - 1)$-dimensional coefficient matrix of these generators. The maximal minors of A_F are, of course, precisely the Plücker coordinates for $V_h^{aff}(n)$ in $\mathbb{G}(\binom{n}{2}, q^\perp)$, or equivalently in $\mathbb{G}(n-1, T_{2,q})$.
We find the equations of the family by asking that the first order syzygies among the generators of I_{Γ_p} lift to the entire family. By [Art76, Proposition 3.1], this is precisely the requirement for the perturbation to define a flat family. We use a standard basis approach (cf. [Sch91]). The syzygies for a subscheme Z in the family are all linear, and the initial terms are inherited from Γ_p. Therefore, the difference between syzygies of Z and syzygies of Γ_p are only multiples of x_n. By the division theorem ([Sch91, Theorem A.3]), every syzygy has the initial term $x_k(x_ix_j)$, where $k > j \geq i$, and has the form

$$x_k f_{ij} = \sum_{st} g_{ij}^{st} f_{st}$$

where f_{ij} is the generator with initial term x_ix_j and g_{ij}^{st} is a linear form such that $g_{ij}^{st} f_{st}$ has higher order than $x_k(x_ix_j)$. More precisely, we therefore consider products of the generators (f_{ij}) with a first order syzygy for Γ_p and add precisely those multiples of x_n in the syzygy that eliminates monomials $x_kx_lx_n$ with $k \leq l < n$ in the product. The relations among the parameters required for the lifting of the syzygies can then be read off as the coefficients of the monomials $x_tx_n^2$.

Theorem 5.4. *The equations defining $V_h^{aff}(n)$ all lie in the linear span of the 2×2 minors of the coefficient matrix A_F of the family of equations f_{ij}. In particular $VAPS(Q, n)$ is a linear section of the Grassmannian $\mathbb{G}(n-1, T_{2,q})$.*

Remark 5.5. *This generalizes the result of Mukai in case $n = 3$, cf.* [Muk92]. *For an exposition of different approaches to the $n = 3$ case, see* [Dol12, Section 2.1.3].

Proof. Consider the following first order syzygies of Γ_p of rank 2 and 3:
$R_i \cdot S_i(m), 1 < i < n, m = 1, 2:$

$$\begin{pmatrix} x_1^2 & x_1x_i & x_i^2 - x_1x_n \end{pmatrix} \cdot \begin{pmatrix} -x_i & x_n \\ x_1 & -x_i \\ 0 & x_1 \end{pmatrix} = 0,$$

where $S_i(m)$ is the m-th column vector in the syzygy matrix S_i,

$R_{ij} \cdot S_{ij}(m), 1 < i < j < n, m = 1, \ldots, 4:$

$$\begin{pmatrix} x_i^2 - x_1x_n & x_1x_i & x_1x_j & x_ix_j & x_j^2 - x_1x_n \end{pmatrix} \cdot \begin{pmatrix} x_j & 0 & 0 & 0 \\ 0 & x_n & x_j & 0 \\ x_n & 0 & -x_i & -x_i \\ -x_i & -x_j & 0 & x_1 \\ 0 & x_i & 0 & 0 \end{pmatrix} = 0,$$

and
$R_{ijk} \cdot S_{ijk}(m), 1 < i < j < k < n, m = 1, 2:$

$$\begin{pmatrix} x_ix_j & x_jx_k & x_ix_k \end{pmatrix} \cdot \begin{pmatrix} -x_k & -x_k \\ x_i & 0 \\ 0 & x_j \end{pmatrix} = 0.$$

These syzygies are clearly linearly independent, and their cardinality, $2\binom{n}{3}$, coincides with the dimension of the space of first order syzygies, according to Proposition 2.1, so they form a basis.
We lift these syzygies by adding the multiples of x_n in the syzygy matrix, that reduces the product to cubic polynomials with monomials only of the form $x_ix_n^2$. We denote by $\tilde{S}_i(j)$ the syzygies obtained from $S_i(j)$ this way. Likewise we denote by $\tilde{R}_i$ the row vector obtained from R_i by substituting the entries x_sx_t by f_{st}. Similarly we get row vectors $\tilde{R}_{ij}, \tilde{R}_{ijk}$ and column vectors $\tilde{S}_{ij}(r)$ and $\tilde{S}_{ijk}(r)$.
For example, with $i = 2$ and $n = 4$ we get

$$\begin{aligned} \tilde{R}_2 \cdot \tilde{S}_2(1) = & \begin{pmatrix} f_{11} & f_{12} & f_{22} & f_{13} & f_{23} & f_{33} \end{pmatrix} \cdot \begin{pmatrix} -x_2 \\ x_1 + a_{(12,2)}x_4 \\ -a_{(11,2)}x_4 \\ a_{(12,3)}x_4 \\ -a_{(11,3)}x_4 \\ 0 \end{pmatrix} \\ = \quad & (-a_{(12,4)} + a_{(11,2)})x_1x_4^2 \\ & +(a_{(11,4)} - a_{(12,2)}^2 + a_{(22,2)}a_{(11,2)} - a_{(13,2)}a_{(12,3)} + a_{(23,2)}a_{(11,3)})x_2x_4^2 \\ & +(-a_{(12,3)}a_{(12,2)} + a_{(22,3)}a_{(11,2)} - a_{(13,3)}a_{(12,3)} + a_{(23,3)}a_{(11,3)})x_3x_4^2 \\ & +(-a_{(12,4)}a_{(12,2)} + a_{(22,4)}a_{(11,2)} - a_{(13,4)}a_{(12,3)} + a_{(23,4)}a_{(11,3)})x_4^3. \end{aligned}$$

For general i and n we get (with the first pair in the index unordered to simplify presentation of the summation)

$$\begin{aligned} \tilde{R}_i \cdot \tilde{S}_i(1) = \quad & (a_{(11,i)} - a_{(1i,n)})x_1x_n^2 + a_{(11,n)}x_ix_n^2 \\ & + \textstyle\sum_{j=2}^{n}(\sum_{k=2}^{n-1}(a_{(11,k)}a_{(ik,j)} - a_{(1i,k)}a_{(1k,j)})x_jx_n^2. \end{aligned}$$

