The Topological Singer Construction

Sverre Lunøe–Nielsen and John Rognes

Received: January 21, 2011

Communicated by Lars Hesselholt

Abstract. We study the continuous (co-)homology of towers of spectra, with emphasis on a tower with homotopy inverse limit the Tate construction X^{tG} on a G-spectrum X. When $G = C_p$ is cyclic of prime order and $X = B^{\wedge p}$ is the p-th smash power of a bounded below spectrum B with $H_*(B; \mathbb{F}_p)$ of finite type, we prove that $(B^{\wedge p})^{tC_p}$ is a topological model for the Singer construction $R_+(H^*(B; \mathbb{F}_p))$ on $H^*(B; \mathbb{F}_p)$. There is a stable map $\epsilon_B \colon B \to (B^{\wedge p})^{tC_p}$ inducing the $\mathrm{Ext}_{\mathscr{A}}$-equivalence $\epsilon \colon R_+(H^*(B; \mathbb{F}_p)) \to H^*(B; \mathbb{F}_p)$. Hence ϵ_B and the canonical map $\Gamma \colon (B^{\wedge p})^{C_p} \to (B^{\wedge p})^{hC_p}$ are p-adic equivalences.

2010 Mathematics Subject Classification: 55P42, 55P91, 55S10, 55T15

Keywords and Phrases: Singer construction, Tate construction, limit of Adams spectral sequences, Ext-equivalence.

1. Introduction

Let p be a fixed prime. We study homology and cohomology groups with $\mathbb{F}_p$-coefficients associated to towers of spectra

$$Y = \operatorname*{holim}_{n \to -\infty} Y_n \longrightarrow \cdots \longrightarrow Y_{n-1} \longrightarrow Y_n \longrightarrow \cdots\,, \tag{1.1}$$

where each Y_n is bounded below and of finite type over $\mathbb{F}_p$, and Y is equal to the homotopy inverse limit of the tower. By a result of Caruso, May and Priddy [11], there exists an *inverse limit of Adams spectral sequences* that calculates the homotopy groups of the p-completion $Y_p^\wedge = F(S^{-1}/p^\infty, Y)$ of Y, where $F(-,-)$ denotes the function spectrum and S^{-1}/p^∞ is the Moore spectrum with homology $\mathbb{Z}/p^\infty$ in degree -1. The E_2-term for this spectral sequence is given by the Ext-groups of the direct limit of cohomology groups

$$H^*_c(Y; \mathbb{F}_p) = \operatorname*{colim}_{n \to -\infty} H^*(Y_n; \mathbb{F}_p)\,, \tag{1.2}$$

arising from the tower (1.1), considered as a module over the mod p Steenrod algebra $\mathscr{A}$. We shall refer to this colimit as the *continuous cohomology groups* of Y.

1.1. The algebraic Singer construction. A natural question is: how well do we understand the structure of (1.2) as an $\mathscr{A}$-module? There is an interesting example of a tower of spectra where this question has the answer: very well. In fact, this question appeared in the study of Segal's Burnside ring conjecture for cyclic groups of prime order. At the heart of W. H. Lin's proof of the case $p = 2$, published in [20], lies a careful study of the $\mathscr{A}$-module

$$P = H^*_c(\mathbb{R}P^\infty_{-\infty}; \mathbb{F}_2) = \operatorname*{colim}_{m\to\infty} H^*(\mathbb{R}P^\infty_{-m}; \mathbb{F}_2)\,,$$

and its associated Ext-groups $\operatorname{Ext}^{*,*}_{\mathscr{A}}(P, \mathbb{F}_2)$. It turns out that $P = P(x, x^{-1}) = \mathbb{F}_2[x, x^{-1}]$ is isomorphic to the so-called *Singer construction* $R_+(M)$ on the trivial $\mathscr{A}$-module $M = \mathbb{F}_2$, up to a degree shift. The Singer construction has an explicit description as a module over $\mathscr{A}$. More importantly, it has the property that there is a natural $\mathscr{A}$-module homomorphism $\epsilon\colon R_+(M) \to M$ that induces an isomorphism

$$\epsilon^*\colon \operatorname{Ext}^{*,*}_{\mathscr{A}}(M, \mathbb{F}_2) \xrightarrow{\cong} \operatorname{Ext}^{*,*}_{\mathscr{A}}(R_+(M), \mathbb{F}_2)\,.$$

1.2. The topological Singer construction. Our objective is to present a topological realization and a useful generalization of these results. For any bounded below spectrum B of finite type over $\mathbb{F}_p$, we construct a tower of spectra

$$(B^{\wedge p})^{tC_p} = \operatorname*{holim}_{n\to-\infty} (B^{\wedge p})^{tC_p}[n] \to \cdots \to (B^{\wedge p})^{tC_p}[n{-}1] \to (B^{\wedge p})^{tC_p}[n] \to \cdots$$

as in (1.1). Here C_p is the cyclic group of order p, and $B^{\wedge p}$ is a C_p-equivariant model of the p-th smash power of B, specified in Definition 5.3. The Tate construction $(B^{\wedge p})^{tC_p}$ is recalled in Definition 4.1, and the spectra $(B^{\wedge p})^{tC_p}[n]$ in the Tate tower are introduced in Definition 4.3.

Remark 1.1. Note that the naively equivariant homotopy type of the p-fold smash power $B^{\wedge p}$, as a spectrum with C_p-action, is the same in all models for the stable homotopy category, but that its genuinely equivariant homotopy type, as a C_p-spectrum, depends on the choice of model for the smash product. In particular, the meaning of the fixed point spectrum $(B^{\wedge p})^{C_p}$ and the geometric fixed point spectrum $\Phi^{C_p}(B^{\wedge p})$ is model-dependent. On the other hand, the homotopy orbit spectrum $(B^{\wedge p})_{hC_p}$, the homotopy fixed point spectrum $(B^{\wedge p})^{hC_p}$ and the Tate construction $(B^{\wedge p})^{tC_p}$ only depend on the naively equivariant homotopy type of $B^{\wedge p}$. For definiteness, we work with Lewis–May equivariant spectra, as in [18].

In Definition 5.8 we introduce the notation

$$R_+(B) = (B^{\wedge p})^{tC_p}$$

for the *topological Singer construction* on B, and in Theorem 5.9 we prove that there is a natural isomorphism

$$R_+(H^*(B; \mathbb{F}_p)) \cong H^*_c(R_+(B); \mathbb{F}_p) = \operatorname*{colim}_{n\to-\infty} H^*((B^{\wedge p})^{tC_p}[n]; \mathbb{F}_p)$$

of $\mathscr{A}$-modules, relating the algebraic and topological Singer constructions.

THEOREM 5.9. *Let B be a symmetric spectrum that is bounded below and of finite type over $\mathbb{F}_p$. There are natural isomorphisms*

$$\omega\colon H^*_c(R_+(B)) \xrightarrow{\cong} R_+(H^*(B))$$

and

$$\omega_*\colon R_+(H_*(B)) \xrightarrow{\cong} H^c_*(R_+(B))$$

of $\mathscr{A}$-modules and complete $\mathscr{A}_*$-comodules, respectively.

We define a natural stable map $\epsilon_B\colon B \to (B^{\wedge p})^{tC_p}$ in Definition 5.10, and prove in Proposition 5.12 that it realizes the algebraically defined $\mathscr{A}$-module homomorphism $\epsilon\colon R_+(H^*(B;\mathbb{F}_p)) \to H^*(B;\mathbb{F}_p)$ in continuous cohomology. It is in the definition of ϵ_B that the choice of genuinely equivariant model for $B^{\wedge p}$ begins to play a role.

PROPOSITION 5.12. *Let B be a bounded below spectrum of finite type over $\mathbb{F}_p$. Then the homomorphism*

$$(\epsilon_B)^*\colon H^*_c(R_+(B)) \longrightarrow H^*(B)$$

induced on continuous cohomology by the spectrum map $\epsilon_B\colon B \to R_+(B)$ is equal to Singer's homomorphism

$$\epsilon_{H^*(B)}\colon R_+(H^*(B)) \longrightarrow H^*(B)$$

associated to the $\mathscr{A}$-module $H^*(B)$, via the identification $\omega : H^*_c(R_+(B)) \cong R_+(H^*(B))$ of Theorem 5.9.

The Segal conjecture for C_p follows as a special case of this, when $B = S$ is the sphere spectrum, since $S^{\wedge p}$ is a model for the genuinely C_p-equivariant sphere spectrum. More generally, we prove in Theorem 5.13 that for any bounded below spectrum B with $H_*(B;\mathbb{F}_p)$ of finite type the canonical map

$$\Gamma\colon (B^{\wedge p})^{C_p} \longrightarrow (B^{\wedge p})^{hC_p}\,,$$

relating the C_p-fixed points and C_p-homotopy fixed points for $B^{\wedge p}$, becomes a homotopy equivalence after p-completion.

THEOREM 5.13. *Let B be a bounded below spectrum of finite type over $\mathbb{F}_p$. Then the natural maps*

$$\epsilon_B\colon B \longrightarrow R_+(B) = (B^{\wedge p})^{tC_p}$$

and

$$\Gamma\colon (B^{\wedge p})^{C_p} \longrightarrow (B^{\wedge p})^{hC_p}$$

are p-adic equivalences of spectra.

In [7, 1.7], with M. Bökstedt and R. R. Bruner, we deduce from this result that there are p-adic equivalences $\Gamma_n\colon (B^{\wedge p^n})^{C_{p^n}} \longrightarrow (B^{\wedge p^n})^{hC_{p^n}}$ for all $n \geq 1$.

1.3. Outline of the paper. In §2, we discuss towers of spectra as above, and the associated limit systems obtained by applying homology or cohomology with $\mathbb{F}_p$-coefficients. When dealing with towers of ring spectra it is convenient to work in homology, while for the formation of the Ext-groups mentioned above it is convenient to work in cohomology. In order to be able to switch back and forth between cohomology and homology we discuss linear topologies arising from filtrations, and continuous dualization. Then we see how an $\mathscr{A}$-module structure in cohomology dualizes to an $\mathscr{A}_*$-comodule structure in a suitably completed sense.

In §3 we recall the algebraic Singer construction on an $\mathscr{A}$-module M, in its cohomological form, and study a dual homological version, defined for $\mathscr{A}_*$-comodules M_* that are bounded below and of finite type. We give the details for the form of the Singer construction that is related to the group C_p, since most references only consider the smaller version related to the symmetric group Σ_p.

Then, in §4, we define a specific tower of spectra $\{X^{tG}[n]\}_n$ with homotopy inverse limit equivalent to the Tate construction $X^{tG} = [\widetilde{EG} \wedge F(EG_+, X)]^G$ on a G-spectrum X. We consider the associated (co-)homological Tate spectral sequences, and compare our approach of working with homology groups to earlier papers that focused directly on homotopy groups.

In §5, we specialize to the case when $X = B^{\wedge p}$. This is also where we discuss the genuinely C_p-equivariant model of the spectrum $B^{\wedge p}$, given by the p-fold smash product of (symmetric) spectra introduced by Bökstedt [6] in his definition of topological Hochschild homology. It is for this particular C_p-equivariant model that we can define the natural stable map $\epsilon_B \colon B \to R_+(B)$ realizing Singer's homomorphism ϵ.

1.4. Notation. Spectra will usually be named B, X or Y. Here B will be a bounded below spectrum or S-algebra of finite type over $\mathbb{F}_p$. Spectra denoted by X will be equipped with an equivariant structure. The main examples we have in mind are the p-fold smash product $X = B^{\wedge p}$ treated here, and the topological Hochschild homology spectrum $X = \mathrm{THH}(B)$ treated in the sequel [22]. When dealing with generic towers of spectra, we will use Y. The example of main interest is the Tate construction $Y = X^{tG}$ on some G-equivariant spectrum X. We write $\mathscr{A}$ for the mod p Steenrod algebra and $\mathscr{A}_*$ for its $\mathbb{F}_p$-linear dual. We will work with left modules over $\mathscr{A}$ and left comodules under $\mathscr{A}_*$. In the body of the paper we write $H_*(B) = H_*(B; \mathbb{F}_p)$ and $H^*(B) = H^*(B; \mathbb{F}_p)$, for brevity. Unlabeled Hom means $\mathrm{Hom}_{\mathbb{F}_p}$, and $\otimes$ means $\otimes_{\mathbb{F}_p}$.

1.5. History and notation of the Singer construction. The Singer construction appeared originally for $p = 2$ in [25] and [26], and for p odd in [19]. The work presented here concentrates on its relation to the calculations by Lin and Gunawardena and their work on the Segal conjecture for groups of prime order. A published account for the case of the group of order 2 is found in [20]. A further study appears in [1], where a more conceptual definition of the Singer construction is given.

In W. Singer's paper [25], the following problem is posed: Let M be an unstable $\mathscr{A}$-module and let $\Delta = \mathbb{F}_2\{\mathrm{Sq}^r \mid r \in \mathbb{Z}\}$ be the graded $\mathbb{F}_2$-vector space with a formal generator Sq^r in each degree. There is a map of graded $\mathbb{F}_2$-vector spaces

$$d\colon \Delta \otimes M \longrightarrow M$$

taking $\mathrm{Sq}^r \otimes m$ to $\mathrm{Sq}^r(m)$ for $r \geq 0$, and to 0 for $r < 0$. Does there exist a natural $\mathscr{A}$-module structure on the source of d rendering this map an $\mathscr{A}$-linear homomorphism? Singer answers this question affirmatively, by using an idea of Wilkerson [27] to construct the $\mathscr{A}$-module that he denotes $R_+(M)$, an $\mathscr{A}$-module map $d\colon R_+(M) \to M$ of degree 1, and an isomorphism $R_+(M) \cong \Delta \otimes M$, also of degree 1, that makes the two maps called d correspond. In the end, the construction does not depend on M being unstable.

In Li and Singer's paper [19], the odd-primary version of this problem is solved, with $\Delta = \mathbb{F}_p\{\beta^i \mathrm{P}^r \mid i \in \{0,1\}, r \in \mathbb{Z}\}$. Starting with that paper there is a degree shift in the notation: $R_+(M)$ now denotes the suspension of $R_+(M)$ from Singer's original paper, so that the $\mathscr{A}$-module map $d\colon R_+(M) \to M$ is of degree 0.

In connection with the Segal conjecture, Adams, Gunawardena and Miller [1] published an algebraic account of the Singer construction, for all primes p. They write $T'(M)$ for the $\mathscr{A}$-module denoted $R_+(M)$ in [25] and [26], and let $T''(M) = \Sigma T'(M)$ be its suspension, denoted $R_+(M)$ in [19]. For the trivial $\mathscr{A}$-module $\mathbb{F}_p$, $T''(\mathbb{F}_p)$ is isomorphic to the Tate homology $\widehat{H}_{-*}(\Sigma_p; \mathbb{F}_p)$, which can be obtained by localization from $\Sigma H^*(\Sigma_p; \mathbb{F}_p)$. Hence $T'(\mathbb{F}_p) = \widehat{H}^*(\Sigma_p; \mathbb{F}_p)$ is a localized form of $H^*(\Sigma_p; \mathbb{F}_p)$. Adams, Gunawardena and Miller are not only concerned with the Segal conjecture for the groups of prime order, but also for the elementary abelian groups $(C_p)^n$, so the cross product in cohomology is important for them. They therefore prefer T' over T''. In fact, they really work with an extended functor $T(M) = T(\mathbb{F}_p) \otimes_{T'(\mathbb{F}_p)} T'(M)$, where $T(\mathbb{F}_p) = \widehat{H}^*(C_p; \mathbb{F}_p)$.

In our context, for the cohomological study of towers of ring spectra it will be the coproduct in Tate homology that is most important, which is why we prefer T'' over T'. Again, our emphasis is on C_p instead of Σ_p, so that the Singer-type functor we shall work with is the extension

$$\widehat{H}_{-*}(C_p; \mathbb{F}_p) \otimes_{T''(\mathbb{F}_p)} T''(M)$$

of $T''(M)$, which is a direct sum of $(p-1)$ shifted copies of the $T''(M) = R_+(M)$ of Li and Singer. This is the functor we shall denote $R_+(M)$, so that $R_+(\mathbb{F}_p) = \widehat{H}_{-*}(C_p; \mathbb{F}_p)$, and $R_+(M) = \Sigma T(M)$ in the notation of [1].

The connection between the Singer construction and the continuous cohomology of a tower of spectra, displayed below as (5.1) for $G = \Sigma_p$, was found by Haynes Miller, and explained in [9, II.5.1]. There the functor denoted R_+ is the same as in [19] for odd p, shifted up one degree from [26] for $p = 2$.

In the following we will make use of the fact that the Singer construction on an $\mathscr{A}$-module M comes equipped with the homomorphism of $\mathscr{A}$-modules $\epsilon\colon R_+(M) \to M$. In Singer's work [26], this map has degree $+1$ (and was

named d), whereas in [19] and [9] it has degree 0. We choose to follow the latter conventions, because this is the functor that with no shift of degrees describes our continuous (co-)homology groups. Furthermore, the homomorphism ϵ will be realized by a map of spectra, and will therefore be of degree zero.
We choose to write $R_+(M)$ instead of $T(M)$ or any of its variants, because the letter T is heavily overloaded by the presence of THH, the Tate construction and the circle group $\mathbb{T}$. To add to the confusion, the letter T is also used in Singer's [26] work, but with a different meaning than the T appearing in [1].

1.6. ACKNOWLEDGMENTS. This work started out as a part of the first author's PhD thesis [21] at the University of Oslo, supervised by the second author. The first author wishes to express the deepest gratitude to Prof. Rognes for offering ideas, help and support during the work of his thesis. Thanks are also due to R. Bruner, G. Carlsson, I. C. Borge, S. Galatius, T. Kro, M. Bökstedt, M. Brun and B. I. Dundas for interest, comments and discussions, and to the referee for his careful reading of the manuscript.

2. LIMITS OF SPECTRA

We introduce our conventions regarding towers of spectra and their associated (co-)homology groups. Our motivation is the result of Caruso, May and Priddy, saying that there is an inverse limit of Adams spectral sequences arising from such towers. The input for this inverse limit of Adams spectral sequences will give us the definition of continuous (co-)homology groups.

2.1. INVERSE LIMITS OF ADAMS SPECTRAL SEQUENCES.

DEFINITION 2.1. Let R be a (Noetherian) ring. A graded R-module M_* is *bounded below* if there is an integer ℓ such that $M_* = 0$ for all $* < \ell$. It is of *finite type* if it is finitely generated over R in each degree.
A spectrum B is *bounded below* if its homotopy $\pi_*(B)$ is bounded below as a graded abelian group. It is of *finite type over* $\mathbb{F}_p$ if its mod p homology $H_*(B) = H_*(B; \mathbb{F}_p)$ is of finite type as a graded $\mathbb{F}_p$-vector space. The spectrum B is of *finite type over* $\widehat{\mathbb{Z}}_p$ if its homotopy $\pi_*(B)$ is of finite type as a graded $\widehat{\mathbb{Z}}_p$-module.

Let $\{Y_n\}_{n\in\mathbb{Z}}$ be a sequence of spectra, with maps $f_n\colon Y_{n-1} \to Y_n$ for all integers n. Assume that each Y_n is bounded below and of finite type over $\mathbb{F}_p$, and let Y be the homotopy inverse limit of this system:

$$Y \longrightarrow \cdots \longrightarrow Y_{n-1} \xrightarrow{f_n} Y_n \longrightarrow \cdots \tag{2.1}$$

In general, Y will neither be bounded below nor of finite type over $\mathbb{F}_p$.
For each n there is an Adams spectral sequence $\{E_r^{*,*}(Y_n)\}_r$ with E_2-term

$$E_2^{s,t}(Y_n) = \mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(Y_n), \mathbb{F}_p) \Longrightarrow \pi_{t-s}((Y_n)^{\wedge}_p)\,,$$

converging strongly to the homotopy groups of the p-completion of Y_n. Each map in the tower (2.1) induces a map of Adams spectral sequences

$f_n\colon \{E_r^{*,*}(Y_{n-1})\}_r \to \{E_r^{*,*}(Y_n)\}_r$. For every r let

$$E_r^{*,*}(\underline{Y}) = \lim_{n\to-\infty} E_r^{*,*}(Y_n)\,,$$

and similarly for the d_r-differentials. We now state and prove a slightly sharper version of [11, 7.1].

PROPOSITION 2.2. *Let $\{Y_n\}_n$ be a tower of spectra such that each Y_n is bounded below and of finite type over $\mathbb{F}_p$, and let $Y = \operatorname{holim}_n Y_n$. Then the bigraded groups $\{E_r^{*,*}(\underline{Y})\}_r$ are the terms of a spectral sequence, with E_2-term*

$$E_2^{s,t}(\underline{Y}) \cong \operatorname{Ext}_{\mathscr{A}}^{s,t}(\operatorname*{colim}_{n\to-\infty} H^*(Y_n), \mathbb{F}_p) \Longrightarrow \pi_{t-s}(Y_p^\wedge)\,,$$

converging strongly to the homotopy groups of the p-completion of Y.

The difference between this statement and the statement in [11] lies in the hypothesis on the Y_n: we do not assume that Y_n is p-complete, and weaken the condition that Y_n should be of finite type over $\widehat{\mathbb{Z}}_p$ to the condition that $H_*(Y_n;\mathbb{F}_p)$ should be of finite type. We refer to the spectral sequence $\{E_r^{*,*}(\underline{Y})\}_r$ as the *inverse limit of Adams spectral sequences* associated to the tower $\{Y_n\}_n$.

