On a Positive Equicharacteristic Variant of the p-Curvature Conjecture

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Abstract. Our aim is to formulate and prove a weak form in equal characteristic $p > 0$ of the p-curvature conjecture. We also show the existence of a counterexample to a strong form of it.

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Introduction

If (E,∇) is a vector bundle with an algebraic integrable connection over a smooth complex variety X, then it is defined over a smooth scheme S over $\mathrm{Spec}\,\mathbb{Z}[\frac{1}{N}]$ for some positive integer N, so $(E,\nabla) = (E_S,\nabla_S)\otimes_S \mathbb{C}$ over $X = X_S \otimes_S \mathbb{C}$ for a geometric generic point $\mathbb{Q}(S) \subset \mathbb{C}$. Grothendieck-Katz's p-curvature conjecture predicts that if for all closed points s of some non-trivial open $U \subset S$, the p-curvature of $(E_S,\nabla_S)\times_S s$ is zero, then (E,∇) is trivialized by a finite étale cover of X (see e.g. [An, Conj.3.3.3]). Little is known about it. N. Katz proved it for Gauß-Manin connections [Ka], for S finite over $\mathrm{Spec}\,\mathbb{Z}[\frac{1}{N}]$ (i.e., if X can be defined over a number field), D. V. Chudnovsky and G. V. Chudnovsky in [CC] proved it in the rank 1 case and Y. André in [An] proved it in case the Galois differential Lie algebra of (E,∇) at the generic point of S is solvable (and for extensions of connections satisfying the conjecture). More recently, B. Farb and M. Kisin [FK] proved it for certain locally symmetric varieties X. In general, one is lacking methods to think of the problem.

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Y. André in [An, II] and E. Hrushovsky in [Hr, V] formulated the following equal characteristic 0 analog of the conjecture: if $X \to S$ is a smooth morphism of smooth connected varieties defined over a characteristic 0 field k, then if (E_S, ∇_S) is a relative integrable connection such that for all closed points s of some non-trivial open $U \subset S$, $(E_S, \nabla_S) \times_S s$ is trivialized by a finite étale cover of $X \times_S s$, then $(E, \nabla)|_{X_{\bar{\eta}}}$ should be trivialized by a finite étale cover, where $\bar{\eta}$ is a geometric generic point and $X_{\bar{\eta}} = X \times_S \bar{\eta}$. So the characteristic 0 analogy to integrable connections is simply integrable connections, and to the p-curvature condition is the trivialization of the connection by a finite étale cover. André proved it [An, Prop. 7.1.1], using Jordan's theorem and Simpson's moduli of flat connections, while Hrushovsky [Hr, p.116] suggested a proof using model theory.

It is tempting to formulate an equal characteristic $p > 0$ analog of Y. André's theorem. A main feature of integrable connections over a field k of characteristic 0 is that they form an abelian, rigid, k-linear tensor category. In characteristic $p > 0$, the category of bundles with an integrable connection is only $\mathcal{O}_{X^{(1)}}$-linear, where $X^{(1)}$ is the relative Frobenius twist of X, and the notion is too weak. On the other hand, in characteristic 0, the category of bundles with a flat connection is the same as the category of $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules. In characteristic $p > 0$, $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules over a smooth variety X defined over a field k form an abelian, rigid, k-linear tensor category (see [Gi]). It is equivalent to the category of stratified bundles. It bears strong analogies with the category of bundles with an integrable connection in characteristic 0. For example, if X is projective smooth over an algebraically closed field, the triviality of the étale fundamental group forces all such $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules to be trivial ([EM]).

So we raise the QUESTION 1: let $f : X \to S$ be a smooth projective morphism of smooth connected varieties, defined over an algebraically closed characteristic $p > 0$ field, let (E, ∇) be a stratified bundle relative to S, such that for all closed point s of some non-trivial open $U \subset S$, the stratified bundle $(E, \nabla)|_{X_s}$ is trivialized by a finite étale cover of $X_s := X \times_S s$. Is it the case that the stratified bundle $(E, \nabla)|_{X_{\bar{\eta}}}$ is trivialized by a finite étale cover of $X_{\bar{\eta}}$?.

In this form, this is not true. Y. Laszlo [Ls] constructed a one dimensional non-trivial family of bundles over a curve over $\mathbb{F}_2$ which is fixed by the square of Frobenius, as a (negative) answer to a question of J. de Jong concerning the behavior of representations of the étale fundamental group over a finite field $\mathbb{F}_q$, $q = p^a$, with values in $GL(r, \mathbb{F}((t)))$, where $\mathbb{F} \supset \mathbb{F}_2$ is a finite extension. In fact, Laszlo's example yields also a counter-example to the question as stated above. We explain this in Sections 1 and 4 (see Corollary 4.3). We remark that if E is a bundle on X, such that the bundle $E|_{X_s}$ is stable, numerically flat (see Definition 3.2) and moves in the moduli, then $E_{\bar{\eta}}$ cannot be trivialized by a finite étale cover (see Proposition 4.2). In contrast, we show that if the family $X \to S$ is trivial (as it is in Laszlo's example), thus $X = Y \times_k S$, if k is algebraically closed, and if $(F_Y^n \times \mathrm{identity}_s)^*(E)|_{Y \times_k s} \cong E|_{Y \times_k s}$ for all closed points s of some non-trivial open in S and some fixed natural number n, then the moduli points of $E|_{Y \times_k s}$ are constant (see Proposition 4.4). Here $F_Y : Y \to Y$ is the absolute Frobenius of Y. In Laszlo's example, one does have $(F_Y^2 \times \mathrm{identity}_s)^*(E)|_{Y \times_k s} \cong E|_{Y \times_k s}$ but only over $k = \mathbb{F}_2$ (i.e., S is also defined over $\mathbb{F}_2$). When one extends the family to the algebraic closure of $\mathbb{F}_2$, to go from the absolute Frobenius over $\mathbb{F}_2$, that is the relative Frobenius over k, to the absolute one, one needs to replace the power 2 with a higher power $n(s)$, which depends on the field of definition of s, and is not bounded.

So we modify question 1 in QUESTION 2: let $f : X \to S$ be a smooth projective morphism of smooth connected varieties, defined over an algebraically closed characteristic field k of characteristic $p > 0$, let E be a bundle such that for all closed points s of some non-trivial open $U \subset S$, the bundle $E|_{X_s}$ is trivialized by a finite Galois étale cover of $X_s := X \times_S s$ of order prime to p. Is it the case that the bundle $E|_{X_{\bar\eta}}$ is trivialized by a finite étale cover of $X_{\bar\eta}$?.

The answer is nearly yes: it is the case if k is not algebraic over its prime field (Theorem 5.1 2)). If $k = \bar{\mathbb{F}}_p$, it might be wrong (Remarks 5.4 2), but what remains true is that there exists a finite étale cover of $X_{\bar\eta}$ over which the pull-back of E is a direct sum of line bundles (Theorem 5.1 1)). The idea of the proof is borrowed from the proof of Y. André's theorem [An, Thm 7.2.2]. The assumption on the degrees of the Galois covers of X_s trivializing $E|_{X_s}$ is necessary (as follows from Laszlo's example) and it allows us to apply Brauer-Feit's theorem [BF, Theorem] in place of Jordan's theorem used by André. However, there is no direct substitute for Simpson's moduli spaces of flat bundles. Instead, we use the moduli spaces constructed in [La1] and we carefully analyze subloci containing the points of interest, that is the numerically flat bundles. The necessary material needed on moduli is gathered in Section 3.

Finally we raise the general QUESTION 3: let $f : X \to S$ be a smooth projective morphism of smooth connected varieties, defined over an algebraically closed characteristic $p > 0$ field, let (E, ∇) be a stratified bundle relative to S, such that for all closed points s of some non-trivial open $U \subset S$, the stratified bundle $(E, \nabla)|_{X_s}$ is trivialized by a finite Galois étale cover of $X_s := X \times_S s$ of order prime to p. Is it the case that the bundle $(E, \nabla)|_{X_{\bar\eta}}$ is trivialized by a finite étale cover of $X_{\bar\eta}$?

We give the following not quite complete answer. If the rank of E is 1, (in which case the assumption on the degrees of the Galois covers is *automatically fulfilled*), then the answer is yes provided S is projective, and for any $s \in U$, $\mathrm{Pic}^{\tau}(X_s)$ is reduced (see Theorem 7.1). The proof relies on (a variant of) an idea of M. Raynaud [Ra], using the height function associated to a symmetric line bundle (that is the reason for our assumption on S) on the abelian scheme and its dual, to show that an infinite Verschiebung-divisible point has height equal to 0 (Theorem 6.2). If E has any rank, then the answer is yes if k is not $\bar{\mathbb{F}}_p$ (Theorem 7.2 2)). In general, there is a prime to p-order Galois cover of $X_{\bar\eta}$ such that the pull-back of E becomes a sum of stratified line bundles (Theorem 7.2 1)).

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1 Preliminaries on relative stratified sheaves

Let S be a scheme of characteristic p (i.e., $\mathscr{O}_S$ is an $\mathbb{F}_p$-algebra). By $F_S^r : S \to S$ we denote the r-th *absolute Frobenius morphism* of S which corresponds to the p^r-th power mapping on $\mathscr{O}_S$.
If X is an S-scheme, we denote by $X_S^{(r)}$ the fiber product of X and S over the r-th Frobenius morphism of S. If it is clear with respect to which structure X is considered, we simplify the notation to $X^{(r)}$. Then the r-th absolute Frobenius morphism of X induces the *relative Frobenius morphism* $F^r_{X/S} : X \to X^{(r)}$. In particular, we have the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "F^r_{X/S}"'] \arrow[rr, bend left, "F^r_X"] \arrow[dr] & X^{(r)} \arrow[r, "W^r_X"'] \arrow[d] & X \arrow[d] \\
& S \arrow[r, "F^r_S"'] & S
\end{tikzcd}$$

which defines $W^r_{X/S} : X^{(r)} \to X$.
Making $r = 1$ and replacing X by $X^{(i)}$, this induces the similar diagram

$$\begin{tikzcd}
X^{(i)} \arrow[r, "F_{X^{(i)}/S}"'] \arrow[rr, bend left, "F_{X^{(i)}}"] \arrow[dr] & X^{(i+1)} \arrow[r, "W_{X^{(i)}}"'] \arrow[d] & X^{(i)} \arrow[d] \\
& S \arrow[r, "F_S"'] & S
\end{tikzcd}$$

We assume that X/S is smooth. A *relative stratified sheaf* on X/S is a sequence $\{E_i, \sigma_i\}_{i \in \mathbb{N}}$ of locally free coherent $\mathscr{O}_{X^{(i)}}$-modules E_i on $X^{(i)}$ and isomorphisms $\sigma_i : F^*_{X^{(i)}/S} E_{i+1} \to E_i$ of $\mathscr{O}_{X^{(i)}}$-modules. A *morphism of relative stratified sheaves* $\{\alpha_i\} : \{E_i, \sigma_i\} \to \{E'_i, \sigma'_i\}$ is a sequence of $\mathscr{O}_{X^{(i)}}$-linear maps $\alpha_i : E_i \to E'_i$ compatible with the σ_i, that is such that $\sigma'_i \circ F^*_{X^{(i)}/S} \alpha_{i+1} = \alpha_i \circ \sigma_i$.
This forms a category $\mathsf{Strat}(X/S)$, which is contravariant for morphisms $\varphi : T \to S$: to $\{E_i, \sigma_i\} \in \mathsf{Start}(X/S)$ one assigns $\varphi^*\{E_i, \sigma_i\} \in \mathsf{Strat}(X \times_S T/T)$ in the obvious way: φ induces $1_{X^{(i)}} \times \varphi : X^{(i)} \times_S T \to X^{(i)}$ and $(\varphi^*\{E_i, \sigma_i\})_i = \{(1_{X^{(i)}} \times \varphi)^* E_i, (1_{X^{(i)}} \times \varphi)^*(\sigma_i)\}$.

