Framed Sheaves over Treefolds and Symmetric Obstruction Theories

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Abstract. We note that open moduli spaces of sheaves over local Calabi-Yau surface geometries framed along the divisor at infinity admit symmetric perfect obstruction theories. We calculate the corresponding Donaldson-Thomas weighted Euler characteristics (as well as the topological Euler characteristics). Furthermore, for blowup geometries, we discuss the contribution of exceptional curves.

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1 Introduction

Moduli spaces of sheaves over threefolds admit virtual fundamental classes in a lot of examples, yielding Donaldson-Thomas invariants [T]. The rank 1 case is particularly interesting, bearing connections with virtual curve counts [MNOP].

In this note, we study *open* moduli spaces of higher rank sheaves over local Calabi-Yau surface geometries, framed along the divisor at infinity. We prove that the moduli spaces admit symmetric perfect obstruction theories, and in this context, we compute the ensuing Donaldson-Thomas Euler characteristics. In addition, we find the topological Euler characteristics of the *compactified* moduli spaces of *framed modules*. We also discuss a "blowup" formula. Finally, we point out other geometries which can be studied by the same methods.

This way, we extend previously known results in two directions [1].

[1] The search for such generalizations motivated our interest this topic.

(i) First, there is quite a bit of literature on moduli spaces of framed sheaves over *surfaces*. An exhaustive survey is not our intention here, but we refer the reader to [BPT], [N] for calculations which we partially carry out in the higher dimensional setting, and also for a more comprehensive bibliography.

(ii) Second, as suggested above, we partially generalize to *higher rank* results about the Hilbert scheme of points over threefolds. For these, the Donaldson-Thomas Euler characteristics, in the form needed here, were calculated in [BF].

2 Framed sheaves over local Calabi-Yau surface geometries

We now detail the discussion. Let S be a smooth complex projective surface, and let X° denote the total space of the canonical bundle $K_S \to S$. We are concerned with moduli spaces of sheaves over the open Calabi-Yau threefold X°. The noncompact geometry does not allow for a good moduli space of semistable sheaves. Instead, we will consider the compact threefold

$$\pi : X = \mathbb{P}(K_S + \mathcal{O}_S) \to S.$$

This comes equipped with two divisors

$$S_\infty = \mathbb{P}(K_S + 0) \text{ and } S_0 = \mathbb{P}(0 + \mathcal{O}_S)$$

corresponding to the summands K_S and $\mathcal{O}_S$. Clearly,

$$X \setminus S_\infty = X^\circ.$$

We form the moduli space $\mathfrak{M}_n$ of semistable *framed modules* (E, ϕ) of rank r with

$$c_1(E) = c_2(E) = 0, \ \ \chi(F) = N := r\chi(\mathcal{O}_S) - n,$$

with a non-zero framing over S_∞:

$$\phi : E \to \mathcal{O}_{S_\infty} \otimes \mathbb{C}^r.$$

The moduli space $\mathfrak{M}_n$ was constructed in [HL]. Semistability was defined with respect to a polynomial δ of degree ≤ 2 with positive leading coefficient, as well as an ample divisor H on X. We will pick

$$H = \pi^\star H_0 + \epsilon c_1(\mathcal{O}_{\mathbb{P}}(1)),$$

for an ample divisor H_0 on S and a sufficiently small rational $\epsilon > 0$. By definition, (E, ϕ) is semistable provided that

(i) for all proper subsheaves F of E, the Hilbert polynomials satisfy

$$P_F - \delta \leq \frac{\text{rk}F}{\text{rk}E}(P_E - \delta);$$

(ii) if F is contained in the kernel of ϕ, then

$$P_F \leq \frac{\mathrm{rk} F}{\mathrm{rk} E}(P_E - \delta).$$

Semistable framed modules admit Harder-Narasimhan filtrations, yielding the notion of S-equivalence. There is a projective moduli space $\mathfrak{M}_n$ of S-equivalence classes of *framed modules*, cf. [HL].

We will consider the open subset

$$\mathfrak{M}_n^\circ \hookrightarrow \mathfrak{M}_n$$

corresponding to what are called *framed sheaves* in [L], [N]. These are stable framed modules (E, ϕ) such that

(iii) E is torsion free, locally free near S_∞, and ϕ is an isomorphism along S_∞.

Over curves and certain surfaces and for special framings, the stability conditions (i) and (ii) are automatic for the framed sheaves of (iii), cf. [BM], but for threefolds stability is not yet known to follow on general grounds.

Example We describe the moduli space in perhaps the simplest example, that of the Hilbert polynomial

$$P_E = P_{\mathcal{O}_X} - \ell.$$

Intuitively, in this case we should get the Hilbert scheme of points. This is not entirely obvious because framed modules are not required to be torsion free and because of the stability condition. The exact description will be determined by comparing δ to the polynomial

$$\Delta = \chi(mH|_{S_\infty}) = \frac{m^2 H_0^2}{2} + \text{l.o.t.}$$

To avoid strictly semistables, we assume that $\Delta - \delta$ is not a constant a with $0 \leq a \leq \ell$.

The sheaves E in the moduli space have rank 1 and can be written in the form

$$0 \to T \to E \to E^\circ \to 0$$

where T is torsion and E° is torsion free. In fact,

$$E^\circ = I_Z \otimes L$$

for some line bundle $L \to X$, and some subscheme Z of dimension at most 1. Now, by stability, or using Lemma 1.2 of [HL], the kernel K of the restricted framing

$$\phi_\infty : T \to \mathcal{O}_{S_\infty}$$

must satisfy

$$p_K \leq 0 \implies K = 0.$$

Therefore ϕ_∞ gives an inclusion of the torsion module T into the framing $\mathcal{O}_{S_\infty}$, showing that

$$T = 0 \text{ or } T = i_\star(I_W \otimes M),$$

for some line bundle M over $i : S_\infty \to X$, and a subscheme $W \subset S_\infty$ of dimension zero.

