Cycle Classes for p-Adic Étale Tate Twists and the Image of p-Adic Regulators

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Abstract. In this paper, we construct Chern class maps and cycle class maps with values in p-adic étale Tate twists [Sa2]. We also relate the p-adic étale Tate twists with the finite part of Bloch-Kato. As an application, we prove that the integral part of p-adic regulator maps has values in the finite part of Galois cohomology under certain assumptions.

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1 Introduction

Let p be a rational prime number. Let A be a Dedekind ring whose fraction field has characteristic zero and which has a residue field of characteristic p. Let X be a regular scheme of pure-dimension which is flat of finite type over $S := \mathsf{Spec}(A)$ and which is a smooth or semistable family around its fibers over S of characteristic p. Extending the idea of Schneider [Sc], the author defined in [Sa2] the objects $\mathfrak{T}_n(r)_X$ $(r, n \geq 0)$ of the derived category of étale $\mathbb{Z}/p^n$-sheaves on X playing the role of the r-th Tate twist with $\mathbb{Z}/p^n$-coefficients, which are endowed with a natural product structure with respect to r and both contravariantly and covariantly functorial (i.e., there exist natural pull-back and trace morphisms) for arbitrary separated morphisms of finite type of such schemes. Those pull-back and trace morphisms satisfy the projection formula.

The first aim of this paper is to construct the following Chern class map and cycle class map for $m, r \geq 0$:

$$\begin{tikzcd}
K_m(X) \arrow[d, dotted, "\text{(missing)}"'] \arrow[dr, "\mathsf{c}_{r,m}\ \text{(Chern class)}"] & \\
\mathsf{CH}^r(X,m) \arrow[r, "\mathsf{cl}_X^{r,m}", "\text{(cycle class)}"'] & H^{2r-m}_{\text{ét}}(X, \mathfrak{T}_n(r)_X),
\end{tikzcd}$$

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where $K_m(X)$ denotes the algebraic *K*-group [Q] and $\mathsf{CH}^r(X,m)$ denotes the higher Chow group [B2]. The Chern classes with values in higher Chow groups have not been defined in this arithmetic situation for the lack of a product structure on them, which we do not deal with in this paper. We will prepare the following results to construct the above maps:

(a) *The Dold-Thom isomorphism, i.e., the projective bundle formula for p-adic étale Tate twists. See Theorem* 4.1 *below for details. We note here that the p-adic étale Tate twists do not satisfy homotopy invariance.*

(b) *For a regular closed immersion $i : X_\star \hookrightarrow X'_\star$ of codimension c of simplicial schemes for which the p-adic étale Tate twists are defined, we construct a Gysin morphism*

$$\mathsf{gys}_i : \mathfrak{T}_n(r)_{X_\star} \longrightarrow Ri^!\mathfrak{T}_n(r+c)_{X'_\star}[2c] \quad \textit{in} \quad D((X_\star)_{\text{ét}}, \mathbb{Z}/p^n)$$

satisfying transitivity and projection formula. See Proposition 5.4 *below for details.*

(c) *We introduce a version of p-adic étale Tate twists with log poles along horizontal normal crossing divisors* (*see* §3 *below*)*, and prove a p-adic analogue of the usual homotopy invariance* (*see Corollary* 4.3 *below*) *and a semi-purity property* (*see Theorem* 6.5 *below*) *for this new coefficient.*

The existence of $\mathsf{C}_{r,m}$ will be verified by (a) and the general framework due to Gillet [Gi1]. The additivity of $\mathsf{C}_{r,m}$ for $m \geq 1$ will follow from (b) (cf. §5). On the other hand, we will need the results in (c) to construct $\mathsf{cl}_X^{r,m}$ (cf. §7). This 'higher' cycle class map will be a fundamental object to study in 'higher' higher classfield theory [Sai]. We will mention a local behavior of $\mathsf{cl}_X^{r,m}$ in Remark 7.2 below.

The second aim of this paper is to relate the p-adic étale Tate twists with the finite part of Galois cohomology [BK2], using the Fontaine-Jannsen conjecture proved by Hyodo, Kato and Tsuji ([HK], [K4], [Ts1], cf. [Ni2]). We assume here that A is a p-adic integer ring and that X is projective over A with strict semistable reduction. Let K be the fraction field of A and put $X_K := X \otimes_A K = X[p^{-1}]$. We define

$$H^i(X, \mathfrak{T}_{\mathbb{Q}_p}(r)_X) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_{n \geq 1} H^i_{\text{ét}}(X, \mathfrak{T}_n(r)_X),$$
$$H^i(X_K, \mathbb{Q}_p(r)) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_{n \geq 1} H^i_{\text{ét}}(X_K, \mu_{p^n}^{\otimes r}),$$

where μ_{p^n} denotes the étale sheaf of p^n-th roots of unity on X_K, i.e, the usual Tate twist on X_K. We have a natural restriction map $H^i(X, \mathfrak{T}_{\mathbb{Q}_p}(r)_X) \to H^i(X_K, \mathbb{Q}_p(r))$ and a canonical descending filtration $F^\bullet$ on $H^i(X_K, \mathbb{Q}_p(r))$ resulting from the Hochschild-Serre spectral sequence for the covering $X_{\overline{K}} := X_K \otimes_K \overline{K} \to X_K$ (cf. (9.0.1)). We define a (not necessarily exhaustive) filtration $F^\bullet$ on $H^i(X, \mathfrak{T}_{\mathbb{Q}_p}(r)_X)$ as the inverse image of $F^\bullet$ on $H^i(X_K, \mathbb{Q}_p(r))$, which induces obvious inclusions for $m \geq 0$

$$\mathsf{gr}_F^m H^i(X, \mathfrak{T}_{\mathbb{Q}_p}(r)_X) \hookrightarrow \mathsf{gr}_F^m H^i(X_K, \mathbb{Q}_p(r)) \simeq H^m(K, H^{i-m}(X_{\overline{K}}, \mathbb{Q}_p(r))).$$

Here $H^*(K,-)$ denotes the continuous Galois cohomology of the absolute Galois group $G_K = \mathsf{Gal}(\overline{K}/K)$ defined by Tate [Ta]. We will prove that

$$\mathsf{gr}^1_F H^i(X, \mathfrak{T}_{\mathbb{Q}_p}(r)_X) \subset H^1_f(K, H^{i-1}(X_{\overline{K}}, \mathbb{Q}_p(r))) \quad (f = \text{ finite part})$$

assuming that p is sufficiently large and that the monodromy-weight conjecture [Mo] holds for the log crystalline cohomology of the reduction of X in degree $i-1$ (see Theorem 9.1 below). This result is an extension of the p-adic point conjecture ([Sc], [LS], [Ne1]) to the semistable reduction case and gives an 'unramified' version of results of Langer [La] and Nekovář [Ne2] relating the log syntomic cohomology of X with the geometric part $H^1_g(K, H^{i-1}(X_{\overline{K}}, \mathbb{Q}_p(r)))$.

There is an application of the above results as follows. Let K be a number field and let V be a proper smooth geometrically integral variety over K. Put $\overline{V} := V \otimes_K \overline{K}$. Let i and r be non-negative integers with $2r \geq i+1$, and let p be a prime number. The étale Chern characters (cf. [So]) induce p-adic regulator maps

$$\begin{aligned} \mathsf{reg}_p^{2r-i-1,r} : K_{2r-i-1}(V)_{\mathfrak{o}} &\longrightarrow H^1(K, H^i(\overline{V}, \mathbb{Q}_p(r))) \qquad (2r > i+1), \\ \mathsf{reg}_p^{0,r} : K_0(V)_{\mathsf{hom}} &\longrightarrow H^1(K, H^{2r-1}(\overline{V}, \mathbb{Q}_p(r))) \qquad (2r = i+1). \end{aligned}$$

Here $K_0(V)_{\mathsf{hom}}$ denotes the homologically trivial part of $K_0(V)$, and $K_m(V)_{\mathfrak{o}}$ denotes the integral part of $K_m(V)$ in the sense of Scholl (see §10 below). Motivated by the study of special values of L-functions, Bloch and Kato [BK2] conjecture that the image of $\mathsf{reg}_p^{2r-i-1,r}$ is contained in the finite part $H^1_f(K, H^i(\overline{V}, \mathbb{Q}_p(r)))$ and spans it over $\mathbb{Q}_p$. In the direction of this conjecture, we will prove the following result, which extends a result of Nekovář [Ne2] Theorem 3.1 on $\mathsf{reg}_p^{0,r}$ to the case $2r > i+1$ and extends a result of Niziol [Ni1] on the potentially good reduction case to the general case:

Theorem 1.1 (§10) *Assume $r \leq p-2$ and the monodromy-weight conjecture for the log crystalline cohomology of degree i of projective strict semistable varieties over $\mathbb{F}_p$. Then we have*

$$\mathsf{Im}(\mathsf{reg}_p^{2r-i-1,r}) \subset H^1_f(K, H^i(\overline{V}, \mathbb{Q}_p(r))).$$

Here a projective strict semistable variety over $\mathbb{F}_p$ means the reduction of a regular scheme which is projective flat over a p-adic integer ring with strict semistable reduction.

We use the alteration theorem of de Jong [dJ] to prove Theorem 1.1, and the projective strict semistable varieties concerned in the assumption mean those obtained from alterations of scalar extensions of V to the completion of K at places dividing p.

This paper is organized as follows. In §2, we introduce cohomological and homological logarithmic Hodge-Witt sheaves with horizontal log poles on normal crossing varieties over a field of characteristic $p > 0$. In §3, we define p-adic étale Tate twists with horizontal log poles, and construct a localization sequence using this object (Theorem 3.12). In §4 and §5, we prove the Dold-Thom isomorphisms and define the Chern class maps for p-adic étale Tate twists. The sections 6 and 7 will be devoted to the construction of cycle class maps for p-adic étale Tate twists. In §8 we will introduce Hodge-Witt cohomology and homology of normal crossing varieties and prove that the monodromy-weight conjecture implies a certain invariant cycle theorem. In §9, we establish the comparison between p-adic étale Tate twists and the finite part of Bloch-Kato. We will prove Theorem 1.1 in §10. In the appendix, we will formulate a continuous crystalline cohomology and a continuous syntomic cohomology to prove several technical compatibility results which will have been used in §9.

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Notation

For an abelian group M and a positive integer n, ${}_nM$ and M/n denote the kernel and the cokernel of the map $M \xrightarrow{\times n} M$, respectively. For a field k, $\overline{k}$ denotes a fixed separable closure, and G_k denotes the absolute Galois group $\mathsf{Gal}(\overline{k}/k)$. For a topological G_k-module M, $H^*(k, M)$ denote the continuous Galois cohomology groups $H^*_{\mathsf{cont}}(G_k, M)$ in the sense of Tate [Ta]. If M is discrete, then $H^*(k, M)$ agree with the étale cohomology groups of $\mathsf{Spec}(k)$ with coefficients in the étale sheaf associated with M.

Unless indicated otherwise, all cohomology groups of schemes are taken over the étale topology. For a scheme X, an abelian sheaf $\mathscr{F}$ on $X_{\text{ét}}$ (or more generally an object in the derived category of abelian sheaves on $X_{\text{ét}}$) and a point $x \in X$, we often write $H^*_x(X, \mathscr{F})$ for $H^*_x(\mathsf{Spec}(\mathscr{O}_{X,x}), \mathscr{F})$. For a positive integer m which is invertible on X, we write μ_m for the étale sheaf of the m-th roots of unity on X. For a prime number p which is invertible on X and integers $m, r \geq 0$, we define

$$H^m(X, \mathbb{Z}_p(r)) := \varprojlim_{n\geq 1} H^i(X, \mu_{p^n}^{\otimes r}),$$
$$H^m(X, \mathbb{Q}_p(r)) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^m(X, \mathbb{Z}_p(r)).$$

For an equi-dimensional scheme X and a non-negative integer q, we write X^q for the set of all points on X of codimension q.

For a scheme (or a simplicial scheme) X and an integer $n \geq 2$, *Shv*$(X_{\text{ét}}, \mathbb{Z}/n)$ denotes the category of étale $\mathbb{Z}/n$-sheaves on $X_{\text{ét}}$, and $D(X_{\text{ét}}, \mathbb{Z}/n)$ denotes the derived category of complexes of objects of *Shv*$(X_{\text{ét}}, \mathbb{Z}/n)$.

2 Logarithmic Hodge-Witt sheaves

We first fix the following terminology.

Definition 2.1 (1) *A normal crossing varity over a field k is a pure-dimensional scheme which is separated of finite type over k and everywhere étale locally* *isomorphic to*

$$\mathsf{Spec}\big(k[t_0,\dots,t_N]/(t_0\cdots t_a)\big) \quad \textit{for some } 0\le a\le N:=\dim(Y).$$

(2) *We say that a normal crossing variety Y is simple if all irreducible components of Y are smooth over k.*

(3) *An admissible divisor on a normal crossing varity Y is a reduced effective Cartier divisor D such that the immersion $D\hookrightarrow Y$ is everywhere étale locally isomorphic to*

$$\mathsf{Spec}\big(k[t_0,\dots,t_N]/(t_0\cdots t_a, t_{a+1}\cdots t_{a+b})\big)\hookrightarrow \mathsf{Spec}\big(k[t_0,\dots,t_N]/(t_0\cdots t_a)\big)$$

for some $a,b\ge 0$ with $a+b\le N=\dim(Y)$.

Let p be a prime number, and let k be a field of characteristic p. Let Y be a normal crossing variety over k, and let D be an admissible divisor on Y. Put $V:=Y-D$ and let f and g be as follows:

$$D \xhookrightarrow{\ f\ } Y \xhookleftarrow{\ g\ } V=Y-D.$$

For $r\in\mathbb{Z}$, we define étale sheaves $\nu^r_{(Y,D),n}$ and $\lambda^r_{(Y,D),n}$ on Y as follows:

$$\begin{aligned}\lambda^r_{(Y,D),n} &:= \mathsf{Im}\left(d\log:(g_*\mathscr{O}_V^\times)^{\otimes r}\to \bigoplus\nolimits_{x\in V^0} i_{x*}W_n\Omega^r_{x,\log}\right)\\ \nu^r_{(Y,D),n} &:= \mathsf{Ker}\left(\partial:\bigoplus\nolimits_{x\in V^0} i_{x*}W_n\Omega^r_{x,\log}\to \bigoplus\nolimits_{x\in V^1} i_{x*}W_n\Omega^{r-1}_{x,\log}\right).\end{aligned}$$

Here for $x\in V$, i_x denotes the composite map $x\hookrightarrow V\hookrightarrow Y$, and $W_n\Omega^q_{x,\log}$ denotes the étale subsheaf of the logarithmic part of the Hodge-Witt sheaf $W_n\Omega^q_x$ (cf. [B1], [I1]), which means the zero sheaf for $q<0$. The map ∂ is the sum of boundary maps due to Kato [K1]. By definition, we have

$$\lambda^r_{(Y,D),n}=\nu^r_{(Y,D),n}=0 \quad \text{for} \quad r<0.$$

When $D=\emptyset$, we put

$$\nu^r_{Y,n}:=\nu^r_{(Y,\emptyset),n} \quad \text{and} \quad \lambda^r_{Y,n}:=\lambda^r_{(Y,\emptyset),n},$$

which are considered in [Sa1]. Although we assumed the perfectness of k in [Sa1], the results in loc. cit. §§2.1–2.4 and §§3.1–3.2 are extended to the case that k is not necessarily perfect by results of Shiho [Sh] Theorems 3.2, 4.1 and the compatibility mentioned in [AS] Remark 7.3 (1). We have $\nu^r_{(Y,D),n}=g_*\nu^r_{V,n}$ by the left exactness of g_*, and $\nu^{r-1}_{D,n}\simeq R^1f^!\nu^r_{Y,n}$ by the purity of $\nu^r_{Y,n}$ ([Sa1] Theorem 2.4.2). Hence there is a short exact sequence

$$0\longrightarrow \nu^r_{Y,n}\longrightarrow \nu^r_{(Y,D),n}\longrightarrow f_*\nu^{r-1}_{D,n}\longrightarrow 0. \tag{2.1.1}$$

By this fact, we have

$$\lambda^r_{(Y,D),n} = \nu^r_{(Y,D),n} = g_* W_n\,\Omega^r_{V,\mathsf{log}}$$

if Y is smooth (loc. cit. (2.4.9)), which we denote by $W_n\,\Omega^r_{(Y,D),\mathsf{log}}$. The following proposition will be useful later, where Y is not necessarily smooth:

PROPOSITION 2.2 *Assume that Y is simple, and let $Y_1, Y_2, \cdots, Y_q$ be the distinct irreducible components of Y. Then there is an exact sequence on $Y_{\text{ét}}$*

$$\begin{aligned} 0 \longrightarrow \lambda^r_{(Y,D),n} \xrightarrow{\check{r}^0} \bigoplus_{|I|=1} W_n\,\Omega^r_{(Y_I,D_I),\mathsf{log}} \xrightarrow{\check{r}^1} \bigoplus_{|I|=2} W_n\,\Omega^r_{(Y_I,D_I),\mathsf{log}} \xrightarrow{\check{r}^2} \\ \cdots \xrightarrow{\check{r}^{q-1}} \bigoplus_{|I|=q} W_n\,\Omega^r_{(Y_I,D_I),\mathsf{log}} \longrightarrow 0, \end{aligned}$$

where the notation '$|I| = t$' means that I runs through all subsets of $\{1, 2, \ldots, q\}$ consisting of t elements, and for such $I = \{i_1, i_2, \ldots, i_t\}$ (i_j's are pair-wise distinct), we put

$$Y_I := Y_{i_1} \cap Y_{i_2} \cap \cdots \cap Y_{i_t} \quad \textit{and} \quad D_I := D \times_Y Y_I.$$

The arrow $\check{r}^0$ denotes the natural restriction map. For (I, I') with $I = \{i_1, i_2, \ldots, i_t\}$ ($i_1 < i_2 < \cdots < i_t$) and $|I'| = t + 1$, the (I, I')-factor of $\check{r}^t$ is defined as

$$\begin{cases} 0 & (\textit{if } I \not\subset I') \\ (-1)^{t-a} \cdot (\beta_{I'I})^* & (\textit{if } I' = I \cup \{i_{t+1}\} \textit{ and } i_1 < \cdots < i_a < i_{t+1} < i_{a+1} < \cdots < i_t), \end{cases}$$

where $\beta_{I'I}$ denotes the closed immersion $Y_{I'} \hookrightarrow Y_I$.

We need the following lemma to prove this proposition:

LEMMA 2.3 *Assume that (Y, D, V) fits into cartesian squares of schemes*

$$\begin{tikzcd} D \arrow[r, hook, "f"] \arrow[d, "\iota"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "i"] \arrow[dr, phantom, "\square"] & V \arrow[l, hook', "g"'] \arrow[d, "i'"] \\ \mathscr{D} \arrow[r, hook] & \mathscr{Y} & \mathscr{V} \arrow[l, hook', "h"'] \end{tikzcd}$$

such that $\mathscr{Y}$ is regular, such that the vertical arrows are closed immersions and such that Y, $\mathscr{D}$ and $Y \cup \mathscr{D}$ are simple normal crossing divisors on $\mathscr{Y}$, where we put $\mathscr{V} := \mathscr{Y} - \mathscr{D}$. Then the pull-back map $i^* h_* \mathscr{O}^\times_{\mathscr{V}} \to g_* \mathscr{O}^\times_V$ on $Y_{\text{ét}}$ is surjective.

Proof of Lemma 2.3. We use the same notation as in Proposition 2.2. For a Cartier divisor E on a scheme Z, let $\mathsf{c}^Z_1(E) \in H^1_{|E|}(Z, \mathscr{O}^\times_Z)$ be the localized first Chern class of the invertible sheaf $\mathscr{O}_Z(E)$. Since $\mathscr{Y}$ is regular, $h_*\mathscr{O}^\times_{\mathscr{V}}$ is generated by $\mathscr{O}^\times_{\mathscr{Y}}$ and local uniformizers of the irreducible components $\{\mathscr{D}_j\}_{j\in J}$ of $\mathscr{D}$. Put

$$D_j := Y \times_{\mathscr{Y}} \mathscr{D}_j \qquad (j \in J),$$

which is an admissible divisor on Y. Since $(g_*\mathscr{O}_V^\times)/\mathscr{O}_Y^\times \simeq R^1f^!\mathscr{O}_Y^\times$, it is enough to show that the Gysin map

$$\varphi : \bigoplus_{j\in J} \mathbb{Z}_{D_j} \longrightarrow R^1f^!\mathscr{O}_Y^\times$$

sending $1 \in \mathbb{Z}_{D_j}$ to $\mathsf{c}_1^Y(D_j)$ is bijective on $D_{\text{ét}}$. By [Sa1] Lemma 3.2.2, there is an exact sequence on $Y_{\text{ét}}$

$$0 \longrightarrow \mathscr{O}_Y^\times \xrightarrow{\check{r}^0} \bigoplus_{|I|=1} \mathscr{O}_{Y_I}^\times \xrightarrow{\check{r}^1} \bigoplus_{|I|=2} \mathscr{O}_{Y_I}^\times \xrightarrow{\check{r}^2} \cdots \xrightarrow{\check{r}^{q-1}} \bigoplus_{|I|=q} \mathscr{O}_{Y_I}^\times \longrightarrow 0, \qquad (2.3.1)$$

where $\check{r}$'s are defined in the same way as $\check{r}$'s in Proposition 2.2, and q denotes the number of the distinct irreducible components of Y. Since $f^!\mathscr{O}_{Y_I}^\times = 0$ for any non-empty subset $I \subset \{1,2,\dots,q\}$, the exactness of (2.3.1) implies that of the lower row of the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{j\in J} \mathbb{Z}_{D_j} \arrow[r, "\check{r}^0"] \arrow[d, "\varphi"'] & \bigoplus_{j\in J}\bigoplus_{|I|=1} \mathbb{Z}_{D_j\cap Y_I} \arrow[r, "\check{r}^1"] \arrow[d, "\wr"'] & \bigoplus_{j\in J}\bigoplus_{|I|=2} \mathbb{Z}_{D_j\cap Y_I} \arrow[d, "\wr"'] \\
0 \arrow[r] & R^1f^!\mathscr{O}_Y^\times \arrow[r, "\check{r}^0"] & \bigoplus_{|I|=1} R^1f_1^!\mathscr{O}_{Y_I}^\times \arrow[r, "\check{r}^1"] & \bigoplus_{|I|=2} R^1f_1^!\mathscr{O}_{Y_I}^\times ,
\end{tikzcd}$$

where $D_j \cap Y_I$ is regular for each $j \in J$ and $I \subset \{1,2,\dots,q\}$ by the assumption that $Y \cup \mathscr{D}$ has simple normal crossings on $\mathscr{X}$. The middle and the right vertical arrows are defined in the same way as for φ, and bijective by the standard purity for $\mathscr{O}^\times$ ([Gr1] III §6). Hence φ is bijective as well. $\square$

Proof of Proposition 2.2. Since the problem is étale local on Y, we may assume that (Y, D, V) fits into a diagram as in Lemma 2.3. Then there is an exact sequence

$$W_n\Omega^r_{(\mathscr{Y},\mathscr{D}),\log} \longrightarrow \bigoplus_{|I|=1} W_n\Omega^r_{(Y_I,D_I),\log} \xrightarrow{\check{r}^1} \cdots \xrightarrow{\check{r}^{q-1}} \bigoplus_{|I|=q} W_n\Omega^r_{(Y_I,D_I),\log} \longrightarrow 0$$

on $\mathscr{Y}_{\text{ét}}$ by Lemma 2.3 and induction on the number of components of Y which is similar as for [Sa1] Lemma 3.2.2. The assertion follows from this exact sequence. $\square$

3 p-ADIC ÉTALE TATE TWISTS WITH LOG POLES

In §§3–7, we are mainly concerned with the following setting.

SETTING 3.1 *Let A be a Dedekind domain whose fraction field has characteristic* 0 *and which has a maximal ideal of positive characteristic. Put*

$$S := \mathsf{Spec}(A).$$

Let p be a prime number which is not invertible in A. Let X be a regular scheme which is flat of finite type over A and whose fibers over the closed points of S of characteristic p are empty or reduced normal crossing divisors on X. We write $Y \subset X$ for the union of those fibers, which may be empty.

Let $D \subset X$ be a normal crossing divisor such that $D \cup Y$ has normal crossings on X (D may be empty). Put $U := X - (Y \cup D)$ and $V := Y - (Y \cap D)$, and consider a diagram of immersions

$$\begin{tikzcd} V \arrow[r, hook] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & X - D \arrow[d, "h"] & U \arrow[l, hook'] \arrow[dl, "j"] \\ Y \arrow[r, hook, "i"] & X & \end{tikzcd}$$

Let n and r be positive integers. We first state the Bloch-Kato-Hyodo theorem on the structure of the sheaf $M_n^r := i^*R^r j_*\mu_{p^n}^{\otimes r}$, which will be useful in this paper. We define the étale sheaf $\mathscr{K}_r^{\mathsf{M}}$ on Y as

$$\mathscr{K}_r^{\mathsf{M}} := (i^*j_*\mathscr{O}_U^\times)^{\otimes n}/J,$$

where J denotes the subsheaf of $(i^*j_*\mathscr{O}_U^\times)^{\otimes n}$ generated by local sections of the form $a_1 \otimes a_2 \otimes \cdots \otimes a_r$ $(a_1, a_2, \dots, a_r \in i^*j_*\mathscr{O}_U^\times)$ with $a_s + a_t = 0$ or 1 for some $1 \le s < t \le n$. There is a homomorphism of étale sheaves ([BK1] 1.2)

$$\mathscr{K}_r^{\mathsf{M}} \longrightarrow M_n^r, \tag{3.1.1}$$

which is a geometric version of Tate's Galois symbol map. For local sections $a_1, a_2, \dots, a_r \in i^*j_*\mathscr{O}_U^\times$, we denote the class of $a_1 \otimes a_2 \otimes \cdots \otimes a_r$ in $\mathscr{K}_r^{\mathsf{M}}$ by $\{a_1, a_2, \dots, a_r\}$, and denote the image of $\{a_1, a_2, \dots, a_r\} \in \mathscr{K}_r^{\mathsf{M}}$ under (3.1.1) again by $\{a_1, a_2, \dots, a_r\}$. We define filtrations $\mathscr{U}^\bullet$ and $\mathscr{V}^\bullet$ on M_n^r as follows.

Definition 3.2 *Put* $\mathfrak{p} := \mathsf{Ker}(\mathscr{O}_X \to i_*\mathscr{O}_Y)$ *and* $1 + \mathfrak{p}^q := \mathsf{Ker}(\mathscr{O}_X^\times \to (\mathscr{O}_X/\mathfrak{p}^q)^\times)$ *for* $q \ge 1$.

(1) *We define $\mathscr{U}^0\mathscr{K}_r^{\mathsf{M}}$ as the full sheaf $\mathscr{K}_n^{\mathsf{M}}$. For $q \ge 1$, we define $\mathscr{U}^q\mathscr{K}_r^{\mathsf{M}} \subset \mathscr{K}_r^{\mathsf{M}}$ as the image of $i^*(1 + \mathfrak{p}^q) \otimes (i^*j_*\mathscr{O}_U^\times)^{\otimes r-1}$.*

(2) *For $q \ge 0$, we define $\mathscr{U}^qM_n^r$ as the image of $\mathscr{U}^q\mathscr{K}_r^{\mathsf{M}}$ under the map* (3.1.1)*.*

(3) *When A is local and its residue field k has characteristic p, we fix a prime element $\pi \in A$ and define $\mathscr{V}^qM_n^r \subset M_n^r$ as the part generated by $\mathscr{U}^{q+1}M_n^r$ and the image of $\mathscr{U}^q\mathscr{K}_{r-1}^{\mathsf{M}} \otimes \langle\pi\rangle$ under* (3.1.1)*.*

Let L_{X° be the log structure on X associated with the normal crossing divisor $Y \cup D$ ([K3]), and let L_{Y° be its inverse image log structure onto $Y_{\text{ét}}$ (loc. cit. (1.4)). The following theorem is a variant of theorems of Bloch-Kato-Hyodo ([BK1] Theorem 1.4, [Hy1] Theorem 1.6), and the case $D = \emptyset$ corresponds to their theorems.

Theorem 3.3 (1) *The symbol map* (3.1.1) *is surjective, i.e., $\mathscr{U}^0M_n^r = M_n^r$.*

(2) *Assume that A is local and that its residue field k has characteristic p. Then there are isomorphisms*

$$M_n^r/\mathscr{V}^0 M_n^r \xrightarrow{\sim} W_n\,\omega^r_{Y^\circ,\mathsf{log}}\,,$$
$$\mathscr{V}^0 M_n^r/\mathscr{U}^1 M_n^r \xrightarrow{\sim} W_n\,\omega^{r-1}_{Y^\circ,\mathsf{log}}\,,$$

where $W_n\,\omega^m_{Y^\circ,\mathsf{log}}$ denotes the image of the logarithmic differential map

$$d\mathsf{log} : (L^{\mathsf{gp}}_{Y^\circ})^{\otimes m} \longrightarrow \bigoplus_{y \in Y^0} i_{y*} W_n\,\Omega^m_{y,\mathsf{log}}\,,$$

and for a point $y \in Y$, i_y denotes the natural map $y \hookrightarrow Y$.

(3) *Under the same assumption as in* (2)*, let e be the absolute ramification index of A, and let L_k be the log structure on $\mathsf{Spec}(k)$ associated with the pre-log structure $\mathbb{N} \to k$ sending $1 \mapsto 0$. Put $e' := pe/(p-1)$. Then for $1 \le q < e'$, there are isomorphisms*

$$\mathscr{U}^q M_1^r/\mathscr{V}^q M_1^r \xrightarrow{\sim} \begin{cases} \omega^{r-1}_{Y^\circ}/\mathscr{B}^{r-1}_{Y^\circ} & (p \nmid q), \\ \omega^{r-1}_{Y^\circ}/\mathscr{Z}^{r-1}_{Y^\circ} & (p \mid q), \end{cases}$$
$$\mathscr{V}^q M_1^r/\mathscr{U}^{q+1} M_1^r \xrightarrow{\sim} \omega^{r-2}_{Y^\circ}/\mathscr{Z}^{r-2}_{Y^\circ}\,.$$

Here $\omega^m_{Y^\circ}$ denotes the differential module of (Y, L_{Y°) over $(\mathsf{Spec}(k), L_k)$ ([K3] (1.7)), and $\mathscr{B}^m_{Y^\circ}$ (resp. $\mathscr{Z}^m_{Y^\circ}$) denotes the image of $d : \omega^{m-1}_{Y^\circ} \to \omega^m_{Y^\circ}$ (resp. the kernel of $d : \omega^m_{Y^\circ} \to \omega^{m+1}_{Y^\circ}$).

(4) *Under the same assumption and notation as in* (3)*, we have $\mathscr{U}^q M_1^r = \mathscr{V}^q M_1^r = 0$ for $q \ge e'$.*

Proof. Note that the irreducible components of D are semistable families around the fibers of characteristic p by the assumption that $Y \cup D$ has normal crossings on X. The assertions (1) and (2) are reduced to the case that X is smooth over S and that $D = \emptyset$ (i.e., the Bloch-Kato theorem) by Tsuji's trick in [Ts2] Proof of Theorem 5.1 and a variant of Hyodo's lemma [Hy1] Lemma 3.5, whose details will be explained in a forthcoming paper [KSS]. The assertion (4) follows from [BK1] Lemma 5.1.

We prove (3). Let $\pi \in A$ be the fixed prime element. Let $\overline{\pi} \in L_{Y^\circ}$ be the image of π, and let $[\overline{\pi}] \subset L_{Y^\circ}$ be the subsheaf of monoids generated by $\overline{\pi}$. The quotient $L_{Y^\circ}/[\overline{\pi}]$ is a subsheaf of monoids of $\mathscr{O}_Y$ (with respect to the multiplication of functions) generated by $\mathscr{O}_Y^\times$ and local equations defining D and irreducible components of Y. There is a surjective homomorphism

$$\delta_m : \mathscr{O}_Y \otimes \{(L_{Y^\circ}/[\overline{\pi}])^{\mathsf{gp}}\}^{\otimes m} \longrightarrow \omega^m_{Y^\circ}$$

defined by the local assignment

$$z \otimes y_1 \otimes \cdots \otimes y_m \longmapsto z \cdot d\mathsf{log}(y_1) \wedge \cdots \wedge d\mathsf{log}(y_m),$$

with $z \in \mathscr{O}_Y$ and each $y_i \in (L_{Y^\circ}/[\overline{\pi}])^{\mathsf{gp}}$. The kernel of δ_m is generated by local sections of the following forms (cf. [Hy1] Lemma 2.2):

(1) $z \otimes y_1 \otimes \cdots \otimes y_m$ such that y_s belongs to $\mathscr{O}^{\times}_{\mathsf{Spec}(k)}|_Y$ for some $1 \leq s \leq m$.

(2) $z \otimes y_1 \otimes \cdots \otimes y_m$ such that $y_s = y_t$ for some $1 \leq s < t \leq m$.

(3) $\sum_{i=1}^{\ell} (a_i \otimes a_i \otimes y_1 \otimes \cdots \otimes y_{m-1}) - \sum_{j=1}^{\ell'} (b_j \otimes b_j \otimes y_1 \otimes \cdots \otimes y_{m-1})$ with each $a_i, b_j \in L_{Y^\circ}/[\overline{\pi}]$ such that the sums $\sum_{i=1}^{\ell} a_i$ and $\sum_{j=1}^{\ell'} b_j$ taken in $\mathscr{O}_Y$ belong to $L_{Y^\circ}/[\overline{\pi}]$ and satisfy $\sum_{i=1}^{\ell} a_i = \sum_{j=1}^{\ell'} b_j$.