If $S = \operatorname{Spec} k$ where k is a field, $\mathsf{Strat}(X/k)$ is an abelian, rigid, tensor category. Giving a rational point $x \in X(k)$ defines a fiber functor via $\omega_x : \mathsf{Strat}(X/k) \to \mathsf{Vec}_k$, $\omega_x(\{E_i, \sigma_i\}) = (E_0)|_x$ in the category of finite dimensional vector spaces over k, thus a k-group scheme $\pi(\mathsf{Strat}(X/k), \omega_x) = \mathrm{Aut}^{\otimes}(\omega_x)$. Tannaka duality implies that $\mathsf{Strat}(X/k)$ is equivalent via ω_x to the representation category of $\pi(\mathsf{Strat}(X/k), \omega_x)$ with values in Vec_k. For any object $\mathbb{E} := \{E_i, \sigma_i\} \in \mathsf{Strat}(X/k)$, we define its *monodromy group* to be the k-affine group scheme $\pi(\langle \mathbb{E} \rangle, \omega_x)$, where $\langle \mathbb{E} \rangle \subset \mathsf{Strat}(X/k)$ is the full subcategory spanned by $\mathbb{E}$. This is the image of $\pi(\mathsf{Strat}(X/k), \omega_x)$ in $GL(\omega_x(\mathbb{E}))$ ([DM, Proposition 2.21 a)]). We denote by $\mathbb{I}_{X/k} \in \mathsf{Strat}(X/k)$ the trivial object, with $E^i = \mathscr{O}_{X^{(i)}}$ and $\sigma_i =$ Identity.

Lemma 1.1. *With the notation above*

1) *If $h : Y \to X$ is a finite étale cover such that $h^*\mathbb{E}$ is trivial, then $h_*\mathbb{I}_{Y/k}$ has finite monodromy group and one has a faithfully flat homomorphism $\pi(\langle h_*\mathbb{I}_{Y/k} \rangle, \omega_x) \to \pi(\langle \mathbb{E} \rangle, \omega_x)$. Thus in particular, $\mathbb{E}$ has finite monodromy group as well.*

2) *If $\mathbb{E} \in \mathsf{Strat}(X/k)$ has finite monodromy group, then there exists a $\pi(\langle \mathbb{E} \rangle, \omega_x)$-torsor $h : Y \to X$ such that $h^*\mathbb{E}$ is trivial in $\mathsf{Strat}(Y/k)$. Moreover, one has an isomorphism $\pi(\langle h_*\mathbb{I}_{Y/k} \rangle, \omega_x) \xrightarrow{\cong} \pi(\langle \mathbb{E} \rangle, \omega_x)$.*

Proof. We first prove 2). Assume $\pi(\langle \mathbb{E} \rangle, \omega_x) =: G$ is a finite group scheme over k. One applies Nori's method [No, Chapter I, II]: the regular representation of G on the affine k-algebra $k[G]$ of regular function defines the Artin k-algebra $k[G]$ as a k-algebra object of the representation category of G on finite dimensional k-vector spaces, (such that $k \subset k[G]$ is the maximal trivial subobject). Thus by Tannaka duality, there is an object $\mathbb{A} = (A^i, \tau_i) \in \mathsf{Strat}(X/k)$, which is an $\mathbb{I}_{X/k}$-algebra object, (such that $\mathbb{I}_{X/k} \subset \mathbb{A}$ is the maximal trivial subobject). We define $h_i : Y_i = \operatorname{Spec}_{X^{(i)}} A^i \to X^{(i)}$. Then the isomorphism τ_i yields an $\mathscr{O}_{X^{(i)}}$-isomorphism between $Y^{(i)} \xrightarrow{h^{(i)}} X^{(i)}$ and $Y_i \xrightarrow{h_i} X^{(i)}$, (see, e.g., [SGA5, Exposé XV, § 1, Proposition 2]), and via this isomorphism, $\mathbb{A}$ is isomorphic to $h_*\mathbb{I}_{Y/k}$. On the other hand, $\omega_x(\mathbb{E})$ is a sub G-representation of $k[G]^{\oplus n}$ for some $n \in \mathbb{N}$, thus $\mathbb{E} \subset \mathbb{A}^{\oplus n}$ in $\mathsf{Strat}(X/k)$, thus there is an inclusion $\mathbb{E} \subset (h_*\mathbb{I}_{Y/k})^{\oplus n}$ in $\mathsf{Strat}(X/k)$, thus $h^*\mathbb{E} \subset (h^*h_*\mathbb{I}_{Y/k})^{\oplus n}$ in $\mathsf{Strat}(Y/k)$. Since $(h^*h_*\mathbb{I}_{Y/k})$ is isomorphic to $\oplus_{\mathrm{length}_k k[G]} \mathbb{I}_{Y/k}$ in $\mathsf{Strat}(Y/k)$ (recall that by [dS, Proposition 13], G is an étale group scheme), then $h^*\mathbb{E}$ is isomorphic to $\oplus_r \mathbb{I}_{Y/k}$, where r is the rank of $\mathbb{E}$. This shows the first part of the statement, and shows the second part as well: indeed, $\mathbb{E}$ is then a subobject of $\oplus_r h_*\mathbb{I}_{Y/k}$, thus $\langle \mathbb{E} \rangle \subset \langle h_*\mathbb{I}_{Y/K} \rangle$ is a full subcategory. One applies [DM, Proposition 2.21 a)] to show that the induced homomorphism $\pi(\langle h_*\mathbb{I}_{Y/k} \rangle, \omega_x) \to \pi(\langle \mathbb{E} \rangle, \omega_x) = G$ is faithfully flat. So $\pi(\langle h_*\mathbb{I}_{Y/k} \rangle, \omega_x)$ acts on $\omega_x(h_*\mathbb{I}_Y) = k[G]$ via its quotient G and the regular representation $G \subset GL(k[G])$. Thus the homomorphism is an isomorphism.

We show 1). Assume that there is a finite étale cover $h : Y \to X$ such that $h^*\mathbb{E}$ is isomorphic in $\mathsf{Strat}(Y/k)$ to $\oplus_r \mathbb{I}_{Y/k}$ where r is the rank of $\mathbb{E}$. Then $\mathbb{E} \subset \oplus_r h_*\mathbb{I}_{Y/k}$, thus $\pi(\langle h_*\mathbb{I}_{Y/k} \rangle, \omega_x) \to \pi(\langle \mathbb{E} \rangle, \omega_x)$ is faithfully flat [DM, loc. cit.], so we are reduced to showing that $\langle h_*\mathbb{I}_{Y/k} \rangle$ has finite monodromy. But, by the same argument as on $\mathbb{E}$, any of its objects of rank r' lies in $\oplus_{r'} h_* \mathbb{I}_{Y/k}$. So we apply [DM, Proposition 2.20 a)] to conclude that the monodromy of $h_* \mathbb{I}_{Y/k}$ is finite. $\square$

Corollary 1.2. *With the notations as in 1.1, if $\mathbb{E} \in \mathsf{Strat}(X/k)$ has finite monodromy group, then for any field extension $K \supset k$, $\mathbb{E} \otimes K \in \mathsf{Strat}(X \otimes K/K)$ has finite monodromy group.*

Let E be an $\mathscr{O}_X$-module. We say that *E has a stratification relative to S* if there exists a relative stratified sheaf $\{E_i, \sigma_i\}$ such that $E_0 = E$.
Let us consider the special case $S = \operatorname{Spec} k$, where k is a perfect field, and X/k is smooth. An *(absolute) stratified sheaf* on X is a sequence $\{E_i, \sigma_i\}_{i \in \mathbb{N}}$ of coherent $\mathscr{O}_X$-modules E_i on X and isomorphisms $\sigma_i : F_X^* E_{i+1} \to E_i$ of $\mathscr{O}_X$-modules.
As k is perfect, the $W_{X^{(i)}}$ are isomorphisms, thus giving an absolute stratified sheaf is equivalent to giving a stratified sheaf relative to $\operatorname{Spec} k$.
We now go back to the general case and we assume that S is an integral k-scheme, where k is a field. Let us set $K = k(S)$ and let $\eta : \operatorname{Spec} K \to S$ be the generic point of S. Let us fix an algebraic closure $\bar{K}$ of K and let $\bar{\eta}$ be the corresponding generic geometric point of S.
By contravariance, a relative stratified sheaf $\{E_i, \sigma_i\}$ on X/S restricts to a relative stratified sheaf $\{E_i, \sigma_i\}|_{X_s}$ in fibers X_s for s a point of S. We are interested in the relation between $\{E_i, \sigma_i\}|_{X_{\bar{\eta}}}$ and $\{E_i, \sigma_i\}|_{X_s}$ for closed points $s \in |S|$. More precisely, we want to understand under which assumptions the finiteness of $\langle \{E_i, \sigma_i\}|_{X_s} \rangle$ for all closed points $s \in |S|$ implies the finiteness of $\langle \{E_i, \sigma_i\}|_{X_{\bar{\eta}}} \rangle$. Recall that finiteness of $\mathbb{E} \subset \mathsf{Strat}(X_s)$ means that all objects of $\langle \mathbb{E} \rangle$ are subquotients in $\mathsf{Strat}(X_s)$ of direct sums of a single object, which is equivalent to saying that after the choice of a rational point, the monodromy group of $\mathbb{E}$ is finite ([DM, Proposition 2.20 (a)]).

Let X be a smooth variety defined over $\mathbb{F}_q$ with $q = p^r$. For all $n \in \mathbb{N} \setminus \{0\}$, one has the commutative diagram

$$\begin{tikzcd} X \arrow[r, "F^{rn}_{X/\mathbb{F}_q}"'] \arrow[rr, bend left, "(F^r_X)^n = F^{rn}_X"] \arrow[dr] & X^{(rn)} \arrow[r, "W^{rn}_{X/\mathbb{F}_q}"'] \arrow[d] & X \arrow[d] \\ & \operatorname{Spec} \mathbb{F}_q \arrow[r, "F^{rn}_{\mathbb{F}_q} = \mathrm{id}"'] & \operatorname{Spec} \mathbb{F}_q \end{tikzcd} \tag{1}$$

which allows us to identify $X^{(rn)}$ with X (as an $\mathbb{F}_q$-scheme).
Let S be an $\mathbb{F}_q$ connected scheme, with field of constants k, i.e. k is the normal closure of $\mathbb{F}_q$ in $H^0(S, \mathscr{O}_X)$. We define $X_S := X \times_{\mathbb{F}_q} S$.

Proposition 1.3. *Let E be a vector bundle on X_S. Assume that there exists a positive integer n such that we have an isomorphism*

$$\tau : ((F^r \times_{\mathbb{F}_q} \mathrm{id}_S)^n)^* E \simeq E. \tag{2}$$

Then E has a natural stratification $\mathbb{E}_\tau = \{E_i, \sigma_i\}$, $E_0 = E$ relative to S.

Proof. We define

$$E_{rn} = (W^{rn}_{X/\mathbb{F}_q} \times_{\mathbb{F}_q} \mathrm{id}_S)^* E. \tag{3}$$

Then we use the factorization

$$\begin{tikzcd}
X \arrow[r, "F_{X/\mathbb{F}_q}"] \arrow[drrrr] & X^{(1)} \arrow[r, "F_{X^{(1)}/\mathbb{F}_q}"] \arrow[drrr] & \cdots \arrow[r] & X^{(rn-1)} \arrow[r, "F_{X^{(rn-1)}/\mathbb{F}_q}"] \arrow[dr] & X^{(rn)} \arrow[d] \\
 & & & & \operatorname{Spec}\mathbb{F}_q
\end{tikzcd} \tag{4}$$

of $F^{rn}_{X/\mathbb{F}_q}$ and we define

$$E_{nr-1} = (F_{X^{(rn-1)}/\mathbb{F}_q} \times_{\mathbb{F}_q} \mathrm{id}_S)^* E_{rn}, \ldots, E_1 = (F_{X^{(1)}/\mathbb{F}_q} \times_{\mathbb{F}_q} \mathrm{id}_S)^* E_2 \tag{5}$$

with identity isomorphisms $\sigma_{nr-1}, \ldots, \sigma_1$. Then we use the isomorphism τ to define

$$\sigma_0 : E \simeq (F_{X/\mathbb{F}_q} \times_{\mathbb{F}_q} \mathrm{id}_S)^* E_1. \tag{6}$$

Assume we constructed the bundles E_i on $X^{(i)}$ for all $i \leq arn$ for some integer $a \geq 1$. We now replace the diagram (1) by the diagram

$$\begin{tikzcd}
X^{(arn)} \arrow[r, "F^{rn}_{X^{(arn)}/\mathbb{F}_q}"'] \arrow[rr, bend left, "(F^r_{X^{(arn)}})^n"] \arrow[dr] & X^{((a+1)rn)} \arrow[r, "W^{rn}_{X^{(arn)}/\mathbb{F}_q}"'] \arrow[d] & X^{(arn)} \arrow[d] \\
 & \operatorname{Spec}\mathbb{F}_q \arrow[r, "F^{rn}_{\mathbb{F}_q}=1"'] & \operatorname{Spec}\mathbb{F}_q
\end{tikzcd} \tag{7}$$