In the first case, since $c_1(E) = c_2(E) = 0$, we must have $c_1(L) = 0$ and Z is zero dimensional. We claim that $L = \mathcal{O}_X$. Indeed, L restricts trivially to the fibers of $X \to S$, hence it must be a pullback

$$L = \pi^\star N$$

of a degree zero line bundle N on S. The framing condition implies that there must exist a nonzero morphism $N \to \mathcal{O}_S$, hence N must be trivial. Therefore, up to isomorphisms, the only framed modules are

$$(E, \iota) : I_Z \to \mathcal{O}_{S_\infty}$$

for zero dimensional subschemes Z of length ℓ. We analyze semistability. The kernel of ι takes the form $I_U(-S_\infty)$ for some zero dimensional scheme U. Thus, we must have

$$\chi(I_U(mH - S_\infty)) \leq \chi(mH) - \ell - \delta \iff \ell - \ell(U) \leq \Delta - \delta.$$

If $\Delta - \delta$ has negative leading term, the inequality cannot be satisfied. If $\Delta - \delta$ has positive leading term, then the inequality is automatic and stability follows. Hence, the moduli space is either empty, or isomorphic to $X^{[\ell]}$. [2]

We claim the second case cannot occur under our assumptions. If it did, then

$$0 \to i_\star(I_W \otimes M) \to E \to L \otimes I_Z \to 0.$$

Calculating the Chern class $c_1(E) = 0$, we find

$$L = \mathcal{O}(-S_\infty) \otimes \pi^\star N,$$

for some degree 0 line bundle N overs S. Therefore

$$0 \to i_\star(I_W \otimes M) \to E \to I_Z(-S_\infty) \otimes \pi^\star N \to 0, \;\; \phi : E \to \mathcal{O}_{S_\infty}.$$

[2] We also remark here that if $\Delta - \delta = a$ for some $a \in \{0, \ldots, \ell\}$, then there are strictly semistable framed modules. Indeed, pairs of subschemes Z° of X° and Z_∞ of S_∞ with $\ell(Z_\infty) = a$ yield the strictly semistable framed modules $(I_{Z^\circ}(-S_\infty), 0) \oplus (i_\star I_{Z_\infty / S_\infty}, \iota)$.

We already argued above that the restriction ϕ_∞ of ϕ to the torsion module $i_\star(I_W \otimes M)$ must be injective. Since $\phi_\infty \neq 0$, there must exist a non-zero morphism $M \to \mathcal{O}_S$, hence $M^\vee$ must be effective. Since $c_2(E) = 0$, we have

$$\iota_\star c_1(M) = [Z].$$

Therefore, M is trivial and Z is of dimension zero. The Hilbert polynomial gives

$$\ell(Z) + \ell(W) = \ell.$$

Furthermore, up to scalars, ϕ_∞ must be the natural inclusion. Semistability implies that

$$P_{i_\star I_W} - \delta \leq 0 \implies \Delta - \delta \leq \ell(W).$$

Since ϕ_∞ is the natural inclusion, the image of ϕ in $\mathcal{O}_{S_\infty}$ must be the ideal sheaf I_U of a scheme $U \subset W$. Let K be the kernel of ϕ. We have

$$P_K = P_E - P_{i_\star I_U} = P_E - (\Delta - \ell(U)).$$

Semistability implies

$$P_K \leq P_E - \delta \implies \Delta - \delta \geq \ell(U).$$

Therefore, we conclude that

$$\Delta - \delta = a$$

for some constant a such that

$$\ell(W) \geq a \geq \ell(U).$$

In particular $0 \leq a \leq \ell$ which contradicts our assumption.

To summarize, when $\Delta - \delta$ is not equal to a constant between 0 and ℓ, we obtain the following description of the moduli space:

(a) if (the leading term of) $\Delta - \delta < 0$, then we get $\emptyset$;

(b) if (the leading term of) $\Delta - \delta > 0$, then the moduli space is the Hilbert scheme $X^{[\ell]}$.

3 Obstruction theory

We note now that the obstruction theory of framed *sheaves* is symmetric. To this end, we assume that δ is *good*, i.e. it satisfies the following conditions:

- if $\deg \delta = 0$, then $\delta > (r-1)n$;
- if $\deg \delta = 2$, the quadratic term of δ is sufficiently small compared to that of Δ.

In particular, any δ of degree 1 is good for all n.

Theorem 1. *When δ is good, the moduli space $\mathfrak{M}_n^\circ$ admits a symmetric perfect obstruction theory at the stable points (E, ϕ).*

Proof. The deformation theory of stable framed sheaves was worked out in [HL], [S]. Write

$$(\mathcal{E}, \Phi) \to \mathfrak{M}_n^\circ \times X$$

for the universal family, which exists by [HL], and let p and q be the natural projections. The complex

$$\mathbf{F} = \mathbf{R}p_\star \mathbf{R}Hom(\mathcal{E}(-S_\infty), \mathcal{E} \otimes q^\star K_X)[2]$$

is an obstruction theory over $\mathfrak{M}_n^\circ$. The obstruction theory is symmetric in the sense that there is a symmetric isomorphism

$$\mathbf{F} \to \mathbf{F}^\vee[1].$$

This is a consequence of Grothendieck duality and of the the crucial observation that

$$K_X = \mathcal{O}(-2S_\infty).$$

The calculation of the canonical bundle standardly follows from the Euler sequence of the projective bundle X.