Hence the assertion follows from the arguments in loc. cit. p. 551. □

We define the étale subsheaf $FM_n^r \subset M_n^r$ as the part generated by $\mathscr{U}^1 M_n^r$ and the image of $(i^* h_* \mathscr{O}_{X-D}^{\times})^{\otimes r}$, where h denotes the open immersion $X - D \hookrightarrow X$. By Theorem 3.3 (2), Proposition 2.2 and the same arguments as in [Sa2] §3.4, we obtain the following theorem:

Theorem 3.4 *There are short exact sequences of sheaves on $Y_{\text{ét}}$*

$$0 \longrightarrow FM_n^r \longrightarrow M_n^r \xrightarrow{\sigma} \nu^{r-1}_{(Y,D\cap Y),n} \longrightarrow 0,$$
$$0 \longrightarrow \mathscr{U}^1 M_n^r \longrightarrow FM_n^r \xrightarrow{\tau} \lambda^r_{(Y,D\cap Y),n} \longrightarrow 0,$$

where σ is induced by the boundary map of Galois cohomology groups due to Kato [K1], *and τ is given by the local assignment*

$$\{a_1, a_2, \ldots, a_r\} \longmapsto d\log(\overline{a_1} \otimes \overline{a_2} \otimes \cdots \otimes \overline{a_r}).$$

Here $a_1, a_2, \ldots, a_r$ are local sections of $i^* h_* \mathscr{O}_{X-D}^{\times}$, and for $a \in i^* h_* \mathscr{O}_{X-D}^{\times}$, $\overline{a}$ denotes its residue class in $g_* \mathscr{O}_V^{\times}$.

Now we define the p-adic étale Tate twists.

Definition 3.5 *For $n \geq 1$ and $r \in \mathbb{Z}$, we define a cochain complex $C_n(r)^{\bullet}_{(X,D)}$ of sheaves on $X_{\text{ét}}$ as follows.*

(1) *For $r = 0$, we define $C_n(0)^{\bullet}_{(X,D)} := \mathbb{Z}/p^n$, the constant sheaf $\mathbb{Z}/p^n$ placed in degree 0. For $r < 0$, we define $C_n(r)^{\bullet}_{(X,D)} := j_! \mathscr{H}om_U(\mu_{p^n}^{\otimes -r}, \mathbb{Z}/p^n)$, the sheaf $j_! \mathscr{H}om_U(\mu_{p^n}^{\otimes -r}, \mathbb{Z}/p^n)$ placed in degree 0.*

(2) *Assume $r \geq 1$, and let $\mathscr{I}^{\bullet}$ be the Godement resolution of $\mu_{p^n}^{\otimes r}$ on $U_{\text{ét}}$. We define $C_n(r)^{\bullet}_{(X,D)}$ as*

$$\begin{aligned} & j_* \mathscr{I}^0 \longrightarrow j_* \mathscr{I}^1 \longrightarrow \cdots \\ & \qquad \longrightarrow j_* \mathscr{I}^{r-1} \longrightarrow \mathsf{Ker}\big(d : j_* \mathscr{I}^r \to j_* \mathscr{I}^{r+1}\big) \xrightarrow{\sigma_n^r} i_* \nu^{r-1}_{(Y,D\cap Y),n}, \end{aligned} \tag{3.5.1}$$

where $j_* \mathscr{I}^0$ is placed in degree 0 and $i_* \nu^{r-1}_{(Y,D\cap Y),n}$ is placed in degree $r+1$. The last arrow σ_n^r is defined as the composite map

$$\sigma_n^r : \mathsf{Ker}\big(d : j_* \mathscr{I}_U^r \to j_* \mathscr{I}_U^{r+1}\big) \longrightarrow\!\!\!\!\!\rightarrow R^r j_* \mu_{p^n}^{\otimes r} \xrightarrow{i_* \sigma} i_* \nu^{r-1}_{(Y,D\cap Y),n},$$

and σ denotes the surjective map in Theorem 3.4.

We often write $\mathfrak{T}_n(r)_{(X,D)}$ for $C_n(r)^\bullet_{(X,D)}$ regarded as an object of $D^b(X_{\text{ét}}, \mathbb{Z}/p^n)$. When $D = \emptyset$, we often write $C_n(r)^\bullet_X$ and $\mathfrak{T}_n(r)_X$ for $C_n(r)^\bullet_{(X,\emptyset)}$ and $\mathfrak{T}_n(r)_{(X,\emptyset)}$, respectively.

PROPOSITION 3.6 *For $r \geq 0$, $\mathfrak{T}_n(r)_{(X,D)}$ is concentrated in $[0, r]$, and there is a distinguished triangle in $D^b(X_{\text{ét}}, \mathbb{Z}/p^n)$*

$$i_*\nu^{r-1}_{(Y,D\cap Y),n}[-r-1] \longrightarrow \mathfrak{T}_n(r)_{(X,D)} \xrightarrow{\ t\ } \tau_{\leq r}Rj_*\mu_{p^n}^{\otimes r} \xrightarrow{\sigma_n^r[-r]} i_*\nu^{r-1}_{(Y,D\cap Y),n}[-r].$$

Proof. The first assertion follows from the surjectivity of σ_n^r in the definition of $C_n(r)^\bullet_{(X,D)}$. The second assertion is straight-forward. $\square$

REMARK 3.7 *Proposition* 3.6 *implies that $\mathfrak{T}_n(r)_X$ defined here agrees with that in* [Sa2] §4 *by a unique isomorphism compatible with the identity map of $\mu_{p^n}^{\otimes r}$ on U* (*loc. cit. Lemma* 4.2.2). *Although we assumed that the residue fields of A of characteristic p are perfect in loc. cit., the results in loc. cit.* §4 – §7.2 *hold true without this assumption by results of Shiho* [Sh] *Theorems* 3.2, 4.1 *and the compatibility result in* [AS] *Remark* 7.3 (1).

PROPOSITION 3.8 *The complex $C_n(r)^\bullet_{(X,D)}$ is contravariantly functorial in the pair (X, D). Here a morphism of pairs $(X, D) \to (X', D')$ means a morphism of schemes $f : X \to X'$ satisfying $f(X - D) \subset X' - D'$.*

Proof. The case $r \leq 0$ is clear. As for the case $r \geq 1$, it is enough to show that the map

$$i_*\sigma : R^r j_*\mu_{p^n}^{\otimes r} \longrightarrow i_*\nu^{r-1}_{(Y,D\cap Y),n}$$

is contravariant in (X, D). Let $f : (X, D) \to (X', D')$ be a morphism of pairs, and consider the following diagram of immersions:

$$Y' \overset{i'}{\hookrightarrow} X' \overset{j'}{\hookleftarrow} U' := X' - D',$$

where Y' denotes the union of the fibers of $X' \to S$ of characteristic p. By the first exact sequence in Theorem 3.4, σ' ($:= \sigma$ for (X', D')) is surjective and $\mathrm{Ker}(i'_*\sigma')$ maps into $\mathrm{Ker}(i_*\sigma)$ under the base-change map

$$f^* : f^*R^r j'_*\mu_{p^n}^{\otimes r} \longrightarrow R^r j_*\mu_{p^n}^{\otimes r}.$$

Hence this map induces a pull-back map

$$f^* : f^*i'_*\nu^{r-1}_{(Y',D'\cap Y'),n} \longrightarrow i_*\nu^{r-1}_{(Y,D\cap Y),n}. \tag{3.8.1}$$

These maps are obviously compatible with σ's and satisfy transitivity. Thus we obtain the proposition. $\square$

Corollary 3.9 (1) *Let $\mathcal{V}$ be the category whose objects are schemes X as in Setting* 3.1 *and whose morphisms are morphisms of schemes. Then the complexes $C_n(r)^\bullet_X = C_n(r)^\bullet_{(X,\emptyset)}$ with $X \in \mathsf{Ob}(\mathcal{V})$ form a complex $C_n(r)^\bullet$ of sheaves on the big étale site $\mathcal{V}_{\text{ét}}$.*

(2) *The Godement resolution $G_n(r)^\bullet_{(X,D)}$ on $X_{\text{ét}}$ of $C_n(r)^\bullet_{(X,D)}$ is contravariantly functorial in (X,D).*

Remark 3.10 *The object $\mathfrak{T}_n(r)_{(X,D)}$ is also contravariantly functorial in the pair (X,D). More precisely, for a morphism of pairs $f : (X,D) \to (X',D')$, there is a natural morphism*

$$f^* : f^*\mathfrak{T}_n(r)_{(X',D')} \longrightarrow \mathfrak{T}_n(r)_{(X,D)} \quad \textit{in} \quad D^b(X_{\text{ét}}, \mathbb{Z}/p^n)$$

by Proposition 3.8, which is, in fact, the unique morphism that extends the pull-back isomorphism of $\mu_{p^n}^{\otimes r}$ for $U \to U'$ (*cf.* [Sa2] *Proposition* 4.2.8).

Remark 3.11 *Let $\mathcal{V}$ and $C_n(r)^\bullet$ be as in Corollary* 3.9*, and let $\mathfrak{T}_n(r)$ be the complex $C_n(r)^\bullet$ regarded as an object of the derived category $D^b(\mathcal{V}_{\text{ét}}, \mathbb{Z}/p^n)$. Let $\mathcal{V}' \subset \mathcal{V}$ be the full subcategory consisting of schemes $X \in \mathsf{Ob}(\mathcal{V})$ with $p^{-1} \in \Gamma(X, \mathscr{O}_X)$. The following facts will be useful later in* §5:

(1) *There exists a unique product structure*

$$\mathfrak{T}_n(q) \otimes^{\mathbb{L}} \mathfrak{T}_n(r) \longrightarrow \mathfrak{T}_n(q+r) \quad \textit{in} \quad D(\mathcal{V}_{\text{ét}}, \mathbb{Z}/p^n)$$

that extends the isomorphism $\mu_{p^n}^{\otimes q} \otimes \mu_{p^n}^{\otimes r} \xrightarrow{\sim} \mu_{p^n}^{\otimes q+r}$ on the big étale site $(\mathcal{V}')_{\text{ét}}$, which follows from the same arguments as in [Sa2] *Proposition* 4.2.6.

(2) *There exists a unique isomorphism*

$$\mathbb{G}_{\mathrm{m}} \otimes^{\mathbb{L}} \mathbb{Z}/p^n[-1] \xrightarrow{\sim} \mathfrak{T}_n(1) \quad \textit{in} \quad D(\mathcal{V}_{\text{ét}}, \mathbb{Z}/p^n)$$

that extends the canonical isomorphism $\mathbb{G}_{\mathrm{m}} \otimes^{\mathbb{L}} \mathbb{Z}/p^n[-1] \longrightarrow \mu_{p^n}$ on $(\mathcal{V}')_{\text{ét}}$, which follows from the same arguments as in loc. cit. Proposition 4.5.1.

Theorem 3.12 *When D is regular, there is a canonical morphism*

$$\delta : \mathfrak{T}_n(r)_{(X,D)} \longrightarrow \alpha_*\mathfrak{T}_n(r-1)_D[-1] \quad \textit{in} \quad D^b(X_{\text{ét}}, \mathbb{Z}/p^n)$$

fitting into a distinguished triangle

$$\alpha_*\mathfrak{T}_n(r-1)_D[-2] \xrightarrow{\alpha_*} \mathfrak{T}_n(r)_X \xrightarrow{\beta^*} \mathfrak{T}_n(r)_{(X,D)} \xrightarrow{\delta} \alpha_*\mathfrak{T}_n(r-1)_D[-1], \quad (3.12.1)$$

where α denotes the closed immersion $D \hookrightarrow X$, and β denotes the natural morphism of pairs $(X,D) \to (X,\emptyset)$. The arrow α_* denotes the Gysin morphism [Sa2] *Theorem* 6.1.3.

Proof. The case $r < 0$ immediately follows from the absolute purity [FG]. To prove the case $r \geq 0$, we first construct the morphism δ. Consider a diagram of immersions

$$\begin{tikzcd} & & U \arrow[dl, "j"'] \arrow[d, "u"] \\ Y \arrow[r, hook, "i"] & X \arrow[r, phantom, "\square"] & X[p^{-1}] \arrow[l, hook', "w"'] \\ E \arrow[u] \arrow[r, hook, "i'"] \arrow[ur, phantom, "\square"] & D \arrow[u, "\alpha"] & D[p^{-1}]. \arrow[u, "v"] \arrow[l, hook', "\varphi"'] \end{tikzcd}$$

There is a distinguished triangle on $X[p^{-1}]_{\text{ét}}$

$$v_*\mu_{p^n}^{\otimes r-1}[-2] \xrightarrow{v_*} \mu_{p^n}^{\otimes r} \xrightarrow{u^*} Ru_*\mu_{p^n}^{\otimes r} \xrightarrow{\delta_1} v_*\mu_{p^n}^{\otimes r-1}[-1],$$

where δ_1 is defined as the composite

$$\delta_1 : Ru_*\mu_{p^n}^{\otimes r} \xrightarrow{-\delta^{\mathsf{loc}}} v_*Rv^!\mu_{p^n}^{\otimes r}[1] \xleftarrow{\sim} v_*\mu_{p^n}^{\otimes r-1}[-1]$$

and δ^{loc} denotes the connecting morphism of a localization sequence. We used the absolute purity [FG] for the last isomorphism. Applying Rw_*, we get a distinguished triangle on $X_{\text{ét}}$

$$\alpha_*R\varphi_*\mu_{p^n}^{\otimes r-1}[-2] \xrightarrow{v_*} Rw_*\mu_{p^n}^{\otimes r} \xrightarrow{u^*} Rj_*\mu_{p^n}^{\otimes r} \xrightarrow{\delta_2} \alpha_*R\varphi_*\mu_{p^n}^{\otimes r-1}[-1]. \qquad (3.12.2)$$

Consider a diagram

$$\begin{tikzcd} \mathfrak{T}_n(r)_{(X,D)} \arrow[r, "t"] \arrow[d, dotted, "\delta"'] & \tau_{\leq r}Rj_*\mu_{p^n}^{\otimes r} \arrow[d, "\delta_3 := \tau_{\leq r}(\delta_2)"] & \\ \alpha_*\mathfrak{T}_n(r-1)_D[-1] \arrow[r] & \tau_{\leq r}(\alpha_*R\varphi_*\mu_{p^n}^{\otimes r-1}[-1]) \arrow[r, "b'"] & \gamma_*\nu_{E,n}^{r-2}[-r], \end{tikzcd}$$

where the lower row is a part of a distinguished triangle (cf. Proposition 3.6 for $(D, \emptyset)$) and γ denotes the composite morphism $\alpha \circ i' : E \hookrightarrow X$. Since $\mathfrak{T}_n(r)_{(X,D)}$ is concentrated in $[0, r]$, we see that the composite $b' \circ \delta_3 \circ t$ is zero by Theorem 3.4 and a simple computation on symbols. On the other hand, we have

$$\mathsf{Hom}_{D^b(X_{\text{ét}},\mathbb{Z}/p^n)}(\mathfrak{T}_n(r)_{(X,D)}, \gamma_*\nu_{E,n}^{r-2}[-r-1]) = 0,$$

again by the fact that $\mathfrak{T}_n(r)_{(X,D)}$ is concentrated in $[0, r]$. Hence there exists a unique morphism δ fitting into the above diagram (cf. [Sa2] Lemma 2.1.2 (1)), which is the desired morphism. Finally the triangle (3.12.1) is distinguished by (3.12.2) and a commutative diagram with exact rows on $X_{\text{ét}}$ ($\gamma = \alpha \circ i'$)

$$\begin{tikzcd} & R^rw_*\mu_{p^n}^{\otimes r} \arrow[r] \arrow[d, "\sigma_X"] & R^rj_*\mu_{p^n}^{\otimes r} \arrow[r, "\delta_3"] \arrow[d, "\sigma_{(X,D)}"] & \alpha_*R^{r-1}\varphi_*\mu_{p^n}^{\otimes r-1} \arrow[r] \arrow[d, "\sigma_D"] & 0 \\ 0 \arrow[r] & i_*\nu_{Y,n}^{r-1} \arrow[r] & i_*\nu_{(Y,E),n}^{r-1} \arrow[r] & \gamma_*\nu_{E,n}^{r-2} \arrow[r] & 0, \end{tikzcd}$$

where the surjectivity of δ_3 in the upper row follows from Theorem 3.3 (1) for $(D, \emptyset)$, and the exactness of the lower row follows from (2.1.1). $\square$

Remark 3.13 *Let ζ_p be a primitive p-th root of unity, and let A' be the normalization of $A[\zeta_p]$. Then all the definitions and results for the pair (X, D) in this section are extended to the scalar extension $(X \otimes_A A', D \otimes_A A')$. Indeed, Theorem* 3.3 (1) *and* (2) *will be proved in* [KSS] *for this generalized situation. One can check Theorem* 3.3 (3) *and* (4) *for $(X \otimes_A A', D \otimes_A A')$ by the same arguments as for (X, D). See* [Sa2] §3.5 *for an argument to extend Theorem* 3.4.

4 Dold-Thom isomorphism

Let S, p and X be as in Setting 3.1, and let n and r be integers with $n \geq 1$. In this section we prove the Dold-Thom isomorphism for p-adic étale Tate twists. Let E be a vector bundle of rank $a+1$ on X, and let $f : \mathbb{P} := \mathbb{P}(E) \to X$ be the associated projective bundle, which is a projective smooth morphism of relative dimension a. Let $\mathscr{O}(1)_E$ be the tautological invertible sheaf on $\mathbb{P}$, and let $\xi \in H^2(\mathbb{P}, \mathfrak{T}_n(1)_{\mathbb{P}})$ be the value of the first Chern class $\mathsf{c}_1(\mathscr{O}(1)_E) \in H^1(\mathbb{P}, \mathscr{O}_{\mathbb{P}}^{\times})$ under the connecting map associated with the Kummer distinguished triangle

$$\mathscr{O}_{\mathbb{P}}^{\times} \longrightarrow \mathscr{O}_{\mathbb{P}}^{\times} \longrightarrow \mathfrak{T}_n(1)_{\mathbb{P}}[1] \longrightarrow \mathscr{O}_{\mathbb{P}}^{\times}[1]$$

(cf. [Sa2] Proposition 4.5.1). The composite morphisms

$$\mathfrak{T}_n(r-q)_X[-2q] \xrightarrow{f^*} Rf_*\mathfrak{T}_n(r-q)_{\mathbb{P}}[-2i] \xrightarrow{-\cup\xi^q} Rf_*\mathfrak{T}_n(r)_{\mathbb{P}} \qquad (0 \leq q \leq a)$$

induce a canonical morphism

$$\gamma_E : \bigoplus_{q=0}^{a} \mathfrak{T}_n(r-q)_X[-2q] \longrightarrow Rf_*\mathfrak{T}_n(r)_{\mathbb{P}} \quad \text{in} \quad D^b(X_{\text{ét}}, \mathbb{Z}/p^n).$$

Theorem 4.1 (Dold-Thom isomorphism) *γ_E is an isomorphism.*

Proof. γ_E is an isomorphism outside of Y by [M] VI Theorem 10.1. The case $r < 0$ follows from this fact and the proper base-change theorem. To prove the case $r \geq 0$, we consider a diagram of schemes

$$\begin{tikzcd} P \arrow[r, hook, "\gamma"] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & \mathbb{P} \arrow[d, "f"] & \\ Y \arrow[r, hook, "i"] & X & X[p^{-1}]. \arrow[l, hook', "j"'] \end{tikzcd}$$

We have to show that $i^*(\gamma_E)$ is an isomorphism:

$$i^*(\gamma_E) : \bigoplus_{q=0}^{a} i^*\mathfrak{T}_n(r-q)_X[-2q] \xrightarrow{\sim} Rg_*\gamma^*\mathfrak{T}_n(r)_{\mathbb{P}}, \tag{4.1.1}$$

where we identified $i^*Rf_*\mathfrak{T}_n(r)_{\mathbb{P}}$ with $Rg_*\gamma^*\mathfrak{T}_n(r)_{\mathbb{P}}$ by the proper base-change theorem. By a standard norm argument (cf. [Sa2] §10.3) using Bockstein triangles (loc. cit. §4.3), we are reduced to the case that $n = 1$ and that $\Gamma(X, \mathscr{O}_X)$ contains a primitive p-th root of unity (see Remark 3.13). We need the following lemma:

LEMMA 4.2 *Let $\overline{\xi} \in H^1(P, \lambda^1_{P,1})$ be the image of ξ under the pull-back map*

$$\gamma^* : H^2(\mathbb{P}, \mathfrak{T}_1(1)_{\mathbb{P}}) \longrightarrow H^1(P, \lambda^1_{P,1})$$

(*cf.* [Sa2] *Proposition* 4.4.10). *Then there are isomorphisms in $D^b(Y_{\text{ét}}, \mathbb{Z}/p)$*

$$\bigoplus_{q=0}^{a} \overline{\xi}^{\,q} \cup - \ : \ \bigoplus_{q=0}^{a} \lambda^{r-q}_{Y,1}[-q] \xrightarrow{\ \sim\ } Rg_*\lambda^r_{P,1} \tag{1}$$

$$\bigoplus_{q=0}^{a} \overline{\xi}^{\,q} \cup - \ : \ \bigoplus_{q=0}^{a} \nu^{r-q}_{Y,1}[-q] \xrightarrow{\ \sim\ } Rg_*\nu^r_{P,1} \tag{2}$$

$$\bigoplus_{q=0}^{a} \overline{\xi}^{\,q} \cup - \ : \ \bigoplus_{q=0}^{a} \omega^{r-q}_{Y}[-q] \xrightarrow{\ \sim\ } Rg_*\omega^r_{P} \tag{3}$$

$$\bigoplus_{q=0}^{a} \overline{\xi}^{\,q} \cup - \ : \ \bigoplus_{q=0}^{a} \mathscr{Z}^{r-q}_{Y}[-q] \xrightarrow{\ \sim\ } Rg_*\mathscr{Z}^r_{P}. \tag{4}$$

Here $\mathscr{Z}^m_Y$ (resp. $\mathscr{Z}^m_P$) denotes the kernel of $d : \omega^m_Y \to \omega^{m+1}_Y$ (resp. $d : \omega^m_P \to \omega^{m+1}_P$).

Proof of Lemma 4.2. Note that $\overline{\xi}$ agrees with the first Chern class of the tautological invertible sheaf on $P = \mathbb{P}(i^*E)$. Since the problems are étale local on Y, we may assume that Y is simple. If Y is smooth, then (1) and (2) are due to Gros [Gr1] I Théorème 2.1.11. The general case is reduced to the smooth case by [Sa1] Proposition 3.2.1, Corollary 2.2.7. As for (3), since we have

$$Rg_*\omega^r_P \simeq \bigoplus_{q=0}^{r} Rg_*(\Omega^q_{P/Y} \otimes_{\mathscr{O}_Y} \omega^{r-q}_Y) \simeq \bigoplus_{q=0}^{r} (Rg_*\Omega^q_{P/Y}) \otimes^{\mathbb{L}}_{\mathscr{O}_Y} \omega^{r-q}_Y$$

by projection formula, the assertion follows from the isomorphisms

$$\overline{\xi}^{\,q} \cup - : \mathscr{O}_Y[-q] \xrightarrow{\ \sim\ } Rg_*\Omega^q_{P/Y} \quad (0 \le q \le a).$$

(4) follows from the same arguments as for [Gr1] I (2.2.3). □

We return to the proof of (4.1.1) for $r \geq 0$. The case $r = 0$ follows from Lemma 4.2 (1) with $r = 0$. To prove the case $r > 1$, we use the objects $\mathbb{K}(r-q)_X \in D^b(Y_{\text{ét}}, \mathbb{Z}/p)$ and $\mathbb{K}(r)_{\mathbb{P}} \in D^b(P_{\text{ét}}, \mathbb{Z}/p)$ defined in [Sa2] Lemma 10.4.1, which fit into distinguished triangles

$$\mathbb{K}(r-q)_X[-1] \to \mu' \otimes^{\mathbb{L}} i^*\mathfrak{T}_1(r-q-1)_X \to i^*\mathfrak{T}_1(r-q)_X \to \mathbb{K}(r-q)_X,$$
$$\mathbb{K}(r)_{\mathbb{P}}[-1] \longrightarrow g^*\mu' \otimes^{\mathbb{L}} \gamma^*\mathfrak{T}_1(r-1)_{\mathbb{P}} \longrightarrow \gamma^*\mathfrak{T}_1(r)_{\mathbb{P}} \longrightarrow \mathbb{K}(r)_{\mathbb{P}}.$$

Here μ' denotes the constant sheaf $i^*j_*\mu_p(\simeq \mathbb{Z}/p)$ on $Y_{\text{ét}}$ and the central arrows are induced by the product structure of Tate twists. By induction on $r \geq 0$, our task is to show that the morphism

$$\bigoplus_{q=0}^{a} \xi^q \cup - \ : \ \bigoplus_{q=0}^{a} \mathbb{K}(r-q)_X[-2q] \longrightarrow Rg_*\mathbb{K}(r)_{\mathbb{P}} \tag{4.2.1}$$

is an isomorphism, where we have used the pull-back morphisms

$$f^* : \mathbb{K}(r-q)_X \longrightarrow \mathbb{K}(r-q)_{\mathbb{P}} \qquad (0 \leq q \leq a)$$

induced by the pull-back morphisms for Tate twists (loc. cit. Proposition 4.2.8, Lemma 2.1.2 (2)). By loc. cit. Lemma 10.4.1 (2), we have

$$\mathscr{H}^m(\mathbb{K}(r-q)_X) \simeq \begin{cases} \mu' \otimes \nu_Y^{r-q-2} & (m = r-q-1) \\ FM_1^{r-q} & (m = r-q) \\ 0 & (\text{otherwise}) \end{cases}$$

and similar facts holds for $\mathbb{K}(r)_P$ (see §3 for FM_n^q). Therefore (4.2.1) is an isomorphism by Lemma 4.2 and Theorems 3.3 (3), (4) and 3.4 with $D = \emptyset$ (see also the projection formula in loc. cit. 4.4.10). This completes the proof of Theorem 4.1. □

The following corollary 4.3 follows immediately from Theorems 3.12, 4.1 and the projection formulra ([Sa2] Corollary 7.2.4), which is a p-adic version of homotopy invariance and plays an important role in our construction of cycle class maps (see §7 below).

Corollary 4.3 *Let E be a vector bundle on X of ranki a, and put $\mathbb{P} = \mathbb{P}(E \oplus \mathbf{1})$, the projective completion of E. Let $\mathbb{P}'$ be the projective bundle $\mathbb{P}(E)$ regarded as the infinite hyperplane, and let $f : \mathbb{P} \to X$ be the natural projection. Then the composite morphism*

$$\mathfrak{T}_n(r)_X \xrightarrow{f^*} Rf_*\mathfrak{T}_n(r)_{\mathbb{P}} \longrightarrow Rf_*\mathfrak{T}_n(r)_{(\mathbb{P},\mathbb{P}')}$$

is an isomorphism in $D^b(X_{\text{ét}}, \mathbb{Z}/p^n)$.

5 Chern class

The main aim of this section is to construct the Chern class map (5.6.1) below. Let S and p be as in Setting 3.1, and let $\mathcal{V}$ be the category whose objects are S-schemes satisfying the conditions in Setting 3.1 for X and whose morphisms are morphisms of schemes. Let $X_\star$ be a simplicial object in $\mathcal{V}$, i.e., a contravariant functor

$$X_\star : \mathit{\Delta}^{\mathsf{op}} \longrightarrow \mathcal{V},$$

where $\mathit{\Delta}$ denotes the simplex category. For a morphism $\gamma : [a] \to [b]$ in $\mathit{\Delta}$, we write

$$\gamma^X : X_b \longrightarrow X_a \qquad (X_a := X_\star([a]))$$

for $X_\star(\gamma)$, which is a morphism in $\mathcal{V}$. For $0 \le i \le a$, let d^i be the coface map in Δ:

$$d^i : [a] \longrightarrow [a+1], \qquad j \mapsto \begin{cases} j & (0 \le j < i) \\ j+1 & (i \le j \le a). \end{cases}$$

For integers $0 \le i \le a$, we often write

$$d_i : X_{a+1} \to X_a$$

for $(d^i)^X$. See [Fr] §1 for the definition of the small étale site $(X_\star)_{\text{ét}}$ on $X_\star$.

DEFINITION 5.1 (1) *For $r \in \mathbb{Z}$, we define a complex $C_n(r)^\bullet_{X_\star}$ of sheaves on $(X_\star)_{\text{ét}}$ by restricting the complex $C_n(r)^\bullet$ on $\mathcal{V}_{\text{ét}}$, cf. Corollary* 3.9. *We will often write $\mathfrak{T}_n(r)_{X_\star}$ for the complex $C_n(r)^\bullet_{X_\star}$ regarded as an object of $D^b((X_\star)_{\text{ét}}, \mathbb{Z}/p^n)$.*

(2) *We define a canonical morphism*

$$\varrho : \mathbb{G}_{\mathrm{m}}[-1] \longrightarrow \mathfrak{T}_n(1)_{X_\star} \quad \text{in} \quad D^b((X_\star)_{\text{ét}})$$

by the composite morphism

$$\mathbb{G}_{\mathrm{m}}[-1] \longrightarrow \mathbb{G}_{\mathrm{m}} \otimes^{\mathbb{L}} \mathbb{Z}/p^n[-1] \xrightarrow{\sim} \mathfrak{T}_n(1)_{X_\star},$$

where the left arrow denotes the canonical morphism induced by $\mathbb{Z} \to \mathbb{Z}/p^n$ and the right arrow is the restriction of the isomorphism in Remark 3.11 (2).

We next review the following basic notions:

DEFINITION 5.2 (1) *A vector bundle over $X_\star$ is a morphism $f : E_\star \to X_\star$ of simplicial schemes such that $f_a : E_a \to X_a$ is a vector bundle for any $a \ge 0$ and such that the commutative diagram*

$$\begin{array}{ccc} E_b & \xrightarrow{f_b} & X_b \\ {\scriptstyle \gamma^E}\downarrow & & \downarrow{\scriptstyle \gamma^X} \\ E_a & \xrightarrow{f_a} & X_a \end{array} \tag{5.2.1}$$

induces an isomorphism $E_b \cong \gamma^{X*}E_a := E_a \times_{X_a} X_b$ of vector bundles over X_b for any morphism $\gamma : [a] \to [b]$ in Δ (cf. [Gi2] *Example* 1.1).

(2) *A regular closed immersion of simplicial schemes is a morphism $i : X_\star \to Y_\star$ such that $i_a : X_a \to Y_a$ is a regular closed immersion for any $a \ge 0$ and such that the diagram*

$$\begin{array}{ccc} X_b & \xrightarrow{i_b} & Y_b \\ {\scriptstyle \gamma^X}\downarrow & & \downarrow{\scriptstyle \gamma^Y} \\ X_a & \xrightarrow{i_a} & Y_a \end{array} \tag{5.2.2}$$

is cartesian for any morphism $\gamma : [a] \to [b]$ in Δ. An effective Cartier divisor $X_\star$ on $Y_\star$ is a regular closed immersion $X_\star \to Y_\star$ of pure codimension 1.

We now define the first Chern classes of effective Cartier divisors and line bundles.

Definition 5.3 (1) *For an effective Cartier divisor $D_\star$ on $X_\star$, we define the first Chern class* $\mathsf{c}_1(D_\star) \in H^2_{D_\star}(X_\star, \mathfrak{T}_n(1))$ *as the value of the first Chern class* $\mathsf{c}_1(D_\star) \in H^1_{D_\star}((X_\star)_{\mathsf{Zar}}, \mathscr{O}^\times)$ *under the composite map*

$$H^1_{D_\star}((X_\star)_{\mathsf{Zar}}, \mathscr{O}^\times) \xrightarrow{\epsilon^*} H^1_{D_\star}(X_\star, \mathbb{G}_{\mathsf{m}}) \xrightarrow{\varrho} H^2_{D_\star}(X_\star, \mathfrak{T}_n(1)),$$

where $\epsilon : (X_\star)_{\text{ét}} \to (X_\star)_{\mathsf{Zar}}$ *denotes the continuous map of small sites. The arrow* ϱ *denotes that in Definition* 5.1 (2).

(2) *For a line bundle $L_\star$ on $X_\star$, we define the first Chern class*

$$\mathsf{c}_1(L_\star) \in H^2(X_\star, \mathfrak{T}_n(1))$$

as the value of the isomorphism class $[L_\star] \in H^1((X_\star)_{\mathsf{Zar}}, \mathscr{O}^\times)$ (*cf.* [Gi2] *Example* 1.1) *under the composite map*

$$H^1((X_\star)_{\mathsf{Zar}}, \mathscr{O}^\times) \xrightarrow{\epsilon^*} H^1(X_\star, \mathbb{G}_{\mathsf{m}}) \xrightarrow{\varrho} H^2(X_\star, \mathfrak{T}_n(1)). \tag{5.3.1}$$

The following proposition plays a key role in the proof of the Whitney sum formula in Proposition 5.5 (3) below.

Proposition 5.4 *Let $f : X_\star \hookrightarrow X'_\star$ be a regular closed immersion of simplicial objects in $\mathcal{V}$ of pure codimension $c \geq 1$. Let r be a non-negative integer. Then there exists a Gysin morphism*

$$\mathsf{gys}_f : \mathfrak{T}_n(r)_{X_\star} \longrightarrow Rf^!\mathfrak{T}_n(r+c)_{X'_\star}[2c] \quad \textit{in} \quad D^+((X_\star)_{\text{ét}}, \mathbb{Z}/p^n)$$

satisfying the following three properties:

(a) (*Consistency with the first Chern class*) *If $r = 1$, then the value of $1 \in \mathbb{Z}/p^n = H^0(X_\star, \mathfrak{T}_n(0))$ under the Gysin map*

$$\mathsf{gys}_f : H^0(X_\star, \mathfrak{T}_n(0)) \longrightarrow H^2_X(X'_\star, \mathfrak{T}_n(1))$$

agrees with the first Chern class $\mathsf{c}_1(X_\star)$ *in Definition* 5.3 (1).

(b) (*Transitivity*) *For another regular closed immersion $g : X'_\star \hookrightarrow X''_\star$ of simplicial objects in $\mathcal{V}$ of pure codimension $c' \geq 1$, the composite morphism*

$$\begin{aligned}\mathfrak{T}_n(r)_{X_\star} \xrightarrow{\mathsf{gys}_f} {} & Rf^!\mathfrak{T}_n(r+c)_{X'_\star}[2c] \\ \xrightarrow{Rf^!(\mathsf{gys}_g)} {} & Rf^!Rg^!\mathfrak{T}_n(r+c+c')_{X''_\star}[2c+2c'] \\ = \!\!= {} & R(g\circ f)^!\mathfrak{T}_n(r+c+c')_{X''_\star}[2(c+c')]\end{aligned}$$

agrees with $\mathsf{gys}_{g\circ f}$.