We then define

$$E_{(a+1)rn} = (W^{rn}_{X^{(arn)}/\mathbb{F}_q} \times_{\mathbb{F}_q} \mathrm{id}_S)^* E_{arn} \tag{8}$$

(which is equal to E under identification of $X^{(arn)}$ with X). Then we use the factorization

$$\begin{tikzcd}
X^{(arn)} \arrow[r, "F_{X^{(arn)}/\mathbb{F}_q}"] \arrow[drrrr] & X^{(arn+1)} \arrow[r, "F_{X^{(arn+1)}/\mathbb{F}_q}"] \arrow[drrr] & \cdots \arrow[r] & X^{((a+1)rn-1)} \arrow[r, "F_{X^{((a+1)rn-1)}/\mathbb{F}_q}"] \arrow[dr] & X^{((a+1)rn)} \arrow[d] \\
 & & & & \operatorname{Spec}\mathbb{F}_q
\end{tikzcd} \tag{9}$$

of $F^{rn}_{X^{(arn)}/\mathbb{F}_q}$ to define

$$\begin{aligned} E_{(a+1)rn-1} = (F_{X^{((a+1)rn-1)}/\mathbb{F}_q} \times_{\mathbb{F}_q} \mathrm{id}_S)^* E_{(a+1)rn}, \ldots, \\ E_{arn+1} = (F_{X^{(arn+1)}/\mathbb{F}_q} \times_{\mathbb{F}_q} \mathrm{id}_S)^* E_{arn+2} \end{aligned} \tag{10}$$

with identity isomorphisms $\sigma_{(a+1)nr-1},\dots,\sigma_{arn+1}$. Then we again use τ to define

$$\sigma_{arn}: E_{arn} \simeq (F^{rn}_{X^{(arn)}/\mathbb{F}_q})^* E_{arn+1}. \tag{11}$$

$\square$

The above construction and [Gi, Proposition 1.7] imply

Proposition 1.4. *Assume in addition to* (2) *that X is proper and $\mathbb{F}_q \subset k \subset \bar{\mathbb{F}}_q$. Fix a rational point $x \in X_S(k)$. Then for any closed point $s \in |S|$, the Tannaka group scheme $\pi(\mathbb{E}_{\tau_s}, \omega_{x\otimes_k k(s)})$ of $\mathbb{E}_{\tau_s} := \mathbb{E}_\tau|_{X_s}$ over the residue field $k(s)$ of s is finite.*

Proof. The bundle E is base changed of a bundle E^0 defined over $X \times_{\mathbb{F}_q} S_0$ for some form S_0 of S defined over a finite extension $\mathbb{F}_{q^a}$ of $\mathbb{F}_q$ such that x is base change of an $\mathbb{F}_{q^a}$-rational point x_0 of $X \times_{\mathbb{F}_q} S_0$. We can also assume that τ comes by base change from $\tau_0 : ((F^r \times_{\mathbb{F}_q} \mathrm{id}_{S_0})^n)^* E^0 \simeq E^0$. Proposition 1.3 yields then a relative stratification $\mathbb{E}^0_{\tau_0} = (E^0_i, \sigma^0_i)$ of E^0 defined over $\mathbb{F}_{q^a}$, with $E_i = E^0_i \otimes_{\mathbb{F}^a_q} k$. A closed point s of $S = S_0 \otimes_{\mathbb{F}^a_q} k$ is a base change of some closed point s_0 of S_0 of degree b say over $\mathbb{F}_{q^a}$. By Corollary 1.2 we just have to show that $\pi(\mathbb{E}_{(\tau_0)_{s_0}}, \omega_{x_0\otimes_{\mathbb{F}_{q^a}} k(s_0)})$ is finite. So we assume that $k = \mathbb{F}_{q^a},\ S = S_0,\ s = s_0$. The underling bundles of $\mathbb{E}_\tau$ and $\mathbb{E}_{\tau^m}$ are by construction all isomorphic for $m = ab$. Thus by [Gi, Proposition 1.7], $\mathbb{E}_\tau \simeq \mathbb{E}_{\tau^m}$ in $\mathsf{Strat}(X/k)$. But this implies that $F^{mn}_{X\times_{\mathbb{F}_{q^a}} s}(\mathbb{E}_{\tau_s}) \cong \mathbb{E}_{\tau_s}$. Thus E is algebraically trivializable on the Lang torsor $h: Y \to X \times_{\mathbb{F}_{q^a}} \mathbb{F}_{q^m}$ and the bundles E_i are trivializable on $Y \times_{X\times_{\mathbb{F}_{q^a}} \mathbb{F}_{q^m}} X^{(i)} = Y^{(i)}/\mathbb{F}_{q^m}$. Thus the stratified bundle $h^*\mathbb{E}_\tau$ on Y relative to $\mathbb{F}_{q^m}$ is trivial. We apply Lemma 1.1 to finish the proof. $\square$

2 Étale trivializable bundles

Let X be a smooth projective variety over an algebraically closed field k. Let $F_X : X \to X$ be the absolute Frobenius morphism.

A locally free sheaf on X is called *étale trivializable* if there exists a finite étale covering of X on which E becomes trivial.

Note that if E is étale trivializable then it is numerically flat (see Definition 3.2 and the subsequent discussion). In particular, stability and semistability for such bundles are independent of a polarization (and Gieseker and slope stability and semistability are equivalent). More precisely, such E is stable if and only if it does not contain any locally free subsheaves of smaller rank and degree 0 (with respect to some or equivalently to any polarization).

Proposition 2.1. (see [LSt]) *If there exists a positive integer n such that $(F^n_X)^*E \simeq E$ then E is étale trivializable. Moreover, if $k = \bar{\mathbb{F}}_p$ then E is étale trivializable if and only if there exists a positive integer n and an isomorphism $(F^n_X)^*E \simeq E$.*

Proposition 2.2. (see [BD]) *If there exists a finite degree d étale Galois covering $f : Y \to X$ such that f^*E is trivial and E is stable, then one has an isomorphism $\alpha : (F^d_X)^*E \simeq E$.*

As a corollary we see that a line bundle on X/k is étale trivializable if and only if it is torsion of order prime to p. One implication follows from the above proposition. The other one follows from the fact that $(F_X^d)^*L \simeq L$ is equivalent to $L^{\otimes(p^d-1)} \simeq \mathscr{O}_X$ and for any integer n prime to p we can find d such that p^d-1 is divisible by n.
We recall that if E is any vector bundle on X such that there is a $d \in \mathbb{N}\setminus\{0\}$ and an isomorphism $\alpha : (F_X^d)^*(E) \cong E$, then E carries an *absolute* stratified structure $\mathbb{E}_\alpha$, i.e. a stratified structure relative to $\mathbb{F}_p$ by the procedure of Proposition 1.3. On the other hand, any stratified stratified structure $\{E_i, \sigma_i\}$ relative to $\mathbb{F}_p$ induces in an obvious way a stratified structure relative to k: the absolute Frobenius $F_X^n : X \to X$ factors through $W^n_{X/k} : X^{(n)} \to X$, so $\{(W^n_{X/k})^*E_n, (W^n_{X/k})^*\sigma_n\}$ is the relative stratified structure, denoted by $\mathbb{E}_{\alpha/k}$. Proposition 2.2 together with Lemma 1.1 2) show

Corollary 2.3. *Under the assumptions of Proposition 2.2, we can take* $d = \mathrm{length}_k k[\pi(\langle \mathbb{E}_{\alpha/k}\rangle, \omega_x)]$.

Let us also recall that there exist examples of étale trivializable bundles such that $(F_X^n)^*E \not\simeq E$ for every positive integer n (see Laszlo's example in [BD]).

Proposition 2.4. (Deligne; see [Ls, 3.2]) *Let X be an $\mathbb{F}_{p^n}$-scheme. If G is a connected linear algebraic group defined over a finite field $\mathbb{F}_{p^n}$ then the embedding $G(\mathbb{F}_{p^n}) \hookrightarrow G$ induces an equivalence of categories between the category of $G(\mathbb{F}_{p^n})$-torsors on X and G-torsors P over X with an isomorphism $(F_X^n)^*P \simeq P$.*

In particular, if G is a connected reductive algebraic group defined over an algebraically closed field k and P is a principal G-bundle on X/k such that there exists an isomorphism $(F_X^n)^*P \simeq P$ for some natural number $n > 0$, then there exists a Galois étale cover $f : Y \to X$ with Galois group $G(\mathbb{F}_{p^n})$ such that f^*P is trivial. Indeed, every reductive group has a $\mathbb{Z}$-form so we can use the above proposition.

3 Preliminaries on relative moduli spaces of sheaves

Let S be a scheme of finite type over a universally Japanese ring R. Let $f : X \to S$ be a projective morphism of R-schemes of finite type with geometrically connected fibers and let $\mathscr{O}_X(1)$ be an f-very ample line bundle.
A *family of pure Gieseker semistable sheaves on the fibres of* $X_T = X \times_S T \to T$ is a T-flat coherent $\mathscr{O}_{X_T}$-module E such that for every geometric point t of T the restriction of E to the fibre X_t is pure (i.e., all its associated points have the same dimension) and Gieseker semistable (which is semistability with respect to the growth of the Hilbert polynomial of subsheaves defined by $\mathscr{O}_X(1)$ (see [HL, 1.2]). We introduce an equivalence relation $\sim$ on such families in the following way. $E \sim E'$ if and only if there exist filtrations $0 = E_0 \subset E_1 \subset \ldots \subset E_m = E$ and $0 = E'_0 \subset E'_1 \subset \ldots \subset E'_m = E'$ by coherent $\mathscr{O}_{X_T}$-modules such that $\oplus_{i=0}^m E_i/E_{i-1}$ is a family of pure Gieseker semistable sheaves on the fibres of X_T and there exists an invertible sheaf L on T such that $\oplus_{i=1}^m E'_i/E'_{i-1} \simeq \left(\oplus_{i=1}^m E_i/E_{i-1}\right) \otimes_{\mathscr{O}_T} L$.
Let us define the moduli functor

$$\mathscr{M}_P(X/S) : (\mathrm{Sch}/S)^o \to \mathrm{Sets}$$

from the category of locally noetherian schemes over S to the category of sets by

$$\mathscr{M}_P(X/S)(T) = \left\{ \begin{array}{l} \sim \text{equivalence classes of families of pure Gieseker} \\ \text{semistable sheaves on the fibres of } T \times_S X \to T, \\ \text{which have Hilbert polynomial } P. \end{array} \right\}.$$

Then we have the following theorem (see [La1, Theorem 0.2]).

THEOREM 3.1. *Let us fix a polynomial P. Then there exists a projective S-scheme $M_P(X/S)$ of finite type over S and a natural transformation of functors*

$$\theta : \mathscr{M}_P(X/S) \to \mathrm{Hom}_S(\cdot, M_P(X/S)),$$

which uniformly corepresents the functor $\mathscr{M}_P(X/S)$. For every geometric point $s \in S$ the induced map $\theta(s)$ is a bijection. Moreover, there is an open scheme $M^s_{X/S}(P) \subset M_P(X/S)$ that universally corepresents the subfunctor of families of geometrically Gieseker stable sheaves.

Let us recall that $M_P(X/S)$ *uniformly corepresents* $\mathscr{M}_P(X/S)$ means that for every flat base change $T \to S$ the fiber product $M_P(X/S) \times_S T$ corepresents the fiber product functor $\mathrm{Hom}_S(\cdot, T) \times_{\mathrm{Hom}_S(\cdot,S)} \mathscr{M}_P(X/S)$. For the notion of corepresentability, we refer to [HL, Definition 2.2.1]. In general, for every S-scheme T we have a well defined morphism $M_P(X/S) \times_S T \to M_P(X_T/T)$ which for a geometric point $T = \mathrm{Spec}\, k(s) \to S$ is bijection on points.
The moduli space $M_P(X/S)$ in general depends on the choice of polarization $\mathscr{O}_X(1)$.

DEFINITION 3.2. Let k be a field and let Y be a projective k-variety. A coherent $\mathscr{O}_Y$-module E is called *numerically flat*, if it is locally free and both E and its dual $E^* = \mathscr{H}om(E, \mathscr{O}_Y)$ are numerically effective on $Y \otimes \bar{k}$, where $\bar{k}$ is an algebraic closure of k.

