On the Generalized Semi-Relativistic Schrödinger-Poisson System in $\mathbb{R}^n$

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Abstract. The Cauchy problem for the semi-relativistic Schrödinger-Poisson system of equations is studied in $\mathbb{R}^n$, $n \geq 1$, for a wide class of nonlocal interactions. Furthermore, the asymptotic behavior of the solution as the mass tends to infinity is rigorously discussed, and compared with solutions to the non-relativistic Schrödinger-Poisson system.

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1 Introduction

1.1 Motivation and heuristic discussion

In this article, we study the global Cauchy problem for the semi-relativistic Schrödinger-Poisson system in $\mathbb{R}^n$, $n \geq 1$, for a wide class of nonlocal interactions, both in the attractive and repulsive cases. This system is relevant to the description of many-body semi-relativistic quantum particles in the mean-field limit. We consider a system of N semi-relativistic quantum particles in $\mathbb{R}^n$, $n \geq 1$ with long-range two-body interactions $g\frac{1}{N}\sum_{1\leq i<j\leq N}\frac{1}{|x_i-x_j|^\gamma}$, with $0<\gamma\leq 1$ if $n \geq 2$, and $0<\gamma<1$ if $n=1$, and with $g \in \mathbb{R}$. In the mean-field limit, one can formally show that the density matrix that describes the *mixed* state of the system satisfies the Hartree-von Neumann equation

$$\begin{cases} i\partial_t \rho(t) = [H_m + w_\gamma \star n(t), \rho(t)], \quad x \in \mathbb{R}^n, \quad n \geq 1, \quad t \geq 0 \\ H_m = \sqrt{m^2 - \Delta} - m, \quad w_\gamma = g\frac{1}{|x|^\gamma}, \quad n(t,x) = \rho(t,x,x), \ \rho(0) = \rho_0 \end{cases}, \tag{1.1}$$

where Δ stands for the n-dimensional Laplacian, $\star$ stands for convolution in $\mathbb{R}^n$, and $m \geq 0$ is the mass.[1] Since ρ_0 is a positive, self-adjoint trace-class operator acting on $L^2(\mathbb{R}^n)$, its kernel can be decomposed with respect to an orthonormal basis of $L^2(\mathbb{R}^n)$,

$$\rho_0(x,y) = \sum_{k \in \mathbb{N}} \lambda_k \psi_k(x) \overline{\psi_k(y)} \tag{1.2}$$

where $\{\psi_k\}_{k \in \mathbb{N}}$ denotes an orthonormal basis of $L^2(\mathbb{R}^n)$. Furthermore,

$$\underline{\lambda} := \{\lambda_k\}_{k \in \mathbb{N}} \in l^1 \ , \ \ \lambda_k \geq 0 \ , \ \ \sum_k \lambda_k = 1.$$

We will show that there exists a one-parameter family of complete orthonormal bases of $L^2(\mathbb{R}^n)$, $\{\psi_k(t)\}_{k \in \mathbb{N}}$, for $t \in \mathbb{R}_+$, such that the kernel of the solution $\rho(t)$ to (1.1) can be represented as

$$\rho(t,x,y) = \sum_{k \in \mathbb{N}} \lambda_k \psi_k(t,x) \overline{\psi_k(t,y)}. \tag{1.3}$$

Substituting (1.3) in (1.1), the one-parameter family of orthonormal vectors $\{\psi_k(t)\}_{k \in \mathbb{N}}$ is seen to satisfy the semi-relativistic Schrödinger-Poisson system

$$i\frac{\partial \psi_k}{\partial t} = H_m \psi_k + V\psi_k, \quad k \in \mathbb{N} \tag{1.4}$$

$$V[\Psi] = w_\gamma \star n[\Psi], \quad \Psi := \{\psi_k\}_{k=1}^\infty, \tag{1.5}$$

$$n[\Psi(x,t)] = \sum_{k=1}^\infty \lambda_k |\psi_k|^2 \,. \tag{1.6}$$

The purpose of this note is to show global well-posedness of (1.4) in a suitable Banach space (to be specified below), and to study the asymptotics of the solution as the mass m tends to ∞, which we compare to solutions to the non-relativistic Schrödinger-Poisson system, see [11]. The semi-relativistic Schrödinger-Poisson system of equations in a finite domain of $\mathbb{R}^3$ and with repulsive Coulomb interactions has been studied recently in [1, 2]. Here, we generalize the result of [1] in several ways. First, the problem is studied in $\mathbb{R}^n$, $n \geq 1$.

[1] The rigorous derivation of the semi-relativistic Hartree-von Neumann equation is a topic of future work, see [3, 4] for a derivation of this system of equations in the non-relativistic case.

Second, we consider a wide class of nonlocal interactions in both the attractive and repulsive cases, and which includes the repulsive Coulomb case in three spatial dimensions. Third, in the infinite mass limit $m \to \infty$, we prove that solutions to the semi-relativistic, and to the non-relativistic Schrödinger-Poisson systems become indistinguishable; the latter has been studied extensively, see for example [5, 8] and references therein. In the special case when the initial density matrix is a pure state $\rho_0 = |\psi_0\rangle\langle\psi_0|$, the Schrödinger-Poisson system becomes a single Hartree equation

$$i\partial_t\psi = (\sqrt{m^2-\Delta}-m)\psi + (w_\gamma \star |\psi|^2)\psi, \quad \psi(0) = \psi_0.$$

In that sense, our analysis generalizes the results of [10, 7] to the effective dynamics of a *mixed state* of a semi-relativistic system.
The organization of this paper is as follows. In Subsection 1.3 we state our main results. We prove local and global well-posedness in Section 2. Finally, in Section 3, we discuss the asymptotic behavior of the solutions as the mass tends to infinity. For the benefit of a general reader, we recall some useful results about fractional integration and fractional Leibniz rule in Appendix A.