Similarly

$$\begin{aligned}
\tilde{R}_i\cdot\tilde{S}_i(2) = \quad & (a_{(1i,i)}-a_{(ii,n)})x_1x_n^2+a_{(1i,n)}x_ix_n^2\\
&+\textstyle\sum_{j=2}^n(-a_{(11,j)}+\sum_{k=2}^{n-1}(a_{(1i,k)}a_{(ik,j)}-a_{(ii,k)}a_{(1k,j)}))x_jx_n^2,\\
\tilde{R}_{ij}\cdot\tilde{S}_{ij}(1) = \quad & (a_{(ij,i)}-a_{(ii,j)})x_1x_n^2+a_{(ij,n)}x_ix_n^2-a_{(ii,n)}x_jx_n^2\\
&+\textstyle\sum_{k=2}^n(-a_{(1j,k)}+\sum_{m=2}^{n-1}(a_{(ij,m)}a_{(im,k)}-a_{(ii,m)}a_{(jm,k)}))x_kx_n^2,\\
\tilde{R}_{ij}\cdot\tilde{S}_{ij}(2) = \quad & (a_{(ij,j)}-a_{(jj,i)})x_1x_n^2-a_{(jj,n)}x_ix_n^2+a_{(ij,n)}x_jx_n^2\\
&+\textstyle\sum_{k=2}^n(-a_{(1i,k)}+\sum_{m=2}^{n-1}(a_{(ij,m)}a_{(jm,k)}-a_{(jj,m)}a_{(im,k)}))x_kx_n^2,\\
\tilde{R}_{ij}\cdot\tilde{S}_{ij}(3) = \quad & (a_{(1j,i)}-a_{(1i,j)})x_1x_n^2+a_{(1j,n)}x_ix_n^2-a_{(1i,n)}x_jx_n^2\\
&+\textstyle\sum_{k=2}^n(\sum_{m=2}^{n-1}(a_{(1j,m)}a_{(im,k)}-a_{(1i,m)}a_{(jm,k)}))x_kx_n^2,\\
\tilde{R}_{ij}\cdot\tilde{S}_{ij}(4) = \quad & (a_{(1j,i)}-a_{(ij,n)})x_1x_n^2-a_{(1j,n)}x_ix_n^2\\
&+\textstyle\sum_{k=2}^n(\sum_{m=2}^{n-1}(a_{(1j,m)}a_{(im,k)}-a_{(ij,m)}a_{(1m,k)}))x_kx_n^2,\\
\tilde{R}_{ijk}\cdot\tilde{S}_{ijk}(1) = \quad & (a_{(ij,k)}-a_{(jk,i)})x_1x_n^2-a_{(jk,n)}x_ix_n^2+a_{(ij,n)}x_kx_n^2\\
&+\textstyle\sum_{l=2}^n(\sum_{m=2}^{n-1}(a_{(ij,m)}a_{(km,l)}-a_{(jk,m)}a_{(im,l)}))x_lx_n^2,\\
\tilde{R}_{ijk}\cdot\tilde{S}_{ijk}(2) = \quad & (a_{(ij,k)}-a_{(ik,j)})x_1x_n^2-a_{(ik,n)}x_jx_n^2+a_{(ij,n)}x_kx_n^2\\
&+\textstyle\sum_{l=2}^n(\sum_{m=2}^{n-1}(a_{(ij,m)}a_{(km,l)}-a_{(ik,m)}a_{(jm,l)}))x_lx_n^2.
\end{aligned}$$

The linear relations in the parameters of the family $V_h^{\mathit{aff}}(n)$ are precisely the coefficients of $x_1x_n^2$ in these products:

LEMMA 5.6. *The space of linear forms in the ideal of $V_h^{\mathit{aff}}(n)$ is generated by the following forms, where $\{i,j,k\}$ is any subset of distinct elements in $\{2,\dots,n-1\}$*

$$a_{(11,i)}-a_{(1i,n)},\ a_{(1i,i)}-a_{(ii,n)},\ a_{(1j,i)}-a_{(ij,n)},\ a_{(ij,j)}-a_{(jj,i)},\ a_{(ij,k)}-a_{(jk,i)}.$$

□

Notice that only the first two occur when $n=3$, and only the first four occur when $n=4$.

Using these linear relations, the quadratic ones all become linear in the 2×2 minors of the matrix A_F of coefficients $a_{(ij,k)}$, i.e. linear in the Plücker coordinates. In fact, by a straightforward but tedious derivation from the above presentation, we may write the generators of the ideal $V_h^{\mathit{aff}}(n)$ as linear combinations of 2×2 minors in the coefficient matrix A_F (cf. the documented computer algebra code to perform the computation of ideal generators [RS11]):

LEMMA 5.7. *Modulo the linear forms the ideal of $V_h^{\mathit{aff}}(3)$ is generated by*

$$a_{(11,3)}+a_{(12,3)}a_{(22,2)}-a_{(12,2)}a_{(22,3)}.$$

□

LEMMA 5.8. *Modulo the linear forms the ideal of $V_h^{\mathit{aff}}(4)$ is generated by*

$$\begin{gathered}
-a_{(12,2)}-a_{(13,3)}+(a_{(23,2)}a_{(22,3)}-a_{(22,2)}a_{(23,3)})+(a_{(33,2)}a_{(23,3)}-a_{(23,2)}a_{(33,3)}),\\
-a_{(11,2)}+(a_{(12,3)}a_{(23,2)}-a_{(23,3)}a_{(12,2)})+(a_{(13,3)}a_{(33,2)}-a_{(33,3)}a_{(13,2)}),\\
-a_{(11,3)}+(a_{(12,2)}a_{(22,3)}-a_{(22,2)}a_{(12,3)})+(a_{(13,2)}a_{(23,3)}-a_{(23,2)}a_{(13,3)}),\\
(a_{(11,2)}a_{(22,3)}-a_{(12,2)}a_{(12,3)})+(a_{(11,3)}a_{(23,3)}-a_{(12,3)}a_{(13,3)}),\\
a_{(11,4)}+(a_{(12,4)}a_{(22,2)}-a_{(22,4)}a_{(12,2)})+(a_{(13,4)}a_{(23,2)}-a_{(23,4)}a_{(13,2)}),\\
a_{(11,4)}+(a_{(12,4)}a_{(23,3)}-a_{(23,4)}a_{(12,3)})+(a_{(13,4)}a_{(33,3)}-a_{(33,4)}a_{(13,3)}).
\end{gathered}$$

$\square$

Lemma 5.9. *Modulo the linear forms the ideal of $V_h^{aff}(n)$, when $n > 4$, is generated by the following forms:*
For $i \in \{2, \ldots, n-1\}$,

$$a_{(11,n)} - \sum_{m=2}^{n-1} (a_{(im,n)} a_{(1m,i)} - a_{(1m,n)} a_{(im,i)}),$$

for any subset $\{i, j\} \subset \{2, \ldots, n-1\}$,

$$a_{(11,i)} - \sum_{m=2}^{n-1} (a_{(jm,n)} a_{(im,j)} - a_{(im,n)} a_{(jm,j)}),$$

$$\sum_{m=2}^{n-1} (a_{(1m,n)} a_{(im,j)} - a_{(im,n)} a_{(1m,j)})$$

and

$$a_{(1i,i)} + a_{(1j,j)} - \sum_{m=2}^{n-1} (a_{(jm,i)} a_{(im,j)} - a_{(im,i)} a_{(jm,j)}),$$

for any subset $\{i, j, k\} \subset \{2, \ldots, n-1\}$,

$$a_{(1j,k)} - \sum_{m=2}^{n-1} (a_{(jm,i)} a_{(im,k)} - a_{(im,i)} a_{(jm,k)})$$

and

$$\sum_{m=2}^{n-1} (a_{(jm,n)} a_{(im,k)} - a_{(im,n)} a_{(jm,k)}),$$

and for any subset $\{i, j, k, l\} \subset \{2, \ldots, n-1\}$,

$$\sum_{m=2}^{n-1} (a_{(im,j)} a_{(km,l)} - a_{(km,j)} a_{(im,l)}).$$

$\square$

Since the open affine sets $V_h^{aff}(n)$ cover $VAPS(Q, n)$, we conclude that $VAPS(Q, n)$ is a linear section of the Grassmannian $\mathbb{G}(\binom{n}{2}, q^{\perp})$. Equivalently, $VAPS(Q, n)$ is projectively equivalent to a linear section of $\mathbb{G}(n-1, T_{2,q})$ in its Plücker embedding. This concludes the proof of Theorem 5.4. $\square$