Assume that Y is smooth. Then a numerically flat sheaf is strongly slope semistable of degree 0 with respect to any polarization (see [La2, Proposition 5.1]). But such a sheaf has a filtration with quotients which are numerically flat and slope stable (see [La2, Theorem 4.1]). Let us recall that a slope stable sheaf is Gieseker stable and any extension of Gieseker semistable sheaves with the same Hilbert polynomial is Gieseker semistable. Thus a numerically flat sheaf is Gieseker semistable with respect to any polarization.
Let P be the Hilbert polynomial of the trivial sheaf of rank r. In case S is a spectrum of a field we write $M_X(r)$ to denote the subscheme of the moduli space $M_P(X/k)$ corresponding to locally free sheaves. For a smooth projective morphism $X \to S$ we also define the moduli subscheme $M(X/S, r) \to S$ of the relative moduli space $M_P(X/S)$ as a union of connected components which contains points corresponding to numerically flat sheaves of rank r. Note that in positive characteristic numerical flatness is not an open condition. More precisely, on a smooth projective variety Y with an ample divisor H, a locally free sheaf with *numerically trivial Chern classes, that is with Chern classes c_i in the Chow group of codimension i cycles intersecting trivially $H^{\dim(Y)-i}$*

for all $i \geq 1$, is numerically flat if and only if it is strongly slope semistable (see [La2, Proposition 5.1]).
By definition for every family E of pure Gieseker semistable sheaves on the fibres of X_T we have a well defined morphism $\varphi_E = \theta([E]) : T \to M_P(X/S)$, which we call a *classifying morphism.*

Proposition 3.3. *Let X be a smooth projective variety defined over an algebraically closed field k of positive characteristic. Let S be a k-variety and let E be a rank r locally free sheaf on $X \times_k S$ such that for every $s \in S(k)$ the restriction E_s is Gieseker semistable with numerically trivial Chern classes. Assume that the classifying morphism $\varphi_E : S \to M_X(r)$ is constant and for a dense subset $S' \subset S(k)$ the bundle E_s is étale trivializable for $s \in S'$. Then $E_{\bar{\eta}}$ is étale trivializable.*

Proof. If E_s is stable for some k-point $s \in S$ then there exists an open neighbourhood U of $\varphi_E(s)$, a finite étale morphism $U' \to U$ and a locally free sheaf $\mathscr{U}$ on $X \times_k U'$ such that the pull backs of E and $\mathscr{U}$ to $X \times_k (\varphi_E^{-1}(U) \times_U U')$ are isomorphic (this is called existence of a universal bundle on the moduli space in the étale topology). But $\varphi_E(S)$ is a point, so this proves that there exists a vector bundle on X such that E is its pull back by the projection $X \times_k S \to X$. In this case the assertion is obvious.
Now let us assume that E_s is not stable for all $s \in S(k)$. If $0 = E_0^s \subset E_1^s \subset ... \subset E_m^s = E_s$ is a Jordan–Hölder filtration (in the category of slope semistable torsion free sheaves), then by assumption the isomorphism classes of semi-simplifications $\oplus_{i=1}^m E_i^s/E_{i-1}^s$ do not depend on $s \in S(k)$. Let $(r_1,...,r_m)$ denote the sequence of ranks of the components E_i^s/E_{i-1}^s for some $s \in S(k)$. Since there is only finitely many such sequences (they differ only by permutation), we choose some permutation that appears for a dense subset $S'' \subset S'$.
Now let us consider the scheme of relative flags $f : \mathrm{Flag}(E/S;P_1,...,P_m) \to S$, where P_i is the Hilbert polynomial of $\mathscr{O}_X^{r_i}$. By our assumption the image of f contains S''. Therefore by Chevalley's theorem it contains an open subscheme U of S. Let us recall that the scheme of relative flags $\mathrm{Flag}(E|_{X\times_k U}/U;P_1,...,P_m) \to U$ is projective. In particular, using Bertini's theorem (k is algebraically closed) we can find a generically finite morphism $W \to U$ factoring through this flag scheme. Let us consider pull back of the universal filtration $0 = F_0 \subset F_1 \subset ... \subset F_m = E_W$ to $X \times_k W$. Note that the quotients $F^i = F_i/F_{i-1}$ are W-flat and by shrinking W we can assume that they are families of Gieseker stable locally free sheaves (since by assumption F_s^i is Gieseker stable and locally free for some points $s \in W(k) \cap S'$). This and the first part of the proof implies that $E_{\bar{\eta}}$ has a filtration by subbundles such that the associated graded sheaf is étale trivializable. By Lemma 5.2 this implies that $E_{\bar{\eta}}$ is étale trivializable. ☐

4 Laszlo's example

Let us describe Laszlo's example of a line in the moduli space of bundles on a curve fixed by the second Verschiebung morphism (see [Ls, Section 3]).
Let us consider a smooth projective genus 2 curve X over $\mathbb{F}_2$ with affine equation

$$y^2 + x(x+1)y = x^5 + x^2 + x.$$

In this case the moduli space $M_X(2,\mathcal{O}_X)$ of rank 2 vector bundles on X with trivial determinant is an $\mathbb{F}_2$-scheme isomorphic to $\mathbb{P}^3$. The pull back of bundles by the relative Frobenius morphism defines the Verschiebung map

$$V : M_{X^{(1)}}(2,\mathcal{O}_{X^{(1)}}) \simeq \mathbb{P}^3 \dashrightarrow M_X(2,\mathcal{O}_X) \simeq \mathbb{P}^3$$

which in appropriate coordinates can be described as

$$[a:b:c:d] \to [a^2+b^2+c^2+d^2 : ab+cd : ac+bd : ad+bc].$$

The restriction of V to the line $\Delta \simeq \mathbb{P}^1$ given by $b=c=d$ is an involution and it can be described as $[a:b] \to [a+b:b]$.
Using a universal bundle on the moduli space (which exists locally in the étale topology around points corresponding to stable bundles) and taking a finite covering $S \to \Delta$ we obtain the following theorem:

THEOREM 4.1. ([Ls, Corollary 3.2]) *There exist a smooth quasi-projective curve S defined over some finite extension of $\mathbb{F}_2$ and a locally free sheaf E of rank 2 on $X \times S$ such that $(F^2 \times \mathrm{id}_S)^*E \simeq E$, $\det E \simeq \mathcal{O}_{X\times S}$ and the classifying morphism $\varphi_E : S \to M_X(2,\mathcal{O}_X)$ is not constant. Moreover, one can choose S so that E_s is stable for every closed point s in S.*

Now note that the map $(F_X)^* : M_X(2,\mathcal{O}_X) \dashrightarrow M_X(2,\mathcal{O}_X)$ defined by pulling back bundles by the absolute Frobenius morphism can be described on Δ as $[a:b] \to [a^2+b^2 : b^2]$. In particular, the map $(F_X^{2n})^*|_\Delta$ is described as $[a:b] \to [a^{2n}, b^{2n}]$. It follows that if a stable bundle E corresponds to a modular point of $\Delta(\mathbb{F}_2^n)\backslash\Delta(\mathbb{F}_2^{n-1})$ (or, equivalently, E is defined over $\mathbb{F}_{2^n}$) then $(F_X^{2n})^*E \simeq E$ and $(F_X^m)^*E \not\simeq E$ for $0<m<2n$.
This implies that for $k=\bar{\mathbb{F}}_2$ and for every $s \in S(k)$, the bundle E_s which is the restriction to $X \times_{\mathbb{F}_2} s$ of the bundle E from Theorem 4.1, is étale trivializable.

Let X, S be varieties defined over an algebraically closed field k of positive characteristic. Assume that X is projective. Let us set $K = k(S)$. Let $\bar{\eta}$ be a generic geometric point of S.

PROPOSITION 4.2. *Let E be a bundle on $X_S = X \times_k S \to S$ which is numerically flat on the closed fibres of $X_S = X \times_k S \to S$. Assume that for some $s \in S$ the bundle E_s is stable and the classifying morphism $\varphi_E : S \to M_X(r)$ is not constant. Then $E_{\bar\eta} = E|_{X_{\bar\eta}}$ is not étale trivializable.*

Proof. Assume that there exists a finite étale cover $\pi' : Y' \to X_{\bar\eta}$ such that $(\pi')^*E_{\bar\eta} \simeq \mathcal{O}_{Y'}^r$. As k is algebraically closed, one has the base change $\pi_1(X) \xrightarrow{\cong} \pi_1(X_{\bar K})$ for the étale fundamental group ([SGA1, Exp. X, Cor.1.8]), so there exists a finite étale cover $\pi : Y \to X$ such that $\pi' = \pi \otimes \bar K$. Hence there exists a finite morphism $T \to U$ over some open subset U of S, such that $\pi_T^*(E_T)$ is trivial where $\pi_T = \pi \times_k \mathrm{id}_T : Y \times_k T \to X \times_k T$ and E_T =pull back by $X \times_k T \to X \times_k U$ of $E|_{X\times_k U}$.
So for any k-rational point $t \in T$, one has $\pi^*E_t \subset \mathcal{O}_Y^r$, where r is the rank of E. Hence $E_t \subset \pi_*\pi^*E_t \subset \pi_*\mathcal{O}_Y^r$, i.e., all the bundles E_t lie in one fixed bundle $\pi_*\mathcal{O}_Y^r$.

Since π is étale, the diagram

$$\begin{tikzcd} Y \arrow[r, "F_Y"] \arrow[d, "\pi"] & Y \arrow[d, "\pi"] \\ X \arrow[r, "F_X"] & X \end{tikzcd}$$

is cartesian (see, e.g., [SGA5, Exp. XIV, §1, Prop. 2]). Since X is smooth, F_X is flat. By flat base change we have isomorphisms $F_X^*(\pi_*\mathscr{O}_Y)\simeq \pi_*(F_Y^*\mathscr{O}_Y)\simeq \pi_*\mathscr{O}_Y$. In particular, this implies that $\pi_*\mathscr{O}_Y$ is strongly semistable of degree 0. Therefore if E_t is stable then it appears as one of the factors in a Jordan–Hölder filtration of $\pi_*\mathscr{O}_Y$. Since the direct sum of factors in a Jordan–Hölder filtration of a semistable sheaf does not depend on the choice of the filtration, there are only finitely many possibilities for the isomorphism classes of stable sheaves E_t for $t\in T(k)$.

It follows that in $U\subset S$ there is an infinite sequence of k-rational points s_i with the property that E_{s_i} is stable (since stability is an open property) and $E_{s_i}\cong E_{s_{i+1}}$. This contradicts our assumption that the classifying morphism φ_E is not constant. ☐

Corollary 4.3. *There exist smooth curves X and S defined over an algebraic closure k of $\mathbb{F}_2$ such that X is projective and there exists a locally free sheaf E on $X\times_k S\to S$ such that for every $s\in S(k)$, the bundle E_s is étale trivializable but $E_{\bar\eta}$ is not étale trivializable. Moreover, on E there exists a structure of a relatively stratified sheaf $\mathbb{E}$ such that for every $s\in S(k)$, the bundle $\mathbb{E}_s$ has finite monodromy but the monodromy group of $\mathbb{E}_{\bar\eta}$ is infinite.*

The second part of the corollary follows from Proposition 1.3. The above corollary should be compared to the following fact:

Proposition 4.4. *Let X be a projective variety defined over an algebraically closed field k of positive characteristic. Let S be a k-variety and let E be a rank r locally free sheaf on $X\times_k S$. Assume that there exists a positive integer n such that for every $s\in S(k)$ we have $(F_X^n)^*E_s\simeq E_s$, where F_X denotes the absolute Frobenius morphism. Then the classifying morphism $\varphi_E : S\to M_X(r)$ is constant and $E_{\bar\eta}$ is étale trivializable.*

Proof. By Proposition 2.1, if $(F_X^n)^*E_s\simeq E_s$ then there exists a finite étale Galois cover $\pi_s : Y_s\to X$ with Galois group $G=\mathrm{GL}_r(\mathbb{F}_{p^n})$ such that $\pi_s^*E_s$ is trivial (in this case it is essentially due to Lange and Stuhler; see [LSt]). This implies that $E_s\subset(\pi_s)_*\pi_s^*E_s\simeq((\pi_s)_*\mathscr{O}_Y)^{\oplus r}$ and hence $\mathrm{gr}_{JH}E_s\subset(\mathrm{gr}_{JH}(\pi_s)_*\mathscr{O}_Y)^{\oplus r}$.

Since X is proper, the étale fundamental group of X is topologically finitely generated and hence there exists only finitely many finite étale coverings of X of fixed degree (up to an isomorphism). This theorem is known as the Lang–Serre theorem (see [LS, Théorème 4]). Let $\mathscr{S}$ be the set of all Galois coverings of X with Galois group G. Then for every closed k-point s of S the semi-simplification of E_s is contained in $(\mathrm{gr}_{JH}\,\alpha_*\mathscr{O}_Y)^{\oplus r}$ for some $\alpha\in\mathscr{S}$. Therefore there are only finitely many possibilities for images of k-points s in $M_X(r)$. Since S is connected, it follows that $\varphi_E : S\to M_X(r)$ is constant.