1.2 Notation

- $A \lesssim B$ means that there exists a positive constant C independent mass m such that $A \leq C\, B$.
- L^p stands for the standard Lebesgue space. Furthermore, $L^p_I B = L^p(I;B)$. $\langle\cdot,\cdot\rangle_{L^2}$ denotes the $L^2(\mathbb{R}^n)$ inner product. We will often use the abbreviated notation L^p_T for $L^p_{[0,T]}$, in the situation where $[0,T]$ denotes a time interval.
- $l^1 = \{\{a_l\}_{l\in\mathbb{N}} | \ \ \sum_{l\geq 1}|a_l| < \infty\}$.
- $W^{s,p} = (-\Delta+1)^{-\frac{s}{2}}L^p$, the standard (complex) Sobolev space. When $p=2$, $W^{s,2} = H^s$. $\dot{H}^s$ denotes the homogeneous Sobolev space with norm $\|\psi\|_{\dot{H}^s} = (\langle\psi,(-\Delta)^s\psi\rangle_{L^2})^{\frac{1}{2}}$.
- For fixed $\underline{\lambda} \in l^1$, $\lambda_k \geq 0$, and for sequences of functions $\Phi := \{\phi_k\}_{k\in\mathbb{N}}$ and $\Psi := \{\psi_k\}_{k\in\mathbb{N}}$, we define the inner product

  $$\langle\Phi,\Psi\rangle_{\mathcal{L}^2} := \sum_{k\geq 1}\lambda_k\langle\phi_k,\psi_k\rangle_{L^2},$$

  which induces the norm

  $$\|\Phi\|_{\mathcal{L}^2} = (\sum_{k\geq 1}\lambda_k\|\phi_k\|^2_{L^2})^{\frac{1}{2}}.$$

  The corresponding Hilbert space is $\mathcal{L}^2$.

- For fixed $\underline{\lambda} \in l^1$, $\lambda_k \geq 0$,

$$\mathcal{H}^s = \{\Psi = \{\psi_k\}_{k\in\mathbb{N}} | \ \ \psi_k \in H^s, \ \sum_{k\geq 1} \lambda_k \|\psi_k\|^2_{H^s} < \infty\}$$

is a Banach space with norm $\|\Psi\|_{\mathcal{H}^s} = (\sum_{k\geq 1} \lambda_k \|\psi_k\|^2_{H^s})^{\frac{1}{2}}$.

- For fixed $\underline{\lambda} \in l^1$, $\lambda_k \geq 0$,

$$\dot{\mathcal{H}}^s = \{\Psi = \{\psi_k\}_{k\in\mathbb{N}} | \ \ \psi_k \in \dot{H}^s, \ \sum_{k\geq 1} \lambda_k \|\psi_k\|^2_{\dot{H}^s} < \infty\}$$

is a Banach space with norm $\|\Psi\|_{\dot{\mathcal{H}}^s} = (\sum_{k\geq 1} \lambda_k \|\psi_k\|^2_{\dot{H}^s})^{\frac{1}{2}}$.

1.3 Statement of main results

For $s > 0$, we define the state space for the Schrödinger-Poisson system by

$$\mathcal{S}^s := \{(\Psi, \underline{\lambda}) | \ \ \Psi = \{\psi_k\}_{k=1} \in \mathcal{H}^s \ \ \textit{is a complete orthonormal system in } L^2(\mathbb{R}^n),$$

$$\underline{\lambda} = \{\lambda_k\}_{k\in\mathbb{N}} \in l^1, \ \lambda_k \geq 0\}.$$

The following is our first main result about the global Cauchy problem.

Theorem 1.1. *Consider the system of equations (1.4)-(1.6), with* $m \geq 0$, *with* $0 < \gamma \leq 1$ *if* $n \geq 2$, *and* $0 < \gamma < 1$ *if* $n = 1$. *Suppose that* $(\Psi(0), \underline{\lambda}) \in \mathcal{S}^s$, $s \geq 1/2$. *If* $g \geq 0$, *or* $g < 0$ *with* $\|\Psi(0)\|_{\mathcal{L}^2}$ *small enough, then there is a unique mild solution* $(\Psi, \underline{\lambda}) \in C([0, \infty], \mathcal{S}^s)$.

Remark 1.2. $\underline{\lambda}$ *is time-independent, and hence the evolution can be thought as that of* $\Psi \in \mathcal{H}^s$.

Remark 1.3. *Local well-posedness requires less regularity, in particular,* $s \geq \gamma/2$, *see Proposition 2.2 in Section 2.1. On the other hand, in order to enhance local to global well-posedness, energy conservation is used, and consequently,* $s \geq \frac{1}{2}$ *is assumed to ensure finiteness of the energy.*

Remark 1.4. *It follows from the proof of local well-posedness (Proposition 2.2 in Section 2.1) that there exists a positive time* T *independent of* $m \geq 0$ *such that* $\|\Psi\|_{L^\infty_T \mathcal{H}^s} \leq C\|\Psi(0)\|_{\mathcal{H}^s}$, $s \geq \gamma/2$, *where* $C > 0$ *is independent of* m.

Remark 1.5. *The solution is continuous in the mass* m. *In particular, as* $m \searrow 0$, *and for* $T > 0$ *fixed,* $\Psi \to \Psi^{(0)}$ *strongly in* $L^\infty_T(\mathcal{H}^s)$, $s \geq 1/2$, *where* $\Psi^{(0)}$ *satisfies (1.4)-(1.6) with* $m = 0$ *and initial condition* $\Psi(0)$, *see Proposition 2.6 in Sect. 2.*

The second result is about the infinite mass limit. Let Γ satisfy the nonrelativistic Schrödinger-Poisson system of equations

$$i\frac{\partial \psi_k}{\partial t} = -\frac{1}{2m}\Delta\psi_k + V\psi_k, \quad k \in \mathbb{N}$$
$$V[\Psi] = w_\gamma \star n[\Psi], \quad \Psi := \{\psi_k\}_{k=1}^{\infty},$$
$$n[\Psi(x,t)] = \sum_{k=1}^{\infty} \lambda_k |\psi_k|^2,$$

with initial condition $\Psi(0) = \{\psi_k(0)\}_{k\in\mathbb{N}}$.