Proof. For any bounded below spectrum B of finite type over $\mathbb{F}_p$, the Adams spectral sequence converges strongly to $\pi_*(B_p^\wedge)$. Since the E_2- and E_∞-terms of this spectral sequence are of finite type, the abelian groups $\pi_*(B_p^\wedge)$ are compact and Hausdorff in the topology given by the Adams filtration.

The category of compact Hausdorff abelian groups is an abelian category, as is the category of discrete abelian groups. The Pontryagin duality functor assigns to each abelian group G its character group $\operatorname{Hom}(G,\mathbb{T})$, where $\mathbb{T} = S^1$ is the circle group. It induces a contravariant equivalence between the category of compact Hausdorff abelian groups and the category of discrete abelian groups. The functor taking a filtered diagram of discrete abelian groups to its colimit is well-known to be exact. It follows that the functor taking a filtered diagram of compact Hausdorff abelian groups to its inverse limit is also an exact functor. In particular, passing to filtered inverse limits commutes with the formation of kernels, images, cokernels and homology, in the abelian category of compact Hausdorff abelian groups.

We now adapt the proof of [11, 7.1], using this version of the exactness of the inverse limit functor. First, we construct a double tower diagram (2.2) of spectra, where the n-th row is an Adams resolution of Y_n, such that each $Z_{n,s}$ is a bounded below spectrum of finite type over $\mathbb{F}_p$. The n-th row can be obtained in a functorial way by smashing Y_n with a fixed Adams resolution for the sphere spectrum S. The top row consists of the homotopy inverse limits

$Z_s = \operatorname{holim}_n Z_{n,s}$ for all $s \geq 0$.

$$
\begin{tikzcd}
\cdots \arrow[r] & Z_s \arrow[r] \arrow[d] & \cdots \arrow[r] & Z_0 = Y \arrow[d] \\
& \vdots \arrow[d] & & \vdots \arrow[d] \\
\cdots \arrow[r] & Z_{n-1,s} \arrow[r] \arrow[d] & \cdots \arrow[r] & Z_{n-1,0} = Y_{n-1} \arrow[d, "f_n"] \\
\cdots \arrow[r] & Z_{n,s} \arrow[r] & \cdots \arrow[r] & Z_{n,0} = Y_n
\end{tikzcd}
\tag{2.2}
$$

By assumption, the homotopy cofiber $K_{n,s}$ of the map $Z_{n,s+1} \to Z_{n,s}$ is a wedge sum of suspended copies of the Eilenberg–Mac Lane spectrum $H = H\mathbb{F}_p$, also bounded below and of finite type over $\mathbb{F}_p$. The exact couple

$$
\begin{tikzcd}
\pi_*(Z_{n,s+1}) \arrow[r, "i"] & \pi_*(Z_{n,s}) \arrow[d, "j"] \\
& \pi_*(K_{n,s}) \arrow[ul, dashed, "k"]
\end{tikzcd}
$$

generates the Adams spectral sequence $\{E_r^{*,*}(Y_n)\}_r$, with $E_1^{s,t}(Y_n) = \pi_{t-s}(K_{n,s})$. The dashed arrow has degree -1.

Now consider the p-completion of diagram (2.2). The n-th row becomes

$$
\ldots \longrightarrow (Z_{n,s})_p^{\wedge} \longrightarrow \ldots \longrightarrow (Z_{n,0})_p^{\wedge} = (Y_n)_p^{\wedge} \tag{2.3}
$$

and the homotopy cofiber of $(Z_{n,s+1})_p^{\wedge} \to (Z_{n,s})_p^{\wedge}$ is the p-completion of $K_{n,s}$, which is just $K_{n,s}$ again. There is therefore a second exact couple

$$
\begin{tikzcd}
\pi_*((Z_{n,s+1})_p^{\wedge}) \arrow[r, "i"] & \pi_*((Z_{n,s})_p^{\wedge}) \arrow[d, "j"] \\
& \pi_*(K_{n,s}) \arrow[ul, dashed, "k"]
\end{tikzcd}
\tag{2.4}
$$

for each n, which generates the same spectral sequence as the first one. Furthermore, in the second exact couple all the (abelian) homotopy groups are compact Hausdorff, since the spectra $Z_{n,s}$ and $K_{n,s}$ are all bounded below and of finite type over $\mathbb{F}_p$.

The p-completion of the top row in (2.2) is the homotopy inverse limit over n of the p-completed rows (2.3). The exactness of filtered limits in the category of compact Hausdorff abelian groups now implies that there are isomorphisms $\pi_*((Z_s)_p^{\wedge}) \cong \lim_n \pi_*((Z_{n,s})_p^{\wedge})$ for all s. Furthermore, the inverse limit over n of the exact couples (2.4) defines a third exact couple (of compact Hausdorff abelian groups)

$$
\begin{tikzcd}
\pi_*((Z_{s+1})^\wedge_p) \arrow[r, "i"] & \pi_*((Z_s)^\wedge_p) \arrow[d, "j"] \\
& \lim_n \pi_*(K_{n,s}) \arrow[ul, dashed, "k"] ,
\end{tikzcd}
\tag{2.5}
$$

which generates the spectral sequence $\{E_r^{*,*}(\underline{Y})\}_r$ that we are after. Here $E_1^{s,t}(\underline{Y}) \cong \lim_n E_1^{s,t}(Y_n)$, and by induction on r the same isomorphism holds for each E_r-term, since $E_{r+1}^{*,*}$ is the homology of $E_r^{*,*}$ with respect to the d_r-differentials, and we have seen that the formation of these limits commutes with homology. In particular, each abelian group $E_r^{s,t}(\underline{Y})$ is compact Hausdorff.
The identification of the E_2-term for $s = 0$ amounts to the isomorphism

$$\lim_n \mathrm{Hom}_{\mathscr{A}}(H^*(Y_n), N) \cong \mathrm{Hom}_{\mathscr{A}}(\operatorname*{colim}_n H^*(Y_n), N)$$

for $N = \Sigma^t \mathbb{F}_p$. The general case follows, since we can compute $\mathrm{Ext}_{\mathscr{A}}^s$ by means of an injective resolution of $\mathbb{F}_p$.
We must now check the convergence of this spectral sequence, which we (and [11]) do following Boardman [3]. The Adams resolution for each Y_n is constructed so that

$$\lim_s \pi_*((Z_{n,s})^\wedge_p) = \operatorname*{Rlim}_s \pi_*((Z_{n,s})^\wedge_p) = 0 \,.$$

These two conditions ensure that the Adams spectral sequence for Y_n converges conditionally [3, 5.10]. The standard interchange of limits isomorphism gives

$$\lim_s \pi_*((Z_s)^\wedge_p) \cong \lim_n \lim_s \pi_*((Z_{n,s})^\wedge_p) = 0 \,.$$

Moreover, the exactness of the inverse limit functor in this case implies that the derived limit

$$\operatorname*{Rlim}_s \pi_*((Z_s)^\wedge_p) = 0$$

vanishes, too. Hence the inverse limit Adams spectral sequence generated by (2.5) is conditionally convergent to $\pi_*(Y^\wedge_p)$. This is a half-plane spectral sequence with entering differentials, in the sense of Boardman. For such spectral sequences, strong convergence follows from conditional convergence together with the vanishing of the groups

$$RE_\infty^{s,t} = \operatorname*{Rlim}_r E_r^{s,t}(\underline{Y}) \,,$$

see [3, 5.1, 7.1]. Again, the vanishing of this Rlim is ensured by the exactness of lim for the compact Hausdorff abelian groups $E_r^{s,t}(\underline{Y})$. □

2.2. CONTINUOUS (CO-)HOMOLOGY. The spectral sequence in Proposition 2.2 is central to the proof of the Segal conjecture for groups of prime order and will be the foundation for the present work. Our work will, in analogy with Lin's proof of the Segal conjecture, focus on the properties of the E_2-term of the above spectral sequence.

Definition 2.3. Let $\{Y_n\}_n$ be a tower of spectra such that each Y_n is bounded below and of finite type over $\mathbb{F}_p$, and let $Y = \operatorname{holim}_n Y_n$. Define the *continuous cohomology* of Y as the colimit

$$H^*_c(Y) = \operatorname*{colim}_{n\to-\infty} H^*(Y_n)\,.$$

Dually, define the *continuous homology* of Y as the inverse limit

$$H_*^c(Y) = \lim_{n\to-\infty} H_*(Y_n)\,.$$

Note that we choose to suppress from the notation the tower of which Y is a homotopy inverse limit, even if the continuous cohomology groups do depend on the choice of inverse system. For example, let $p = 2$ and let $Y = S^\wedge_2$ be the 2-completed sphere spectrum. Since Y is bounded below and of finite type over $\mathbb{F}_2$, we may express Y by the constant tower of spectra. But by W. H. Lin's theorem, $S^\wedge_2 \simeq \operatorname{holim}_n \Sigma\mathbb{R}P^\infty_n$, where each $\Sigma\mathbb{R}P^\infty_n$ is also bounded below and of finite type over $\mathbb{F}_2$. Now $\operatorname{colim}_n H^*(\Sigma\mathbb{R}P^\infty_n) = \Sigma P(x, x^{-1}) = R_+(\mathbb{F}_2)$ is much larger than $H^*(S^\wedge_2) = \mathbb{F}_2$.

By the universal coefficient theorem and our finite type assumptions, the $\mathbb{F}_p$-linear dual of $H^*_c(Y)$ is naturally isomorphic to $H^c_*(Y)$. The continuous homology of Y will often not be of finite type, so its dual is in general not isomorphic to the continuous cohomology. However, if we take into account the linear topology on the inverse limit, given by the kernel filtration induced from the tower, we do get that the continuous dual of the continuous homology is isomorphic to the continuous cohomology. We discuss this in §2.4.

Note that the continuous cohomology is a direct limit of bounded below $\mathscr{A}$-modules. The direct limit might of course not be bounded below, but we do get a natural $\mathscr{A}$-module structure on $H^*_c(Y)$ in the category of all $\mathscr{A}$-modules. Dually, the continuous homology is an inverse limit of bounded below $\mathscr{A}_*$-comodules, but the inverse limit might be neither bounded below nor an $\mathscr{A}_*$-comodule in the usual, algebraic, sense. Instead we get a completed coaction of $\mathscr{A}_*$

$$H^c_*(Y) \longrightarrow \mathscr{A}_* \mathbin{\widehat{\otimes}} H^c_*(Y)\,,$$

where $\widehat{\otimes}$ is the tensor product completed with respect to the above-mentioned linear topology on the continuous homology. We discuss this in §2.5.

2.3. Filtrations. For every $n \in \mathbb{Z}$, let A^n be a graded $\mathbb{F}_p$-vector space and assume that these vector spaces fit into a sequence

$$0 \longrightarrow \cdots \longrightarrow A^n \longrightarrow A^{n-1} \longrightarrow \cdots \longrightarrow A^{-\infty} \tag{2.6}$$

with trivial inverse limit, and colimit denoted by $A^{-\infty}$. We assume further that each A^n is of finite type. Let $A_n = \operatorname{Hom}(A^n, \mathbb{F}_p)$ be the dual of A^n. The diagram above dualizes to a sequence

$$A_{-\infty} \longrightarrow \cdots \longrightarrow A_{n-1} \longrightarrow A_n \longrightarrow \cdots \longrightarrow 0 \tag{2.7}$$

with inverse limit

$$A_{-\infty} = \lim_n A_n = \lim_n \operatorname{Hom}(A^n, \mathbb{F}_p) \cong \operatorname{Hom}(\operatorname*{colim}_n A^n, \mathbb{F}_p) = \operatorname{Hom}(A^{-\infty}, \mathbb{F}_p)$$

isomorphic to the dual of $A^{-\infty}$, and trivial colimit. The last fact follows from the assumption that A^n is finite dimensional in each degree. Indeed,

$$\lim_n A^n \cong \lim_n \mathrm{Hom}(A_n, \mathbb{F}_p) \cong \mathrm{Hom}(\operatorname*{colim}_n A_n, \mathbb{F}_p)$$

and thus $\lim_n A^n = 0$ implies that $\mathrm{colim}_n A_n$ is trivial, since the latter injects into its double dual. Furthermore, the derived inverse limits $\mathrm{Rlim}_n A^n$ and $\mathrm{Rlim}_n A_n$ are zero, again because A^n and A_n are degreewise finite.
Adapting Boardman's notation [3, 5.4], we define filtrations of the colimit of (2.6) and the inverse limit of (2.7) using the corresponding sequential limit systems.

Definition 2.4. For each $n \in \mathbb{Z}$, let

$$F^n A^{-\infty} = \mathrm{im}(A^n \to A^{-\infty})$$

and

$$F_n A_{-\infty} = \ker(A_{-\infty} \to A_n)\,.$$

Then

$$\cdots \subset F^n A^{-\infty} \subset F^{n-1} A^{-\infty} \subset \cdots \subset A^{-\infty} \tag{2.8}$$

and

$$\cdots \subset F_{n-1} A_{-\infty} \subset F_n A_{-\infty} \subset \cdots \subset A_{-\infty} \tag{2.9}$$

define a decreasing (resp. increasing) sequence of subspaces of $A^{-\infty}$ (resp. $A_{-\infty}$).

The filtration (2.8) clearly *exhausts* $A^{-\infty}$, in the terminology of Boardman [3, 2.1]. Since each A^n and $\ker(A^n \to A^{-\infty})$ is of finite type, the right derived limits $\mathrm{Rlim}_n A^n$ and $\mathrm{Rlim}_n \ker(A^n \to A^{-\infty})$ are both zero. By assumption $\lim_n A^n = 0$, hence both $\lim_n F^n A^{-\infty}$ and $\mathrm{Rlim}_n F^n A^{-\infty}$ vanish. This means that the filtration (2.8) is *Hausdorff* and *complete*, also in the sense of Boardman [3, 2.1], and that the canonical map $A^{-\infty} \to \lim_n (A^{-\infty}/F^n A^{-\infty})$ is an isomorphism. Completeness is equivalent to saying that Cauchy sequences converge in the linear topology given by the filtration. That the filtration is Hausdorff is saying that Cauchy sequences have unique limits.
For the filtration (2.9), the proof of [3, 5.4(b)]) shows, without any hypotheses, that the filtration is Hausdorff and complete. It also shows that the filtration is exhaustive, since the colimit of (2.7) is trivial. We collect these facts in the following lemma.

Lemma 2.5. *Assume that the inverse limit* $\lim_n A^n$ *in* (2.6) *is trivial and that each* A^n *is of finite type. Then both filtrations given in Definition 2.4 (of* $\mathrm{colim}_n A^n = A^{-\infty}$ *resp. its dual* $A_{-\infty}$*) are exhaustive, Hausdorff and complete.*

2.4. DUALIZATION. The dual of the inverse limit $A_{-\infty}$ of (2.7) is the double dual of the colimit $A^{-\infty}$ of (2.6). It contains this colimit in a canonical way, but is often strictly bigger, since $A^{-\infty}$ need not be of finite type. To remedy this, we take into account the linear topology on the limit induced by the inverse system, and dualize by considering the continuous $\mathbb{F}_p$-linear dual.

In this topology on $A_{-\infty}$, an open neighborhood basis of the origin is given by the collection of subspaces $\{F_n A_{-\infty}\}_n$. A continuous homomorphism $A_{-\infty} \to \mathbb{F}_p$ is thus an $\mathbb{F}_p$-linear function whose kernel contains $F_n A_{-\infty}$ for some n. The set of these forms an $\mathbb{F}_p$-vector space $\mathrm{Hom}^c(A_{-\infty}, \mathbb{F}_p)$, which we call the *continuous dual* of $A_{-\infty}$.

LEMMA 2.6. *There is a natural isomorphism*

$$\mathrm{Hom}(A^{-\infty}, \mathbb{F}_p) \cong A_{-\infty} \,.$$

Give $A_{-\infty}$ the linear topology induced by the system of neighborhoods $\{F_n A_{-\infty}\}_n$. Then there is a natural isomorphism

$$\mathrm{Hom}^c(A_{-\infty}, \mathbb{F}_p) \cong A^{-\infty} \,.$$

Proof. The first isomorphism has already been explained. For the second, we wish to compute

$$\mathrm{Hom}^c(A_{-\infty}, \mathbb{F}_p) \cong \operatorname*{colim}_n \mathrm{Hom}(A_{-\infty}/F_n A_{-\infty}, \mathbb{F}_p) \,.$$

The dual of the image $F^n A^{-\infty} = \mathrm{im}(A^n \to A^{-\infty})$ is the image

$$\mathrm{Hom}(F^n A^{-\infty}, \mathbb{F}_p) \cong \mathrm{im}(A_{-\infty} \to A_n) \cong A_{-\infty}/F_n A_{-\infty} \,, \tag{2.10}$$

and $F^n A^{-\infty}$ is of finite type, so the canonical homomorphism

$$F^n A^{-\infty} \xrightarrow{\cong} \mathrm{Hom}(A_{-\infty}/F_n A_{-\infty}, \mathbb{F}_p)$$

into its double dual is an isomorphism. Passing to the colimit as $n \to -\infty$ we get the desired isomorphism, since $\mathrm{colim}_n F^n A^{-\infty} \cong A^{-\infty}$. □

2.5. LIMITS OF $\mathscr{A}_*$-COMODULES. Until now, the objects of our discussion have been graded vector spaces over $\mathbb{F}_p$. We will now add more structure, and assume that (2.6) is a diagram of modules over the Steenrod algebra $\mathscr{A}$. It follows that the finite terms A_n in the dual tower (2.7) are comodules under the dual Steenrod algebra $\mathscr{A}_*$. We need to discuss in what sense these comodule structures carry over to the inverse limit $A_{-\infty}$.

Let M_* be a graded vector space, with a linear topology given by a system $\{U_\alpha\}_\alpha$ of open neighborhoods, with each U_α a graded subspace of M_*. We say that M_* is *complete Hausdorff* if the canonical homomorphism $M_* \to \lim_\alpha(M_*/U_\alpha)$ is an isomorphism. Let V_* be a graded vector space, bounded below and given the discrete topology. By the *completed tensor product* $V_* \mathbin{\widehat{\otimes}} M_*$ we mean the limit $\lim_\alpha(V_* \otimes (M_*/U_\alpha))$, with the linear topology given by the kernels of the surjections $V_* \mathbin{\widehat{\otimes}} M_* \to V_* \otimes (M_*/U_\alpha)$. The completed tensor product is complete Hausdorff by construction. Given a second graded vector space W_*, discrete and bounded below, there is a canonical isomorphism $(V_* \otimes W_*) \mathbin{\widehat{\otimes}} M_* \cong V_* \mathbin{\widehat{\otimes}} (W_* \mathbin{\widehat{\otimes}} M_*)$.