Using the linear relations we may reduce the number of variables, when $n > 4$, and use as indices the following unordered three element sets:

$$\mathcal{I} = \{\{11k\} | 1 < k \leq n\} \cup \{\{1jk\} | 1 < j \leq k < n\} \cup \{\{ijk\} | 1 < i \leq j \leq k < n\}.$$

Let $R = \mathbb{C}[a_I | I \in \mathcal{I}]$. We substitute $a_{11k} = a_{11,k}, a_{1jk} = a_{1j,k}, a_{ijk} = a_{ij,k}$ and get:

LEMMA 5.10. *The ideal of $V_h^{aff}(n)$, when $n > 4$, is generated by the following polynomials in R: For $i \in \{2,\dots,n-1\}$,*

$$a_{11n} - \sum_{m=2}^{n-1}(a_{1im}^2 - a_{11m}a_{iim}),$$

for any subset $\{i,j\} \subset \{2,\dots,n-1\}$,

$$a_{11i} - \sum_{m=2}^{n-1}(a_{1jm}a_{ijm} - a_{1im}a_{jjm}),$$

$$\sum_{m=2}^{n-1}(a_{11m}a_{ijm} - a_{1im}a_{1jm}), \quad a_{1ii} + a_{1jj} - \sum_{m=2}^{n-1}(a_{ijm}^2 - a_{iim}a_{jjm}),$$

for any subset $\{i,j,k\} \subset \{2,\dots,n-1\}$,

$$a_{1jk} - \sum_{m=2}^{n-1}(a_{ijm}a_{ikm} - a_{iim}a_{jkm}), \quad \sum_{m=2}^{n-1}(a_{1jm}a_{ikm} - a_{1im}a_{jkm}),$$

and for any subset $\{i,j,k,l\} \subset \{2,\dots,n-1\}$,

$$\sum_{m=2}^{n-1}(a_{ijm}a_{klm} - a_{jkm}a_{ilm}).$$

□

Notice that these generators are all homogeneous in the weights introduced above. The linear parts of the ideal generators define the tangent space of the family $V_h^{aff}(n)$ at $[\Gamma_p]$, so another consequence of our computations is the tangent space dimension.

PROPOSITION 5.11. *Let $L(n)$ be the space of linear forms spanned by the linear parts of the generators in the ideal of $V_h^{aff}(n)$. Then $L(3)$ is spanned by*

$$a_{(11,3)}, \quad a_{(11,2)} - a_{(12,3)} \quad \textit{and} \quad a_{(12,2)} - a_{(22,3)}.$$

$L(4)$ is spanned by

$$a_{(11,4)}, \quad a_{(11,2)}, \quad a_{(12,4)}, \quad a_{(11,3)}, \quad a_{(13,4)},$$
$$a_{(12,2)} - a_{(22,4)}, \quad a_{(13,3)} - a_{(33,4)}, \quad a_{(12,2)} + a_{(13,3)},$$
$$a_{(12,3)} - a_{(23,4)}, \quad a_{(13,2)} - a_{(12,3)}, \quad a_{(23,3)} - a_{(33,2)} \quad \textit{and} \quad a_{(22,3)} - a_{(23,2)}.$$

$L(n)$, when $n > 4$, is spanned by $a_{(11,n)}$ and for any $i \in \{2,\dots,n-1\}$,

$$a_{(11,i)}, a_{(1i,n)}, a_{(1i,i)}, a_{(ii,n)},$$

for any subset $\{i,j\} \subset \{2,\dots,n-1\}$,

$$a_{(1j,i)}, a_{(ij,n)}, a_{(ii,j)} - a_{(ij,i)},$$

and for any subset $\{i,j,k\} \subset \{2,\dots,n-1\}$,

$$a_{(ij,k)} - a_{(jk,i)}.$$

In particular

$$V_h^{aff}(3) \cong \mathbf{A}^3,\ V_p^{aff}(4) \cong \mathbf{A}^6,\ V_h^{aff}(5) \cong \mathbf{A}^{10}.$$

Proof. The linear parts of the generators can be read off Lemma 5.6 and Lemmas 5.7, 5.8 and 5.9. Notice only that the two term forms

$$a_{(12,2)} - a_{(22,4)}, \quad a_{(13,3)} - a_{(33,4)}, \quad a_{(12,2)} + a_{(13,3)}$$

span a three dimensional space, while

$$a_{(1i,i)} - a_{(ii,n)}, \quad a_{(1j,j)} - a_{(jj,n)}, \quad a_{(1i,i)} + a_{(1j,j)}, \quad \text{for } 1 < i < j < n$$

span the space generated by

$$a_{(1i,i)}, \quad a_{(ii,n)}, \quad 1 < i < n$$

when $n > 4$.
For $n = 3$, the family $V_h^{aff}(3)$ has 6 parameters, while there are three independent linear forms in the relations so the tangent space at $[\Gamma_p]$ has dimension $6 - 3 = 3$ as expected. In fact $V_h^{aff}(3) \cong \mathbf{A}^3$ with parameters

$$a_{(11,2)}, a_{(12,2)}, a_{(22,2)}.$$

For $n = 4$ the family $V_h^{aff}(4)$ has 18 parameters, while the linear forms in the relations are generated by 12 independent forms, so the tangent space at $[\Gamma_p]$ has dimension $18 - 12 = 6$. In fact $V_h^{aff}(4) \cong \mathbf{A}^6$ with parameters

$$a_{(12,2)}, a_{(12,3)}, a_{(23,3)}, a_{(22,3)}, a_{(22,2)}, a_{(33,3)}.$$

For $n > 4$ we see that all parameters with a 1 or an n in the index are independent forms in the space of linear parts of ideal generators in $V_h^{aff}(n)$. Furthermore, the other linear parts, simply expresses that $\{(ijk) | 1 < i \leq j \leq k < n\}$ form a natural index set for representatives of the parameters. The cardinality of this index set is simply the cardinality of monomials of degree 3 in $n - 2$ variables, i.e. $\binom{n}{3}$. In case $n = 5$ we again conclude that $V_h^{aff}(5) \cong \mathbf{A}^{10}$ with parameters

$$\{a_{ijk} | 2 \leq i \leq j \leq k \leq 4\}.$$

□

Corollary 5.12. *The tangent space dimension of $VAPS(Q, n)$ at $[\Gamma_p]$ is $\binom{n}{3}$ when $n > 5$. When $n \leq 5$, $VAPS(Q, n)$ has a finite cover of affine spaces, in particular $VAPS(Q, n)$ is smooth and coincides with $VPS(Q, n)$.* □

Remark 5.13. *Let Γ be a smooth apolar subscheme to Q consisting of n distinct points. Any subset of $n - 2$ points in Γ is contained in a pencil of apolar subschemes that form a line in $VPS(Q, n)$ through $[\Gamma]$. Thus $\binom{n}{2}$ lines in $VPS(Q, n)$ through $[\Gamma]$ is contained in the tangent space at $[\Gamma]$.*

We extend this remark and give a conceptual reason for the dimension of the tangent space to $VAPS(Q, n)$ at $[\Gamma_p]$.