THEOREM 1.6. *Suppose that the hypotheses of Theorem 1.1 hold. Then there exists $\tau > 0$ such that $\Psi - \Gamma \to 0$ in $L^\infty_\tau(\mathcal{H}^s)$, $s \geq \gamma/2$, as $m \to \infty$.*

In other words, when the mass tends to infinity, the solutions of the semi-relativistic, and of the non-relativistic Schrödinger-Poisson systems of equations asymptotically become indistinguishable.

REMARK 1.7. *The proof of Theorem 1.6 relies on local well-posedness, and this is why the result holds for $s \geq \gamma/2$.*

2 WELL-POSEDNESS

2.1 LOCAL WELL-POSEDNESS

In what follows, we fix $\underline{\lambda} \in l^1$, $\lambda_l \geq 0$, $l \in \mathbb{N}$. We start by showing that the nonlinearity $V[\Psi]\Psi$ is locally Lipschitz.

LEMMA 2.1. *For $\Psi, \Phi \in \mathcal{H}^s$, $s \geq \gamma/2$,*

$$\|V[\Psi]\Psi - V[\Phi]\Phi\|_{\mathcal{H}^s} \lesssim (\|\Psi\|^2_{\mathcal{H}^s} + \|\Phi\|^2_{\mathcal{H}^s})\|\Psi - \Phi\|_{\mathcal{H}^s}.$$

Proof. The proof relies on the fractional Leibniz rule and fractional integration, see Appendix A. From the Minkowski inequality,

$$\|V[\Psi]\Psi - V[\Phi]\Phi\|_{\mathcal{H}^s} \lesssim \|(V[\Psi] - V[\Phi])\Psi\|_{\mathcal{H}^s} + \|V[\Phi](\Psi - \Phi)\|_{\mathcal{H}^s} \tag{2.1}$$

We begin by estimating the first term on the right.

$$\begin{aligned}
&\|(V[\Psi] - V[\Phi])\Psi\|_{\mathcal{H}^s} \lesssim \sum_{k,l\geq 1} \lambda_k \lambda_l \|w_\gamma \star (|\psi_l|^2 - |\phi_l|^2)\psi_k\|_{H^s} \\
&\lesssim \sum_{k,l\geq 1} \lambda_k\lambda_l\{\|w_\gamma \star (|\psi_l|^2 - |\phi_l|^2)\|_{L^\infty}\|\psi_k\|_{H^s} \\
&\qquad\qquad + \|w_\gamma \star (|\psi_l|^2 - |\phi_l|^2)\|_{W^{s,\frac{2n}{\gamma}}}\|\psi_k\|_{L^{\frac{2n}{n-\gamma}}}\} \\
&\lesssim \sum_{k,l\geq 1} \lambda_k\lambda_l\{\|\psi_l - \phi_l\|_{H^{\frac{\gamma}{2}}}(\|\psi_l\|_{H^{\frac{\gamma}{2}}} + \|\psi_l\|_{H^{\frac{\gamma}{2}}})\|\psi_k\|_{H^s} \\
&\qquad\qquad + \||\psi_l|^2 - |\phi_l|^2\|_{L^{\frac{2n}{2n-\gamma}}}\|\psi_k\|_{H^{\frac{\gamma}{2}}}\} \\
&\lesssim (\|\Psi\|^2_{\mathcal{H}^s} + \|\Phi\|^2_{\mathcal{H}^s})\|\Psi - \Phi\|_{\mathcal{H}^s}.
\end{aligned} \tag{2.2}$$

Here, we used Minkowski inequality in the first line, fractional Leibniz rule (Lemma A.1 in the Appendix) in the second line, Hölder's inequality, fractional integration (Lemma A.2) and Lemma A.3 in the third line. Similarly,

$$\begin{aligned}
&\|V[\Phi](\Psi-\Phi)\|_{\mathcal{H}^s} \lesssim \sum_{k,l\geq 1}\lambda_k\lambda_l\|w_\gamma\star|\phi_l|^2(\psi_k-\phi_k)\|_{H^s}\\
&\lesssim \sum_{k,l\geq 1}\lambda_k\lambda_l\{\|w_\gamma\star|\phi_l|^2\|_{L^\infty}\|\psi_k-\phi_k\|_{H^s}+\|w_\gamma\star|\phi_l|^2\|_{W^{s,\frac{2n}{\gamma}}}\|\psi_k-\phi_k\|_{L^{\frac{2n}{n-\gamma}}}\}\\
&\lesssim \sum_{k,l\geq 1}\lambda_k\lambda_l\{\|\phi_l\|^2_{H^{\frac{\gamma}{2}}}\|\psi_k-\phi_k\|_{H^s}+\||\phi_l|^2\|_{L^{\frac{2n}{2n-\gamma}}}\|\psi_k-\phi_k\|_{H^{\frac{\gamma}{2}}}\}\\
&\lesssim \|\Phi\|^2_{\mathcal{H}^s}\|\Psi-\Phi\|_{\mathcal{H}^s}. \qquad (2.3)
\end{aligned}$$

The claim of the lemma follows from inequalities (2.1), (2.2) and (2.3). ☐

Using a standard contraction map argument, the generalized semi-relativistic Schrödinger-Poisson system of equations is locally well-posed.

Proposition 2.2. *Consider the system of equations (1.4)-(1.6), with* $m\geq 0$, $0<\gamma\leq 1$ *if* $n\geq 2$, *and* $0<\gamma<1$ *if* $n=1$. *Suppose that* $(\Psi(0),\underline{\lambda})\in\mathcal{S}^s$, $s\geq\gamma/2$. *Then there exists a positive time* T *such that the unique solution* $\Psi\in C([0,T];\mathcal{H}^s)$. *Furthermore, there exists a maximal time* $\tau^*\in(0,\infty]$ *such that* $\lim_{t\nearrow\tau^*}\|\Psi(t)\|_{\mathcal{H}^s}=\infty$.