DEFINITION 2.7. Let M_* be a complete Hausdorff graded $\mathbb{F}_p$-vector space. We say that M_* is a *complete $\mathscr{A}_*$-comodule* if there is a continuous graded homomorphism $\nu\colon M_* \to \mathscr{A}_* \mathbin{\widehat{\otimes}} M_*$ such that the diagrams

$$\begin{tikzcd} M_* \arrow[r, "\nu"] \arrow[dr, "\cong"'] & \mathscr{A}_* \mathbin{\widehat{\otimes}} M_* \arrow[d, "\epsilon \widehat{\otimes} 1"] \\ & \mathbb{F}_p \mathbin{\widehat{\otimes}} M_* \end{tikzcd}$$

and

$$\begin{tikzcd} M_* \arrow[rr, "\nu"] \arrow[d, "\nu"'] & & \mathscr{A}_* \mathbin{\widehat{\otimes}} M_* \arrow[dr, "\psi \widehat{\otimes} 1"] & \\ \mathscr{A}_* \mathbin{\widehat{\otimes}} M_* \arrow[r, "1 \widehat{\otimes} \nu"] & \mathscr{A}_* \mathbin{\widehat{\otimes}} (\mathscr{A}_* \mathbin{\widehat{\otimes}} M_*) \arrow[rr, "\cong"] & & (\mathscr{A}_* \otimes \mathscr{A}_*) \mathbin{\widehat{\otimes}} M_* \end{tikzcd} \tag{2.11}$$

commute. Here $\epsilon\colon \mathscr{A}_* \to \mathbb{F}_p$ and $\psi\colon \mathscr{A}_* \to \mathscr{A}_* \otimes \mathscr{A}_*$ denote the counit and coproduct in the dual Steenrod algebra, respectively. Let N_* be another complete $\mathscr{A}_*$-comodule and let $f\colon N_* \to M_*$ be a continuous graded homomorphism. Then $f \in \operatorname{Hom}^c_{\mathscr{A}_*}(N_*, M_*)$ if the diagram

$$\begin{tikzcd} N_* \arrow[r, "\nu"] \arrow[d, "f"'] & \mathscr{A}_* \mathbin{\widehat{\otimes}} N_* \arrow[d, "1 \widehat{\otimes} f"] \\ M_* \arrow[r, "\nu"] & \mathscr{A}_* \mathbin{\widehat{\otimes}} M_* \end{tikzcd}$$

commutes. Hence there is an equalizer diagram
(2.12)

$$\operatorname{Hom}^c_{\mathscr{A}_*}(N_*, M_*) \longrightarrow \operatorname{Hom}^c(N_*, M_*) \underset{f \mapsto \nu \circ f}{\overset{f \mapsto (1 \widehat{\otimes} f) \circ \nu}{\rightrightarrows}} \operatorname{Hom}^c(N_*, \mathscr{A}_* \mathbin{\widehat{\otimes}} M_*)\,.$$

LEMMA 2.8. *Suppose given a sequence of graded $\mathbb{F}_p$-vector spaces, as in (2.6), with each A^n bounded below and of finite type. Suppose also that A^n is an $\mathscr{A}$-module and that $A^n \to A^{n-1}$ is $\mathscr{A}$-linear, for each finite n. Then, with notation as above, $A^{-\infty}$ is an $\mathscr{A}$-module, and the topological $\mathbb{F}_p$-vector space $A_{-\infty}$ is a complete $\mathscr{A}_*$-comodule.*

Proof. The category of $\mathscr{A}$-modules is closed under direct limits, so the first claim of the lemma is immediate. For each n we get a commutative diagram

$$\begin{tikzcd} \mathscr{A} \otimes A^n \arrow[r, two heads] \arrow[d, "\lambda^n"'] & \mathscr{A} \otimes F^n A^{-\infty} \arrow[r, tail] \arrow[d] & \mathscr{A} \otimes A^{-\infty} \arrow[d, "\lambda"] \\ A^n \arrow[r, two heads] & F^n A^{-\infty} \arrow[r, tail] & A^{-\infty}\,, \end{tikzcd}$$

where the vertical arrows are the $\mathscr{A}$-module action maps. For every finite n, the dual of the $\mathscr{A}$-module action map $\lambda^n\colon \mathscr{A} \otimes A^n \to A^n$ defines an $\mathscr{A}_*$-comodule coaction map $\nu_n\colon A_n = \operatorname{Hom}(A^n, \mathbb{F}_p) \to \operatorname{Hom}(\mathscr{A} \otimes A^n, \mathbb{F}_p) \cong \operatorname{Hom}(\mathscr{A}, \mathbb{F}_p) \otimes$

$\mathrm{Hom}(A^n, \mathbb{F}_p) = \mathscr{A}_* \otimes A_n$, where the middle isomorphism uses that $\mathscr{A}$ and A^n are bounded below and of finite type over $\mathbb{F}_p$. Similarly, the dual of the diagram above gives a commutative diagram

$$\begin{tikzcd}
\mathscr{A}_* \otimes A_n & \mathscr{A}_* \otimes A_{-\infty}/F_n A_{-\infty} \arrow[l, tail] & \mathrm{Hom}(\mathscr{A} \otimes A^{-\infty}, \mathbb{F}_p) \arrow[l, two heads] \\
A_n \arrow[u, "\nu_n"] & A_{-\infty}/F_n A_{-\infty} \arrow[l, tail] \arrow[u] & A_{-\infty} \arrow[l, two heads] \arrow[u]
\end{tikzcd}$$

where we use the identification from (2.10). Passing to limits over n, we get the diagram

$$\begin{tikzcd}
\lim_n (\mathscr{A}_* \otimes A_n) & \mathscr{A}_* \widehat{\otimes} A_{-\infty} \arrow[l, "\cong"'] & \mathrm{Hom}(\mathscr{A} \otimes A^{-\infty}, \mathbb{F}_p) \arrow[l, "\cong"'] \\
\lim_n A_n \arrow[u, "\lim_n \nu_n"] & A_{-\infty} \arrow[l, "\cong"'] \arrow[u, "\nu"] & A_{-\infty} . \arrow[l, "="'] \arrow[u, "\mathrm{Hom}(\lambda, \mathbb{F}_p)"']
\end{tikzcd}$$

The middle vertical coaction map ν is continuous, as it is realized as the inverse limit of a homomorphism of towers. It is clear that the upper left hand horizontal map is injective, but we claim that it is also surjective.

To see this, let Z_n be the cokernel of $A_{-\infty}/F_n A_{-\infty} \hookrightarrow A_n$. We know from Lemma 2.5 that $\lim_n(A_{-\infty}/F_n A_{-\infty}) \cong \lim_n A_n$ and $\mathrm{Rlim}_n(A_{-\infty}/F_n A_{-\infty}) = 0$, so $\lim_n Z_n = 0$. This implies that $\lim_n(\mathscr{A}_* \otimes Z_n) = 0$, since there are natural injective maps $\mathscr{A}_* \otimes Z_n \hookrightarrow \mathrm{Hom}(\mathscr{A}, Z_n)$, and $\lim_n(\mathscr{A}_* \otimes Z_n) \hookrightarrow \lim_n \mathrm{Hom}(\mathscr{A}, Z_n) \cong \mathrm{Hom}(\mathscr{A}, \lim_n Z_n) = 0$. Now $\mathscr{A}_* \otimes Z_n$ is the cokernel of $\mathscr{A}_* \otimes A_{-\infty}/F_n A_{-\infty} \hookrightarrow \mathscr{A}_* \otimes A_n$, hence in the limit $\mathscr{A}_* \widehat{\otimes} A_{-\infty} \to \lim_n(\mathscr{A}_* \otimes A_n)$ is surjective.

The commutativity of the diagrams in Definition 2.7 is immediate since they are obtained as the inverse limits of the corresponding diagrams involving A_n and ν_n. Thus $A_{-\infty}$ is a complete $\mathscr{A}_*$-comodule. □

COROLLARY 2.9. *Let $\{Y_n\}_n$ be a tower of spectra as in (2.1), each bounded below and of finite type over $\mathbb{F}_p$, with homotopy inverse limit Y. Then the continuous cohomology $H^*_c(Y) = \mathrm{colim}_n H^*(Y_n)$ is an $\mathscr{A}$-module, the continuous homology $H^c_*(Y) = \lim_n H_*(Y_n)$ is a complete $\mathscr{A}_*$-comodule, and there are natural isomorphisms $\mathrm{Hom}(H^*_c(Y), \mathbb{F}_p) \cong H^c_*(Y)$ and $\mathrm{Hom}^c(H^c_*(Y), \mathbb{F}_p) \cong H^*_c(Y)$, in the respective categories.* □

3. THE ALGEBRAIC SINGER CONSTRUCTIONS

Classically, the algebraic Singer construction is an endofunctor on the category of modules over the Steenrod algebra. In §3.1 we recall its definition, and a key property proved by Adams, Gunawardena and Miller. We then dualize the construction in §3.2.

Later, we will see how the algebraic Singer construction arises in its cohomological (resp. homological) form as the continuous cohomology (resp. continuous homology) of a certain tower of truncated Tate spectra. This tower of spectra induces a natural filtration on the Singer construction. We introduce this filtration in purely algebraic terms in the present section, and will show in §5.4 that the algebraic and topological definitions agree.

3.1. The cohomological Singer construction.

Definition 3.1. Let M be an $\mathscr{A}$-module. The *Singer construction* $R_+(M)$ on M is a graded $\mathscr{A}$-module given additively by the formulas

$$\Sigma^{-1}R_+(M) = P(x, x^{-1}) \otimes M$$

for $p = 2$, and

$$\Sigma^{-1}R_+(M) = E(x) \otimes P(y, y^{-1}) \otimes M$$

for p odd. Here $\deg(x) = 1$, $\deg(y) = 2$, and Σ^{-1} denotes desuspension by one degree. The action of the Steenrod algebra is given, for $r \in \mathbb{Z}$ and $a \in M$, by the formula

$$\mathrm{Sq}^s(x^r \otimes a) = \sum_j \binom{r-j}{s-2j} x^{r+s-j} \otimes \mathrm{Sq}^j(a) \tag{3.1}$$

for $p = 2$, and the formulas

$$\begin{aligned}\mathrm{P}^s(y^r \otimes a) &= \sum_j \binom{r-(p-1)j}{s-pj} y^{r+(p-1)(s-j)} \otimes \mathrm{P}^j(a) \\ &\quad + \sum_j \binom{r-(p-1)j-1}{s-pj-1} xy^{r+(p-1)(s-j)-1} \otimes \beta\,\mathrm{P}^j(a) \\ \mathrm{P}^s(xy^{r-1} \otimes a) &= \sum_j \binom{r-(p-1)j-1}{s-pj} xy^{r+(p-1)(s-j)-1} \otimes \mathrm{P}^j(a)\end{aligned}$$

and

$$\begin{aligned}\beta(y^r \otimes a) &= 0 \\ \beta(xy^{r-1} \otimes a) &= y^r \otimes a\end{aligned}$$

for p odd.

This is the form of the Singer construction that is related to the cyclic group C_p. The cohomology of the classifying space of this group is $H^*(BC_p) \cong E(x) \otimes P(y)$ for p odd, with $\deg(x) = 1$, $\deg(y) = 2$ and $\beta(x) = y$, as above. The natural $\mathscr{A}$-module structure on $H^*(BC_p)$ extends to the localization $H^*(BC_p)[y^{-1}] = E(x) \otimes P(y, y^{-1})$, and letting $M = \mathbb{F}_p$ we get that $\Sigma^{-1}R_+(\mathbb{F}_p)$ is isomorphic to $H^*(BC_p)[y^{-1}]$ as an $\mathscr{A}$-module. The case $p = 2$ is similar.

When p odd there is a second form of the Singer construction, related to the symmetric group Σ_p. Following [19, p. 272] we identify $H^*(B\Sigma_p)$ with the subalgebra $E(u) \otimes P(v)$ of $H^*(BC_p)$ generated by $u = -xy^{p-2}$ and $v = -y^{p-1}$, with $\deg(u) = 2p-3$ and $\deg(v) = 2p-2$. The smaller form of the Singer construction then corresponds to the direct summand $E(u) \otimes P(v, v^{-1}) \otimes M$ of index $(p-1)$ in $E(x) \otimes P(y, y^{-1}) \otimes M$. Explicit formulas for the action of the

Steenrod operations on the smaller form of the Singer construction are given in [26, (3.2)], [19, §2] and [9, p. 47].
In our work, we are only concerned with the version of the Singer construction related to the group C_p. The exact form of the formulas in Definition 3.1 is justified by Theorem 5.2 below.

3.1.1. *The cohomological ϵ-map.* An important property of $R_+(M)$ is that there exists a natural homomorphism $\epsilon\colon R_+(M) \to M$ of $\mathscr{A}$-modules. In Singer's original definition for $p = 2$, the map is given by the formula

$$\epsilon(\Sigma x^{r-1} \otimes a) = \begin{cases} \mathrm{Sq}^r(a) & \text{for } r \geq 0, \\ 0 & \text{for } r < 0. \end{cases} \tag{3.2}$$

For p odd, the $\mathscr{A}$-submodule spanned by elements of the form $\Sigma y^{(p-1)r} \otimes a$ or $\Sigma xy^{(p-1)r-1} \otimes a$ is a direct summand in $R_+(M)$. The homomorphism ϵ is given by first projecting onto this direct summand and then composing with the map

$$\begin{aligned} \Sigma y^{(p-1)r} \otimes a &\longmapsto -(-1)^r \beta\, \mathrm{P}^r(a) \\ \Sigma xy^{(p-1)r-1} \otimes a &\longmapsto (-1)^r\, \mathrm{P}^r(a) \end{aligned} \tag{3.3}$$

for $r \geq 0$, still mapping to 0 for $r < 0$. See [9, p. 50]. It is clear that ϵ is surjective.
We recall the key property of ϵ. Adams, Gunawardena and Miller [1] make the following definition.

Definition 3.2. An $\mathscr{A}$-module homomorphism $L \to M$ is a Tor-*equivalence* if the induced map

$$\mathrm{Tor}^{\mathscr{A}}_{*,*}(\mathbb{F}_p, L) \longrightarrow \mathrm{Tor}^{\mathscr{A}}_{*,*}(\mathbb{F}_p, M) \tag{3.4}$$

is an isomorphism.

The relevance of this condition is:

Proposition 3.3 ([1, 1.2]). *If $L \to M$ is a* Tor*-equivalence, then for any $\mathscr{A}$-module N that is bounded below and of finite type the induced map*

$$\mathrm{Ext}^{*,*}_{\mathscr{A}}(M, N) \longrightarrow \mathrm{Ext}^{*,*}_{\mathscr{A}}(L, N) \tag{3.5}$$

is an isomorphism.

Here is their key result, proved in [1, 1.3].

Theorem 3.4 (Gunawardena, Miller). *The Singer homomorphism*

$$\epsilon\colon R_+(M) \to M$$

is a Tor*-equivalence.*

We will later encounter instances of $\mathscr{A}$-module homomorphisms $R_+(M) \to N$ induced by maps of spectra. It is often possible to determine those homomorphisms by the following corollary.

Corollary 3.5. *Let M, N be $\mathscr{A}$-modules such that N is bounded below and of finite type. Then*

$$\epsilon^*\colon \operatorname{Hom}_{\mathscr{A}}(M,N) \longrightarrow \operatorname{Hom}_{\mathscr{A}}(R_+(M),N)$$

is an isomorphism, so any $\mathscr{A}$-linear homomorphism $f\colon R_+(M) \to N$ factors as $g \circ \epsilon$ for a unique $\mathscr{A}$-linear homomorphism $g\colon M \to N$:

$$\begin{tikzcd} R_+(M) \arrow[rr, "\epsilon"] \arrow[dr, "f"'] & & M \arrow[dl, "g"] \\ & N & \end{tikzcd}$$

Proof. This is clear from Theorem 3.4 and Proposition 3.3. □

Remark 3.6. A special case of this occurs when $M = N$ is a non-trivial cyclic $\mathscr{A}$-module. Then

$$\mathbb{F}_p \cong \operatorname{Hom}_{\mathscr{A}}(M,M) \cong \operatorname{Hom}_{\mathscr{A}}(R_+(M),M)\,,$$

so any $\mathscr{A}$-linear homomorphism $R_+(M) \to M$ is equal to a scalar multiple of ϵ.

3.2. The homological Singer construction. Before we define the homological version of the Singer construction on an $\mathscr{A}_*$-comodule M_*, we need to discuss a natural filtration on the cohomological Singer construction. For a bounded below $\mathscr{A}$-module M of finite type over $\mathbb{F}_p$, let $F^nR_+(M)$ be equal to

$$\mathbb{F}_2\{\Sigma x^r \otimes a \mid r \in \mathbb{Z},\ \deg(a) = q,\ 1 + r - q \geq n\}$$

for $p = 2$, and

$$\mathbb{F}_p\{\Sigma x^i y^r \otimes a \mid i \in \{0,1\},\ r \in \mathbb{Z},\ \deg(a) = q,\ 1 + i + 2r - (p-1)q \geq n\}$$

for p odd. In each case a runs through an $\mathbb{F}_p$-basis for M. Then

$$\cdots \subset F^nR_+(M) \subset F^{n-1}R_+(M) \subset \cdots \subset R_+(M) \tag{3.6}$$

is an exhaustive filtration of $R_+(M)$, which is clearly Hausdorff. Because M is bounded below and of finite type, each $F^nR_+(M)$ is bounded below and of finite type, so $\operatorname{Rlim}_n F^nR_+(M)$ is trivial. Hence the filtration is complete.

For reasons made clear in Corollary 5.16, we will refer to this filtration as the *Tate filtration*. When M is the cohomology of a bounded below spectrum of finite type over $\mathbb{F}_p$, we will see how (3.6) is induced from topology. In this case, it will be immediate that the filtration is one of $\mathscr{A}$-modules. For a general $\mathscr{A}$-module M, this can be checked directly using the explicit formulas in Definition 3.1.

We are now in the situation discussed in the previous section, with $A^n = F^nR_+(M)$ and $A^{-\infty} = R_+(M)$. Letting $F^nR_+(M)_* = \operatorname{Hom}(F^nR_+(M),\mathbb{F}_p) = A_n$ we get an inverse system

$$\cdots \longrightarrow F^{n-1}R_+(M)_* \longrightarrow F^nR_+(M)_* \longrightarrow \cdots \tag{3.7}$$

as in (2.7), dual to the direct system (3.6). We are interested in the inverse limit $A_{-\infty} = \lim_n A_n$, with the linear topology given by this tower of surjections. Recall Definition 2.7 of a complete $\mathscr{A}_*$-comodule.

Definition 3.7. Let M_* be a bounded below $\mathscr{A}_*$-comodule of finite type. Its dual $M = \mathrm{Hom}(M_*, \mathbb{F}_p)$ is a bounded below $\mathscr{A}$-module of finite type, and $M_* \cong \mathrm{Hom}(M, \mathbb{F}_p)$. We define the *homological Singer construction* on M_* to be the complete $\mathscr{A}_*$-comodule given by

$$R_+(M_*) = \mathrm{Hom}(R_+(M), \mathbb{F}_p)\,.$$

It is isomorphic to the inverse limit $\lim_n F^n R_+(M)_*$.

A more explicit description can be given. For $p = 2$ the $\mathbb{F}_2$-linear dual of

$$\widehat{H}_{-*}(C_2; \mathbb{F}_2) \cong \Sigma H^*(C_2; \mathbb{F}_2)[x^{-1}] = \Sigma P(x, x^{-1})$$

is isomorphic to the ring of Laurent polynomials $\widehat{H}^{-*}(C_2; \mathbb{F}_2) = P(u, u^{-1})$, where $\deg(u) = -1$ and u^{-r} is dual to Σx^{r-1}. For p odd, the $\mathbb{F}_p$-linear dual of

$$\widehat{H}_{-*}(C_p; \mathbb{F}_p) \cong \Sigma H^*(C_p; \mathbb{F}_p)[y^{-1}] = \Sigma E(x) \otimes P(y, y^{-1})$$

is isomorphic to $\widehat{H}^{-*}(C_p; \mathbb{F}_p) = E(u) \otimes P(t, t^{-1})$, where $\deg(u) = -1$, $\deg(t) = -2$ and $u^{1-i}t^{-r}$ is dual to $\Sigma x^i y^{r-1}$. These notations are compatible with those from [5]. We get an identification of $F^n R_+(M)_*$ with

$$\mathbb{F}_2\{u^r \otimes \alpha \mid r \in \mathbb{Z},\ \deg(\alpha) = q,\ r + q \le -n\}$$

for $p = 2$, and with

$$\mathbb{F}_p\{u^i t^r \otimes \alpha \mid i \in \{0, 1\},\ r \in \mathbb{Z},\ \deg(\alpha) = q,\ i + 2r + (p-1)q \le -n\}$$

for p odd. In each case α ranges over an $\mathbb{F}_p$-basis for M_*. The maps of (3.7) are given by the obvious projections. Thus, $R_+(M_*)$ is isomorphic to the graded vector space of formal series

$$\sum_{r=-\infty}^{\infty} u^r \otimes \alpha_r$$

for $p = 2$, and

$$\sum_{r=-\infty}^{\infty} t^r \otimes \alpha_{0,r} + \sum_{r=-\infty}^{\infty} ut^r \otimes \alpha_{1,r}$$

for p odd. In each of these sums r is bounded below, but not above, since M_* is bounded below.

Using the linear topology on $R_+(M_*)$ given by the kernel filtration coming from (3.7), we may reformulate this as follows: Let

$$\Lambda = \widehat{H}^{-*}(C_p; \mathbb{F}_p) = \begin{cases} P(u, u^{-1}) & \text{for } p = 2, \\ E(u) \otimes P(t, t^{-1}) & \text{for } p \text{ odd.} \end{cases}$$

Consider $\Lambda \otimes M_* \subset R_+(M_*)$. For every n the composition $\Lambda \otimes M_* \subset R_+(M_*) \to F^n R_+(M)_*$ is surjective, so the completed tensor product $\Lambda \widehat{\otimes} M_*$ (for the linear topology on Λ derived from the grading) is canonically isomorphic to $R_+(M_*)$.

3.2.1. *The homological ϵ_*-map.* Let

$$\epsilon_* \colon M_* \longrightarrow R_+(M_*)$$

be the dual of $\epsilon\colon R_+(M) \to M$. Then ϵ_* is a continuous homomorphism of complete $\mathscr{A}_*$-comodules. Continuity is trivially satisfied since the source of ϵ_* has the discrete topology.

Dualizing (3.2) and (3.3), we see that ϵ_* is given by the formulas

$$\epsilon_*(\alpha) = \sum_{r=0}^{\infty} u^{-r} \otimes \mathrm{Sq}^r_*(\alpha) \tag{3.8}$$

for $p = 2$, and

$$\epsilon_*(\alpha) = \sum_{r=0}^{\infty} (-1)^r t^{-(p-1)r} \otimes \mathrm{P}^r_*(\alpha) - \sum_{r=0}^{\infty} (-1)^r u t^{-(p-1)r-1} \otimes (\beta\,\mathrm{P}^r)_*(\alpha) \tag{3.9}$$

for p odd. This expression may be compared with [1, (3.6)]. It is clear that ϵ_* is injective.

Lemma 3.8. *Let M and N be bounded below $\mathscr{A}$-modules of finite type, and let M_* and N_* be the dual $\mathscr{A}_*$-comodules. Then*

$$\epsilon_* \colon \operatorname{Hom}_{\mathscr{A}_*}(N_*, M_*) \longrightarrow \operatorname{Hom}^c_{\mathscr{A}_*}(N_*, R_+(M_*))$$

is an isomorphism, so any continuous $\mathscr{A}_*$-comodule homomorphism $f_*\colon N_* \to R_+(M_*)$ factors as $f_* = \epsilon_* \circ g_*$ for a unique $\mathscr{A}_*$-comodule homomorphism $g_*\colon N_* \to M_*$.