The obstruction theory is perfect with amplitude contained in $[-1, 0]$. Indeed, the amplitude is clearly contained in $[-2, 1]$. By symmetry, it suffices to explain that the degree -2 term is zero. In turn, this is implied by the vanishing

$$\mathrm{Hom}(E, E(-S_\infty)) = 0$$

which holds for all sheaves E in $\mathfrak{M}_n^\circ$. Indeed, assuming there is a non-zero morphism

$$E \to E(-S_\infty),$$

we let K and I denote its kernel and image, and write r_K and r_I for their ranks. We have $r_I > 0$. By stability

$$P_K - \delta \leq \frac{r_K}{r}(P_E - \delta), \ P_{I(S_\infty)} - \delta \leq \frac{r_I}{r}(P_E - \delta).$$

Considering the quadratic terms of these inequalities, we obtain

$$c_1(K) \cdot H^2 \leq \delta_0 \left(1 - \frac{r_K}{r}\right), \ c_1(I) \cdot H^2 + r_I S_\infty \cdot H^2 \leq \delta_0 \left(1 - \frac{r_I}{r}\right),$$

where δ_0 is half the leading term of δ. We also have

$$c_1(K) + c_1(I) = 0.$$

Adding, we obtain

$$r_I S_\infty \cdot H^2 \leq \delta_0$$

which is impossible when the leading term $\delta_0 < S_\infty \cdot H^2$ is sufficiently small. This completes the proof. $\square$

EXAMPLE We determine the obstruction theory for the previous example. We consider case (b) corresponding to $\Delta - \delta > 0$, $\Delta - \delta$ does not equal a constant a with $0 \leq a \leq \ell$. The tangent space at the ideal sheaf I_Z was found in [HL] to be

$$T_Z\mathfrak{M} = \mathrm{Ext}^1(I_Z, [I_Z \to \mathcal{O}_{S_\infty}]).$$

This can be calculated from the exact triangle

$$[I_Z \to \mathcal{O}_{S_\infty}] \to [\mathcal{O}_X \to \mathcal{O}_{S_\infty}] \cong \mathcal{O}(-S_\infty) \to \mathcal{O}_Z.$$

We have

$$\mathrm{Ext}^0(I_Z, \mathcal{O}_X(-S_\infty)) = \mathrm{Ext}^3(\mathcal{O}_X, I_Z(-S_\infty))^\vee = H^3(I_Z(-S_\infty))^\vee$$

$$= H^3(\mathcal{O}_X(-S_\infty))^\vee = 0,$$

and similarly for Ext^1. From the exact triangle, we obtain

$$0 \to \mathrm{Ext}^0(I_Z, \mathcal{O}_Z) \cong T_Z X^{[\ell]} \to T_Z\mathfrak{M} \to 0.$$

In particular, this agrees with the identification

$$\mathfrak{M} \cong X^{[\ell]}.$$

Thus, by symmetry, the obstruction theory of $\mathfrak{M}$ coincides with the usual obstruction theory for the Hilbert scheme *only* along the open part $(X^\circ)^{[\ell]}$.

4 CALCULATIONS

Symmetric perfect obstruction theories have associated Behrend functions [B]. In particular, the open moduli space

$$\mathfrak{M}_n^\circ \hookrightarrow \mathfrak{M}_n$$

is endowed with a constructible function

$$\nu : \mathfrak{M}_n^\circ \to \mathbb{Z}.$$

We will calculate the Donaldson-Thomas weighted Euler characteristic

$$\widetilde{\chi}(\mathfrak{M}_n^\circ) = \sum_k k\chi(\nu^{-1}(k)).$$

Since the obstruction theory is not perfect symmetric over the boundary, these weighted Euler characteristics do not calculate intersection theoretic Donaldson-Thomas invariants of $\mathfrak{M}_n$.

4.1 Virtual localization

Our computation is via equivariant localization. The following result was proved in [BF] for torus actions with isolated fixed points, and in [LQ] in arbitrary generality. Let $\mathfrak{M}$ be a moduli space admitting a $\mathbf{C}^\star$-action compatible with the symmetric perfect obstruction theory. Then the fixed point set $\mathfrak{M}^{\mathbf{C}^\star}$ also inherits a symmetric perfect obstruction theory. Furthermore, the Behrend functions of $\mathfrak{M}$ and $\mathfrak{M}^{\mathbf{C}^\star}$ at torus fixed points p are related by

$$\nu_{\mathfrak{M}}(p) = (-1)^{\epsilon_p} \nu_{\mathfrak{M}^{\mathbf{C}^\star}}(p),$$

where ϵ_p is given by the difference in the dimension of the Zariski tangent spaces

$$\epsilon_p = \dim T_p \mathfrak{M} - \dim T_p \mathfrak{M}^{\mathbf{C}^\star}.$$

This observation is used in [LQ] as follows. The torus acts on the subscheme

$$\{p \in \mathfrak{M} \setminus \mathfrak{M}^{\mathbf{C}^\star} : \nu_{\mathfrak{M}}(p) = k\}$$

with no fixed points, hence its Euler characteristic must be zero, cf. [LY]. Therefore,

$$\chi(\{p \in \mathfrak{M} : \nu_{\mathfrak{M}}(p) = k\}) = \chi(\{p \in \mathfrak{M}^{\mathbf{C}^\star} : \nu_{\mathfrak{M}}(p) = k\})$$

yielding

$$\widetilde{\chi}(\mathfrak{M}) = \sum_k k \chi(\{p \in \mathfrak{M}^{\mathbf{C}^\star} : \nu_{\mathfrak{M}^{\mathbf{C}^\star}}(p) = k(-1)^{\epsilon_p}\}).$$

We will apply these remarks to the action of $\mathbf{C}^\star$ on $\mathfrak{M}_n$ induced by the scaling action in the fibers of the projective bundle $X \to S$ and the scaling action on the framing coming from a generic embedding

$$\mathbf{C}^\star \hookrightarrow \mathbf{GL}_r.$$

We will find the torus fixed points in $\mathfrak{M}_n$.