(c) (*Projection formula*) *The following diagram commutes in* $D((X_\star)_{\text{ét}}, \mathbb{Z}/p^n)$:

$$\begin{CD}
Rf^!\mathfrak{T}_n(q)_{X'_\star} \otimes^{\mathbb{L}} \mathfrak{T}_n(r)_{X_\star} @>{\mathrm{id}\otimes\mathrm{gys}_f}>> Rf^!\mathfrak{T}_n(q)_{X'_\star} \otimes^{\mathbb{L}} Rf^!\mathfrak{T}_n(n+r)_{X'_\star}[2r] \\
@V{\pi}VV @VV{\textit{product}}V \\
\mathfrak{T}_n(q+r)_{X_\star} @>{\mathrm{gys}_f}>> Rf^!\mathfrak{T}_n(q+r+c)_{X'_\star}[2c]\,,
\end{CD}$$

where the left vertical arrow π *is the composite morphism*

$$\begin{aligned} Rf^!\mathfrak{T}_n(q)_{X'_\star} \otimes^{\mathbb{L}} \mathfrak{T}_n(r)_{X_\star} &\xrightarrow{f^*\otimes\mathrm{id}} \mathfrak{T}_n(q)_{X_\star} \otimes^{\mathbb{L}} \mathfrak{T}_n(r)_{X_\star} \\ &\xrightarrow{\textit{product}} \qquad \mathfrak{T}_n(q+r)_{X_\star} \end{aligned}$$

and the product structures on $\{\mathfrak{T}_n(r)_{X_\star}\}_{r\in\mathbb{Z}}$ *and* $\{\mathfrak{T}_n(r)_{X_\star}\}_{r\in\mathbb{Z}}$ *are obtained from that in Remark* 3.11.

Proof. Put $U_\star := X_\star \otimes \mathbb{Z}[p^{-1}]$ and $V_\star := X'_\star \otimes \mathbb{Z}[p^{-1}]$. Let $\varphi : U_\star \hookrightarrow V_\star$ be the regular closed immersion induced by f. By the absolute purity [FG] and the spectral sequence

$$E_1^{a,b} = H^b_{U_a}(V_a, \mu_{p^n}^{\otimes c}) \Longrightarrow H^{a+b}_{U_\star}(V_\star, \mu_{p^n}^{\otimes c}),$$

we have

$$H^{2c}_{U_\star}(V_\star, \mu_{p^n}^{\otimes c}) \simeq \mathsf{Ker}(d_0^* - d_1^* : H^{2c}_{U_0}(V_0, \mu_{p^n}^{\otimes c}) \longrightarrow H^{2c}_{U_1}(V_1, \mu_{p^n}^{\otimes c})).$$

By the definition of regular closed immersions in Definition 5.2 (2), the cycle class $\mathsf{cl}_{V_0}(U_0) \in H^{2c}_{U_0}(V_0, \mu_{p^n}^{\otimes c})$ lies in the group on the right hand side, loc. cit. Proposition 1.1.3. We thus define the cycle class

$$\mathsf{cl}_{V_\star}(U_\star) \in H^{2c}_{U_\star}(V_\star, \mu_{p^n}^{\otimes c})$$

as the element corresponding to $\mathsf{cl}_{V_0}(U_0)$. Since $\varphi^*\mu^{\otimes r}_{p^n,V_\star} \simeq \mu^{\otimes r}_{p^n,U_\star}$ on $(U_\star)_{\text{ét}}$, the cup product with $\mathsf{cl}_{V_\star}(U_\star)$ defines a Gysin morphism

$$\mathsf{gys}_\varphi : \mu^{\otimes r}_{p^n,U_\star} \simeq \varphi^*\mu^{\otimes r}_{p^n,V_\star} \xrightarrow{\mathsf{cl}_{V_\star}(U_\star)\,\cup\,-} R\varphi^!\mu^{\otimes r+c}_{p^n,V_\star}[2c] \quad \text{in} \quad D^+((U_\star)_{\text{ét}}, \mathbb{Z}/p^n),$$

which satisfies the three properties (a) – (c) listed above (see loc. cit. Proposition 1.2.1 for (b)). We show that there exists a unique morphism

$$\mathsf{gys}_f : \mathfrak{T}_n(r)_{X_\star} \longrightarrow Rf^!\mathfrak{T}_n(r+c)_{X'_\star}[2c] \quad \text{in} \quad D^+((X_\star)_{\text{ét}}, \mathbb{Z}/p^n)$$

that extends gys_φ. Put

$$Y_\star := X_\star \otimes \mathbb{Z}/p\mathbb{Z}, \quad \mathfrak{L} := \mathfrak{T}_n(r)_{X_\star} \quad \text{and} \quad \mathfrak{M} := Rf^!\mathfrak{T}_n(r+c)_{X'_\star}[2c].$$

and let α and β be as follows:

$$U_\star \overset{\beta}{\hookrightarrow} X_\star \overset{\alpha}{\hookleftarrow} Y_\star\,.$$

Consider the following diagram in $D^+((X_\star)_{\text{ét}}, \mathbb{Z}/p^n)$ whose lower row is distinguished:

$$\begin{tikzcd}
 & \mathfrak{L} \arrow[r, "t"] & \tau_{\leq r} R\beta_* \mu_{p^n}^{\otimes r} \arrow[d, "R\beta_*(\mathsf{gys}_\varphi)"] & \\
\alpha_* R\alpha^! \mathfrak{M} \arrow[r, "\alpha_*"] & \mathfrak{M} \arrow[r, "\beta^*"] & R\beta_*\beta^*\mathfrak{M} \arrow[r, "-\delta"] & \alpha_* R\alpha^! \mathfrak{M}[1].
\end{tikzcd} \tag{5.4.1}$$

Here the upper horizontal arrow is the canonical morphism (cf. Propositions 3.6 and 3.8), and the lower row is the localization distinguished triangle for $\mathfrak{M}$ (cf. [Sa2] (1.9.2)). We have

$$\tau_{\leq r} \alpha_* R\alpha^! \mathfrak{M} = 0 \tag{5.4.2}$$

by the purity in loc. cit. Theorem 4.4.7, which implies that

$$\begin{aligned}
&\mathsf{Hom}_{D^+((X_\star)_{\text{ét}}, \mathbb{Z}/p^n)}(\tau_{\leq r} R\beta_* \mu_{p^n}^{\otimes r}, \alpha_* R\alpha^! \mathfrak{M}[1]) \\
&\qquad\qquad = \mathsf{Hom}_{\mathit{Shv}((X_\star)_{\text{ét}}, \mathbb{Z}/p^n)}(R^r\beta_* \mu_{p^n}^{\otimes r}, \alpha_* R^{r+1}\alpha^! \mathfrak{M}),
\end{aligned}$$

where $\mathit{Shv}((X_\star)_{\text{ét}}, \mathbb{Z}/p^n)$ denotes the category of étale sheaves of $\mathbb{Z}/p^n$-modules on $X_\star$. By this fact and the compatibility fact in loc. cit. Theorem 6.1.1, one can easily check that the composite $(-\delta) \circ R\beta_*(\mathsf{gys}_\varphi)$ is zero in $D^+((X_\star)_{\text{ét}}, \mathbb{Z}/p^n)$. Therefore we obtain a unique morphism gys_f that extends gys_φ again by (5.4.2) and by loc. cit. Lemma 2.1.2 (1). The property (a) of gys_f is straight-forward, and the property (b) follows from the uniqueness of gys_f. The property (c) follows from the same argument as for loc. cit. Corollary 7.2.4. □

Following the method of Grothendieck [G] and Gillet [Gi1], we define Chern classes

$$\mathsf{c}(E_\star) = (\mathsf{c}_i(E_\star))_{i \geq 0} \in \bigoplus_{i \geq 0} H^{2i}(X_\star, \mathfrak{T}(i))$$

of a vector bundle $E_\star$ over $X_\star$ as follows. Let $E_\star$ be of rank a, and let f be the natural projection $\mathbb{P}(E_\star) \to X_\star$. Let $\xi \in H^2(\mathbb{P}(E_\star), \mathfrak{T}_n(1))$ be the value of the first Chern class of the tautological line bundle on $\mathbb{P}(E_\star)$, cf. Definition 5.3 (2). Noting the Dold-Thom isomorphism

$$\bigoplus_{i=1}^{a} H^{2i}(X_\star, \mathfrak{T}_n(i)) \simeq H^{2a}(\mathbb{P}(E_\star), \mathfrak{T}_n(a)), \quad (b_i)_{i=1}^a \mapsto \sum_{i=1}^{a} f^*(b_i) \cup \xi^{a-i} \tag{5.4.3}$$

obtained from Theorem 4.1 and [Gi1] Lemma 2.4, we define

$$\mathsf{c}_0(E_\star) := 1 \quad \text{and} \quad \mathsf{c}_i(E_\star) := 0 \quad \text{for} \quad i > a,$$

and define $\mathsf{c}_i = \mathsf{c}_i(E_\star)$ for $i = 1, 2, \ldots, a$ by the equation

$$\xi^a + f^*(\mathsf{c}_1) \cup \xi^{a-1} + \cdots + f^*(\mathsf{c}_{a-1}) \cup \xi + f^*(\mathsf{c}_a) = 0$$

in $H^{2a}(\mathbb{P}(E_\star), \mathfrak{T}_n(a))$.

Proposition 5.5 *Let $X_\star$ be a simplicial object in $\mathcal{V}$. Then the Chern classes $\mathsf{c}(E_\star)$ of vector bundles $E_\star$ over $X_\star$ satisfy the following properties*:

(1) (*Normalization*) *If $E_\star$ is a line bundle, then we have $\mathsf{c}_0(E_\star) = 1$ and $\mathsf{c}_i(E_\star) = 0$ for $i > 1$, and $\mathsf{c}_1(E_\star)$ is the value of the first Chern class $\mathsf{c}_1(E_\star) \in H^1((X_\star)_{\mathsf{Zar}}, \mathscr{O}^\times)$ under the map* (5.3.1).

(2) (*Functoriality*) *For a morphism $f : X_\star \to X'_\star$ of simplicial objects in $\mathcal{V}$ and a vector bundle $E_\star$ on $X'_\star$, we have*

$$\mathsf{c}(f^*E_\star) = f^*\mathsf{c}(E_\star),$$

where f^* on the right hand side denotes the pull-back map obtained from the contravariant functoriality of $\mathfrak{T}_n(r)$, cf. Remark 3.10.

(3) (*Whitney sum*) *For a short exact sequence $0 \to E'_\star \to E_\star \to E''_\star \to 0$ of vector bundles on $X_\star$, we have*

$$\mathsf{c}_r(E_\star) = \sum_{s+t=r} \mathsf{c}_s(E'_\star) \cup \mathsf{c}_t(E''_\star) \in H^{2r}(X_\star, \mathfrak{T}_n(r))$$

for each $r \geq 0$, where the cup product is taken with respect to the product structure of $\{\mathfrak{T}_n(r)_{X_\star}\}_{r\geq 0}$ in the derived category of $\mathbb{Z}/p^n$-sheaves on $(X_\star)_{\text{ét}}$, cf. Remark 3.11 (1).

Moreover, the Chern classes $\mathsf{c}(E_\star)$ are characterized by these three properties.

Proof. The properties (1) and (2) immediately follow from this definition of Chern classes. The last assertion on the uniqueness follows from the splitting principle of vector bundles, whose details are straight-forward and left to the reader.

We prove the property (3) using the arguments of Grothendieck in [G] p. 144 Theorem 1 (iii), as follows. Let $\pi' : D'_\star \to X_\star$ and $\pi'' : D''_\star \to X_\star$ be the (simplicial) flag schemes of $E'_\star$ and $E''_\star$, respectively, and put

$$D_\star := D'_\star \times_{X_\star} D''_\star ,$$

which is identified with the flag scheme of $\pi''^*E'_\star$ over $D''_\star$. Let $f : D_\star \to X_\star$ be the natural projection. Since the pull-back map

$$f^* : H^{2i}(X_\star, \mathfrak{T}_n(i)) \longrightarrow H^{2i}(D_\star, \mathfrak{T}_n(i)),$$

is injective by (5.4.3), we may replace $(X_\star, E_\star, E'_\star, E''_\star)$ with $(D_\star, f^*E_\star, f^*E'_\star, f^*E''_\star)$ to assume that $E_\star$ has a filtration by subbundles

$$E_\star = E^0_\star \supset E^1_\star \supset \cdots \supset E^a_\star = 0 \qquad (a := \mathsf{rank}(E_\star))$$

such that the quotient $E^i_\star/E^{i+1}_\star$ is a line bundle over $X_\star$ for $0 \leq i \leq a-1$ and such that $E^b_\star = E'_\star$ for $b = \mathsf{rank}(E''_\star)$. Now let

$$g : X'_\star := \mathbb{P}(E_\star) \longrightarrow X_\star$$

be the projective bundle associated with $E_\star$. Let $L_\star$ be the tautological line bundle over $X' = \mathbb{P}(E_\star)$ and let $s : X'_\star \to g^*E_\star \otimes L_\star =: F_\star$ be the section induced by the canonical inclusion $(L_\star)^\vee \hookrightarrow g^*E_\star$:

$$s : X'_\star \xrightarrow{1} \mathbb{A}^1_{X'_\star} \longrightarrow g^*E_\star \otimes L_\star = F_\star\,.$$

Put $F^i_\star := g^*E^i_\star \otimes L$ and $V^i_\star := s^{-1}(F^i_\star)$ for $0 \le i \le a$. Then $V^a_\star$ is empty because s does not vanish. Moreover $V^i_\star$ is a simplicial object in $\mathcal{V}$ and the section

$$s_i : V^i_\star \xrightarrow{s} F^i_\star|_{V^i_\star} \longrightarrow (F^i_\star/F^{i+1}_\star)|_{V^i_\star}$$

meets the zero section transversally for $0 \le i \le a-1$, cf. [G] p. 147. On the other hand, to prove the Whitney sum formula, it is enough to show

$$\prod_{i=0}^{m+n-1} \mathsf{c}_1(F^i_\star/F^{i+1}_\star) = 0.$$

We are thus reduced to the following simplicial analogue of loc. cit. p.141 Lemma 2:

LEMMA 5.6 *Let $X_\star$ be a simplicial object in $\mathcal{V}$. Let $E_\star$ be a vector bundle of rank a over $X_\star$. Let $(E^i_\star)_{0\le i\le a}$ be a descending filtration consisting of subbundles on $E_\star$ such that $E^i_\star$ has rank $a-i$. For $1 \le i \le a$, put*

$$\xi_i := \mathsf{c}_1(E^{i-1}_\star/E^i_\star) \in H^2(X_\star, \mathfrak{T}_n(1)).$$

Let $s : X_\star \to E_\star$ be a section of $E_\star \to X_\star$. For $0 \le i \le a$, put

$$V^i_\star := s^{-1}(E^i_\star), \qquad L^i_\star := (E^i_\star/E^{i+1}_\star)|_{V^i_\star} \quad (\textit{restriction of } E^i_\star/E^{i+1}_\star \textit{ onto } V^i_\star),$$

and let $s_i : V^i_\star \to L^i_\star$ be the section induced by s. Assume the following condition:

- *$V^i_\star$ is a simplicial object in $\mathcal{V}$ for $0 \le i \le a$, and s_i intersects the zero section transversally for $0 \le i \le a-1$.*

Then we have

$$\mathsf{cl}_{X_\star}(V^a_\star) = \prod_{i=1}^{a} \xi_i \quad \textit{in} \quad H^{2a}(X_\star, \mathfrak{T}_n(a))\,,$$

where $\mathsf{cl}_{X_\star}(V^a_\star)$ denotes the value of 1 under the Gysin map

$$\mathbb{Z}/p^n = H^0(V^a_\star, \mathfrak{T}_n(0)) \longrightarrow H^{2a}(X_\star, \mathfrak{T}_n(a))\,.$$

One can easily check this lemma by the properties of the Gysin morphisms in Proposition 5.4 and the arguments in loc. cit. p.141 Lemma 2. This completes the proof of Proposition 5.5. □

Now let X be a scheme which belongs to $\mathcal{V}$. Applying the construction of Chern classes to the case $X_\star = \mathsf{B}_\star\mathsf{GL}_{s,X}$ and $E_\star =$ universal rank s bundle over $\mathsf{B}_\star\mathsf{GL}_s/X$, we obtain Chern classes

$$\mathsf{c}_j(E_\star) \in H^{2j}(\mathsf{B}_\star\mathsf{GL}_{s,X}, \mathfrak{T}_n(j)) \qquad (j \ge 0),$$

which are called the *universal rank s Chern classes*. On the other hand, let $I_n(j)^{\bullet}_X$ be an injective resolution of the complex $C_n(j)^{\bullet}_X$ on $X_{\text{ét}}$ defined in Definition 3.5, and consider the following complex of abelian sheaves on X_{Zar}:

$$\cdots \longrightarrow \epsilon_* I_n(j)^{2j-2}_X \xrightarrow{d^{2j-2}} \epsilon_* I_n(j)^{2j-1}_X \xrightarrow{d^{2j-1}} \mathsf{Ker}(d^{2j} : \epsilon_* I_n(j)^{2j}_X \to \epsilon_* I_n(j)^{2j+1}_X).$$

Here $\epsilon : X_{\text{ét}} \to X_{\mathsf{Zar}}$ denotes the continuous map of small sites, and we regarded this sequence as a chain complex with the most right term placed in degree 0. We apply the Dold-Puppe construction [DP] to this complex to obtain a sheaf of simplicial abelian groups, which we denote by $\mathsf{K}(\mathfrak{T}_n(j), 2j)$. For a closed subset Z of X and non-negative integers $i, r \geq 0$, we define the Chern class map

$$\mathsf{C}^Z_{r,i} : K^Z_i(X) \longrightarrow H^{2r-i}_Z(X, \mathfrak{T}_n(r)) \tag{5.6.1}$$

as the following composite map (cf. [Gi1] Definition 2.22):

$$\begin{aligned} K^Z_i(X) &\longrightarrow H^{-i}_Z(X_{\mathsf{Zar}}, \mathbb{Z} \times \mathbb{Z}_\infty \mathsf{B}_\star \mathsf{GL}(\mathscr{O}_X)) \\ &\xrightarrow{\mathsf{pr}} H^{-i}_Z(X_{\mathsf{Zar}}, \mathbb{Z}_\infty \mathsf{B}_\star \mathsf{GL}(\mathscr{O}_X)) \\ &\xrightarrow{\pi_i(\mathbb{Z}_\infty \mathsf{c}_r)} H^{-i}_Z(X_{\mathsf{Zar}}, \mathbb{Z}_\infty \mathsf{K}(\mathfrak{T}_n(r), 2r)) \cong H^{2r-i}_Z(X_{\text{ét}}, \mathfrak{T}_n(r)). \end{aligned}$$

Here $\mathbb{Z}_\infty$ denotes the Bousfield-Kan completion [BoK], and we have used the universal rank s Chern class $\mathsf{c}_r(E_\star)$ for a sufficiently large s to define the arrow $\pi_i(\mathbb{Z}_\infty \mathsf{c}_r)$. See [Gi1] Proposition 2.15 for the first arrow and see loc. cit. p. 226 for the last isomorphism. The map $\mathsf{C}^X_{r,0}$ agrees with c_r for $X_\star = X$ (constant simplicial scheme) defined before.

Theorem 5.7 (1) *$\mathsf{C}^Z_{r,i}$ is contravariantly functorial in the pair (X, Z), that is, for a morphism $f : X \to X'$ in $\mathcal{V}$ and a closed subset $Z' \subset X'$ with $f^{-1}(Z') \subset Z$, there is a commutative diagram*

$$\begin{array}{ccc} K^{Z'}_i(X') & \xrightarrow{\mathsf{C}^{Z'}_{r,i}} & H^{2r-i}_{Z'}(X', \mathfrak{T}_n(r)) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ K^Z_i(X) & \xrightarrow{\mathsf{C}^Z_{r,i}} & H^{2r-i}_Z(X, \mathfrak{T}_n(r)). \end{array}$$

(2) *$\mathsf{C}^Z_{r,i}$ is additive for $i > 0$.*

(3) *The induced Chern character*

$$\mathsf{ch} : \bigoplus_{i \geq 0} K_i(X) \longrightarrow \prod_{i,r \geq 0} H^i(X, \mathfrak{T}_{\mathbb{Q}_p}(r))$$

with $H^i(X, \mathfrak{T}_{\mathbb{Q}_p}(r)) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_{n \geq 1} H^i(X, \mathfrak{T}_n(r))$ is a ring homomorphism.

Proof. (1) follows from the functoriality results in Proposition 5.5 (2) and Proposition 3.8. The assertion (2) follows from Proposition 5.5 (3) and [Gi1] Lemma 2.26. See loc. cit. Definition 2.34 and Proposition 2.35 for the assertion (3). □

6 Purity along log poles

This and the next section are devoted to the construction of cycle class maps from higher Chow groups to p-adic étale Tate twists.

Let k be a field of characteristic $p > 0$, and let Y be a normal crossing variety over k. Let c be a positive integer. Let D be a non-empty admissible divisor on Y, and let Z be a reduced closed subscheme of Y which has codimension $\geq c$ and contained in D. Let $i : Z \hookrightarrow Y$ be the natural closed immersion. In this section, we first prove the following purity result:

Theorem 6.1 (Purity) *We have $R^q i^! \nu^r_{(Y,D),n} = 0$ for $q \leq c$.*

See Theorem 6.5 below for a consequence of this theorem. Note that the assertion for $q = c$ does not follow directly from Gros' purity [Gr1] II Théorème 3.5.8, even when Y is smooth. The case that Y is smooth has been considered and proved independently by Mieda ([Mi1] Proof of Theorem 2.4, [Mi2]). Although the proof of Theorem 6.1 given below is essentially a variant of Gros' proof of [Gr1] II Théorème 3.5.8, we include detailed computations for the convenience of the reader.

Proof of Theorem 6.1. Since the problem is étale local, we may assume that Y is simple and that there exists a pair $(\mathscr{Y}, \mathscr{D})$ satisfying the following (1) and (2):

(1) $\mathscr{Y}$ is smooth over k and contains Y as a normal crossing divisor,

(2) $\mathscr{D}$ is a normal crossing divisor on $\mathscr{Y}$ such that $Y \cup \mathscr{D}$ has normal crossings on $\mathscr{Y}$ and such that $Y \cap \mathscr{D} = D$.

Then we have a short exact sequence on $\mathscr{Y}_{\text{ét}}$

$$0 \longrightarrow W_n\Omega^{r+1}_{(\mathscr{Y},\mathscr{D}),\log} \longrightarrow W_n\Omega^{r+1}_{(\mathscr{Y},Y\cup\mathscr{D}),\log} \longrightarrow \nu^r_{(Y,D),n} \longrightarrow 0$$

(a variant of (2.1.1)), which reduces the assertion to the case that Y is smooth over k and that D has normal crossings on Y. Furthermore, we may assume that Z is *regular of pure codimension* c by a standard devissage argument. In what follows, we prove

$$R^q i^! W_n\Omega^r_{(Y,D),\log} = 0 \quad \text{for} \quad q \neq c+1, \tag{6.1.1}$$

assuming that Y is smooth over k (and that Z is regular and contained in D). By the short exact sequences on $Y_{\text{ét}}$

$$0 \longrightarrow W_{n-1}\Omega^r_{(Y,D),\log} \longrightarrow W_n\Omega^r_{(Y,D),\log} \longrightarrow \Omega^r_{(Y,D),\log} \longrightarrow 0,$$
$$0 \longrightarrow \Omega^r_{(Y,D),\log} \longrightarrow \mathscr{Z}^r_Y(\log D) \xrightarrow{1-C} \Omega^r_Y(\log D) \longrightarrow 0$$

(cf. (2.1.1) and [Sa1] Corollary 2.2.5 (2), Lemma 2.4.6), it is enough to show that

$$R^q i^! \mathscr{Z}^r_Y(\log D) = 0 = R^q i^! \Omega^r_Y(\log D) \quad \text{for} \quad q \neq c, \tag{6.1.2}$$
$$1 - C : R^c i^! \mathscr{Z}^r_Y(\log D) \longrightarrow R^c i^! \Omega^r_Y(\log D) \quad \text{is injective.} \tag{6.1.3}$$

The assertion (6.1.2) follows from the smoothness of Y and the fact that $\Omega^r_Y(\log D)$ and $\mathscr{Z}^r_Y(\log D)$ are locally free over $\mathscr{O}_Y$ and $(\mathscr{O}_Y)^p$, respectively.

We prove (6.1.3) in what follows. Since the problem is étale local on Z, we may assume the following condition:

($\star$) *Y is affine, and there exists a regular sequence $t_1, \cdots, t_c \in \Gamma(Y, \mathscr{O}_Y)$ such that the ideal $(t_1, \cdots, t_c) \subset \mathscr{O}_Y$ defines Z and such that $t_1, \cdots, t_a$ $(1 \le a \le c)$ are uniformizers of the irreducible components of D. Moreover, Ω^1_Y is free over $\mathscr{O}_Y$ and has a basis $\{dt_\lambda\}_{\lambda\in\Lambda}$ which contains $dt_1, \ldots, dt_c$.*

Let τ be the natural open immersion

$$\tau : Y[t_1^{-1}, \ldots, t_c^{-1}] \hookrightarrow Y.$$

For $j = 1, \ldots, c$, let σ_j be the natural open immersion

$$\sigma_j : Y[t_1^{-1}, \ldots, t_{j-1}^{-1}, t_{j+1}^{-1}, \ldots, t_c^{-1}] \hookrightarrow Y.$$

We recall here the following standard fact (cf. [Gr1] II (3.3.6)):

Lemma 6.2 *We have isomorphisms of sheaves on $Z_{\text{ét}}$*

$$R^c i^! \Omega^r_Y(\log D) \simeq \tau_*\tau^*\Omega^r_Y(\log D) \Big/ \textstyle\sum_{j=1}^c \ \sigma_{j*}\sigma_j^*\Omega^r_Y(\log D) \tag{6.2.1}$$
$$= \tau_*\tau^*\Omega^r_Y \Big/ \Big(\textstyle\sum_{j=1}^a \ \sigma_{j*}\sigma_j^*\Omega^r_Y(\log D) + \sum_{j=a+1}^c \ \sigma_{j*}\sigma_j^*\Omega^r_Y \Big),$$
$$R^c i^! \mathscr{Z}^r_Y(\log D) \simeq \tau_*\tau^*\mathscr{Z}^r_Y(\log D) \Big/ \textstyle\sum_{j=1}^c \ \sigma_{j*}\sigma_j^*\mathscr{Z}^r_Y(\log D) \tag{6.2.2}$$
$$= \tau_*\tau^*\mathscr{Z}^r_Y \Big/ \Big(\textstyle\sum_{j=1}^a \ \sigma_{j*}\sigma_j^*\mathscr{Z}^r_Y(\log D) + \sum_{j=a+1}^c \ \sigma_{j*}\sigma_j^*\mathscr{Z}^r_Y \Big),$$

where we regarded the sheaves on the right hand side as sheaves on $Z_{\text{ét}}$ naturally.

We define an ascending filtration Fil_m $(m \ge 0)$ on $\tau_*\tau^*\Omega^r_Y$ as

$$\mathsf{Fil}_m(\tau_*\tau^*\Omega^r_Y) := \left\{ \frac{1}{(t_1 t_2 \cdots t_c)^{pm}}\, \omega \in \tau_*\tau^*\Omega^r_Y \;\middle|\; \omega \in \Omega^r_Y \right\}.$$

Let $\mathsf{Fil}_\bullet(R^c i^! \Omega^r_Y(\log D))$ be the induced filtration, and let $\mathsf{Fil}_\bullet(R^c i^! \mathscr{Z}^r_Y(\log D))$ be its inverse image under the canonical map

$$R^c i^! \mathscr{Z}^r_Y(\log D) \longrightarrow R^c i^! \Omega^r_Y(\log D)) \tag{6.2.3}$$

induced by the natural inclusion $\mathscr{Z}^r_Y(\log D) \hookrightarrow \Omega^r_Y(\log D))$. Note that

$$R^c i^! \mathscr{Z}^r_Y(\log D) = \bigcup_{m=0}^{\infty} \mathsf{Fil}_m(R^c i^! \mathscr{Z}^r_Y(\log D)).$$

We prove here the following lemma.

Lemma 6.3 (1) *The map* (6.2.3) *is injective.*

(2) $\mathsf{Fil}_m(R^c i^! \mathscr{Z}_Y^r(\log D))$ *is generated by elements of the form*

$$\left[\frac{1}{(t_1t_2\cdots t_c)^{pm}}\,\omega\right] \quad \textit{with} \quad \omega \in \mathscr{Z}_Y^r,$$

where $[-]$ *denotes the residue class in* $\mathsf{Fil}_m(R^c i^! \mathscr{Z}_Y^r(\log D))$ *via* (6.2.2)*.*

(3) *The kernel of the projection via* (6.2.1)

$$\mathsf{Fil}_m(\tau_*\tau^*\Omega_Y^r) \longrightarrow \mathsf{Fil}_m(R^c i^! \Omega_Y^r(\log D))$$

agrees with the subgroup

$$\left\{ \sum_{j=1}^{c} \frac{1}{(t_1\cdots t_{j-1}t_{j+1}\cdots t_c)^{pm}}\,\omega_j \;\middle|\; \begin{array}{ll} \omega_j \in \Omega_Y^r(\log D_j) & (1 \le j \le a) \\ \omega_j \in \Omega_Y^r & (a < j \le c) \end{array} \right\},$$

where $D_j \subset Y$ *denotes the regular divisor defined by* t_j *for* $1 \le j \le a$*.*

Proof. We prove (1). By the short exact sequence

$$0 \longrightarrow \mathscr{Z}_Y^r(\log D) \longrightarrow \Omega_Y^r(\log D) \xrightarrow{d} \mathscr{B}_Y^{r+1}(\log D) \longrightarrow 0,$$

we have a long exact sequence on $Z_{\text{ét}}$

$$\cdots \longrightarrow R^{c-1}i^! \mathscr{B}_Y^{r+1}(\log D) \longrightarrow R^c i^! \mathscr{Z}_Y^r(\log D) \longrightarrow R^c i^! \Omega_Y^r(\log D) \longrightarrow \cdots .$$

The assertion follows from the smoothness of Y and the fact that $\mathscr{B}_Y^{r+1}(\log D)$ is locally free over $(\mathscr{O}_Y)^p$. The assertion (2) is a consequence of (1), and (3) is straightforward. $\square$

We return to the proof of (6.1.3), and compute the map $1-C$ in (6.1.3):

$$1 - C : R^c i^! \mathscr{Z}_Y^r(\log D) \longrightarrow R^c i^! \Omega_Y^r(\log D)), \tag{6.3.1}$$

using the filtration $\mathsf{Fil}_\bullet$. Put

$$\mathsf{gr}_m(-) := \mathsf{Fil}_m(-)/\mathsf{Fil}_{m-1}(-).$$

Since we have

$$C\left(\frac{1}{(t_1t_2\cdots t_c)^{pm}}\,\omega\right) = \frac{1}{(t_1t_2\cdots t_c)^{m}}\,C(\omega) \quad \text{for} \quad \omega \in \mathscr{Z}_Y^r,$$

the Cartier operator preserves $\mathsf{Fil}_\bullet$ (by Lemma 6.3 (2)) and induces a map

$$\mathsf{gr}_m(C) : \mathsf{gr}_m(R^c i^! \mathscr{Z}_Y^r(\log D)) \longrightarrow \mathsf{gr}_m(R^c i^! \Omega_Y^r(\log D)),$$

which is the zero map for $m \geq 2$. Hence we have

$$\text{the kernel of (6.3.1)} \subset \mathsf{Fil}_1(R^c i^! \mathscr{Z}_Y^r(\log D)).$$

Let $D_j \subset Y$ be as we defined in Lemma 6.3 (3) for $1 \leq j \leq a$. Fix an arbitrary $\omega \in \mathscr{Z}_Y^r$, and assume that

$$x := \left[\frac{1}{(t_1 t_2 \cdots t_c)^p} \, \omega \right] \in \mathsf{Fil}_1(R^c i^! \mathscr{Z}_Y^r(\log D))$$

belongs to the kernel of (6.3.1). To show (6.1.3), we have to prove that

$$\omega = \sum_{j=1}^{c} t_j^p \, \eta_j \quad \text{for some } \begin{cases} \eta_j \in \Omega_Y^r(\log D_j) & (1 \leq j \leq a) \\ \eta_j \in \Omega_Y^r & (a < j \leq c) \end{cases} \tag{6.3.2}$$

in $\Omega_Y^r(\log D)$, which is equivalent to that $x = 0$ in $R^c i^! \mathscr{Z}_Y^r(\log D)$ by Lemma 6.3 (1) and (6.2.1). Since $x = C(x)$ in $\mathsf{Fil}_1(R^c i^! \Omega_Y^r(\log D))$ by assumption, we have

$$\frac{1}{(t_1 t_2 \cdots t_c)^p} \, \omega - \frac{1}{t_1 t_2 \cdots t_c} \, C(\omega) = \sum_{j=1}^{c} \frac{1}{(t_1 t_2 \cdots t_{j-1} t_{j+1} \cdots t_c)^p} \, \alpha_j$$

in $\tau_* \tau^* \Omega_Y^r$ for some $\alpha_j \in \Omega_Y^r(\log D_j)$ $(1 \leq j \leq a)$ and $\alpha_j \in \Omega_Y^r$ $(a < j \leq c)$ by Lemma 6.3 (3). This implies

$$\omega = (t_1 t_2 \cdots t_c)^{p-1} C(\omega) + \sum_{j=1}^{c} t_j^p \, \alpha_j \quad \text{ in } \quad \Omega_Y^r, \tag{6.3.3}$$

and our task is to prove that

$$(t_1 t_2 \cdots t_c)^{p-1} C(\omega) = \sum_{j=1}^{c} t_j^p \, \zeta_j \quad \text{for some } \begin{cases} \zeta_j \in \Omega_Y^r(\log D_j) & (1 \leq j \leq a) \\ \zeta_j \in \Omega_Y^r & (a < j \leq c). \end{cases} \tag{6.3.4}$$

Take a basis $\{dt_\lambda\}_{\lambda \in \Lambda}$ of Ω_Y^1 over $\mathscr{O}_Y$ which contain $dt_1, \dots, dt_c$ (see the condition $(\star)$). Fix an ordering on Λ and let J be the set of all r-tuples $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_r)$ of elements of Λ satisfying

$$\lambda_1 < \lambda_2 < \cdots < \lambda_r.$$

For $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_r) \in J$, put

$$dt_{\boldsymbol{\lambda}} := dt_{\lambda_1} \wedge dt_{\lambda_2} \wedge \cdots \wedge dt_{\lambda_r} \in \Omega_Y^r.$$

Using the basis $\{dt_{\boldsymbol{\lambda}}\}_{\boldsymbol{\lambda} \in J}$ of Ω_Y^r over $\mathscr{O}_Y$, we decompose $C(\omega)$ as

$$C(\omega) = \epsilon + \delta \qquad (\epsilon, \delta \in \Omega_Y^r). \tag{6.3.5}$$

Here ϵ is an $\mathscr{O}_Y$-linear combination of $dt_{\boldsymbol{\lambda}}$'s which contain dt_j for some $1 \le j \le a$, and δ is an $\mathscr{O}_Y$-linear combination of $dt_{\boldsymbol{\lambda}}$'s which contain none of $dt_1, \dots, dt_a$. We have

$$(t_1 t_2 \cdots t_c)^{p-1} \epsilon = \sum_{j=1}^{a} t_j^p \, \beta_j \quad \text{for some } \beta_j \in \Omega_Y^r(\log D_j) \quad (1 \le j \le a), \tag{6.3.6}$$

which implies (6.3.4) if $\delta = 0$. Otherwise, we proceed as follows. Put

$$y := (t_2 t_3 \cdots t_c)^{p-1} \delta \in \Omega_Y^r.$$

Since $d\omega = 0$, the equalities (6.3.3) and (6.3.6) imply

$$t_1^{p-2} \, dt_1 \wedge y = t_1^{p-1} \, dy + \sum_{j=1}^{c} t_j^p \, \beta_j' \quad \text{for some } \begin{cases} \beta_j' \in \Omega_Y^r(\log D_j) & (1 \le j \le a) \\ \beta_j' \in \Omega_Y^r & (a < j \le c). \end{cases}$$

By this equality and the assumption on δ, one can easily check that

$$t_1^{p-2} \, dt_1 \wedge y = \sum_{j=1}^{c} t_j^p \, \beta_j'' \quad \text{for some } \begin{cases} \beta_1'' \in \Omega_Y^r(\log D_1) & \\ \beta_j'' \in \Omega_Y^r & (2 \le j \le c). \end{cases} \tag{6.3.7}$$

To proceed with the proof of (6.3.4), we need the following lemma:

LEMMA 6.4 *Let f and g be the following $\mathscr{O}_Y$-linear maps, respectively*:

$$f : \Omega_Y^r \longrightarrow \Omega_Y^{r+1}, \quad z \mapsto dt_1 \wedge z,$$
$$g : \Omega_Y^r(\log D_1) \longrightarrow \Omega_Y^{r+1}(\log D_1), \quad z \mapsto \frac{dt_1}{t_1} \wedge z.$$

Then the following holds:

(1) *For $m \ge 1$, we have*

$$\mathrm{Im}(f) \cap \left(t_1^m \, \Omega_Y^{r+1}(\log D_1) + \sum_{j=2}^{c} t_j^m \, \Omega_Y^{r+1} \right) = t_1^m \, \mathrm{Im}(g) + \sum_{j=2}^{c} t_j^m \, \mathrm{Im}(f).$$

(2) *For $m, n \ge 0$, we have*

$$t_1^n \, \Omega_Y^{r+1} \cap \left(\sum_{j=2}^{c} t_j^m \, \mathrm{Im}(f) \right) = \sum_{j=2}^{c} t_1^n \, t_j^m \, \mathrm{Im}(f).$$

(3) *For $m \ge 0$, the sequence*

$$t_1^m \, \Omega_Y^{r-1} \xrightarrow{\ f\ } t_1^m \, \Omega_Y^r \xrightarrow{\ f\ } t_1^m \, \Omega_Y^{r+1}$$

is exact.