Proof. Notice that $\operatorname{Hom}_{\mathscr{A}_*}(N_*, M_*) = \operatorname{Hom}^c_{\mathscr{A}_*}(N_*, M_*)$ and $\mathscr{A}_* \otimes N_* = \mathscr{A}_* \widehat{\otimes} N_*$, since M_* and N_* are discrete. Applying $\operatorname{Hom}(-, \mathbb{F}_p)$ to a commutative square

$$\begin{tikzcd}
\mathscr{A} \otimes R_+(M) \arrow[r, "\lambda"] \arrow[d, "1\otimes f"'] & R_+(M) \arrow[d, "f"] \\
\mathscr{A} \otimes N \arrow[r, "\lambda"] & N
\end{tikzcd}$$

we get a commutative square

$$\begin{tikzcd}
\mathscr{A}_* \widehat{\otimes} R_+(M_*) & R_+(M_*) \arrow[l, "\nu"'] \\
\mathscr{A}_* \otimes N_* \arrow[u, "1\widehat{\otimes} f_*"] & N_* \arrow[l, "\nu"] \arrow[u, "f_*"']
\end{tikzcd}$$

of continuous homomorphisms, where $R_+(M_*)$ and $\mathscr{A}_* \widehat{\otimes} R_+(M_*)$ have the limit topologies, while N_* and $\mathscr{A}_* \otimes N_*$ are discrete. Applying $\operatorname{Hom}^c(-, \mathbb{F}_p)$ to the latter square we recover the first, by Lemma 2.6. Hence the right hand vertical map in the commutative square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{A}}(M,N) \arrow[r, "\epsilon^*", "\cong"'] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathscr{A}}(R_+(M),N) \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathscr{A}_*}(N_*,M_*) \arrow[r, "\epsilon_*"] & \mathrm{Hom}^c_{\mathscr{A}_*}(N_*,R_+(M_*))
\end{tikzcd}$$

is an isomorphism. It is easy to see that the left hand vertical map is an isomorphism, and the upper horizontal map is an isomorphism by Corollary 3.5. □

3.2.2. *Various remarks on the homological Singer construction.* The following remarks are not necessary for our immediate applications, but we include them to shed some light on the coaction $\nu\colon R_+(M_*) \to \mathscr{A}_* \mathbin{\widehat{\otimes}} R_+(M_*)$ and the dual Singer map $\epsilon_*\colon M_* \to R_+(M_*)$, and their relations to the completions introduced so far.
Dualizing (3.1), we get that the dual Steenrod operations on classes $u^r \otimes \alpha$ in $\Lambda \otimes M_* \subset R_+(M_*)$ are given by

$$\mathrm{Sq}^s_*(u^r \otimes \alpha) = \sum_j \binom{-r-s-1}{s-2j} u^{r+s-j} \otimes \mathrm{Sq}^j_*(\alpha) \tag{3.10}$$

for $p = 2$, and similarly for p odd. This sum is finite, since M_* is assumed to be bounded below, so we have the following commutative diagram:

$$\begin{tikzcd}
R_+(M_*) \arrow[r, "\nu"] & \mathscr{A}_* \mathbin{\widehat{\otimes}} R_+(M_*) \\
\Lambda \otimes M_* \arrow[r] \arrow[u, hook] & \mathscr{A}_* \mathbin{\widehat{\otimes}} (\Lambda \otimes M_*) \arrow[u, hook]
\end{tikzcd} \tag{3.11}$$

Two remarks are in order. First, $\Lambda \otimes M_*$ is not complete with respect to the subspace topology from $R_+(M_*)$. Hence $\Lambda\otimes M_*$ is not a complete $\mathscr{A}_*$-comodule in the sense explained above. Second, there are elements $u^r \otimes \alpha$ in $\Lambda \otimes M_*$ with the property that $\mathrm{Sq}^s_*(u^r \otimes \alpha)$ is nonzero for infinitely many s, and similarly for p odd. For example, $\mathrm{Sq}^s_*(u^{-1} \otimes \alpha)$ contains the term

$$\binom{-s}{s} u^{s-1} \otimes \alpha = \binom{2s-1}{s} u^{s-1} \otimes \alpha$$

for $j = 0$, according to (3.10). This equals $u^{s-1} \otimes \alpha$ whenever $s = 2^e$ is a power of 2, so $\nu(u^{-1} \otimes \alpha)$ is an infinite sum. Hence $\Lambda \otimes M_*$ is not an algebraic $\mathscr{A}_*$-comodule, either.
We will now identify the image of the homological version of the Singer map

$$\epsilon_*\colon M_* \longrightarrow R_+(M_*)$$

with the maximal algebraic $\mathscr{A}_*$-comodule contained in $R_+(M_*)$.

Definition 3.9. Given a complete $\mathscr{A}_*$-comodule N_*, let $N_*^{\mathrm{alg}} \subseteq N_*$ be given by the pullback

$$\begin{tikzcd} N_*^{\mathrm{alg}} \arrow[r, "\nu|N_*^{\mathrm{alg}}"] \arrow[d, tail] & \mathscr{A}_* \otimes N_* \arrow[d, tail] \\ N_* \arrow[r, "\nu"] & \mathscr{A}_* \widehat{\otimes} N_* \end{tikzcd}$$

in graded $\mathbb{F}_p$-vector spaces. In other words, N_*^{alg} consists of the $\alpha \in N_*$ whose coaction $\nu(\alpha) = \sum_I \mathrm{Sq}_*^I \otimes \alpha_I$ (in the notation for $p = 2$) is a finite sum, rather than a formal infinite sum. Here I runs over the admissible sequences, so that $\{\mathrm{Sq}_*^I\}_I$ is a basis for $\mathscr{A}_*$, and $\alpha_I = \mathrm{Sq}_*^I(\alpha)$.

Lemma 3.10. *The restricted coaction map $\nu|N_*^{\mathrm{alg}}$ factors (uniquely) through the inclusion $\mathscr{A}_* \otimes N_*^{\mathrm{alg}} \subseteq \mathscr{A}_* \otimes N_*$, hence defines a map*

$$\nu^{\mathrm{alg}} \colon N_*^{\mathrm{alg}} \longrightarrow \mathscr{A}_* \otimes N_*^{\mathrm{alg}}$$

that makes N_*^{alg} an $\mathscr{A}_*$-comodule in the algebraic sense.

Proof. The composite

$$N_*^{\mathrm{alg}} \xrightarrow{\nu|N_*^{\mathrm{alg}}} \mathscr{A}_* \otimes N_* \xrightarrow{1\otimes\nu} \mathscr{A}_* \otimes (\mathscr{A}_* \widehat{\otimes} N_*)$$

factors as

$$N_*^{\mathrm{alg}} \xrightarrow{\nu|N_*^{\mathrm{alg}}} \mathscr{A}_* \otimes N_* \xrightarrow{\psi\otimes 1} \mathscr{A}_* \otimes \mathscr{A}_* \otimes N_* \subseteq \mathscr{A}_* \otimes (\mathscr{A}_* \widehat{\otimes} N_*)$$

by coassociativity (2.11) of the complete coaction. Hence $\nu|N_*^{\mathrm{alg}}$ factors through the pullback $\mathscr{A}_* \otimes N_*^{\mathrm{alg}}$ in

$$\begin{tikzcd} \mathscr{A}_* \otimes N_*^{\mathrm{alg}} \arrow[r, "1\otimes\nu|N_*^{\mathrm{alg}}"] \arrow[d, tail] & \mathscr{A}_* \otimes \mathscr{A}_* \otimes N_* \arrow[d, tail] \\ \mathscr{A}_* \otimes N_* \arrow[r, "1\otimes\nu"] & \mathscr{A}_* \otimes (\mathscr{A}_* \widehat{\otimes} N_*) \,. \end{tikzcd}$$

Algebraic counitality and coassociativity of the lifted map ν^{alg} follow from the corresponding properties of ν displayed in Definition 2.7. □

The following identification stems from a conversation with M. Bökstedt.

Proposition 3.11. *The image of the injective homomorphism $\epsilon_* \colon M_* \to R_+(M_*)$ equals the maximal algebraic sub $\mathscr{A}_*$-comodule $R_+(M_*)^{\mathrm{alg}} \subset R_+(M_*)$.*

Proof. Let L_* be any algebraic $\mathscr{A}_*$-comodule. Given any $\alpha \in L_*$, with coaction $\nu(\alpha) = \sum_I \mathrm{Sq}_*^I \otimes \alpha_I$, let $\langle\alpha\rangle \subseteq L_*$ be the graded vector subspace spanned by the $\alpha_I = \mathrm{Sq}_*^I(\alpha)$. Here we are using the notation appropriate for $p = 2$; the case p odd is completely similar. Since $\nu(\alpha)$ is a finite sum, $\langle\alpha\rangle$ is a finite dimensional subspace. Furthermore, it is a sub $\mathscr{A}_*$-comodule, since

$\nu(\alpha_I) = \sum_J \mathrm{Sq}_*^J \otimes \mathrm{Sq}_*^J(\alpha_I)$ and $\mathrm{Sq}_*^J(\alpha_I) = (\mathrm{Sq}^I \mathrm{Sq}^J)_*(\alpha)$ is a finite sum of terms $\mathrm{Sq}_*^K(\alpha) = \alpha_K$.

Now consider the case $L_* = R_+(M_*)^{\mathrm{alg}}$. It is clear that $\epsilon_*(M_*) \subseteq R_+(M_*)^{\mathrm{alg}}$, since M_* is an algebraic $\mathscr{A}_*$-comodule and ϵ_* respects the coaction. Let $\alpha \in R_+(M_*)^{\mathrm{alg}}$ be any element, and consider the linear span

$$N_* = \epsilon_*(M_*) + \langle \alpha \rangle \subseteq R_+(M_*)^{\mathrm{alg}} .$$

It is bounded below and of finite type, so by Lemma 3.8 there is a unique lift g_*

$$\begin{tikzcd} M_* \arrow[rr, "\epsilon_*"] & & R_+(M_*) \\ & N_* \arrow[ul, "g_*"] \arrow[ur, tail, "f_*"'] & \end{tikzcd}$$

of the inclusion $f_*\colon N_* \to R_+(M_*)$. Hence $N_* \subseteq \epsilon_*(M_*)$, so in fact $\alpha \in \epsilon_*(M_*)$. $\square$

4. The Tate construction

We recall the Tate construction of Greenlees [12], and its relation with homotopy orbit and homotopy fixed point spectra. We then show how it can be expressed as the homotopy inverse limit of bounded below spectra, in two equivalent ways. This lets us make sense of the continuous (co-)homology groups of the Tate construction.

We then describe the homological Tate spectral sequences. There are two types, one converging to the continuous homology of the Tate construction and one converging to the continuous cohomology. The terms of these spectral sequences will be linearly dual to each other, but, as already noted in §2.2, their target groups will only be dual in a topologized sense. The main properties of these spectral sequences are summarized in Propositions 4.14, 4.15 and 4.17.

4.1. Equivariant spectra and various fixed point constructions.

We review some notions from stable equivariant homotopy theory, in the framework of Lewis–May spectra [18]. Let G be a compact Lie group, quite possibly finite, and let $\mathscr{U}$ be a complete G-universe. We fix an identification $\mathscr{U}^G = \mathbb{R}^\infty$, and write $i\colon \mathbb{R}^\infty \to \mathscr{U}$ for the inclusion.

Let $G\mathscr{S}\mathscr{U}$ be the category of genuine G-spectra, and let $G\mathscr{S}\mathbb{R}^\infty$ be the category of naive G-spectra. Similarly, let $\mathscr{S}\mathbb{R}^\infty$ be the category of (non-equivariant) spectra. The restriction of universe functor $i^*\colon G\mathscr{S}\mathscr{U} \to G\mathscr{S}\mathbb{R}^\infty$ has a left adjoint, the extension of universe functor $i_*\colon G\mathscr{S}\mathbb{R}^\infty \to G\mathscr{S}\mathscr{U}$, see [18, §II.1].

The functor $\mathscr{S}\mathbb{R}^\infty \to G\mathscr{S}\mathbb{R}^\infty$, giving a spectrum the trivial G-action, has a left adjoint taking a naive G-spectrum Y to the orbit spectrum Y/G, as well as a right adjoint taking Y to the fixed point spectrum Y^G. For a genuine G-spectrum X, the orbit spectrum $X/G = (i^*X)/G$ and fixed point spectrum $X^G = (i^*X)^G$ are defined by first restricting to the underlying naive G-spectra.

Let EG be a free, contractible G-CW complex. Let $c\colon EG_+ \to S^0$ be the collapse map that sends EG to the non-base point of S^0, and let $\widetilde{EG}$ be its mapping cone, so that we have a (homotopy) cofiber sequence

$$EG_+ \xrightarrow{c} S^0 \longrightarrow \widetilde{EG} \tag{4.1}$$

of based G-CW complexes. The n-skeleton $\widetilde{EG}^{(n)}$ of $\widetilde{EG}$ is then the mapping cone of the restricted collapse map $EG_+^{(n-1)} \to S^0$, for each $n \geq 0$. We may and will assume that each skeleton $EG^{(n-1)}$ is a finite G-CW complex.

DEFINITION 4.1. For each naive G-spectrum Y let $Y_{hG} = (EG_+ \wedge Y)/G$ be the *homotopy orbit spectrum*, and let $Y^{hG} = F(EG_+, Y)^G$ be the *homotopy fixed point* spectrum. For each genuine G-spectrum X let

$$X_{hG} = (EG_+ \wedge i^*X)/G = (i^*X)_{hG}$$

and

$$X^{hG} = F(EG_+, X)^G = (i^*X)^{hG}$$

be defined by first restricting to the G-trivial universe. Furthermore, let

$$X^{tG} = [\widetilde{EG} \wedge F(EG_+, X)]^G$$

be the *Tate construction* on X. This is the spectrum denoted $\widehat{\mathbb{H}}(G, X)$ by Bökstedt and Madsen [5] and $t_G(X)^G$ by Greenlees and May [14].

The Segal conjecture is concerned with the map $\Gamma\colon X^G \to X^{hG}$ induced by $F(c, 1)\colon X \cong F(S^0, X) \to F(EG_+, X)$ by passing to fixed points. By smashing the cofiber sequence (4.1) with $F(c, 1)$ and passing to G-fixed points, we can embed this map in the following diagram, consisting of two horizontal cofiber sequences:

$$\begin{tikzcd}
{[EG_+ \wedge X]^G} \arrow[r] \arrow[d, "\simeq"'] & X^G \arrow[r] \arrow[d, "\Gamma"] & {[\widetilde{EG} \wedge X]^G} \arrow[d, "\hat{\Gamma}"] \\
{[EG_+ \wedge F(EG_+, X)]^G} \arrow[r] & F(EG_+, X)^G \arrow[r] & {[\widetilde{EG} \wedge F(EG_+, X)]^G}
\end{tikzcd}$$

The adjunction counit $\epsilon\colon i_* i^* X \to X$ and the map $F(c, 1)$ are both G-maps and non-equivariant equivalences. By the G-Whitehead theorem, both maps

$$1 \wedge \epsilon\colon i_*(EG_+ \wedge i^*X) = EG_+ \wedge i_* i^* X \longrightarrow EG_+ \wedge X$$

and

$$1 \wedge F(c, 1)\colon EG_+ \wedge X \longrightarrow EG_+ \wedge F(EG_+, X)$$

are genuine G-equivalences. Hence we have the equivalence indicated on the left. Furthermore, there is an Adams transfer equivalence

$$\tilde{\tau}\colon (\Sigma^{\operatorname{ad} G} EG_+ \wedge i^*X)/G \xrightarrow{\simeq} [i_*(EG_+ \wedge i^*X)]^G\,, \tag{4.2}$$

where $\operatorname{ad} G$ denotes the adjoint representation of G. See [18, §II.2] and [14, Part I] for further details.

In the cases of interest to us, when G is discrete or abelian, the adjoint representation is trivial so that $\Sigma^{\mathrm{ad}\, G} = \Sigma^{\dim G}$. Hence we may rewrite the diagram above as the following *norm–restriction* diagram

$$
\begin{tikzcd}
\Sigma^{\dim G} X_{hG} \arrow[r, "N"] \arrow[d, "="'] & X^G \arrow[r, "R"] \arrow[d, "\Gamma"] & {[\widetilde{EG} \wedge X]^G} \arrow[d, "\hat{\Gamma}"] \\
\Sigma^{\dim G} X_{hG} \arrow[r, "N^h"] & X^{hG} \arrow[r, "R^h"] & X^{tG}
\end{tikzcd}
\tag{4.3}
$$

for any genuine G-spectrum X. We note that the adjunction counit $\epsilon \colon i_* i^* X \to X$ induces equivalences $(i_* i^* X)_{hG} \simeq X_{hG}$ and $(i_* i^* X)^{hG} \simeq X^{hG}$, hence $(i_* i^* X)^{tG} \simeq X^{tG}$, so the Tate construction on X only depends on the naive G-spectrum underlying X.

The spectra in the lower row have been studied by means of spectral sequences converging to their homotopy groups, e.g. in [5], [15], [23], [2] and [16]. These spectral sequences arise in the case of the homotopy orbit and fixed point spectra by choosing a filtration of EG, and by a filtration of $\widetilde{EG}$ introduced by Greenlees [12] in the case of the Tate spectrum X^{tG}. We shall instead be concerned with the spectral sequences that arise by applying homology in place of homotopy.

4.2. Tate cohomology and the Greenlees filtration of $\widetilde{EG}$. We recall the definition of the Tate cohomology groups from [10, §XII.3], and the associated Tate homology groups. Let G be a finite group, let $\mathbb{F}_p G = \mathbb{F}_p[G]$ be its group algebra, and let (P_*, d_*) be a *complete resolution* of the trivial $\mathbb{F}_p G$-module $\mathbb{F}_p$ by free $\mathbb{F}_p G$-modules. This is a commutative diagram

$$
\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "d_0"] \arrow[d, two heads] & P_{-1} \arrow[r, "d_{-1}"] & P_{-2} \arrow[r] & \cdots \\
& & \mathbb{F}_p \arrow[ur, tail] & & &
\end{tikzcd}
$$

of $\mathbb{F}_p G$-modules, where the P_n's are free and the horizontal sequence is exact. The image of d_0 is identified with $\mathbb{F}_p$, as indicated.

Definition 4.2. Given an $\mathbb{F}_p G$-module M the *Tate cohomology* and *Tate homology groups* are defined by

$$\widehat{H}^n(G; M) = H^n(\mathrm{Hom}_{\mathbb{F}_p G}(P_*, M))$$

and

$$\widehat{H}_n(G; M) = H_n(P_* \otimes_{\mathbb{F}_p G} M)\,,$$

respectively, where (P_*, d_*) is a complete $\mathbb{F}_p G$-resolution. (To form the balanced tensor product, we turn P_* into a complex of right $\mathbb{F}_p G$-modules by means of the group inverse.) These groups are independent of the chosen complete $\mathbb{F}_p G$-resolution, and there are isomorphisms

$$\widehat{H}^n(G; M) \cong \widehat{H}_{-n-1}(G; M)$$

and

$$\operatorname{Hom}(\widehat{H}_n(G;M),\mathbb{F}_p) \cong \widehat{H}^n(G;\operatorname{Hom}(M,\mathbb{F}_p))$$

for all integers n. Note that we do not follow the shifted grading convention for Tate homology given in [14, 11.2].

The topological analogue of a complete resolution is a bi-infinite filtration of $\widetilde{EG}$, in the category of G-spectra, which was introduced by Greenlees [12]. We recall the details of the construction. For brevity we shall not distinguish notationally between a based G-CW complex and its suspension G-CW spectrum. For integers $n \geq 0$ we let $\widetilde{E}_n = \widetilde{EG}^{(n)}$ be (the suspension spectrum of) the n-skeleton of $\widetilde{EG}$, while $\widetilde{E}_{-n} = D(\widetilde{E}_n) = F(\widetilde{EG}^{(n)}, S)$ is its functional dual. These definitions agree for $n = 0$, as $\widetilde{E}_0 = S$ is the sphere spectrum. Splicing the skeleton filtration of $\widetilde{EG}$ with its functional dual, we get the following diagram

$$D(\widetilde{EG}) \longrightarrow \ldots \longrightarrow \widetilde{E}_{-1} \longrightarrow \widetilde{E}_0 = S \longrightarrow \widetilde{E}_1 \longrightarrow \widetilde{E}_2 \longrightarrow \ldots \longrightarrow \widetilde{EG}\,, \tag{4.4}$$

which we call the *Greenlees filtration*. Both $\widetilde{EG} \simeq \operatorname{hocolim}_n \widetilde{E}_n$ and $D(\widetilde{EG}) \simeq \operatorname{holim}_n \widetilde{E}_n$ are non-equivariantly contractible.