Lemma 2. *Assume δ is good. The $\mathbf{C}^\star$-fixed framed modules in $\mathfrak{M}_n$ take the form*

$$E = \bigoplus_{i=1}^r I_{Z_i}$$

where Z_i are zero dimensional subschemes of X invariant under the action of the torus, of total length n. The framing ϕ is the natural composition $E \hookrightarrow \mathcal{O}_X^r \to \mathcal{O}_{S_\infty}^r$.

Proof. We first prove that all invariant framed modules are torsion free. Indeed, the torsion module T of E is $\mathbf{C}^\star$-fixed. By stability, the framing ϕ gives a $\mathbf{C}^\star$-invariant injection

$$\phi_\infty : T \hookrightarrow \mathcal{O}_{S_\infty} \otimes \mathbb{C}^r.$$

Therefore, the torsion module splits

$$T = \oplus_{j=1}^{\ell} i_\star (I_{W_j} \otimes M_j),$$

for zero dimensional subschemes W_j of S_∞ and line bundles M_j over S_∞. Again by stability applied to the torsion submodule T we find

$$p_T - \delta \leq 0 \implies \ell \cdot H^2 \cdot S_\infty \leq \delta_0$$

where δ_0 is half the quadratic coefficient of δ. By assumption, we may take $\delta_0 < H^2 \cdot S_\infty$, implying that $\ell = 0$ and showing that the torsion module vanishes.

Now, since E is torsion free and $\mathbf{C}^\star$-invariant, the argument of [BPT] shows that

$$E = \oplus_{i=1}^{r} I_{Z_i} \otimes L_i$$

where L_i are line bundles over X and Z_i are subschemes of dimension at most 1. The subschemes Z_i must be torus invariant. Since $c_1(E) = c_2(E) = 0$, we find

$$\sum c_1(L_i) = 0 \tag{1}$$

and furthermore

$$\sum_{i=1}^{r} c_1(L_i)^2 = 2 \sum_{i=1}^{r} [Z_i]. \tag{2}$$

Since the framed module (E, ϕ) is semistable, for all submodules F of E of positive rank we must have

$$\frac{c_1(F) \cdot H^2 - \delta_0}{\text{rk } F} \leq \frac{c_1(E) \cdot H^2 - \delta_0}{\text{rk } E}.$$

Taking $F = I_{Z_i} \otimes L_i$ we find

$$c_1(L_i) \cdot H^2 \leq \delta_0 \left(1 - \frac{1}{r}\right).$$

Now, since δ_0 is sufficiently small compared to the denominator of the rational divisor H, we conclude

$$c_1(L_i) \cdot H^2 \leq 0.$$

In fact, $c_1(L_i) \cdot H^2 = 0$ for all i, because of (1). We argue that L_i are trivial and Z_i are zero dimensional.

Write

$$c_1(L_i) = \pi^\star D_i + d_i \zeta,$$

where D_i are divisors on the surface S and $\zeta = c_1(\mathcal{O}_{\mathbb{P}}(1))$. We calculate

$$\sum_i c_1(L_i)^2 = \pi^\star \left(\sum_i D_i^2\right) + 2 \sum_i d_i \left(\pi^\star D_i \cdot \zeta\right) + (\sum_i d_i^2)\zeta^2 = 2 \sum_i [Z_i], \tag{3}$$

Set
$$M = 2\sum_i d_i D_i - \left(\sum_i d_i^2\right) K_S.$$
Using
$$\zeta^2 + K_S \cdot \zeta = 0,$$
we conclude from (3) that
$$\pi^\star\left(\sum_i D_i^2\right) + \pi^\star M \cdot \zeta = 2\sum_i [Z_i]. \tag{4}$$
Pushing (4) forward under π we find
$$M = 2\sum_i \pi_\star[Z_i].$$
As a consequence, M is effective. The requirement that the slopes of L_i are trivial translates into the condition
$$(\pi^\star D_i + d_i\zeta)(\pi^\star H_0 + \epsilon\zeta)^2 = 0$$
which rewrites as
$$(D_i - d_i K_S) \cdot \Sigma = -d_i$$
where
$$\Sigma = \frac{\epsilon(2H_0 - \epsilon K_S)}{H_0^2}$$
is an ample rational curve class on S for small ϵ. Write
$$F_i = D_i - d_i K_S$$
so that
$$F_i \cdot \Sigma = -d_i.$$
Since
$$M = 2\sum_i d_i F_i + (\sum_i d_i^2) K_S$$
is effective, its intersection with Σ must be positive. This gives
$$-2\sum_i d_i^2 + (\sum_i d_i^2) K_S \cdot \Sigma \geq 0.$$
For small ϵ we have $K_S \cdot \Sigma < 2$. We conclude from here that $d_i = 0$ for all i. Therefore $M = 0$, and by (4) we must have
$$\pi^\star\left(\sum_i D_i^2\right) = 2\sum_i [Z_i]$$

is effective. Note that the left hand side is supported on fibers. Therefore,

$$\sum_i D_i^2 \geq 0. \tag{5}$$

We moreover proved

$$F_i \cdot \Sigma = 0 \implies D_i \cdot \Sigma = 0.$$

Since Σ is ample, by Hodge index theorem we have

$$D_i^2 \leq 0,$$

with equality only if D_i is numerically equivalent to 0. In fact equality must occur because of (5). This yields $c_1(L_i) = 0$. In turn,

$$L_i = \pi^\star N_i$$

for some line bundles $N_i \to S$ of first Chern class 0. Furthermore, from (3) we find $[Z_i] = 0$ hence Z_i must be zero dimensional.