Proposition 5.14. *Let $[\Gamma_p] \in VPS(Q,n) \subset \mathbb{G}(n-1,T_{2,q})$ be a point on the subvariety TQ^{-1} in its Grassmannian embedding. Then $VPS(Q,n)$ contains the cone over a 3-uple embedding of $\mathbf{P}^{n-3}$ with vertex at $[\Gamma_p]$.*

Proof. We first identify a cone over a 3-uple embedding of $\mathbf{P}^{n-3}$ inside $VAPS(Q,n)$, and then give an explicit description of the apolar subschemes parameterized by this cone in order to show that the cone is contained in $VPS(Q,n)$.
Consider the subvariety $V_p^{vero}(n) \subset V_h^{aff}(n)$ parameterizing ideals I_Γ with coefficient matrix $A_F(\Gamma) = (I\ A)$ where the submatrix $A = (a_{(ij,k)})$ has rank at most 1 and has nonzero entries only in the submatrix $A_0 \subset A$ with entries $\{a_{(ij,k)} | 1 < i \le j < n, 1 < k < n\}$. As above, using the linear relations, we may substitute the parameters $a_{(ij,k)}$ with parameters a_{ijk} whose indices are unordered triples (ijk). In these new parameters the matrix A_0 takes the form:

$$A_0 = \begin{pmatrix} a_{222} & a_{223} & \cdots & a_{22(n-1)} \\ a_{223} & a_{233} & \cdots & a_{23(n-1)} \\ \cdots & \cdots & \cdots & \cdots \\ a_{22(n-1)} & a_{23(n-1)} & \cdots & a_{2(n-1)(n-1)} \\ \cdots & \cdots & \cdots & \cdots \\ a_{2(n-1)(n-1)} & a_{3(n-1)(n-1)} & \cdots & a_{(n-1)(n-1)(n-1)} \end{pmatrix}.$$

By Theorem 5.4, the equations of $V_h^{aff}(n)$ are linear in the 2×2 minors of the coefficient matrix A, so any rank 1 matrix A_0 defines a point on $V_p^{vero}(n)$. The symmetry in the indices explains why the 2×2 minors of the matrix define the 3-uple embedding of $\mathbf{P}^{n-3}$. Since the ideal of Γ_p correspond to the zero matrix, we conclude that the subvariety $V_p^{vero}(n)$ in $VAPS(Q,n)$ is the cone over this 3-uple embedding.
To see that $V_p^{vero}(n)$ is contained in $VPS(Q,n)$ we show that a general point on $V_p^{vero}(n)$ lies in the closure of smooth apolar subscheme to Q. For this, we describe for each general point $s \in \mathbf{P}^{n-3}$ an apolar subscheme Γ_s belonging to $V_p^{vero}(n)$. It has two components $\Gamma_s = \Gamma_{s,0} \cup p_s$, the first one $\Gamma_{s,0}$ of length $n-1$ and supported at p, while the second component p_s is a closed point. We shall show that q has a decomposition $q = q_l + q(l)^2 \in T_2$ where $[q(l)] = p_s \in \mathbf{P}(T_1)$ and $q_l \in (l^\perp)^2$. The subscheme $\Gamma_{s,0}$ is apolar to q_l and contains the first order neighborhood of p inside the quadric $\{q_l^{-1} = 0\} \subset \mathbf{P}(l^\perp)$ in the hyperplane polar to p_s. Then $\Gamma_{s,0}$ lies in the closure of smooth apolar subschemes to q_l. We conclude by applying Proposition 2.4.
Let $s = [s_2 : .. : s_{n-1}] \in \mathbf{P}^{n-3}$ and let

$$||s||^2 = s_2^2 + \ldots + s_{n-1}^2, \quad \langle s,x\rangle = \sum_{i=2}^{n-1} s_i x_i, \quad \langle s,y\rangle = \sum_{i=2}^{n-1} s_i y_i,$$

then

$$x_1^2, \ldots, x_1 x_{n-1},$$

$$x_i^2 - x_1 x_n - s_i^2 \langle s,x\rangle x_n \quad 1 < i < n, \quad x_i x_j - s_i s_j \langle s,x\rangle x_n \quad 1 < i < j < n$$

defines a subscheme Γ_s that belongs to $V_p^{vero}(n)$. When $||s||^2 \neq 0$, then Γ_s contains the point

$$p_s = [q(||s||^2\langle s, x\rangle + x_1)] = [||s||^2\langle s, y\rangle + y_n] \in \mathbf{P}(T_1)$$

Consider the linear subspace $L_s = \{x_1 = 0\} \cap \{\langle s, x\rangle = 0\}$. The intersection $\Gamma_s \cap L_s$ is the subscheme defined by

$$x_2^2 = x_2 x_3 = \ldots = x_{n-1}^2 = 0.$$

This subscheme has length $n-2$. The union $p_s \cup (\Gamma_s \cap L_s)$ spans the hyperplane $\{x_1 = 0\}$, so the residual point in Γ_s is the pole, with respect to Q^{-1}, of this hyperplane, i.e. the point p. Therefore the subscheme $\Gamma_{s,0} = \Gamma_s \setminus p_s$ has length $n-1$, is supported in p, and contains the first order neighborhood of p in the codimension two linear space L_s.

The subscheme $\Gamma_{s,0}$ is apolar to the quadric

$$q_s = (||s||^2\langle s, y\rangle + y_n)^2 - ||s||^6(2y_1y_n + y_2^2 + \ldots + y_{n-1}^2).$$

Let $l = ||s||^2\langle s, x\rangle + x_1$. Then $p_s = [q(l)]$, while

$$l^\perp = \langle y_2 - ||s||^2 s_i y_1, \ldots, y_{n-1} - ||s||^2 s_i y_1, y_n\rangle.$$

Then $q_s \in (l^\perp)^2 < 2$ and

$$(q(l))^2 - q_s = ||s||^6 \cdot q \in T_2.$$

According to Proposition 2.4 a subscheme Γ_0 in $\mathbf{P}(l^\perp)$ of length $n-1$ is apolar to q_s if and only if $\Gamma = \Gamma_0 \cup p_s$ is apolar to q. Now, $\Gamma_{s,0}$ is apolar to q_s and contains a first order neighborhood of a point on the smooth quadric $\{{q_s}^{-1} = 0\}$ in $\mathbf{P}(l^\perp) \subset \mathbf{P}(T_1)$. By Remark 3.5, the subscheme $\Gamma_{s,0}$ is a subscheme like Γ_p, with respect to q_s. Therefore $\Gamma_{s,0}$ lies in the closure of smooth apolar subschemes to q_s. But then Γ_s must lie in the closure of smooth apolar subschemes to q. Hence $[\Gamma_s] \in VPS(q, n)$.