Applying homology to the non-limiting terms of this filtration (i.e., the $\widetilde{E}_n$ for $n \in \mathbb{Z}$) gives a spectral sequence with $E^1_{s,t} = H_{s+t}(\widetilde{E}_s/\widetilde{E}_{s-1})$ that converges to $H_*(\widetilde{EG}, D(\widetilde{EG})) = 0$. It is concentrated on the horizontal axis, since $\widetilde{E}_n/\widetilde{E}_{n-1}$ is a finite wedge sum of G-free n-sphere spectra $G_+ \wedge S^n$ for each integer n. Hence the spectral sequence collapses at the E^2-term, and we get a long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & H_2(\widetilde{E}_2/\widetilde{E}_1) \arrow[r, "d^1_{2,0}"] & H_1(\widetilde{E}_1/\widetilde{E}_0) \arrow[r, "d^1_{1,0}"] \arrow[d, two heads] & H_0(\widetilde{E}_0/\widetilde{E}_{-1}) \arrow[r] & \cdots \\ & & H_0(S) \arrow[ur, tail] & & \end{tikzcd} \tag{4.5}$$

of finitely generated free $\mathbb{F}_pG$-modules. Letting

$$P_n = H_{n+1}(\widetilde{E}_{n+1}/\widetilde{E}_n)$$

and $d_n = d^1_{n+1,0}$ for all integers n yields a complete resolution (P_*, d_*) of $\mathbb{F}_p = H_0(S)$.

4.3. Continuous homology of the Tate construction. Let G be a finite group and let X be a genuine G-spectrum. By means of the Greenlees filtration, we may filter the Tate construction X^{tG} by a tower of spectra.

Definition 4.3. For each integer n let $\widetilde{EG}/\widetilde{E}_{n-1}$ be the homotopy cofiber of the map $\widetilde{E}_{n-1} \to \widetilde{EG}$, and define

$$X^{tG}[-\infty, n-1] = [\widetilde{E}_{n-1} \wedge F(EG_+, X)]^G$$
$$X^{tG}[n] = X^{tG}[n,\infty] = [\widetilde{EG}/\widetilde{E}_{n-1} \wedge F(EG_+, X)]^G\,.$$

Smashing the cofiber sequence $\widetilde{E}_{n-1} \to \widetilde{EG} \to \widetilde{EG}/\widetilde{E}_{n-1}$ with $F(EG_+, X)$ and taking G-fixed points, we get a cofiber sequence

$$X^{tG}[-\infty, n{-}1] \longrightarrow X^{tG} \longrightarrow X^{tG}[n, \infty]$$

for each integer n. The maps $\widetilde{E}_{n-1} \to \widetilde{E}_n$ in the Greenlees filtration (4.4) induce maps between these cofiber sequences, which combine to the non-limiting terms of the following horizontal tower of vertical cofiber sequences:

(4.6)

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & \cdots \arrow[r] & X^{tG}[-\infty, n{-}1] \arrow[r] \arrow[d] & X^{tG}[-\infty, n] \arrow[r] \arrow[d] & \cdots \arrow[r] & X^{tG} \arrow[d, "="] \\
X^{tG} \arrow[r, "="] \arrow[d, "="'] & \cdots \arrow[r, "="] & X^{tG} \arrow[r, "="] \arrow[d] & X^{tG} \arrow[r, "="] \arrow[d] & \cdots \arrow[r, "="] & X^{tG} \arrow[d] \\
X^{tG} \arrow[r] & \cdots \arrow[r] & X^{tG}[n, \infty] \arrow[r] & X^{tG}[n{+}1, \infty] \arrow[r] & \cdots \arrow[r] & *
\end{tikzcd}$$

Lemma 4.4. *Let X be a G-spectrum. Then*

$$\operatorname*{holim}_{n\to-\infty} X^{tG}[-\infty, n] \simeq * \qquad \text{and} \qquad \operatorname*{hocolim}_{n\to\infty} X^{tG}[n, \infty] \simeq *$$

so

$$\operatorname*{holim}_{n\to-\infty} X^{tG}[n, \infty] \simeq X^{tG} \qquad \text{and} \qquad \operatorname*{hocolim}_{n\to\infty} X^{tG}[-\infty, n] \simeq X^{tG}\,.$$

Proof. For negative n, $\widetilde{E}_n = D(\widetilde{E}_m)$ for $m = -n$, and there is a G-equivariant equivalence $\nu\colon D(\widetilde{E}_m) \wedge Z \xrightarrow{\simeq} F(\widetilde{E}_m, Z)$ for any G-spectrum Z, since the finite G-CW spectrum $\widetilde{E}_m$ is dualizable [18, III.2.8]. Hence

$$\begin{aligned}
\operatorname*{holim}_{n\to-\infty} X^{tG}[-\infty, n] &= \operatorname*{holim}_{n\to-\infty} [\widetilde{E}_n \wedge F(EG_+, X)]^G \\
&= \operatorname*{holim}_{m\to\infty} [D(\widetilde{E}_m) \wedge F(EG_+, X)]^G \simeq \operatorname*{holim}_{m\to\infty} F(\widetilde{E}_m \wedge EG_+, X)^G \\
&\cong F(\operatorname*{hocolim}_{m\to\infty} \widetilde{E}_m \wedge EG_+, X)^G \simeq F(\widetilde{EG} \wedge EG_+, X)^G\,,
\end{aligned}$$

which is contractible because $\widetilde{EG} \wedge EG_+$ is G-equivariantly contractible.
For the second claim we use that $\widetilde{EG}/\widetilde{E}_n$ is a free G-CW spectrum. Indeed, for $n \ge 0$, $\widetilde{EG}/\widetilde{E}_n \simeq \Sigma(EG/EG^{(n-1)})$. Thus, by the G-Whitehead theorem and the Adams transfer equivalence (4.2) we have

$$\begin{aligned}
\operatorname*{hocolim}_{n\to\infty} X^{tG}[n{+}1, \infty] &= \operatorname*{hocolim}_{n\to\infty} [\widetilde{EG}/\widetilde{E}_n \wedge F(EG_+, X)]^G \\
&\simeq \operatorname*{hocolim}_{n\to\infty} [\widetilde{EG}/\widetilde{E}_n \wedge X]^G \simeq \operatorname*{hocolim}_{n\to\infty} [\widetilde{EG}/\widetilde{E}_n \wedge i_* i^* X]^G \\
&\simeq \operatorname*{hocolim}_{n\to\infty} (\widetilde{EG}/\widetilde{E}_n \wedge i^* X)/G \cong (\operatorname*{hocolim}_{n\to\infty} \widetilde{EG}/\widetilde{E}_n \wedge i^* X)/G\,,
\end{aligned}$$

which is contractible since $\operatorname{hocolim}_{n\to\infty} (\widetilde{EG}/\widetilde{E}_n)$ is G-equivariantly contractible. The remaining claims follow, since the homotopy limit of a fiber sequence is a fiber sequence, and the homotopy colimit of a cofiber sequence is a cofiber sequence. □

Hereafter we abbreviate $X^{tG}[n,\infty]$ to $X^{tG}[n]$. Then the lower horizontal tower $\{X^{tG}[n]\}_n$ in (4.6) becomes

$$X^{tG} \longrightarrow \dots \longrightarrow X^{tG}[n] \longrightarrow X^{tG}[n+1] \longrightarrow \dots \longrightarrow * \,, \tag{4.7}$$

which we will refer to as the *Tate tower*.
The following two lemmas should be compared with the sequences (2.6) and (2.7).

Lemma 4.5. *Let X be a G-spectrum. Then*

$$\lim_{n\to\infty} H^*(X^{tG}[n]) = \operatorname*{Rlim}_{n\to\infty} H^*(X^{tG}[n]) = \operatorname*{colim}_{n\to\infty} H_*(X^{tG}[n]) = 0\,.$$

Proof. In cohomology, we have a Milnor lim-Rlim short exact sequence

$$0 \longrightarrow \operatorname*{Rlim}_n H^{*-1}(X^{tG}[n]) \longrightarrow H^*(\operatorname*{hocolim}_n X^{tG}[n]) \longrightarrow \lim_n H^*(X^{tG}[n]) \longrightarrow 0\,.$$

By Lemma 4.4 the middle term is zero, hence so are the other two terms. In homology, we have the isomorphism

$$\operatorname*{colim}_{n\to\infty} H_*(X^{tG}[n]) \cong H_*(\operatorname*{hocolim}_{n\to\infty} X^{tG}[n])\,.$$

By the same lemma the right hand side is zero. □

Lemma 4.6. *Suppose that X is bounded below and of finite type over $\mathbb{F}_p$. Then each spectrum $X^{tG}[n]$ is bounded below and of finite type over $\mathbb{F}_p$. Hence*

$$\operatorname*{Rlim}_{n\to-\infty} H_*(X^{tG}[n]) = 0\,.$$

Proof. Let $X^{tG}[n,m] = [\widetilde{E}_m/\widetilde{E}_{n-1} \wedge F(EG_+, X)]^G$. For $m \ge n$ there is a cofiber sequence

$$X^{tG}[n, m-1] \longrightarrow X^{tG}[n,m] \longrightarrow \bigvee \Sigma^m X\,,$$

with one copy of $\Sigma^m X$ in the wedge sum for each of the finitely many G-free m-cells in $\widetilde{EG}$. Since the connectivity of $\Sigma^m X$ grows to infinity with m, the first claim of the lemma follows by induction on m. The derived limit of any tower of finite groups is zero, which gives the second conclusion. □

We use the Tate tower (4.7) to define the continuous (co-)homology of X^{tG}, as in Definition 2.3 and Corollary 2.9.

Definition 4.7. Let G be a finite group and X a G-spectrum whose underlying non-equivariant spectrum is bounded below and of finite type over $\mathbb{F}_p$. By the *continuous cohomology* of X^{tG} we mean the $\mathscr{A}$-module

$$H^*_c(X^{tG}) = \operatorname*{colim}_{n\to-\infty} H^*(X^{tG}[n])\,.$$

By the *continuous homology* of X^{tG} we mean the complete $\mathscr{A}_*$-comodule

$$H_*^c(X^{tG}) = \lim_{n\to-\infty} H_*(X^{tG}[n])\,.$$

There is a natural isomorphism $\mathrm{Hom}(H^*_c(X^{tG}), \mathbb{F}_p) \cong H^c_*(X^{tG})$ of complete $\mathscr{A}_*$-comodules, as well as a natural isomorphism $\mathrm{Hom}^c(H^c_*(X^{tG}), \mathbb{F}_p) \cong H^*_c(X^{tG})$ of $\mathscr{A}$-modules.

Remark 4.8. Different G-CW structures on EG will give rise to different Greenlees filtrations and Tate towers, but by cellular approximation any two choices give pro-isomorphic towers in homology. The continuous homology, as a complete $\mathscr{A}_*$-comodule, is therefore independent of the choice. Likewise for continuous cohomology.

We end this subsection by giving a reformulation of the Tate construction, known as Warwick duality [13, §4], which will be used in §5.2 when making a topological model for the Singer construction.

Proposition 4.9 ([14, 2.6]). *There is a natural chain of equivalences*

$$\Sigma F(\widetilde{EG}, EG_+ \wedge X)^G \simeq [\widetilde{EG} \wedge F(EG_+, X)]^G = X^{tG}\,.$$

Proof. We have a commutative diagram of G-spectra

$$\begin{tikzcd} EG_+ \wedge X \arrow[r, "F(c,1\wedge 1)"] & F(EG_+, EG_+ \wedge X) \\ EG_+ \wedge EG_+ \wedge X \arrow[u, "c\wedge 1\wedge 1"', "\simeq"] \arrow[r] \arrow[d, "1\wedge F(c,c\wedge 1)"', "\simeq"] & F(EG_+, EG_+ \wedge X) \arrow[u, "="'] \arrow[d, "\simeq"', "F(1,c\wedge 1)"] \\ EG_+ \wedge F(EG_+, X) \arrow[r, "c\wedge 1"] & F(EG_+, X)\,. \end{tikzcd} \tag{4.8}$$

The maps labeled $\simeq$ are G-equivalences. The proposition follows by taking horizontal homotopy cofibers and fixed points. □

We now strengthen this to a statement about towers.

Lemma 4.10. *For each integer m there is a natural chain of equivalences*

$$\Sigma F(\widetilde{E}_m, EG_+ \wedge X)^G \simeq [\widetilde{EG}/\widetilde{E}_{-m} \wedge F(EG_+, X)]^G = X^{tG}[1-m]$$

connecting the tower

$$\Sigma F(\widetilde{EG}, EG_+ \wedge X)^G \to \dots \longrightarrow \Sigma F(\widetilde{E}_{m+1}, EG_+ \wedge X)^G \\ \longrightarrow \Sigma F(\widetilde{E}_m, EG_+ \wedge X)^G \longrightarrow \dots$$

to the Tate tower

$$X^{tG} \longrightarrow \dots \longrightarrow X^{tG}[-m] \longrightarrow X^{tG}[1-m] \longrightarrow \dots\,.$$

Proof. First observe that there is a natural chain of G-homotopy equivalences

$$\begin{tikzcd} \Sigma F(\widetilde{E}_m, EG_+ \wedge X) \\ \Sigma D(\widetilde{E}_m) \wedge EG_+ \wedge X \arrow[u, "\nu"', "\simeq"] \arrow[d, "1\wedge 1\wedge F(c,1)", "\simeq"'] \\ \Sigma D(\widetilde{E}_m) \wedge EG_+ \wedge F(EG_+, X) \\ \Sigma \widetilde{E}_{-m} \wedge EG_+ \wedge F(EG_+, X)\,. \arrow[u, "\simeq"] \end{tikzcd}$$

The upper map ν is an equivalence because $\widetilde{E}_m$ is dualizable. The middle map is an equivalence because EG_+ is G-free and $F(c,1)$ is a non-equivariant equivalence. The lower map is the identity when $m \geq 0$ and the map induced by the natural equivalence $\widetilde{E}_{-m} \to D(D(\widetilde{E}_{-m})) = D(\widetilde{E}_m)$ when $m \leq 0$. The latter two maps agree for $m = 0$, since $\widetilde{E}_0 = S$.

We proceed to prove that there is a chain of G-equivalences

$$\Sigma \widetilde{E}_{-m} \wedge EG_+ \simeq \widetilde{EG}/\widetilde{E}_{-m}\,,$$

natural for varying $m \in \mathbb{Z}$. This will prove the lemma after passage to G-fixed points.

For $m \geq 0$, let $c_m \colon EG_+^{(m-1)} \to S^0$ be the restricted collapse map. Consider the following commutative diagram of G-spectra:

$$\begin{tikzcd} * \arrow[r] \arrow[d] & D(\widetilde{E}_m) \arrow[r, "="] \arrow[d] & \widetilde{E}_{-m} \arrow[d] \\ EG_+ \arrow[r, "c"] \arrow[d, "="'] & S \arrow[r] \arrow[d, "Dc_m"] & \widetilde{EG} \arrow[d] \\ EG_+ \arrow[r, "Dc_m \circ c"] & D(EG_+^{(m-1)}) \arrow[r] & \widetilde{EG}/\widetilde{E}_{-m} \\ EG_+ \arrow[r, "Dc_m \wedge 1"] \arrow[u, "="] & D(EG_+^{(m-1)}) \wedge EG_+ \arrow[r] \arrow[u, "1\wedge c"', "\simeq"] & \Sigma \widetilde{E}_{-m} \wedge EG_+ \arrow[u, "\simeq"] \end{tikzcd}$$

All four rows are cofiber sequences. If we disregard the lower row, then all three remaining columns are also cofiber sequences. Hence, the homotopy cofiber of $Dc_m \circ c$ is equivalent to $\widetilde{EG}/\widetilde{E}_{-m}$, and we have a G-equivalence $\Sigma \widetilde{E}_{-m} \wedge EG_+ \xrightarrow{\simeq} \widetilde{EG}/\widetilde{E}_{-m}$, natural for varying $m \geq 0$.

For $m \le 0$ let $n = -m \ge 0$, let $i_n \colon EG_+^{(n-1)} \to EG_+$ be the inclusion, let $\delta_n \colon EG_+^{(n-1)} \to EG_+^{(n-1)} \wedge EG_+$ be the diagonal, and let $c_n \colon EG_+^{(n-1)} \to S^0$ be as above. Consider the following commutative diagram of G-spectra:

$$\begin{tikzcd}
S \arrow[r, "="] \arrow[d] & S \arrow[r] \arrow[d] & * \arrow[d] \\
\widetilde{E}_n \arrow[r] \arrow[d] & \widetilde{EG} \arrow[r] \arrow[d] & \widetilde{EG}/\widetilde{E}_n \arrow[d, "\simeq"] \\
\Sigma EG_+^{(n-1)} \arrow[r, "\Sigma i_n"] \arrow[d, "\Sigma\delta_n", "\simeq"'] & \Sigma EG_+ \arrow[r] \arrow[d, "="'] & \Sigma(EG/EG^{(n-1)}) \arrow[d, "\simeq"] \\
\Sigma EG_+^{(n-1)} \wedge EG_+ \arrow[r, "\Sigma c_n \wedge 1"] & \Sigma EG_+ \arrow[r] & \Sigma\widetilde{E}_n \wedge EG_+
\end{tikzcd}$$

Again, all four rows are cofiber sequences. If we disregard the lower row, then all three remaining columns are also cofiber sequences. Hence, the middle vertical map on the right is a G-equivalence, and we have a G-equivalence $\widetilde{EG}/\widetilde{E}_{-m} \xrightarrow{\simeq} \Sigma\widetilde{E}_{-m} \wedge EG_+$, natural for varying $m \le 0$.
The direction of the G-equivalence changes when we pass from $m \ge 0$ to $m \le 0$, but for $m = 0$ the two equivalences are mutually inverse identifications of ΣEG_+ and $\widetilde{EG}/S$. Hence we can connect $\Sigma\widetilde{E}_{-m} \wedge EG_+$ to $\widetilde{EG}/\widetilde{E}_{-m}$ by a chain of two G-equivalences, where the first uses the maps $\Sigma\widetilde{E}_{-m} \wedge EG_+ \to \widetilde{EG}/\widetilde{E}_{-m}$ for $m \ge 0$ and identity maps for $m < 0$, and the second uses identity maps for $m \ge 0$ and the maps $\widetilde{EG}/\widetilde{E}_{-m} \to \Sigma\widetilde{E}_{-m} \wedge EG_+$ for $m < 0$. These are then compatible for all $m \in \mathbb{Z}$. □

Corollary 4.11. *The continuous (co-)homology of X^{tG} may be computed from the tower*

$$X^{tG} \longrightarrow \dots \longrightarrow \Sigma F(\widetilde{E}_{m+1}, EG_+ \wedge X)^G \longrightarrow \Sigma F(\widetilde{E}_m, EG_+ \wedge X)^G \longrightarrow \dots$$

as

$$H_c^*(X^{tG}) \cong \operatorname*{colim}_{m\to\infty} \Sigma H^*(F(\widetilde{E}_m, EG_+ \wedge X)^G)$$

and

$$H_*^c(X^{tG}) \cong \lim_{m\to\infty} \Sigma H_*(F(\widetilde{E}_m, EG_+ \wedge X)^G)\,.$$

4.4. The (co-)homological Tate spectral sequences. Let G be a finite group and X a G-spectrum. The cofiber sequence $\widetilde{E}_s/\widetilde{E}_{s-1} \to \widetilde{EG}/\widetilde{E}_{s-1} \to \widetilde{EG}/\widetilde{E}_s$ induces a cofiber sequence

$$[\widetilde{E}_s/\widetilde{E}_{s-1} \wedge F(EG_+, X)]^G \longrightarrow X^{tG}[s] \xrightarrow{i} X^{tG}[s+1]$$

for every integer s. The left hand term is equivalent to

$$[\widetilde{E}_s/\widetilde{E}_{s-1} \wedge X]^G \simeq (\widetilde{E}_s/\widetilde{E}_{s-1} \wedge i^*X)/G$$

since $\widetilde{E}_s/\widetilde{E}_{s-1}$ is G-free.

Applying cohomology, we get an exact couple of $\mathscr{A}$-modules

$$
\begin{tikzcd}
A^{s+1,*} \arrow[r, "i"] & A^{s,*} \arrow[d] \\
& \widehat{E}_1^{s,*} \arrow[ul, dashed]
\end{tikzcd}
\tag{4.9}
$$

with

$$A^{s,t} = H^{s+t}(X^{tG}[s]) \qquad \text{and} \qquad \widehat{E}_1^{s,t} = H^{s+t}((\widetilde{E}_s/\widetilde{E}_{s-1} \wedge i^*X)/G)\,.$$

By Lemma 4.5, $\lim_s A^s = \mathrm{Rlim}_s A^s = 0$, so this spectral sequence converges conditionally to the colimit $H_c^*(X^{tG})$, in the first sense of [3, 5.10]. Applying homology instead, we get an exact couple of algebraic $\mathscr{A}_*$-comodules

$$
\begin{tikzcd}
A_{s,*} \arrow[r, "i"] & A_{s+1,*} \arrow[dl, dashed] \\
\widehat{E}^1_{s,*} \arrow[u]
\end{tikzcd}
\tag{4.10}
$$

with

$$A_{s,t} = H_{s+t}(X^{tG}[s]) \qquad \text{and} \qquad \widehat{E}^1_{s,t} = H_{s+t}((\widetilde{E}_s/\widetilde{E}_{s-1} \wedge i^*X)/G)\,.$$

By Lemma 4.5, $\mathrm{colim}_s A_s = 0$, so this spectral sequence converges conditionally to the limit $H^c_*(X^{tG})$, in the second sense of [3, 5.10].