Proof. Given $\rho, T>0$, consider the Banach space

$$\mathcal{B}^s_{T,\rho}=\{\Psi\in L^\infty_T(\mathcal{H}^s):\ \|\Psi\|_{L^\infty_T\mathcal{H}^s}\leq\rho\}.$$

Let $U^{(m)}=e^{-itH_m}$, the unitary operator generated by the semi-relativistic Hamiltonian $H_m=\sqrt{-\Delta+m^2}-m$. We define the mapping $\mathcal{N}$ by

$$\mathcal{N}(\Psi)(t)=U^{(m)}(t)\Psi(0)-i\int_0^t U^{(m)}(t-t')V[\Psi(t')]\Psi(t')dt',$$

which is the solution given by the Duhamel formula. First we show that $\mathcal{N}$ is a mapping from $\mathcal{B}^s_{T,\rho}$ into itself.

$$\begin{aligned}
\|\mathcal{N}(\Psi)\|_{L^\infty_T\mathcal{H}^s} &\leq \|\Psi(0)\|_{\mathcal{H}^s}+T\|V[\Psi]\Psi\|_{L^\infty_T\mathcal{H}^s}\\
&\leq \|\Psi(0)\|_{\mathcal{H}^s}+T\sum_{k,l\geq 1}\lambda_k\lambda_l\|w_\gamma\star|\psi_l|^2\psi_k\|_{L^\infty_T H^s}\\
&\leq \|\Psi(0)\|_{\mathcal{H}^s}+T\sum_{k,l\geq 1}\lambda_k\lambda_l\{\|w_\gamma\star(|\psi_l|^2)\|_{L^\infty_T L^\infty}\|\psi_k\|_{L^\infty_T H^s}+\\
&\quad +\|w_\gamma\star(|\psi_l|^2)\|_{L^\infty_T W^{s,\frac{2n}{\gamma}}}\|\psi_k\|_{L^\infty_T L^{\frac{2n}{n-\gamma}}}\},
\end{aligned}$$

where we have used fractional Leibniz rule (Lemma A.1) in the last inequality. It follows from fractional integration (Lemma A.2) and Sobolev embedding

$H^{\frac{\gamma}{2}} \hookrightarrow L^{\frac{2n}{n-\gamma}}$ that

$$\begin{aligned}
\|\mathcal{N}(\Psi)\|_{L^\infty_T \mathcal{H}^s} &\leq \|\Psi(0)\|_{\mathcal{H}^s} + T \sum_{k,l\geq 1} \lambda_k \lambda_l \{\|\psi_l\|^2_{L^\infty_T H^{\frac{\gamma}{2}}} \|\psi_k\|_{L^\infty_T H^s} + \\
&+ \|\psi_l\|^2_{L^\infty_T L^{\frac{2n}{n-\gamma}}} \|\psi_k\|_{L^\infty_T H^s}\} \\
&\leq \|\Psi(0)\|_{\mathcal{H}^s} + T \sum_{k,l\geq 1} \lambda_k \lambda_l \{\|\psi_l\|^2_{L^\infty_T H^{\frac{\gamma}{2}}} \|\psi_k\|_{L^\infty_T H^s}\} \\
&\leq \|\Psi(0)\|_{\mathcal{H}^s} + T (\sum_{l\geq 1} \lambda_l \{\|\psi_l\|^2_{L^\infty_T H^{\frac{\gamma}{2}}}) (\sum_{k\geq 1} \lambda_k \|\psi_k\|^2_{L^\infty_T H^s})^{\frac{1}{2}} \\
&\leq \|\Psi(0)\|_{\mathcal{H}^s} + T \|\Psi\|^2_{L^\infty_T \mathcal{H}^{\frac{\gamma}{2}}} \|\Psi\|_{L^\infty_T \mathcal{H}^s},
\end{aligned}$$

where we have used the fact that $\lambda_k \geq 1$ and $\sum_{k\geq 1} \lambda_k = 1$ before the last inequality.
Since $s \geq \frac{\gamma}{2}$, and since by assumption, $\Psi \in B^s_{T,\rho}$, we can choose T and ρ such that

$$\|\Psi(0)\|_{\mathcal{H}^s} \leq \frac{\rho}{2}, \quad T\rho^2 < \frac{1}{2},$$

it follows from the last inequality and the Duhamel formula that

$$\|\Psi\|_{L^\infty_T \mathcal{H}^s} \leq 2\|\Psi(0)\|_{\mathcal{H}^s} \leq \rho.$$

Second, since the nonlinearity is locally Lipschitz (Lemma 2.1), $\mathcal{N}$ is a contraction map for sufficiently small T.

$$\begin{aligned}
&\|\mathcal{N}(\Psi) - \mathcal{N}(\Phi)\|_{L^\infty_T \mathcal{H}^s} \leq T\|V[\Psi]\Psi - V[\Phi]\Phi\|_{L^\infty_T \mathcal{H}^s} \\
&\lesssim T\rho^2 \|\Psi - \Phi\|_{L^\infty_T \mathcal{H}^s}.
\end{aligned}$$

Local well-posedness follows from a standard contraction mapping argument, see for example, [6]. □

It follows from local well-posedness that for every $k \in \mathbb{N}$, $\|\psi_k\|_{L^2}$ is conserved.

LEMMA 2.3. *Suppose that the hypotheses of Proposition 2.2 hold. Then* $\|\psi_k(t)\|_{L^2} = \|\psi_k(0)\|_{L^2}, \quad t \in [0, \tau^*)$.