□

Corollary 5.15. *$VPS(Q, n)$ is singular when $n \geq 6$.*

Proof. The cone with vertex at $[\Gamma_p] \in TQ^{-1}$ over the 3-uple embedding of $\mathbf{P}^{n-3}$ is contained in the tangent space of $VPS(Q, n)$ at $[\Gamma_p]$, i.e. also in the tangent space of $VAPS(Q, n)$. Since the span of the cone and the tangent space of the latter have the same dimension, they coincide. In particular the tangent space of $VPS(Q, n)$ at $[\Gamma_p]$ has dimension $\binom{n}{3}$. When $n \geq 6$, then $\binom{n}{3} > \binom{n}{2} = \dim VPS(Q, n)$ so $VPS(Q, n)$ is singular. □

We pursue the case $n = 6$ a bit further and show that $VAPS(Q, 6)$ and $VPS(Q, 6)$ coincide. We use the symmetric variables

$$a_{ijk} = a_{(ij,k)},\ 1 \leq i, j, k \leq 6$$

for any permutation of the letters i, j, k. According to Lemma 5.10 we may list the generators explicitly. This list is however not minimal. In fact, a minimal set of generators is given by the following twenty generators in weight 2, four generators in weight 3 and one generator in weight 4. The twenty generators of weight 2 are the generators of weight 2 in Lemma 5.10: For each $1 < k < 6$, and each pair $\{i, j\} \subset \{2, 3, 4, 5\} \setminus \{k\}$ the generator

$$-a_{ij6} + \sum_{m=2}^{5} (a_{ikm}a_{jkm} - a_{ijm}a_{kkm}),$$

for each pair $\{i, j\} \subset \{2, 3, 4, 5\}$ the generator

$$-a_{ii6} - a_{jj6} + \sum_{m=2}^{5} (a_{ijm}a_{ijm} - a_{iim}a_{jjm}),$$

and additionally the two generators

$$\sum_{m=2}^{5} (a_{23m}a_{45m} - a_{24m}a_{35m}) \quad \text{and} \quad \sum_{m=2}^{5} (a_{23m}a_{45m} - a_{25m}a_{34m}).$$

The last five generators are computed from the list of Lemma 5.10 using *Macaulay2* [GS], see the documented code in [RS11].
Of weight 3 we find, for $i = 2, 3, 4$:

$$a_{11i} - \sum_{m=2}^{5} (a_{im6}a_{m55} - a_{m56}a_{im5})$$

and

$$a_{115} - \sum_{m=2}^{5} (a_{m46}a_{m45} - a_{m56}a_{m44}).$$

The generator of weight 4 is

$$a_{116} - \sum_{m=2}^{5} a_{m56}^2 + \sum_{m=2}^{5} a_{11m}a_{m55}.$$

The ten parameters with 6 in the index appear linearly in the 20 generators of weight 2, while the five parameters with 11 in the index appear linearly in the five generators of weights 3 and 4. The remaining 10 generators of weight 2 therefore depend only on 20 parameters a_I. In fact they depend only on 16 linear forms. It is a remarkable fact that these ten quadratic forms define the 10-dimensional spinor variety. To see this we choose and rename the following 16 forms:

$$x_{1234} = -a_{353} + a_{252},\ x_{15} = -a_{555} + a_{454} + a_{353} + a_{252},\ x_{34} = a_{453},$$

$$x_{1235} = a_{554} - a_{444} + a_{343} + a_{242},\ x_{14} = a_{343} - a_{242},\ x_{35} = a_{553} - a_{232},$$

$$x_{1245} = -a_{553} - a_{443} + a_{333} - a_{232},\ x_{13} = a_{553} - a_{443},\ x_{24} = a_{452},$$

$$x_{1345} = a_{552} + a_{442} + a_{332} - a_{222},\ x_{12} = a_{552} - a_{442},\ x_{23} = a_{352},$$

$$x_{2345} = a_{454} - a_{353},\ x_{45} = a_{554} - a_{242},\ x_{25} = -a_{442} + a_{332},\ x_0 = a_{342}.$$

In these variables the ten quadratic generators takes the form

$$q_0 = x_{25}x_{34} - x_{35}x_{24} + x_{45}x_{23} + x_{2345}x_0,$$

$$q_1 = -x_{45}x_{13} + x_{14}x_{35} - x_{15}x_{34} + x_{1345}x_0,$$

$$\begin{aligned}
q_2 &= x_{45}x_{12} + x_{14}x_{25} + x_{15}x_{24} + x_{1245}x_0,\\
q_3 &= -x_{35}x_{12} + x_{13}x_{25} - x_{15}x_{23} + x_{1235}x_0,\\
q_4 &= x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} + x_{1234}x_0,\\
q_5 &= x_{1345}x_{12} + x_{1245}x_{13} + x_{1235}x_{14} + x_{15}x_{1234},\\
q_6 &= -x_{2345}x_{12} + x_{1245}x_{23} + x_{1235}x_{24} + x_{1234}x_{25},\\
q_7 &= -x_{2345}x_{13} - x_{1345}x_{23} + x_{1235}x_{34} + x_{1234}x_{35},\\
q_8 &= -x_{2345}x_{14} - x_{1345}x_{24} - x_{1245}x_{34} + x_{1234}x_{45},\\
q_9 &= -x_{15}x_{2345} - x_{1345}x_{25} - x_{1245}x_{35} - x_{1235}x_{45}.
\end{aligned}$$

The first five express (when $x_0 = 1$) the variables x_{ijkl} as quadratic Pfaffians in the x_{st}, while the last five quadrics express the linear syzygies among these Pfaffians. The ten quadratic forms satisfy the following quadratic relation

$$q_0q_5 + q_1q_6 + q_2q_7 + q_3q_8 + q_4q_9 = 0.$$

In fact the ten quadratic forms generate the ideal of the 10-dimensional spinor variety embedded in $\mathbf{P}^{15}$ by its spinor coordinates [RS00, Section 6],[Muk95].

Corollary 5.16. *$V_h^{aff}(6)$ is isomorphic to a cone over the ten-dimensional spinor variety embedded in $\mathbf{P}^{15}$ by its spinor coordinates. In particular $VAPS(Q,6)$ is singular, irreducible and coincides with $VPS(Q,6)$.* □

We end this section summarizing some computational results, for small n, of some natural subschemes of $VAPS(Q,n)$. The first is the punctual part $V_p^{loc}(n)$ of $VAPS(Q,n)$, i.e. the variety of apolar subschemes in $VAPS(Q,n)$ with support at a single point p. The support p of a local apolar subscheme must lie on Q^{-1} by Lemma 2.3. Therefore we may assume that $p = [0 : 0 : ... : 1]$, and use the equations 5.1. Of course, $[\Gamma_p]$ is then in $V_p^{loc}(n)$. Furthermore, $V_p^{loc}(n)$ is naturally contained in a second natural subscheme of $VAPS(Q,n)$, namely $V_p^{sec}(n)$, the variety of apolar subschemes in $V_h^{aff}(n)$ that contains the point p. We will do the explicit computation in the cases where $VAPS(Q,n) = VPS(Q,n)$ is smooth, i.e. when $n < 6$. An apolar subscheme in $V_h^{aff}(n)$ lies in $V_p^{sec}(n)$ if and only if the term x_n^2 does not appear in any equation, so $V_p^{sec}(n)$ is defined by the equations $a_{(ij,n)} = 0$ for $1 \le i \le j < n$ in $V_h^{aff}(n)$. The linear relations then imply that $a_{(1i,j)} = 0$ for all i and j, and as before that each parameter $a_{(ij,k)}$ with $1 < i,j,k < n$ may be represented by a parameter

$$a_{ijk} \text{ with } 1 < i \le j \le k < n.$$

In particular for $n = 3$ the only parameter left is a_{222}, and $V_p^{sec}(3)$ is isomorphic to the affine line. The equations $x_1^2 = x_1x_2 = x_2^2 - x_1x_3 - a_{222}x_2x_3$ define a scheme supported at p only if $a_{222} = 0$, so $V_p^{loc}(n)$ is a point in the case $n = 3$. The computation of $V_p^{sec}(n)$ follow the same procedure for every n. For a local scheme Γ in $V_p^{loc}(n)$ we may set $x_n = 1$ in the equations.