Proof. These assertions follow from linear algebra over $\mathscr{O}_Y$. The details are straightforward and left to the reader. $\square$

By (6.3.7) and Lemma 6.4 (1) for $m = p$, we have

$$t_1^{p-2}\, dt_1 \wedge y = t_1^{p-1}\, dt_1 \wedge \gamma_1 + \sum_{j=2}^{c} t_j^p\, dt_1 \wedge \gamma_j \tag{6.4.1}$$

for some $\gamma_1 \in \Omega_Y^r(\log D_1)$ and some $\gamma_j \in \Omega_Y^r$ $(2 \le j \le c)$. If $p = 2$, this equality implies

$$t_1^{p-2}\, dt_1 \wedge y = t_1^{p-2}\, dt_1 \wedge \left(t_1\gamma_1 + \sum_{j=2}^{c} t_j^p\, \gamma_j \right). \tag{6.4.2}$$

If $p \ge 3$, we obtain (6.4.2) from (6.4.1) by replacing $\gamma_2, \dots, \gamma_c$ suitably in Ω_Y^r $(2 \le j \le c)$, where we used Lemma 6.4 (2) for $(m, n) = (p, p-2)$. Noting that $t_1\gamma_1$ and $t_j^p\, \gamma_j$ $(2 \le j \le c)$ belong to Ω_Y^r, we have

$$t_1^{p-2}\, y = t_1^{p-2} dt_1 \wedge \vartheta + t_1^{p-1}\gamma_1 + \sum_{j=2}^{c} t_j^p\, (t_1^{p-2}\gamma_j) \quad \text{for some } \vartheta \in \Omega_Y^{r-1}$$

by (6.4.2) and Lemma 6.4 (3) for $m = p - 2$. Thus we have

$$(t_1 t_2 \cdots t_c)^{p-1}\delta = t_1^{p-1}\, y = t_1^p \left(\frac{dt_1}{t_1} \wedge \vartheta + \gamma_1 \right) + \sum_{j=2}^{c} t_j^p\, (t_1^{p-1}\gamma_j),$$

which implies (6.3.4) (see also (6.3.5), (6.3.6)). This completes the proof of (6.3.2), (6.1.3) and Theorem 6.1. $\square$

THEOREM 6.5 *Let S, p and X be as in Setting* 3.1*, and let c be a positive integer. Let D be a non-empty normal crossing divisor on X, and let Z be a closed subset of X which has codimension $\ge c$. Put $U := X - D$. Then we have*

$$H_Z^q(X, \mathfrak{T}_n(r)_{(X,D)}) \simeq \begin{cases} 0 & (q < r + c) \\ H_{Z\cap U}^{r+c}(U, \mathfrak{T}_n(r)_U) & (q = r + c). \end{cases} \tag{6.5.1}$$

In particular, when Z has pure codimension r on X, we have

$$H_Z^q(X, \mathfrak{T}_n(r)_{(X,D)}) \simeq \begin{cases} 0 & (q < 2r) \\ \mathbb{Z}/p^n[Z^0 \cap U] & (q = 2r), \end{cases} \tag{6.5.2}$$

where $\mathbb{Z}/p^n[Z^0 \cap U]$ means the free $\mathbb{Z}/p^n$-module generated by the set $Z^0 \cap U$.

Proof. Admitting (6.5.1), we obtain (6.5.2) from the purity of $\mathfrak{T}_n(r)_U$ ([Sa2] Theorem 4.4.7, Corollary 4.4.9) and the absolute purity of $\mu_{p^n}^{\otimes r}$ on $U[p^{-1}]$ ([FG]). To show (6.5.1), we divide the problem into the following 4 cases:

(1) $Z \subset D \cap Y$ (2) $Z \subset D$ and $Z \not\subset Y$

(3) Z is arbitrary and $q < r + c$ (4) Z is arbitrary and $q = r + c$.

The case (1) follows from Theorems 3.3, 6.1 (see also Remark 3.13) and the same arguments as in [Sa2] Theorem 4.4.7. The case (2) follows from the case (1) and the same arguments as in loc. cit. Corollary 4.4.9. The case (3) also follows from similar arguments as for the previous cases. To prove the case (4), write Z as the union of closed subsets

$$Z = Z_1 \cup Z_2,$$

where Z_1 has pure codimension c on X, Z_2 has codimension $\geq c + 1$ on X and we suppose that no irreducible components of Z_2 are contained in Z_1. Since $Z_1 \cap Z_2$ has codimension $\geq c + 2$ on X, the assertion in the cases (1)–(3) and a Mayer-Vietoris long exact sequence

$$\begin{aligned} \cdots \longrightarrow H^{r+c}_{Z_1 \cap Z_2}(X, \mathfrak{T}_n(r)_{(X,D)}) \longrightarrow \bigoplus_{j=1,2} H^{r+c}_{Z_j}(X, \mathfrak{T}_n(r)_{(X,D)}) \\ \longrightarrow H^{r+c}_{Z}(X, \mathfrak{T}_n(r)_{(X,D)}) \longrightarrow H^{r+c+1}_{Z_1 \cap Z_2}(X, \mathfrak{T}_n(r)_{(X,D)}) \longrightarrow \cdots \end{aligned}$$

imply isomorphisms

$$H^{r+c}_{Z}(X, \mathfrak{T}_n(r)_{(X,D)}) \simeq \bigoplus_{j=1,2} H^{r+c}_{Z_j}(X, \mathfrak{T}_n(r)_{(X,D)}) \simeq H^{r+c}_{Z_1}(X, \mathfrak{T}_n(r)_{(X,D)}).$$

Similarly, we have $H^{r+c}_{Z \cap U}(U, \mathfrak{T}_n(r)_U) \simeq H^{r+c}_{Z_1 \cap U}(U, \mathfrak{T}_n(r)_U)$ by the purity of $\mathfrak{T}_n(r)_U$ ([Sa2] Corollary 4.4.9). Hence we may assume that Z has pure codimension c on X. Moreover we may assume that no irreducible components of Z are contained in D by the cases (1)+(2) and a similar devissage argument. Then noting that $Z \cap D$ has codimension $\geq c + 1$ on X, we obtain the case (4) from the cases (1)+(2) and a long exact sequence

$$\begin{aligned} \cdots \longrightarrow H^{r+c}_{Z \cap D}(X, \mathfrak{T}_n(r)_{(X,D)}) \longrightarrow H^{r+c}_{Z}(X, \mathfrak{T}_n(r)_{(X,D)}) \longrightarrow H^{r+c}_{Z \cap U}(U, \mathfrak{T}_n(r)_U) \\ \longrightarrow H^{r+c+1}_{Z \cap D}(X, \mathfrak{T}_n(r)_{(X,D)}) \longrightarrow \cdots . \end{aligned}$$

This completes the proof of Theorem 6.5. $\square$

DEFINITION 6.6 *Let X and D be as in Theorem* 6.5*, and let C be a cycle on $U := X - D$ of codimension r. Let $W \subset U$ be the support of C and let $\overline{W}$ be its closure in X. Then we define the cycle class*

$$\mathsf{cl}_X(C) \in H^{2r}_{\overline{W}}(X, \mathfrak{T}_n(r)_{(X,D)})$$

as the inverse image of $\mathsf{cl}_U(C)$ ([Sa2] 5.1.2) *under the isomorphism in Theorem* 6.5

$$H^{2r}_{\overline{W}}(X, \mathfrak{T}_n(r)_{(X,D)}) \xrightarrow{\ \sim\ } H^{2r}_{W}(U, \mathfrak{T}_n(r)_U).$$

7 Cycle class map

Let S, p and X be as in Setting 3.1. In this section, we construct a cycle class morphism

$$\mathsf{cl}_X : \mathbb{Z}(r)_X^{\text{ét}} \otimes^{\mathbb{L}} \mathbb{Z}/p^n \longrightarrow \mathfrak{T}_n(r)_X \quad \text{in} \quad D(X_{\text{ét}}, \mathbb{Z}/p^n), \tag{7.0.1}$$

following the method of Bloch [B3] §4 (cf. [GL] §3, see also Remark 7.3 below). Here $\mathbb{Z}(r)_X^{\text{ét}}$ denotes the étale sheafification on X of the presheaf of cochain complexes

$$U \longmapsto z^r(U, \star)[-2r],$$

and $z^r(U, \star)$ denotes Bloch's cycle complex

$$\cdots \longrightarrow \underset{(\text{degree } -q)}{z^r(U, q)} \xrightarrow{d_q} \underset{(\text{degree } -q+1)}{z^r(U, q-1)} \xrightarrow{d_{q-1}} \cdots \xrightarrow{d_1} \underset{(\text{degree } 0)}{z^r(U, 0)}.$$

We review the definition of this complex briefly, which will be useful later. Let $\Delta^\star$ be the standard cosimplicial scheme over $\mathsf{Spec}(\mathbb{Z})$:

$$\Delta^q := \mathsf{Spec}(\mathbb{Z}[t_0, t_1, \ldots, t_q]/(t_0 + t_1 + \cdots + t_q = 1)).$$

A face of Δ^q (of codimension $a \geq 1$) is a closed subscheme defined by the equation

$$t_{i_1} = t_{i_2} = \cdots = t_{i_a} = 0 \quad \text{for some} \quad 0 \leq i_1 < i_2 < \cdots < i_a \leq q.$$

Now $z^r(U, q)$ is defined as the free abelian group generated by the set of all integral closed subschemes on $U \times \Delta^q$ of codimension r which meet all faces of $U \times \Delta^q$ properly. Here a face of $U \times \Delta^q$ means the product of U and a face of Δ^q. Noting that the faces of codimension 1 are effective Cartier divisors, we define the differential map d_q as the alternating sum of pull-back maps along the faces of codimension 1, which defines the above complex $z^r(U, \star)$.

We fix a projective completion $\overline{\Delta^q}$ of Δ^q as follows:

$$\overline{\Delta^q} := \mathsf{Proj}\,(\mathbb{Z}[T_0, T_1, \ldots, T_q, T_\infty]/(T_0 + T_1 + \cdots + T_q = T_\infty)).$$

Let $D^q \subset \overline{\Delta^q}$ be the hyperplane at infinity:

$$D^q : T_\infty = 0.$$

The following proposition plays a key role in our construction of the morphism (7.0.1).

Proposition 7.1 *Let q and r be integers with $q, r \geq 0$, and let U be étale of finite type over X. Let $\Sigma^{r,q}$ be the set of all closed subsets on $U \times \Delta^q$ of pure codimension r which meet the faces of $U \times \Delta^q$ properly. For $W \in \Sigma^{r,q}$, let $\overline{W}$ be the closure of W in $U \times \overline{\Delta^q}$. Then:*

(1) *There is an isomorphism*

$$\mathsf{cl}^{r,q} : z^r(U, q) \otimes \mathbb{Z}/p^n \xrightarrow{\sim} \varinjlim_{W \in \Sigma^{r,q}} H^{2r}_{\overline{W}}\left(U \times \overline{\Delta^q}, \mathfrak{T}_n(r)_{(U \times \overline{\Delta^q}, U \times D^q)}\right)$$

sending a cycle $C \in z^r(U, q)$ to the cycle class $\mathsf{cl}_{U \times \overline{\Delta^q}}(C)$ (see Definition 6.6).

(2) *For $W \in \Sigma^{r,q}$, the natural morphism*

$$\tau_{\leq 2r}\, R\Gamma_{\overline{W}}\left(U \times \overline{\Delta^q}, \mathfrak{T}_n(r)_{(U\times\overline{\Delta^q}, U\times D^q)}\right) \longrightarrow H^{2r}_{\overline{W}}\left(U \times \overline{\Delta^q}, \mathfrak{T}_n(r)_{(U\times\overline{\Delta^q}, U\times D^q)}\right)[-2r]$$

is an isomorphism in the derived category of $\mathbb{Z}/p^n$-modules.

(3) *Let C be a cycle which belongs to $z^r(U, q)$, and let W be the support of C (note that W belongs to $\Sigma^{r,q}$). Let $\overline{i} : U \times \overline{\Delta^{q-1}} \hookrightarrow U \times \overline{\Delta^q}$ be the closure of a face map $i : U \times \Delta^{q-1} \hookrightarrow U \times \Delta^q$. Then the pull-back map*

$$\overline{i}^* : H^{2r}_{\overline{W}}\left(U \times \overline{\Delta^q}, \mathfrak{T}_n(r)_{(U\times\overline{\Delta^q}, U\times D^q)}\right) \longrightarrow H^{2r}_{\overline{i}^{-1}(\overline{W})}\left(U \times \overline{\Delta^{q-1}}, \mathfrak{T}_n(r)_{(U\times\overline{\Delta^{q-1}}, U\times D^{q-1})}\right)$$

sends the cycle class $\mathsf{cl}_{U\times\overline{\Delta^q}}(C)$ to $\mathsf{cl}_{U\times\overline{\Delta^{q-1}}}(i^*C)$, where i^*C denotes the pull-back of the cycle C along i.

Proof. (1) and (2) follow from Theorem 6.5 and the definition of $z^r(U, q)$. We show (3). By Theorem 6.5, it is enough to show that the pull-back map

$$i^* : H^{2r}_W(U \times \Delta^q, \mathfrak{T}_n(r)_{U\times\Delta^q}) \longrightarrow H^{2r}_{i^{-1}(W)}\left(U \times \Delta^{q-1}, \mathfrak{T}_n(r)_{U\times\Delta^{q-1}}\right)$$

sends the cycle class $\mathsf{cl}_{U\times\Delta^q}(C)$ to $\mathsf{cl}_{U\times\Delta^{q-1}}(i^*C)$. Put

$$\mathscr{U} := U \times \Delta^q, \quad \mathscr{D} := i(U \times \Delta^{q-1}) \subset \mathscr{U} \quad \text{and} \quad \mathscr{V} := \mathscr{U} - \mathscr{D}$$

and let $t \in \Gamma(\mathscr{U}, \mathscr{O}_{\mathscr{U}})$ be a defining equation of $\mathscr{D}$. Noting that $i^{-1}(W) = W \cap \mathscr{D}$, consider the following diagram:

$$\begin{tikzcd}
H^{2r}_W(\mathscr{U}, \mathfrak{T}_n(r)_{\mathscr{U}}) \arrow[r, "i^*"] \arrow[d, "\alpha"'] & H^{2r}_{W\cap\mathscr{D}}(\mathscr{D}, \mathfrak{T}_n(r)_{\mathscr{D}}) \arrow[d, "i_*", "\wr"'] \\
H^{2r+1}_{W\cap\mathscr{V}}(\mathscr{V}, \mathfrak{T}_n(r+1)_{\mathscr{V}}) \arrow[r, "-\delta"] \arrow[d] & H^{2r+2}_{W\cap\mathscr{D}}(\mathscr{U}, \mathfrak{T}_n(r+1)_{\mathscr{U}}) \arrow[d, "\wr"] \\
\bigoplus_{y\in W^0} H^{2r+1}_y(\mathscr{V}, \mathfrak{T}_n(r+1)_{\mathscr{V}}) \arrow[r, "-\delta"] & \bigoplus_{x\in(W\cap\mathscr{D})^0} H^{2r+2}_x(\mathscr{U}, \mathfrak{T}_n(r+1)_{\mathscr{U}}) \\
\bigoplus_{y\in W^0} \kappa(y)^\times/p^n \arrow[r, "\mathsf{div}"] \arrow[u, "g", "\wr"'] & \bigoplus_{x\in(W\cap\mathscr{D})^0} \mathbb{Z}/p^n. \arrow[u, "g'"', "\wr"]
\end{tikzcd}$$

Here we defined α by sending $\omega \in H^{2r}_W(\mathscr{U}, \mathfrak{T}_n(r)_{\mathscr{U}})$ to $\{t|_{\mathscr{V}}\} \cup \omega|_{\mathscr{V}}$, where $\{t|_{\mathscr{V}}\}$ denotes the class of $t|_{\mathscr{V}} \in \Gamma(\mathscr{V}, \mathscr{O}^\times_{\mathscr{V}})$ in $H^1(\mathscr{V}, \mathfrak{T}_n(1)_{\mathscr{V}})$. The arrows i_*, g and g' are

Gysin isomorphisms (cf. [Sa2] Theorem 4.4.7). The arrows δ are boundary maps of localization long exact sequences, and the central square commutes obviously. The arrow div denotes the divisor map, and the bottom square commutes by the compatibility in loc. cit. Theorem 6.1.1 and [JSS] Theorem 3.1.1 (see also Remark 3.7 of this paper). The top square commutes by the following equalities ($\omega \in H^{2r}_W(\mathscr{U}, \mathfrak{T}_n(r)_{\mathscr{U}})$):

$$i_* \circ i^*(\omega) = \mathsf{cl}_{\mathscr{U}}(\mathscr{D}) \cup \omega = -\delta(\{t|_{\mathscr{V}}\}) \cup \omega = -\delta(\{t|_{\mathscr{V}}\} \cup \omega|_{\mathscr{V}}) = -\delta \circ \alpha(\omega),$$

where the first (resp. second) equality follows from the projection formula in [Sa2] Corollary 7.2.4 (resp. the same compatibility as mentioned before). Hence the assertion follows from the transitivity of Gysin maps for $W \cap \mathscr{D} \hookrightarrow \mathscr{D} \hookrightarrow \mathscr{U}$ (loc. cit. Corollary 6.3.3) and the fact that the divisor map div sends $t|_y \in \kappa(y)^\times/p^n$ to the cycle $i^*[y]$, where $[y]$ means the cycle on $\mathscr{U}$ given by the closure of y. This completes the proof. $\square$

We construct the morphism (7.0.1). For U as in the proposition, there is a diagram of cochain complexes concerning $\bullet$ (see Corollary 3.9 (2) for $G_n(r)^\bullet$):

$$\begin{aligned} z^r(U,q) \otimes \mathbb{Z}/p^n[-2r] &\xrightarrow{\mathsf{cl}^{r,q}} \varinjlim_{W \in \Sigma^{r,q}} H^{2r}_{\overline{W}}\left(U \times \overline{\Delta^q}, \mathfrak{T}_n(r)_{(U \times \overline{\Delta^q}, U \times D^q)}\right)[-2r] \\ &\xleftarrow{\alpha^{r,q}} \varinjlim_{W \in \Sigma^{r,q}} \tau_{\le 2r}\, \Gamma_{\overline{W}}\left(U \times \overline{\Delta^q}, G_n(r)^\bullet_{(U \times \overline{\Delta^q}, U \times D^q)}\right) \\ &\xrightarrow{\beta^{r,q}} \Gamma\left(U \times \overline{\Delta^q}, G_n(r)^\bullet_{(U \times \overline{\Delta^q}, U \times D^q)}\right). \end{aligned}$$

Here $\alpha^{r,q}$ and $\beta^{r,q}$ are natural maps of complexes, which are obviously contravariant for the face maps $U \times \Delta^{q-1} \hookrightarrow U \times \Delta^q$. The map $\mathsf{cl}^{r,q}$ is bijective and contravariant for these face maps by Proposition 7.1 (1) and (3). Hence we get homomorphisms of double complexes concerning $(\star, \bullet)$

$$\begin{aligned} z^r(U,\star) \otimes \mathbb{Z}/p^n[-2r] &\xleftarrow{(\mathsf{cl}^{r,\star})^{-1} \circ \alpha^{r,\star}} \varinjlim_{W \in \Sigma^{r,\star}} \tau_{\le 2r}\, \Gamma_{\overline{W}}\left(U \times \overline{\Delta^\star}, G_n(r)^\bullet_{(U \times \overline{\Delta^\star}, U \times D^\star)}\right) \\ &\xrightarrow{\beta^{r,\star}} \Gamma\left(U \times \overline{\Delta^\star}, G_n(r)^\bullet_{(U \times \overline{\Delta^\star}, U \times D^\star)}\right) \\ &\longleftarrow \Gamma(U, G_n(r)^\bullet_U), \end{aligned}$$

where the differentials in the $\star$-direction are alternating sums of pull-back maps along the faces of codimension 1, and the last arrow is the inclusion to the factor of $\star = 0$. The first and the last arrows are quasi-isomorphisms on the associated total complexes by Proposition 7.1 (2) and Corollary 4.3, respectively. We thus obtain the desired morphism (7.0.1) in $D(X_{\text{ét}}, \mathbb{Z}/p^n)$ by sheafifying the diagram of total complexes.

Remark 7.2 *The Rost-Voevodsky theorem* ([Vo1], [Vo2], [We], cf. [BK1] §3) *and the Suslin-Voevodsky theorem* [SV] (*cf.* [GL]) *imply that the morphism* (7.0.1) *induces isomorphisms* (*cf.* [Sa2] *Conjecture* 1.4.1)

$$\begin{aligned} \tau_{\le r}\left(\mathbb{Z}(r)^{\text{ét}}_X \otimes^{\mathbb{L}} \mathbb{Z}/p^n\right) &\xrightarrow{\sim} \mathfrak{T}_n(r)_X && \text{in } D^b(X_{\text{ét}}, \mathbb{Z}/p^n), \\ \tau_{\le r}\left(\mathbb{Z}(r)^{\mathsf{Zar}}_X \otimes^{\mathbb{L}} \mathbb{Z}/p^n\right) &\xrightarrow{\sim} \tau_{\le r} R\varepsilon_* \mathfrak{T}_n(r)_X && \text{in } D^b(X_{\mathsf{Zar}}, \mathbb{Z}/p^n), \end{aligned}$$

which one can easily check by similar arguments as in [SS] §A.2. *Here* $\mathbb{Z}(r)_X^{\mathsf{Zar}}$ *denotes the complex of Zariski sheaves on* X *defined as* $U \mapsto z^r(U,\star)[-2r]$, *and* ε *denotes the continuous map of sites* $X_{\text{ét}} \to X_{\mathsf{Zar}}$.

REMARK 7.3 *If* X *is smooth over* S, *then* $\mathbb{Z}(r)_X^{\mathsf{Zar}}$ *is concentrated in degrees* $\leq r$ *by a result of Geisser* [Ge] *Corollary* 4.4. *Hence his arguments in loc. cit.* §6, *Proof of Theorem* 1.3 *gives an alternative construction of* (7.0.1) *in this case.*

8 DE RHAM-WITT COHOMOLOGY AND HOMOLOGY

In this section we define two kinds of de Rham-Witt complexes for normal crossing varieties playing the roles of cohomology and homology, and relate them with the modified de Rham-Witt complex of Hyodo [Hy2]. The main results of this section are Theorems 8.8 and 8.12 below.

Let k be a field of characteristic $p > 0$, and let Y be a normal crossing variety over k. Let $U \subset Y$ be a dense open subset which is smooth over k, and let $\sigma : U \hookrightarrow Y$ be the natural open immersion.

DEFINITION 8.1 *We define the complex* $(W_n\Lambda_Y^\bullet, d)$ *of étale sheaves on* Y *as the differential graded subalgebra of* $\sigma_* W_n\Omega_U^\bullet$ *generated by*

$$W_n\mathscr{O}_Y \quad \textit{and} \quad \lambda_{Y,n}^1 (\subset \sigma_* W_n\Omega_U^1),$$

which we call the cohomological de Rham-Witt complex of Y. *It is easy to see that* $W_n\Lambda_Y^\bullet$ *does not depend on the choice of* U. *For* $n = 1$, $W_1\Lambda_Y^\bullet$ *is the same as the complex* $\Lambda_Y^\bullet$ *defined in* [Sa1] §3.3.

By the relations

$$\begin{cases} V(aR(x)) = V(a)x \in W_{n+1}\Omega_U^r & (a \in W_n\mathscr{O}_U,\, x \in W_{n+1}\Omega_{U,\log}^r) \\ F(ax) = F(a)R(x) \in W_n\Omega_U^r & (a \in W_{n+1}\mathscr{O}_U,\, x \in W_{n+1}\Omega_{U,\log}^r) \\ R(ax) = R(a)R(x) \in W_n\Omega_U^r & (a \in W_{n+1}\mathscr{O}_U,\, x \in W_{n+1}\Omega_{U,\log}^r) \end{cases} \tag{8.1.1}$$

(cf. [I1] I Théorème 2.17, Proposition 2.18), the operators V, F, R on $W_n\Omega_U^r$ induce operators

$$\begin{aligned} V &: W_n\Lambda_Y^r \longrightarrow W_{n+1}\Lambda_Y^r, \\ F &: W_{n+1}\Lambda_Y^r \longrightarrow W_n\Lambda_Y^r, \\ R &: W_{n+1}\Lambda_Y^r \longrightarrow W_n\Lambda_Y^r, \end{aligned}$$

which satisfy relations

$$FV = VF = p, \quad FdV = d, \quad dF = pFd, \quad Vd = pdV, \tag{8.1.2}$$

$$RV = VR, \qquad RF = FR, \qquad Rd = dR. \tag{8.1.3}$$

The local structure of $W_n\Lambda_Y^r$ can be written in terms of the usual Hodge-Witt sheaves of the strata of Y.

PROPOSITION 8.2 (1) *Assume that Y is simple, and let $Y_1, Y_2, \cdots, Y_q$ be the irreducible components of Y. Then there is an exact sequence on $Y_{\text{ét}}$*

$$0 \longrightarrow W_n \Lambda^r_Y \xrightarrow{\check{r}^0} \bigoplus_{|I|=1} W_n \Omega^r_{Y_I} \xrightarrow{\check{r}^1} \bigoplus_{|I|=2} W_n \Omega^r_{Y_I} \xrightarrow{\check{r}^2} \cdots \xrightarrow{\check{r}^{q-1}} \bigoplus_{|I|=q} W_n \Omega^r_{Y_I} \longrightarrow 0,$$

where the notation is the same as in Proposition 2.2*.*

(2) *Assume that Y is embedded into a smooth k-variety $\mathscr{Y}$ as a normal crossing divisor. Let $i : Y \hookrightarrow \mathscr{Y}$ be the closed immersion. Then there is a short exact sequence on $\mathscr{Y}_{\text{ét}}$*

$$0 \longrightarrow W_n \Omega^r_{\mathscr{Y}}(-\log Y) \longrightarrow W_n \Omega^r_{\mathscr{Y}} \xrightarrow{i^*} i_* W_n \Lambda^r_Y \longrightarrow 0.$$

Proof. (1) follows from the contravariant functoriality of Hodge-Witt sheaves of smooth varieties and the same arguments as for [Sa1] Proposition 3.2.1. The assertion (2) follows from (1) and [Mo] Lemma 3.15.1. □

We next introduce homological Hodge-Witt sheaves.

DEFINITION 8.3 *For $r \geq 0$, we define the étale sheaf $W_n \Xi^r_Y$ on Y as the $W_n \mathscr{O}_Y$-submodule of $\sigma_* W_n \Omega^r_U$ generated by $\nu^r_{Y,n} (\subset \sigma_* W_n \Omega^r_U)$. Similarly as for $W_n \Lambda^r_Y$, the sheaf $W_n \Xi^r_Y$ does not depend on the choice of U. For $n = 1$, $W_1 \Xi^r_Y$ agrees with the sheaf Ξ^r_Y defined in* [Sa1] §2.5.

Since $\lambda^r_{Y,n} \subset \nu^r_{Y,n}$ by definition, we have $W_n \Lambda^r_Y \subset W_n \Xi^r_Y$, which agree with $W_n \Omega^r_Y$ when Y is smooth (cf. [Sa1] Remark 3.1.4). The following local description will be useful later.

PROPOSITION 8.4 (1) *Under the same setting and notation as in Proposition* 8.2 (1)*, there is a canonical ascending filtration F_a $(a \geq 0)$ on $W_n \Xi^r_Y$ satisfying*

$$\mathsf{F}_0(W_n \Xi^r_Y) = 0 \quad \text{and} \quad \mathsf{gr}^{\mathsf{F}}_a W_n \Xi^r_Y \simeq \bigoplus_{|I|=a} W_n \Omega^{r-a+1}_{Y_I},$$

where the isomorphism for $a \geq 2$ depends on the fixed ordering of the irreducible components of Y.

(2) *Under the same setting and notation as in Proposition* 8.2 (2)*, there is a short exact sequence on $\mathscr{Y}_{\text{ét}}$*

$$0 \longrightarrow W_n \Omega^{r+1}_{\mathscr{Y}} \longrightarrow W_n \Omega^{r+1}_{\mathscr{Y}}(\log Y) \xrightarrow{\varrho} i_* W_n \Xi^r_Y \longrightarrow 0,$$

where ϱ is induced by the Poincaré residue mapping

$$\varrho_0 : W_n \Omega^{r+1}_{\mathscr{Y}}(\log Y) \longrightarrow (i\sigma)_* W_n \Omega^r_U.$$

Proof. We first recall the following fact due to Mokrane [Mo] Proposition 1.4.5. Under the setting of (2), let $\mathsf{F}_a W_n\,\Omega^m_{\mathscr{Y}}(\log Y)$ $(a \geq 0)$ be the image of the product

$$W_n\,\Omega^a_{\mathscr{Y}}(\log Y) \times W_n\,\Omega^{m-a}_{\mathscr{Y}} \longrightarrow W_n\,\Omega^m_{\mathscr{Y}}(\log Y).$$

Put $\mathsf{F}_{-1} W_n\,\Omega^m_{\mathscr{Y}}(\log Y) := 0$. Then there are isomorphisms

$$\mathsf{gr}^{\mathsf{F}}_a\, W_n\,\Omega^m_{\mathscr{Y}}(\log Y) \simeq \begin{cases} W_n\,\Omega^m_{\mathscr{Y}} & (a=0) \\ \bigoplus_{|I|=a}\ W_n\,\Omega^{m-a}_{Y_I} & (a \geq 1), \end{cases} \tag{8.4.1}$$

which are induced by Poincaré residue mappings for $a \geq 1$ and depend on the fixed ordering of the irreducible components of Y for $a \geq 2$. We prove (2). Because the complement $\Sigma := Y - U$ has codimension ≥ 2 on $\mathscr{Y}$ and the sheaves $W_n\,\Omega^{r+1}_{\mathscr{Y}}$ and $W_n\,\Omega^{r+1}_{\mathscr{Y}}(\log Y)$ are finitely successive extensions of locally free $\mathscr{O}_{\mathscr{Y}}$-modules of finite rank ([I1] I Corollaire 3.9), the sequence

$$0 \longrightarrow W_n\,\Omega^{r+1}_{\mathscr{Y}} \longrightarrow W_n\,\Omega^{r+1}_{\mathscr{Y}}(\log Y) \xrightarrow{\ \varrho_0\ } (i\sigma)_* W_n\,\Omega^r_U$$

is exact by (8.4.1) for $\mathscr{Y} - \Sigma$. Hence it is enough to show that $\mathsf{Im}(\varrho_0) = i_* W_n\,\Xi^r_Y$. Let $\mathsf{F}_a(\nu^r_{Y,n})$ $(a \geq 0)$ be the filtration on $\nu^r_{Y,n}$ in [Sa1] Proposition 2.2.1, which is defined under the assumption that Y is simple and satisfies

$$\mathsf{F}_0(\nu^r_{Y,n}) = 0 \quad \text{and} \quad \mathsf{gr}^{\mathsf{F}}_a\,\nu^r_{Y,n} \simeq \bigoplus_{|I|=a} W_n\,\Omega^{r-a+1}_{Y_I,\log} \quad \text{(non-canonically).}$$

Hence comparing this local description of $\nu^r_{Y,n}$ with (8.4.1) for $m = r+1$, we see that $\mathsf{Im}(\varrho_0) = i_* W_n\,\Xi^r_Y$. As for (1), we define the desired filtration $\mathsf{F}_a(W_n\,\Xi^r_Y)$ as the $W_n\mathscr{O}_Y$-submodule generated by $\mathsf{F}_a(\nu^r_{Y,n})$. This filtration satisfies the required properties by (2) and again by (8.4.1). □

Since the residue mapping ϱ_0 in Proposition 8.4 (2) commutes with the operators d, V, F, R (cf. (8.1.1)), these operators on $W_n\,\Omega^{\bullet}_U$ induce operators

$$\begin{aligned} d &: W_n\,\Xi^r_Y \longrightarrow W_n\,\Xi^{r+1}_Y, \\ V &: W_n\,\Xi^r_Y \longrightarrow W_{n+1}\,\Xi^r_Y, \\ F &: W_{n+1}\,\Xi^r_Y \longrightarrow W_n\,\Xi^r_Y, \\ R &: W_{n+1}\,\Xi^r_Y \longrightarrow W_n\,\Xi^r_Y \end{aligned}$$

by Proposition 8.4 (2) (without the assumptions in Proposition 8.4 (1) or (2)), which satisfy the relations listed in (8.1.2) and (8.1.3). We call the resulting complex $(W_n\,\Xi^{\bullet}_Y, d)$ *the homological de Rham-Witt complex of* Y.

