Several Variables p-Adic L-Functions for Hida Families of Hilbert Modular Forms

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Abstract. After formulating Conjecture A for p-adic L-functions defined over ordinary Hilbert modular Hida deformations on a totally real field F of degree d, we construct two p-adic L-functions of $d+1$-variable depending on the parity of weight as a partial result on Conjecture A. We will also state Conjecture B which is a corollary of Conjecture A but is important by itself. Main issues of the construction are the study of Hida theory of Hilbert modular forms by using Hilbert modular varieties (without using Shimura curves), the study of higher dimensional modular symbols on Hilbert modular varieties and delicate treatments on archimedean and p-adic periods.

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Contents

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1. Introduction

1.1. General overview. Let p be an odd prime number fixed throughout the paper. In the spirit of Iwasawa theory, our interest is in constructing and studying p-adic analytic L-functions which interpolate special values of Hecke L-functions of various automorphic forms (or Hasse-Weil L-functions of various motives). According to the philosophy of Iwasawa Main Conjecture, the p-adic analytic L-function is expected to coincide with its algebraic counterpart encoding the behaviour of generalized class groups or Selmer groups.

In the classical situation, we fix an automorphic form f for a certain algebraic group G on a number field F and we study the p-adic analytic L-function $L_p(f)$ interpolating special values of the Hecke L-function $L(f, \phi, s)$ twisted by Hecke characters ϕ on F of p-power order and p-power conductor. Note that, by the class field theory, Hecke characters as above are identified with characters of the Galois group $\mathrm{Gal}(F_{\{p\}}/F)$ where $F_{\{p\}}$ is the maximal abelian pro-p extension of F unramified outside primes above p. The most fundamental case is the case of a totally real number field F, in which case Leopoldt conjecture predicts that $F_{\{p\}}$ is almost equal to the cyclotomic $\mathbb{Z}_p$-extension F_∞ of F. Thus, $L_p(f)$ is an element of the cyclotomic Iwasawa algebra $\mathcal{O}[[\mathrm{Gal}(F_\infty/F)]]$ over the ring of integers $\mathcal{O}$ of a finite extension of $\mathbb{Q}_p$, which is non-canonically isomorphic to a power series in one variable $\mathcal{O}[[T]]$. When $F = \mathbb{Q}$ and $G = \mathbb{G}_m$, the p-adic L-function $L_p(f) \in \mathcal{O}[[\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]]$ of an automorphic form on G (that is a Hecke character of F) is constructed by Kubota-Leopoldt, Iwasawa and Coleman. When $F = \mathbb{Q}$ and $G = \mathrm{GL}_2$, $L_p(f) \in \mathcal{O}[[\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]]$ is constructed by Mazur-Tate-Teitelbaum [MTT]. There are some known work of p-adic L-functions over the cyclotomic Iwasawa algebra $\mathcal{O}[[\mathrm{Gal}(F_\infty/F)]]$ for other algebraic groups G of higher rank. We do not try to give a list of previous work on such constructions.

We are interested in a vast generalization of Iwasawa theory from the theory over the cyclotomic Iwasawa algebra to the theory over the whole algebra $R^{\mathrm{n.o}}_{\overline{\rho}}$ of nearly ordinary Galois deformations (with suitable local conditions) of a given mod p representation $\overline{\rho}$ (see [G] and [Oc2] for the project of an Iwasawa theory over deformation algebras). We are particularly interested in constructing p-adic analytic L-functions $L_p(\overline{\rho}) \in R^{\mathrm{n.o}}_{\overline{\rho}}$. Let $\rho_{f,p}$ be a modular p-adic representation lifting $\overline{\rho}$. By the universal property, $R^{\mathrm{n.o}}_{\overline{\rho}}$ parameterizes in particular twists of $\rho_{f,p}$ by Hecke characters on F of p-power order and p-power conductor. Hence, there is a surjection $R^{\mathrm{n.o}}_{\overline{\rho}} \twoheadrightarrow \mathcal{O}[[\mathrm{Gal}(F_\infty/F)]]$ and we expect that the specialization of $L_p(\overline{\rho}) \in R^{\mathrm{n.o}}_{\overline{\rho}}$ to be the classical cyclotomic p-adic L-function $L_p(f) \in \mathcal{O}[[\mathrm{Gal}(F_\infty/F)]]$. In this way, our project is really a generalization which contains previous work.

When a given mod p representation $\overline{\rho}$ of $\mathrm{Gal}(\overline{F}/F)$ has rank one, $R^{\mathrm{n.o}}_{\overline{\rho}}$ is isomorphic to $\mathcal{O}[[\mathrm{Gal}(F_\infty/F)]]$. Hence, in this case, the theory is the same as the classical cyclotomic theory. The first important new case arises when $\mathrm{rank}(\overline{\rho}) = 2$. In this case, the Krull dimension of $R^{\mathrm{n.o}}_{\overline{\rho}}$ is greater than that of $\mathcal{O}[[\mathrm{Gal}(F_\infty/F)]]$. We have a natural surjection from $R^{\mathrm{n.o}}_{\overline{\rho}}$ to the ($\overline{\rho}$-component of) Hida's nearly ordinary Hecke algebra $\mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}}$ and this surjection is conjectured to be an isomorphism.

Up to now, we took the viewpoint of Galois deformation rings since we believe that this is the appropriate framework for construction of more general p-adic L-functions. However, from now on, we will try to define the p-adic analytic L-function $L_p(\overline{\rho})$ in $\mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}}$ rather than in $R^{\mathrm{n.o}}_{\overline{\rho}}$ since $\mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}}$ is more closely related to the L-values.

For the rest of the introduction, we fix a rank-two mod p representation $\overline{\rho}$ of $\mathrm{Gal}(\overline{F}/F)$ which is modular and nearly ordinary. When $F = \mathbb{Q}$, there is a canonical isomorphism:

$$\mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}} \cong \mathbb{T}^{\mathrm{ord}}_{\overline{\rho}} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[[\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]] = \mathbb{T}^{\mathrm{ord}}_{\overline{\rho}}[[\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]]$$

where $\mathbb{T}^{\mathrm{ord}}_{\overline{\rho}}$ is (the $\overline{\rho}$-component of) nearly ordinary Hecke algebra, which is finite and flat over $\mathcal{O}[[1+p\,\mathbb{Z}_p]]$. The algebras $\mathbb{T}^{\mathrm{ord}}_{\overline{\rho}}$ and $\mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}}$ depend on a certain tame conductor N. However, if there is no confusion, we will omit N in the notations. By Hida's theory, for every p-stabilized elliptic (nearly) ordinary eigen cuspform f of weight $k \geq 2$ and conductor Np^* ($* \in \mathbb{N}$) such that the residual representation $\overline{\rho}_f$ associated to ρ_f is isomorphic to $\overline{\rho}$, there exists a unique $\kappa = \kappa_f : \mathbb{T}^{\mathrm{ord}}_{\overline{\rho}} \longrightarrow \overline{\mathbb{Q}}_p$ such that the q-expansion of f equals $\sum_n \kappa_f(T_n)q^n$, where T_n is the n-th Hecke operator in $\mathbb{T}^{\mathrm{ord}}_{\overline{\rho}}$. Hence, for every pair (f,ϕ) of ordinary p-stabilized eigen cuspform f as above and a character ϕ of $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$, there is an unique $\kappa = \kappa_{f,\phi} : \mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}} \longrightarrow \overline{\mathbb{Q}}_p$ such that the q-expansion of $f \otimes \phi$ equals $\sum_n \kappa_{f,\phi}(T_n)q^n$. The algebra $\mathbb{T}^{\mathrm{ord}}_{\overline{\rho}}$ is a local algebra which might contain zero divisors in general. A quotient $\mathcal{R} = \mathbb{T}^{\mathrm{ord}}_{\overline{\rho}}/\mathfrak{A}$ by a prime ideal $\mathfrak{A}$ of height zero is called a branch of $\mathbb{T}^{\mathrm{ord}}_{\overline{\rho}}$. A branch of $\mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}}$ is defined

exactly in the same way. For a branch $\mathcal{R}$ of $\mathbb{T}^{\mathrm{ord}}_{\overline{\rho}}$, $\mathcal{R}\widehat{\otimes}_{\mathcal{O}}\mathcal{O}[[\mathrm{Gal}(\mathbb{Q}_\infty / \mathbb{Q})]]$ is a branch of $\mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}}$ and this correspondence gives a bijection between the set of branches of $\mathbb{T}^{\mathrm{ord}}_{\overline{\rho}}$ and the set of branches of $\mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}}$.
The p-adic L-function $L_p(f) \in \mathcal{O}[[\mathrm{Gal}(\mathbb{Q}_\infty / \mathbb{Q})]]$ of an elliptic modular form f has been constructed in the 70's and extensively studied by the method of modular symbols. Kitagawa refined Mazur's method of Λ-adic modular symbols which is a family of modular symbols associated to Hida family. We recall the following theorem of [Ki].