Lemma 5.17. *A local scheme Γ supported at p, that belongs to $V_p^{loc}(n)$, is Gorenstein. The maximal ideal of its affine coordinate ring is spanned by $x_2,\dots,x_{n-1},x_1$, and its socle is generated by x_1.*

Proof. The scheme Γ is Gorenstein by Lemma 2.6. The maximal ideal is certainly generated by $x_1, x_2, \dots, x_{n-1}$, and since Γ is nondegenerate these are linearly independent. Finally, $x_1 x_i = 0$ for all i by the apolarity condition as soon as $p \in \Gamma$, so the socle is generated by x_1. $\square$

We may now get explicit equations for $V_p^{loc}(n)$. If $[\Gamma] \in V_p^{loc}(n)$, then by definition $m_p^n = 0$. But the maximal ideal is generated by $x_1, x_2, \dots, x_{n-1}$, so this means that any monomial of degree n in the x_i must vanish in the coordinate ring of Γ.
On the other hand, the equations for Γ define the products

$$x_i x_j = \sum_{k=2}^{n-1} a_{ijk} x_k \quad \text{and} \quad x_i^2 = x_1 + \sum_{k=2}^{n-1} a_{iik} x_k$$

in this ring. Therefore, by iteration, we get polynomial relations in the parameters a_{ijk}. Imposing the apolarity condition, symmetrizing the parameters and adding the equations for $V_p^{sec}(n)$, then after, possibly, saturation we get set theoretic equations for $V_p^{loc}(n)$.
When $n = 4$ we have the parameters $a_{222}, a_{223}, a_{233}, a_{333}$ for $V_p^{sec}(4)$ and the relation

$$a_{223}^2 - a_{222}a_{233} + a_{233}^2 - a_{223}a_{333}.$$

Thus $V_p^{sec}(4)$ is a quadric hypersurface in $\mathbf{A}^4$.
For $V_p^{loc}(4)$ we first get the subscheme defined by the equations

$$x_1^2 = x_1 x_2 = x_1 x_3 = 0$$

and

$$x_2^2 = x_1 + a_{222}x_2 + a_{223}x_3, x_2x_3 = a_{223}x_2 + a_{233}x_3, x_3^2 = x_1 + a_{233}x_2 + a_{333}x_3.$$

The coefficient of x_1 using these relations iteratively to compute $x_2^4, \dots, x_3^4$, must vanish, so it yields the equations $a_{222} + a_{233} = a_{223} + a_{333} = a_{233}^2 + a_{223}^2 = 0$. The other coefficients give no additional relations, and neither does the equations for $V_p^{sec}(4)$, so $V_p^{loc}(4)$ is 1-dimensional and consists of a pair of affine intersecting lines.
When $n = 5$, the computation becomes a bit more involved. There are ten parameters a_{ijk}. The equations of $V_p^{sec}(5)$ are

$$\begin{gathered}
a_{234}^2 - a_{233}a_{244} + a_{334}^2 - a_{333}a_{344} + a_{344}^2 - a_{334}a_{444} = 0,\\
a_{224}a_{234} - a_{223}a_{244} + a_{234}a_{334} - a_{233}a_{344} + a_{244}a_{344} - a_{234}a_{444} = 0,\\
a_{224}a_{233} - a_{223}a_{234} + a_{234}a_{333} - a_{233}a_{334} + a_{244}a_{334} - a_{234}a_{344} = 0,\\
a_{234}^2 + a_{224}^2 - a_{222}a_{244} + a_{244}^2 - a_{223}a_{344} - a_{224}a_{444} = 0,\\
a_{223}a_{224} - a_{222}a_{234} + a_{233}a_{234} + a_{234}a_{244} - a_{223}a_{334} - a_{224}a_{344} = 0,\\
a_{223}^2 - a_{222}a_{233} + a_{233}^2 - a_{223}a_{333} - a_{224}a_{334} + a_{234}^2 = 0.
\end{gathered}$$

They define in $\mathbf{A}^{10}$ the affine cone over the intersection of the Grassmannian variety $\mathbb{G}(2,5)$ with a quadric. For $V_p^{loc}(5)$ there are additional equations defining the cone over the tangent developable of a rational normal sextic curve, a

n	$V_p^{loc}(n)$		$V_p^{sec}(n)$		$V_h^{aff}(n)$	
	dim	degree	dim	degree	dim	degree
3	0	1	1	1	3	1
	point		$\mathbf{A}^1$		$\mathbf{A}^3$	
4	1	2	3	2	6	1
	two lines		Quadric 3-fold		$\mathbf{A}^6$	
5	3	10	6	10	10	1
	cone over tangent developable of a rational sextic curve		cone over $\mathbb{G}(2,5) \cap Q$		$\mathbf{A}^{10}$	
6					15	12
					cone over S_{10}	

TABLE 1.

codimension 3 linear section of $V_p^{sec}(5)$. The cone over the rational normal curve parameterizes local apolar subschemes that are not curvilinear. For the computations in *Macaulay2* [GS], see the documented code in [RS11].
The findings are summarized in Table 1.

6. GLOBAL INVARIANTS OF $VPS(Q, n)$

We consider $VPS(Q, n)$ as a subscheme of $\mathbb{G}(n-1, T_{2,q})$, and the incidence

$$I_Q^{VPS} = \{([q'], [\Gamma]) | [q'] \subset \langle \Gamma \rangle\} \subset \mathbf{P}(T_{2,q}) \times VPS(Q, n).$$

The incidence is a projective bundle,

$$I_Q^{VPS} = \mathbf{P}(E_Q) \xrightarrow{\pi} VPS(Q, n),$$

while the first projection is birational (the rational map $\gamma : \mathbf{P}(T_{2,q}) \dashrightarrow VPS(Q, n)$ factors through the inverse of this projection). Denote by L the tautological divisor on $\mathbf{P}(E_Q)$. It is the pullback of the hyperplane divisor on $\mathbf{P}(T_{2,q})$. When $VPS(Q, n)$ is smooth,

$$\mathrm{Pic}(I_Q^{VPS}) \cong \mathrm{Pic}(VPS(Q, n)) \oplus \mathbb{Z}[L].$$

Recall from Lemma 4.9, that the set $H_h \subset VPS(Q, n)$ of subschemes Γ that intersects a hyperplane $h \subset \mathbf{P}(T_1)$ form a Plücker divisor restricted to $VPS(Q, n)$. Therefore the class of the Plücker divisor coincides with the first Chern class $c_1(E_Q)$.

THEOREM 6.1. *i)* $\mathrm{Pic}(VPS(Q, 4) \cong \mathrm{Pic}(VPS(Q, 5)) \cong \mathbb{Z}$.
ii) The ample generator H is very ample, and $VPS(Q, 4)$ and $VPS(Q, 5)$ are Fano-manifolds of index 2.
iii) The boundary in $VPS(Q, n)$ consisting of singular apolar subschemes is, when $n \leq 5$, an anticanonical divisor.