Remark 8.5 *There are natural injective homomorphisms of complexes*

$$\bigoplus_{r=0}^{\dim(Y)} \lambda^r_{Y,n}[-r] \hookrightarrow W_n\,\Lambda^{\bullet}_Y, \qquad \bigoplus_{r=0}^{\dim(Y)} \nu^r_{Y,n}[-r] \hookrightarrow W_n\,\Xi^{\bullet}_Y,$$

where the differentials on the complexes on the left hand side are defined as zero. Indeed the differentials on $W_n\,\Omega^{\bullet}_U$ *are zero on* $W_n\,\Omega^{\bullet}_{U,\log}$.

The following Proposition 8.6 and Theorem 8.8 explain a fundamental relationship between $W_n\Lambda_Y^\bullet$ and $W_n\Xi_Y^\bullet$.

Proposition 8.6 *The complex $W_n\Xi_Y^\bullet$ is a $W_n\Lambda_Y^\bullet$-module, that is, there is a natural $W_n\mathscr{O}_Y$-bilinear pairing*

$$W_n\Lambda_Y^r \times W_n\Xi_Y^s \longrightarrow W_n\Xi_Y^{r+s} \qquad (r,s \geq 0) \tag{8.6.1}$$

satisfying the Leibniz rule

$$d(x\cdot y) = (dx)\cdot y + (-1)^r x\cdot dy \qquad (x\in W_n\Lambda_Y^r,\ y\in W_n\Xi_Y^s).$$

Moreover this pairing is compatible with R and satisfies relations

$$\begin{aligned} F(x\cdot y) &= F(x)\cdot F(y) \\ x\cdot V(y) &= V(F(x)\cdot y) \\ V(x)\cdot y &= V(x\cdot F(y)) \end{aligned} \qquad (x\in W_n\Lambda_Y^r,\ y\in W_n\Xi_Y^s).$$

Proof. The pairing (8.6.1) is induced by the product of $W_n\Omega_U^\bullet$ and the biadditive pairing

$$\lambda_{Y,n}^r \times \nu_{Y,n}^s \longrightarrow \nu_{Y,n}^{r+s}$$

defined in [Sa1] Definition 3.1.1. The properties of the pairing (8.6.1) follow from the corresponding properties of the product of $W_n\Omega_U^\bullet$ ([I1] I Théorème 2.17, Proposition 2.18). □

Remark 8.7 *When k is perfect and Y is the special fiber of a regular semistable family X over a discrete valuation ring with residue field k, we have*

$$W_n\Lambda_Y^\bullet \subset W_n\omega_Y^\bullet \subset W_n\Xi_Y^\bullet$$

by [Sa1] *Proposition* 4.2.1. *Here $W_n\omega_Y^\bullet$ denotes the modified de Rham-Witt complex associated with X* [Hy2]. *The complex $W_n\Xi_Y^\bullet$ does not in general have a product structure unless Y is smooth.*

Theorem 8.8 *Assume that k is perfect and that Y is proper over k. Put $b := \dim(Y)$ and $W_n := W_n(k)$. Then*:

(1) *There is a canonical trace map $\mathsf{tr}_n : \mathbb{H}^{2b}(Y, W_n\Xi_Y^\bullet) \to W_n$, which is bijective if Y is geometrically connected over k.*

(2) *The pairing*

$$\mathbb{H}^i(Y, W_n\Lambda_Y^\bullet) \times \mathbb{H}^{2b-i}(Y, W_n\Xi_Y^\bullet) \xrightarrow{(8.6.1)} \mathbb{H}^{2b}(Y, W_n\Xi_Y^\bullet) \xrightarrow{\mathsf{tr}_n} W_n$$

is a non-degenerate pairing of finitely generated W_n-modules for any $i \geq 0$.

Proof. (1) Let f_n be the canonical morphism $W_nY \to W_n$, where W_nY means the scheme consisting of the topological space Y and the structure sheaf $W_n\mathscr{O}_Y$. Since $\mathbb{H}^{2b}(Y, W_n\,\Xi_Y^\bullet) \simeq H^b(Y, W_n\,\Xi_Y^b)$, our task is to show that

$$W_n\,\Xi_Y^b \xrightarrow{\ \sim\ } f_n^! W_n \quad \text{in} \quad D^+_{\mathsf{qc}}(W_nY, W_n\mathscr{O}_Y), \tag{8.8.1}$$

where the subscript qc means the triangulated subcategory consisting of complexes with quasi-coherent cohomology sheaves and $f_n^!$ means the twisted inverse image functor of Hartshorne [Ha]. If Y is smooth, (8.8.1) is a theorem of Ekedahl [E]. In the general case, (8.8.1) is reduced to the smooth case by Proposition 8.4 (2) and the same arguments as for [Sa1] Proposition 2.5.9.

(2) The case $n = 1$ follows from [Sa1] Corollary 2.5.11, Proposition 3.3.5. The case $n \geq 2$ follows from a standard induction argument on n and the following lemma: $\square$

Lemma 8.9 (1) *There are injective homomorphisms*

$$\underline{p} : W_{n-1}\,\Lambda_Y^r \hookrightarrow W_n\,\Lambda_Y^r \quad \textit{and} \quad \underline{p} : W_{n-1}\,\Xi_Y^r \hookrightarrow W_n\,\Xi_Y^r$$

induced by the multiplication by p *on* $W_n\,\Lambda_Y^r$ *and* $W_n\,\Xi_Y^r$*, respectively.*

(2) *The following natural projections of complexes are quasi-isomorphisms*:

$$W_n\,\Lambda_Y^\bullet/\underline{p}(W_{n-1}\,\Lambda_Y^\bullet) \longrightarrow \Lambda_Y^\bullet \quad \textit{and} \quad W_n\,\Xi_Y^\bullet/\underline{p}(W_{n-1}\,\Xi_Y^\bullet) \longrightarrow \Xi_Y^\bullet.$$

Proof. When Y is smooth, the assertions are due to Illusie [I1] I Proposition 3.4 and Corollaire 3.15. In the general case, (1) follows from that for the dense open subset $U \subset Y$ we fixed before. To prove (2), we may assume that Y is simple. Then the assertions follow from those for the strata of Y and Propositions 8.2 (1) and 8.4 (1). $\square$

In the rest of this section, we work under the following setting. Let A be a discrete valuation ring with perfect residue field k, and let X be a regular semistable family over A. Put $Y := X \otimes_A k$. Recall that

$$W_n\,\Lambda_Y^\bullet \subset W_n\,\omega_Y^\bullet \subset W_n\,\Xi_Y^\bullet$$

by Remark 8.7. We define a Frobenius endomorphism φ on these complexes by p^mF on degree $m \geq 0$. There is a short exact sequence of complexes with Frobenius action

$$0 \longrightarrow W_n\,\omega_Y^\bullet(-1)[-1] \xrightarrow{\frac{dt}{t}\wedge} W_n\,\widetilde{\omega}_Y^\bullet \longrightarrow W_n\,\omega_Y^\bullet \longrightarrow 0 \tag{8.9.1}$$

by [Hy2] (1.4.3), where the complex $W_n\,\omega_Y^\bullet(-1)$ means the complex $W_n\,\omega_Y^\bullet$ with Frobenius endomorphism $p \cdot \varphi$. The monodromy operator

$$N : W_n\,\omega_Y^\bullet \longrightarrow W_n\,\omega_Y^\bullet(-1)$$

is defined as the connecting morphism associated with this sequence.

Proposition 8.10 *There is a short exact sequence of complexes with Frobenius action*

$$0 \longrightarrow W_n\,\Lambda_Y^\bullet \longrightarrow W_n\,\widetilde{\omega}_Y^\bullet \longrightarrow W_n\,\Xi_Y^\bullet(-1)[-1] \longrightarrow 0 \tag{8.10.1}$$

fitting into a commutative diagram of complexes

$$\begin{tikzcd}
 & W_n\,\omega_Y^\bullet(-1)[-1] \arrow[d, "\frac{dt}{t}\wedge"'] \arrow[dr, hook] & \\
W_n\,\Lambda_Y^\bullet \arrow[r] \arrow[dr, hook] & W_n\,\widetilde{\omega}_Y^\bullet \arrow[r] \arrow[d] & W_n\,\Xi_Y^\bullet(-1)[-1] \\
 & W_n\,\omega_Y^\bullet. &
\end{tikzcd}$$

Consequently, we obtain a complex of $W_n(k)$-modules with Frobenius action for $i \geq 0$

$$\begin{aligned}&\mathbb{H}^i(Y, W_n\,\Lambda_Y^\bullet) \longrightarrow \mathbb{H}^i(Y, W_n\,\omega_Y^\bullet)\\ &\qquad\qquad \xrightarrow{N} \mathbb{H}^i(Y, W_n\,\omega_Y^\bullet)(-1) \longrightarrow \mathbb{H}^i(Y, W_n\,\Xi_Y^\bullet)(-1).\end{aligned} \tag{8.10.2}$$

Proof. Let $\sigma : U \hookrightarrow Y$ be as before and define a homomorphism $\partial : W_n\,\widetilde{\omega}_Y^r \to \sigma_* W_n\,\Omega_U^{r-1}$ as the composite

$$\partial : W_n\,\widetilde{\omega}_Y^r \longrightarrow \sigma_*\sigma^* W_n\,\widetilde{\omega}_Y^r \xrightarrow{\ \sim\ } \sigma_*(W_n\,\Omega_U^r \oplus W_n\,\Omega_U^{r-1}) \xrightarrow{\mathsf{pr}_2} \sigma_* W_n\,\Omega_U^{r-1},$$

where the second arrow is obtained from the fact that the sequence (8.9.1) splits on U. It is easy to see that ∂ satisfies

$$d\partial = \partial d \qquad \text{and} \qquad \partial F = pF\partial,$$

which induces a map of complexes $\partial^\bullet : W_n\,\widetilde{\omega}_Y^\bullet \longrightarrow \sigma_* W_n\,\Omega_U^\bullet(-1)[-1]$. To show that $\partial^\bullet$ induces the exact sequence (8.10.1), we may assume the assumption in Proposition 8.2 (2). Then we have an isomorphism of complexes

$$W_n\,\widetilde{\omega}_Y^\bullet \simeq W_n\,\Omega_{\mathscr{Y}}^\bullet(\log Y)/W_n\,\Omega_{\mathscr{Y}}^\bullet(-\log Y) \qquad \text{([Hy2] p. 247 Lemma)}$$

and the assertion follows from Propositions 8.2 (2) and 8.4 (2). The commutativity of the diagram in the proposition is straight-forward and left to the reader. □

We prove that the monodromy-weight conjecture for log crystalline cohomology implies an invariant cycle 'theorem' (cf. [I2] 2.4.5), which will be useful in §§9–10 below. Let A be a discrete valuation ring with finite residue field k, and X be a regular scheme which is projective flat over A with strict semistable reduction. Put $K_0 := \mathsf{Frac}(W(k))$ and $Y := X \otimes_A k$. For integer $i \geq 0$, we define

$$C^i := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_{n\geq 1} \mathbb{H}^i(Y, W_n\,\Lambda_Y^\bullet), \tag{8.10.3}$$

$$D^i := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_{n\geq 1} \mathbb{H}^i(Y, W_n\,\omega_Y^\bullet), \tag{8.10.4}$$

$$E^i := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_{n\geq 1} \mathbb{H}^i(Y, W_n\,\Xi_Y^\bullet), \tag{8.10.5}$$

which are finite-dimensional over K_0 and F-isocrystals over $\mathsf{Spec}(k)$. The content of the monodromy-weight conjecture due to Mokrane is the following:

Conjecture 8.11 ([Mo] Conjecture 3.27) *The filtration on D^i induced by the weight spectral sequence* (*loc. cit.* 3.25, *cf.* [Na] (2.0.9;p)) *agrees with the monodromy filtration induced by the monodromy operator $N : D^i \to D^i(-1)$* ([Mo] 3.26).

Theorem 8.12 *Let $i \geq 0$ be an integer, and assume Conjecture* 8.11 *for D^i. Then the following sequence of F-isocrystals over* $\mathsf{Spec}(k)$ *induced by* (8.10.2) *is exact*:

$$C^i \longrightarrow D^i \xrightarrow{N} D^i(-1) \longrightarrow E^i(-1).$$

Proof. Suppose that $\#(k) = p^a$. Then φ^a is K_0-linear and we use the notion of weights concerning φ^a. Because the groups

$$\mathbb{H}^i(Y, W_n \Lambda_Y^\bullet), \quad \mathbb{H}^i(Y, W_n \omega_Y^\bullet), \quad \mathbb{H}^i(Y, W_n \Xi_Y^\bullet), \quad \mathbb{H}^i(Y, W_n \widetilde{\omega}_Y^\bullet)$$

are finite for any $n \geq 1$, there is a diagram of F-isocrystals with exact rows by (8.9.1) and Proposition 8.10

$$\begin{tikzcd}
 & \widetilde{D}^i \arrow[r] \arrow[d, equal] & D^i \arrow[r, "N"] & D^i(-1) \arrow[r] \arrow[drr, dotted] & \widetilde{D}^{i+1} \arrow[d, equal] & \\
C^i \arrow[r] \arrow[urr, dotted] & \widetilde{D}^i \arrow[r, "\partial"] & E^{i-1}(-1) \arrow[r] & C^{i+1} \arrow[r] & \widetilde{D}^{i+1} \arrow[r, "\partial"] & E^i(-1),
\end{tikzcd}$$

where we put

$$\widetilde{D}^i := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_{n \geq 1} \mathbb{H}^i(Y, W_n \widetilde{\omega}_Y^\bullet).$$

By the assumption on D^i, $\mathsf{Ker}(N)$ has weights $\leq i$ and $\mathsf{Coker}(N)$ has weights $\geq i+2$. Hence for the reason of weights, it is enough to show the following lemma, where we do not need Conjecture 8.11:

Lemma 8.13 *For integers $i \geq 0$, C^i has weights $\leq i$ and E^i has weights $\geq i$.*

Proof of Lemma 8.13. When Y is smooth, the lemma is a consequence of the Katz-Messing theorem [KM]. In the general case, C^i has weights $\leq i$ by the spectral sequence of F-isocrystals obtained from Proposition 8.2 (1)

$$E_1^{s,t} = \bigoplus_{|I|=s+1} D_I^t \Longrightarrow C^{s+t},$$

where we put

$$D_I^t := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_{n \geq 1} \mathbb{H}^t(Y_I, W_n \Omega_{Y_I}^\bullet).$$

Moreover E^i has weights $\geq i$ by the duality result in Theorem 8.8. Thus we obtain Lemma 8.13 and Theorem 8.12. $\square$

REMARK 8.14 *One can show Lemma* 8.13 *for* E^i *alternatively by the Katz-Messing theorem and the spectral sequence obtained from Proposition* 8.4 (1)

$$E_1^{s,t} = \bigoplus_{|I|=1-s} D_I^{2s+t}(s) \Longrightarrow E^{s+t}.$$

9 COMPARISON WITH THE FINITE PART OF BLOCH-KATO

In this section we relate the cohomology of p-adic étale Tate twists with the finite part of Bloch-Kato [BK2]. Let p be a prime number, and let K be a p-adic local field, i.e., a finite extension of $\mathbb{Q}_p$. Let O_K be the integer ring of K, and let k be the residue field of O_K. In this section we consider Tate twists only in the sense of G_K-$\mathbb{Z}_p$-modules (and not in the sense of F-crystals). For a topological G_K-$\mathbb{Z}_p$-module M and an integer $r \in \mathbb{Z}$, we define

$$M(r) := \begin{cases} M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)^{\otimes r} & (\text{if } r \geq 0), \\ \mathsf{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p(1)^{\otimes(-r)}, M) & (\text{if } r < 0), \end{cases}$$

where $\mathbb{Z}_p(1)$ denotes the topological G_K-module $\varprojlim_{n\geq 1} \mu_{p^n}(\overline{K})$.

Let X be a regular scheme which is projective flat over O_K with strict semistable reduction, and put $X_K := X \otimes_{O_K} K$ and $Y := \mathscr{X} \otimes_{O_K} k$. Let $j : X_K \hookrightarrow X$ be the natural open immersion, and let i and r be non-negative integers. Put $\mathscr{V}^i := H^i(X_{\overline{K}}, \mathbb{Q}_p)$ (see also Notation) and

$$H^{i+1}(X, \mathfrak{T}_{\mathbb{Q}_p}(r)) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_{n\geq 1} H^{i+1}(X, \mathfrak{T}_n(r)_X),$$

which are finite-dimensional over $\mathbb{Q}_p$. Let $F^\bullet$ be the filtration on $H^{i+1}(X_K, \mathbb{Q}_p(r))$ resulting from the Hochschild-Serre spectral sequence

$$E_2^{a,b} = H^a(K, \mathscr{V}^b(r)) \Longrightarrow H^{a+b}(X_K, \mathbb{Q}_p(r)). \tag{9.0.1}$$

Let $F^qH^{i+1}(X, \mathfrak{T}_{\mathbb{Q}_p}(r))$ be the inverse image of $F^qH^{i+1}(X_K, \mathbb{Q}_p(r))$ under the canonical map $j^* : H^{i+1}(X, \mathfrak{T}_{\mathbb{Q}_p}(r)) \to H^{i+1}(X_K, \mathbb{Q}_p(r))$. We have

$$\Phi^{i,r} := \mathsf{gr}^1_F H^{i+1}(X, \mathfrak{T}_{\mathbb{Q}_p}(r)) \hookrightarrow \mathsf{gr}^1_F H^{i+1}(X_K, \mathbb{Q}_p(r)) = H^1(K, \mathscr{V}^i(r))$$

by definition. We relate $\Phi^{i,r}$ with the finite part of Bloch-Kato [BK2]:

$$H^1_f(K, \mathscr{V}^i(r)) := \mathsf{Ker}(\iota : H^1(K, \mathscr{V}^i(r)) \to H^1(K, \mathscr{V}^i \otimes_{\mathbb{Q}_p} B_{\mathsf{crys}})).$$

where B_{crys} denotes the period ring of crystalline representations defined by Fontaine [Fo1], and ι is induced by the natural inclusion $\mathbb{Q}_p(r) \hookrightarrow B_{\mathsf{crys}}$. The main result of this section is as follows

THEOREM 9.1 *Assume* $0 \leq r \leq p-2$ *and Conjecture* 8.11 *for* $D^i := D^i(Y)$. *Then we have*

$$\Phi^{i,r} \subset H^1_f(K, \mathscr{V}^i(r)).$$

Further if $r \geq \dim(X_K) - p + 3$, *then we have* $\Phi^{i,r} = H^1_f(K, \mathscr{V}^i(r))$.

Remark 9.2 *When X is smooth over O_K, Theorem* 9.1 *is a part of the p-adic point conjecture raised by Schneider* [Sc] *and proved by Langer-Saito* [LS] *and Nekovář* [Ne1].

Put

$$H^{i+1}(X, \tau_{\leq r} Rj_* \mathbb{Q}_p(r)) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_{n \geq 1} H^{i+1}(X, \tau_{\leq r} Rj_* \mu_{p^n}^{\otimes r})$$

and let $F^q H^{i+1}(X, \tau_{\leq r} Rj_* \mu_{p^n}^{\otimes r})$ be the inverse image of $F^q H^{i+1}(X_K, \mathbb{Q}_p(r))$ under the map

$$j^* : H^{i+1}(X, \tau_{\leq r} Rj_* \mu_{p^n}^{\otimes r}) \to H^{i+1}(X_K, \mathbb{Q}_p(r)).$$

We first prepare the following key commutative diagram to prove Theorem 9.1, assuming $0 \leq r \leq p-2$:

$$\begin{tikzcd} H^1(K, \mathscr{V}^i(r)) \arrow[d, "\iota"'] & & F^1 H^{i+1}(X, \tau_{\leq r} Rj_* \mathbb{Q}_p(r)) \arrow[ll] \arrow[d, "\alpha"] \arrow[dr, "\sigma'"] \arrow[dll, phantom, "\text{(A)}"] & \\ H^1(K, \mathscr{V}^i \otimes_{\mathbb{Q}_p} B_{\mathsf{crys}}) & & D^i/ND^i \arrow[ll, "\delta"] \arrow[r, "\epsilon"'] \arrow[ur, phantom, "\text{(B)}"] & E^i, \end{tikzcd} \tag{9.2.1}$$

where $E^i := E^i(Y)$ is as in (8.10.5), and D^i/ND^i denotes the cokernel of the monodromy operator $N : D^i \to D^i$. The top arrow is induced by j^* and an edge homomorphism of the spectral sequence (9.0.1). The arrow ϵ is induced by the complex (8.10.2), and the arrow σ' is induced by the composite map

$$H^{i+1}(X, \tau_{\leq r} Rj_* \mu_{p^n}^{\otimes r}) \xrightarrow{\sigma} H^{i+1-r}(Y, \nu_{Y,n}^{r-1}) \longrightarrow \mathbb{H}^i(Y, W_n \Xi_Y^\bullet) \quad (n \geq 1).$$

See Theorem 3.4 for σ and see Remark 8.5 for the last map. The constructions of the arrows α and δ and the commutativity of the square (A) deeply rely on the p-adic Hodge theory. The commutativity of the triangle (B) is rather straight-forward once we define α (see Proposition 9.10 below). The proof of Theorem 9.1 will be given after Proposition 9.10.

Remark 9.3 *In his paper* [La], *Langer considered the diagram* (A) *assuming that O_K is absolutely unramified* (*loc. cit. Proposition* 2.9). *We remove this assumption and the assumption* ($*$) *stated in loc. cit. p.* 191, *using a continuous version of crystalline cohomology* (*see Appendix* A *below*). *For this purpose, we include a detailed construction of α and a proof of the commutativity of* (A).

Construction of δ. We first construct the map δ. There is an exact sequence of topological G_K-$\mathbb{Q}_p$-modules

$$0 \longrightarrow \mathscr{V}^i \otimes_{\mathbb{Q}_p} B_{\mathsf{crys}} \longrightarrow \mathscr{V}^i \otimes_{\mathbb{Q}_p} B_{\mathsf{st}} \xrightarrow{1 \otimes N} \mathscr{V}^i \otimes_{\mathbb{Q}_p} B_{\mathsf{st}} \longrightarrow 0, \tag{9.3.1}$$

where N denotes the monodromy operator on B_{st}. Taking continuous Galois cohomology groups, we obtain a long exact sequence

$$\begin{aligned}0 \longrightarrow (\mathscr{V}^i \otimes_{\mathbb{Q}_p} B_{\mathsf{crys}})^{G_K} &\longrightarrow (\mathscr{V}^i \otimes_{\mathbb{Q}_p} B_{\mathsf{st}})^{G_K} \xrightarrow{1\otimes N} (\mathscr{V}^i \otimes_{\mathbb{Q}_p} B_{\mathsf{st}})^{G_K} \\ &\longrightarrow H^1(G_K, \mathscr{V}^i \otimes_{\mathbb{Q}_p} B_{\mathsf{crys}}) \longrightarrow \cdots .\end{aligned}$$

By the C_{st}-conjecture proved by Hyodo, Kato and Tsuji ([HK], [K4], [Ts1], cf. [Ni2]), the last arrow of the top row is identified with the map $N : D^i \to D^i$ (see (9.3.3) below for the construction of C_{st}-isomorphisms). We thus define the desired map δ as the connecting map of this sequence.

C_{st}-ISOMORPHISM. Before constructing α, we recall the construction of C_{st}-isomorphisms, which will be useful later ([K4], [Ts1] §4.10). See [K3] for the general framework of log structures and log schemes. Put

$$W := W(k) \quad \text{and} \quad K_0 := \mathsf{Frac}(W) = W[p^{-1}].$$

For a log scheme (Z, M_Z) and an integer $n \geq 1$, we define

$$(Z_n, M_{Z_n}) := (Z, M_Z) \times_{\mathsf{Spec}(\mathbb{Z})} \mathsf{Spec}(\mathbb{Z}/p^n),$$

where we regarded $\mathsf{Spec}(\mathbb{Z})$ and $\mathsf{Spec}(\mathbb{Z}/p^n)$ as log schemes by endowing them with the trivial log structures and the fiber product is taken in the category of log schemes. Let M be the log structure on X associated with the normal crossing divisor Y. Let $\overline{O_K}$ be the integral closure of O_K in $\overline{K}$, and let $\overline{M}$ be the log structure on $\overline{X} := X \otimes_{O_K} \overline{O_K}$ define by base-change (see (A.0.2) below). Put $\overline{Y} := Y \otimes_k \overline{k}$. For $n \geq 1$ and $s \geq 0$, let $\mathscr{S}_n^{\sim}(s)_{(\overline{X},\overline{M})} \in D^+(\overline{Y}_{\text{ét}}, \mathbb{Z}/p^n)$ be Tsuji's version of syntomic complex of $(\overline{X}, \overline{M})$ (§A.5). Let $\overline{j} : X_{\overline{K}} \hookrightarrow \overline{X}$ be the natural open immersion. For $m, s \geq 0$, we define the syntomic cohomology of $(\overline{X}, \overline{M})$ as

$$H^m_{\mathsf{syn}}((\overline{X}, \overline{M}), \mathscr{S}^{\sim}_{\mathbb{Q}_p}(s)) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_{n\geq 1} H^m\big(\overline{Y}, \mathscr{S}_n^{\sim}(s)_{(\overline{X},\overline{M})}\big).$$

The following isomorphism due to Tsuji ([Ts1] Theorem 3.3.4 (2)) plays a crucial role:

$$H^m_{\mathsf{syn}}((\overline{X}, \overline{M}), \mathscr{S}^{\sim}_{\mathbb{Q}_p}(s)) \simeq H^m(\overline{X}, \tau_{\leq s} R\overline{j}_* \mathbb{Q}_p(s)). \tag{9.3.2}$$

The right hand side is isomorphic to $\mathscr{V}^m(s) = H^m(X_{\overline{K}}, \mathbb{Q}_p(s))$ if $s \geq \dim(X_K)$ or $s \geq m$. We now introduce the crystalline cohomology of $(\overline{X}, \overline{M})$ over W:

$$H^m_{\mathsf{crys}}((\overline{X}, \overline{M})/W)_{\mathbb{Q}_p} := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_{n\geq 1} H^m_{\mathsf{crys}}((\overline{X}_n, \overline{M}_n)/W_n),$$

where $H^m_{\mathsf{crys}}((\overline{X}_n, \overline{M}_n)/W_n)$ denotes the crystal cohomology of $(\overline{X}_n, \overline{M}_n)$ over W_n (see §A.1 below). We define a G_K-equivariant homomorphism

$$g^m : \mathscr{V}^m \longrightarrow D^m \otimes_{K_0} B_{\mathsf{st}}$$

as the composite map ($d := \dim(X_K)$)

$$\begin{aligned} \mathscr{V}^m &\xrightarrow[\sim]{(1)} H^m_{\mathsf{syn}}((\overline{X},\overline{M}), \mathscr{S}^{\sim}_{\mathbb{Q}_p}(d)) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(-d) \\ &\xrightarrow{(2)} H^m_{\mathsf{crys}}((\overline{X},\overline{M})/W)_{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(-d) \\ &\xrightarrow[\sim]{(3)} (D^m \otimes_{K_0} B^+_{\mathsf{st}})^{N=0} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(-d) \\ &\xrightarrow{(4)} D^m \otimes_{K_0} B_{\mathsf{st}}\,, \end{aligned} \tag{9.3.3}$$

where (1) is obtained from the isomorphism (9.3.2) with $s = d$, the arrow (2) is the p^d-times of the canonical map induced by (A.5.2) below, and (3) is the crystalline interpretation of $(D^m \otimes_{K_0} B^+_{\mathsf{st}})^{N=0}$ due to Kato ([K4] §4, cf. Proposition A.3.1 (3) below). The map (4) is induced by the natural inclusion $\mathbb{Q}_p(-d) \hookrightarrow B_{\mathsf{st}}$. The B_{st}-linear extension of g^m is an isomorphism by [Ts2] Theorem 4.10.2:

$$\mathscr{V}^m \otimes_{\mathbb{Q}_p} B_{\mathsf{st}} \xrightarrow{\sim} D^m \otimes_{K_0} B_{\mathsf{st}}\,, \tag{9.3.4}$$

which we call the C_{st}-isomorphism. Now we have a commutative diagram of G_K-modules for $m, s \geq 0$

$$\begin{tikzcd} H^m_{\mathsf{syn}}((\overline{X},\overline{M}), \mathscr{S}^{\sim}_{\mathbb{Q}_p}(s)) \arrow[r, "(1)"'] \arrow[d, "(9.3.2)"'] & H^m_{\mathsf{crys}}((\overline{X},\overline{M})/W)_{\mathbb{Q}_p} \arrow[r, "\sim", "(2)"'] & (D^m \otimes_{K_0} B^+_{\mathsf{st}})^{N=0} \arrow[d, hook] \\ \mathscr{V}^m(s) \arrow[r, hook, "\mathbb{Q}_p(s) \hookrightarrow B_{\mathsf{st}}"] & \mathscr{V}^m \otimes_{\mathbb{Q}_p} B_{\mathsf{st}} \arrow[r, "\sim", "(9.3.4)"'] & D^m \otimes_{K_0} B_{\mathsf{st}}, \end{tikzcd} \tag{9.3.5}$$

where the arrow (1) is the p^r-times of the canonical map (A.5.2), and (2) is the crystalline interpretation of $(D^m \otimes_{K_0} B^+_{\mathsf{st}})^{N=0}$ mentioned in (9.3.3). This diagram commutes by the construction of (9.3.4) and the compatibility facts in [Ts1] Corollaries 4.8.8 and 4.9.2.