THEOREM 1.1 (Kitagawa-Mazur). *Let $\mathcal{R}^{\mathrm{ord}}$ be a branch of $\mathbb{T}^{\mathrm{ord}}_{\overline{\rho}}$. Assume that $\mathcal{R}^{\mathrm{ord}}$ is a Gorenstein ring. Then, there exists $L_p(\mathcal{R}^{\mathrm{ord}}) \in \mathcal{R}^{\mathrm{ord}}[[\mathrm{Gal}(\mathbb{Q}_\infty / \mathbb{Q})]]$ which is characterized by the interpolation property:*

$$\frac{\kappa_{f,\chi^{j-1}\phi}(L_p(\mathcal{R}^{\mathrm{ord}}))}{C_{f,p}} = (j-1)!G(\phi,j)A_p(f)\frac{L(f,\phi\omega^{1-j},j)}{(-2\pi\sqrt{-1})^{j-1}C_{f,\infty}}$$

for every pair $(f,\chi^{j-1}\phi)$ as follows:

- *The form f is a p-stabilized ordinary eigen cuspform $f \in S_k(\Gamma_1(Np^*))$ satisfying $\overline{\rho}_f \cong \overline{\rho}$ such that $\kappa_f : \mathbb{T}^{\mathrm{ord}}_{\overline{\rho}} \longrightarrow \overline{\mathbb{Q}}_p$ factors through $\mathcal{R}$.*
- *The character $\chi^{j-1}\phi$ consists of a finite character ϕ of $\mathrm{Gal}(\mathbb{Q}_\infty / \mathbb{Q})$ and an integer j such that $1 \leq j \leq k-1$.*

Here, the notation in the above equation is as follows.
The element $C_{p,f} \in \overline{\mathbb{Z}}_p^\times / \overline{\mathbb{Z}}_{(p)}^\times$ is a p-adic period for f and $C_{f,\infty} \in \mathbb{C}^\times / \overline{\mathbb{Z}}_{(p)}^\times$ is a complex period for f where $\overline{\mathbb{Z}}_p$ (resp. $\overline{\mathbb{Z}}$) is the ring of integers of $\overline{\mathbb{Q}}_p$ (resp. $\overline{\mathbb{Q}}$) and $\overline{\mathbb{Z}}_{(p)}$ is the localization of $\overline{\mathbb{Z}}$ at the valuation induced by a fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.
The symbol $G(\phi,j)$ is the Gauss sum for the Dirichlet character $(\phi\omega^{a+1-j})^{-1}$ where a is an integer such that $\overline{\rho} \otimes \omega^{-a}$ is ordinary.
The term $A_p(f)$ is given by

$$A_p(f) = \begin{cases} \left(1 - \dfrac{p^{j-1}}{a_p(f \otimes \omega^{-a})}\right) & \textit{if } \phi\omega^{a+1-j} \textit{ is trivial,} \\ \left(\dfrac{p^{j-1}}{a_p(f \otimes \omega^{-a})}\right)^{c(\phi,j)} & \textit{if } \phi\omega^{a+1-j} \textit{ has conductor } p^{c(\phi,j)}. \end{cases}$$

By abuse of notation, $a_p(f \otimes \omega^{-a})$ is the Hecke eigenvalue at p of the p-stabilized newform associated to $f \otimes \omega^{-a}$.

REMARK 1.2. Kitagawa [Ki] constructs the p-adic L-functions on each branch $\mathbb{T}^{\mathrm{ord}}_{\overline{\rho}}/\mathfrak{A}$ assuming that $\mathbb{T}^{\mathrm{ord}}_{\overline{\rho}}/\mathfrak{A}$ is Gorenstein. As explained above, we believe that the construction on the whole $\mathbb{T}^{\mathrm{ord}}_{\overline{\rho}}$ is more universal. It seems to us that such a universal construction will be possible if we assume that $\mathbb{T}^{\mathrm{ord}}_{\overline{\rho}}$ is Gorenstein as well as some conditions.

For other results related to Theorem 1.1, we give the following remark:

Remark 1.3. (1) Results similar to that of Theorem 1.1 are obtained using a similar method of Λ-adic modular symbol by Greenberg-Stevens ([GS]) and by Ohta (unpublished). Later was found another construction based on the method of Rankin-Selberg and it has been generalized to obtain results similar to Kitagawa's result by Fukaya[Fu], Ochiai[Oc1] and Panchishkin[P2]. (Panchishkin's construction is done for a p-adic family of positive slope, but the same construction works for a Hida family)

(2) The p-adic L-functions obtained at [GS], [Fu], [Oc1] and [P2] are weaker than the one by Kitagawa since they are sometimes defined only at a localization of a branch $\mathcal{R}$ but not on the whole of $\mathcal{R}$. In [Fu], [Oc1] and [P2], the complex periods $C_{f,\infty}$ cannot be optimally normalized and are defined at $\mathbb{C}^\times/\overline{\mathbb{Q}}^\times$. In [GS], the analogue of $C_{f,p}$ is not a p-adic unit as in the Kitagawa's one and we only know that $C_{f,p}$ are non zero except finitely many f's.

We fix a totally real number field F with $d := [F : \mathbb{Q}] > 1$ throughout the paper. We would like to introduce the nearly ordinary Hecke algebra $\mathbb{T}^{\mathfrak{n}.\mathrm{o}}_{\overline{\rho}}$ over a general totally real field F of degree $d > 1$ and construct a p-adic L-function which generalizes Theorem 1.1 (Everything which will be discussed in this paper works for $d = 1$ except that we omit the condition of holomorphy at infinity in Definition 2.1 which is always valid for $d > 1$ thanks to the Kocher princicple).

Remark 1.4. In order to work in the setting of Hilbert modular forms, it is more efficient to switch to the terminology of nearly ordinary modular forms rather than ordinary modular forms twisted by characters. Let I_F be the set of embeddings of F into $\mathbb{R}$. It will also be more efficient to cinsider modular forms with double-digit weight $w = (w_1, w_2) \in \mathbb{Z}[I_F] \times \mathbb{Z}[I_F]$ as explained in [H6]. When $F = \mathbb{Q}$, a cusp form of weight k in the usual sense is of weight $(0, k-1)$. For an ordinary elliptic cusp form of weight $(0, k-1)$, the twist by a Hecke character of weight j is a nearly ordinary cuspform of weight $(-j, k-1-j)$. Conversely, every elliptic cuspform of weight $(-j, k-1-j)$ which is nearly ordinary at p is obtained as a twist of an ordinary cuspform of weight $(0, k-1)$. For general totally real fields F, the situation is a bit more complicated. Since all global Hecke character on totally real fields coincide with an integral power of the Norm character multiplied by a finite order global Hecke character, a cuspform of weight (w_1, w_2) and a cuspform of weight (w_1', w_2') nearly ordinary at p are a twist of the other only when there exists an integer j such that $w_1 - w_1' = w_2 - w_2' = jt$.
From now on, we will switch to the notation of nearly ordinary forms with double-digit weight.