Proof. i) Let $n = 4$ or $n = 5$. Then the Plücker divisor H is very ample by the above. Furthermore, the complement V_p^{aff} of the special Plücker divisor defined by a tangent hyperplane to $Q^{-1} \subset \mathbf{P}(T_1)$, the divisor $H_{\{x_n=0\}}$ in the above notation, is isomorphic to affine space by Proposition 5.11. Therefore the Picard group has rank 1 as soon as this special Plücker divisor is irreducible. The tangent hyperplanes to Q^{-1} cover all of $\mathbf{P}(T_1)$, so the corresponding Plücker divisors cover $VPS(Q,n)$. Furthermore, for any subscheme Γ in $VPS(Q,n)$, there is tangent hyperplane that does not meet Γ, so these special Plücker divisors have no common point on $VPS(Q,n)$. Assume that the special Plücker divisors are reducible, then we may write $H = H_1 + H_2$, where both H_1 and H_2 moves without base points on $VPS(Q,n)$. Since $H \cdot l = 1$ for every line on $VPS(Q,n)$, only one of the two components can have positive intersection with a line. The other, say H_2, must therefore contain every line that it intersects. But this is impossible, since H_2 must contain all of $VPS(Q,n)$, by the following lemma:

Lemma 6.2. *Any two polar simplices* Γ *and* Γ' *are connected by a sequence of lines in* $VPS(Q,n)$.

Proof. This is immediate when $n = 2$. For $n > 2$, let $[l] \in \Gamma$ and $[l'] \in \Gamma'$, and let $\mathbf{P}(U) = h_l \cap h_{l'} \subset \mathbf{P}(T_1)$ be the intersection of their polar hyperplanes. Then $q = l^2 + l_1^2 + q_U = (l')^2 + (l'_1)^2 + q_U$ for $q_U \in U^2$ and suitable l_1 and l'_1. Let Γ_U be a polar simplex for q_U. Then Γ is line connected to $\Gamma_U \cup \{[l_1],[l]\}$ by induction hypothesis. Likewise Γ is line connected to $\Gamma_U \cup \{[l'_1],[l']\}$. Finally $\Gamma_U \cup \{[l_1],[l]\}$ and $\Gamma_U \cup \{[l'_1],[l']\}$ span a line in $VPS(Q,n)$, which completes the induction. $\square$

ii) Since $\mathrm{Pic}(I_Q^{VPS}) \cong \mathrm{Pic}(VPS(Q,n)) \oplus \mathbb{Z}[L]$ we deduce from i) that the birational morphism

$$\sigma\colon I_Q^{VPS} \to \mathbf{P}(T_{2,q})$$

has an irreducible exceptional divisor. Let $E \in \mathrm{Pic}(I_Q^{VPS})$ be the class of this exceptional divisor. Then, since the map $\gamma : \mathbf{P}(T_{2,q}) \dashrightarrow \mathbb{G}(n-1, T_{2,q})$ is defined by polynomials of degree $\binom{n}{2}$, the size of the minors in the Mukai form, we have

$$\pi^*H = \binom{n}{2}L - E \text{ and } K_{I_Q^{VPS}} = -(\binom{n+1}{2} - 1)L + E.$$

On the other hand $H = c_1(E_Q)$ where $I_Q^{VPS} = \mathbf{P}(E_Q)$ is a projective bundle over $VPS(Q,n)$ so

$$-(\binom{n+1}{2} - 1)L + E = K_{I_Q^{VPS}} = \pi^*K_{VPS} + \pi^*(c_1(E_Q)) - (n-1)L.$$

Therefore $-K_{VPS(Q,n)} = 2H$. Finally, since $VPS(Q,n) \subset \mathbb{G}(n-1, T_{2,q})$ contains lines, H is not divisible.
iii) The boundary in $VPS(Q,n)$ consisting of singular apolar subschemes, coincides, by Lemma 2.3, with the set of subschemes $\Gamma \subset \mathbf{P}(T_1)$ that intersect quadric Q^{-1}. The Plücker divisor H is represented by the divisor of subschemes Γ that intersect a hyperplane in $\mathbf{P}(T_1)$, so $-K = 2H$ is represented by the boundary. $\square$

THEOREM 6.3. *Let $n > 2$ and let $VPS(Q,n) \subset \mathbb{G}(n-1, T_{2,q})$ be the variety of polar simplices in its Grassmannian embedding, with Plücker divisor H. The $VPS(Q,n)$ has degree*

$$H^m = \sum_{\lambda \vdash m} \binom{m}{\lambda} / (\lambda^*!) \cdot d_\lambda$$

where the sum runs over all partitions $\lambda = (\lambda_1, \dots, \lambda_n)$ of $m = \binom{n}{2} = \dim VPS(Q,n)$ into integers $n-1 \geq \lambda_1 \geq \dots \geq \lambda_n \geq 0$. Here $\lambda^* = (\lambda_1^*, \dots, \lambda_{n-1}^*)$ denotes the sequence $\lambda_i^* = \mid \{j \mid \lambda_j = i\} \mid$ and $\lambda^*! = \Pi \lambda_i^*!$. Finally

$$d_\lambda = \prod_{1 \leq i < j \leq n} (D_i + D_j)$$

is the intersection number of m divisors on the product

$$\mathbf{P}^{n-1-\lambda_1} \times \ldots \times \mathbf{P}^{n-1-\lambda_n}$$

with D_i the pullback of the hyperplane class on the i^{th} component.

Proof. We first show that for $\binom{n}{2}$ general hyperplanes $h_i \subset \mathbf{P}(T_1)$, the corresponding Plücker divisors H_{h_i} has a proper transverse intersection on the smooth part of $VPS(Q,n)$. Therefore, by properness, the intersection is finite, and, by transversality, it is smooth, so it is a finite set of points. The cardinality is the degree of $VPS(Q,n)$.

First, let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition of m and consider the partition $h_{11}, \dots, h_{1\lambda_1}, \dots, h_{n1}, \dots, h_{n\lambda_n}$ of m general hyperplanes into n sets of size $\lambda_1, \dots, \lambda_n$. Let $L_i = \cap_j h_{ij}$, it is a linear space of dimension $n-1-\lambda_i$. Consider the product of these linear spaces in the product $\mathbf{P}(T_1)^n$:

$$L_1 \times \cdots \times L_n \subset \mathbf{P}(T_1) \times \cdots \times \mathbf{P}(T_1).$$

Let $\Delta \in \mathbf{P}(T_1)^n$ be the union of all diagonals and let $L^o = L_1 \times \cdots \times L_n \setminus \Delta \subset \mathbf{P}(T_1)^n$. Then L^o parameterizes n-tuples of points $\Gamma = \{p_1, \dots, p_n\} \subset \mathbf{P}(T_1)$, with $p_i \in L_i$. Of course, L^o has a natural map to the Hilbert scheme of $\mathbf{P}(T_1)$ that forgets the ordering, so we will identify elements in L^o with their image in the Hilbert scheme.