Remark 9.4 *We modified the C_{st}-isomorphisms in* [Ts1] *by constant multiplications to make them consistent with the canonical map from syntomic cohomology of Kato to* (*continuous*) *crystalline cohomology* (*see the map* (1) *of* (9.5.1) *and the equality* (A.5.4) *below*). *Under this modification, the element t_{syn} in* [Ts1] *p.* 378 *is replaced with $p^{-1}t_{\mathsf{syn}}$, and Corollary* 4.8.8 *of loc. cit. remains true.*

Construction of α. We start the construction of α assuming $0 \leq r \leq p-2$, which will be complete in Corollary 9.8 below. For $n \geq 1$, let $\mathscr{S}_n(r)_{(X,M)} \in D^+(Y_{\text{ét}}, \mathbb{Z}/p^n)$ be the syntomic complex of Kato (see §A.5 below). Since $0 \leq r \leq p-2$, there is an isomorphism

$$(\tau_{\leq r} Rj_*\mu_{p^n})|_Y \simeq \mathscr{S}_n(r)_{(X,M)} \quad \text{in } D^+(Y_{\text{ét}}, \mathbb{Z}/p^n) \tag{9.4.1}$$

for each $n \geq 1$ ([Ts2] Theorem 5.1), and $H^{i+1}(X, \tau_{\leq r} Rj_*\mathbb{Q}_p(r))$ is isomorphic to the syntomic cohomology

$$H^{i+1}_{\mathsf{syn}}((X,M), \mathscr{S}_{\mathbb{Q}_p}(r)) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_{n\geq 1} H^{i+1}(Y, \mathscr{S}_n(r)_{(X,M)})$$

by the proper base-change theorem. Therefore we consider $H^{i+1}_{\mathsf{syn}}((X,M),\mathscr{S}_{\mathbb{Q}_p}(r))$ instead of $H^{i+1}(X,\tau_{\le r}Rj_*\mathbb{Q}_p(r))$, which is the major reason we assume $r \le p-2$. We first prove:

LEMMA 9.5 *The kernel of the composite map*

$$\xi : H^m_{\mathsf{syn}}((X,M),\mathscr{S}_{\mathbb{Q}_p}(r)) \longrightarrow H^m_{\mathsf{crys}}((X,M)/W)_{\mathbb{Q}_p} \longrightarrow H^m_{\mathsf{crys}}((\overline{X},\overline{M})/W)_{\mathbb{Q}_p}$$

agrees with that of the composite map

$$\eta : H^m_{\mathsf{syn}}((X,M),\mathscr{S}_{\mathbb{Q}_p}(r)) \overset{(9.4.1)}{\simeq} H^m(X,\tau_{\le r}Rj_*\mathbb{Q}_p(r)) \\ \longrightarrow \mathscr{V}^m(r) = H^m(X_{\overline{K}},\mathbb{Q}_p(r)).$$

Proof. The map η factor through (9.3.2) with $s=r$, and the map ξ factors through the map (1) in (9.3.5) with $s=r$ by (A.5.4) below. Hence the diagram

$$\begin{tikzcd} H^m_{\mathsf{syn}}((X,M),\mathscr{S}_{\mathbb{Q}_p}(r)) \arrow[r, "\xi"] \arrow[d, "\eta"'] & H^m_{\mathsf{crys}}((\overline{X},\overline{M})/W)_{\mathbb{Q}_p} \arrow[d, hook] \\ \mathscr{V}^m(r) \arrow[r, hook] & D^m \otimes_{\mathbb{Q}_p} B_{\mathsf{st}} \end{tikzcd}$$

commutes by (9.3.5), which implies the assertion. $\square$

Let $H^m_{\mathsf{cont\text{-}cr}}((X,M)/W)$ be the continuous crystalline cohomology of (X,M) over W (see §A.2 below). Put

$$H^m_{\mathsf{syn}}((X,M),\mathscr{S}_{\mathbb{Q}_p}(r))^0 := \mathsf{Ker}(\eta : H^m_{\mathsf{syn}}((X,M),\mathscr{S}_{\mathbb{Q}_p}(r)) \longrightarrow \mathscr{V}^m(r)),$$
$$H^m_{\mathsf{cont\text{-}cr}}((X,M)/W)_{\mathbb{Q}_p} := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^m_{\mathsf{cont\text{-}cr}}((X,M)/W),$$
$$H^m_{\mathsf{cont\text{-}cr}}((X,M)/W)^0_{\mathbb{Q}_p} := \mathsf{Ker}(H^m_{\mathsf{cont\text{-}cr}}((X,M)/W)_{\mathbb{Q}_p} \to H^m_{\mathsf{crys}}((\overline{X},\overline{M})/W)_{\mathbb{Q}_p}).$$

There are canonical maps

$$\begin{aligned} H^{i+1}_{\mathsf{syn}}((X,M),\mathscr{S}_{\mathbb{Q}_p}(r))^0 &\xrightarrow{(1)} H^{i+1}_{\mathsf{cont\text{-}cr}}((X,M)/W)^0_{\mathbb{Q}_p} \\ &\xrightarrow{(2)} H^1(K, H^i_{\mathsf{crys}}((\overline{X},\overline{M})/W)_{\mathbb{Q}_p}) \\ &\xrightarrow[\sim]{(3)} H^1(K, (D^i \otimes_{K_0} B^+_{\mathsf{st}})^{N=0}) \\ &\xrightarrow{(4)} H^1(K, \mathscr{V}^i \otimes_{\mathbb{Q}_p} B_{\mathsf{crys}}), \end{aligned} \tag{9.5.1}$$

where (1) is the canonical map obtained from Lemma 9.5 and Proposition A.6.1 (2) below, the arrow (3) is the isomorphism (2) in (9.3.5) for $m=i$, and (4) is obtained from the C_{st}-isomorphism (9.3.4) and (9.3.1). The most crucial map (2) is obtained from an edge homomorphism of the spectral sequence in Theorem A.4.2 (see also Corollary A.4.5 below). To proceed with the construction of α, we need to introduce some notation.

Definition 9.6 *Put* $S := \mathsf{Spec}(O_K)$, *and fix a prime* π *of* O_K. *Let* M_S *be the log structure on* S *associated with the closed point* $\{\pi = 0\}$. *Put* $V := \mathsf{Spec}(W[t])$, *and let* M_V *be the log structure on* V *associated with the divisor* $\{t = 0\}$. *Let*

$$\iota : (S, M_S) \longrightarrow (V, M_V)$$

be the exact closed immersion over W *sending* $t \mapsto \pi$. *For each* $n \geq 1$, *let* $(\mathscr{E}_n, M_{\mathscr{E}_n})$ *be the PD-envelope of the exact closed immersion* $\iota_n : (V_n, M_{V_n}) \hookrightarrow (S_n, M_{S_n})$ *compatible with the canonical PD-structure on* pW_n. *We define*

$$H^m_{\mathsf{crys}}((X,M)/(\mathscr{E}, M_{\mathscr{E}}))_{\mathbb{Q}_p} := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_{n \geq 1} H^m_{\mathsf{crys}}((X_n, M_n)/(\mathscr{E}_n, M_{\mathscr{E}_n})),$$

which is an $R_{\mathscr{E}} := \varprojlim_{n\geq 1} \Gamma(\mathscr{E}_n, \mathscr{O}_{\mathscr{E}_n})$*-module endowed with a* $\varphi_{\mathscr{E}}$*-semilinear endomorphism* φ *and a* K_0*-linear endomorphism* N *satisfying* $N\varphi = p\varphi N$. *Here* $\varphi_{\mathscr{E}}$ *is a certain canonical Frobenius operator on* $R_{\mathscr{E}}$. *See* [Ts1] *p.* 253 *for* $\varphi_{\mathscr{E}}$, *loc. cit.* §4.3 *for* N *and* φ, *and see loc. cit. Remark* 4.3.9 *for* N.

The following lemma plays a key role in our construction of α, where we do not need to assume $0 \leq r \leq p-2$ (compare with the assumption $(*)$ in [La] p. 191).

Lemma 9.7 (1) *There exists a long exact sequence*

$$\begin{aligned} \cdots \longrightarrow H^i_{\mathsf{cont\text{-}cr}}((X,M)/W)_{\mathbb{Q}_p} &\longrightarrow H^i_{\mathsf{crys}}((X,M)/(\mathscr{E}, M_{\mathscr{E}}))_{\mathbb{Q}_p} \\ \xrightarrow{N} H^i_{\mathsf{crys}}((X,M)/(\mathscr{E}, M_{\mathscr{E}}))_{\mathbb{Q}_p} &\xrightarrow{\partial} H^{i+1}_{\mathsf{cont\text{-}cr}}((X,M)/W)_{\mathbb{Q}_p} \longrightarrow \cdots \end{aligned}$$

and the image of ∂ *agrees with* $H^{i+1}_{\mathsf{cont\text{-}cr}}((X,M)/W)^0_{\mathbb{Q}_p}$.

(2) *There is a commutative diagram whose lower row is exact*

$$\begin{tikzcd} & H^i_{\mathsf{crys}}((X,M)/(\mathscr{E}, M_{\mathscr{E}}))_{\mathbb{Q}_p} \arrow[r, two heads, "\partial"] \arrow[d] & H^{i+1}_{\mathsf{cont\text{-}cr}}((X,M)/W)^0_{\mathbb{Q}_p} \arrow[d] \\ D^i \arrow[r, "N"] & D^i \arrow[r, "(-1)^{i+1}\delta"] & H^1(K, \mathscr{V}^i \otimes_{\mathbb{Q}_p} B_{\mathsf{crys}}). \end{tikzcd}$$

Here the right vertical arrow is the composite of (2) – (4) *in* (9.5.1)*, and the left vertical arrow is the specialization maps with* $t \mapsto 0$ (*cf.* [Ts1] (4.4.4)).

Proof. (1) follows from Proposition A.2.4 and Corollaries A.3.2 (1) and A.3.4 below. As for (2), the exactness of the lower row has been explained in the construction of δ. See Corollary A.4.5 below for the commutativity of the square. □

Corollary 9.8 *The composite map* (9.5.1) *factors as follows*:

$$H^{i+1}_{\mathsf{syn}}((X,M), \mathscr{S}_{\mathbb{Q}_p}(r))^0 \longrightarrow D^i/ND^i \overset{\delta}{\hookrightarrow} H^1(K, \mathscr{V}^i \otimes_{\mathbb{Q}_p} B_{\mathsf{crys}}).$$

We define α *as the first arrow of this decomposition.*

Commutativity of the diagram (9.2.1). We first prove that the square (A) in (9.2.1) is commutative. By the construction of α and δ, it is enough to show

Theorem 9.9 *The following square is commutative*:

$$\begin{tikzcd} H^{i+1}_{\mathsf{syn}}((X,M),\mathscr{S}_{\mathbb{Q}_p}(r))^0 \arrow[r, "\sim"] \arrow[d, "(9.5.1)"'] & F^1H^{i+1}(X,\tau_{\leq r}Rj_*\mathbb{Q}_p(r)) \arrow[d] \\ H^1(K,\mathscr{V}^i\otimes_{\mathbb{Q}_p}B_{\mathsf{crys}}) & H^1(K,\mathscr{V}^i(r)), \arrow[l, "B_{\mathsf{crys}}\hookleftarrow\mathbb{Q}_p(r)"'] \end{tikzcd} \tag{9.9.1}$$

where the right vertical arrow is the top arrow in (9.2.1).

Proof. Put $d:=\dim(X_K)$. We define a G_K-homomorphism

$$\beta^{i,r}:\mathscr{V}^i(r)\longrightarrow\begin{cases}H^i_{\mathsf{crys}}((\overline{X},\overline{M})/W)_{\mathbb{Q}_p}(r-d) & (\text{if } r<d)\\ H^i_{\mathsf{crys}}((\overline{X},\overline{M})/W)_{\mathbb{Q}_p} & (\text{if } r\geq d)\end{cases}$$

as the following composite map for $r\geq d$:

$$\beta^{i,r}:\mathscr{V}^i(r)\xleftarrow[\sim]{(9.3.2)}H^i_{\mathsf{syn}}((\overline{X},\overline{M}),\mathscr{S}_{\widetilde{\mathbb{Q}_p}}(r))\xrightarrow{(9.3.5)\,(1)}H^i_{\mathsf{crys}}((\overline{X},\overline{M})/W)_{\mathbb{Q}_p},$$

and as the following map for $r<d$:

$$\beta^{i,r}:\mathscr{V}^i(r)\simeq\mathscr{V}^i(d)\otimes_{\mathbb{Q}_p}\mathbb{Q}_p(r-d)\xrightarrow{\beta^{i,d}\otimes\mathsf{id}}H^i_{\mathsf{crys}}((\overline{X},\overline{M})/W)_{\mathbb{Q}_p}(r-d).$$

We next define a homomorphism

$$\begin{aligned}\gamma^{i,r}:\ & H^{i+1}_{\mathsf{syn}}((X,M),\mathscr{S}_{\mathbb{Q}_p}(r))^0\\ &\longrightarrow\begin{cases}H^1(K,H^i_{\mathsf{crys}}((\overline{X},\overline{M})/W)_{\mathbb{Q}_p}(r-d)) & (\text{if } r<d)\\ H^1(K,H^i_{\mathsf{crys}}((\overline{X},\overline{M})/W)_{\mathbb{Q}_p}) & (\text{if } r\geq d)\end{cases}\end{aligned}$$

as the following composite map, if $r<d$:

$$\begin{aligned}H^{i+1}_{\mathsf{syn}}((X,M),\mathscr{S}_{\mathbb{Q}_p}(r))^0 &\xrightarrow{\text{(i)}} H^1(K,H^i_{\mathsf{crys}}((\overline{X},\overline{M})/W)_{\mathbb{Q}_p})\\ &\simeq\ H^1(K,H^i_{\mathsf{crys}}((\overline{X},\overline{M})/W)_{\mathbb{Q}_p}(r-d)\otimes_{\mathbb{Q}_p}\mathbb{Q}_p(d-r))\\ &\xrightarrow{\text{(ii)}} H^1(K,H^i_{\mathsf{crys}}((\overline{X},\overline{M})/W)_{\mathbb{Q}_p}(r-d)),\end{aligned}$$

where (i) is the composite of (1) and (2) in (9.5.1), and (ii) is induced by the inclusion $\mathbb{Q}_p(d-r)\hookrightarrow H^0_{\mathsf{crys}}((\overline{X},\overline{M})/W)_{\mathbb{Q}_p}$ (cf. (A.6.5) below) and the cup product of crystalline cohomology. If $r\geq d$, then we define $\gamma^{i,r}$ as the composite of (1) and (2) in (9.5.1).

We prove the commutativity of (9.9.1) assuming $r < d$. In this case, the following triangle commutes by Theorem A.6.7 below:

$$\begin{tikzcd} H^{i+1}_{\mathsf{syn}}((X,M),\mathscr{S}_{\mathbb{Q}_p}(r))^0 \arrow[d] \arrow[dr, "\gamma^{i,r}"] & \\ H^1(K,\mathscr{V}^i(r)) \arrow[r, "\beta^{i,r}"] & H^1(K,H^i_{\mathsf{crys}}((\overline{X},\overline{M})/W)_{\mathbb{Q}_p}(r-d)), \end{tikzcd} \tag{9.9.2}$$

where the left vertical arrow is the composite of the top horizontal and the right vertical arrows of (9.9.1). Moreover the following square commutes by the construction of the C_{st}-isomorphism (9.3.4) (compare with (9.3.5)):

$$\begin{tikzcd} \mathscr{V}^i(r) \arrow[r, "\beta^{i,r}"] \arrow[d, "\mathbb{Q}_p(r)\hookrightarrow B_{\mathsf{crys}}"'] & H^i_{\mathsf{crys}}((\overline{X},\overline{M})/W)_{\mathbb{Q}_p}(r-d) \arrow[d] \\ \mathscr{V}^i \otimes_{\mathbb{Q}_p} B_{\mathsf{crys}} \arrow[r, "\text{(9.3.1), (9.3.4)}", "\sim"'] & (D^i \otimes_{\mathbb{Q}_p} B_{\mathsf{st}})^{N=0}, \end{tikzcd} \tag{9.9.3}$$

where the right vertical arrow is given by the isomorphism (2) of (9.3.5) and the inclusion $\mathbb{Q}_p(r-d) \hookrightarrow B_{\mathsf{crys}}$. The commutativity of (9.9.1) follows from these commutative diagrams and the fact that the composite map (9.5.1) agrees with the composite of $\gamma^{i,r}$ and the map

$$H^1(K, H^i_{\mathsf{crys}}((\overline{X},\overline{M})/W)_{\mathbb{Q}_p}(r-d)) \longrightarrow H^1(K, \mathscr{V}^i \otimes_{\mathbb{Q}_p} B_{\mathsf{crys}})$$

induced by the right vertical arrow and the bottom isomorphism in (9.9.3).

When $r \geq d$, the following triangle commutes by Corollary A.6.3 below:

$$\begin{tikzcd} H^{i+1}_{\mathsf{syn}}((X,M),\mathscr{S}_{\mathbb{Q}_p}(r))^0 \arrow[d] \arrow[dr, "\gamma^{i,r}"] & \\ H^1(K,\mathscr{V}^i(r)) \arrow[r, "\beta^{i,r}"] & H^1(K,H^i_{\mathsf{crys}}((\overline{X},\overline{M})/W)_{\mathbb{Q}_p}), \end{tikzcd}$$

where the left vertical arrow is defined in the same way as for that of (9.9.2). The commutativity of (9.9.1) follows from this diagram and the diagram (9.3.5). This completes the proof of Theorem 9.9. $\square$

Proposition 9.10 *The triangle* (B) *in* (9.2.1) *is commutative.*

Proof. There is a commutative diagram by the definition of ∂ in Proposition A.2.4

$$\begin{tikzcd} H^i_{\mathsf{crys}}((X,M)/(\mathscr{E},M_{\mathscr{E}}))_{\mathbb{Q}_p} \arrow[r, "t\mapsto 0"] \arrow[d, "\partial"'] & D^i \arrow[d, "\frac{dt}{t}\wedge"] \\ H^{i+1}_{\mathsf{cont\text{-}cr}}((X,M)/W)_{\mathbb{Q}_p} \arrow[r, "t\mapsto 0"] & \widetilde{D}^{i+1}, \end{tikzcd}$$

where $\widetilde{D}^{i+1} = \widetilde{D}^{i+1}(Y)$ is as in the proof of Theorem 8.12, the left (resp. right) vertical arrow is as in Lemma 9.7 (1) (resp. given by the product with $d\log(t)$, cf. (8.9.1)), and the horizontal arrows are the specialization maps by $t \mapsto 0$. Therefore in view of the constructions of α and σ', it is enough to check that the following diagram of canonical morphisms commutes in $D^+(Y_{\text{ét}}, \mathbb{Z}/p^n)$ for each $n \geq 0$:

$$\begin{tikzcd}
\mathscr{S}_n(r)_{(X,M)} \arrow[r] \arrow[d, "\sigma"'] & \mathbb{J}^{[r]}_{(X_n,M_n)/W_n} \arrow[r, "s"] & W_n\widetilde{\omega}_Y^{\bullet \geq r} \arrow[r] \arrow[d, "(8.10.1)"'] & W_n\widetilde{\omega}_Y^{\bullet} \arrow[d, "(8.10.1)"] \\
\nu^{r-1}_{Y,n}[-r] \arrow[rr, "\text{Remark 8.5}"] & & W_n\,\varXi_Y^{\bullet \geq r-1}[-1] \arrow[r] & W_n\,\varXi_Y^{\bullet}[-1],
\end{tikzcd}$$

where $\mathbb{J}^{[r]}_{(X_n,M_n)/W_n}$ is as in §A.5 below, and s denotes the specialization map by $t \mapsto 0$. The right square commutes obviously in the category of complexes of sheaves. As for the left square, we have only to show the commutativity on the diagram of the r-th cohomology sheaves because $W_n\,\varXi_Y^{\bullet \geq r-1}[-1]$ is concentrated in degrees $\geq r$ and $\mathscr{S}_n(r)_{(X,M)}$ is concentrated in degrees $\leq r$ by (9.4.1). Since we have

$$\mathscr{H}^r(\mathscr{S}_n(r)_{(X,M)}) \overset{(9.4.1)}{\simeq} (R^r j_* \mu_{p^n}^{\otimes r})|_Y$$

and the sheaf on the right hand side is generated by symbols (cf. the case $D = \emptyset$ of Theorem 3.3 (1)), one can check this commutativity of the diagram of sheaves by a straight-forward computation on symbols using the compatibility in [Ts1] Proposition 3.2.4 (2). □

PROOF OF THEOREM 9.1. We have the following commutative diagram by (9.2.1):

$$\begin{tikzcd}
F^1H^{i+1}(X, \mathfrak{T}_{\mathbb{Q}_p}(r)) \arrow[r] \arrow[dr] & F^1H^{i+1}(X, \tau_{\leq r}Rj_*\mathbb{Q}_p(r)) \arrow[rr, "\sigma'"] \arrow[d] \arrow[drr, "\alpha"'] & \arrow[d, phantom, "\text{(B)}"] & E^i \\
& H^1(K, \mathscr{V}^i(r)) \arrow[r, phantom, "\text{(A)}"] \arrow[drr, "\iota"'] & & D^i/ND^i \arrow[u, hook, "\epsilon"'] \arrow[d, "\delta"] \\
& & & H^1(K, \mathscr{V}^i \otimes_{\mathbb{Q}_p} B_{\mathsf{crys}}),
\end{tikzcd}$$

where the left triangle commutes obviously. The kernel of ι is $H^1_f(K, \mathscr{V}^i(r))$ by definition. The top row is a complex by the definitions of σ' and $\mathfrak{T}_n(r)_X$. The map ϵ is injective by Theorem 8.12 and Conjecture 8.11 for D^i. We obtain the first assertion

$$\varPhi^{i,r} \subset H^1_f(K, \mathscr{V}^i(r)) \tag{9.10.1}$$

by a simple diagram chase on this diagram. To obtain the second assertion, we repeat the same arguments for $\varPhi^{2d-i,d-r+1}$, where we put $d := \dim(X_K)$. The map

$$\epsilon : D^{2d-i}/ND^{2d-i} \longrightarrow E^{2d-i}$$

is injective by Conjecture 8.11 for D^i, Theorems 8.12 and 8.8 and the Poincaré duality between D^i and D^{2d-i} [Hy2]. Hence we have

$$\Phi^{2d-i,d-r+1} \subset H^1_f(K, \mathscr{V}^{2d-i}(d-r+1)). \tag{9.10.2}$$

The results (9.10.1) and (9.10.2) imply $\Phi^{i,r} = H^1_f(K, \mathscr{V}^i(r))$, because $\Phi^{i,r}$ and $\Phi^{2d-i,d-r+1}$ (resp. $H^1_f(K, \mathscr{V}^i(r))$ and $H^1_f(K, \mathscr{V}^{2d-i}(d-r+1))$) are the exact annihilators of each other under the non-degenerate pairing of the Tate duality

$$H^1(K, \mathscr{V}^i(r)) \times H^1(K, \mathscr{V}^{2d-i}(d-r+1)) \longrightarrow \mathbb{Q}_p$$

by [Sa1] Corollary 10.6.1 (resp. [BK2] Proposition 3.8). □

10 Image of p-adic regulators

In this section, we prove Theorem 1.1. We first review the definitions of the finite part of Galois cohomology [BK2] and the integral part of algebraic *K*-groups [Sch] over number fields.

Let K be a number field and let $\mathscr{V}$ be a topological G_K-$\mathbb{Q}_p$-module which is finite-dimensional over $\mathbb{Q}_p$. Let $\mathfrak{o}$ be the integer ring of K. For a place v of K, let K_v (resp. $\mathfrak{o}_v$) be the completion of K (resp. $\mathfrak{o}$) at v. The finite part $H^1_f(K, \mathscr{V}) \subset H^1(K, \mathscr{V})$ is defined as

$$H^1_f(K, \mathscr{V}) := \mathsf{Ker}\left(H^1(K, \mathscr{V}) \longrightarrow \prod_{v\,:\,\text{finite}} \frac{H^1(K_v, \mathscr{V})}{H^1_f(K_v, \mathscr{V})} \right),$$

where v runs through all finite places of K, and $H^1_f(K_v, \mathscr{V})$ is defined as follows:

$$H^1_f(K_v, \mathscr{V}) := \begin{cases} \mathsf{Ker}(H^1(K_v, \mathscr{V}) \to H^1(I_v, \mathscr{V})) & (\text{if } v \nmid p), \\ \mathsf{Ker}(H^1(K_v, \mathscr{V}) \to H^1(K_v, \mathscr{V} \otimes_{\mathbb{Q}_p} B_{\mathsf{crys}})) & (\text{if } v | p), \end{cases}$$

where I_v denotes the inertia subgroup of G_{K_v}. We next review the definition of the integral part of algebraic *K*-groups. Let V be a proper smooth variety over the number field K. First fix a finite place v of K. By de Jong's alteration theorem [dJ], there is a proper generically finite morphism

$$\pi : V' \longrightarrow V_v := V \otimes_K K_v$$

such that V' has a projective regular model X' with strict semistable reduction over the integer ring of some finite extension of K_v. The integral part $K_m(V_v)_{\mathfrak{o}_v} \subset K_m(V_v) \otimes \mathbb{Q}$ is the kernel of the composite map

$$K_m(V_v) \otimes \mathbb{Q} \xrightarrow{\pi^*} K_m(V') \otimes \mathbb{Q} \longrightarrow \frac{K_m(V') \otimes \mathbb{Q}}{\text{Image of } K_m(X') \otimes \mathbb{Q}},$$

which is in fact independent of X' ([Sch] §1). The integral part $K_m(V)_{\mathfrak{o}}$ is defined as

$$K_m(V)_{\mathfrak{o}} := \mathsf{Ker}\left(K_m(V) \otimes \mathbb{Q} \longrightarrow \prod_{v\,:\,\text{finite}} \frac{K_m(V_v) \otimes \mathbb{Q}}{K_m(V_v)_{\mathfrak{o}_v}} \right),$$

where v runs through all finite places of K. If V admits a regular model which is proper flat over the integer ring of K, then $K_m(V)_{\mathfrak{o}}$ agrees with the image of the K-group of the model, i.e., the integral part considered by Beilinson. By these definitions of $H^1_f(K, \mathscr{V})$ and $K_{2r-i-1}(V)_{\mathfrak{o}}$, Theorem 1.1 is immediately reduced to Theorem 10.1 below, which is an analogue of Theorem 1.1 over local fields.

Let ℓ and p be prime numbers. We change the setting slightly, and let K be an ℓ-adic local field, i.e., a finite extension of $\mathbb{Q}_\ell$. Let O_K be the integer ring of K, and let V be a proper smooth variety over K. Let i and r be integers with $2r \geq i+1 \geq 1$. There is a p-adic regulator map obtained from étale Chern character

$$\mathsf{reg}_p : K_{2r-i-1}(V)_{O_K,\,\mathsf{hom}} \longrightarrow H^1(K, H^i(\overline{V}, \mathbb{Q}_p(r))),$$

where $\overline{V}$ denotes $V \otimes_K \overline{K}$, and $K_{2r-i-1}(V)_{O_K,\,\mathsf{hom}}$ denotes the subspace of $K_{2r-i-1}(V)_{O_K}$ consisting of all elements which vanish in $H^{i+1}(\overline{V}, \mathbb{Q}_p(r))$ under the Chern character.

THEOREM 10.1 *If $\ell = p$, assume that $r \leq p-2$ and that Conjecture* 8.11 *holds for projective strict semistable varieties over $\mathbb{F}_p$ in degree i. Then $\mathsf{Im}(\mathsf{reg}_p)$ is contained in $H^1_f(K, H^i(\overline{V}, \mathbb{Q}_p(r)))$.*

REMARK 10.2 *When $\ell = p$, we need Conjecture* 8.11 *for the reduction of an alteration of V. When $\ell \neq p$, we do not need the monodromy-weight conjecture, but use Deligne's proof of the Weil conjecture* [D] *to show that reg_p is zero.*

Proof of Theorem 10.1. We first reduce the problem to the case that V has a regular model which is projective flat over O_K with strict semistable reduction. By de Jong's alteration theorem [dJ], there exists a proper generically finite morphism $\pi : V' \to V$ such that V' has a projective regular model with strict semistable reduction over the integer ring O_L of $L := \Gamma(V', \mathscr{O}_{V'})$. Then there is a commutative diagram

$$\begin{array}{ccccc}
K_{2r-i-1}(V)_{O_K,\,\mathsf{hom}} & \xrightarrow{\mathsf{reg}_p} & H^1(K, H^i(\overline{V}, \mathbb{Q}_p(r))) & \longrightarrow & \dfrac{H^1(K, H^i(\overline{V}, \mathbb{Q}_p(r)))}{H^1_f(K, H^i(\overline{V}, \mathbb{Q}_p(r)))} \\
{\scriptstyle \pi^*}\big\downarrow & & {\scriptstyle \pi^*}\big\downarrow & & {\scriptstyle \pi^*}\big\downarrow \\
K_{2r-i-1}(V')_{O_L,\,\mathsf{hom}} & \xrightarrow{\mathsf{reg}_p} & H^1(L, H^i(\overline{V'}, \mathbb{Q}_p(r))) & \longrightarrow & \dfrac{H^1(L, H^i(\overline{V'}, \mathbb{Q}_p(r)))}{H^1_f(L, H^i(\overline{V'}, \mathbb{Q}_p(r)))}
\end{array}$$

where $\overline{V'}$ denotes $V' \otimes_L \overline{K}$, and the right (and the middle) vertical arrows are split injective by a standard argument using a corestriction map of Galois cohomology and a trace map of étale cohomology. By this diagram and the definition of $K_*(V)_{O_K}$, Theorem 10.1 for V is reduced to that for V'. Thus we may assume that V has a projective regular model X with strict semistable reduction over O_K. Then the case $\ell \neq p$ follows from [Ne1] II Theorem 2.2 (cf. [D]). We prove the case $\ell = p$. Let Y be the reduction of X. Put

$$H^{i+1}(X, \mathfrak{T}_{\mathbb{Q}_p}(r)) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_{n \geq 1} H^{i+1}(X, \mathfrak{T}_n(r)),$$

and let $K_{2r-i-1}(X)_{\mathsf{hom}}$ (resp. $H^{i+1}(X,\mathfrak{T}_{\mathbb{Q}_p}(r))^0$) be the kernel of the composite map

$$\begin{aligned}&K_{2r-i-1}(X) \longrightarrow K_{2r-i-1}(V) \xrightarrow{\mathsf{ch}} H^{i+1}(V,\mathbb{Q}_p(r)) \to H^{i+1}(\overline{V},\mathbb{Q}_p(r))\\ &(\text{resp. } \ H^{i+1}(X,\mathfrak{T}_{\mathbb{Q}_p}(r)) \longrightarrow H^{i+1}(V,\mathbb{Q}_p(r)) \to H^{i+1}(\overline{V},\mathbb{Q}_p(r)) \ ).\end{aligned}$$

There is a commutative diagram by Corollary 5.7

$$\begin{tikzcd} K_{2r-i-1}(X)_{\mathsf{hom}} \arrow[r, two heads] \arrow[d, "\mathsf{ch}"'] & K_{2r-i-1}(V)_{O_K,\,\mathsf{hom}} \arrow[d, "\mathsf{reg}_p"] \\ H^{i+1}(X,\mathfrak{T}_{\mathbb{Q}_p}(r))^0 \arrow[r] & H^1(K,H^i(\overline{V},\mathbb{Q}_p(r))). \end{tikzcd}$$

The image of the bottom arrow is contained in $H^1_f(K,H^i(\overline{V},\mathbb{Q}_p(r)))$ by Theorem 9.1 and Conjecture 8.11 for $D^i = D^i(Y)$, which implies Theorem 10.1. This completes the proof of Theorems 10.1 and 1.1. $\square$

A Continuous crystalline cohomology

In this appendix, we formulate continuous versions of crystalline and syntomic cohomology of log schemes, combining the methods of Jannsen, Kato and Tsuji ([J], [K2], [K3], [Ts1], [Ts2]). The results of this appendix have been used in §9 of this paper. See [K3] for the general framework of log structures and log schemes.

Let p be a prime number, and let K be a complete discrete valuation field of characteristic 0 whose residue field k is a perfect field of characteristic p. Let O_K be the integer ring of K. Put $W := W(k)$, $W_n := W_n(k)$ $(n \geq 1)$ and $K_0 := \mathsf{Frac}(W)$. Let X be a regular scheme which is projective flat over O_K with semistable reduction, and put

$$\begin{aligned}X_K := X \otimes_{O_K} K, \qquad Y := X \otimes_{O_K} k\\ X_{\overline{K}} := X_K \otimes_K \overline{K}, \quad \overline{Y} := Y \otimes_k \overline{k} \quad \text{and} \quad \overline{X} := X \otimes_{O_K} \overline{O_K},\end{aligned}$$

where $\overline{O_K}$ denotes the integral closure of O_K in $\overline{K}$. Let M be the log structure on X associated with the normal crossing divisor Y. We endow $\overline{X}$ with a log structure $\overline{M}$ as follows. For a finite field extension L/K, put $S_L := \mathsf{Spec}(O_L)$, and let M_{S_L} be the log structure on S_L associated with its closed point. We denote (S_K, M_{S_K}) simply by (S, M_S), and define (X_{O_L}, M_{O_L}) by base-change in the category of log schemes

$$(X_{O_L}, M_{O_L}) := (X,M) \times_{(S,M_S)} (S_L, M_{S_L})\,. \tag{A.0.1}$$

We then define the log structure $\overline{M}$ on $\overline{X}$ as that associated with the pre-log structure

$$\varinjlim_{K\subset L\subset \overline{K}} M_{O_L}|_{\overline{X}}, \tag{A.0.2}$$

where L runs through all finite field extensions of K contained in $\overline{K}$, $M_{O_L}|_{\overline{X}}$ denotes the topological inverse image of M_{O_L} onto $\overline{X}$, and the inductive limit is taken in the category of étale sheaves of monoids on $\overline{X}$.

For a log scheme (Z, M_Z) and an integer $n \geq 1$, we define

$$(Z_n, M_{Z_n}) := (Z, M_Z) \times_{\mathsf{Spec}(\mathbb{Z})} \mathsf{Spec}(\mathbb{Z}/p^n),$$

where we regarded $\mathsf{Spec}(\mathbb{Z})$ and $\mathsf{Spec}(\mathbb{Z}/p^n)$ as log schemes by endowing them with the trivial log structures and the fiber product is taken in the category of log schemes.

We denote by $\mathit{Mod}(\mathbb{Z}/p^\bullet)$ the category of the projective systems $\{F_n\}_{n\geq 1}$ of $\mathbb{Z}_p$-modules such that F_n is a $\mathbb{Z}/p^n$-module for each $n \geq 1$. For a pro-finite group G, $\mathit{Mod}(G\text{-}\mathbb{Z}/p^\bullet)$ denotes the category of the projective systems $\{F_n\}_{n\geq 1}$ of discrete G-modules such that F_n is a $\mathbb{Z}/p^n$-module.

For a scheme T, $\mathit{Shv}(T_{\text{ét}}, \mathbb{Z}/p^\bullet)$ denotes the category of the projective systems $\{\mathscr{F}_n\}_{n\geq 1}$ of étale sheaves on T such that $\mathscr{F}_n$ is a $\mathbb{Z}/p^n$-sheaf for each $n \geq 1$. For a pro-finite group G acting on T, $\mathit{Shv}(T_{\text{ét}}, G\text{-}\mathbb{Z}/p^\bullet)$ denotes the category of the projective systems $\{\mathscr{F}_n\}_{n\geq 1}$ of étale G-sheaves on T such that $\mathscr{F}_n$ is a $\mathbb{Z}/p^n$-sheaf for each $n \geq 1$. We write $D(T_{\text{ét}}, \mathbb{Z}/p^\bullet)$ (resp. $D(T_{\text{ét}}, G\text{-}\mathbb{Z}/p^\bullet)$) for the derived category of $\mathit{Shv}(T_{\text{ét}}, \mathbb{Z}/p^\bullet)$ (resp. $\mathit{Shv}(T_{\text{ét}}, G\text{-}\mathbb{Z}/p^\bullet)$).

For an additive category $\mathscr{C}$, we write $\mathbb{Q} \otimes \mathscr{C}$ for the $\mathbb{Q}$-tensor category of $\mathscr{C}$, i.e., the category whose objects are the same as $\mathscr{C}$ and such that for objects $A, B \in \mathscr{C}$, the group of morphisms $\mathsf{Hom}_{\mathbb{Q}\otimes\mathscr{C}}(A, B)$ is given by $\mathbb{Q} \otimes \mathsf{Hom}_{\mathscr{C}}(A, B)$. We often write $\mathbb{Q} \otimes A$ for $A \in \mathscr{C}$ regarded as an object of $\mathbb{Q} \otimes \mathscr{C}$ to avoid confusions.