We denote by $\widetilde{F}_\infty$ the composition of all $\mathbb{Z}_p$-extensions of F. Note that Leopoldt's conjecture predicts that $\widetilde{F}_\infty = F_\infty$. We fix a residual modular Galois representation $\overline{\rho}$ and an integral ideal $\mathfrak{n}$ of F which is prime to p. We

note that, over a general totally real field F of degree $d > 1$, there are two different approaches to introduce the ($\overline{\rho}$-component of) nearly ordinary Hecke algebra $\mathbb{T}^{\mathrm{n.o}}$ of tame conductor $\mathfrak{n}$.
The original approach due to Hida ([H1], [H2], [H3]) uses a quarternion algebra B over F which is unramified at all finite primes of F. He chooses a quarternion algebra B such that Shimura varieties $Y^B_{K_1(\mathfrak{n})\cap K_{11}(p^n)}$ associated to $K_1(\mathfrak{n}) \cap K_{11}(p^n)$ (see Section 2.1 for the definition of $K_1(\mathfrak{n})$ and $K_{11}(p^n)$) are of dimension $e = 1$ (resp. $e = 0$) when $[F : \mathbb{Q}]$ is odd (resp. even). The nearly ordinary part $\mathcal{H}^B_{\mathrm{n.o}}(w_1, w_2)$ of the limit $\varprojlim_n H^e(Y^B_{K_1(\mathfrak{n})\cap K_{11}(p^n)}; \mathcal{L}(w_1, w_2; \mathcal{O}))$ for a certain local system $\mathcal{L}(w_1, w_2; \mathcal{O})$ is a finitely generated free module over a certain Iwasawa algebra $\Lambda_{\mathrm{n.o}}$ (see Definition 4.2). The nearly ordinary Hecke algebra in this context $\mathbb{T}^{\mathrm{n.o}}_B$ is defined to be a $\Lambda_{\mathrm{n.o}}$-subalgebra in $\mathrm{End}_{\Lambda_{\mathrm{n.o}}}(\mathcal{H}^B_{\mathrm{n.o}}(w_1, w_2))$ generated by Hecke operators. Let $\overline{\rho}$ be a mod p representation of F. In general, for a local domain R whose residue field $R/\mathfrak{M}_R$ is a finite field of characteristic p and an R-module M with R-linear action of Hecke correspondence, the $\overline{\rho}$-part $M_{\overline{\rho}}$ is defined by

$$(1) \qquad M_{\overline{\rho}} = \{x \in M \mid T_\lambda x \equiv \mathrm{Tr}(\overline{\rho}(\mathrm{Frob}_\lambda))x \bmod \mathfrak{M}_R M \text{ for every } \lambda \nmid \mathfrak{n}p\},$$

where T_λ is the Hecke correspondence at the prime λ and $\overline{\rho}(\mathrm{Frob}_\lambda)$ means the action on the inertia fixed part when λ divides $\mathfrak{n}$. Then, $\overline{\rho}$-component $(\mathbb{T}^{\mathrm{n.o}}_B)_{\overline{\rho}}$ is defined to be a $\Lambda_{\mathrm{n.o}}$-subalgebra in $\mathrm{End}_{\Lambda_{\mathrm{n.o}}}(\mathcal{H}^B_{\mathrm{n.o}}(w_1, w_2)_{\overline{\rho}})$ generated by Hecke operators. The algebra $(\mathbb{T}^{\mathrm{n.o}}_B)_{\overline{\rho}}$ is one of the local components of the semi-local $\Lambda_{\mathrm{n.o}}$-algebra $\mathbb{T}^{\mathrm{n.o}}_B$.
The second approach to introduce the ($\overline{\rho}$-component of) nearly ordinary Hecke algebra, which is essential to our work, is the one which uses Hilbert modular varieties. Taking the matrix algebra $M_2(F)$ over F in place of B, the Shimura varieties $Y^{M_2(F)}_{K_1(\mathfrak{n})\cap K_{11}(p^n)}$ are so called Hilbert modular varieties, which are of dimension d. The nearly ordinary part $\mathcal{H}^{M_2(F)}_{\mathrm{n.o}}(w_1, w_2)$ of the limit $\varprojlim_n H^d(Y^{M_2(F)}_{K_1(\mathfrak{n})\cap K_{11}(p^n)}; \mathcal{L}(w_1, w_2; \mathcal{O}))$ for the standard local system $\mathcal{L}(w_1, w_2; \mathcal{O})$ (cf. §2.4) is not known to be a finitely generated free module over $\Lambda_{\mathrm{n.o}}$ (see Definition 4.2). However, we will prove later in this article, that $\mathcal{H}^{M_2(F)}_{\mathrm{n.o}}(w_1, w_2)_{\overline{\rho}}$ for a suitable mod p representation $\overline{\rho}$ is a finitely generated free module over $\Lambda_{\mathrm{n.o}}$. Hence, $(\mathbb{T}^{\mathrm{n.o}}_{M_2(F)})_{\overline{\rho}}$ is also defined to be a $\Lambda_{\mathrm{n.o}}$-subalgebra in $\mathrm{End}_{\Lambda_{\mathrm{n.o}}}(\mathcal{H}^{M_2(F)}_{\mathrm{n.o}}(w_1, w_2)_{\overline{\rho}})$ generated by Hecke operators. See Section 4 for the precise result.

REMARK 1.5. In this way, for a mod p representation $\overline{\rho}$ satisfying suitable conditions, there are two different definitions of nearly ordinary Hecke algebras $(\mathbb{T}^{\mathrm{n.o}}_B)_{\overline{\rho}}$ and $(\mathbb{T}^{\mathrm{n.o}}_{M_2(F)})_{\overline{\rho}}$ which are both local rings finite and torsion free over $\Lambda_{\mathrm{n.o}}$. There should be a comparison between these two different definitions, but it seems unknown at the moment. However, in order to discuss modular symbol method, it is necessary to work on Hilbert modular varieties $Y^{M_2(F)}_{K_1(\mathfrak{n})\cap K_{11}(p^n)}$ which are non-compact and have cusps at infinity. Thus, we only take the second approach in this paper.

From now on through the introduction, we assume that our nearly oridinary modular mod p representation $\overline{\rho}$ satisfies a certain condition so that $(\mathbb{T}_B^{\mathrm{ord}})_{\overline{\rho}}$ and $(\mathbb{T}^{\mathrm{n.o}}_{M_2(F)})_{\overline{\rho}}$ are defined and (the $\overline{\rho}$-component of) the nearly ordinary (resp. ordinary) Hecke algebra $\mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}}$ (resp. $\mathbb{T}^{\mathrm{ord}}_{\overline{\rho}}$) means always the one obtained by the second approach.

As is shown in Theorem 4.15 later, for every p-stabilized Hilbert nearly ordinary eigen cuspform φ of cohomological weight (w_1, w_2), level $\mathfrak{n}\, p^*$ satisfying $\overline{\rho}_\varphi \cong \overline{\rho}$, there exists a unique algebra homomorphism $\kappa = \kappa_\varphi : \mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}} \longrightarrow \overline{\mathbb{Q}}_p$ such that the q-expansion of φ equals to $\displaystyle\sum_{\mathfrak{a}} \kappa_\varphi(T_\mathfrak{a}) q^\mathfrak{a}$ where $T_\mathfrak{a}$ is the $\mathfrak{a}$-th Hecke operator in $\mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}}$ for each integral ideal $\mathfrak{a}$ of F. The algebra $\mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}}$ is finite and torsion free over $\mathcal{O}[[(1+p\,\mathbb{Z}_p)^{d+1+\delta_{F,p}}]]$, where $\delta_{F,p}$ is the Leopoldt defect (conjectured to be zero by Leopoldt conjecture). For each prime ideal $\mathfrak{A}$ of height zero on $\mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}}$, $\mathcal{R} = \mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}}/\mathfrak{A}$ is called a branch of $\mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}}$ as is the previous page. Every arithmetic specialization $\kappa : \mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}} \longrightarrow \overline{\mathbb{Q}}_p$ factors through a unique branch $\mathcal{R}$ in which case we say that the arithmetic specialization κ is on the branch $\mathcal{R}$ of $\mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}}$.

A double-digit weight (w_1, w_2) is called *critical* if we have the inequality $w_{1,\tau} < 0 \leq w_{2,\tau}$ for every $\tau \in I_F$, where $w_{i,\tau} \in \mathbb{Z}$ is the coefficient given by $w_i = \displaystyle\sum_{\tau \in I_F} w_{i,\tau}\tau$. Put $t = \sum_{\tau \in I_F} \tau \in \mathbb{Z}[I_F]$ where $I_F = \{\tau : F \hookrightarrow \mathbb{R}\}$ is the set of embeddings. A double-digit weight (w_1, w_2) is called a cohomological double-digit weight if $w_2 - w_1 \geq t$ and $w_1 + w_2 \in \mathbb{Z}t$. As a dictionary in this paper, we recall that $k = w_2 - w_1 + t$ plays a role of the weight of modular form in the classical sense.