We can rewrite the $\widehat{E}^1$-term as

$$\widehat{E}^1_{s,t} \cong H_s(\widetilde{E}_s/\widetilde{E}_{s-1}) \otimes_{\mathbb{F}_pG} H_t(X) = P_{s-1} \otimes_{\mathbb{F}_pG} H_t(X)\,,$$

and the d^1-differential is induced by the differential in the complete resolution (P_*, d_*), so

$$\widehat{E}^2_{s,t} \cong \widehat{H}_{s-1}(G; H_t(X)) \cong \widehat{H}^{-s}(G; H_t(X))\,.$$

Dually, the $\widehat{E}_1$-term is

$$\widehat{E}_1^{s,t} \cong \mathrm{Hom}(P_{s-1} \otimes_{\mathbb{F}_pG} H_t(X), \mathbb{F}_p) \cong \mathrm{Hom}_{\mathbb{F}_pG}(P_{s-1}, H^t(X))$$

and

$$\widehat{E}_2^{s,t} \cong \widehat{H}^{s-1}(G; H^t(X)) \cong \widehat{H}_{-s}(G; H^t(X))\,.$$

DEFINITION 4.12. Let G be a finite group and X a G-spectrum. The *cohomological Tate spectral sequence* of X is the conditionally convergent spectral sequence

$$\widehat{E}_2^{s,t} = \widehat{H}_{-s}(G; H^t(X)) \Longrightarrow H_c^{s+t}(X)$$

associated with the exact couple of $\mathscr{A}$-modules (4.9). Dually, the *homological Tate spectral sequence* of X is the conditionally convergent spectral sequence

$$\widehat{E}^2_{s,t} = \widehat{H}^{-s}(G; H_t(X)) \Longrightarrow H^c_{s+t}(X)$$

associated with the exact couple of $\mathscr{A}_*$-comodules (4.10).

Remark 4.13. There is a natural isomorphism $\widehat{E}_r^{s,t} \cong \mathrm{Hom}(\widehat{E}^r_{s,t}, \mathbb{F}_p)$ for all finite r, s and t, so that the d_r-differential $d_r^{s,t} \colon \widehat{E}_r^{s,t} \to \widehat{E}_r^{s+r,t-r+1}$ is the linear dual of the d^r-differential $d^r_{s+r,t-r+1} \colon \widehat{E}^r_{s+r,t-r+1} \to \widehat{E}^r_{s,t}$. In this sense the cohomological Tate spectral sequence is dual to the homological Tate spectral sequence.

To get strong convergence, we need X to be bounded below in the cohomological case, and that X is bounded below and of finite type over $\mathbb{F}_p$ in the homological case.

Proposition 4.14. *Let G be a finite group and X a G-spectrum. Assume that X is bounded below. Then X^{tG} is the homotopy inverse limit of a tower $\{X^{tG}[s]\}_s$ of bounded below spectra, and the cohomological Tate spectral sequence*

$$\widehat{E}_2^{s,t}(X) = \widehat{H}_{-s}(G; H^t(X)) \Longrightarrow H_c^{s+t}(X^{tG})$$

converges strongly to the continuous cohomology of X^{tG} as an $\mathscr{A}$-module.

Proof. When $H^*(X)$ is bounded below, the cohomological Tate spectral sequence has exiting differentials in the sense of Boardman, so the spectral sequence is automatically strongly convergent by [3, 6.1]. In other words, the filtration of $H_c^{s+t}(X^{tG})$ by the sub $\mathscr{A}$-modules

$$F^s H_c^*(X^{tG}) = \mathrm{im}(H^*(X^{tG}[s]) \to H_c^*(X^{tG}))$$

is exhaustive, complete and Hausdorff, and there are $\mathscr{A}$-module isomorphisms

$$F^s H_c^*(X^{tG}) / F^{s+1} H_c^*(X^{tG}) \cong \widehat{E}_\infty^{s,*} \,.$$

□

Proposition 4.15. *Let G be a finite group and X a G-spectrum. Assume that X is bounded below and of finite type over $\mathbb{F}_p$. Then X^{tG} is the homotopy inverse limit of a tower $\{X^{tG}[s]\}_s$ of bounded below spectra of finite type over $\mathbb{F}_p$, and the homological Tate spectral sequence*

$$\widehat{E}^2_{s,t}(X) = \widehat{H}^{-s}(G; H_t(X)) \Longrightarrow H^c_{s+t}(X^{tG})$$

converges strongly to the continuous homology of X^{tG} as a complete $\mathscr{A}_*$-comodule.

Proof. When $H_*(X)$ is bounded below, the homological Tate spectral sequence has entering differentials in the sense of Boardman. The derived limit $R\widehat{E}^\infty_{*,*} = \mathrm{Rlim}_r \widehat{E}^r_{*,*}$ vanishes since $H_*(X)$, and thus $\widehat{E}^2_{*,*}$, is finite in each (bi-)degree. Hence the spectral sequence is strongly convergent by [3, 7.4]. In other words, the filtration of $H^c_{s+t}(X^{tG})$ by the complete sub $\mathscr{A}_*$-comodules

$$F_s H_*^c(X^{tG}) = \ker(H_*^c(X^{tG}) \to H_*^c(X^{tG}[s]))$$

is exhaustive, complete and Hausdorff, and there are algebraic $\mathscr{A}_*$-comodule isomorphisms

$$F_{s+1} H_*^c(X^{tG}) / F_s H_*^c(X^{tG}) \cong \widehat{E}^\infty_{s,*} \,.$$

□

4.4.1. *Homotopy vs. Homology.* We used the Tate tower (4.7) to define our homological Tate spectral sequence, by applying homology with $\mathbb{F}_p$-coefficients. When studying the *homotopy groups* of the Tate construction, it has been customary to apply $\pi_*(-)$ to the upper tower in (4.6):
(4.11)

$$* \longrightarrow \cdots \longrightarrow X^{tG}[-\infty, n-1] \longrightarrow X^{tG}[-\infty, n] \longrightarrow \cdots \longrightarrow X^{tG}$$

Applying a homological functor to these two towers of spectra gives two different exact couples with isomorphic spectral sequences. If we are working with homotopy, we get two spectral sequences converging to the same groups. Using (4.11) yields a spectral sequence converging to the colimit

$$\operatorname*{colim}_n \pi_*(X^{tG}[-\infty, n]) \cong \pi_*(X^{tG}),$$

while using (4.7) yields an isomorphic spectral sequence converging to the inverse limit

$$\lim_n \pi_*(X^{tG}[n]) \cong \pi_*(X^{tG}). \tag{4.12}$$

The latter isomorphism assumes that X is bounded below and of suitably finite type, so that the right derived limit $\operatorname{Rlim}_n \pi_*(X^{tG}[n]) = 0$. For example, it suffices if $\pi_*(X)$ is of finite type over $\mathbb{Z}$ or $\widehat{\mathbb{Z}}_p$.
When working with homology with $\mathbb{F}_p$-coefficients, instead of homotopy groups, the failure of the isomorphism we made use of in (4.12) makes the situation more interesting. Applying $H_*(-)$ to the tower (4.11) will produce a sequence of homology groups whose inverse limit is not trivial in general. This means that the associated spectral sequence will not be conditionally convergent to the direct limit

$$\operatorname*{colim}_n H_*(X^{tG}[-\infty, n]) \cong H_*(X^{tG}).$$

In fact, we have seen that the (isomorphic) homological Tate spectral sequence, arising from (4.7), converges strongly to

$$\lim_n H_*(X^{tG}[n]) = H^c_*(X^{tG}),$$

which is only rarely isomorphic to $H_*(X^{tG})$, since inverse limits and homology do not commute in general.
We end this discussion by noticing that the continuous homology groups of the Tate construction on X can be thought of as the homotopy groups of the Tate construction on $H \wedge X$, where $H = H\mathbb{F}_p$ is the Eilenberg–Mac Lane spectrum of $\mathbb{F}_p$. In other words, continuous homology of X^{tG} is a special case of homotopy.

Proposition 4.16. *For any bounded below G-spectrum X of finite type over $\mathbb{F}_p$ there is a natural isomorphism*

$$\pi_*(H \wedge X)^{tG} \cong H^c_*(X^{tG}),$$

where H has the trivial G-action.

Proof. For all integers m we have

$$\begin{aligned}(H \wedge X)^{tG}[1{-}m] &\simeq \Sigma F(\widetilde{E}_m, EG_+ \wedge H \wedge X)^G \\ &\simeq H \wedge \Sigma F(\widetilde{E}_m, EG_+ \wedge X)^G \simeq H \wedge X^{tG}[1{-}m]\,.\end{aligned}$$

The first and last equivalences follow from Lemma 4.10, while the middle equivalence follows from the fact that $\widetilde{E}_m$ is G-equivariantly dualizable. Thus, we have that $\pi_*(H \wedge X)^{tG}[n] \cong H_*(X^{tG}[n])$ for all integers n. For a general G-spectrum X, we then have the following surjective maps:

$$\begin{aligned}\pi_*(H \wedge X)^{tG} &\longrightarrow \lim_{n\to-\infty} \pi_*(H \wedge X)^{tG}[n] \\ &\cong \lim_{n\to-\infty} H_*(X^{tG}[n]) = H_*^c(X^{tG})\,.\end{aligned} \tag{4.13}$$

Since X was assumed to be bounded below and of finite type over $\mathbb{F}_p$, the groups in the first inverse limit system are all of finite type over $\mathbb{F}_p$, so their Rlim vanishes and the map in (4.13) is an isomorphism. □

The previous proposition and discussion tells us that the continuous homology of X^{tG} can be considered both as the direct limit $\operatorname{colim}_n \pi_*(H \wedge X)^{tG}[-\infty, n]$ or as the inverse limit $\lim_n \pi_*(H \wedge X)^{tG}[n, \infty]$. In both cases, the filtration of the two groups given by their defining towers are the same.

4.5. Multiplicative structure. We now discuss how the treatment in [16, §4.3] of multiplicative structure in the homotopical Tate spectral sequence carries over to the homological Tate spectral sequence.

Let X be a bounded below G-equivariant ring spectrum of finite type over $\mathbb{F}_p$. We assume that the unit map $\eta\colon S \to X$ and the multiplication map $\mu\colon X \wedge X \to X$ are equivariant with respect to the trivial G-action on S and the diagonal G-action on $X \wedge X$. By [14, 3.5], the homotopy cartesian square

$$\begin{tikzcd} X^G \arrow[r, "R"] \arrow[d, "\Gamma"] & {[\widetilde{EG} \wedge X]^G} \arrow[d, "\hat\Gamma"] \\ X^{hG} \arrow[r, "R^h"] & X^{tG} \end{tikzcd} \tag{4.14}$$

in (4.3) is a diagram of ring spectra and ring spectrum maps. Up to homotopy there is a unique G-equivalence $f\colon \widetilde{EG} \wedge \widetilde{EG} \xrightarrow{\simeq} \widetilde{EG}$, compatible with the inclusion $S^0 \to \widetilde{EG}$ and the homeomorphism $S^0 \wedge S^0 \cong S^0$. Using f, the multiplication map on X^{tG} is given by the composition

$$\begin{array}{c}
[\widetilde{EG} \wedge F(EG_+, X)]^G \wedge [\widetilde{EG} \wedge F(EG_+, X)]^G \\
\Big\downarrow \\
[\widetilde{EG} \wedge \widetilde{EG} \wedge F(EG_+ \wedge EG_+, X \wedge X)]^G \\
\Big\downarrow{\scriptstyle f \wedge F(\Delta,\mu)} \\
[\widetilde{EG} \wedge F(EG_+, X)]^G .
\end{array}$$

The other multiplication maps arise by replacing $\widetilde{EG}$, EG_+ or both by S^0. The unit map to X^G is adjoint to the G-map $S \to X$, since S has the trivial G-action. The other unit maps arise by composition with the maps in (4.14). The non-equivariant ring spectrum structure on the Eilenberg–Mac Lane spectrum H makes $H \wedge X$ a naively G-equivariant ring spectrum, so $(H \wedge X)^{tG}$ is a ring spectrum. The induced graded ring structure on $\pi_*(H \wedge X)^{tG}$ then gives a graded ring structure on the continuous homology $H^c_*(X^{tG})$, by the isomorphism of Proposition 4.16.

Proposition 4.17. *Let G be a finite group and X a G-equivariant ring spectrum. Assume that X is bounded below and of finite type over $\mathbb{F}_p$. Then the homological Tate spectral sequence*

$$\widehat{E}^2_{s,t} = \widehat{H}^{-s}(G; H_t(X)) \Longrightarrow H^c_{s+t}(X^{tG})$$

is a strongly convergent $\mathscr{A}_*$-comodule algebra spectral sequence, whose product at the $\widehat{E}^2$-term is given by the cup product in Tate cohomology and the Pontryagin product on $H_*(X)$.

Proof. Using the Greenlees filtration, we may filter $(H \wedge X)^{tG}$ by the tower (4.11). This produces a homotopical Tate spectral sequence with $\widehat{E}^2$-term

$$\widehat{E}^2_{s,t} = \widehat{H}^{-s}(G; \pi_t(H \wedge X)) = \widehat{H}^{-s}(G; H_t(X)) ,$$

converging strongly to the homotopy $\pi_{s+t}(H \wedge X)^{tG} \cong H^c_{s+t}(X^{tG})$. By the proof of Proposition 4.16 it is additively isomorphic to the homological Tate spectral sequence of Proposition 4.15. By [16, 4.3.5], it is also an algebra spectral sequence, with differentials being derivations with respect to the product. The $\widehat{E}^\infty$-term is the associated graded of the multiplicative filtration of $\pi_{s+t}(H \wedge X)^{tG}$ given by the images

$$\mathrm{im}(\pi_*(H \wedge X)^{tG}[-\infty, s] \to \pi_*(H \wedge X)^{tG}) .$$

□

5. The topological Singer construction

5.1. Realizing the Singer construction as continuous cohomology. As observed by Miller, and presented by Bruner, May, McClure and Steinberger in [9, §II.5], there is a particular inverse system of spectra whose continuous cohomology realizes the Singer construction in the version related to the symmetric group Σ_p. We go through the adjustments needed to realize the version of the Singer construction related to the subgroup C_p generated by the cyclic permutation $(1\ 2\ \cdots\ p)$.

Let B be a non-equivariant spectrum that is bounded below and of finite type over $\mathbb{F}_p$. For each subgroup $G \subseteq \Sigma_p$ there is an extended power construction [9, §I.2]

$$D_G(B) = EG \ltimes_G B^{(p)} ,$$

well defined in the stable homotopy category. Here $B^{(p)}$ denotes the external p-th smash power of B, and G permutes the p copies of B. It follows from [9, I.2.4] that $D_G(B)$ is bounded below and of finite type over $\mathbb{F}_p$. More precisely, we have the following calculation.

Lemma 5.1 ([9, I.2.3]). *There is a natural isomorphism*

$$H_*(D_G(B)) \cong H_*(G; H_*(B)^{\otimes p}) ,$$

where G permutes the p copies of $H_*(B)$.

The p-fold diagonal map $S^1 \to S^1 \wedge \cdots \wedge S^1 \cong S^p$ induces maps $\Delta : \Sigma D_G(B) \to D_G(\Sigma B)$ as in [9, §II.3]. Applied to desuspensions of B, these assemble to an inverse system

$$\dots \longrightarrow \Sigma^{n+1} D_G(\Sigma^{-n-1} B) \xrightarrow{\Sigma^n \Delta} \Sigma^n D_G(\Sigma^{-n} B) \longrightarrow \dots \xrightarrow{\Delta} D_G(B) \tag{5.1}$$

in the stable homotopy category. This is a tower of bounded below spectra of finite type over $\mathbb{F}_p$, so it makes sense to talk about its associated continuous cohomology.

We now follow [9, §II.5], but focus on the case $G = C_p$ instead of $G = \Sigma_p$. There is an additive isomorphism

$$H_*(D_{C_p}(B)) \cong \mathbb{F}_p\{e_0 \otimes \alpha_1 \otimes \cdots \otimes \alpha_p\} \oplus \mathbb{F}_p\{e_j \otimes \alpha^{\otimes p} \mid j \geq 0\}$$

where the α_i and α range over a basis for $H_*(B)$, the α_i are not all equal, and only one representative is taken from each C_p-orbit of the tensors $\alpha_1 \otimes \cdots \otimes \alpha_p$. The grading is determined by $\deg(e_j) = j$. Dually, there is an isomorphism

$$H^*(D_{C_p}(B)) \cong \mathbb{F}_p\{w_0 \otimes a_1 \otimes \cdots \otimes a_p\} \oplus \mathbb{F}_p\{w_j \otimes a^{\otimes p} \mid j \geq 0\}$$

where the a_i and a range over the dual basis for $H^*(B)$, and $w_j \otimes a^{\otimes p}$ is dual to $e_j \otimes \alpha^{\otimes p}$ when a is dual to α. It follows that

$$\begin{aligned} H^*(\Sigma^n D_{C_p}(\Sigma^{-n} B)) \cong \mathbb{F}_p\{\Sigma^n w_0 \otimes \Sigma^{-n} a_1 \otimes \cdots \otimes \Sigma^{-n} a_p\} \\ \oplus \mathbb{F}_p\{\Sigma^n w_j \otimes (\Sigma^{-n} a)^{\otimes p} \mid j \geq 0\} . \end{aligned} \tag{5.2}$$

By [9, II.5.6], the map Δ in (5.1) is given in cohomology as

$$\Delta^*(w_j \otimes a^{\otimes p}) = (-1)^{j+1} \alpha(q) \cdot \Sigma w_{j+p-1} \otimes (\Sigma^{-1} a)^{\otimes p}$$

where $\deg(a) = q$, $m = (p-1)/2$ and $\alpha(q) = -(-1)^{mq} m!$. For $p = 2$ the numerical coefficient should be read as 1. The other classes $w_0 \otimes a_1 \otimes \cdots \otimes a_p$ map to zero. It follows that

$$(\Sigma^n \Delta)^*(\Sigma^n w_j \otimes (\Sigma^{-n} a)^{\otimes p}) = (-1)^{j+1} \alpha(q-n) \cdot \Sigma^{n+1} w_{j+p-1} \otimes (\Sigma^{-n-1} a)^{\otimes p} . \tag{5.3}$$

The action of the Steenrod algebra $\mathscr{A}$ on $H^*(D_{C_p}(B))$ is given by the Nishida relations. Together with the explicit formula above for the maps $(\Sigma^n \Delta)^*$, this determines the direct limit of cohomology groups as an $\mathscr{A}$-module.
Miller observed that this direct limit can be described in closed form by the Singer construction on the $\mathscr{A}$-module $H^*(B)$, up to a single degree shift. We now give the C_p-equivariant extension of the Σ_p-equivariant case discussed in [9, II.5.1].

Theorem 5.2. *For each spectrum B that is bounded below and of finite type over $\mathbb{F}_p$, there is a natural isomorphism of $\mathscr{A}$-modules*

$$\omega \colon \operatorname*{colim}_{n\to\infty} H^*(\Sigma^n D_{C_p}(\Sigma^{-n} B)) \xrightarrow{\cong} \Sigma^{-1} R_+(H^*(B)) .$$

Proof. For $p = 2$ the isomorphism is given by

$$\omega(\Sigma^n w_{r+n} \otimes (\Sigma^{-n} a)^{\otimes 2}) = x^{r+q} \otimes a$$

where $\deg(a) = q$. For p odd, the isomorphism is given by

$$\omega(\Sigma^n w_{2(r+mn)} \otimes (\Sigma^{-n} a)^{\otimes p}) = \frac{(-1)^{q-n}}{\nu(q-n)} \cdot y^{r+mq} \otimes a$$

and

$$\omega(\Sigma^n w_{2(r+mn)-1} \otimes (\Sigma^{-n} a)^{\otimes p}) = \frac{(-1)^{q}}{\nu(q-n)} \cdot x y^{r+mq-1} \otimes a ,$$

where $\deg(a) = q$, $m = (p-1)/2$ and $\nu(2j+\epsilon) = (-1)^j (m!)^\epsilon$ for $\epsilon \in \{0,1\}$.
It follows from (5.3) and the relation $\alpha(q)\nu(q-1)^{-1} = \nu(q)^{-1}$ that these homomorphisms are compatible under $(\Sigma^n \Delta)^*$. It is then clear from (5.2) that ω is an additive isomorphism.
This isomorphism commutes with the Bockstein and Steenrod operations on the extended powers, described in [18, VIII.3.2] and [9, II.5.5], and the explicitly defined Bockstein and Steenrod operations on the Singer construction, given in Definition 3.1. Here are the details: For $p = 2$, the Steenrod reduced square Sq^s takes $\omega(\Sigma^n w_{r+n} \otimes (\Sigma^{-n} a)^{\otimes 2}) = x^{r+q} \otimes a$ to

$$\sum_j \binom{r+q-j}{s-2j} x^{r+q+s-j} \otimes \mathrm{Sq}^j(a) ,$$

by Definition 3.1, while ω takes

$$\begin{aligned} &\mathrm{Sq}^s(\Sigma^n w_{r+n} \otimes (\Sigma^{-n} a)^{\otimes 2}) \\ &\qquad = \sum_j \binom{r+n+q-n-j}{s-2j} \Sigma^n w_{r+n+s-2j} \otimes (\Sigma^{-n} \mathrm{Sq}^j(a))^{\otimes 2} \end{aligned}$$

(given by [9, II.5.5(i)]) to

$$\sum_j \binom{r+q-j}{s-2j} x^{r+s-2j+q+j} \otimes \mathrm{Sq}^j(a)\,,$$

and these expressions are equal.