Thus

$$E = \oplus_{i=1}^r I_{Z_i} \otimes \pi^\star N_i,$$

where

$$\sum \ell(Z_i) = n.$$

Clearly, Z_i must be torus invariant and $\phi = \oplus \phi_i$ where

$$\phi_i : I_{Z_i} \otimes \pi^\star N_i \to \mathcal{O}_{S_\infty}.$$

We next claim that $\phi_i \neq 0$ for all i. Indeed, if $\phi_i = 0$ for some i, then $I_{Z_i} \otimes \pi^\star N_i$ is in the kernel of ϕ, yielding by stability

$$\chi(I_{Z_i}(mH) \otimes \pi^\star N_i) \leq \frac{1}{r}\left(\sum_j \chi(I_{Z_j}(mH) \otimes \pi^\star N_j) - \delta\right).$$

This gives

$$n \geq \ell(Z_i) \geq \frac{n}{r} + \frac{\delta}{r}.$$

This is a contradiction since $\delta > (r-1)n$. Therefore $\phi_i \neq 0$, showing that there exists a non-zero morphism $N_i \to \mathcal{O}_S$. Therefore N_i must be trivial, completing the proof. $\square$

Over the open moduli space $\mathfrak{M}_n^\circ$, the same result holds without any restrictions on δ:

Lemma 2A. *The $\mathbf{C}^\star$-fixed framed sheaves E in $\mathfrak{M}_n^\circ$ must split*

$$E = \bigoplus_{i=1}^{r} I_{Z_i}$$

where Z_i are zero dimensional subschemes of X° invariant under the action of the torus, of total length n.

Proof. By assumption E is torsion free, hence

$$E = \oplus_{i=1}^{r} I_{Z_i} \otimes L_i.$$

Since the framing is an isomorphism, we conclude Z_i is contained in X° and L_i is trivial on S_∞. Hence,

$$L_i = \mathcal{O}(d_i S_0)$$

for some integers d_i. We claim that $d_i = 0$ for all i. This in turn implies that Z_i are zero dimensional by using $c_2(E) = 0$.

Assume first that the quadratic term of δ is sufficiently small. This case is already covered by Lemma 2, but a simpler argument is possible over $\mathfrak{M}_n^\circ$; we record it here for future reference. Indeed, the stability condition applied to $L_i \otimes I_{Z_i}$ gives

$$d_i S_0 \cdot H^2 = c_1(L_i) \cdot H^2 \leq \delta_0 \left(1 - \frac{1}{r}\right) \implies d_i \leq 0.$$

Since $c_1(E) = 0$, we have $\sum_i d_i = 0$. Hence $d_i = 0$ for all i, as claimed.

We now give the general argument. Using that $c_1(E) = c_2(E) = 0$, we find

$$(\sum_i d_i^2) S_0^2 = 2 \sum_i [Z_i]. \tag{6}$$

Assume not all d_i are equal to 0. Since the Z_i's are torus invariant and disjoint from S_∞, their cohomology classes are supported on the surface S_0. Using that

$$S_0^2 = (K_S^2) f + \zeta \cdot \pi^\star K_S,$$

from equation (6) we find $K_S^2 = 0$. From here, pushing forward under π, we conclude

$$(\sum_i d_i^2) K_S = 2 \sum_i \pi_\star [Z_i]$$

which is effective. Hence

$$K_S \cdot H_0 \geq 0.$$

Now, $I_{Z_i}(d_i S_0 - S_\infty)$ is contained in the kernel of ϕ. Hence by stability

$$(d_i S_0 - S_\infty) \cdot H^2 \leq -\frac{\delta_0}{r}.$$

Pick an index i such that $d_i \geq 1$. The above inequality implies

$$(S_0 - S_\infty) \cdot H^2 \leq -\frac{\delta_0}{r} \implies \pi^\star K_S \cdot H^2 \leq -\frac{\delta_0}{r}.$$

However,

$$\pi^\star K_S \cdot H^2 = \epsilon \cdot (2H_0 - \epsilon K_S) K_S = 2\epsilon \cdot K_S \cdot H_0 \geq 0.$$

Therefore, $K_S \cdot H_0 = \delta_0 = 0$. Since the quadratic term of δ is 0, the previous paragraph applies, showing that in fact all $d_i = 0$. $\square$

Lemma 3. *If δ is good, all torus fixed framed sheaves E in $\mathfrak{M}_n$ described above are stable.*

Proof. Let F be a subsheaf of $E = \oplus I_{Z_i}$ of rank r'. Since F is a subsheaf of $\mathcal{O}_X^r$, by Gieseker semistability we have

$$P_F \leq r'\chi(mH) < \frac{r'}{r} P_E + \frac{r - r'}{r}\delta,$$

at least when $r' \neq r$, using that $\delta > (r-1)n$. When $r' = r$, induction on r yields the claim. For the inductive step, consider the non-zero map $F \to I_{Z_r}$, and write F' for the kernel. Then, apply the induction hypothesis to F' which is contained in $\oplus_{i=1}^{r-1} I_{Z_i}$.