We propose the following conjecture:

Conjecture A . Let $\mathcal{R}$ be a branch of $\mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}}$. There exists an element $L_p(\mathcal{R}) \in \mathcal{R}$ which satisfies the interpolation property:

$$\frac{\kappa_\varphi(L_p(\mathcal{R}))}{C_{\varphi,p}} = \prod_{\mathfrak{p}\,|(p)} A_\mathfrak{p}(\varphi) \frac{L(\varphi,0)}{C_{\varphi,\infty}},$$

for every p-stabilized nearly ordinary eigen cuspform $\varphi \in S_{w_1,w_2}(K_1(\mathfrak{n}p^*))$ of critical cohomological double-digit weight $w = (w_1, w_2)$ on $\mathcal{R}$, where the term $A_\mathfrak{p}(\varphi)$ is defined as follows:

$$A_\mathfrak{p}(\varphi) = \begin{cases} \left(1 - \dfrac{1}{N_{F/\mathbb{Q}}(\mathfrak{p}) a_\mathfrak{p}(\varphi)}\right) & \text{if } a_\mathfrak{p}(\varphi) \neq 0, \\ \left(\dfrac{1}{N_F(\mathfrak{p}) a_\mathfrak{p}(\varphi^0)}\right)^{\mathrm{ord}_\mathfrak{p}\ \mathrm{Cond}(\phi_0)} & \text{if } a_\mathfrak{p}(\varphi) = 0, \end{cases} \tag{2}$$

where φ^0 is the nearly ordinary form of weight (w_1, w_2) which is of minimal conductor among twists of φ by finite order Hecke characters of F with p-primary conductor and ϕ_0 is the unique finite order character of $\mathrm{Gal}(\widetilde{F}_\infty/F)$ such that $\varphi = \varphi^0 \otimes \phi_0$. The number $N_F(\mathfrak{p})$ is the absolute norm of the prime ideal $\mathfrak{p}$. Further, $C_{\varphi,p} \in \overline{\mathbb{Z}}_p^\times/\overline{\mathbb{Z}}_{(p)}^\times$ and $C_{\varphi,\infty} \in \mathbb{C}^\times/\overline{\mathbb{Z}}_{(p)}^\times$ are a p-adic period and a complex period for φ (cf. Remark 1.6 (1)). For any ordinary eigen cuspform φ of critical double-digit weight (w_1, w_2) and for any nearly ordinary eigen cuspform φ' of critical double-digit weight $(w_1 - r, w_2 - r)$ respectively satisfying $\varphi' = \varphi \otimes N_F^r \phi$ with the norm character N_F of F and a finite Hecke character ϕ, we have

$$C_{\varphi,p,v} = C_{\varphi',p,v'} \tag{3}$$

$$C_{\varphi,\infty,v} = C_{\varphi',\infty,v'} \frac{G(\phi^{-1})}{(-2\pi\sqrt{-1})^{dr}} \prod_{\tau \in I_F} \frac{\Gamma(-w'_{1,\tau})}{\Gamma(-w_{1,\tau})} \tag{4}$$

where $G(\phi)$ is the Gauss sum for ϕ defined in Definition 3.6 and $\Gamma(s)$ is the Gamma function.

Remark 1.6. (1) The complex period $C_{\varphi,\infty}$ is a usual motivic complex period defined via the comparison isomorphism between the de Rham realization and Betti realization. On the other hand, the p-adic period $C_{\varphi,p}$ which appears here is not expected to be a motivic p-adic period obtained via the comparison theorem. Though it is not motivic, we call $C_{\varphi,p}$ a p-adic period following Greenberg. Since motivic p-adic periods will probably transcendental over $\mathbb{Q}_p$ there should be some modifications if we state Conjecture A using motivic p-adic periods. The author has some speculations on these possible different formulations of Conjectures depending the choices of p-adic periods, but it will be discussed elsewhere.

(2) Though we do not have a canonical lift of $C_{\varphi,p}$ (resp. $C_{\varphi,\infty}$) to $\overline{\mathbb{Z}}_p^\times$ (resp. $\mathbb{C}^\times$), the ratio "$C_{\varphi,p}/C_{\varphi,\infty}$" should be well-defined. (cf. Definition 4.16, Remark 4.17)

(3) When $F = \mathbb{Q}$ and when $\mathcal{R}$ is a Gorenstein ring, Conjecture A is equivalent to Theorem 1.1.

(4) We expect that a more general conjecture by replacing $\mathcal{R}$ by the whole local component of the Hecke algebra $\mathbb{T}_{\overline{\rho}}^{\mathrm{n.o}}$ should be true.

The ordinary Hecke algebra $\mathbb{T}_{\overline{\rho}}^{\mathrm{ord}}$ is finite and torsion-free over $\mathcal{O}[[(1 + p\,\mathbb{Z}_p)^{1+\delta_{F,p}}]]$. Since $d + 1 + \delta_{F,p} > 2 + \delta_{F,p}$, the natural surjection $\mathbb{T}_{\overline{\rho}}^{\mathrm{n.o}} \twoheadrightarrow \mathbb{T}_{\overline{\rho}}^{\mathrm{ord}}[[\mathrm{Cl}_F^+(p^\infty)_p]]$ cannot be an isomorphism by looking at the Krull dimensions. As is also shown in Theorem 4.15 later, for every p-stabilized Hilbert ordinary eigen cuspform φ of parallel weight $k \geq 2$ and of level Np^* satisfying $\overline{\rho}_\varphi \cong \overline{\rho}$, there exists an unique algebra homomorphism $\kappa = \kappa_\varphi : \mathbb{T}_{\overline{\rho}}^{\mathrm{ord}} \longrightarrow \overline{\mathbb{Q}}_p$ such that the q-expansion of φ equals $\sum_{\mathfrak{a}} \kappa_\varphi(T_{\mathfrak{a}}) q^{\mathfrak{a}}$. We propose another conjecture:

Conjecture B . Let $\mathcal{R}^{\mathrm{ord}}$ be a branch of $\mathbb{T}_{\overline{\rho}}^{\mathrm{ord}}$. There exists $L_p(\mathcal{R}^{\mathrm{ord}}) \in \mathcal{R}^{\mathrm{ord}}[[\mathrm{Cl}_F^+(p^\infty)_p]]$ which has the interpolation property:

$$\frac{\kappa_\varphi(L_p(\mathcal{R}^{\mathrm{ord}}))}{C_{\varphi,p}} = \prod_{\mathfrak{p}\,|(p)} A_{\mathfrak{p}}(\varphi) \frac{L(\varphi,0)}{C_{\varphi,\infty}}$$

for every p-stabilized ordinary eigen cuspform of critical parallel weight $k \geq 2$ and of level $\mathfrak{n}p^*$ on $\mathcal{R}^{\mathrm{ord}}$. Here, $C_{\varphi,p} \in \overline{\mathbb{Z}}_p^\times/\overline{\mathbb{Z}}_{(p)}^\times$ and $C_{\varphi,\infty} \in \mathbb{C}^\times/\overline{\mathbb{Z}}_{(p)}^\times$ are a p-adic period and a complex period for φ satisfying the same relation as (2) and (3) of Conjecture A.

Since, for every $\mathcal{R}^{\mathrm{ord}}$, $\mathcal{R}^{\mathrm{ord}}[[\mathrm{Gal}(F_\infty/F)]]$ is a quotient of some branch $\mathcal{R}$ of $\mathbb{T}_{\overline{\rho}}^{\mathfrak{n},\mathrm{o}}$, it is clear that Conjecture B is an immediate corollary of Conjecture A. We remark that Mok [Mo] constructs a two-variable p-adic L-function related to Conjecture B by the method of Rankin-Selberg and he obtains an application to the problem of trivial zeros of the one-variable p-adic L-functions of Hilbert modular forms. However, his construction is slightly weaker than the p-adic L-function of Conjecture B since his complex periods $C_{\varphi,\infty}$ are Rankin-Selberg type which are different from p-optimal modular symbol type periods required in Conjecture B. In fact, with Rankin-Selberg type period, the p-valuation of the special values which appear in the interpolation formula might not match with the value expected by generalized Birch and Swinneton-Dyer conjecture. It seems difficult to improve this point of the method of Rankin-Selberg (see Remark 1.3. 3. for a similar problem). Also, it is not clear if Mok's method works for Conjecture A.

1.2. Main results and technical difficulty on the work. In 1976, Manin [M2] generalized the method of modular symbol on modular curves in the setting of Hilbert modular varieties and constructed the cyclotomic p-adic L-function $L_p(\varphi) \in \mathcal{O}[[\mathrm{Gal}(F_\infty/F)]]$ of a Hilbert modular form φ. In this paper, we prove a weaker version of Conjecture A by generalizing the method of Λ-adic modular symbols on Hilbert modular varieties (see Theorem 5.1 for the precise statement of Theorem A below). In order to state the result, we introduce the following conditions ($\mathrm{Van}_{\overline{\rho}}$) and ($\mathrm{Ir}_{\overline{\rho}}$) for our fixed $\overline{\rho}$ and $\mathfrak{N}$.

($\mathrm{Van}_{\overline{\rho}}$) The module $H^i(Y_{K_1(\mathfrak{n})\cap K_{11}(p^n)}, \mathcal{L}(w_1,w_2,\mathcal{O}))_{\overline{\rho}}$ vanishes for any $i \neq d$, any $n \in \mathbb{N}$ and any cohomological double-digit weight (w_1,w_2), where $Y_{K_1(\mathfrak{n})\cap K_{11}(p^n)}$ is a Hilbert modular variety of level $K_1(\mathfrak{n}) \cap K_{11}(p^n)$.