A.1 CRYSTALLINE COMPLEXES

We construct the following objects:

$$\begin{gathered}\mathbb{E}_{(X_\bullet, M_\bullet)/W_\bullet}, \mathbb{E}_{(X_\bullet, M_\bullet)/(\mathscr{E}_\bullet, M_{\mathscr{E}_\bullet})} \in D^+(Y_{\text{ét}}, \mathbb{Z}/p^\bullet) \\ \mathbb{E}_{(\overline{X}_\bullet, \overline{M}_\bullet)/W_\bullet}, \mathbb{E}_{(\overline{X}_\bullet, \overline{M}_\bullet)/(\mathscr{E}_\bullet, M_{\mathscr{E}_\bullet})} \in D^+(\overline{Y}_{\text{ét}}, G_K\text{-}\mathbb{Z}/p^\bullet),\end{gathered}$$

where W_n means $\mathsf{Spec}(W_n)$ endowed with the trivial log structure for each $n \geq 1$. See Definition 9.6 for $(\mathscr{E}_n, M_{\mathscr{E}_n})$. To construct $\mathbb{E}_{(X_\bullet, M_\bullet)/W_\bullet}$, we fix an étale hypercovering $(X^\star, M^\star) \to (X, M)$ and a closed immersion $(X^\star, M^\star) \hookrightarrow (Z^\star, M_{Z^\star})$ of simplicial fine log schemes over W such that (Z^i, M_{Z^i}) is smooth over W for each $i \in \mathbb{N}$ (cf. [HK] (2.18)). Put

$$Y^\star := X^\star \otimes_{O_K} k,$$

which is an étale hypercovering of Y. For $n \geq 1$, let $(\mathscr{D}_n^\star, M_{\mathscr{D}_n^\star})$ be the PD-envelope of $(X_n^\star, M_n^\star) \hookrightarrow (Z_n^\star, M_{Z_n^\star})$ with respect to the canonical PD-structure on $(p) \subset W_n$ ([K3] Definition 5.4), and we define a complex $\mathbb{E}_{(X_n^\star, M_n^\star)/W_n}$ of sheaves on $Y_{\text{ét}}^\star$ as

$$\mathscr{O}_{\mathscr{D}_n^\star} \xrightarrow{d} \mathscr{O}_{\mathscr{D}_n^\star} \otimes_{\mathscr{O}_{Z_n^\star}} \omega^1_{(Z_n^\star, M_{Z_n^\star})/W_n} \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{O}_{\mathscr{D}_n^\star} \otimes_{\mathscr{O}_{Z_n^\star}} \omega^q_{(Z_n^\star, M_{Z_n^\star})/W_n} \xrightarrow{d} \cdots,$$

where the first term is placed in degree 0 and $\omega^q_{(Z_n^\star, M_{Z_n^\star})/W_n}$ denotes the q-th differential module of $(Z_n^\star, M_{Z_n^\star})$ over W_n cf. [K3] (1.7). See loc. cit. Theorem 6.2 for d.

Regarding this complex as a complex of projective systems (with respect to $n \geq 1$) of sheaves on $Y^\star_{\text{ét}}$, we obtain a complex $\mathbb{E}_{(X^\star_\bullet, M^\star_\bullet)/W_\bullet}$ of objects of $Shv(Y^\star_{\text{ét}}, \mathbb{Z}/p^\bullet)$. We then define

$$\mathbb{E}_{(X_\bullet, M_\bullet)/W_\bullet} := R\theta_*(\mathbb{E}_{(X^\star_\bullet, M^\star_\bullet)/W_\bullet}) \in D^+(Y_{\text{ét}}, \mathbb{Z}/p^\bullet),$$

where $\theta : Shv(Y^\star_{\text{ét}}, \mathbb{Z}/p^\bullet) \to Shv(Y_{\text{ét}}, \mathbb{Z}/p^\bullet)$ denotes the natural morphism of topoi. The resulting object $\mathbb{E}_{(X_\bullet, M_\bullet)/W_\bullet}$ is independent of the choice of the pair $((X^\star, M^\star), (Z^\star, M_{Z^\star}))$ by a standard argument (cf. [K2] p. 212).

To construct $\mathbb{E}_{(X_\bullet, M_\bullet)/(\mathscr{E}_\bullet, M_{\mathscr{E}_\bullet})}$, we put $V := \mathsf{Spec}(W[t])$, and define M_V as the log structure on V associated with the divisor $\{t = 0\}$. We regard (X, M) as a log scheme over (V, M_V) by the composite map

$$(X, M) \to (S, M_S) \to (V, M_V),$$

where the last map is given by $T \mapsto \pi$, the prime element of O_K we fixed in Definition 9.6 to define $(\mathscr{E}_n, M_{\mathscr{E}_n})$. We then apply the same construction as for $\mathbb{E}_{(X_\bullet, M_\bullet)/W_\bullet}$ to the morphism $(X, M) \to (V, M_V)$, i.e, fix an étale hypercovering $(X^\star, M^\star) \to (X, M)$ and a closed immersion $i^\star : (X^\star, M^\star) \hookrightarrow (Z^\star, M_{Z^\star})$ of simplicial log schemes over (V, M_V) such that (Z^i, M_{Z^i}) is smooth over W for each $i \in \mathbb{N}$. We define $\mathbb{E}_{(X_\bullet, M_\bullet)/(\mathscr{E}_\bullet, M_{\mathscr{E}_\bullet})}$ by replacing $\omega^q_{(Z^\star_n, M_{Z^\star_n})/W_n}$ with $\omega^q_{(Z^\star_n, M_{Z^\star_n})/(V_n, M_{V_n})}$ in the above definition of $\mathbb{E}_{(X_\bullet, M_\bullet)/W_\bullet}$.

We construct $\mathbb{E}_{(\overline{X}_\bullet, \overline{M}_\bullet)/W_\bullet}$ as follows. Fix an étale hypercovering $(X^\star, M^\star) \to (X, M)$ and for each finite extension L/K contained in $\overline{K}$, fix a closed immersion $(X^\star_{O_L}, M^\star_{O_L}) \hookrightarrow (Z^\star_L, M_{Z^\star_L})$ ($(X^\star_{O_L}, M^\star_{O_L}) := (X^\star, M^\star) \times_{(S, M_S)} (S_L, M_{S_L})$) of simplicial fine log schemes over W such that $(Z^i_L, M_{Z^i_L})$ is smooth over W for each $i \in \mathbb{N}$ and L/K, and such that for finite extensions L'/L there are morphisms $\tau_{L'/L} : (Z^\star_{L'}, M_{Z^\star_{L'}}) \to (Z^\star_L, M_{Z^\star_L})$ satisfying transitivity. For a finite extension L/K, let k_L be the residue field of L, and put

$$Y^\star_L := X^\star \otimes_{O_K} k_L \quad \text{and} \quad \overline{Y}^\star := X^\star \otimes_{O_K} \overline{k},$$

which are étale hypercoverings of $Y_L := Y \otimes_k k_L$ and $\overline{Y}$, respectively. We define a complex $\mathbb{E}_{(X^\star_{O_L,\bullet}, M^\star_{O_L,\bullet})/W_\bullet}$ on $Y^\star_L$ applying the same construction as for $\mathbb{E}_{(X^\star_\bullet, M^\star_\bullet)/W_\bullet}$ to the embedding $(X^\star_{O_L}, M^\star_{O_L}) \hookrightarrow (Z^\star_L, M_{Z^\star_L})$, whose inverse image onto $\overline{Y}^\star$ yields an inductive system of complexes of objects of $Shv(\overline{Y}^\star_{\text{ét}}, G_K\text{-}\mathbb{Z}/p^\bullet)$ with respect to finite extensions L/K. We then define

$$\mathbb{E}_{(\overline{X}^\star_\bullet, \overline{M}^\star_\bullet)/W_\bullet} := \varinjlim_{K \subset L \subset \overline{K}} \mathbb{E}_{(X^\star_{O_L,\bullet}, M^\star_{O_L,\bullet})/W_\bullet}\big|_{\overline{Y}^\star},$$
$$\mathbb{E}_{(\overline{X}_\bullet, \overline{M}_\bullet)/W_\bullet} := R\overline{\theta}_*(\mathbb{E}_{(\overline{X}^\star_\bullet, \overline{M}^\star_\bullet)/W_\bullet}) \in D^+(\overline{Y}_{\text{ét}}, G_K\text{-}\mathbb{Z}/p^\bullet),$$

where $\overline{\theta} : Shv(\overline{Y}^\star_{\text{ét}}, G_K\text{-}\mathbb{Z}/p^\bullet) \to Shv(\overline{Y}_{\text{ét}}, G_K\text{-}\mathbb{Z}/p^\bullet)$ denotes the natural morphism of topoi. We construct $\mathbb{E}_{(\overline{X}_\bullet, \overline{M}_\bullet)/(\mathscr{E}_\bullet, M_{\mathscr{E}_\bullet})}$ applying similar arguments to the complexes $\mathbb{E}_{(X^\star_{O_L,\bullet}, M^\star_{O_L,\bullet})/(\mathscr{E}_\bullet, M_{\mathscr{E}_\bullet})}\big|_{\overline{Y}^\star}$.

We define the following objects of $\mathit{Mod}(\mathbb{Z}/p^\bullet)$:

$$H^i_{\mathsf{crys}}((X_\bullet, M_\bullet)/W_\bullet) := R^i\Gamma(\mathbb{E}_{(X_\bullet,M_\bullet)/W_\bullet}), \tag{A.1.1}$$

$$H^i_{\mathsf{crys}}((X_\bullet, M_\bullet)/(\mathscr{E}_\bullet, M_{\mathscr{E}_\bullet})) := R^i\Gamma(\mathbb{E}_{(X_\bullet,M_\bullet)/(\mathscr{E}_\bullet,M_{\mathscr{E}_\bullet})}), \tag{A.1.2}$$

$$H^i_{\mathsf{crys}}((\overline{X}_\bullet, \overline{M}_\bullet)/W_\bullet) := R^i\overline{\Gamma}\big(\mathbb{E}_{(\overline{X}_\bullet,\overline{M}_\bullet)/W_\bullet}\big) \tag{A.1.3}$$

$$H^i_{\mathsf{crys}}((\overline{X}_\bullet, \overline{M}_\bullet)/(\mathscr{E}_\bullet, M_{\mathscr{E}_\bullet})) := R^i\overline{\Gamma}\big(\mathbb{E}_{(\overline{X}_\bullet,\overline{M}_\bullet)/(\mathscr{E}_\bullet,M_{\mathscr{E}_\bullet})}\big), \tag{A.1.4}$$

where Γ and $\overline{\Gamma}$ denote the following left exact functors, respectively:

$$\Gamma := \Gamma(Y, -) : \mathit{Shv}(Y_{\text{ét}}, \mathbb{Z}/p^\bullet) \to \mathit{Mod}(\mathbb{Z}/p^\bullet), \tag{A.1.5}$$

$$\overline{\Gamma} := \Gamma(\overline{Y}, -) : \mathit{Shv}(\overline{Y}_{\text{ét}}, G_K\text{-}\mathbb{Z}/p^\bullet) \to \mathit{Mod}(G_K\text{-}\mathbb{Z}/p^\bullet). \tag{A.1.6}$$

We define 'naive' crystalline cohomology groups

$$\begin{array}{lcl} H^i_{\mathsf{crys}}((X,M)/W), & & H^i_{\mathsf{crys}}((X,M)/(\mathscr{E}, M_{\mathscr{E}})), \\ H^i_{\mathsf{crys}}((\overline{X},\overline{M})/W) & \text{and} & H^i_{\mathsf{crys}}((\overline{X},\overline{M})/(\mathscr{E}, M_{\mathscr{E}})) \end{array}$$

as the projective limit of (A.1.1)–(A.1.4), respectively.

Remark A.1.6 *Let $\pi_n : \mathit{Shv}(Y_{\text{ét}}, \mathbb{Z}/p^\bullet) \to \mathit{Shv}(Y_{\text{ét}}, \mathbb{Z}/p^n)$ be the natural functor sending $\{\mathscr{F}_m\}_{m\geq 1}$ to $\mathscr{F}_n$. Since π_n is exact, it extends to a triangulated functor*

$$\pi_n : D^+(Y_{\text{ét}}, \mathbb{Z}/p^\bullet) \longrightarrow D^+(Y_{\text{ét}}, \mathbb{Z}/p^n),$$

which sends $\mathbb{E}_{(X_\bullet,M_\bullet)/W_\bullet} \mapsto \mathbb{E}_{(X_n,M_n)/W_n}$, an object computing the crystalline cohomology of $(X_n, M_n)/W_n$, because π_n is compatible with the gluing functor $R\theta_*$. Moreover by [J] *Proposition* 1.1 (b)*, we have*

$$H^i_{\mathsf{crys}}((X_\bullet, M_\bullet)/W_\bullet) \simeq \{H^i_{\mathsf{crys}}((X_n, M_n)/W_n)\}_{n\geq 1}.$$

We have similar facts for (A.1.2)*,* (A.1.3) *and* (A.1.4) *as well.*

A.2 Continuous crystalline cohomology

We define continuous crystalline cohomology groups as follows:

$$H^i_{\mathsf{cont\text{-}cr}}((X,M)/W) := R^i\big(\varprojlim \circ \Gamma\big)\big(\mathbb{E}_{(X_\bullet,M_\bullet)/W_\bullet}\big), \tag{A.2.1}$$

$$H^i_{\mathsf{cont\text{-}cr}}((X,M)/(\mathscr{E}, M_{\mathscr{E}})) := R^i\big(\varprojlim \circ \Gamma\big)\big(\mathbb{E}_{(X_\bullet,M_\bullet)/(\mathscr{E}_\bullet,M_{\mathscr{E}_\bullet})}\big), \tag{A.2.2}$$

where Γ is as in (A.1.5). Because Γ has an exact left adjoint, it preserves injectives and there exists a spectral sequence

$$E_2^{a,b} = R^a \varprojlim\, H^b_{\mathsf{crys}}((X_\bullet, M_\bullet)/W_\bullet) \Longrightarrow H^{a+b}_{\mathsf{cont\text{-}cr}}((X,M)/W).$$

Because $R^a\varprojlim = 0$ for $a \geq 2$, this spectral sequence breaks up into short exact sequences

$$\begin{aligned} 0 \longrightarrow R^1\varprojlim\, H^{i-1}_{\mathsf{crys}}((X_\bullet, M_\bullet)/W_\bullet) \longrightarrow H^i_{\mathsf{cont\text{-}cr}}&((X,M)/W) \\ \longrightarrow H^i_{\mathsf{crys}}&((X,M)/W) \longrightarrow 0. \end{aligned} \tag{A.2.3}$$

We have similar exact sequences for (A.2.2) by the same arguments.

Proposition A.2.4 *There exists a distinguished triangle*

$$\mathbb{E}_{(X_\bullet,M_\bullet)/W_\bullet} \xrightarrow{\mathsf{can}} \mathbb{E}_{(X_\bullet,M_\bullet)/(\mathscr{E}_\bullet,M_{\mathscr{E}_\bullet})} \xrightarrow{\nu} \mathbb{E}_{(X_\bullet,M_\bullet)/(\mathscr{E}_\bullet,M_{\mathscr{E}_\bullet})} \xrightarrow{\frac{dt}{t}\wedge} \mathbb{E}_{(X_\bullet,M_\bullet)/W_n}[1]$$

in $D^+(Y_{\text{ét}}, \mathbb{Z}/p^\bullet)$. *Consequently we obtain a long exact sequence*

$$\begin{aligned} \cdots \longrightarrow H^i_{\mathsf{cont\text{-}cr}}((X,M)/W) \longrightarrow H^i_{\mathsf{cont\text{-}cr}}((X,M)/(\mathscr{E},M_{\mathscr{E}})) \\ \xrightarrow{\nu} H^i_{\mathsf{cont\text{-}cr}}((X,M)/(\mathscr{E},M_{\mathscr{E}})) \xrightarrow{\partial} H^{i+1}_{\mathsf{cont\text{-}cr}}((X,M)/W) \longrightarrow \cdots. \end{aligned}$$

Proof. The assertion follows from the same arguments as in the proof of [K4] Lemma 4.2 with $\mathcal{F} = \mathscr{O}^{\mathsf{crys}}$. □

There is a commutative diagram (see Definition 9.6 for N)

$$\begin{array}{ccc} H^i_{\mathsf{cont\text{-}cr}}((X,M)/(\mathscr{E},M_{\mathscr{E}})) & \xrightarrow{\nu} & H^i_{\mathsf{cont\text{-}cr}}((X,M)/(\mathscr{E},M_{\mathscr{E}})) \\ \downarrow & & \downarrow \\ H^i_{\mathsf{crys}}((X,M)/(\mathscr{E},M_{\mathscr{E}})) & \xrightarrow{N} & H^i_{\mathsf{crys}}((X,M)/(\mathscr{E},M_{\mathscr{E}})). \end{array}$$

We will show later that the vertical arrows are bijective in Corollary A.3.2 (1) below.

A.3 Comparison of projective systems of crystalline cohomology

We recall here the following comparison facts on projective systems of crystalline cohomology groups, which will be useful later. Note that for $A_\bullet \in \mathit{Mod}(\mathbb{Z}/p^\bullet)$, both $\varprojlim A_\bullet$ and $R^1 \varprojlim A_\bullet$ have finite exponents if $\mathbb{Q} \otimes (A_\bullet) \simeq 0$ in $\mathbb{Q} \otimes \mathit{Mod}(\mathbb{Z}/p^\bullet)$.

Proposition A.3.1 *Let i be a non-negative integer, and put* $D^i_n := \mathbb{H}^i(Y, W_n\omega^\bullet_Y)$. *Let P_n be the ring defined in* [Ts1] §1.6, *and put* $R_{\mathscr{E}_n} := \Gamma(\mathscr{E}_n, \mathscr{O}_{\mathscr{E}_n})$.

(1) *There is an isomorphism in* $\mathbb{Q} \otimes \mathit{Mod}(\mathbb{Z}/p^\bullet)$

$$\mathbb{Q} \otimes \left(H^i_{\mathsf{crys}}((X_\bullet,M_\bullet)/(\mathscr{E}_\bullet,M_{\mathscr{E}_\bullet}))\right) \simeq \mathbb{Q} \otimes \{R_{\mathscr{E}_n} \otimes_{W_n} D^i_n\}_{n\geq 1}$$

and $R^1\varprojlim H^i_{\mathsf{crys}}((X_\bullet,M_\bullet)/(\mathscr{E}_\bullet,M_{\mathscr{E}_\bullet}))$ *has a finite exponent.*

(2) *There is an isomorphism in* $\mathbb{Q} \otimes \mathit{Mod}(G_K\text{-}\mathbb{Z}/p^\bullet)$

$$\mathbb{Q} \otimes \left(H^i_{\mathsf{crys}}((\overline{X}_\bullet,\overline{M}_\bullet)/(\mathscr{E}_\bullet,M_{\mathscr{E}_\bullet}))\right) \simeq \mathbb{Q} \otimes \{P_n \otimes_{W_n} D^i_n\}_{n\geq 1}$$

and $R^1\varprojlim H^i_{\mathsf{crys}}((\overline{X}_\bullet,\overline{M}_\bullet)/(\mathscr{E}_\bullet,M_{\mathscr{E}_\bullet}))$ *has a finite exponent.*

(3) *There is an isomorphism in* $\mathbb{Q} \otimes \mathit{Mod}(G_K\text{-}\mathbb{Z}/p^\bullet)$

$$\mathbb{Q} \otimes \left(H^i_{\mathsf{crys}}((\overline{X}_\bullet,\overline{M}_\bullet)/W_\bullet)\right) \simeq \mathbb{Q} \otimes \left\{(P_n \otimes_{W_n} D^i_n)^{N=0}\right\}_{n\geq 1}$$

and $R^1\varprojlim H^i_{\mathsf{crys}}((\overline{X}_\bullet,\overline{M}_\bullet)/W_\bullet)$ *has a finite exponent. Here N acts on* $P_n \otimes_{W_n} D^i_n$ *by* $N_{P_n} \otimes 1 + 1 \otimes N$ *and* N_{P_n} *denotes the monodromy operator on* P_n, *cf.* [Ts1] *p.* 253.

Proof. (1) The first assertion follows from [HK] Lemma 5.2 (see also $(*)$ in the proof of [Ts1] Proposition 4.4.6). To show the second assertion, we check that the projective system $\{R_{\mathscr{E}_n} \otimes_{W_n} D_n^i\}_{n\geq 1}$ satisfies the Mittag-Leffler condition. Indeed, D_n^i is finitely generated over W_n, and the projection $R_{\mathscr{E}_{n+1}} \to R_{\mathscr{E}_n}$ is surjective by the presentation

$$R_{\mathscr{E}_n} = W[t,\, t^{e\nu}/\nu!\ (\nu \geq 1)] \otimes_W W_n \quad (e := [K : K_0])$$

obtained from the definition of $(\mathscr{E}_n, M_{\mathscr{E}_n})$ (see the proof of [Ts1] Proposition 4.4.6).

(2) The first assertion follows from that of (1) and [Ts1] Proposition 4.5.4. The second assertion follows from the fact that $\{P_n \otimes_{W_n} D_n^i\}_{n\geq 1}$ satisfies the Mittag-Leffler condition. Indeed, the natural projection $P_{n+1} \to P_n$ is surjective (loc. cit. Lemma 1.6.7) and D_n^i is finitely generated over W_n.

(3) See the proof of loc. cit. Proposition 4.5.6 for the first assertion.. We show the second assertion. Note that P_n is flat over W_n, because $R_{\mathscr{E}_n}$is flat over W_n and P_n is flat over $R_{\mathscr{E}_n}$ by the above presentation of $R_{\mathscr{E}_n}$ and loc. cit. Proposition 4.1.5. Let $\{D_n\}_{n\geq 1}$ be a projective system of W-modules such that D_n is finitely generated over W_n for each n, and let $N : \{D_n\}_{n\geq 1} \to \{D_n\}_{n\geq 1}$ be a nilpotent W-endomorphism. Our task is to show that

$$R^1 \varprojlim \{\big(P_n \otimes_{W_n} D_n\big)^{N=0}\}_{n\geq 1} = 0.$$

Consider a short exact sequence of projective systems

$$0 \longrightarrow \{(D_n)^{N=0}\}_{n\geq 1} \longrightarrow \{D_n\}_{n\geq 1} \longrightarrow \{N(D_n)\}_{n\geq 1} \longrightarrow 0.$$

Note that $(D_n)^{N=0}$ and $N(D_n)$ are finitely generated over W_n for each n. Let b be the minimal integer for which $N^b = 0$ on $\{D_n\}_{n\geq 1}$. By [K4] Lemma 4.3 and the flatness of P_n over W_n, we have a short exact sequence for each $n \geq 1$

$$0 \to (P_n \otimes_{W_n} (D_n)^{N=0})^{N=0} \to (P_n \otimes_{W_n} D_n)^{N=0} \to (P_n \otimes_{W_n} N(D_n))^{N=0} \to 0,$$

which yields a short exact sequence of projective systems with respect to $n \geq 1$. Since $N^{b-1} = 0$ on $\{N(D_n)\}_{n\geq 1}$, we may assume $b = 1$ by induction on $b \geq 1$. Now let B_n be as in [Ts1] §1.6 and let A_{crys} be as in loc. cit. §1.1. Then we have isomorphisms

$$(P_n \otimes_{W_n} D_n)^{N=0} \overset{(1)}{\simeq} (P_n)^{N=0} \otimes_{W_n} D_n \overset{(2)}{\simeq} B_n \otimes_{W_n} D_n \overset{(3)}{=} (A_{\mathsf{crys}}/p^n) \otimes_{W_n} D_n,$$

where (1) follows from the assumption $b = 1$ and the flatness of P_n over W_n. The isomorphism (2) (resp. (3)) follows from loc. cit. Corollary 1.6.6 (resp. the definition of B_n in loc. cit. §1.6). Thus $R^1 \varprojlim \{(P_n \otimes_{W_n} D_n)^{N=0}\}_{n\geq 1}$ is zero, and we obtain the assertion. $\square$

COROLLARY A.3.2 (1) *For $i \geq 0$, we have*

$$H^i_{\mathsf{crys}}((X, M)/(\mathscr{E}, M_{\mathscr{E}}))_{\mathbb{Q}_p} \simeq H^i_{\mathsf{cont\text{-}cr}}((X, M)/(\mathscr{E}, M_{\mathscr{E}}))_{\mathbb{Q}_p}\,.$$

(2) *The torsion subgroups of* $H^i_{\mathsf{crys}}((\overline{X},\overline{M})/(\mathscr{E},M_{\mathscr{E}}))$ *and* $H^i_{\mathsf{crys}}((\overline{X},\overline{M})/W)$ *have finite exponents for any* $i \geq 0$.

Proof. (1) follows from Proposition A.3.1 (1) and the remark after (A.2.3). Since $\varprojlim_{n\geq 1} P_n$ is p-torsion free, the assertion (2) follows from the isomorphisms in Proposition A.3.1 (3). □

Remark A.3.3 *If K is absolutely unramified, then we have*

$$H^i_{\mathsf{crys}}((X_n,M_n)/W_n) \simeq \mathbb{H}^i(Y, W_n\widetilde{\omega}^{\bullet}_Y),$$

which is finitely generated over W_n by Theorem 8.8 (2) *and* (8.10.1) (*see also* [Hy2] (1.4.3), (2.4.2)). *Consequently, the projective system $H^i_{\mathsf{crys}}((X_\bullet,M_\bullet)/W_\bullet)$ satisfies the Mittag-Leffler condition and we obtain a long exact sequence*

$$\begin{aligned}\cdots \longrightarrow H^i_{\mathsf{crys}}((X,M)/W)_{\mathbb{Q}_p} &\longrightarrow H^i_{\mathsf{crys}}((X,M)/(\mathscr{E},M_{\mathscr{E}}))_{\mathbb{Q}_p}\\ &\xrightarrow{N} H^i_{\mathsf{crys}}((X,M)/(\mathscr{E},M_{\mathscr{E}}))_{\mathbb{Q}_p} \longrightarrow H^{i+1}_{\mathsf{crys}}((X,M)/W)_{\mathbb{Q}_p} \longrightarrow \cdots,\end{aligned}$$

which removes the assumption $(*)$ *in* [La] *p.* 191. *On the other hand, the author does not know if $H^i_{\mathsf{crys}}((X_\bullet,M_\bullet)/W_\bullet)$ satisfies the Mittag-Leffler condition even up to torsion, when K is not absolutely unramified.*

The following corollary has been used in the proof of Lemma 9.7 (1):

Corollary A.3.4 *In the following commutative diagram of canonical maps, the arrows* (3) *and* (4) *are injective*:

$$\begin{array}{ccc} H^i_{\mathsf{cont\text{-}cr}}((X,M)/W)_{\mathbb{Q}_p} & \xrightarrow{(1)} & H^i_{\mathsf{crys}}((X,M)/(\mathscr{E},M_{\mathscr{E}}))_{\mathbb{Q}_p} \\ {\scriptstyle (2)}\downarrow & & \downarrow{\scriptstyle (3)} \\ H^i_{\mathsf{crys}}((\overline{X},\overline{M})/W)_{\mathbb{Q}_p} & \xrightarrow{(4)} & H^i_{\mathsf{crys}}((\overline{X},\overline{M})/(\mathscr{E},M_{\mathscr{E}}))_{\mathbb{Q}_p}. \end{array}$$

In particular, the kernel of (1) *agrees with that of* (2).

Proof. The injectivity of (3) follows from Proposition A.3.1 (1) and (2) and the injectivity of the natural maps $R_{\mathscr{E}_n} \to P_n$ for $n \geq 1$ ([Ts1] Proposition 4.1.5). The injectivity of (4) follows from Proposition A.3.1 (2) and (3). □

A.4 continuous-Galois crystalline cohomology

We define the continuous-Galois crystalline cohomology as follows:

$$H^i_{\mathsf{cG\text{-}cr}}((X,M)/W) := R^i\big(\varprojlim \varGamma_{\mathsf{Gal}}\overline{\varGamma}\big)\big(\mathbb{E}_{(\overline{X}_\bullet,\overline{M}_\bullet)/W_\bullet}\big),$$

where $\overline{\Gamma}$ is as in (A.1.6), and Γ_{Gal} denotes the functor taking G_K-invariant subgroups:

$$\Gamma_{\mathsf{Gal}} := \Gamma(G_K, -) : \mathit{Mod}(G_K\text{-}\mathbb{Z}/p^\bullet) \longrightarrow \mathit{Mod}(\mathbb{Z}/p^\bullet).$$

There is a natural map

$$H^i_{\mathsf{cont\text{-}cr}}((X,M)/W) \longrightarrow H^i_{\mathsf{cG\text{-}cr}}((X,M)/W) \tag{A.4.1}$$

by definition.

THEOREM A.4.2 *There exists a Hochschild-Serre spectral sequence*

$$E_2^{a,b} = H^a(K, H^b_{\mathsf{crys}}((\overline{X},\overline{M})/W)_{\mathbb{Q}_p}) \Longrightarrow H^{a+b}_{\mathsf{cG\text{-}cr}}((X,M)/W)_{\mathbb{Q}_p}.$$

Proof. Because $\overline{\Gamma}$ has an exact left adjoint functor, it preserves injectives and there exists a spectral sequence

$$E_2^{a,b} = H^a(K, H^b_{\mathsf{crys}}((\overline{X}_\bullet,\overline{M}_\bullet)/W_\bullet)) \Longrightarrow H^{a+b}_{\mathsf{cG\text{-}cr}}((X,M)/W),$$

where $H^*(K,\{F_n\}_{n\geq 1})$ for a projective system $\{F_n\}_{n\geq 1}$ of discrete G_K-modules denotes the continuous Galois cohomology of G_K in the sense of Jannsen [J] §2. We show that the canonical map

$$\alpha^{a,b} : H^a(K, H^b_{\mathsf{crys}}((\overline{X},\overline{M})/W)_{\mathbb{Q}_p}) \longrightarrow H^a(K, H^b_{\mathsf{crys}}((\overline{X}_\bullet,\overline{M}_\bullet)/W_\bullet))_{\mathbb{Q}_p}$$

(see the proof of loc. cit. Theorem 2.2) is bijective for $a, b \geq 0$. Put

$$L^b_\bullet := \{P_n \otimes_{W_n} D^b_n\}_{n\geq 1}, \qquad L^b := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim L^b_\bullet = B^+_{\mathsf{st}} \otimes_{K_0} D^b,$$
$$T^b_\bullet := H^b_{\mathsf{crys}}((\overline{X}_\bullet,\overline{M}_\bullet)/W_\bullet) \quad \text{and} \quad T^b := H^b_{\mathsf{crys}}((\overline{X},\overline{M})/W)_{\mathbb{Q}_p}.$$

Since the torsion part of $\varprojlim L^b_\bullet$ has a finite exponent, the canonical map

$$\beta^{a,b} : H^a(K, L^b) \longrightarrow H^a(K, L^b_\bullet)_{\mathbb{Q}_p} \tag{A.4.3}$$

is bijective by loc. cit. Theorem 5.15 (c). We have a short exact sequence of objects in $\mathbb{Q}\otimes \mathit{Mod}(G_K\text{-}\mathbb{Z}/p^\bullet)$

$$0 \longrightarrow \mathbb{Q}\otimes(T^b_\bullet) \longrightarrow \mathbb{Q}\otimes(L^b_\bullet) \xrightarrow{N} \mathbb{Q}\otimes(L^b_\bullet) \longrightarrow 0$$

and a short exact sequence of topological G_K-modules

$$0 \longrightarrow T^b \longrightarrow L^b \xrightarrow{N} L^b \longrightarrow 0 \tag{A.4.4}$$

by Proposition A.3.1 (3) and [K4] Lemma 4.3. These exact sequences yield a commutative diagram with exact rows

$$\begin{tikzcd}
\cdots \arrow[r] & H^{a-1}(K,L^b) \arrow[r] \arrow[d, "\beta^{a-1,b}"'] & H^a(K,T^b) \arrow[r] \arrow[d, "\alpha^{a,b}"'] & H^a(K,L^b) \arrow[r, "N"] \arrow[d, "\beta^{a,b}"'] & H^a(K,L^b) \arrow[r] \arrow[d, "\beta^{a,b}"'] & \cdots \\
\cdots \arrow[r] & H^{a-1}(K,L^b_\bullet)_{\mathbb{Q}_p} \arrow[r] & H^a(K,T^b_\bullet)_{\mathbb{Q}_p} \arrow[r] & H^a(K,L^b_\bullet)_{\mathbb{Q}_p} \arrow[r, "N"] & H^a(K,L^b_\bullet)_{\mathbb{Q}_p} \arrow[r] & \cdots,
\end{tikzcd}$$

where the arrows $\beta^{a-1,b}$ and $\beta^{a,b}$ are bijective by (A.4.3). Hence the assertion follows from the five lemma. $\square$

Corollary A.4.5 *Put*

$$H^{i+1}_{\textsf{cont-cr}}((X,M)/W)^0_{\mathbb{Q}_p} := \mathsf{Ker}\big(H^{i+1}_{\textsf{cont-cr}}((X,M)/W)_{\mathbb{Q}_p} \to H^{i+1}_{\textsf{crys}}((\overline{X},\overline{M})/W)_{\mathbb{Q}_p}\big).$$

Then the canonical homomorphism

$$H^{i+1}_{\textsf{cont-cr}}((X,M)/W)^0_{\mathbb{Q}_p} \longrightarrow H^1(K, H^i_{\textsf{crys}}((\overline{X},\overline{M})/W)_{\mathbb{Q}_p}),$$

obtained from (A.4.1) *and an edge homomorphism of the spectral sequence in Theorem* A.4.2*, fits into a commutative diagram*

$$\begin{tikzcd} H^i_{\textsf{crys}}((X,M)/(\mathscr{E},M_{\mathscr{E}}))_{\mathbb{Q}_p} \arrow[r, two heads, "\partial"] \arrow[d] & H^{i+1}_{\textsf{cont-cr}}((X,M)/W)^0_{\mathbb{Q}_p} \arrow[d] \\ (B^+_{\textsf{st}} \otimes_{K_0} D^i)^{G_K} \arrow[r, "(-1)^{i+1}\delta"] & H^1(K, H^i_{\textsf{crys}}((\overline{X},\overline{M})/W)_{\mathbb{Q}_p}), \end{tikzcd}$$

where ∂ is the connecting map induced by that in Proposition A.2.4 (*see Corollary* A.3.4 *for the existence and the surjectivity*)*, and δ denotes the connecting map of continuous Galois cohomology associated with* (A.4.4) *with $b = i$. The left vertical arrow is obtained from Proposition* A.3.1 (2).