Next, assume F is in the kernel of ϕ. The kernel of ϕ is contained in $\mathcal{O}_X(-S_\infty)^r$ (and it is isomorphic to $\oplus_j I_{Z_j}(-S_\infty)$ for E in $\mathfrak{M}_n^\circ$). By Gieseker-semistability, we have

$$P_F \leq r'\chi(mH - S_\infty) < \frac{r'}{r}(r\chi(mH) - n - \delta) = \frac{r'}{r}(P_E - \delta),$$

using that δ is good. $\square$

Lemma 4. *For all torus fixed sheaves E in $\mathfrak{M}_n^\circ$, we have*

$$\dim T_E \mathfrak{M}_n \equiv rn \mod 2.$$

Proof. Since $E = \oplus I_{Z_i}$ is stable, the tangent space is calculated in [HL]:

$$T_E \mathfrak{M}_n = \mathrm{Ext}^1(E, E(-S_\infty)) = \sum_{i,j} \mathrm{Ext}^1(I_{Z_i}, I_{Z_j}(-S_\infty)).$$

We consider first the contributions of terms corresponding to pairs of indices (i,j) and (j,i) for $i \neq j$:

$$\mathrm{Ext}^1(I_{Z_i}, I_{Z_j}(-S_\infty)) + \mathrm{Ext}^1(I_{Z_j}, I_{Z_i}(-S_\infty))$$

$$= \operatorname{Ext}^1(I_{Z_i}, I_{Z_j}(-S_\infty)) + \operatorname{Ext}^2(I_{Z_i}, I_{Z_j}(-S_\infty))$$

by Serre duality. Now, considering the above expression modulo 2 we obtain

$$\chi(I_{Z_i}, I_{Z_j}(-S_\infty)) + \operatorname{Ext}^0(I_{Z_i}, I_{Z_j}(-S_\infty)) + \operatorname{Ext}^3(I_{Z_i}, I_{Z_j}(-S_\infty)).$$

Next, it is easily seen that Ext^0 vanishes, and same for Ext^3 by duality. Thus, we are left with

$$\begin{aligned}
\chi(I_{Z_i}, I_{Z_j}(-S_\infty)) &= \chi(\mathcal{O}_X, \mathcal{O}_X(-S_\infty)) - \chi(\mathcal{O}_X, \mathcal{O}_{Z_j}) - \chi(\mathcal{O}_{Z_i}, \mathcal{O}_X(-S_\infty)) \\
&= \chi(\mathcal{O}_X(-S_\infty)) - \ell(Z_j) + \ell(Z_i) \\
&= \ell(Z_i) + \ell(Z_j) \mod 2.
\end{aligned}$$

We consider now the terms with $i = j$:

$$\operatorname{Ext}^1(I_{Z_i}, I_{Z_i}(-S_\infty)).$$

This term was already worked out in the deformation theory of Example 1. We obtained

$$\operatorname{Ext}^1(I_{Z_i}, I_{Z_i}(-S_\infty)) = \operatorname{Ext}^0(I_{Z_i}, \mathcal{O}_{Z_i}) \equiv \ell(Z_i) \mod 2,$$

where for the last congruence we used [BF] or [MNOP]. The lemma follows by collecting the above facts. ☐

We can now put together the calculation of Lemma 4 and the remarks about Behrend functions in Subsection 4.1 to calculate the Donaldson-Thomas Euler characteristic of $\mathfrak{M}_n^\circ$. We write $\mathfrak{X}_\ell$ for the subset of the Hilbert scheme of points in X° which parametrizes torus fixed Z's of length ℓ. For each partition $\vec{\ell}$ into r parts $(\ell_1, \ldots, \ell_r)$ with

$$\ell_1 + \ldots + \ell_r = n$$

we write

$$\mathfrak{X}^{\vec{\ell}} = \mathfrak{X}^{\ell_1} \times \ldots \times \mathfrak{X}^{\ell_r}.$$

Then, $\mathfrak{X}^{\vec{\ell}}$ are the $\mathbf{C}^\star$-fixed loci of $\mathfrak{M}_n^\circ$. With the convention that

$$\vec{Z} = (Z_1, \ldots, Z_r)$$

represents an r-tuple of schemes in $\mathfrak{X}^{\vec{\ell}}$, we calculate

$$\begin{aligned}
&\widetilde{\chi}\,(\mathfrak{M}_n^\circ) = \\
&= \sum_{\vec{\ell}} \sum_k k\, \chi(\{\vec{Z} \in \mathfrak{X}^{\vec{\ell}} : \nu_{\mathfrak{X}^{\vec{\ell}}}(\vec{Z}) = k(-1)^{rn - \dim T_{\vec{Z}} \mathfrak{X}^{\vec{\ell}}}\}) \\
&= (-1)^{(r-1)n} \sum_{\vec{\ell}} \sum_{\vec{k}} \prod_{i=1}^r k_i\, \chi(\{Z_i \in \mathfrak{X}^{\ell_i} : \nu_{\mathfrak{X}^{\ell_i}}(Z_i) = k_i(-1)^{\ell_i - \dim T_{Z_i} \mathfrak{X}^{\ell_i}}\}) \\
&= (-1)^{(r-1)n} \sum_{\vec{\ell}} \prod_{i=1}^r \left(\sum_k k\chi(\{Z : \nu_{\mathfrak{X}^{\ell_i}}(Z) = k(-1)^{\ell_i - \dim T_Z \mathfrak{X}^{\ell_i}}\} \right)
\end{aligned}$$