Proof. Multiplying (1.4) by $\overline{\psi_k}$ and integrating over space yields

$$\frac{i}{2}\partial_t \|\psi_l\|^2 = \langle \psi_l, H_m \psi_l\rangle + \langle \psi_l, V[\Psi]\psi_l\rangle.$$

Taking the imaginary part of both sides of the equation yields $\partial_t \|\psi_l\|^2 = 0$. □

The energy functional associated with the semi-relativistic Schrödinger-Poisson system is

$$\mathcal{E}(\Psi) = \frac{1}{2}\langle \Psi, H_m \Psi\rangle_{\mathcal{L}^2} + \frac{1}{4}\langle \Psi, V[\Psi]\Psi\rangle_{\mathcal{L}^2}.$$

Formally, conservation of energy follows from multiplying (1.4) by $\lambda_l \partial_t \overline{\psi_k}$, integrating over space, and summing over $k \geq 1$. To make the argument precise, we need a regularization procedure.

Lemma 2.4. *Suppose that the hypotheses of Proposition 2.2 hold. Then* $\mathcal{E}(\Psi(t)) = \mathcal{E}(\Psi(0))$, $t \in [0, \tau^*)$, *is satisfied for solutions* $\Psi \in C([0,\tau^*), \mathcal{H}^s)$ *with* $s \geq \frac{1}{2}$.

Proof. Let

$$\mathcal{J}_\epsilon = (\epsilon H_m + 1)^{-1}, \quad \epsilon > 0,$$

act on the sequence of embedding spaces

$$\cdots \quad \mathcal{H}^{\frac{3}{2}} \hookrightarrow \mathcal{H}^{\frac{1}{2}} \hookrightarrow \mathcal{H}^{\frac{-1}{2}} \hookrightarrow \mathcal{H}^{\frac{-3}{2}} \quad \cdots$$

It follows from fractional calculus that

(i) $\mathcal{J}_\epsilon$ is a bounded operator from $\mathcal{H}^s$ to $\mathcal{H}^{s+1}$,

(ii) $\|\mathcal{J}_\epsilon \Psi\|_{\mathcal{H}^s} \leq \|\Psi\|_{\mathcal{H}^s}$, and

(iii) $\mathcal{J}_\epsilon \Psi \to \Psi$ strongly in $\mathcal{H}^s$ as $\epsilon \to 0$.

Now,

$$\begin{aligned}
\mathcal{E}(\mathcal{J}_\epsilon \Psi(t_2)) - \mathcal{E}(\mathcal{J}_\epsilon \Psi(t_1)) &= \int_{t_1}^{t_2} \partial_t \mathcal{E}(\mathcal{J}_\epsilon \Psi(t))\, dt \\
&= \mathrm{Re}\Big\{ \int_{t_1}^{t_2} -i\langle H_m \mathcal{J}_\epsilon \Psi(t), H_m \mathcal{J}_\epsilon \Psi(t)\rangle_{\mathcal{L}^2} + \\
&\quad + \langle H_m \mathcal{J}_\epsilon \Psi(t), \mathcal{J}_\epsilon V[\Psi(t)]\Psi(t)\rangle_{\mathcal{L}^2} + \\
&\quad + \langle \mathcal{J}_\epsilon V[\mathcal{J}_\epsilon \Psi(t)]\mathcal{J}_\epsilon \Psi(t), H_m \mathcal{J}_\epsilon \Psi(t)\rangle_{\mathcal{L}^2} + \\
&\quad + \langle \mathcal{J}_\epsilon V[\mathcal{J}_\epsilon \Psi(t)]\mathcal{J}_\epsilon \Psi(t), \mathcal{J}_\epsilon V[\Psi(t)]\Psi(t)\rangle_{\mathcal{L}^2} \Big\}.
\end{aligned}$$

The first term is trivially zero, since $H_m \mathcal{J}_\epsilon = \mathcal{J}_\epsilon H_m$. Let

$$\begin{aligned}
g_\epsilon(t) = \mathrm{Re}\{ &\langle H_m \mathcal{J}_\epsilon \Psi(t), \mathcal{J}_\epsilon V[\Psi(t)]\Psi(t)\rangle_{\mathcal{L}^2} + \\
&+ \langle \mathcal{J}_\epsilon V[\mathcal{J}_\epsilon \Psi(t)]\mathcal{J}_\epsilon \Psi(t), H_m \mathcal{J}_\epsilon \Psi(t)\rangle_{\mathcal{L}^2} + \\
&+ \langle \mathcal{J}_\epsilon V[\mathcal{J}_\epsilon \Psi(t)]\mathcal{J}_\epsilon \Psi(t), \mathcal{J}_\epsilon V[\Psi(t)]\Psi(t)\rangle_{\mathcal{L}^2} \}.
\end{aligned}$$

Then

$$\mathcal{E}(\mathcal{J}_\epsilon \Psi(t_2)) - \mathcal{E}(\mathcal{J}_\epsilon \Psi(t_1)) = \int_{t_1}^{t_2} g_\epsilon(t) dt.$$

It follows from the above properties (i)-(iii) of $\mathcal{J}_\epsilon$ that $\lim_{\epsilon \to 0} g_\epsilon(t) = 0$. Furthermore,

$$g_\epsilon(t) \leq \|V[\Psi(t)]\Psi(t)\|_{\mathcal{L}^2} \|H_m \Psi(t)\|_{\mathcal{L}^2} + \|V[\Psi(t)]\Psi(t)\|^2_{\mathcal{L}^2}. \tag{2.4}$$

Using Lemma A.3, we have

$$\|V[\Psi]\Psi\|_{\mathcal{L}^2} \lesssim \sum_{k,l \geq 1} \lambda_k \lambda_l \|\psi_l\|^2_{\dot{H}^{\frac{\gamma}{2}}} \|\psi_k\|_{L^2}.$$