For p odd, the Bockstein operation β takes $\omega(\Sigma^n w_{2(r+mn)} \otimes (\Sigma^{-n}a)^{\otimes p}) = (-1)^{q-n}\nu(q-n)^{-1} \cdot y^{r+mq} \otimes a$ to 0 by Definition 3.1. This equals the image under ω of $\beta(\Sigma^n w_{2(r+mn)} \otimes (\Sigma^{-n}a)^{\otimes p}) = 0$.

The Steenrod reduced power P^s takes $\omega(\Sigma^n w_{2(r+mn)} \otimes (\Sigma^{-n}a)^{\otimes p})$ to $(-1)^{q-n}\nu(q-n)^{-1}$ times the sum

$$\sum_j \binom{r+mq-(p-1)j}{s-pj} y^{r+mq+(p-1)(s-j)} \otimes \mathrm{P}^j(a)$$
$$+ \sum_j \binom{r+mq-(p-1)j-1}{s-pj-1} xy^{r+mq+(p-1)(s-j)-1} \otimes \beta\,\mathrm{P}^j(a)\,,$$

according to Definition 3.1. The homomorphism ω takes $\mathrm{P}^s(\Sigma^n w_{2(r+mn)} \otimes (\Sigma^{-n}a)^{\otimes p})$, which equals the sum

$$\sum_j \binom{r+mq-(p-1)j}{s-pj} \Sigma^n w_I \otimes (\Sigma^{-n}\,\mathrm{P}^j(a))^{\otimes p}$$
$$+ \alpha(q-n) \sum_j \binom{r+mq-(p-1)j-1}{s-pj-1} \Sigma^n w_{I-p} \otimes (-1)^n (\Sigma^{-n}\beta\,\mathrm{P}^j(a))^{\otimes p}$$

where $I = 2(r+mn)+2(s-pj)(p-1)$ (given by [9, II.5.5(ii)] and the fact that β anti-commutes with suspension), to

$$\sum_j \binom{r+mq-(p-1)j}{s-pj} \frac{(-1)^{q-n}}{\nu(q-n)} \cdot y^J \otimes \mathrm{P}^j(a)$$
$$+ \alpha(q-n) \sum_j \binom{r+mq-(p-1)j-1}{s-pj-1} \frac{(-1)^{q+1-n}}{\nu(q+1-n)} \cdot xy^{J-1} \otimes \beta\,\mathrm{P}^j(a)$$

where $J = r+(s-j)(p-1)+mq$. These expressions are equal because of the relation $\alpha(q-n)\nu(q-n-1)^{-1} = \nu(q-n)^{-1}$ mentioned above, and the fact that $\nu(q-n-1) = -\nu(q+1-n)$.

Also for p odd, β takes $\omega(\Sigma^n w_{2(r+mn)-1} \otimes (\Sigma^{-n}a)^{\otimes p}) = (-1)^q\nu(q-n)^{-1} \cdot xy^{r+mq-1} \otimes a$ to $(-1)^q\nu(q-n)^{-1} \cdot y^{r+mq} \otimes a$ by Definition 3.1. This equals the image under ω of $\beta(\Sigma^n w_{2(r+mn)-1} \otimes (\Sigma^{-n}a)^{\otimes p}) = (-1)^n\Sigma^n w_{2(r+mn)} \otimes (\Sigma^{-n}a)^{\otimes p}$ (given in [18, VIII.3.2(iii)]).

Finally, P^s takes $\omega(\Sigma^n w_{2(r+mn)-1} \otimes (\Sigma^{-n}a)^{\otimes p})$ to $(-1)^q\nu(q-n)^{-1}$ times the sum

$$\sum_j \binom{r+mq-(p-1)j-1}{s-pj} xy^{r+mq+(p-1)(s-j)-1} \otimes \mathrm{P}^j(a)\,,$$

according to Definition 3.1, while ω takes $\mathrm{P}^s(\Sigma^n w_{2(r+mn)-1} \otimes (\Sigma^{-n} a)^{\otimes p})$, which equals the sum

$$\sum_j \binom{r+mq-1-(p-1)j}{s-pj} \Sigma^n w_{2(r+mn)-1+2(s-pj)(p-1)} \otimes (\Sigma^{-n}\,\mathrm{P}^j(a))^{\otimes p}$$

(given by [9, II.5.5(ii)]), to

$$\sum_j \binom{r+mq-1-(p-1)j}{s-pj} \frac{(-1)^q}{\nu(q-n)} \cdot xy^{r+(s-j)(p-1)+mq-1} \otimes \mathrm{P}^j(a)\,.$$

These expressions are obviously equal. □

5.2. The relationship between the Tate and Singer constructions. We now show that the inverse system (5.1) for $G = C_p$ and B a bounded below spectrum can be realized, up to a single suspension, as the Tate tower (4.7) for a C_p-spectrum $X = B^{\wedge p}$.

As a naive C_p-spectrum, $B^{\wedge p}$ is equivalent to the p-fold smash product $B \wedge \cdots \wedge B$, with the C_p-action given by cyclic permutation of the factors. The genuinely equivariant definition of $B^{\wedge p}$ is obtained by specialization from Bökstedt's definition in [6], [4] of the topological Hochschild homology $\mathrm{THH}(B)$ of a symmetric ring spectrum B, in the sense of [17]. Namely, $B^{\wedge p} = \mathrm{sd}_p\,\mathrm{THH}(B)_0 = \mathrm{THH}(B)_{p-1}$ equals the 0-simplices of the p-fold edgewise subdivision of $\mathrm{THH}(B)$, which in turn equals the $(p-1)$-simplices of $\mathrm{THH}(B)$.

The ring structure on B is only relevant for the simplicial structure on $\mathrm{THH}(B)$, and is not needed for the formation of its $(p-1)$-simplices. However, it is necessary to assume that the spectrum B is realized as a symmetric spectrum. We now make a review of definitions, in order to compare the Bökstedt-style smash powers of symmetric spectra with the external powers of Lewis–May spectra.

From now on, let $\mathscr{U}$ be the complete C_p-universe

$$\mathscr{U} = \mathbb{R}^\infty \oplus \cdots \oplus \mathbb{R}^\infty = (\mathbb{R}^\infty)^p$$

with C_p-action given by cyclic permutation of summands. The inclusion $i\colon \mathbb{R}^\infty \to \mathscr{U}$ is the diagonal embedding $\Delta\colon \mathbb{R}^\infty \to (\mathbb{R}^\infty)^p$. Let $\mathfrak{A} = \{\mathbb{R}^n \mid n \ge 0\}$ be the sequential indexing set in $\mathbb{R}^\infty$, and let $\mathfrak{A}^p = \{\mathbb{R}^n \oplus \cdots \oplus \mathbb{R}^n = (\mathbb{R}^n)^p \mid n \ge 0\}$ be the associated diagonal indexing set [18, §VI.5] in $\mathscr{U}$. Recall that an equivariant prespectrum D is Σ-*cofibrant* in the sense of [18, I.8.7], hence *good* in the sense of Hesselholt and Madsen [15, Def. A.1], if each structure map $\Sigma^{W-V} D(V) \to D(W)$ is an equivariant cofibration for $V \subseteq W$ in the indexing set. There is a functorial thickening D^τ of equivariant prespectra, which produces Σ-cofibrant, hence good, prespectra, and there is a natural spacewise equivariant equivalence $D^\tau \to D$, see [15, Lem. A.1].

Let B be a symmetric spectrum of topological spaces, with n-th space B_n for each $n \ge 0$. Recall that B is S-cofibrant in the sense of [17, 5.3.6] if the natural map $\nu_n\colon L_n B \to B$ is a cofibration for each n, where the latching space $L_n B$ is the n-th space in the spectrum $B \wedge \bar{S}$. Here $\bar{S}$ is the symmetric spectrum with 0-th space $*$ and n-th space S^n, for $n > 0$. We prefer to follow the terminology in [24, III.1.2] and refer to the S-cofibrant symmetric spectra as being *flat*. Every symmetric spectrum is level equivalent, hence stably equivalent, to a flat symmetric spectrum, so any spectrum may be modeled by a flat symmetric spectrum. Each symmetric spectrum B has an underlying sequential prespectrum indexed on $\mathfrak{A}$, with $B(\mathbb{R}^n) = B_n$ equal to the n-th space of B. The structure map $\sigma\colon B(\mathbb{R}^{n-1}) \wedge S^1 \to B(\mathbb{R}^n)$ factors as $B_{n-1} \wedge S^1 \xrightarrow{\iota_n} L_n B \xrightarrow{\nu_n} B_n$, where ι_n is always a cofibration. Hence the underlying prespectrum of a flat symmetric spectrum is Σ-cofibrant.

Definition 5.3. Let B be a symmetric spectrum, with n-th space B_n for each $n \ge 0$, and let I be Bökstedt's category of finite sets $\mathbf{n} = \{1, 2, \dots, n\}$ for $n \ge 0$ and injective functions. Let $B^{\wedge p}_{\mathrm{pre}}$ be the C_p-equivariant prespectrum with V-th space

$$B^{\wedge p}_{\mathrm{pre}}(V) = \operatorname*{hocolim}_{(\mathbf{n}_1,\dots,\mathbf{n}_p)\in I^p} \mathrm{Map}(S^{n_1} \wedge \dots \wedge S^{n_p}, B_{n_1} \wedge \dots \wedge B_{n_p} \wedge S^V)$$

for each finite dimensional $V \subset \mathscr{U}$. Here C_p acts by cyclically permuting the $\mathbf{n}_i$, the S^{n_i} and the B_{n_i}, as well as acting on S^V. Let

$$B^{\wedge p} = L((B^{\wedge p}_{\mathrm{pre}})^\tau)$$

be the genuine C_p-spectrum in $C_p\mathscr{S}\mathscr{U}$ obtained by spectrification from the functorial good thickening $(B^{\wedge p}_{\mathrm{pre}})^\tau$ of this prespectrum. The natural maps

$$B^{\wedge p}_{\mathrm{pre}}(V) \xleftarrow{\simeq} (B^{\wedge p}_{\mathrm{pre}})^\tau(V) \xrightarrow{\simeq} B^{\wedge p}(V)$$

are C_p-equivariant equivalences by the proof of [15, Prop. 2.4].

Definition 5.4. Let B be a prespectrum indexed on $\mathfrak{A}$. The p-fold external smash product $B^{(p)}$ is the spectrification in $C_p\mathscr{S}\mathscr{U}$ of the C_p-equivariant prespectrum

$$B^{(p)}_{\mathrm{pre}}((\mathbb{R}^n)^p) = B(\mathbb{R}^n) \wedge \dots \wedge B(\mathbb{R}^n) = B(\mathbb{R}^n)^{\wedge p}$$

indexed on $\mathfrak{A}^p$. When B is Σ-cofibrant, so is $B^{(p)}_{\mathrm{pre}}$, hence the V-th space of the spectrification is given by the colimit

$$B^{(p)}(V) = \operatorname*{colim}_{V \subseteq (\mathbb{R}^n)^p} \mathrm{Map}(S^{(\mathbb{R}^n)^p - V}, B(\mathbb{R}^n)^{\wedge p}) .$$

Here the colimit runs over the $n \in \mathbb{N}_0$ such that $V \subseteq (\mathbb{R}^n)^p$, and $(\mathbb{R}^n)^p - V$ denotes the orthogonal complement of V in $(\mathbb{R}^n)^p$. Suspension by V induces an isomorphism

$$B^{(p)}(V) \cong \operatorname*{colim}_{n\in\mathbb{N}_0} \mathrm{Map}((S^n)^{\wedge p}, B(\mathbb{R}^n)^{\wedge p} \wedge S^V) ,$$

with inverse given by suspension by $(\mathbb{R}^m)^p - V$, followed by p instances of the stabilization $B(\mathbb{R}^n) \wedge S^m \to B(\mathbb{R}^{n+m})$, for m sufficiently large.

We say that a symmetric spectrum B is *convergent* if there are integers $\lambda(n)$ that grow to infinity with n, such that B_n is $((n/2)+\lambda(n))$-connected and the structure map $\Sigma B_n \to B_{n+1}$ is $(n+\lambda(n))$-connected, for all sufficiently large n. These hypotheses suffice for the use of Bökstedt's approximation lemma in the following proof. Every symmetric spectrum is stably equivalent to a convergent one.

Lemma 5.5. *Let B be a flat, convergent symmetric spectrum. There is a natural chain of weak equivalences of naive C_p-spectra*

$$i^*B^{(p)} \simeq i^*B^{\wedge p}\,.$$

Proof. For every finite dimensional $V \subset \mathbb{R}^\infty$ we have a natural chain of C_p-equivariant maps

$$\begin{array}{c}
\operatorname*{colim}_{n\in\mathbb{N}_0} \operatorname{Map}((S^n)^{\wedge p}, B(\mathbb{R}^n)^{\wedge p} \wedge S^V) \\
\big\uparrow \simeq \\
\operatorname*{hocolim}_{n\in\mathbb{N}_0} \operatorname{Map}((S^n)^{\wedge p}, B(\mathbb{R}^n)^{\wedge p} \wedge S^V) \\
\big\downarrow \simeq \\
\operatorname*{hocolim}_{(n_1,\dots,n_p)\in\mathbb{N}_0^p} \operatorname{Map}(S^{n_1}\wedge\dots\wedge S^{n_p}, B_{n_1}\wedge\dots\wedge B_{n_p}\wedge S^V) \\
\big\downarrow \simeq \\
\operatorname*{hocolim}_{(\mathbf{n}_1,\dots,\mathbf{n}_p)\in I^p} \operatorname{Map}(S^{n_1}\wedge\dots\wedge S^{n_p}, B_{n_1}\wedge\dots\wedge B_{n_p}\wedge S^V)
\end{array}$$

connecting $B^{(p)}(V)$ to $B^{\wedge p}_{\text{pre}}(V)$. The upper map is a weak equivalence because B is flat, hence Σ-cofibrant. The middle map is a weak homotopy equivalence because the diagonal $\mathbb{N}_0 \to \mathbb{N}_0^p$ is (co-)final. The lower map is a weak equivalence by convergence, the fact that $\mathbb{N}_0^p$ is filtering, and Bökstedt's approximation lemma [6, 1.5], see [8, 2.5.1] for a published proof. Applying the thickening construction and spectrifying, we get the desired chain of naively C_p-equivariant weak equivalences. □

Definition 5.6. For $p=2$, let $\mathbb{R}(1)$ be the sign representation of C_2. For p odd, let $\mathbb{C}(1)$ be the standard rank 1 representation of $C_p \subset S^1$, and let $\mathbb{C}(i)$ be its i-th tensor power. For all primes p, let $W \subset \mathbb{R}^p$ be the orthogonal complement of the diagonal copy of $\mathbb{R}$. Then $W \cong \mathbb{R}(1)$ for $p=2$, and $W \cong \mathbb{C}(1)\oplus\cdots\oplus\mathbb{C}(m)$ for p odd, where $m=(p-1)/2$. Let $EC_p = S(\infty W)$, $\widetilde{EC}_p = S^{\infty W}$, and give EC_p a C_p-CW structure so that

$$\widetilde{EC}_p^{((p-1)n)} = S^{nW}$$

for all $n \geq 0$. Then $\widetilde{E}_{(p-1)n} = S^{nW}$ for all integers n.

Proposition 5.7. *Let B be a flat and convergent symmetric spectrum, and give EC_p a free C_p-CW structure as in the definition above. There is a natural weak equivalence*

$$(B^{\wedge p})_{hC_p} = (EC_{p+} \wedge i^* B^{\wedge p})/C_p \simeq EC_p \ltimes_{C_p} B^{(p)} = D_{C_p}(B)\,.$$

More generally, there are weak equivalences

$$(B^{\wedge p})^{tC_p}[1-(p-1)n] \simeq \Sigma^{1+n} D_{C_p}(\Sigma^{-n} B)$$

for all $n \geq 0$, which are compatible with the $(p-1)$-fold composites of maps in the Tate tower (4.7) *for $X = B^{\wedge p}$*

$$\dots \longrightarrow (B^{\wedge p})^{tC_p}[1-(p-1)(n+1)] \longrightarrow (B^{\wedge p})^{tC_p}[1-(p-1)n] \longrightarrow \dots\,,$$

and the suspension of the inverse system (5.1) *for $G = C_p$*

$$\dots \longrightarrow \Sigma^{1+n+1} D_{C_p}(\Sigma^{-n-1} B) \longrightarrow \Sigma^{1+n} D_{C_p}(\Sigma^{-n} B) \longrightarrow \dots\,.$$

Proof. By the untwisting theorem [18, VI.1.17] and Lemma 5.5 there are C_p-equivariant equivalences

$$EC_p \ltimes B^{(p)} \simeq EC_{p+} \wedge i^* B^{(p)} \simeq EC_{p+} \wedge i^* B^{\wedge p}\,,$$

since EC_p is C_p-free. The first claim follows by passage to C_p-orbit spectra. More generally, there are C_p-equivariant equivalences

$$\Sigma^{1+n} EC_p \ltimes (\Sigma^{-n} B)^{(p)} \simeq \Sigma F(S^{nW}, EC_p \ltimes B^{(p)}) \simeq \Sigma F(S^{nW}, EC_{p+} \wedge i^* B^{\wedge p})$$

by [18, VI.1.5], since $S^n \wedge S^{nW} \cong (S^n)^{\wedge p}$. Passing to C_p-orbits, and using the Adams transfer equivalence (4.2), we get the equivalences

$$\Sigma^{1+n} D_{C_p}(\Sigma^{-n} B) \simeq \Sigma F(S^{nW}, EC_{p+} \wedge B^{\wedge p})^{C_p}\,.$$

The right hand side is a model for $(B^{\wedge p})^{tC_p}[1-(p-1)n]$, by Lemma 4.10, since $\widetilde{E}_{(p-1)n} = S^{nW}$. The stabilization of the left hand side given by $\Delta\colon S^1 \to S^p$ is compatible under all of these equivalences with the stabilization of the right hand side given by the inclusion $S^0 \to S^W$, again by [18, VI.1.5]. □

Definition 5.8. For each symmetric spectrum B let the *topological Singer construction* on B be the spectrum

$$R_+(B) = (B^{\wedge p})^{tC_p}\,.$$

The topological Singer construction realizes the algebraic Singer constructions, in the following sense.

Theorem 5.9. *Let B be a symmetric spectrum that is bounded below and of finite type over $\mathbb{F}_p$. There are natural isomorphisms*

$$\omega\colon H_c^*(R_+(B)) \xrightarrow{\cong} R_+(H^*(B))$$

and

$$\omega_*\colon R_+(H_*(B)) \xrightarrow{\cong} H_*^c(R_+(B))$$

of $\mathscr{A}$-modules and complete $\mathscr{A}_*$-comodules, respectively.

Proof. We may replace B by a stably equivalent flat and convergent symmetric spectrum, without changing the (co-)homology of $B^{\wedge p}$ and $R_+(B)$. By Proposition 5.7 there are $\mathscr{A}$-module isomorphisms

$$H^*((B^{\wedge p})^{tC_p}[1-(p-1)n]) \cong \Sigma^{1+n} H^*(D_{C_p}(\Sigma^{-n} B))$$

for each n, which by Theorem 5.2 induce $\mathscr{A}$-module isomorphisms

$$H^*_c((B^{\wedge p})^{tC_p}) \cong R_+(H^*(B))$$

after passage to colimits. The dual $\mathscr{A}_*$-comodule isomorphisms then induce complete $\mathscr{A}_*$-comodule isomorphisms

$$H_*^c((B^{\wedge p})^{tC_p}) \cong \mathrm{Hom}(R_+(H^*(B)), \mathbb{F}_p) = R_+(H_*(B))$$

after passage to limits. □

5.3. The topological Singer ϵ-map. We now turn to the construction of a stable map $\epsilon_B \colon B \to R_+(B)$ that realizes the Singer homomorphism ϵ on passage to cohomology. Having done so will enable us to prove Theorem 5.13. Let B be a symmetric spectrum that is bounded below and of finite type over $\mathbb{F}_p$. The C_p-spectrum $X = B^{\wedge p}$ introduced in Definition 5.3 is then also bounded below and of finite type over $\mathbb{F}_p$. We shall make use of parts of the Hesselholt–Madsen proof that $\mathrm{THH}(B)$ is a *cyclotomic spectrum*. By the first half of the proof of [15, Prop. 2.1] there is a natural equivalence

$$\bar{s}_{C_p} \colon [\widetilde{EC}_p \wedge B^{\wedge p}]^{C_p} \xrightarrow{\simeq} \Phi^{C_p}(B^{\wedge p}) \,,$$

where $\Phi^{C_p}(X)$ denotes the *geometric fixed point spectrum* of X. Furthermore, by the simplicial degree $k = p-1$ part of the proof of [15, Prop. 2.5] there is a natural equivalence

$$r'_{C_p} \colon \Phi^{C_p}(B^{\wedge p}) \xrightarrow{\simeq} B \,.$$

If B is a ring spectrum, both of these equivalences are ring spectrum maps. We abuse notation and write R also for the composite map

$$(B^{\wedge p})^{C_p} \xrightarrow{R} [\widetilde{EC}_p \wedge B^{\wedge p}]^{C_p} \xrightarrow{\simeq} \Phi^{C_p}(B^{\wedge p}) \xrightarrow{\simeq} B \,,$$

which corresponds (at the level of 0-simplices) to the restriction map $R\colon \mathrm{sd}_p \mathrm{THH}(B)^{C_p} \to \mathrm{THH}(B)$ of [4] and [15].