The groups $K_1(\mathfrak{n})$ and $K_{11}(p^n)$ above are defined at §2.1. See (1) for the definition of $\overline{\rho}$-part $H^i(Y_{K_1(\mathfrak{n})\cap K_{11}(p^n)}, \mathcal{L}(w_1,w_2,\mathcal{O}))_{\overline{\rho}}$.

($\mathrm{Ir}_{\overline{\rho}}$) The $[F:\mathbb{Q}]$-th power mod p Galois representation $\overline{\rho}^{\otimes[F:\mathbb{Q}]}$ of $\mathrm{Gal}(\overline{F}/F)$ is irreducible.

The main result of this article is as follows:

Theorem A . Assume that the conditions ($\mathrm{Van}_{\overline{\rho}}$) and ($\mathrm{Ir}_{\overline{\rho}}$) are satisfied for our fixed $\overline{\rho}$ and $\mathfrak{n}$. Let $\mathcal{R}$ be a branch of $\mathbb{T}_{\overline{\rho}}^{\mathfrak{n},\mathrm{o}}$. Assume that $\mathcal{R}$ is a Gorenstein algebra.
Then, there exists a p-adic L-function $L_p^{\mathrm{even}}(\mathcal{R}) \in \mathcal{R}$ (resp. $L_p^{\mathrm{odd}}(\mathcal{R}) \in \mathcal{R}$) which satisfies the interpolation property

$$\frac{\kappa_\varphi(L_p^*(\mathcal{R}))}{C_{\varphi,p}} = \prod_{\mathfrak{p}\,|(p)} A_{\mathfrak{p}}(\varphi)\frac{L(\varphi,0)}{C_{\varphi,\infty}} \quad (* = \text{even (resp. odd)}),$$

for every p-stabilized nearly ordinary eigen cuspform $\varphi \in S_{w_1,w_2}(K_1(\mathfrak{n}\, p^*))$ of critical cohomological weight $w = (w_1, w_2)$ on $\mathcal{R}$ such that $w_1 + w_2 = rt$ are multiples of $t = \sum_{\tau \in I_F} \tau$ by even (resp. odd) integers r. Here, all terms in the above interpolation are the same as explained in Conjecture A.

Remark 1.7. (1) Since $w_1 + w_2$ is a multiple of t for every critical cohomological weight $w = (w_1, w_2)$, each of $L_p^{\mathrm{even}}(\mathcal{R})$ and $L_p^{\mathrm{odd}}(\mathcal{R})$ satisfy the half of the desired interpolation property. Hence, Conjecture A is true if $L_p^{\mathrm{even}}(\mathcal{R})$ is equal to $L_p^{\mathrm{odd}}(\mathcal{R})$ as an element of $\mathcal{R}$. Though we prove only the half of Conjecture A, a phenomenon of such a partial interpolation and such a theorem were quite new as far as we know. We believe that our work shed a new light on this area.

(2) The assumption that $\mathcal{R}$ is Gorenstein might not be satisfied in general. When $F = \mathbb{Q}$, there are some known explicit Eisenstein local components which are not Gorenstein. If $\mathcal{R}$ is not Gorenstein, we have some technical difficulties to construct the p-adic L-function in $\mathcal{R}$. Probably, we can construct $L_p(\mathcal{R})$ only as an element in a localization of $\mathcal{R}$ and we have a similar interpolation only by means p-adic periods $C_{\varphi,p}$ which are not necessarily p-adic units.

Theorem A immediately implies a corollary which gives two p-adic L-functions for the interpolation predicted by Conjecture B depending on the parity of weight. However, we will prove in [DO] the following full interpolation result which is stronger than the corollary of Theorem A.

Theorem B . Assume that the conditions ($\mathrm{Van}_{\overline{\rho}}$) and ($\mathrm{Ir}_{\overline{\rho}}$) are satisfied for our fixed $\overline{\rho}$ and $\mathfrak{n}$. Let $\mathcal{R}^{\mathrm{ord}}$ be a branch of $\mathbb{T}^{\mathrm{ord}}_{\overline{\rho}}$. Assume that $\mathcal{R}^{\mathrm{ord}}$ is a Gorenstein algebra. Then, there exists a p-adic L-function $L_p(\mathcal{R}^{\mathrm{ord}}) \in \mathcal{R}^{\mathrm{ord}}[[\mathrm{Cl}^+_F(p^\infty)_p]]$ which satisfies the same interpolation property as stated in Conjecture B.

Remark 1.8. We remark that the p-adic L-function in Theorem B interpolates not only the half of the desired specializations depending on the parity but all desired specializations. This is because we can simply use the space of ordinary modular symbols for the proof of Theorem B and we do not have to use the space of modular symbols for the level structure ZK_{11} which is necessary for the proof of Theorem A. As is remarked in the previous section, Mok [Mo] proves a result which is quite similar as Theorem B.

From the look of the statement of our main theorem (Theorem A), this work might seem to be done by a routine translation of the method of Mazur-Kitagawa form the method of modular symbols over modular curves into the method of modular symbols over Hilbert modular varieties. However, if one tries to establish this kind of result, one immediately finds a lot of difficulties including the ones coming from complicated natures of the generalization of Hida theory to Hilbert modular forms. This is why there has not been a result analogous to Theorem A long time after [Ki] and [M2] and why it took a long time for us to fill the detail of the work. So, it is important for us to explain difficulties of the work and ingredients of the paper.

(1) On the process of Mellin transform and the theory of modular symbols on Hilbert modular varieties, we often have the problem of the action of units of the totally real field F which did not exist in the classical situation of modular curves and where we considered the field of rationals $\mathbb{Q}$.
(2) We have to control the torsion of the étale cohomology of Hilbert modular varieties so that our p-adic L-function interpolates special values of Hecke L-function (cf. the condition ($\mathrm{Van}_{\overline{\rho}}$) stated before Main Theorem A). Such problem as well as the freeness of the Hecke algebra was studied by Dimitrov[Di1] and by Lan-Suh[LS].
(3) By Shimura's work, the special value $L(\varphi, 0)$ of the Hecke L-function of a Hilbert modular form φ of critical weight is equal to a complex period $\Omega_{\varphi,\infty}$ modulo multiplication by elements in $\overline{\mathbb{Q}}^{\times}$. The periods $\Omega_{\varphi,\infty}$ are invariant modulo multiplication by elements in $\overline{\mathbb{Q}}^{\times}$ under twists by Hecke characters. Our p-adic L-function should satisfy an interpolation property which matches well with Shimura's results (and Deligne's conjecture). Hence, in our nearly ordinary Hida deformations (consisting of $d+1$ variables), we have to separate d variables related to the weights of modular forms from the variable related to the twist by Hecke characters. Though the separation of the variables is not difficult for $F = \mathbb{Q}$ in Kitagawa's work, the direct analogue of this construction for a general totally real field gives us only a result for Conjecture B. To have a positive result for Conjecture A, we need a delicate choice of the level $(\mathbb{Z}_p \otimes \mathfrak{o})^{\times} K_{11}(p^n)$ which is between the usual level structures $K_0(p^n)$ and $K_{11}(p^n)$. The use of $(\mathbb{Z}_p \otimes \mathfrak{o})^{\times} K_{11}(p^n)$ already appears in a work of Hida [H5] but, to our best knowledge, its relation to the construction of a p-adic L-function has not been pointed out anywhere. Roughly speaking, when $F = \mathbb{Q}$, the level structure $(\mathbb{Z}_p \otimes \mathfrak{o})^{\times} K_{11}(p^n)$ yields a one-variable Hida family characterized by $j = \frac{k}{2}$ in the two-variable Hida family obtained by $K_{11}(p^n)$ in which the weight k of modular forms and Tate twist j vary freely.

At the end of this introduction, we recall that the algebraic counterpart of the p-adic L-functions on branches $\mathcal{R}$ of $\mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}}$ is given by the Selmer group $\mathrm{Sel}_{\mathcal{T}}$ of the universal Galois deformation $\mathcal{T}$ over $\mathcal{R}$. The group $\mathrm{Sel}_{\mathcal{T}}$ is defined to be a subgroup of the Galois cohomology $H^1(F, \mathcal{T} \otimes_{\mathcal{R}} \mathcal{R}^{\vee})$ and its Pontrjagin dual $(\mathrm{Sel}_{\mathcal{T}})^{\vee}$ is conjectured to be a finitely generated torsion module over $\mathcal{R}$. This conjecture is proposed in [Oc3] and is partially proved in [FO]. We finish our paper by stating the $d+1$-variable Iwasawa Main Conjecture with our p-adic L-functions constructed in this paper.