By applying these results when $r = 1$, and using the identification of the rank 1 moduli space with the Hilbert scheme worked out in Example 1, we obtain

$$\widetilde{\chi}((X^\circ)^{[\ell]}) = \sum_k k\chi(\{Z : \nu_{\mathfrak{X}^\ell}(Z) = k(-1)^{\ell - \dim T_Z \mathfrak{X}_\ell}\}).$$

This yields

$$\widetilde{\chi}(\mathfrak{M}_n^\circ) = (-1)^{(r-1)n} \sum_{\vec{\ell}} \widetilde{\chi}((X^\circ)^{[\ell_1]}) \cdot \ldots \cdot \widetilde{\chi}((X^\circ)^{[\ell_r]}).$$

We form the generating series

$$\sum_n q^n \widetilde{\chi}(\mathfrak{M}_n^\circ) = \left(\sum_\ell ((-1)^{r-1}q)^\ell \widetilde{\chi}((X^\circ)^{[\ell]}) \right)^r.$$

Now, from [BF] we lift the calculation

$$\sum_\ell q^\ell \widetilde{\chi}((X^\circ)^{[\ell]}) = \mathsf{M}(-q)^{e(X^\circ)} = \mathsf{M}(-q)^{e(S)},$$

where $\mathsf{M}(q)$ is the MacMahon function

$$\mathsf{M}(q) = \prod_{k=1}^\infty (1 - q^k)^{-k}.$$

To summarize, for δ good, we proved

Theorem 5. *The following equality holds*

$$\sum_{n \geq 0} q^n \widetilde{\chi}(\mathfrak{M}_n^\circ) = \mathsf{M}((-1)^r q)^{re(S)}. \tag{7}$$

We are unable to define (and calculalte) the virtual motive $[\mathfrak{M}_n^\circ]^{\mathrm{vir}}$, as it is done in rank 1 in [BBS] and for surfaces in [N]. This question may deserve further study.

4.2 Topological Euler characteristics

Lemma 1 also allows us to calculate the topological Euler characteristics of the compact spaces $\mathfrak{M}_n$ via the localization results of [LY]:

$$e(\mathfrak{M}_n) = e(\mathfrak{M}_n^{\mathbf{C}^\star}).$$

The same calculation as above shows that

$$\sum q^n e(\mathfrak{M}_n) = \left(\sum q^n e(\mathfrak{I}_n) \right)^r = \mathsf{M}(q)^{re(X)} \tag{8}$$

where $\mathfrak{I}_n \cong X^{[n]}$ denotes the rank 1 moduli space. The series needed here

$$\sum q^n e(\mathfrak{I}_n) = \mathsf{M}(q)^{e(X)}$$

is computed in [C]. The answer we found is valid whenever δ is good.

4.3 Blow-up surfaces

A slightly more complicated example arises by considering blow-up geometries. Indeed, assume that the surface S contains a (-1)-curve C. Then, $C \hookrightarrow S_0$ is super-rigid in X:

$$N_{C/X} = N_{C/S} \oplus N_{S_0/X}|_C = \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1).$$

We consider the moduli space $\mathfrak{M}_{n,k}$ of rank r modules over X framed by a trivial rank r bundle at infinity, with numerics

$$c_1 = 0,\ c_2 = k[C],\ \chi = r\chi(\mathcal{O}_X) - n.$$

In order not to worry about stability, we assume that δ is good of degree 2. By the argument in the first two paragraphs of Lemma 2A, the torus fixed sheaves in $\mathfrak{M}^{\circ}_{n,k}$ take the form

$$E = \oplus_i I_{Z_i}$$

where Z_i may have at most 1 dimensional components contained in X°. Furthermore,

$$\sum_i [Z_i] = k[C], \quad \sum_i \chi(\mathcal{O}_{Z_i}) = n.$$

In fact, $[Z_i] = k_i[C]$ for non-negative integers k_i adding up to k. Indeed, after projecting to the blowdown surface

$$X \to S \to \bar{S}$$

the classes of the effective curves Z_i add up to 0, hence they must be trivial. This shows that the components of Z_i are supported on the fibers of $X \to S$ or are contained in the Hirzebruch surface

$$\mathbb{F} = \mathbb{P}(K_S|_C \oplus \mathcal{O}_C) \to C.$$

In fact, by torus invariance, all components of Z_i must be supported over fibers or over the zero section $C \hookrightarrow S_0$. Since

$$\sum [Z_i] = k[C]$$

contains no fiber classes, or alternatively since the framing must be an isomorphism along S_∞, we conclude that Z_i has no support over fibers, hence $[Z_i] = k_i[C]$ as claimed.

We carry out the computation of the Donaldson-Thomas Euler characteristics. In the new setting, for all $\mathbb{C}^\star$-fixed sheaves E in $\mathfrak{M}^{\circ}_{n,k}$ we have

$$\dim T_E \mathfrak{M}_{n,k} \equiv rn - k \mod 2.$$

The proof follows that of Lemma 4. The only change is the calculation

$$\dim \operatorname{Ext}^1(I_{Z_i}, I_{Z_i}(-S_\infty)) \equiv \chi(\mathcal{O}_{Z_i}) - k_i \mod 2.$$

To this end, consider the exact sequence

$$0 \to I_{Z_i}(-S_\infty) \to I_{Z_i} \to \mathcal{O}_{S_\infty} \to 0.$$

Since the map $\operatorname{Ext}^0(I_{Z_i}, I_{Z_i}) \to \operatorname{Ext}^0(I_{Z_i}, \mathcal{O}_{S_\infty})$ is an isomorphism, we obtain the exact sequence

$$0 \to \dim \operatorname{Ext}^1(I_{Z_i}, I_{Z_i}(-S_\infty)) \to \operatorname{Ext}^1(I_{Z_i}, I_{Z_i}) \to \operatorname{Ext}^1(I_{Z_i}, \mathcal{O}_{S_\infty}).$$