The remaining part of the proposition follows from Proposition 3.3.

□

Note that by Proposition 4.2 together with Corollary 2.3, the monodromy groups of E_s in Theorem 4.1 for $s \in S(k)$ are not uniformly bounded. In fact, only if k is an algebraic closure of a finite field do we know that the monodromy groups of E_s are finite because then E_s can be defined over some finite subfield of k and the isomorphism $(F^2)^*E_s \simeq E_s$ implies that for some n we have $(F_X^n)^*E_s \simeq E_s$ (see the paragraph following Theorem 4.1).
Moreover, the above proposition shows that in Theorem 4.1, we cannot hope to replace F with the absolute Frobenius morphism F_X.

5 Analogue of the Grothendieck-Katz conjecture in positive equicharacteristic

As Corollary 4.3 shows, the positive equicharacteristic version of the Grothendieck–Katz conjecture which requests a relatively stratified bundle to have finite monodromy group on the geometric generic fiber once it does on all closed fibers, does not hold in general. But one can still hope that it holds for a family of bundles coming from representations of the prime-to-p quotient of the étale fundamental group. In this section we follow André's approach [An, Théorème 7.2.2] in the equicharacteristic zero case to show that this is indeed the case.

Let k be an algebraically closed field of positive characteristic p. Let $f : X \to S$ be a smooth projective morphism of k-varieties (in particular, integral k-schemes). Let η be the generic point of S. In particular, $X_{\bar{\eta}}$ is smooth (see [SGA1, Defn 1.1]).

Theorem 5.1. *Let E be a locally free sheaf of rank r on X. Let us assume that there exists a dense subset $U \subset S(k)$ such that for every s in U, there is a finite Galois étale covering $\pi_s : Y_s \to X_s$ of Galois group of order prime-to-p such that $\pi_s^*(E_s)$ is trivial.*

1) *Then there exists a finite Galois étale covering $\pi_{\bar{\eta}} : Y_{\bar{\eta}} \to X_{\bar{\eta}}$ of order prime-to-p such that $\pi_{\bar{\eta}}^* E_{\bar{\eta}}$ is a direct sum of line bundles.*

2) *If k is not algebraic over its prime field and U is open in S, then $E_{\bar{\eta}}$ is étale trivializable on a finite étale cover $Z_{\bar{\eta}} \to X_{\bar{\eta}}$ which factors as a Kummer (thus finite abelian of order prime to p) cover $Z_{\bar{\eta}} \to Y_{\bar{\eta}}$ and a Galois cover $Y_{\bar{\eta}} \to X_{\bar{\eta}}$ of order prime to p.*

Proof. Without loss of generality, shrinking S if necessary, we may assume that S is smooth. Moreover, by passing to a finite cover of S and replacing U by its inverse image, we can assume that f has a section $\sigma : S \to X$.
By assumption for every $s \in U$ there exists a finite étale Galois covering $\pi_s : Y_s \to X_s$ with Galois group Γ_s of order prime-to-p and such that $\pi_s^* E_s$ is trivial. To these data one can associate a representation $\rho_s : \pi_1^{p'}(X_s, \sigma(s)) \to \Gamma_s \subset \mathrm{GL}_r(k)$ of the prime-to-p quotient of the étale fundamental group.

By the Brauer–Feit version of Jordan's theorem (see [BF, Theorem]) there exist a constant $j(r)$ such that Γ_s contains an abelian normal subgroup A_s of index $\leq j(r)$ (here we use assumption that the p-Sylow subgroup of Γ_s is trivial).
For a k-point s of S we have a homomorphism of specialization

$$\alpha_s : \pi_1(X_{\bar\eta}, \sigma(\bar\eta)) \twoheadrightarrow \pi_1(X_s, \sigma(s)),$$

which induces an isomorphism of the prime-to-p quotients of the étale fundamental groups.
So for every $s \in U$ we can define the composite morphism

$$\tilde\rho_s : \pi_1^{p'}(X_{\bar\eta}, \sigma(\bar\eta)) \xrightarrow{\alpha_s} \pi_1^{p'}(X_s, \sigma(s)) \xrightarrow{\rho_s} \Gamma_s \twoheadrightarrow \Gamma_s/A_s.$$

Let K be the kernel of the canonical homomorphism $\pi_* : \pi_1(X, \sigma(\bar\eta)) \longrightarrow \pi_1(S, \bar\eta)$, let $K^{p'}$ be its maximal pro-p'-quotient. Then by [SGA1, Exp. XIII, Proposition 4.3 and Exemples 4.4], one has $K^{p'} = \pi_1^{p'}(X_{\bar\eta}, \sigma(\bar\eta))$, the maximal pro-$p'$-quotient of $\pi_1(X_{\bar\eta}, \sigma(\bar\eta))$, and one has a short exact sequence

$$\{1\} \to \pi_1^{p'}(X_{\bar\eta}, \sigma(\bar\eta)) \longrightarrow \pi_1'(X, \sigma(\bar\eta)) \xrightarrow{\pi_*} \pi_1(S, \bar\eta) \to \{1\},$$

where $\pi_1'(X, \sigma(\bar\eta))$ is defined as the push-out of $\pi_1(X, \sigma(\bar\eta))$ by $K \to K^{p'}$.
Since $X_{\bar\eta}$ is proper, $\pi_1(X_{\bar\eta}, \sigma(\bar\eta))$ is topologically finitely generated. Therefore $\pi_1^{p'}(X_{\bar\eta}, \sigma(\bar\eta))$ is also topologically finitely generated and hence it contains only finitely many subgroups of indices $\leq j(r)$. Let G be the intersection of all such subgroups in $\pi_1^{p'}(X_{\bar\eta}, \sigma(\bar\eta))$. It is a normal subgroup of finite index. Since $\ker(\tilde\rho_s)$ is a normal subgroup of index $\leq j(r)$ in $\pi_1^{p'}(X_{\bar\eta}, \sigma(\bar\eta))$ we have

$$G \subset \bigcap_{s \in U} \ker(\tilde\rho_s).$$

Now let us consider the commutative diagram

$$\begin{tikzcd}
 & \pi_1(X_{\bar\eta}, \sigma(\bar\eta)) \arrow[r] \arrow[d] & \pi_1(X, \sigma(\bar\eta)) \arrow[r] \arrow[d] & \pi_1(S, \bar\eta) \arrow[r] \arrow[d] & \{1\} \\
\{1\} \arrow[r] & \pi_1^{p'}(X_{\bar\eta}, \sigma(\bar\eta)) \arrow[r] & \pi_1'(X, \sigma(\bar\eta)) \arrow[r] & \pi_1(S, \bar\eta) \arrow[r] & \{1\}
\end{tikzcd}$$

Then $G \cdot \sigma_*(\pi_1(S, \bar\eta)) \subset \pi_1'(X, \sigma(\bar\eta))$ is a subgroup of finite index. It is open by the Nikolov–Segal theorem [NS, Theorem 1.1]. So the pre-image H of this subgroup under the quotient homomorphism $\pi_1(X, \sigma(\bar\eta)) \to \pi_1'(X, \sigma(\bar\eta))$ defines a finite étale covering $h : X' \to X$.
Let us take $s \in S(k)$. Since the composition

$$H \subset \pi_1(X, \sigma(\bar\eta)) \to \pi_1(X, \sigma(s)) \to \pi_1(S, s)$$

is surjective, the geometric fibres of $X' \to S$ are connected. Let us choose a k-point in X' lying over $\sigma(s)$. By abuse of notation we call it $\sigma'(s)$. Similarly, let us choose a geometric point $\sigma'(\bar\eta)$ of $X'_{\bar\eta}$ lying over $\sigma(\bar\eta)$. Then for any $s \in U$ we have the following commutative diagram:

$$\begin{tikzcd} \pi_1^{p'}(X'_{\bar\eta}, \sigma'(\bar\eta)) \arrow[r, "h_*"] \arrow[d, "\simeq"] \arrow[rr, bend left, "0"] & \pi_1^{p'}(X_{\bar\eta}, \sigma(\bar\eta)) \arrow[r] \arrow[d, "\alpha_s \simeq"] & \pi_1^{p'}(X_{\bar\eta}, \sigma(\bar\eta))/G \arrow[d] \\ \pi_1^{p'}(X'_s, \sigma'(s)) \arrow[r, "h_*"] & \pi_1^{p'}(X_s, \sigma(s)) \arrow[r] & \Gamma_s/A_s \end{tikzcd}$$

This diagram shows that $\pi_1^{p'}(X'_s, \sigma'(s)) \to \Gamma_s$ factors through A_s and hence $E'_s = (h^*E)_s$ is trivialized by a finite étale Galois covering $\pi'_s : Y'_s \to X'_s$ with an abelian Galois group of order prime to p, which is a subgroup of A_s. Since

$$E'_s \subset (\pi'_s)_*(\pi'_s)^* E'_s \simeq ((\pi'_s)_* \mathscr{O}_{Y'_s})^{\oplus r},$$

and $(\pi'_s)_*\mathscr{O}_{Y'_s}$ is a direct sum of torsion line bundles of orders prime to p, it follows that for every $s \in U$ the bundle E'_s is also a direct sum of torsion line bundles of order prime to p.

We consider the union $M(X'/S, r)$ of the components of $M_P(X'/S)$ containing moduli points of numerically flat bundles, as defined in Section 3. Let us consider the S-morphism $\psi : M(X'/S, 1)^{\times_S r} \to M(X'/S)$ given by $([L_1], ..., [L_r]) \to [\oplus L_i]$ (in fact we give it by this formula on the level of functors; existence of the morphism follows from the fact that moduli schemes corepresent these functors). The bundle E' gives us a section $\tau : S \to M(X'/S, r)$, and by the above for every k-rational point s of U, the point $\tau(s)$ is contained in the image of ψ. Therefore $\tau(S)$ is contained in the image of ψ as ψ is projective (thus proper).

Let us consider the fibre product

$$\begin{tikzcd} M(X'/S,1)^{\times_S r} \times_{M(X'/S,r)} S \arrow[r] \arrow[d] & S \arrow[d, "\tau"] \\ M(X'/S,1)^{\times_S r} \arrow[r] & M(X'/S,r) \end{tikzcd}$$

Let us recall that in positive characteristic the canonical map $M(X' \times_S S'/S', r) \to M(X'/S, r) \times_S S'$ need not be an isomorphism (although it is an isomorphism for $r = 1$). Anyway we can find an étale morphism $S' \to S$ over some non-empty open subset of S, such that there exists a map $\upsilon : S' \to M(X' \times_S S'/S', 1)^{\times_{S'} r}$ which composed with $M(X' \times_S S'/S', 1)^{\times_{S'} r} \to M(X' \times_S S'/S', r) \to M(X'/S, r)$ gives the composition of $S' \to S$ with τ. This shows that the pull back E'' of E' to $X' \times_S S'$ has a filtration whose quotients are line bundles which are of degree 0 on the fibres of $X' \times_S S' \to S'$.

Now let us note the following lemma:

Lemma 5.2. *Let $f : X \to S$ be a projective morphism of k-varieties. Let $0 \to G_1 \to G \to G_2 \to 0$ be a sequence of locally free sheaves on X. Assume that there exists a dense subset $U \subset S(k)$ such that for each $s \in U$ this sequence splits after restricting to X_s. Then it splits on the fibre X_η over the generic point η of S.*

Proof. By shrinking S if necessary, we may assume that S is affine and the relative cohomology sheaf $R^1 p_* \mathscr{H}om(G_2, G_1)$ is locally free. The above short exact sequence defines a class $\lambda \in \mathrm{Ext}^1(G_2, G_1) \simeq H^0(S, R^1 f_* \mathscr{H}om(G_2, G_1))$, such that $\lambda(s) = 0$ for every k-rational point s of U. It follows that $\lambda = 0$ and hence the sequence is split over the generic point of S. □

Now let us note that on a smooth projective variety every short exact sequence of the form $0 \to G_1 \to G \to G_2 \to 0$ in which G is a direct sum of line bundles of degree 0 and G_2 is a line bundle of degree 0 splits. So the filtration of E'' restricted to the closed fibers splits. Therefore the above lemma and easy induction show that $E''_{\eta'}$ is a direct sum of line bundles, where η' is the generic point of S'. This shows the first part of the theorem.

To prove the second part of the theorem, we may assume that $U = S$. Let us take a line bundle L on X such that for every k-rational point s the line bundle L_s is étale trivializable. We need to prove that there exists a positive integer n prime to p and such that $L_\eta^{\otimes n} \simeq \mathscr{O}_{X_\eta}$.