The Gagliardo-Nirenberg inequality,

$$\|\psi_l\|_{\dot{H}^{\frac{\gamma}{2}}} \lesssim \|\psi_l\|^{\gamma}_{\dot{H}^{\frac{1}{2}}} \|\psi_l\|^{1-\gamma}_{L^2},$$

together with conservation of charge (Lemma 2.3), yields

$$\begin{aligned}\|V[\Psi]\Psi\|_{\mathcal{L}^2} &\lesssim \sum_{l\geq 1} \lambda_l \|\psi_l\|^{2\gamma}_{\dot{H}^{\frac{1}{2}}} \\ &\lesssim (\sum_{l\geq 1} \lambda_l \|\psi_l\|^{2}_{\dot{H}^{\frac{1}{2}}})^{\gamma} \\ &\lesssim \|\Psi\|^{2\gamma}_{\mathcal{H}^{\frac{1}{2}}},\end{aligned}$$

where we have used in the second inequality the fact that $\sum_{l\geq 1} \lambda_l = 1, \;\; \lambda_l \geq 0$, and $f(x) = x^{\gamma}, \;\; 0 < \gamma < 1$, is concave (equality when $\gamma = 1$ is trivially satisfied). Substituting back in (2.4) yields

$$g_\epsilon(t) \lesssim \|\Psi\|^{2\gamma+1}_{\mathcal{H}^{\frac{1}{2}}} + \|\Psi\|^{4\gamma}_{\mathcal{H}^{\frac{1}{2}}},$$

which is finite for $t < \tau^*$. By the Dominated Convergence Theorem,

$$\mathcal{E}(\Psi(t_2)) - \mathcal{E}(\Psi(t_1)) = \int_{t_1}^{t_2} \lim_{\epsilon\to 0} g_\epsilon(t) dt = 0\,,$$

as claimed. $\square$

Global well-posedness in $\mathcal{H}^s$, for $s \geq \frac{1}{2}$, follows from conservation of charge and energy.

Proposition 2.5. *Suppose that the hypotheses of Proposition 2.2 hold. Then, if $g > 0$ or $g < 0$ with $\|\Psi(0)\|_{\mathcal{L}^2}$ small enough, and for $s \geq \frac{1}{2}$,*

$$\|\Psi(t)\|_{\mathcal{H}^s} \leq C \|\Psi(0)\|_{\mathcal{H}^s} e^{\alpha\left(\mathcal{E}(\Psi(0)) + \|\Psi(0)\|^{\delta}_{\mathcal{L}^2}\right)t},$$

where C, α and δ are positive constants that are independent of $m \geq 0$.

Proof. We start by bounding $\|\Psi(t)\|_{\dot{\mathcal{H}}^{\frac{\gamma}{2}}}$ from above, uniformly in time.

$$\begin{aligned}\langle \Psi, V[\Psi]\Psi\rangle_{\mathcal{L}^2} &= \sum_{l\geq 1} \lambda_l \langle \psi_l, V[\Psi]\psi_l\rangle \\ &\leq \|V[\Psi]\|_{L^\infty} \|\Psi\|^2_{\mathcal{L}^2} \\ &\lesssim (\sum_{k\geq 1} \lambda_k \|\psi_k\|^2_{\dot{H}^{\frac{\gamma}{2}}}) \|\Psi\|^2_{\mathcal{L}^2} \\ &\lesssim (\sum_{k\geq 1} \lambda_k \|\psi_k\|^{2\gamma}_{\dot{H}^{\frac{1}{2}}}) \|\Psi\|^2_{\mathcal{L}^2} \\ &\lesssim (\sum_{k\geq 1} \lambda_k \|\psi_k\|^{2}_{\dot{H}^{\frac{1}{2}}})^{\gamma} \|\Psi\|^2_{\mathcal{L}^2} \\ &\lesssim \|\Psi\|^{\gamma}_{\mathcal{H}^{\frac{1}{2}}} \|\Psi\|^2_{\mathcal{L}^2}.\end{aligned}$$

Here, we used Hölder's inequality in the second line, Lemma A.3 in the third line, the Gagliardo-Nirenberg inequality and conservation of charge in the fourth line, and $\sum_{k\geq 1}\lambda_k = 1,\quad \lambda_k \geq 0$, the fact that $x^\gamma,\quad 0<\gamma<1$, is concave in the fifth line (equality when $\gamma = 1$ is trivially satisfied). Together with conservation of energy (Lemma 2.4), this implies that for $g>0$ or $g<0$ with $\|\Psi(0)\|_{\mathcal{L}^2}$ small enough,

$$\|\Psi\|_{\dot{\mathcal{H}}^{\frac{\gamma}{2}}} \leq \alpha\left(\mathcal{E}(\Psi(t)) + \|\Psi(0)\|_{\mathcal{L}^2}^{\delta}\right), \tag{2.5}$$

where α and δ are constants independent of the mass $m\geq 0$. Now, it follows from the Duhamel formula that

$$\begin{aligned}\|\Psi(t)\|_{\mathcal{H}^s} &\leq \|\Psi(0)\|_{\mathcal{H}^s} + \int_0^t \|\Psi(t')\|^2_{\dot{\mathcal{H}}^{\frac{\gamma}{2}}}\|\Psi(t')\|_{\mathcal{H}^s}\, dt' \\ &\leq \|\Psi(0)\|_{\mathcal{H}^s} + \alpha\left(\mathcal{E}(\Psi(t)) + \|\Psi(0)\|^{\delta}_{\mathcal{L}^2}\right)\int_0^t \|\Psi(t')\|_{\mathcal{H}^s}\, dt',\end{aligned}$$

where we used Hölder's and Minkowski inequalities in the first line, and (2.5) in the second line. By Gronwall's lemma,

$$\|\Psi(t)\|_{\mathcal{H}^s} \leq \|\Psi(0)\|_{\mathcal{H}^s} e^{\alpha\left(\mathcal{E}(\Psi(0)) + \|\Psi(0)\|^{\delta}_{\mathcal{L}^2}\right)t}$$

follows. □

Proof of Theorem 1.1. It follows from Propositions 2.2 and 2.5 that $\tau^* = \infty$, i.e., the generalized semi-relativistic Schrödinger-Poisson system of equations is globally well-posed. □

We now prove the claim of Remark 1.5 about the asymptotic behaviour of the system as the mass tends to zero.