To find the last group, we use the local to global spectral sequence

$$H^p(Ext^q(I_{Z_i}, \mathcal{O}_{S_\infty})) \implies \operatorname{Ext}^{p+q}(I_{Z_i}, \mathcal{O}_{S_\infty}).$$

The terms with $q \geq 1$ vanish since Z_i avoids S_∞, while the $q = 0$ terms equal $H^p(\mathcal{O}_S)$. Therefore,

$$\operatorname{Ext}^1(I_{Z_i}, \mathcal{O}_{S_\infty}) = H^1(\mathcal{O}_S).$$

From the exact sequence, we conclude

$$\operatorname{Ext}^1(I_{Z_i}, I_{Z_i}(-S_\infty)) = \operatorname{Ext}^1(I_{Z_i}, I_{Z_i})_0.$$

The dimension of the last vector space was found in [BB] using Theorem 2 of [MNOP]. The answer is

$$\operatorname{Ext}^1(I_{Z_i}, I_{Z_i}(-S_\infty)) \equiv \chi(\mathcal{O}_{Z_i}) - k_i \mod 2$$

as claimed above.

We form the generating series

$$\sum_{n,k} \widetilde{\chi}\,(\mathfrak{M}^\circ_{n,k})\, q^n v^k = \left(\sum_{n,k} ((-1)^{r-1} q)^n v^k\, \widetilde{\chi}(\mathfrak{I}^\circ_{n,k}) \right)^r$$

where $\mathfrak{I}^\circ_{n,k}$ denotes the rank 1 framed moduli space. This is isomorphic to the Hilbert scheme. The rank 1 Donaldson-Thomas invariants of super-rigid curves were calculated in [BB]:

$$\sum_{n,k} q^n v^k \widetilde{\chi}(\mathfrak{I}^\circ_{n,k}) = \mathsf{M}(-q)^{e(X^\circ)} \cdot \prod_{m=1}^{\infty} (1 - (-q)^m v)^m.$$

Therefore, we obtain

$$\sum_{n,k} \widetilde{\chi}\,(\mathfrak{M}^\circ_{n,k})\, q^n v^k = \mathsf{M}((-1)^r q)^{re(S)} \cdot \prod_{m=1}^{\infty} (1 - ((-1)^r q)^m v)^{mr}. \tag{9}$$

4.4 Other geometries

There are other geometries for which the above methods apply. We discuss some of them here. Most straightforwardly, assuming S_∞ is a smooth framing divisor with

$$K_X = -2S_\infty,$$

then our techniques yield

$$\sum_n q^n \widetilde{\chi}(\mathfrak{M}_n^\circ) = \mathsf{M}((-1)^r q)^{re(X^\circ)}. \tag{10}$$

In order to make the proof of Lemma 2A work, we need to assume for instance that the restriction

$$\operatorname{Pic}(X) \cap (H^2)^\perp \to \operatorname{Pic}(S_\infty)$$

is injective. If X is Fano of index 2, this requirement is satisfied by the Lefschetz hyperplane theorem applied to the ample class S_∞. Examples pertinent to this setting include, among others:

- X is a cubic in $\mathbb{P}^4$ or a $(2,2)$ complete intersection in $\mathbb{P}^5$, and S_∞ is a hyperplane section;
- X is a double cover of $\mathbb{P}^3$ branched along a quartic, and S_∞ is the pullback of a hyperplane.

Fano threefolds of index higher than 2 also yield symmetric perfect obstruction theories. This can be checked directly using the well-known classification:

- X is a quadric in $\mathbb{P}^4$ or $X = \mathbb{P}^3$, and S_∞ is a hyperplane section.

In index 3, more examples arise from the curve geometry:

- $X = \mathbb{P}(\mathcal{O}_C + E) \to C$, with $E \to C$ any rank 2 bundle of determinant $\det E = K_C$, and S_∞ the divisor at infinity.

Since the same argument works in all cases above, let us only discuss the rank r sheaves over $\mathbb{P}^3$ framed along the plane at infinity $\mathbb{P}^2 \hookrightarrow \mathbb{P}^3$. Along $\mathfrak{M}_n^\circ$, the obstruction theory is symmetric since

$$T_E\mathfrak{M}_n = \operatorname{Ext}^1(E, E(-1)) = \operatorname{Ext}^1(E, E(-3)) = \operatorname{Ext}^2(E, E(-1))^\vee = \operatorname{ob}_E^\vee.$$

The second isomorphism follows from the short exact sequences

$$0 \to E(-k-1) \to E(-k) \to \mathcal{O}_{\mathbb{P}^2}^r(-k) \to 0$$

for $k = 1$ and $k = 2$, and the vanishings

$$\operatorname{Ext}^0(E, \mathcal{O}_{\mathbb{P}^2}(-k)) = \operatorname{Ext}^1(E, \mathcal{O}_{\mathbb{P}^2}(-k)) = 0 \text{ for } k = 1, 2.$$

The first vanishing is clear. The second follows from the local to global spectral sequence:

$$E_2^{p,q} = H^p(Ext^q(E, \mathcal{O}_{\mathbb{P}^2}(-k)) \to \operatorname{Ext}^{p+q}(E, \mathcal{O}_{\mathbb{P}^2}(-k))$$

with vanishing E_2 terms when $p + q = 1$. This proves the claim about the obstruction theory. Equation (10) still holds by the same methods.

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