Consider the incidence between subschemes $\Gamma \in L^o$ and quadratic forms $q \in T_2$:

$$I_L = \{(\Gamma, [q]) | I_\Gamma \subset q^\perp\} \subset L^o \times \mathbf{P}(T_2).$$

This variety is defined by the equations $h_{ij}(p_i) = 0$ and the apolarity, $q(I_\Gamma) = 0$. Clearly L is a smooth scheme of dimension $\binom{n}{2}$. The fibers of the projection $I_L \to L$ are $(n-1)$-dimensional projective spaces, so I_L is a smooth variety of dimension equal to $\dim \mathbf{P}(T_2)$. The projection $I_L \to \mathbf{P}(T_2)$ is clearly onto, so the fibers are finite. Since both spaces are smooth, the general fiber is smooth. Now, $\Gamma \subset L^o$ lies in the fiber over $[q]$, precisely when $I_\Gamma \subset q^\perp$, i.e. $[\Gamma] \in VPS(Q,n)$ and $h_{ij}(p_i) = 0$, i.e. $[\Gamma]$ lies in the intersection of all the Plücker hyperplanes $H_{h_{ij}}$. Since the general fibers are smooth the divisors $H_{h_{ij}}$ intersect transversally in $VPS(Q,n)$, where $Q = \{q = 0\}$, and have an isolated intersection point at each point $[\Gamma]$. Turning the argument around and

considering all partitions, we get that for general hyperplanes $h_1, \ldots, h_m$ in $\mathbf{P}(T_1)$ the Plücker hyperplanes H_{h_i} has a transversal intersection at a finite number of points in $VPS(Q, n)$ corresponding to smooth apolar subschemes.
We proceed to compute the cardinality of the intersection, i.e. the formula given in the theorem. Let $[\Gamma] = [\{p_1, \ldots, p_n\}] \in VPS(Q, n)$ be a point in the intersection of the hyperplanes H_{h_j}. Then each h_j contains some $p_i \in \Gamma$, by the definition of H_{h_j}. For each i let λ_i be the number of hyperplanes h_j that contains p_i. The set of positive integers $\{\lambda_1, ..., \lambda_n\}$ must add up to m: It is at least m by definition, and at most m by the generality assumption discussed above. Therefore the point $[\Gamma]$ defines a unique partition of the set of hyperplanes $\{h_j\}_{j=1}^m$ into subsets $\{h_{ij}\}_{j=1}^{\lambda_i}$ of cardinality λ_i, as above.
The factor $\binom{m}{\lambda}$ in the degree formula counts the number of ordered partitions of m hyperplanes into subsets of cardinality λ_i, while $\lambda^*!$ counts the permutations of the subsets of the same cardinality, i.e. the number of ordered partitions determined by $[\Gamma]$. Therefore the remaining factor d_λ for each partition should count the number of polar simplices Γ that intersect the n linear subspaces $L_i = \cap_j h_{ij} \subset \mathbf{P}(T_1)$ of codimension $\lambda_i, i = 1, ..., n$.
Let $[\Gamma] = [\{p_1, ..., p_n\}] \in VPS(Q, n)$ and assume that $(p_1, ..., p_n) \in L_1 \times \cdots \times L_n$. For each pair of linear spaces L_i, L_j the bilinear form associated to the quadratic form q restricts to a linear form on the product $L_i \times L_j$ that vanishes on (p_i, p_j). This linear form defines a divisor H_{ij} in the divisor class $H_i + H_j$, where H_i is the pullback to the product of the hyperplane class on L_i. If D_{ij} and D_i are the pullbacks of H_{ij}, respectively H_i, to the product $\Pi_i L_i$, then $\Gamma \subset \Pi_i L_i$ lies in the intersection $\cap_{i<j} D_{ij}$.
Conversely, consider a point $(p_1, \ldots, p_n) \in L_1 \times \cdots \times L_n$ that lies in the intersection of the divisors $\cap_{i<j} D_{ij}$. The projection of this point into $\mathbf{P}(T_1)$ is a collection of n points $\Gamma = \{p_1, \ldots, p_n\}$. Let $p_i = [v_i], v_i \in T_1$, then the bilinear form $q : T_1 \times T_1 \to \mathbb{C}, \quad (q^{-1}(v_i)(q))(v_j) = 0$ for every $i \neq j$, so the hyperplanes $p_i^\perp \subset \mathbf{P}(S_1)$ form a polar simplex to Q. Hence $[\Gamma]$ is a point in $VPS(Q, n)$.
Thus d_λ counts the number of polar simplices Γ that intersect the n linear subspaces L_i and the degree formula follows.

□

The Theorem 1.1 in the introduction follows from Corollary 5.12, Corollary 5.15 and Theorem 6.1. Theorem 1.2 follows from Corollary 2.2, Corollary 5.2, Theorem 6.1 and the degree is computed from Theorem 6.3. Theorem 1.3 follows from Theorem 5.4, Corollary 5.16 and Corollary 2.7.

References

[Art76] Michael Artin: *Lectures on Deformations of Singularities*, TATA Institute of Fundamental Research, Bombay (1976). 488, 490

[Beh81] Kurt Behnke: On Projective Resolution of Frobenius Algebras and Gorenstein Rings. *Math. Ann.* 257 (1981), 219–238. 474

[DZ04] Dragomir Z. Doković, Kaiming Zhao: Tridiagonal normal forms for orthogonal similarity classes of symmetric matrices, *Linear Algebra and its Applications* 384 (2004), 77–84. 483

[Dol12] Igor V. Dolgachev: *Classical algebraic geometry: a modern view.* Cambridge University Press (2012). 480, 490

[ERS81] David Eisenbud, Oswald Riemenschneider, Frank-Olaf Schreyer: Resolutions of Cohen-Macaulay Algebras. *Math. Ann.* 257 (1981), 85–98. 474

[Gan59] Felix Gantmacher: *Theory of Matrices I,II.* Chelsea Publishing (1959). 478, 483

[GS] Daniel R. Grayson and Michael E. Stillman: Macaulay2, a software system for research in algebraic geometry, Available at http://www.math.uiuc.edu/Macaulay2/. 472, 474, 487, 498, 501

[HP52] W.V.D.Hodge & D. Pedoe: *Methods of Algebraic Geometry, Vol 2.* Cambridge University Press, Cambridge (1952). 478

[Mac16] Francis S. Macaulay: *The algebraic theory of modular systems*, Cambridge University Press, London (1916). 477

[Muk92] Shigeru Mukai: Fano 3-folds, *Complex Projective Geometry*, London Math. Soc. Lect. Note Ser. 179, Cambridge University Press, London (1992), 255–263. 470, 480, 490

[Muk95] Shigeru Mukai: Curves and Symmetric Spaces, I, *American J. Math.* 117 (1995), 1627-1644. 499

[RS00] Kristian Ranestad and Frank-Olaf Schreyer: Varieties of Sums of Powers, *J. reine angew. Math.* 525 (2000), 147–181. 474, 499

[RS11] Kristian Ranestad and Frank-Olaf Schreyer: 492, 498, 501 VarietyOfPolarSimplices.m2 a *Macalay2 package* available at http://www.math.uni-sb.de/ag/schreyer/home/computeralgebra.htm.

[Sch91] Frank-Olaf Schreyer: A standard basis approach to syzygies of canonical curves, *J. reine angew. Math.* 421 (1991), 83–123. 490

Kristian Ranestad
Matematisk institutt
Universitetet i Oslo
PO Box 1053
Blindern
NO-0316 Oslo
Norway
ranestad@math.uio.no

Frank-Olaf Schreyer
Mathematik und Informatik
Universität des Saarlandes
Campus E2 4
D-66123 Saarbrücken
Germany
schreyer@math.uni-sb.de