$d+1$-VARIABLE IWASAWA MAIN CONJECTURE (cf. [Oc3])

Under certain conditions (see the above articles for the exact statements), we conjecture that the principal ideals of a $d+1+\delta_{F,p}$-variable nearly ordinary Hecke algebra $\mathcal{R}$ generated by $L_p^{\mathrm{even}}(\mathcal{R})$ and $L_p^{\mathrm{odd}}(\mathcal{R})$ constructed in Theorem A are equal and they coincide with the characteristic ideal of $(\mathrm{Sel}_{\mathcal{T}})^{\vee}$.

Note that we also have 2-variable and 1-variable Iwasawa Main Conjectures corresponding respectively to Theorem B and Theorem C as special cases of the above Iwasawa Main Conjecture which are formulated in the same way.

1.3. LIST OF NOTATIONS. At the last page, we will list the symbols which appear in the article. Here, we list some of the most basic notations in the article.

• Let $\mathfrak{o}$ be the ring of integers of our fixed totally real field F and let $\mathfrak{o}_+^\times$ be the group of totally positive units of $\mathfrak{o}$.
• We denote by $\mathbb{A}_F$ the ring of adèles of F and by $\mathbb{A}_F^{\mathrm{fin}}$ the subring of finite adèles $F\otimes\widehat{\mathbb{Z}}$. We put $\widehat{\mathfrak{o}}=\mathfrak{o}\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}}$, where $\widehat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$.
• $p\geq 5$ is a fixed odd prime number unramified in F.
• $\mathfrak{H}$ is the upper half-plane $\{z\in\mathbb{C}|\mathrm{Im}z>0\}$ and $\mathfrak{H}^{\pm}=\mathbb{C}-\mathbb{R}$.
• Let $\mathcal{O}$ be a discrete valuation ring finite flat over $\mathbb{Z}_p$ which contains all conjugates of $\mathfrak{o}$. We denote by $\mathcal{K}$ the field of fractions of $\mathcal{O}$.
• For any $n\in\mathbb{Z}[I_F]$, we put $n_{\max}=\max\{n_\tau|\tau\in I_F\}$, $n_{\min}=\min\{n_\tau|\tau\in I_F\}$.
• We denote by T the p-Sylow subgroup of the maximal torus in $\mathrm{GL}_2(\mathfrak{o}\otimes_{\mathbb{Z}}\mathbb{Z}_p)$. We refer to 4.1 for various tori $T_{\mathrm{n.o}}$, T_{Z}, T_{ord}, T^{arith}, $T^{\mathrm{arith}}_{\mathrm{n.o}}$, $T^{\mathrm{arith}}_{\mathrm{Z}}$ related to T.
• For a (pro-)abelian group A, we denote by A_p the p-Sylow subgroup of A.
• For a locally compact abelian group A, we denote by A^{PD} the Pontrjagin dual of A.
• Suppose that the ideal $(p)\subset\mathbb{Z}$ decomposes into the product $\mathfrak{p}_1\cdots\mathfrak{p}_s$ in $\mathfrak{o}$. In this paper, if there is no possibility of confusion, we often take the multi-index notation p^α which means $\mathfrak{p}_1^{\alpha_1}\cdots\mathfrak{p}_s^{\alpha_s}$.

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2. Mellin transform of Hilbert modular forms and Hecke L-functions.

2.1. Automorphic forms on GL_2. We define the $\mathbb{C}$-vector space of Hilbert automorphic forms as in [H1], except that our normalization is cohomological.

Definition 2.1. Let K be an open compact subgroup of $\mathrm{GL}_2(\mathbb{A}_F)$. Let us take a cohomological double-digit weight $w = (w_1, w_2) \in \mathbb{Z}[I_F] \times \mathbb{Z}[I_F]$ (as defined in §1.3). The space $M_{w_1,w_2}(K;\mathbb{C})$ of adelic Hilbert modular forms of weight (w_1, w_2), level K is the $\mathbb{C}$-vector space of functions $\varphi : \mathrm{GL}_2(\mathbb{A}_F) \to \mathbb{C}$ satisfying the followings three conditions:
(i) $\varphi(\gamma g y) = \varphi(g)$ for all $\gamma \in \mathrm{GL}_2(F)$, $y \in K$ and $g \in \mathrm{GL}_2(\mathbb{A}_F)$.
(ii) $\varphi(gu) = \det(u)^{w_1-t} \exp(-\sqrt{-1}\sum_{\tau\in I_F} k_\tau\theta_\tau)\varphi(g)$, for all $u \in \mathrm{GL}_2(F\otimes_{\mathbb{Q}}\mathbb{R})$ and $g \in \mathrm{GL}_2(\mathbb{A}_F)$, where $\theta_\tau \in \mathbb{R}$ is such that $u_\tau \in \mathbb{R}^\times \cdot \left(\begin{smallmatrix} \cos(\theta_\tau) & -\sin(\theta_\tau) \\ \sin(\theta_\tau) & \cos(\theta_\tau)\end{smallmatrix}\right)$ and $k \in \mathbb{Z}[I_F]$ is defined to be $k = w_2 - w_1 + t$.

(iii) For all $\delta \in \mathrm{GL}_2(\mathbb{A}_F^{\mathrm{fin}})$,

$$\varphi_\delta : z = (z_\tau) \in \mathfrak{H}^{I_F} \mapsto \varphi\left(\delta \left(\begin{smallmatrix} 1 & \mathrm{Re}\, z \\ 0 & 1\end{smallmatrix}\right) \left(\begin{smallmatrix} \sqrt{\mathrm{Im}\, z} & 0 \\ 0 & \sqrt{\mathrm{Im}\, z}^{-1}\end{smallmatrix}\right)\right)$$

is holomorphic at $z_\tau \in \mathfrak{H}$ for every $\tau \in I_F$.
The space $S_{w_1,w_2}(K)$ of adelic Hilbert automorphic cuspforms is the subspace of $M_{w_1,w_2}(K)$ consisting of functions satisfying the following additional condition:
(iv) $\int_{\mathbb{A}_F/F} \varphi\left(\left(\begin{smallmatrix} 1 & x \\ 0 & 1\end{smallmatrix}\right) g\right) dx = 0$ for all $g \in \mathrm{GL}_2(\mathbb{A}_F)$, where dx denotes an additive Haar measure.

For an ideal $\mathfrak{l}$ of $\mathfrak{o}$, we consider the following open compact subgroups of $\mathrm{GL}_2(\mathbb{A}_F^{\mathrm{fin}})$:

$$K_0(\mathfrak{l}) = \left\{ \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathfrak{o}}) \middle| c \in \mathfrak{l}\,\widehat{\mathfrak{o}} \right\},$$
$$K_1(\mathfrak{l}) = \left\{ \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in K_0(\mathfrak{l}) \middle| d-1 \in \mathfrak{l}\,\widehat{\mathfrak{o}} \right\},$$
$$K_{11}(\mathfrak{l}) = \left\{ \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in K_1(\mathfrak{l}) \middle| a-1 \in \mathfrak{l}\,\widehat{\mathfrak{o}} \right\}.$$

Let Z be the p-Sylow subgroup of $(\mathfrak{o}\otimes\mathbb{Z}_p)^\times$ viewed as a subgroup of the center of $\mathrm{GL}_2(\mathbb{A}_F^{\mathrm{fin}})$. Then, $ZK_{11}(\mathfrak{l})$ is another important example of an open compact subgroups of $\mathrm{GL}_2(\mathbb{A}_F^{\mathrm{fin}})$. We have the following inclusions for every prime $\mathfrak{l}$ of F:

$$\begin{array}{ccc} ZK_{11}(\mathfrak{l}) & \subset & K_0(\mathfrak{l}) \\ \cup & & \cup \\ K_{11}(\mathfrak{l}) & \subset & K_1(\mathfrak{l}), \end{array}$$

For $z = (z_\tau) \in \mathfrak{H}^{I_F}$ and $\gamma = \begin{pmatrix} a_\tau & b_\tau \\ c_\tau & d_\tau\end{pmatrix} \in G(F\otimes_{\mathbb{Q}}\mathbb{R})$ we put $j(\gamma, z) = c_\tau z_\tau + d_\tau$.