Proposition 2.6. *Consider the system of equations (1.4)-(1.6) with initial condition $(\underline{\lambda}, \Psi(0))$. Let $\Psi^{(0)}$ denote the solution of the initial value problem with mass $m = 0$, and fix $T>0$. Under the hypotheses of Proposition 2.5, $\Psi \to \Psi^{(0)}$ strongly in $L^\infty_T(\mathcal{H}^s),\quad s\geq 1/2$, as $m\to 0$.*

Proof. Proposition 2.5 implies that, given $T>0$, there exists finite $\rho>0$ such that

$$\sup_{m\in[0,1]} \|\Psi\|_{L^\infty_T \mathcal{H}^s} < \rho. \tag{2.6}$$

We now compare the norm of the difference of $\Psi(t)$ and $\Psi^{(0)}(t)$, $t\in[0,T]$. It follows from the Duhamel formula that

$$\begin{aligned}\|\Psi(t)-\Psi^{(0)}(t)\|_{\mathcal{H}^s} &\lesssim \|\left(U^{(m)}(t)-U^{(0)}(t)\right)\Psi(0)\|_{\mathcal{H}^s}+\\ &+\int_0^t \{\|V[\Psi(t')]\Psi(t')-V[\Psi^{(0)}(t')]\Psi^{(0)}(t')\|_{\mathcal{H}^s}+\\ &+\|\left(U^{(m)}(t')-U^{(0)}(t')\right)V[\Psi^{(0)}(t')]\Psi^{(0)}(t')\|_{\mathcal{H}^s}\}dt'\\ &\lesssim mT\|\Psi(0)\|_{\mathcal{H}^s}+\int_0^t \|V[\Psi(t')]\Psi(t')-V[\Psi^{(0)}(t')]\Psi^{(0)}(t')\|_{\mathcal{H}^s}\ dt'\\ &+\frac{mT^2}{2}\|V[\Psi^{(0)}]\Psi^{(0)}\|_{L^\infty_T\mathcal{H}^s},\end{aligned}$$

where we used Minkowski inequality in the first inequality and Hölder's inequality in the second. We also used $0\le\sqrt{-\Delta+m^2}-m\le m$.
It follows from the fact that the nonlinearity is locally Lipschitz (Lemma 2.1) and (2.6) that

$$\begin{aligned}&\|V[\Psi(t')]\Psi(t')-V[\Psi^{(0)}(t')]\Psi^{(0)}(t')\|_{\mathcal{H}^s}\lesssim\rho^2\|\Psi(t')-\Psi^{(0)}(t')\|_{\mathcal{H}^s},\\ &\|V[\Psi^{(0)}]\Psi^{(0)}\|_{L^\infty_T\mathcal{H}^s}\lesssim\rho^3.\end{aligned}$$

Hence

$$\|\Psi(t)-\Psi^{(0)}(t)\|_{\mathcal{H}^s}\lesssim m\rho T+m\rho^3T+\rho^2\int_0^t\|\Psi(t')-\Psi^{(0)}(t')\|_{\mathcal{H}^s}\ dt'.$$

By Gronwall's lemma, $\Psi\to\Psi^{(0)}$ strongly in $L^\infty_T(\mathcal{H}^s)$ as $m\to 0$. □

3 Asymptotic behaviour of solutions as mass tends to infinity

In this section, we discuss the asymptotics of the solution as the mass m tends to infinity.

Proof of Theorem 1.6. Recall that from the proof of local well-posedness in Section 2.1, there exists $T>0$ independent of m such that $\|\Psi\|_{L^\infty_T\mathcal{H}^s}\le C\|\Psi(0)\|_{\mathcal{H}^s},\ s\ge\gamma/2$, where C is independent of m. Similarly, one can show that there exists $T'>0$ independent of m such that $\|\Gamma\|_{L^\infty_{T'}\mathcal{H}^s}\le C\|\Psi(0)\|_{\mathcal{H}^s}$, where C is independent of m. Let $\tau=\min(T,T')$. Let $\tilde\Gamma=\{\tilde\gamma_k\}_{k\in\mathbb{N}}$ satisfy the system of equations

$$\begin{cases} i\partial_t\tilde\Gamma=V[\tilde\Gamma]\tilde\Gamma,\\ V[\tilde\Gamma]=w_\gamma\star n[\tilde\Gamma],\quad n[\tilde\Gamma]=\sum_{k=1}^\infty\lambda_k|\tilde\gamma_k|^2,\end{cases}$$

with initial condition $\tilde\Gamma(0)=\Psi(0)$. Alternatively, $\tilde\Gamma$ satisfies the integral equation

$$\tilde\Gamma(t)=\Psi(0)-i\int_0^t V[\tilde\Gamma(t')]\tilde\Gamma(t')dt'.$$

Uniqueness of the solution follows from the fact that the nonlinearity is locally Lipschitz (Lemma 2.1). We are going to compare Ψ to $\tilde{\Gamma}$, and then $\tilde{\Gamma}$ to Γ.

$$\|\Psi(t)-\tilde{\Gamma}(t)\|_{\mathcal{H}^s} \leq \| \left(U^{(m)}(t)-1\right)\Psi(0)\|_{\mathcal{H}^s}+ \tag{3.1}$$

$$+\int_0^t \| \left(U^{(m)}(t-t')-1\right)V[\tilde{\Gamma}(t')]\tilde{\Gamma}(t')\|_{\mathcal{H}^s}dt'+ \tag{3.2}$$

$$+\int_0^t \|V[\Psi(t')]\Psi(t')-V[\tilde{\Gamma}(t')]\tilde{\Gamma}(t')\|_{\mathcal{H}^s}dt'. \tag{3.3}$$