We thank the referee for showing us the following lemma.

Lemma 5.3. *Let $g : A \to S$ be an abelian scheme and let σ be a section of g such that for all $s \in S(k)$, $\sigma(s)$ is torsion of order prime to p. Then σ is torsion of order prime to p.*

Proof. We may assume that S is normal and affine. Let us choose a subfield $k' \subset k$ that is finitely generated and transcendental over $\mathbb{F}_p$ and such that $A \to S$ and σ come by base change $\mathrm{Spec}\, k \to \mathrm{Spec}\, k'$ from an abelian scheme $g' : A' \to S'$ and a section σ' defined over k'. Let $m > 1$ be prime to p and let Γ be the subgroup $A'(S') \cap [m]^{-1}(\mathbb{Z}.\sigma')$ of $A'(S')$. Then Γ is a finitely generated group. Note that assumptions of Néron's specialization theorem [L, Chapter 9, Theorem 6.2] are satisfied and therefore there exists a Hilbert set Σ of points $s' \in S'$ for which the specialization map $A'(S') \to A'_{s'}(k(s'))$ is injective on Γ. Since the Hilbert subset $\Sigma \subset S'$ contains infinitely many closed points (see [L, Chapter 9, Theorems 5.1, 5.2 and 4.2]), there is a closed point $s \in S$ the image of which in S' lies in Σ. The specialization of $\mathbb{Z}.\sigma$ at s is injective and hence σ is torsion of order dividing the order of $\sigma(s)$, which is prime to p. □

Let us first assume that $X \to S$ is of relative dimension 1. By passing to a finite cover of S we can assume that f has a section. The relative Picard scheme $A = \mathrm{Pic}^0(X/S) \to S$ is smooth. Using the above lemma to the section corresponding to the line bundle L we see that there exists some positive integer n prime to p and a line bundle M on S such that $L^{\otimes n} \simeq f^* M$. In particular, $L_\eta^{\otimes n} \simeq \mathscr{O}_{X_\eta}$.

Now we use induction on the relative dimension of $f : X \to S$ to prove the theorem in the general case. Note that our assumptions imply that $L_{\bar\eta}$ is numerically flat and therefore the family $\{L_{\bar\eta}^{\otimes n}\}_{n\in\mathbb{Z}}$ is bounded. Thus for any sufficiently ample divisor H on $X_{\bar\eta}$ we have $H^1(X_{\bar\eta}, L_{\bar\eta}^{\otimes n}(-H)) = 0$ for all integers n. We consider such an H which is defined over η.

Using Bertini's theorem we can find a very ample divisor $Y \subset X$ in the linear system $|H|$ such that $f|_Y : Y \to S$ is smooth (possibly after shrinking S) and such that for every positive integer n we have $H^1(X_\eta, L^{\otimes n}(-Y)|_{Y_\eta}) = 0$. Indeed, shrinking S and using semicontinuity of cohomology, we may assume that H is defined over S, that the function $\dim H^0(X_s, \mathscr{O}_{X_s}(H))$ is constant and S is affine. Let us choose a k-rational point s in S. Then by Grauert's theorem (see [Ha, Chapter III, Corollary 12.9]) the restriction map

$$H^0(X, \mathscr{O}_X(H)) \to H^0(X_s, \mathscr{O}_{X_s}(H))$$

is surjective. By Bertini's theorem in the linear system $|\mathscr{O}_{X_s}(H)|$ there exists a smooth divisor. By the above we can lift it to a divisor $Y \subset X$, which after shrinking S is the required divisor.

Applying our induction assumption to $L|_Y$ on $Y \to X$ we see that there exists a positive integer n prime to p such that $(L|_Y)_\eta^{\otimes n} \simeq \mathscr{O}_{Y_\eta}$. Using the short exact sequence

$$0 \to L_\eta^{\otimes n}(-Y_\eta) \to L_\eta^{\otimes n} \to (L_Y^{\otimes n})_\eta \to 0$$

we see that the map

$$H^0(X_\eta, L_\eta^{\otimes n}) \to H^0(Y_\eta, (L_Y^{\otimes n})_\eta)$$

is surjective. In particular, $L_\eta^{\otimes n}$ has a section and hence it is trivial. □

Remarks 5.4. 1. Laszlo's example shows that the first part of the theorem is false if one does not assume that orders of the monodromy groups of E_s are prime to p (in this example $E_{\bar\eta}$ is a stable rank 2 vector bundle). Note that in this example, E has even the richer structure of a relatively stratified bundle (see Proposition 1.3).

2. Let E be a supersingular elliptic curve defined over $k = \bar{\mathbb{F}}_p$. Let M be a line bundle of degree 0 and of infinite order on $E_{\overline{\mathbb{F}_p(t)}}$. Then one can find a smooth curve S defined over k such that there exists a line bundle L on $X = S \times_k E \to S$ such that $L_{\bar\eta} \simeq M$. In this example the line bundle L_s is torsion for every k-rational point s of S as it is defined over a finite field. Since E is a supersingular elliptic curve, there are no torsion line bundles of order divisible by p. So in this case all line bundles L_s for $s \in S(k)$ are étale trivializable (and the monodromy group has order prime to p).

   This shows that the second part of Theorem 5.1 is no longer true if k is an algebraic closure of a finite field.

Let us keep the notation from the beginning of the section, i.e., k is an algebraically closed field of positive characteristic p and $f : X \to S$ is a smooth projective morphism of k-varieties (in particular connected) with geometrically connected fibers. For simplicity, we also assume that f has a section $\sigma : S \to X$.

Lemma 5.5. *Let E be a locally free sheaf on X. If there exists a point $s_0 \in S(k)$ such that E_{s_0} is numerically flat then $E_{\bar{\eta}}$ is also numerically flat. In particular, if there exists a point $s_0 \in S(k)$ such that there is a finite covering $\pi_{s_0} : Y_{s_0} \to X_{s_0}$ such that $\pi_{s_0}^*(E_{s_0})$ is trivial, then $E_{\bar{\eta}}$ is also numerically flat.*

Proof. Let us fix a relatively ample line bundle. If E_{s_0} is numerically flat then it is strongly semistable with numerically trivial Chern classes (see [La2, Proposition 5.1]). Since E is S-flat, the restriction of E to any fiber has numerically trivial Chern classes (as intersection numbers remain constant on fibres). Now note that for any n the sheaf $(F^n_{X_{s_0}/k})^* E_{s_0}$ is slope semistable. Since slope semistability is an open property, it follows that $(F^n_{X_\eta/K})^* E_\eta$ is also slope semistable. By [HL, Corollary 1.3.8] it follows that $(F^n_{X_{\bar{\eta}}/\bar{K}})^* E_{\bar{\eta}}$ is also slope semistable. Thus $E_{\bar{\eta}}$ is strongly semistable with vanishing Chern classes and hence it is numerically flat by [La2, Proposition 5.1]. □

Let us recall that numerically flat sheaves on a proper k-variety Y form a Tannakian category. A rational point $y \in Y(k)$ neutralizes it. Thus we can define *S-fundamental group scheme of Y at the point y* (see [La2, Definition 6.1]). For a numerically flat sheaf E on Y, we consider the Tannaka k-group $\pi_S(\langle E \rangle, y) := \mathrm{Aut}^{\otimes}(\langle E \rangle, y) \subset \mathrm{GL}(E_y)$, where now $\langle E \rangle$ is the full tensor subcategory of numerically flat bundles spanned by E. We call it the *S-monodromy group scheme*. Using this language we can reformulate Theorem 5.1 in the following way (for simplicity we reformulate only the second part of the theorem).

Theorem 5.6. *Let E be an S-flat family of numerically flat sheaves on the fibres of $X \to S$. Let us assume that k is not algebraic over its prime field and there exists a non-empty open subset $U \subset S(k)$ such that for every s in U, the S-monodromy group scheme $\pi_S(\langle E_s \rangle, \sigma(s))$ is finite étale of order prime-to-p. Then $\pi_S(\langle E_{\bar{\eta}} \rangle, \sigma(\bar{\eta}))$ is also finite étale.*

6 Verschiebung divisible points on abelian varieties: on the theorem by M. Raynaud

Let K be an arbitrary field of positive characteristic p and let A be an abelian variety defined over K. The multiplication by p^n map $[p^n] : A \to A$ factors through the relative Frobenius morphism $F^n_{A/K} : A \to A^{(n)}$ and hence defines the *Verschiebung morphism* $V^n : A^{(n)} \to A$ such that $V^n F^n_{A/K} = [p^n]$.

Definition 6.1. A K-point P of A is said to be *V-divisible* if for every positive integer n there exists a K-point P_n in $A^{(n)}$ such that $V^n(P_n) = P$.

Let T be an integral noetherian separated scheme of dimension 1 with field of rational functions K. Let us recall that a smooth, separated group scheme of finite type $\mathscr{A} \to T$ is called a *Néron model* of A if the general fiber of $\mathscr{A} \to T$ is isomorphic to A and for every smooth morphism $X \to T$, a morphism $X_K \to \mathscr{A}_K$ extends (then uniquely) to a T-morphism $X \to \mathscr{A}$.

Assume that the base field K is the function field of a normal projective variety S defined over a field k of positive characteristic p.
We say that A has *a good reduction* at a codimension 1 point $s \in S$ if the Néron model of A over $\operatorname{Spec} \mathscr{O}_{S,s}$ is an abelian scheme (the usual definition is slightly different as it assumes that the identity component of the special fibre of the Néron model is an abelian variety; it is equivalent to the above one by [BLR, 7.4, Theorem 5]). We say that A has *potential good reduction* at a codimension 1 point $s \in S$ if there exists a finite Galois extension K' of K such that if S' is the normalization of S in K' then $A_{K'}$ has good reduction at every codimension 1 point $s' \in S'$ lying over s.
We say that A has *(potential) good reduction* if it has (potential) good reduction at every codimension 1 point of S. Assume that A has good reduction at every codimension 1 point of S. Then there exists a *big open* subset $U \subset S$ (i.e., the codimension of the complement of U in S is ≥ 2) and an abelian U-scheme $\mathscr{A} \to U$. Note that the group $A(K)$ of K-points of A is isomorphic via the restriction map to the group of rational sections $U \dashrightarrow \mathscr{A}$ of $\mathscr{A} \to U$ defined over some big open subset of U. The section corresponding to $P \in A(K)$ will be denoted by $\tilde{P} : U \dashrightarrow \mathscr{A}$.
Let $c \in \operatorname{Pic} A$ be a class of a line bundle L. By the theorem of the cube c, satisfies the following equality:

$$m_{123}^* c - m_{12}^* c - m_{13}^* c - m_{23}^* c + m_1^* c + m_2^* c + m_3^* c = 0,$$

where m_I for $I \subset \{1,2,3\}$ is the map $A \times_K A \times_K A \to A$ defined by addition over the factors in I. (In particular, m_i is the i-th projection). Combining [MB, Chapter III, 3.1] (relying on [MB, Chapter II, Proposition 1.2.1]), the line bundle $L \in \operatorname{Pic}(A)$ extends uniquely (at least if we fix a rigidification) to a line bundle $\tilde{L}$ over $\mathscr{A}_V$ such that the class $\tilde{c} = [\tilde{L}] \in \operatorname{Pic}(\mathscr{A}_V)$ is cubical, i.e., satisfies the relation

$$\tilde{m}_{123}^* \tilde{c} - \tilde{m}_{12}^* \tilde{c} - \tilde{m}_{13}^* \tilde{c} - \tilde{m}_{23}^* \tilde{c} + \tilde{m}_1^* \tilde{c} + \tilde{m}_2^* \tilde{c} + \tilde{m}_3^* \tilde{c} = 0,$$

where $V \subset U$ is a big open subset and where $\tilde{m}_I$ for $I \subset \{1,2,3\}$ is the map $\mathscr{A} \times_S \mathscr{A} \times_S \mathscr{A} \to \mathscr{A}$ defined by addition over the factors in I.
Now let us choose an ample line bundle H on S. Then the map $\hat{h}_c : A(K) \to \mathbb{Z}$ given by

$$\hat{h}_c(P) = \deg_H (\tilde{P} - \tilde{0})^* \tilde{c}$$

is well defined as $\tilde{P}$ is defined on a big open subset of S and $\tilde{P}^* \tilde{L}$ extends to a rank 1 reflexive sheaf on S. This map is the canonical (Néron–Tate) height of A corresponding to c (see [MB, Chapter III, Section 3]).

The following theorem was suggested to the authors by M. Raynaud (in the good reduction case over a curve S, and with a somewhat different proof).