2.2. HECKE OPERATORS. For an open compact subgroup K of $\mathrm{GL}_2(\mathbb{A}_F^{\mathrm{fin}})$ and an element $\delta \in \mathrm{GL}_2(\mathbb{A}_F^{\mathrm{fin}})$, the Hecke operator $[K\delta K]$ acts on the left on $\varphi \in S_k(K)$ as follows:

$$\varphi|_{[K\delta K]}(x) = \sum_i \varphi(x\delta_i) \text{ , where } [K\delta K] = \coprod_i \delta_i K.$$

Assume that K is factorizable as $\prod_\lambda K_\lambda$ where K_λ is an open compact subgroup of $\mathrm{GL}_2(K_\lambda)$ (λ running over the set of all finite places of F) and K_λ is an open compact subgroup of $\mathrm{GL}_2(\mathfrak{o}_\lambda)$ for almost all λ with $\mathfrak{o}_\lambda$ the completion $\mathfrak{o}$ of at λ. Let ϖ_λ be a uniformizer of $\mathfrak{o}_\lambda$. Then, the Hecke operator $\left[K\begin{pmatrix}\varpi_\lambda & 0\\ 0 & 1\end{pmatrix}K\right]$ is denoted by T_λ, if $K_\lambda = \mathrm{GL}_2(\mathfrak{o}_\lambda)$, and by U_λ, otherwise. Here, we regard $\begin{pmatrix}\varpi_\lambda & 0\\ 0 & 1\end{pmatrix} \in \mathrm{GL}_2(K_\lambda)$ as an element in $\mathrm{GL}_2(\mathbb{A}_F^{\mathrm{fin}})$ which is 1 in $\mathrm{GL}_2(K_{\lambda'})$ at all other finite primes λ'. It is important to observe that U_λ might depend on the choice of ϖ_λ (for example when $K \subset K_{11}(\lambda)$). In order to show the dependence on the choice of a uniformizer, we sometimes denote $\left[K\begin{pmatrix}\varpi_\lambda & 0\\ 0 & 1\end{pmatrix}K\right]$ by U_{ϖ_λ}. The group $\{\pm 1\}^{I_F}$ acts on the space of Hilbert modular forms by the action of the element $\iota = (\iota_\tau)_{\tau\in I_F} \in \mathrm{GL}_2(F\otimes\mathbb{R})$ with $\iota_\tau = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ at every infinite place τ. This action naturally commutes with the Hecke action.

2.3. NEARLY ORDINARY MODULAR FORMS. For an element $m = \sum_{\tau\in I_F} m_\tau \tau \in \mathbb{Z}[I_F]$, we denote by $\mathbb{Q}(m)$ the smallest subfield of F which contains x^m for every $x \in F$. Let $A \subset \mathbb{C}$ be an algebra satisfying the following properties:

(1) The algebra A contains the ring of integers of $\mathbb{Q}(m)$.
(2) For every element $\delta \in (\mathbb{A}_F^{\mathrm{fin}})^\times$, the fractional ideal $\delta^m A$ is a principal ideal.

In fact, for every non-archimedean prime λ of F, we fix a generator $\{\varpi_\lambda^m\} \in A$, which determines a generator $\{\delta^m\} \in A$ for any $\delta \in (\mathbb{A}_F^{\mathrm{fin}})^\times$. We refer to [H1, Sect.3] for the proof of the existence of such algebra A. Let us define modified Hecke operators $T_{0,\lambda}$ and $U_{0,\lambda}$ for non-archimedean prime λ of F when the weight k is not parallel acting on the space $S_{w_1,w_2}(K;A)$. On the space $S_{w_1,w_2}(K;A)$, we define:

$$T_{0,\lambda} = \{\varpi_\lambda^{w_1}\}T_\lambda,\ U_{0,\lambda} = \{\varpi_\lambda^{w_1}\}U_\lambda,$$
$$T_0(p) = \prod_{\mathfrak{p}\,|p} T_{0,\mathfrak{p}},\ U_0(p) = \prod_{\mathfrak{p}\,|p} U_{0,\mathfrak{p}}.$$

By using these modified Hecke operators, we define nearly ordinary modular forms as follows:

DEFINITION 2.2. A Hilbert modular eigenform $\varphi \in M_{w_1,w_2}(K;\overline{\mathbb{Q}})$ is *nearly ordinary* at p if the eigenvalue of φ with respect to $T_{0,\mathfrak{p}}$ (or $U_{0,\mathfrak{p}}$) is a p-adic unit for all primes $\mathfrak{p}$ of F dividing p.

2.4. Hilbert modular varieties and standard local systems. For an open compact subgroup K of $\mathrm{GL}_2(\mathbb{A}_F^{\mathrm{fin}})$, we denote by Y_K the Hilbert modular variety of level K with complex points

$$Y_K(\mathbb{C}) = \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)/KK_\infty^+$$

where K_∞^+ is the group $(\mathrm{SO}_2(\mathbb{R})\mathbb{R}_+^\times)^d$. We will consider the Hilbert modular varieties as analytic varieties. By the strong approximation theorem for GL_2, Y_K has $h_K := |F^\times\backslash \mathbb{A}_F^\times/\det(K)(F\otimes\mathbb{R})_+^\times|$ connected components, each isomorphic to a quotient of $\mathfrak{H}^{I_F}$ by a congruence subgroup of $\mathrm{GL}_2(F)$ of the form $\mathrm{GL}_2(F)\cap \left(\begin{smallmatrix}\mathfrak{c} & 0\\ 0 & 1\end{smallmatrix}\right)^{-1} K\,\mathrm{GL}_2(F\otimes_{\mathbb{Q}}\mathbb{R})_+\left(\begin{smallmatrix}\mathfrak{c} & 0\\ 0 & 1\end{smallmatrix}\right)$ for each $\mathfrak{c}$ which represents a class of $F^\times\backslash\mathbb{A}_F^\times/\det(K)(F\otimes\mathbb{R})_+^\times$. Let us fix a set of representative $\{\mathfrak{c}_i\in\mathbb{A}_F^\times\}_{1\le i\le h_F}$ of the narrow class group $\mathrm{Cl}_F^+ := F^\times\backslash\mathbb{A}_F^\times/\widehat{\mathfrak{o}}^\times(F\otimes\mathbb{R})_+^\times$ of F where $h_F = \sharp\,\mathrm{Cl}_F^+$ is the narrow class number of F. Throughout the paper, we assume the condition:

$$\mathfrak{c}_{i,p} = 1 \text{ for every } i, \tag{5}$$

where $\mathfrak{c}_{i,p}$ is the p-component of $\mathfrak{c}_i$. Note that the number of connected components of $Y_{K_1(\mathfrak{m})}$ equals to h_F and

$$Y_{K_1(\mathfrak{m})} \cong \coprod_{1\le i\le h_F} \Gamma_{K_1(\mathfrak{m})}(\mathfrak{c}_i)\backslash\mathfrak{H}^{I_F},$$

where $\Gamma_{K_1(\mathfrak{m})}(\mathfrak{c}_i) = \mathrm{GL}_2(F)\cap\left(\begin{smallmatrix}\mathfrak{c}_i & 0\\ 0 & 1\end{smallmatrix}\right)^{-1}K_1(\mathfrak{m})\,\mathrm{GL}_2(F\otimes_{\mathbb{Q}}\mathbb{R})_+\left(\begin{smallmatrix}\mathfrak{c}_i & 0\\ 0 & 1\end{smallmatrix}\right)$ for each representative $\mathfrak{c}_i\in\mathbb{A}_F^\times$ of the narrow class group of F. For each $\mathfrak{c}_i$, we have the following description of the group $\Gamma_{K_1(\mathfrak{m})}(\mathfrak{c}_i)$:

$$\Gamma_{K_1(\mathfrak{m})}(\mathfrak{c}_i) =$$
$$= \left\{ \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \mathrm{GL}_2(F)\cap\begin{pmatrix}\mathfrak{o} & \mathfrak{c}_i^{-1}\mathfrak{d}^{-1}\\ \mathfrak{c}_i\mathfrak{d}\mathfrak{m} & \mathfrak{o}\end{pmatrix} \;\middle|\; ad-bc\in\mathfrak{o}_+^\times,\ a\equiv 1 \pmod{\mathfrak{m}}\right\}$$

where $\mathfrak{d}$ is the different of $\mathfrak{o}$.
We fixed an integral ideal $\mathfrak{n}\subset\mathfrak{o}$ prime to p. We denote by $Y_{11}(\mathfrak{n}p^\alpha)$ (resp. $Y_Z(\mathfrak{n}p^\alpha)$, $Y_1(\mathfrak{n}p^\alpha)$) the Hilbert modular variety Y_K for $K = K_1(\mathfrak{n})\cap K_{11}(p^\alpha)$ (resp. $ZK_1(\mathfrak{n})\cap K_{11}(p^\alpha)$, $K_1(\mathfrak{n}p^\alpha)$). Similar analytic descriptions as above holds for these varieties, but we omit them since these are more or less parallel to the above description.