Proof. The assertion follows from Proposition A.3.1 (2) and a simple computation on boundary maps arising from the distinguished triangle

$$\begin{aligned} R\overline{\Gamma}\big(\mathbb{E}_{(\overline{X}_\bullet,\overline{M}_\bullet)/W_\bullet}\big) &\xrightarrow{\textsf{can}} R\overline{\Gamma}\big(\mathbb{E}_{(\overline{X}_\bullet,\overline{M}_\bullet)/(\mathscr{E}_\bullet,M_{\mathscr{E}_\bullet})}\big) \\ &\xrightarrow{\nu} R\overline{\Gamma}\big(\mathbb{E}_{(\overline{X}_\bullet,\overline{M}_\bullet)/(\mathscr{E}_\bullet,M_{\mathscr{E}_\bullet})}\big) \xrightarrow{\frac{dt}{t}\wedge} R\overline{\Gamma}\big(\mathbb{E}_{(\overline{X}_\bullet,\overline{M}_\bullet)/W_\bullet}\big)[1] \end{aligned}$$

in $D^+(\mathit{Mod}(G_K\text{-}\mathbb{Z}/p^\bullet))$ (a variant of Proposition A.2.4). The sign $(-1)^{i+1}$ in the diagram arises from the difference of the orientations between this distinguished triangle and (A.4.4), and the fact that the construction of connecting morphisms associated with short exact sequences of complexes commutes with the shift functor $[i]$ up to the sign $(-1)^i$. The details are straight-forward and left to the reader. □

A.5 Syntomic complexes

We construct the following objects for $r \geq 0$:

$$\mathscr{S}^{\sim}_\bullet(r)_{(X,M)} \in D^+(Y_{\textsf{ét}}, \mathbb{Z}/p^\bullet) \quad \text{and} \quad \mathscr{S}^{\sim}_\bullet(r)_{(\overline{X},\overline{M})} \in D^+(\overline{Y}_{\textsf{ét}}, G_K\text{-}\mathbb{Z}/p^\bullet),$$

and the following objects for r with $0 \leq r \leq p-1$

$$\mathscr{S}_\bullet(r)_{(X,M)} \in D^+(Y_{\textsf{ét}}, \mathbb{Z}/p^\bullet) \quad \text{and} \quad \mathscr{S}_\bullet(r)_{(\overline{X},\overline{M})} \in D^+(\overline{Y}_{\textsf{ét}}, G_K\text{-}\mathbb{Z}/p^\bullet).$$

Definition A.5.1 *Let (T, M_T) be a log scheme over $\mathbb{Z}_p$. A Frobenius endomorphism $\varphi : (T, M_T) \to (T, M_T)$ is a morphism over $\mathbb{Z}_p$ such that $\varphi \otimes \mathbb{Z}/p : (T_1, M_{T_1}) \to (T_1, M_{T_1})$ is the absolute Frobenius endomorphism in the sense of* [K3] *Definition* 4.7.

To construct the objects $\mathscr{S}_\bullet^{\sim}(r)_{(X,M)}$ and $\mathscr{S}_\bullet(r)_{(X,M)}$, we fix an étale hypercovering $(X^\star, M^\star) \to (X,M)$ and a closed immersion $(X^\star, M^\star) \hookrightarrow (Z^\star, M_{Z^\star})$ of simplicial fine log schemes over W such that (Z^i, M_{Z^i}) is smooth over W and has a Frobenius endomorphism for each $i \in \mathbb{N}$. Let $n \geq 1$ be an integer, and let $(\mathscr{D}_n^\star, M_{\mathscr{D}_n^\star})$ be the PD-envelope of $(X_n^\star, M_n^\star)$ in $(Z_n^\star, M_{Z_n^\star})$ with respect to the canonical PD-structure on $(p) \subset W_n$. For $i \geq 1$, let $\mathscr{J}^{[i]} \subset \mathscr{O}_{\mathscr{D}_n^\star}$ be the i-th divided power of the ideal $\mathscr{J} = \mathsf{Ker}(\mathscr{O}_{\mathscr{D}_n^\star} \to \mathscr{O}_{X_n^\star})$. For $i \leq 0$, put $\mathscr{J}^{[i]} := \mathscr{O}_{\mathscr{D}_n^\star}$. Let $\mathbb{J}^{[r]}_{(X_n^\star, M_n^\star)/W_n}$ be the complex of sheaves on $Y^\star_{\text{ét}}$

$$\mathscr{J}^{[r]} \xrightarrow{d} \mathscr{J}^{[r-1]} \otimes_{\mathscr{O}_{Z_n^\star}} \omega^1_{(Z_n^\star, M_{Z_n^\star})/W_n} \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{J}^{[r-q]} \otimes_{\mathscr{O}_{Z_n^\star}} \omega^q_{(Z_n^\star, M_{Z_n^\star})/W_n} \xrightarrow{d} \cdots,$$

where $\mathscr{J}^{[r]}$ is placed in degree 0. See [Ts2] Corollary 1.10 for d. The complex $\mathbb{E}_{(X_n^\star, M_n^\star)/W_n}$ we considered in §A.1 agrees with $\mathbb{J}^{[0]}_{(X_n^\star, M_n^\star)/W_n}$. We define a complex $\mathscr{S}_n^{\sim}(r)_{(X^\star, M^\star)}$ on $Y^\star_{\text{ét}}$ as the mapping fiber of the homomorphism

$$p^r - \varphi_n^* : \mathbb{J}^{[r]}_{(X_n^\star, M_n^\star)} \to \mathbb{E}_{(X_n^\star, M_n^\star)}$$

For $0 \leq r \leq p-1$, the Frobenius endomorphism on $(Z_{n+r}^\star, M_{Z_{n+r}^\star})$ induces a homomorphism of complexes

$$f_r := \overline{p^{-r} \cdot \varphi_{n+r}^*} : \mathbb{J}^{[r]}_{(X_n^\star, M_n^\star)/W_n} \longrightarrow \mathbb{E}_{(X_n^\star, M_n^\star)/W_n}$$

(cf. [Ts2] p. 540). We define a complex $\mathscr{S}_n(r)_{(X^\star, M^\star)}$ on $Y^\star_{\text{ét}}$ as the mapping fiber of the homomorphism

$$1 - f_r : \mathbb{J}^{[r]}_{(X_n^\star, M_n^\star)} \to \mathbb{E}_{(X_n^\star, M_n^\star)}$$

Regarding $\mathscr{S}_n^{\sim}(r)_{(X^\star, M^\star)}$ and $\mathscr{S}_n(r)_{(X^\star, M^\star)}$ as complexes of projective systems (on $n \geq 1$) of sheaves on $Y^\star_{\text{ét}}$, we define

$$\begin{aligned} \mathscr{S}_\bullet^{\sim}(r)_{(X,M)} &:= R\theta_*(\mathscr{S}_\bullet^{\sim}(r)_{(X^\star, M^\star)}) \qquad (r \geq 0), \\ \mathscr{S}_\bullet(r)_{(X,M)} &:= R\theta_*(\mathscr{S}_\bullet(r)_{(X^\star, M^\star)}) \qquad (0 \leq r \leq p-1), \end{aligned}$$

where $\theta : \mathit{Shv}(Y^\star_{\text{ét}}, \mathbb{Z}/p^\bullet) \to \mathit{Shv}(Y_{\text{ét}}, \mathbb{Z}/p^\bullet)$ denotes the natural morphism of topoi. The resulting objects are independent of the pair $((X^\star, M^\star), (Z^\star, M_{Z^\star}))$ (cf. [K2] p. 212).

We construct $\mathscr{S}_\bullet^{\sim}(r)_{(\overline{X}, \overline{M})}$ for $r \geq 0$ and $\mathscr{S}_\bullet(r)_{(\overline{X}, \overline{M})}$ for $0 \leq r \leq p-1$ as follows. Fix an étale hypercovering $(X^\star, M^\star) \to (X, M)$ and for each finite extension L/K contained in $\overline{K}$, fix a closed immersion $(X^\star_{O_L}, M^\star_{O_L}) \hookrightarrow (Z^\star_L, M_{Z^\star_L})$ $((X^\star_{O_L}, M^\star_{O_L}) := (X^\star, M^\star) \times_{(S, M_S)} (S_L, M_{S_L}))$ of simplicial fine log schemes over W such that $(Z^i_L, M_{Z^i_L})$ is smooth over W for each $i \in \mathbb{N}$ and L/K, such that Z^i_L has a Frobenius endomorphism for each $i \in \mathbb{N}$ and L/K, and such that for finite extensions L'/L there are morphisms $\tau_{L'/L} : (Z^\star_{L'}, M_{Z^\star_{L'}}) \to (Z^\star_L, M_{Z^\star_L})$ which satisfy transitivity and compatibility with Frobenius morphisms. For a finite extension L/K, we define $Y^\star_L$ in a similar way as in §A.1. We define complexes

$\mathscr{S}_\bullet^\sim(r)_{(X_{O_L},M_{O_L})}$ and $\mathscr{S}_\bullet(r)_{(X_{O_L},M_{O_L})}$ on $Y_L^\star$ applying the same constructions as for the complexes $\mathscr{S}_\bullet^\sim(r)_{(X,M)}$ and $\mathscr{S}_\bullet(r)_{(X,M)}$, respectively, to the embedding $(X_{O_L}^\star, M_{O_L}^\star) \hookrightarrow (Z_L^\star, M_{Z_L^\star})$, whose inverse images onto $\overline{Y}^\star := X^\star \otimes_{O_K} \overline{k}$ yield inductive systems of complexes of objects of $Shv(\overline{Y}^\star_{\text{ét}}, G_K\text{-}\mathbb{Z}/p^\bullet)$ with respect to finite extensions L/K. We define

$$\mathscr{S}_\bullet^\sim(r)_{(\overline{X}^\star_\bullet,\overline{M}^\star_\bullet)} := \varinjlim_{K\subset L\subset \overline{K}} \mathscr{S}_\bullet^\sim(r)_{(X^\star_{O_L,\bullet},M^\star_{O_L,\bullet})}\big|_{\overline{Y}^\star},$$
$$\mathscr{S}_\bullet^\sim(r)_{(\overline{X}_\bullet,\overline{M}_\bullet)} := R\overline{\theta}_*(\mathscr{S}_\bullet^\sim(r)_{(\overline{X}^\star_\bullet,\overline{M}^\star_\bullet)}) \in D^+(\overline{Y}_{\text{ét}}, G_K\text{-}\mathbb{Z}/p^\bullet),$$

where $\overline{\theta} : Shv(\overline{Y}^\star_{\text{ét}}, G_K\text{-}\mathbb{Z}/p^\bullet) \to Shv(\overline{Y}_{\text{ét}}, G_K\text{-}\mathbb{Z}/p^\bullet)$ denotes the natural morphism of topoi. We construct $\mathscr{S}_\bullet(r)_{(\overline{X}_\bullet,\overline{M}_\bullet)}$ from $\mathscr{S}_\bullet(r)_{(X^\star_{O_L,\bullet},M^\star_{O_L,\bullet})}$'s in a similar way. When $0 \le r \le p-1$, there is a canonical projection

$$\psi_\bullet^r : \mathscr{S}_\bullet^\sim(r)_{(\overline{X},\overline{M})} \longrightarrow \mathscr{S}_\bullet(r)_{(\overline{X},\overline{M})}$$

induced by the identity map of $\mathbb{E}_{(\overline{X}_\bullet,\overline{M}_\bullet)/W_\bullet}$. There are canonical morphisms

$$\widetilde{c}_\bullet^r : \mathscr{S}_\bullet^\sim(r)_{(\overline{X},\overline{M})} \longrightarrow \mathbb{J}^{[r]}_{(\overline{X}_\bullet,\overline{M}_\bullet)/W_\bullet} \xrightarrow{1} \mathbb{E}_{(\overline{X}_\bullet,\overline{M}_\bullet)/W_\bullet} \quad (r \ge 0) \tag{A.5.2}$$
$$c_\bullet^r : \mathscr{S}_\bullet(r)_{(\overline{X},\overline{M})} \longrightarrow \mathbb{J}^{[r]}_{(\overline{X}_\bullet,\overline{M}_\bullet)/W_\bullet} \xrightarrow{1} \mathbb{E}_{(\overline{X}_\bullet,\overline{M}_\bullet)/W_\bullet} \quad (0 \le r \le p-1), \tag{A.5.3}$$

which satisfy

$$p^r \cdot \widetilde{c}_\bullet^r = c_\bullet^r \circ \psi_\bullet^r \qquad (\text{when } 0 \le r \le p-1). \tag{A.5.4}$$

For $r \ge 0$, we define

$$\mathbb{Z}/p^\bullet(r)' := (p^a a!)^{-1}\mathbb{Z}_p(r) \otimes \mathbb{Z}/p^\bullet \in Shv(X_{\overline{K}}, G_K\text{-}\mathbb{Z}/p^\bullet),$$

where a denotes the maximal integer $\le r/(p-1)$. We have $\mathbb{Z}/p^\bullet(r)' = \mu_{p^\bullet}^{\otimes r}$ canonically when $r \le p-2$. Similarly for $r < 0$, we define

$$\mathbb{Z}/p^\bullet(r)' := p^a a!\, \mathbb{Z}_p(r) \otimes \mathbb{Z}/p^\bullet \in Shv(X_{\overline{K}}, G_K\text{-}\mathbb{Z}/p^\bullet) \tag{A.5.5}$$

with a the maximal integer with $a \le -r/(p-1)$, which will be used in (A.6.4) below. Let $\overline{j} : X_{\overline{K}} \hookrightarrow \overline{X}$ and $\overline{\iota} : \overline{Y} \hookrightarrow \overline{X}$ be the natural immersions. Let $R\overline{j}_*$ and $\overline{\iota}^*$ be the following functors, respectively:

$$R\overline{j}_* : D^+((X_{\overline{K}})_{\text{ét}}, G_K\text{-}\mathbb{Z}/p^\bullet) \longrightarrow D^+(\overline{X}_{\text{ét}}, G_K\text{-}\mathbb{Z}/p^\bullet),$$
$$\overline{\iota}^* : D^+(\overline{X}_{\text{ét}}, G_K\text{-}\mathbb{Z}/p^\bullet) \longrightarrow D^+(\overline{Y}_{\text{ét}}, G_K\text{-}\mathbb{Z}/p^\bullet).$$

Theorem A.5.6 ([Ts1] §3.1, [K4] Theorem 5.4) *For $r \ge 0$, there exists a canonical morphism in $D^+(\overline{Y}_{\text{ét}}, G_K\text{-}\mathbb{Z}/p^\bullet)$ compatible with product structures*

$$\eta_\bullet^r : \mathscr{S}_\bullet^\sim(r)_{(\overline{X},\overline{M})} \longrightarrow \overline{\iota}^* R\overline{j}_* \mathbb{Z}/p^\bullet(r)'.$$

If $r \le p-2$, then $\eta_\bullet^r$ factors through an isomorphism

$$\mathscr{S}_\bullet(r)_{(\overline{X},\overline{M})} \xrightarrow{\sim} \tau_{\le r}\, \overline{\iota}^* R\overline{j}_* \mu_{p^\bullet}^{\otimes r}. \tag{A.5.7}$$

Proof. We define $\eta^r_\bullet$ applying the arguments in [Ts1] §3.1 in the category of $\mathbb{Z}/p^\bullet$-sheaves. If $r \le p-2$, then we have $\mathbb{Z}/p^\bullet(r)' = \mu_{p^\bullet}^{\otimes r}$, and $\eta^r_\bullet$ factors through a morphism $\overline{\eta}^r_\bullet : \mathscr{S}_\bullet(r)_{(\overline{X},\overline{M})} \to \overline{\iota}^* R\overline{j}_* \mu_{p^\bullet}^{\otimes r}$ by the construction of $\eta^r_\bullet$ (loc. cit. (3.1.11)). The morphism $\overline{\eta}^r_\bullet$ induces an isomorphism as claimed, because $\mathscr{S}_\bullet(r)_{(\overline{X},\overline{M})}$ is concentrated in $[0,r]$ and $\overline{\eta}^r_\bullet$ induces isomorphisms on the q-th cohomology objects with $0 \le q \le r$ by [K4] Theorem 5.4. $\square$

We define

$$\begin{aligned}
H^i_{\mathsf{syn}}((X,M),\mathscr{S}_\bullet(r)) &:= R^i\Gamma(\mathscr{S}_\bullet(r)_{(X,M)}),\\
H^i_{\mathsf{syn}}((X,M),\mathscr{S}_{\mathbb{Z}_p}(r)) &:= \varprojlim H^i_{\mathsf{syn}}((X,M),\mathscr{S}_\bullet(r)),\\
H^i_{\mathsf{syn}}((X,M),\mathscr{S}_{\mathbb{Q}_p}(r)) &:= \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^i_{\mathsf{syn}}((X,M),\mathscr{S}_{\mathbb{Z}_p}(r)),\\
H^i_{\mathsf{syn}}((\overline{X},\overline{M}),\mathscr{S}_{\mathbb{Q}_p}(r)) &:= \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim R^i\overline{\Gamma}\big(\mathscr{S}_\bullet(r)_{(\overline{X},\overline{M})}\big).
\end{aligned}$$

where Γ and $\overline{\Gamma}$ are as in (A.1.5) and (A.1.6), respectively.

A.6 CONTINUOUS(-GALOIS) SYNTOMIC COHOMOLOGY

We assume $r \le p-2$ in what follows. For $i \ge 0$, we define the continuous syntomic cohomology as follows:

$$H^i_{\mathsf{cont\text{-}syn}}((X,M),\mathscr{S}_{\mathbb{Z}_p}(r)) := R^i\big(\varprojlim \Gamma\big)\big(\mathscr{S}_\bullet(r)_{(X,M)}\big).$$

Similarly, we define the continuous-Galois syntomic cohomology as follows:

$$H^i_{\mathsf{cG\text{-}syn}}((X,M),\mathscr{S}_{\mathbb{Z}_p}(r)) := R^i\big(\varprojlim \Gamma_{\mathsf{Gal}}\overline{\Gamma}\big)\big(\mathscr{S}_\bullet(r)_{(\overline{X},\overline{M})}\big).$$

We put

$$H^i_{\mathsf{cont\text{-}syn}}((X,M),\mathscr{S}_{\mathbb{Q}_p}(r)) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^i_{\mathsf{cont\text{-}syn}}((X,M),\mathscr{S}_{\mathbb{Z}_p}(r)).$$

PROPOSITION A.6.1 *Let $i \ge 0$ be an integer.*

(1) *Let $\eta : H^i_{\mathsf{syn}}((X,M),\mathscr{S}_{\mathbb{Q}_p}(r)) \to H^i(X_{\overline{K}},\mathbb{Q}_p(r))$ be as in Lemma 9.5. Then the kernel of the composite map*

$$H^i_{\mathsf{cont\text{-}syn}}((X,M),\mathscr{S}_{\mathbb{Q}_p}(r)) \longrightarrow H^i_{\mathsf{syn}}((X,M),\mathscr{S}_{\mathbb{Q}_p}(r)) \xrightarrow{\eta} H^i(X_{\overline{K}},\mathbb{Q}_p(r))$$

agrees with that of the composite map

$$H^i_{\mathsf{cont\text{-}syn}}((X,M),\mathscr{S}_{\mathbb{Q}_p}(r)) \to H^i_{\mathsf{syn}}((X,M),\mathscr{S}_{\mathbb{Q}_p}(r)) \to H^i_{\mathsf{crys}}((\overline{X},\overline{M})/W)_{\mathbb{Q}_p},$$

which we denote by $H^i_{\mathsf{cont\text{-}syn}}((X,M),\mathscr{S}_{\mathbb{Q}_p}(r))^0$, in what follows.

(2) *If K is a p-adic local field (i.e., k is finite), then we have*

$$H^i_{\mathsf{cont\text{-}syn}}((X,M),\mathscr{S}_{\mathbb{Z}_p}(r)) \xrightarrow{\sim} H^i_{\mathsf{syn}}((X,M),\mathscr{S}_{\mathbb{Z}_p}(r)).$$

In particular, we have the following canonical map in this case:

$$H^i_{\mathsf{syn}}((X,M),\mathscr{S}_{\mathbb{Z}_p}(r)) \longrightarrow H^i_{\mathsf{cont\text{-}cr}}((X,M)/W).$$

(3) *If* $r \geq d := \dim(X_K)$*, then we have*

$$H^i_{\mathsf{cG\text{-}syn}}((X,M),\mathscr{S}_{\mathbb{Z}_p}(r)) \xrightarrow{\sim} H^i_{\mathsf{cont}}(X_K,\mathbb{Z}_p(r)),$$

where $H^*_{\mathsf{cont}}(X_K,\mathbb{Z}_p(r))$ *denotes the continuous étale cohomology* [J].

Proof. (1) The assertion immediately follows from Lemma 9.5.

(2) There is a short exact sequence analogous to (A.2.3)

$$0 \longrightarrow R^1\varprojlim\ H^{i-1}_{\mathsf{syn}}((X,M),\mathscr{S}_\bullet(r)) \longrightarrow H^i_{\mathsf{cont\text{-}syn}}((X,M),\mathscr{S}_{\mathbb{Z}_p}(r)) \\ \longrightarrow H^i_{\mathsf{syn}}((X,M),\mathscr{S}_{\mathbb{Z}_p}(r)) \longrightarrow 0.$$

Since X is proper over k and k is finite by assumption, $H^{i-1}_{\mathsf{syn}}((X,M),\mathscr{S}_n(r))$ is finite for any $i,n \geq 1$ by [Ts1] Proposition 2.4.1. See also the remark in loc. cit., p. 263. The assertion follows from these facts.

(3) Since $r \geq d$ by assumption, (A.5.7) implies $\mathscr{S}_\bullet(r)_{(\overline{X},\overline{M})} \simeq \bar{\iota}^* R\bar{j}_*\mu_{p^\bullet}^{\otimes r}$. Hence the assertion follows from the isomorphisms in $D^+(\mathit{Mod}(G_K\text{-}\mathbb{Z}/p^\bullet))$

$$R\Gamma\big(X_{\overline{K}},\mu_{p^\bullet}^{\otimes r}\big) = R\Gamma\big(\overline{X},R\bar{j}_*\mu_{p^\bullet}^{\otimes r}\big) \xrightarrow{\sim} R\Gamma\big(\overline{Y},\bar{\iota}^*R\bar{j}_*\mu_{p^\bullet}^{\otimes r}\big), \qquad \text{(A.6.2)}$$

where the last isomorphism is a consequence of the proper base-change theorem for the usual étale cohomology. $\square$

For $i \geq 0$, put

$$H^i(X_K,\mathbb{Q}_p(r))^0 := \mathsf{Ker}(H^i(X_K,\mathbb{Q}_p(r)) \to H^i(X_{\overline{K}},\mathbb{Q}_p(r))),$$
$$H^i_{\mathsf{syn}}((X,M),\mathscr{S}_{\mathbb{Q}_p}(r))^0 := \mathsf{Ker}(\eta : H^i_{\mathsf{syn}}((X,M),\mathscr{S}_{\mathbb{Q}_p}(r)) \to H^i(X_{\overline{K}},\mathbb{Q}_p(r))).$$

We have $H^i(X_K,\mathbb{Q}_p(r)) = H^i_{\mathsf{cont}}(X_K,\mathbb{Q}_p(r))$ when K is a p-adic local field. The following corollary is a consequence of Proposition A.6.1 (1), (2) and the covariant functoriality of Hochschild-Serre spectral sequences in coefficients (see also the diagram in the proof of Theorem A.6.7 below).

Corollary A.6.3 *Assume that* K *is a* p*-adic local field, and let* e *be the composite map*

$$e : H^{i+1}_{\mathsf{syn}}((X,M),\mathscr{S}_{\mathbb{Q}_p}(r))^0 \to H^{i+1}(X_K,\mathbb{Q}_p(r))^0 \to H^1(K,H^i(X_{\overline{K}},\mathbb{Q}_p(r))),$$

where the last map is an edge homomorphism of the Hochschild-Serre spectral sequence (9.0.1)*. If* $r \geq d$*, then* e *fits into a commutative diagram*

$$\begin{CD} H^{i+1}_{\mathsf{syn}}((X,M),\mathscr{S}_{\mathbb{Q}_p}(r))^0 @>>> H^{i+1}_{\mathsf{cont\text{-}cr}}((X,M)/W)^0_{\mathbb{Q}_p} \\ @V{e}VV @VVV \\ H^1(K,H^i(X_{\overline{K}},\mathbb{Q}_p(r))) @>{\beta^{i,r}}>> H^1(K,H^i_{\mathsf{crys}}((\overline{X},\overline{M})/W)_{\mathbb{Q}_p}), \end{CD}$$

where the top arrow is obtained from Proposition A.6.1 (1) *and* (2)*, the right vertical arrow is the map in Corollary* A.4.5 *and* $\beta^{i,r}$ *is as in the proof of Theorem* 9.9.

We next construct a commutative diagram in $\mathbb{Q} \otimes D^+(Mod(G_K\text{-}\mathbb{Z}/p^\bullet))$ assuming $r < d$

$$\begin{tikzcd}
\mathbb{Q} \otimes R\Gamma\big(\overline{Y}, \mathscr{S}_\bullet(r)_{(\overline{X},\overline{M})}\big) \arrow[d, "f^r"'] \arrow[dr, "h^r"] & \\
\mathbb{Q} \otimes R\Gamma\big(X_{\overline{K}}, \mu_{p^\bullet}^{\otimes r}\big) \arrow[r, "g^r"] & \mathbb{Q} \otimes \big(R\Gamma\big(\overline{Y}, \mathbb{E}_{(\overline{X}_\bullet,\overline{M}_\bullet)/W_\bullet}\big) \otimes \mathbb{Z}/p^\bullet(r-d)'\big),
\end{tikzcd} \tag{A.6.4}$$

which is a key ingredient of the commutative diagram (A) of §9 for the case $r < d$. See (A.5.5) for $\mathbb{Z}/p^\bullet(r-d)'$. We define h^r as the composite morphism

$$\begin{aligned}
&\mathbb{Q} \otimes R\Gamma\big(\overline{Y}, \mathscr{S}_\bullet(r)_{(\overline{X},\overline{M})}\big) \\
&\xrightarrow{\text{(A.5.3)}} \mathbb{Q} \otimes R\Gamma\big(\overline{Y}, \mathbb{E}_{(\overline{X}_\bullet,\overline{M}_\bullet)/W_\bullet}\big) \\
&\xrightarrow{\ \sim\ } \mathbb{Q} \otimes \big\{R\Gamma\big(\overline{Y}, \mathbb{E}_{(\overline{X}_\bullet,\overline{M}_\bullet)/W_\bullet}\big) \otimes \mathbb{Z}/p^\bullet(d-r)' \otimes \mathbb{Z}/p^\bullet(r-d)'\big\} \\
&\longrightarrow \mathbb{Q} \otimes \big\{R\Gamma\big(\overline{Y}, \mathbb{E}_{(\overline{X}_\bullet,\overline{M}_\bullet)/W_\bullet}\big) \otimes \mathbb{Z}/p^\bullet(r-d)'\big\},
\end{aligned}$$

where the last arrow is induced by the p^{d-r}-times of the composite map

$$\begin{aligned}
\mathbb{Q} \otimes \mathbb{Z}/p^\bullet(d-r)' &\overset{(1)}{\simeq} \mathbb{Q} \otimes H^0_{\mathsf{syn}}((\overline{X}_\bullet, \overline{M}_\bullet), \mathscr{S}_\bullet^{\sim}(d-r)) \\
&\overset{\text{(A.5.2)}}{\hookrightarrow} \mathbb{Q} \otimes H^0_{\mathsf{crys}}((\overline{X}_\bullet, \overline{M}_\bullet)/W_\bullet)
\end{aligned} \tag{A.6.5}$$

(see (A.6.6) below for the isomorphism (1)) and the product of crystalline complexes. We define f^r as the morphism induced by (A.5.7) and the isomorphisms in (A.6.2). To define g^r, we need an isomorphism

$$\widetilde{f}^d : \mathbb{Q} \otimes R\Gamma\big(\overline{Y}, \mathscr{S}_\bullet^{\sim}(d)_{(\overline{X},\overline{M})}\big) \simeq \mathbb{Q} \otimes R\Gamma(X_{\overline{K}}, \mathbb{Z}/p^\bullet(d)') \tag{A.6.6}$$

induced by $\widetilde{\eta}^d_\bullet$ in Theorem A.5.6 (cf. [Ts1] Theorem 3.3.2 (1)). We define

$$\widetilde{h}^d : \mathbb{Q} \otimes R\Gamma\big(\overline{Y}, \mathscr{S}_\bullet^{\sim}(d)_{(\overline{X},\overline{M})}\big) \longrightarrow \mathbb{Q} \otimes R\Gamma\big(\overline{Y}, \mathbb{E}_{(\overline{X}_\bullet,\overline{M}_\bullet)/W_\bullet}\big)$$

in the same way as h^r (using (A.5.2) instead of (A.5.3)) and define $g^d := p^d \cdot (\widetilde{h}^d \circ (\widetilde{f}^d)^{-1})$. Finally we define g^r as the composite of the natural map

$$R\Gamma\big(X_{\overline{K}}, \mu_{p^\bullet}^{\otimes r}\big) \longrightarrow R\Gamma\big(X_{\overline{K}}, \mathbb{Z}/p^\bullet(d)'\big) \otimes \mathbb{Z}/p^\bullet(r-d)'$$

and $g^d \otimes \mathrm{id}$. The above diagram is commutative by the definition of g^r and the compatibility of $\eta^r_\bullet$ with products (cf. [Ts1] §3.1). Now we prove

THEOREM A.6.7 *Assume that K is a p-adic local field (i.e., k is finite). Then the diagram* (9.9.2) *commutes for $r < d$.*

Proof. We first note the isomorphisms

$$\begin{aligned} H^{i+1}_{\mathsf{cont\text{-}syn}}((X,M),\mathscr{S}_{\mathbb{Q}_p}(r)) &\xrightarrow{\ \sim\ } H^{i+1}_{\mathsf{syn}}((X,M),\mathscr{S}_{\mathbb{Q}_p}(r)),\\ H^{i+1}_{\mathsf{cont}}(X_K,\mathbb{Q}_p(r)) &\xrightarrow{\ \sim\ } H^{i+1}(X_K,\mathbb{Q}_p(r)) \end{aligned}$$

by the assumption that k is finite (cf. Proposition A.6.1 (2)). For integers $m \geq 0$ and $s < 0$, put

$$\begin{aligned} H^m_{\mathsf{cG\text{-}cr}}((X,M)/W;s) &:= R^m\big(\varprojlim \mathit{\Gamma}_{\mathsf{Gal}}\big)\big\{R\mathit{\Gamma}\big(\overline{Y},\mathbb{E}_{(\overline{X}_\bullet,\overline{M}_\bullet)/W_\bullet}\big)\otimes\mathbb{Z}/p^\bullet(s)'\big\},\\ H^m_{\mathsf{cG\text{-}cr}}((X,M)/W;s)^0_{\mathbb{Q}_p} &:= \mathsf{Ker}(H^m_{\mathsf{cG\text{-}cr}}((X,M)/W;s)_{\mathbb{Q}_p}\to H^m_{\mathsf{crys}}((\overline{X},\overline{M})/W)_{\mathbb{Q}_p}(s)). \end{aligned}$$

By (A.6.4), there is a commutative diagram

$$\begin{tikzcd} H^{i+1}_{\mathsf{syn}}((\overline{X},\overline{M}),\mathscr{S}_{\mathbb{Q}_p}(r)) \arrow[d, "f^r"'] \arrow[dr, "h^r"] & \\ H^{i+1}(X_{\overline{K}},\mathbb{Q}_p(r))_{\mathbb{Q}_p} \arrow[r, "g^r"] & H^{i+1}_{\mathsf{crys}}((\overline{X},\overline{M})/W)_{\mathbb{Q}_p}(r-d), \end{tikzcd}$$

where the bottom arrow is the same as $\beta^{i+1,r}$. By this diagram we obtain the arrows $(g^r)^0$ and $(h^r)^0$ in the following commutative diagram:

$$\begin{tikzcd} H^{i+1}_{\mathsf{syn}}((X,M),\mathscr{S}_{\mathbb{Q}_p}(r))^0 \arrow[d, "(f^r)^0"'] \arrow[dr, "(h^r)^0"] & \\ H^{i+1}_{\mathsf{cont}}(X,\mathbb{Q}_p(r))^0 \arrow[r, "(g^r)^0"] \arrow[d, "\mathrm{edge}"'] & H^{i+1}_{\mathsf{cG\text{-}cr}}((X,M)/W;r-d)^0_{\mathbb{Q}_p} \arrow[d, "\mathrm{edge}"] \\ H^1(K,H^i(X_{\overline{K}},\mu_{p^\bullet}^{\otimes r}))_{\mathbb{Q}_p} \arrow[r, "g^r"] & H^1(K,H^i_{\mathsf{crys}}((\overline{X}_\bullet,\overline{M}_\bullet)/W_\bullet)\otimes\mathbb{Z}/p^\bullet(r-d))_{\mathbb{Q}_p} \\ H^1(K,H^i(X_{\overline{K}},\mathbb{Q}_p(r))) \arrow[r, "\beta^{i,r}"] \arrow[u, "c", "\wr"'] & H^1(K,H^i_{\mathsf{crys}}((\overline{X},\overline{M})/W)_{\mathbb{Q}_p}(r-d)), \arrow[u, "c'"', "\wr"] \end{tikzcd}$$

where the top triangle is induced by (A.6.4) and the central square commutes by the functoriality of Hochschild-Serre spectral sequences. The arrows c and c' are canonical maps, and the bottom square commutes by the definitions of $\beta^{i,r}$ and g^r. The map c is bijective by [J] Theorem 5.15 (c) and the finiteness of $H^i(X_{\overline{K}},\mu_{p^n}^{\otimes r})$ for $n \geq 1$. The bijectivity of c' is obtained from a similar argument as for the proof of Theorem A.4.2. Moreover it is easy to see that the composite of the left column agrees with the left vertical arrow of (9.9.2), and that the composite of h^r and the right column agrees with $\gamma^{i,r}$ in (9.9.2). The commutativity in question follows from these facts. $\square$

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