THEOREM 6.2. *Assume that A has potential good reduction. If $P \in A(K)$ is V-divisible and c is symmetric then $\hat{h}_c(P) = 0$.*

Proof. Let us first assume that A has good reduction. By assumption there exists a K-point P_n of $A^{(n)}$ such that $V^n(P_n) = P$. Since $\mathscr{A} \to U$ is an abelian scheme, so is $\mathscr{A}^{(n)} \to U$, thus P_n is the restriction to $\operatorname{Spec} K$ of $\tilde{P}_n \in \mathscr{A}^{(n)}(U)$.

Let us factor the absolute Frobenius morphism F_A^n into the composition of the relative Frobenius morphism $F_{A/K}^n : A \to A^{(n)}$ and $W_n : A^{(n)} \to A$. Let us set $c_n = W_n^* c$. Its cubical extension $\tilde{c}_n \in \mathrm{Pic}\,(\mathscr{A}_{V_n}^{(n)})$, for some big open $V_n \subset U$, together with H allows one to define $\hat{h}_{c_n}(P_n)$ by the corresponding formula. Since $(F_A^n)^* c = p^n c$, we have $(F_{A/K}^n)^* c_n = p^n c$. On the other hand, since c is symmetric, we have $[p^n]^* c = p^{2n} c$ and hence $(F_{A/K}^n)^*((V^n)^* c) = p^{2n} c$. Therefore

$$(F_{A/K}^n)^*((V^n)^* c - p^n c_n) = 0.$$

Since $F_{A/K}^n$ is an isogeny this implies that the class $d = (V^n)^* c - p^n c_n$ is torsion. By additivity and functoriality of the canonical height (see [Se, Theorem, p. 35]) we have

$$\hat{h}_c(P) = \hat{h}_{(V^n)^* c}(P_n) = \hat{h}_{p^n c_n}(P_n) + \hat{h}_d(P_n) = p^n \cdot \hat{h}_{c_n}(P_n)$$

(note that additivity implies that $\hat{h}_{md} = m\hat{h}_d$, so since $md = 0$ for some m, we get $\hat{h}_d = 0$). Therefore if $\hat{h}_c(P) \neq 0$ then $|\hat{h}_c(P)| \geq p^n$ and we get a contradiction if n is sufficiently large.

Now let us consider the general case. Since there exist only finitely many codimension 1 points $s \in S$ at which A has bad reduction, one can find a finite Galois extension K' of K such that if S' is the normalization of S in K' then $A_{K'}$ has good reduction at every codimension 1 point $s' \in S'$. On the other hand, if $P \in A(K)$ is V-divisible on A, $P \otimes K' \in A(K')$ is V divisible on $A_{K'}$. Then by the above we have $\hat{h}_{\pi^* c}(P') = 0$ and functoriality of the canonical height implies that $\hat{h}_c(P) = 0$. □

Remark 6.3. It is an interesting problem whether Theorem 6.2 holds for an arbitrary abelian variety A/K. Its proof shows that one can use the semiabelian reduction theorem to reduce the general statement to the case when A has semiabelian reduction (see [BLR, 7.4, Theorem 1]).

Now assume that S is geometrically connected. Then the extension $k \subset K$ is regular (i.e., K/k is separable and k is algebraically closed in K). Let (B, τ) be the K/k-trace of the abelian K-variety A, where B is an abelian k-variety and $\tau : B_K \to A$ is a homomorphism of abelian K-varieties (it exists by [Co, Theorem 6.2]). Let us recall that by definition (B, τ) is a final object in the category of pairs consisting of an abelian k-variety and a K-map from the scalar K-extension of this variety to A.

Since the extension $k \subset K$ is regular, the kernel K-group scheme of τ is connected (with connected dual) ([Co, Theorem 6.12]). Therefore τ is injective on K-points and in particular we can treat $B(k)$ as a subgroup of $A(K)$.

COROLLARY 6.4. *Assume that A has potential good reduction. If $P \in A(K)$ is V-divisible then $[P] \in (A(K)/B(k))_{\mathrm{tors}}$. In particular, if k is algebraically closed then $P \in B(k) + A(K)_{\mathrm{tors}} \subset A(K)$.*

Proof. We can choose the class $c \in \mathrm{Pic}\,(A)$ so that it is ample and symmetric. Then the first part of the corollary follows from Theorem 6.2 and [Co, Theorem 9.15] (which is true for regular extensions K/k).

To prove the second part take positive integer m such that $mP = Q \in B(k)$. Since k is algebraically closed, the set $B(k)$ is divisible and there exists $Q' \in B(k)$ such that $mQ' = Q$. Then $P = Q' + (P - Q')$, where $m(P - Q') = 0$. $\square$

Let us assume that the field k is algebraically closed. It is an interesting question whether a V-divisible K-point P of A can be written as a sum of $Q+R$, where $Q \in B(k)$ and $R \in A(K)_{\mathrm{tors}}$ is torsion *of order prime-to-p*.
By the Lang–Néron theorem ([Co, Theorem 2.1]), the groups $A^{(i)}(K)/B^{(i)}(k)$ are finitely generated. It follows that the groups $G_i = (A^{(i)}(K)/B^{(i)}(k))_{\mathrm{tors}}$ are finite.
Note that the homomorphism $B(k) \to B^{(i)}(k)$ induced by $F^i_{B/k}$ is a bijection. One has a factorization $F^i_{A/K} : A(K^{1/p^i}) \to A^{(i)}(K) \to A^{(i)}(K^{1/p^i})$, inducing a bijection $A(K^{1/p^i}) \to A^{(i)}(K)$. Thus in particular,

$$F_i : A(K)/B(k) \to A^{(i)}(K)/B^{(i)}(k)$$

is injective.
Moreover, the Verschiebung morphism induces the homomorphisms

$$V_i : A^{(i)}(K)/B^{(i)}(k) \to A(K)/B(k)$$

such that $V_iF_i = p^i$ and $F_iV_i = p^i$. This shows that prime-to-p torsion subgroups of groups G_i are isomorphic and in particular have the same order m.
Now let us assume that orders of the p-primary torsion subgroups of the abelian groups G_i are uniformly bounded by some p^e. Then for all $i \geq e$

$$F_i(m[P]) = F_i(V_i(m[P_i])) = p^i m[P_i] = 0.$$

This implies that $m[P] = 0$, so $mP \in B(k)$. Now $B(k)$ is a divisible group so there exists some $Q' \in B(k)$ such that $mP = mQ$. Then $R = P - Q \in A(K)$ is torsion of order prime to p. So we conclude

Lemma 6.5. *If the order of the G_i is bounded as i goes to infinity, under the assumption the Theorem 6.2, there exists a positive integer m, prime to p and such that $m \cdot P_i \in B(k)$ for every integer i.*

Note that the above assumption on G_i is satisfied, e.g., if A is an elliptic curve over the function field K of a smooth curve over $k = \bar{k}$. If A is isotrivial then the assertion is clear. If A is not isotrivial then the j-invariant of A is transcendental over k. In this case $A(K^{\mathrm{perf}})_{\mathrm{tors}}$ is finite (see [Le]) so orders of the groups $G_i = A^{(i)}(K)_{\mathrm{tors}}$ are uniformly bounded.

7 Stratified bundles

In this section we use the height estimate of the previous section and the fact that torsion stratified line bundles on a perfect field have order prime to p (apply Proposition 2.2 together with Lemma 1.1).

Let k be an algebraically closed field of positive characteristic p. Let $f : X \to S$ be a smooth projective morphism of k-varieties with geometrically connected fibres. Assume that S is projective, which surely is a very strong assumption. Indeed, if $k \neq \mathbb{F}_p$, and in the statement of Theorem 7.1, S' is open, then one obtains the stronger Theorem 7.2. For simplicity, let us also assume that f has a section $\sigma : S \to X$. Consider the torsion component $\mathrm{Pic}^{\tau}(X/S) \to S$ of identity of $\mathrm{Pic}(X/S) \to S$. Let $\varphi_n : \mathrm{Pic}(X/S) \to \mathrm{Pic}(X/S)$ be the multiplication by n map. Then there exists an open subgroup scheme $\mathrm{Pic}^{\tau}(X/S)$ of $\mathrm{Pic}(X/S)$ such that every geometric point s of S the fibre of $\mathrm{Pic}^{\tau}(X/S)$ over s is the union

$$\bigcup_{n>0} \varphi_n^{-1}(\mathrm{Pic}^0(X_s)),$$

where $\mathrm{Pic}^0(X_s)$ is the connected component of the identity of $\mathrm{Pic}(X_s/s)$. It is well known that $\mathrm{Pic}^{\tau}(X/S) \to S$ is also a closed subgroup scheme of $\mathrm{Pic}(X/S)$. Moreover, the morphism $\mathrm{Pic}^{\tau}(X/S) \to S$ is projective and the formation of $\mathrm{Pic}^{\tau}(X/S) \to S$ commutes with a base change of S (see, e.g., [Kl, Theorem 6.16 and Exercise 6.18]). We assume that $\mathrm{Pic}^0(X_s)$ is reduced for every point $s \in S$.

Theorem 7.1. *Let $\mathbb{L} = \{L_i, \sigma_i\}$ be a relatively stratified line bundle on X/S. Assume that there exists a dense subset $S' \subset S(k)$ such that for every $s \in S'$ the stratified bundle $\mathbb{L}_s = \mathbb{L}|_{X_s}$ has finite monodromy. Then $\mathbb{L}_{\bar{\eta}}$ has finite monodromy.*

Proof. Replacing $\mathbb{L}$ by a power $\mathbb{L}^{\otimes N}$, where N is sufficiently large, we may assume that $\mathbb{L}_s \in \mathrm{Pic}^0(X_s)$ for all closed points s in S (see [Kl, Corollary 6.17]).
By assumption $\hat{\pi} : \hat{\mathscr{A}} = \mathrm{Pic}^0(X/S) \to S$ is an abelian scheme. Let us consider the dual abelian scheme $\mathscr{A} \to S$. We have a well defined Albanese morphism $g : (X, \sigma) \to (\mathscr{A}, e)$ (see [FGA, Exposé VI, Théorème 3.3]). Moreover, the map $g^* : \mathrm{Pic}^0(\mathscr{A}/S) \to \hat{\mathscr{A}} = \mathrm{Pic}^0(X/S)$ is an isomorphism of S-schemes. Let us set $\hat{A} = \hat{\mathscr{A}}_\eta$.
Let P_i be the K-point of $\hat{A}^{(i)}$ corresponding to $(L_i)_\eta$. Note that the K-point $P_0 \in \hat{A}$ is V-divisible. Indeed, by the definition of a relative stratification we have $V^n(P_n) = P_0$ for all integers n. Similarly, we see that all the points $P_i \in \hat{A}^{(i)}(K)$ are V-divisible. By Corollary 6.4 it follows that $P_i \in \hat{B}^{(i)}(k) + \hat{A}^{(i)}(K)_{\mathrm{tors}}$, where $(\hat{B}/k, \hat{\tau} : \hat{B}_K \to \hat{A})$ is the K/k-trace of $\hat{A}$ (note that $(\hat{B}^{(i)}/k, \hat{\tau}^{(i)})$ is the K/k-trace of $\hat{A}^{(i)}$). So for every $i \geq 0$ we can write $P_i = Q_i + R_i$ for some $Q_i \in \hat{B}^{(i)}(k)$ and $R_i \in \hat{A}^{(i)}(K)_{\mathrm{tors}}$.
Now we transpose the above by duality. Let A be the dual abelian K-variety of $\hat{A}$ and B the dual abelian k-variety of $\hat{B}$. We have the K/k-images $\tau^{(i)} : A_\eta^{(i)} \to B_K^{(i)}$ and an S-morphism $\tau : \mathscr{A} \to B \times_k S$ (possibly after shrinking S). By abuse of notation we can treat L_i as line bundles on $\mathscr{A}$ because $g^* : \mathrm{Pic}^0(\mathscr{A}/S) \to \mathrm{Pic}^0(X/S)$ is an isomorphism. Let M_i be the line bundle on $B^{(i)}$ corresponding to Q_i and let $\pi_i : B^{(i)} \times_k S \to B^{(i)}$ denote the projection. Let us fix a non-negative integer i and take a positive integer n_i such that $n_i R_i = 0$. Then the line bundle $L_i^{\otimes n_i} \otimes \tau^* \pi_i^* M_i^{\otimes -n_i}$ has degree 0 on every fiber of $\mathscr{A} \to S$. Thus it is trivial after restriction to $\mathscr{A}_\eta$. Hence after shrinking S we can assume that $L_i^{\otimes n_i} \simeq \tau^* \pi_i^* M_i^{\otimes n_i}$.
Let us fix a point $s \in S(k)$ and consider the morphism

$$\pi_i' = (\tau^{(i)} \pi_i)_{\mathscr{A}_s^{(i)}} : \mathscr{A}_s^{(i)} \to B^{(i)}.$$