To estimate the first term on the right-hand-side, we apply the Fourier transform and use Parseval's Theorem,

$$\begin{aligned}
&\| \left(U^{(m)}(t)-1\right)\Psi(0)\|_{\mathcal{H}^s}^2 \\
&= \sum_{l\geq 1}\lambda_l \int_{\mathbb{R}^n} |e^{-it(\sqrt{m^2+|k|^2}-m)}-1|^2(1+|k|^2)^s|\widehat{\psi}_l(0,k)|^2dk \\
&\leq \sum_{l\geq 1}\lambda_l\{\int_{|k|\leq m^{\frac{1}{4}}} |e^{-it(\sqrt{m^2+|k|^2}-m)}-1|^2(1+|k|)^{2s}|\widehat{\psi}_l(0,k)|^2dk+ \\
&+\int_{|k|>m^{\frac{1}{4}}} |e^{-it(\sqrt{m^2+|k|^2}-m)}-1|^2(1+|k|)^{2s}|\widehat{\psi}_l(0,k)|^2dk\} \\
&\leq \sum_{l\geq 1}\lambda_l\{\int_{|k|\leq m^{\frac{1}{4}}} \frac{t^2|k|^4}{(\sqrt{m^2+|k|^2}+m)^2}(1+|k|)^{2s}|\widehat{\psi}_l(0,k)|^2dk+ \\
&+4\int_{|k|>m^{\frac{1}{4}}}(1+|k|)^{2s}|\widehat{\psi}_l(0,k)|^2dk\} \\
&\leq \frac{\tau^2}{4m}\|\Psi(0)\|_{\mathcal{H}^s}^2+4\sum_{l\geq 1}\int_{|k|>m^{\frac{1}{4}}}(1+|k|)^{2s}|\widehat{\psi}_l(0,k)|^2dk \\
&\to 0 \ \text{ as } \ m\to\infty.
\end{aligned}$$

Since $V[\tilde{\Gamma}]\tilde{\Gamma}\in\mathcal{H}^s$, it follows from the Dominated Convergence Theorem that

$$\lim_{m\to\infty}\int_0^t \| \left(U^{(m)}(t-t')-1\right)V[\tilde{\Gamma}(t')]\tilde{\Gamma}(t')\|_{\mathcal{H}^s}dt'=0.$$

To estimate the third term, let $\rho>0$ be a constant such that

$$\sup_{m\geq 1}(\|\Psi\|_{L^\infty_\tau\mathcal{H}^s}+\|\Gamma\|_{L^\infty_\tau\mathcal{H}^s})+\|\tilde{\Gamma}\|_{L^\infty_\tau\mathcal{H}^s}\leq\rho.$$

It follows from the fact that the nonlinearity is locally Lipschitz that

$$\|V[\Psi(t')]\Psi(t')-V[\tilde{\Gamma}(t')]\tilde{\Gamma}(t')\|_{\mathcal{H}^s}\leq C\rho^2\|\Psi(t')-\tilde{\Gamma}(t')\|_{\mathcal{H}^s},$$

where C is a positive constant independent of m.

Therefore,

$$\|\Psi(t)-\tilde{\Gamma}(t)\|_{\mathcal{H}^s}\leq f_m+C\rho^2\int_0^t\|\Psi(t')-\tilde{\Gamma}(t')\|_{\mathcal{H}^s}dt',$$

where f_m bounds the first two terms on the r.h.s. of (3.1). As shown above, $\lim_{m\to\infty}f_m=0$ and C is independent of m, so that it application of Gronwall's lemma yields

$$\lim_{m\to\infty}\|\Psi-\tilde{\Gamma}\|_{L^\infty_\tau\mathcal{H}^s}=0.$$

Similarly, one can show that

$$\|\Gamma(t)-\tilde{\Gamma}(t)\|_{\mathcal{H}^s}\leq g_m+C\rho^2\int_0^t\|\Psi(t')-\tilde{\Gamma}(t')\|_{\mathcal{H}^s}dt',$$

where $\lim_{m\to\infty}g_m=0$ and C is independent of m, and it follows that

$$\lim_{m\to\infty}\|\Gamma-\tilde{\Gamma}\|_{L^\infty_\tau\mathcal{H}^s}=0.$$

Since

$$\|\Psi-\Gamma\|_{L^\infty_\tau\mathcal{H}^s}\leq\|\Psi-\tilde{\Gamma}\|_{L^\infty_\tau\mathcal{H}^s}+\|\Gamma-\tilde{\Gamma}\|_{L^\infty_\tau\mathcal{H}^s},$$

it follows that

$$\lim_{m\to\infty}\|\Psi-\Gamma\|_{L^\infty_\tau\mathcal{H}^s}=0\,,$$

as desired. □

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A APPENDIX

The following result about the fractional Leibniz rule can be found in [9].

LEMMA A.1.

$$\|\mathcal{D}^s(uv)\|_{L^p}\lesssim\|\mathcal{D}^su\|_{L^{q_1}}\|v\|_{L^{r_1}}+\|u\|_{L^{q_2}}\|\mathcal{D}^sv\|_{L^{r_2}},$$

where $\frac{1}{p}=\frac{1}{q_i}+\frac{1}{r_i},\quad i=1,2.$

The second result is about inequality involving fractional integral operators, which can be found, for example, in [12].

Lemma A.2. *Let I_α, for $0 < \alpha < n$, be the fractional integral operator*

$$I_\alpha(u) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n} u(y)\ dy.$$

Then

$$\|I_\alpha(u)\|_{L^p} \lesssim \|u\|_{L^q}, \quad \frac{1}{p} = \frac{1}{q} - \frac{\alpha}{n}.$$

We also recall the following useful Hardy-type inequality.

Lemma A.3. *Let $0 < \gamma < n$. Then,*

$$\sup_{x\in\mathbb{R}^n} | \int_{\mathbb{R}^n} \frac{1}{|x-y|^\gamma} |u(y)|^2 dy| \lesssim \|u\|^2_{\dot{H}^{\frac{\gamma}{2}}}\ .$$

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