Definition 5.10. Let $\epsilon_B \colon B \to R_+(B)$ be the natural stable map given by the composite

$$B \xleftarrow{\simeq} \Phi^{C_p}(B^{\wedge p}) \xleftarrow{\simeq} [\widetilde{EC}_p \wedge B^{\wedge p}]^{C_p} \xrightarrow{\hat{\Gamma}} (B^{\wedge p})^{tC_p} = R_+(B) \,.$$

If B is a ring spectrum then ϵ_B is a ring spectrum map.

With this notation we can rewrite the homotopy cartesian square in (4.3) for $X = B^{\wedge p}$ as follows:

(5.4)
$$\begin{tikzcd} (B^{\wedge p})^{C_p} \arrow[r, "R"] \arrow[d, "\Gamma"'] & B \arrow[d, "\epsilon_B"] \\ (B^{\wedge p})^{hC_p} \arrow[r, "R^h"] & R_+(B) \end{tikzcd}$$

We thank M. Bökstedt for a helpful discussion on the following two results.

Lemma 5.11. *The stable map $\epsilon_B \colon B \to R_+(B)$ commutes with suspension, in the sense that $\epsilon_{\Sigma B} = \Delta\Sigma\epsilon_B$.*

Proof. Consider the commutative diagram:

$$\begin{tikzcd} \Sigma B \arrow[d, "="'] & \Sigma[\widetilde{EC}_p \wedge B^{\wedge p}]^{C_p} \arrow[l, "\simeq"'] \arrow[r, "\Sigma\hat{\Gamma}"] \arrow[d, "\Delta"] & \Sigma R_+(B) \arrow[d, "\Delta"] \\ \Sigma B & {[\widetilde{EC}_p \wedge (\Sigma B)^{\wedge p}]^{C_p}} \arrow[l, "\simeq"'] \arrow[r, "\hat{\Gamma}"] & R_+(\Sigma B) \end{tikzcd}$$

The vertical maps labeled Δ are induced by the diagonal inclusion $S^1 \to S^p$, which on the right hand side is the same map as was used in the interpretation (Proposition 5.7) of $R_+(B)$ as the inverse system of suspended extended power constructions. Hence these maps are weak equivalences. □

Proposition 5.12. *Let B be a bounded below spectrum of finite type over $\mathbb{F}_p$. Then the homomorphism*

$$(\epsilon_B)^* \colon H^*_c(R_+(B)) \longrightarrow H^*(B)$$

induced on continuous cohomology by the spectrum map $\epsilon_B \colon B \to R_+(B)$ is equal to Singer's homomorphism

$$\epsilon_{H^*(B)} \colon R_+(H^*(B)) \longrightarrow H^*(B)$$

associated to the $\mathscr{A}$-module $H^*(B)$, via the identification $\omega : H^*_c(R_+(B)) \cong R_+(H^*(B))$ of Theorem 5.9.

Proof. By Corollary 3.5 there is a unique $\mathscr{A}$-module homomorphism $g_B \colon H^*(B) \to H^*(B)$ that makes the square

$$\begin{tikzcd} R_+(H^*(B)) \arrow[d, "\epsilon_{H^*(B)}"'] & H^*_c(R_+(B)) \arrow[l, "\omega"', "\cong"] \arrow[d, "(\epsilon_B)^*"] \\ H^*(B) \arrow[r, "g_B"] & H^*(B) \end{tikzcd}$$

commute. We must show that g_B equals the identity.

First consider the case $B = H$. The homological Tate spectral sequence

$$\widehat{E}^2_{*,*}(H) = \widehat{H}^{-*}(C_p; H_*(H)^{\otimes p}) \Longrightarrow H^c_*(R_+(H))$$

is an algebra spectral sequence (Proposition 4.17), and $\epsilon_H \colon H \to R_+(H)$ is a ring spectrum map, so the image of the unit $1 \in H_0(H)$ under $(\epsilon_H)_*$ is represented by the unit $1 \otimes 1^{\otimes p}$ in $\widehat{H}^0(C_p; H_0(H)^{\otimes p})$. Hence $(\epsilon_H)^*$ maps the dual class represented by $1 \otimes 1^{\otimes p}$ in $\widehat{H}_0(C_p; H^0(H)^{\otimes p})$ to $1 \in H^0(H)$. Now $\omega(1 \otimes 1^{\otimes p}) = \Sigma xy^{-1} \otimes 1 \in R_+(\mathscr{A})$ and $\epsilon_{\mathscr{A}}(\Sigma xy^{-1} \otimes 1) = 1 \in \mathscr{A}$, so $g_H(1) = 1$. (We replace xy^{-1} by x^{-1} for $p = 2$.) Since g_H is an $\mathscr{A}$-module homomorphism, it must be equal to the identity morphism of $\mathscr{A} = H^*(H)$.

The case $B = \Sigma^n H$ then follows by Lemma 5.11.

In the general case any element in $H^n(B)$ is represented by a map $f \colon B \to \Sigma^n H$, which induces an $\mathscr{A}$-module homomorphism $f^* \colon \Sigma^n \mathscr{A} \to H^*(B)$. By naturality of the isomorphism ω, the Singer homomorphism ϵ, and the spectrum map ϵ, we get a diagram

$$\begin{tikzcd}
R_+(\Sigma^n \mathscr{A}) \arrow[dr, "R_+(f^*)"] \arrow[ddd, "\epsilon_{\Sigma^n \mathscr{A}}"'] & & & H^*_c(R_+(\Sigma^n H)) \arrow[lll, "\omega"', "\cong"] \arrow[dl, "R_+(f)^*"'] \arrow[ddd, "(\epsilon_{\Sigma^n H})^*"] \\
& R_+(H^*(B)) \arrow[d, "\epsilon_{H^*(B)}"'] & H^*_c(R_+(B)) \arrow[l, "\omega"', "\cong"] \arrow[d, "(\epsilon_B)^*"] & \\
& H^*(B) \arrow[r, "g_B"'] & H^*(B) & \\
\Sigma^n \mathscr{A} \arrow[ur, "f^*"] \arrow[rrr, "="] & & & H^*(\Sigma^n H) \arrow[ul, "f^*"']
\end{tikzcd}$$

where the left hand, upper and right hand trapezoids all commute. The inner square commutes by construction, and the outer square commutes by the case $B = \Sigma^n H$. Since $\epsilon_{\Sigma^n \mathscr{A}}$ is surjective, it follows that the lower trapezoid also commutes. Hence g_B equals the identity on the class $f^*(\Sigma^n 1) \in H^n(B)$. Since n and f were arbitrary, this proves that g_B equals the identity on all of $H^*(B)$. □

The following theorem generalizes the Segal conjecture for C_p.

Theorem 5.13. *Let B be a bounded below spectrum of finite type over $\mathbb{F}_p$. Then the natural maps*

$$\epsilon_B \colon B \longrightarrow R_+(B) = (B^{\wedge p})^{tC_p}$$

and

$$\Gamma \colon (B^{\wedge p})^{C_p} \longrightarrow (B^{\wedge p})^{hC_p}$$

are p-adic equivalences of spectra.

Proof. The map ϵ_B induces a map of spectral sequences

$$E_2^{*,*}(B) = \mathrm{Ext}^{*,*}_{\mathscr{A}}(H^*(B), \mathbb{F}_p) \longrightarrow \mathrm{Ext}^{*,*}_{\mathscr{A}}(H^*_c(R_+(B)), \mathbb{F}_p) = E_2^{*,*}(R_+(B))$$

where the first is the Adams spectral sequence of B, and the second is the inverse limit of Adams spectral sequences associated to the Tate tower

$\{(B^{\wedge p})^{tC_p}[n]\}_n$, as in Proposition 2.2. The map converges strongly to the homomorphism

$$\pi_*(\epsilon_B)^\wedge_p \colon \pi_*(B^\wedge_p) \longrightarrow \pi_*(R_+(B)^\wedge_p)\,.$$

By Propositions 5.12 and 3.3 and Theorem 3.4, the map of E_2-terms is an isomorphism, hence so is the map of E_∞-terms and of the abutments. In other words, ϵ_B is a p-adic equivalence. The corresponding assertion for Γ follows immediately, since diagram (5.4) is homotopy cartesian. □

5.4. The Tate spectral sequence for the topological Singer construction. We conclude by relating the homological Tate spectral sequence for $B^{\wedge p}$ to the Tate filtration on the homological Singer construction for $H_*(B)$, and likewise in cohomology.

Proposition 5.14. *Let B be a bounded below spectrum of finite type over $\mathbb{F}_p$. The homological Tate spectral sequence*

$$\widehat{E}^2_{*,*} = \widehat{H}^{-*}(C_p; H_*(B)^{\otimes p}) \Longrightarrow H^c_*((B^{\wedge p})^{tC_p})$$

converging to $H^c_*(R_+(B)) \cong R_+(H_*(B))$ collapses at the $\widehat{E}^2$-term. Hence the $\widehat{E}^2 = \widehat{E}^\infty$-term is given by

$$\widehat{E}^\infty_{*,*} = P(u, u^{-1}) \otimes \mathbb{F}_2\{\alpha^{\otimes 2}\}$$

for $p = 2$, and by

$$\widehat{E}^\infty_{*,*} = E(u) \otimes P(t, t^{-1}) \otimes \mathbb{F}_p\{\alpha^{\otimes p}\}$$

for p odd. In each case α runs through an $\mathbb{F}_p$-basis for $H_*(B)$.
For $p = 2$ and any $r \in \mathbb{Z}$, $\alpha \in H_q(B)$, the element $u^r \otimes \alpha \in R_+(H_*(B))$ is represented in the Tate spectral sequence by

$$u^{r+q} \otimes \alpha^{\otimes 2} \in \widehat{E}^\infty_{-r-q,2q}\,.$$

For p odd and any $i \in \{0, 1\}$, $r \in \mathbb{Z}$ and $\alpha \in H_q(B)$, the element $u^i t^r \otimes \alpha \in R_+(H_*(B))$ is represented in the Tate spectral sequence by

$$\frac{(-1)^q}{\nu(q)} \cdot u^i t^{r+mq} \otimes \alpha^{\otimes p} \in \widehat{E}^\infty_{-i-2r-(p-1)q,pq}\,,$$

where $m = (p-1)/2$ and $\nu(2j+\epsilon) = (-1)^j (m!)^\epsilon$ for $\epsilon \in \{0, 1\}$.

Proof. Consider first the case $B = S^q$. Then the result is trivial for dimensional reasons. The $\widehat{E}^2$-term is concentrated in bidegrees $(*, pq)$, so there is no room for differentials, and there are no extension problems to be solved. The formula for the homological Tate spectral sequence representative for $u^i t^r \otimes \alpha$ will follow by dualization from the cohomological case, given below.
Let H be a model for the mod p Eilenberg–Mac Lane spectrum as a commutative symmetric ring spectrum, and form the spectrum $H^{\wedge p}$ in $C_p\mathscr{S}\mathscr{U}$ as in Definition 5.3. The iterated multiplication on H then induces a naively C_p-equivariant map $H^{\wedge p} \to H$. Let $f \colon S^q \to H \wedge B$ represent a class $\alpha \in H_q(B)$, and consider the naively C_p-equivariant composite map

$$f^p \colon H \wedge S^{pq} \longrightarrow H \wedge (H \wedge B)^{\wedge p} \simeq H \wedge H^{\wedge p} \wedge B^{\wedge p} \longrightarrow H \wedge H \wedge B^{\wedge p} \longrightarrow H \wedge B^{\wedge p}\,.$$

On homotopy groups it induces the homomorphism $H_*(S^q)^{\otimes p} \to H_*(B)^{\otimes p}$ that takes $\iota_q^{\otimes p}$ to $\alpha^{\otimes p}$, where $\iota_q = \Sigma^q 1$ is the fundamental class in $H_q(S^q)$.
By applying the Tate construction to this map, we get a map of spectra

$$(f^p)^{tC_p} \colon (H \wedge S^{pq})^{tC_p} \longrightarrow (H \wedge B^{\wedge p})^{tC_p}$$

and an associated map of homotopical Tate spectral sequences, converging strongly to a homomorphism $\pi_*(H \wedge S^{pq})^{tC_p} \to \pi_*(H \wedge B^{\wedge p})^{tC_p}$. We can rewrite the latter as an $\mathbb{F}_p$-linear map

$$H_*^c(R_+(S^q)) \longrightarrow H_*^c(R_+(B)) \tag{5.5}$$

by Proposition 4.16. For p odd, this map is given at the level of $\widehat{E}^2$-terms as sending $u^i t^r \otimes \iota_q^{\otimes p}$ to $u^i t^r \otimes \alpha^{\otimes p}$, and similarly for $p = 2$. The statement of the proposition then follows by naturality. □

Note that the $\mathbb{F}_p$-linear map (5.5) is not a homomorphism of $\mathscr{A}_*$-comodules in general, because f^p is not of the form $1_H \wedge (-)$.

Proposition 5.15. *Let B be a bounded below spectrum of finite type over $\mathbb{F}_p$. The cohomological Tate spectral sequence*

$$\widehat{E}_2^{*,*} = \widehat{H}_{-*}(C_p; H^*(B)^{\otimes p}) \Longrightarrow H_c^*((B^{\wedge p})^{tC_p})$$

converging to $H_c^*(R_+(B)) \cong R_+(H^*(B))$ collapses at the $\widehat{E}_2$-term, so that

$$\widehat{E}_\infty^{*,*} = \Sigma P(x, x^{-1}) \otimes \mathbb{F}_2\{a^{\otimes 2}\}$$

for $p = 2$, and

$$\widehat{E}_\infty^{*,*} = \Sigma E(x) \otimes P(y, y^{-1}) \otimes \mathbb{F}_p\{a^{\otimes p}\}$$

for p odd. In each case a runs through an $\mathbb{F}_p$-basis for $H^*(B)$.
For $p = 2$ and any $r \in \mathbb{Z}$, $a \in H^q(B)$, the element $\Sigma x^r \otimes \alpha \in R_+(H^*(B))$ is represented in the Tate spectral sequence by

$$\Sigma x^{r-q} \otimes a^{\otimes 2} \in \widehat{E}_\infty^{1+r-q,2q} .$$

For p odd and any $i \in \{0, 1\}$, $r \in \mathbb{Z}$ and $a \in H^q(B)$, the element $\Sigma x^i y^r \otimes a \in R_+(H^*(B))$ is represented in the Tate spectral sequence by

$$(-1)^q \nu(q) \cdot \Sigma x^i y^{r-mq} \otimes a^{\otimes p} \in \widehat{E}_\infty^{1+i+2r-(p-1)q,pq} .$$

Proof. This follows by dualization from the homological case. In the special case $B = S^q$, $\Sigma x^i y^r \otimes a^{\otimes p} \in H_c^*(R_+(B)) \cong \widehat{H}_{-*}(C_p; H^*(S^q)^{\otimes p})$ is represented by $(-1)^q \nu(q)^{-1} \cdot \Sigma x^i y^{r+mq} \otimes a \in R_+(H^*(S^q))$, by the explicit isomorphism given in the proof of Theorem 5.2. The formula for the cohomological Tate spectral sequence representative follows. □

Corollary 5.16. *The Tate filtration*

$$\{F^n R_+(H^*(B))\}_n$$

of the Singer construction $R_+(H^*(B))$ corresponds, under the isomorphism $R_+(H^*(B)) \cong H_c^*(R_+(B))$, to the Boardman filtration

$$\{F^n H_c^*(R_+(B))\}_n$$

of $H^*_c(R_+(B))$.

Proof. For each integer n, the Boardman filtration $F^n H^*_c(R_+(B))$ equals the image of $H^*((B^{\wedge p})^{tC_p}[n])$ in $H^*((B^{\wedge p})^{tC_p})$, which is the part of $H^*((B^{\wedge p})^{tC_p})$ represented in filtrations $\geq n$ at the $\widehat{E}_\infty$-term. This corresponds to the part of the Singer construction $H^*(R_+(B))$ spanned by the monomials $\Sigma x^r \otimes a$ with $1+r-q \geq n$ for $p=2$, and by the monomials $\Sigma x^i y^r \otimes a$ with $1+i+2r-(p-1)q \geq n$ for p odd, which precisely equals the n-th term $F^n R_+(H^*(B))$ of the Tate filtration, as defined in §3.2. □

References

[1] J. F. Adams, J. H. Gunawardena, and H. Miller, *The Segal conjecture for elementary abelian p-groups*, Topology 24 (1985), no. 4, 435–460.

[2] Ch. Ausoni and J. Rognes, *Algebraic K-theory of topological K-theory*, Acta Math. 188 (2002), no. 1, 1–39.

[3] J. M. Boardman, *Conditionally convergent spectral sequences*, Homotopy invariant algebraic structures (Baltimore, MD, 1998), Contemp. Math., vol. 239, Amer. Math. Soc., Providence, RI, 1999, pp. 49–84.

[4] M. Bökstedt, W. C. Hsiang, and I. Madsen, *The cyclotomic trace and algebraic K-theory of spaces*, Invent. Math. 111 (1993), no. 3, 465–539.

[5] M. Bökstedt and I. Madsen, *Topological cyclic homology of the integers*, Astérisque 226 (1994), 7–8, 57–143. K-theory (Strasbourg, 1992).

[6] M. Bökstedt, *Topological Hochschild homology* (ca. 1986). Bielefeld preprint.

[7] M. Bökstedt, R. R. Bruner, S. Lunøe-Nielsen, and J. Rognes, *On cyclic fixed points of spectra* (2007). arXiv:0712.3476 preprint.

[8] M. Brun, *Topological Hochschild homology of* $\mathbf{Z}/p^n$, J. Pure Appl. Algebra 148 (2000), no. 1, 29–76.

[9] R. R. Bruner, J. P. May, J. E. McClure, and M. Steinberger, H_∞ *ring spectra and their applications*, Lecture Notes in Mathematics, vol. 1176, Springer-Verlag, Berlin, 1986.

[10] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton University Press, Princeton, N. J., 1956.

[11] J. Caruso, J. P. May, and S. B. Priddy, *The Segal conjecture for elementary abelian p-groups. II. p-adic completion in equivariant cohomology*, Topology 26 (1987), no. 4, 413–433.

[12] J. P. C. Greenlees, *Representing Tate cohomology of G-spaces*, Proc. Edinburgh Math. Soc. (2) 30 (1987), no. 3, 435–443.

[13] ———, *Tate cohomology in commutative algebra*, J. Pure Appl. Algebra 94 (1994), no. 1, 59–83.

[14] J. P. C. Greenlees and J. P. May, *Generalized Tate cohomology*, Mem. Amer. Math. Soc. 113 (1995), no. 543, viii+178.

[15] L. Hesselholt and I. Madsen, *On the K-theory of finite algebras over Witt vectors of perfect fields*, Topology 36 (1997), no. 1, 29–101.

[16] ———, *On the K-theory of local fields*, Ann. of Math. (2) 158 (2003), no. 1, 1–113.

[17] M. Hovey, B. Shipley, and J. Smith, *Symmetric spectra*, J. Amer. Math. Soc. 13 (2000), no. 1, 149–208.

[18] L. G. Lewis Jr., J. P. May, and M. Steinberger, *Equivariant stable homotopy theory*, Lecture Notes in Mathematics, vol. 1213, Springer-Verlag, Berlin, 1986. With contributions by J. E. McClure.

[19] H. H. Li and W. M. Singer, *Resolutions of modules over the Steenrod algebra and the classical theory of invariants*, Math. Z. 181 (1982), no. 2, 269–286.

[20] W. H. Lin, D. M. Davis, M. E. Mahowald, and J. F. Adams, *Calculation of Lin's Ext groups*, Math. Proc. Cambridge Philos. Soc. 87 (1980), no. 3, 459–469.

[21] S. Lunøe-Nielsen, *The Segal conjecture for topological Hochschild homology of commutative S-algebras*, PhD dissertation, University of Oslo, Norway, 2005.

[22] S. Lunøe-Nielsen and J. Rognes, *The Segal conjecture for topological Hochschild homology of complex cobordism*, Journal of Topology 4 (2011), no. 3, 591-622.

[23] J. Rognes, *Topological cyclic homology of the integers at two*, J. Pure Appl. Algebra 134 (1999), no. 3, 219–286.

[24] S. Schwede, *An untitled book project about symmetric spectra*, 2007. http://www.math.uni-bonn.de/people/schwede/SymSpec.pdf (in preparation).

[25] W. M. Singer, *On the localization of modules over the Steenrod algebra*, J. Pure Appl. Algebra 16 (1980), no. 1, 75–84.

[26] ———, *A new chain complex for the homology of the Steenrod algebra*, Math. Proc. Cambridge Philos. Soc. 90 (1981), no. 2, 279–292.

[27] C. Wilkerson, *Classifying spaces, Steenrod operations and algebraic closure*, Topology 16 (1977), no. 3, 227–237.

Sverre Lunøe–Nielsen
Dept. of Mathematics
University of Oslo
Norway
sverreln@math.uio.no
http://folk.uio.no/sverreln

John Rognes
Dept. of Mathematics
University of Oslo
Norway
rognes@math.uio.no
http://folk.uio.no/rognes