Note that $\tau^{(i)}$ has connected fibres and hence $(\pi_i')_*\mathcal{O}_{\mathscr{A}_s^{(i)}} = \mathcal{O}_{B^{(i)}}$. By assumption there exists a positive integer a_s, such that for every i the order of the line bundle $(L_i)_{\mathscr{A}_s}$ divides a_s. The important point is that a_s is prime to p.
Therefore $(\pi_i')^* M_i^{\otimes a_s n_i} \simeq \mathcal{O}_{A_s}$ and by the projection formula

$$M_i^{\otimes a_s n_i} \simeq (\pi_i')_*(\pi_i')^* M_i^{\otimes a_s n_i} \simeq (\pi_i')_*\mathcal{O}_{A_s} \simeq \mathcal{O}_B.$$

This implies that M_i is a torsion line bundle and hence $Q_i \in \hat{A}^{(i)}(K)_{\mathrm{tors}}$. Therefore

$$P_i = Q_i + R_i \in \hat{A}^{(i)}(K)_{\mathrm{tors}}.$$

Let us recall that the set of p-torsion points of $\hat{A}(K)$ is finite. Assuming it is not empty, we can therefore find a non-empty open subset $U \subset S$ such that for every $s \in U(k)$ and every p-torsion point $T \in \hat{A}(K)$ the section $\tilde{T}$ is defined on U and the point $\tilde{T}(s)$ is non-zero.
Let us write the order of P_i as $m_i p^{e_i}$, where m_i is not divisible by p. If $e_0 \geq 1$ then the point $m_0 p^{e_0-1} P_0$ is p-torsion in $\hat{A}(K)$. If we take $s \in S' \cap U(k)$, then $a_s m_0 p^{e_0-1} \tilde{P}_0(s) = [L_0^{\otimes a_s m_0}]_s = 0$, a contradiction. It follows that $m_0 P_0 = 0$. Similarly, the order of all P_i is prime to p.
As already mentioned in the last section, the homomorphism $\hat{A}(K^{1/p^i}) \to \hat{A}^{(i)}(K)$ induced by $F^i_{A/K}$ is a bijection. So we have an induced injection

$$F_i : \hat{A}(K) \to \hat{A}^{(i)}(K).$$

On the other hand, the Verschiebung morphism induces homomorphisms

$$V_i : \hat{A}^{(i)}(K) \to \hat{A}(K)$$

such that $V_i F_i(P) = p^i P$ and $F_i V_i(Q) = p^i Q$ for all $P \in \hat{A}(K)$ and $Q \in \hat{A}^{(i)}(K)$. Hence

$$p^i m_0 P_i = F_i V_i(m_0 P_i) = F_i(m_0 P_0) = 0$$

and since the order of P_i is prime to p we have $m_0 P_i = 0$ for all $i \geq 0$. Therefore $(L_i)_{\bar{\eta}}^{\otimes m_0} \simeq \mathcal{O}_{X_{\bar{\eta}}}$ for all i and the stratified line bundle $\mathbb{L}_{\bar{\eta}}$ has finite monodromy. □

Now we fix the following notation: k is an algebraically closed field of positive characteristic p and $f : X \to S$ is a smooth projective morphism of k-varieties with geometrically connected fibres.

THEOREM 7.2. *Let $\mathbb{E} = \{E_i, \sigma_i\}$ be a relatively stratified bundle on X/S. Assume that there exists a dense subset $U \subset S(k)$ such that for every $s \in U$ the stratified bundle $\mathbb{E}_s = \mathbb{E}|_{X_s}$ has finite monodromy of order prime to p.*

1) *Then there exists a finite Galois étale covering $\pi_{\bar{\eta}} : Y_{\bar{\eta}} \to X_{\bar{\eta}}$ of order prime-to-p such that $\pi_{\bar{\eta}}^* \mathbb{E}_{\bar{\eta}}$ is a direct sum of stratified line bundles.*

2) *If $k \neq \bar{\mathbb{F}}_p$ and U is open in $S(k)$, then the monodromy group of $\mathbb{E}_{\bar\eta}$ is finite, and $\mathbb{E}_{\bar\eta}$ trivializes on a finite étale cover $Z_{\bar\eta} \to X_{\bar\eta}$ which factors as a Kummer (thus finite abelian of order prime to p) cover $Z_{\bar\eta} \to Y_{\bar\eta}$ and a Galois cover $Y_{\bar\eta} \to X_{\bar\eta}$ of order prime to p.*

Proof. We prove 1). Let us first remark that the schemes $X^{(i)}_{\bar\eta}$, $i \geq 0$, are all isomorphic (as schemes, not as k-schemes). Therefore the relative Frobenius induces an isomorphism on fundamental groups.
By the first part of Theorem 5.1 we know that there exists a finite Galois étale covering $\pi_i : Y_{\bar\eta,i} \to X^{(i)}_{\bar\eta}$ of degree prime to p such that $\pi_i^*(E_i)$ is a direct sum of line bundles $\oplus_1^r L_{ij}$. Note that from the proof of Theorem 5.1 the degree of π_i depends only on $\pi_1^{p'}(X^{(i)}_{\bar\eta}, \sigma^{(i)}(\bar\eta))$ and the Brauer-Feit constant $j(r)$, and therefore it can be bounded independently of i. Using the Lang–Serre theorem (see [LS, Théorème 4]) we can therefore assume that $Y_{\bar\eta,i} = Y^{(i)}_{\bar\eta}$, where $Y_{\bar\eta} = Y_{\bar\eta,0}$. Now we know that

$$\oplus_{j=1}^r L_{ij} \simeq (F^i_{Y^{(i)}_{\bar\eta}/\bar\eta})^* \big(\oplus_{j'=1}^r L_{i+1,j'}\big).$$

By the Krull-Schmidt theorem, the set of isomorphism classes of line bundles $\{L_{ij}\}_j$ is the same as the set of isomorphism classes of lines bundles which come by pull-back $\{(F^i_{Y^{(i)}_{\bar\eta}/\bar\eta})^*(L_{i+1,j'})\}_{j'}$. So we can reorder the indices j' so that

$$(F^i_{Y^{(i)}_{\bar\eta}/\bar\eta})^*(L_{i+1,j}) \cong L_{i,j}.$$

This finishes the proof of 1).
To prove 2), we do the proof 1) replacing $Y_{\bar\eta} \to X_{\bar\eta}$ by $Z_{\bar\eta} \to X_{\bar\eta}$ of Theorem 5.1 2). This finishes the proof of 2).

□

Remarks 7.3. 1) Case 2) of Theorem 7.2 applied to a line bundle extends Theorem 7.1, where S was assumed to be projective, $\mathrm{Pic}^0(X_s)$ reduced for all $s \in S$ closed, $S' \subset S(k)$ dense, to the case when S is not necessarily projective and $S' \subset S(k)$ is open and dense, but we have to assume that k is not algebraic over its prime field.

2) If $Y_{\bar\eta}$ has a good projective model satisfying assumptions of Theorem 7.1 then it follows that $\mathbb{E}_{\bar\eta}$ has finite monodromy.

References

[An] Y. André: Sur la conjecture des p-courbures de Grothendieck-Katz et un problème de Dwork, Geometric aspects of Dwork theory, Vol. I, II, 55–112, Walter de Gruyter GmbH & Co. KG, Berlin, 2004.

[BD] I. Biswas, L. Ducrohet: An analog of a theorem of Lange and Stuhler for principal bundles, *C. R. Math. Acad. Sci. Paris* 345 (2007), 495–497.

[BLR] S. Bosch, W. Lütkebohmert, M. Raynaud: Néron models, *Ergebnisse der Mathematik und ihrer Grenzgebiete* 21, Springer-Verlag, Berlin, 1990. x+325 pp.

[BF] R. Brauer, W. Feit: An analogue of Jordan's theorem in characteristic p, *Ann. of Math.* 84 (1966) 119–131.

[CC] D. V. Chudnovsky, G. V. Chudnovsky: Applications of Padé approximations to the Grothendieck conjecture on linear differential equations, Number theory (New York, 1983-84), 52–100, *Lecture Notes in Math.* 1135, Springer, Berlin, 1985.

[Co] B. Conrad: Chow's K/k-image and K/k-trace, and the Lang-Néron theorem, *Enseign. Math.* (2) 52 (2006), 37–108.

[DM] P. Deligne, J. Milne: Tannakian categories, *Lecture Notes in Mathematics* 100, 101–228. Springer Verlag.

[dS] J.-P. dos Santos: Fundamental group schemes for stratified sheaves, *J. Algebra* 317 (2007), no. 2, 691–713.

[EM] H. Esnault, V. Mehta: Simply connected projective manifolds in characteristic $p > 0$ have no nontrivial stratified bundles, *Inventiones math.* 181 (2010), 449–465.

[FK] B. Farb, M. Kisin: The Grothendieck-Katz conjecture for certain locally symmetric varieties, *Int. Math. Res. Not.* 22 (2009), 4159–4167.

[Gi] D. Gieseker: Flat vector bundles and the fundamental group in non-zero characteristics, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* 2 (1975), 1–31.

[FGA] A. Grothendieck: Fondements de la géométrie algébrique, [Extraits du Séminaire Bourbaki, 1957–1962.] Secrétariat mathématique, Paris 1962.

[SGA1] A. Grothendieck: Revêtements étales et groupe fondamental, Documents Mathématiques (Paris) 3, Séminaire de géométrie algébrique du Bois Marie 1960–61, With two papers by M. Raynaud, Updated and annotated reprint of the 1971 original, Société Mathématique de France, Paris, 2003.

[SGA5] A. Grothendieck: Cohomologie l-adique et fonctions L, Séminaire de Géometrie Algébrique du Bois-Marie 1965–1966, *Lecture Notes in Mathematics*, Vol. 589. Springer-Verlag, Berlin-New York, 1977.

[Ha] R. Hartshorne: Algebraic Geometry, *Graduate Texts in Mathematics* 52, Springer Verlag, 1977.

[Hr] E. Hrushovsky: Computing the Galois group of a linear differential equation, in Differential Galois Theory, *Banach Center Publications* 58, Inst. of Math. Polish Academy of Sciences, Warszawa, 2002, 97–138.

[HL] D. Huybrechts, M. Lehn: The geometry of moduli spaces of sheaves, *Aspects of Mathematics* 31, 1997.

[Ka] N. Katz: Algebraic Solutions of Differential Equations (p-Curvature and the Hodge Filtration), *Invent. Math.* 18 (1972), 1–118.

[Kl] S. L. Kleiman: The Picard scheme, Fundamental algebraic geometry, 235–321, *Math. Surveys Monogr.* 123, Amer. Math. Soc., Providence, RI, 2005.

[L] S. Lang: Fundamentals of Diophantine geometry, Springer-Verlag, New York, 1983.

[LS] S. Lang, J.-P. Serre: Sur les revetements non ramifiés des variétés algébriques, *Amer. J. Math.* 79 (1957), 319–330.

[LSt] H. Lange, U. Stuhler: Vektorbündel auf Kurven und Darstellungen der algebraischen Fundamentalgruppe, *Math. Z.* 156 (1977), 73–83.

[La1] A. Langer: Semistable sheaves in positive characteristic, *Ann. of Math.* 159 (2004), 251–276.

[La2] A. Langer: On the S-fundamental group scheme, *Ann. Inst. Fourier* 61 (2011), 2077–2119.

[Ls] Y. Laszlo: A non-trivial family of bundles fixed by the square of Frobenius, *C. R. Acad. Sci. Paris Sér. I Math.* 333 (2001), 651–656.

[Le] M. Levin: On the group of rational points on elliptic curves over function fields, *Amer. J. Math.* 90 (1968), 456–462.

[MB] L. Moret-Bailly: Pinceaux de variétés abéliennes, *Astérisque* 129 (1985), 266 pp.

[NS] N. Nikolov, D. Segal: On finitely generated profinite groups I. Strong completeness and uniform bounds, *Ann. of Math. (2)* 165 (2007), 171–238.

[No] M. Nori: The fundamental group scheme, *Proc. Indian Acad. Sci.* 91 (1982), 73-122.

[Ra] M. Raynaud: Letter to Hélène Esnault dated November 24, 2009.

[Se] J.-P. Serre: Lectures on the Mordell-Weil theorem, *Aspects of Mathematics* E15, Friedr. Vieweg & Sohn, Braunschweig, 1989.

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