Definition 2.3. Let A be a $\mathbb{Z}$-algebra and let (w_1, w_2) be an element of $\mathbb{Z}[I_F]\times\mathbb{Z}[I_F]$.

(1) For each $\tau\in I_F$, we define a module $L(w_{1,\tau}, w_{2,\tau}; A)$ to be

$$L(w_{1,\tau}, w_{2,\tau}; A) = \bigoplus_{\substack{0\le i,j\le w_{2,\tau}-w_{1,\tau}-1\\ i+j = w_{2,\tau}-w_{1,\tau}-1}} AX_\tau^i Y_\tau^j.$$

Put $\begin{pmatrix} X_\tau\\ Y_\tau\end{pmatrix}^{w_{2,\tau}-w_{1,\tau}-1} =$

$$= {}^{\mathrm{t}}\!\left(X_\tau^{w_{2,\tau}-w_{1,\tau}-1}, X_\tau^{w_{2,\tau}-w_{1,\tau}-2}Y_\tau, \ldots, Y_\tau^{w_{2,\tau}-w_{1,\tau}-1}\right).$$

On the module $L(w_{1,\tau}, w_{2,\tau}; A)$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A)$ acts from the left on the way which sends $\begin{pmatrix} X_\tau \\ Y_\tau \end{pmatrix}^{w_{2,\tau}-w_{1,\tau}-1}$ to $(ad-bc)^{w_{1,\tau}} \begin{pmatrix} aX_\tau + bY_\tau \\ cX_\tau + dY_\tau \end{pmatrix}^{w_{2,\tau}-w_{1,\tau}-1}$.

(2) We define a module $L(w_1, w_2; A)$ to be $\underset{\tau \in I_F}{\otimes} L(w_{1,\tau}, w_{2,\tau}; A)$ where the tensor product is taken over A.

Let K be an open compact subgroup of $\mathrm{GL}_2(\mathbb{A}_F^{\mathrm{fin}})$. Similarly as the case of $K = K_1(\mathfrak{m})$, Y_K is presented as follows:

$$\begin{aligned} Y_K &= \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)/KK_\infty^+ \\ &= \mathrm{GL}_2(F)\backslash\big(\mathrm{GL}_2(\mathbb{A}_F^{\mathrm{fin}}) \times \mathrm{GL}_2(F \otimes_{\mathbb{Q}} \mathbb{R})\big)/KK_\infty^+ \\ &= \mathrm{GL}_2(F)\backslash\big(\mathrm{GL}_2(\mathbb{A}_F^{\mathrm{fin}})/K \times \mathrm{GL}_2(F \otimes_{\mathbb{Q}} \mathbb{R})/K_\infty^+\big) \\ &= \mathrm{GL}_2(F)\backslash\big(\mathrm{GL}_2(\mathbb{A}_F^{\mathrm{fin}})/K \times (\mathfrak{H}^\pm)^{I_F}\big) \\ &\cong \coprod_{1 \le i \le h_F} \Gamma_K(\mathfrak{c}_i)\backslash \mathfrak{H}^{I_F} \end{aligned}$$

where the discrete group $\Gamma_K(\mathfrak{c}_i)$ is defined to be:

$$\Gamma_K(\mathfrak{c}_i) = \mathrm{GL}_2(F) \cap \left(\begin{smallmatrix} \mathfrak{c}_i & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} K\, \mathrm{GL}_2(F \otimes_{\mathbb{Q}} \mathbb{R})_+ \left(\begin{smallmatrix} \mathfrak{c}_i & 0 \\ 0 & 1 \end{smallmatrix}\right)$$

for each representative $\mathfrak{c}_i \in \mathbb{A}_F^\times$ of the ray class group $F^\times\backslash\mathbb{A}_F^\times/\det(K)(F\otimes\mathbb{R})_+^\times$. Each component $\Gamma_K(\mathfrak{c}_i)\backslash(\mathfrak{H}^+)^{I_F}$ is an affine algebraic variety of dimension d.

DEFINITION 2.4. Let A be a subring of $\mathbb{C}$ or $\overline{\mathbb{Q}}_p$ which contains the Galois closure of $\mathfrak{o}[\mathfrak{d}^{-1}\mathfrak{c}_i^{-1}]$ for each $1 \le i \le h_K$. We define the local system $\mathcal{L}(w_1, w_2; A)$ on Y_K to be:

$$\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F) \times L(w_1, w_2; A)/KK_\infty^+. \tag{6}$$

Here, the group KK_∞^+ acts on $L(w_1, w_2; A)$ trivially and (6) is presented as

$$\coprod_{i=1}^{h_K} \Gamma_K(\mathfrak{c}_i)\backslash\big(\mathfrak{H}^{I_F} \times L(w_1, w_2; A)\big),$$

where $\Gamma_K(\mathfrak{c}_i)$ acts on $(\mathfrak{H}^+)^{I_F} \times L(w_1, w_2; A)$ diagonally for each conjugate corresponding to $\tau \in I_F$.

Note that for $K' \subset K$, there is a natural projection $\mathrm{pr} : Y_{K'} \to Y_K$. For $\delta \in \mathrm{GL}_2(\mathbb{A}_F^{\mathrm{fin}})$, we define the Hecke correspondence $[K\delta K]$ on Y_K to be $(\mathrm{pr}_1)_* \circ (\cdot\delta)^* \circ (\mathrm{pr}_2)^*$ from the diagram below:

$$\begin{tikzcd} & Y_{K\cap\delta^{-1}K\delta} \arrow[r, "\cdot\delta"] \arrow[dl, "\mathrm{pr}_1"'] & Y_{\delta K\delta^{-1}\cap K} \arrow[dr, "\mathrm{pr}_2"] & \\ Y_K & & & Y_K, \end{tikzcd}$$

where $(\ )^*$ means the pull-back and $(\ )_*$ means the trace map.
We define the standard Hecke cohomological operators $T_\lambda = \left[K \begin{pmatrix} \varpi_\lambda & 0 \\ 0 & 1 \end{pmatrix} K\right]$, $S_\lambda = \left[K \begin{pmatrix} \varpi_\lambda & 0 \\ 0 & \varpi_\lambda \end{pmatrix} K\right]$, for $\lambda \notin \Sigma_K$, and $U_\lambda = \left[K \begin{pmatrix} \varpi_\lambda & 0 \\ 0 & 1 \end{pmatrix} K\right]$, for $\lambda \in \Sigma_K$, where ϖ_λ is a uniformizer of F_λ. The Betti cohomology groups $\mathrm{H}^\bullet(Y_K, \mathcal{L}(w_1, w_2; A))$ admit a natural action of all Hecke correspondences.

The action of the group $\{\pm 1\}^{I_F}$ on $(Y_K, \mathcal{L}(w_1, w_2; A))$ is induced by the action of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on the τ-component. Note that, for each $\tau \in I_F$, the action of $\{\pm 1\}^{I_F}$ on Y_K is the one induced by sending z_τ to $-\overline{z}_\tau$ on the upper-half plane $\mathfrak{H}^{I_F}$ corresponding to the τ-component of Y_τ. The action of $\{\pm 1\}^{I_F}$ on $(Y_K, \mathcal{L}(w_1, w_2; A))$ induces the action of $\{\pm 1\}^{I_F}$ on the cohomology $\mathrm{H}^\bullet(Y_K, \mathcal{L}(w_1, w_2; A))$. This action commutes with the Hecke action. For a character $\epsilon : \{\pm 1\}^{I_F} \to \{\pm 1\}$ we denote by $\mathrm{H}^\bullet(Y_K, \mathcal{L}(w_1, w_2; A))[\epsilon]$ the ϵ-isotypic part for this action.

2.5. STANDARD q-EXPANSION AND L-FUNCTION. In this paragraph we consider forms of level $K_1(\mathfrak{m})$. We recall the relation between the adelic definition of the modular form and modular forms over Hilbert upper half planes defined as follows:

DEFINITION 2.5. For each i with $1 \leq i \leq h_K$, the space $M_{w_1,w_2}(\Gamma_{K_1(\mathfrak{m})}(\mathfrak{c}_i); \mathbb{C})$ of Hilbert modular forms of cohomological weight (w_1, w_2), level $\Gamma_{K_1(\mathfrak{m})}(\mathfrak{c}_i)$ is the $\mathbb{C}$-vector space of the functions $f_i : \mathfrak{H}^{I_F} \to \mathbb{C}$ which are holomorphic at z_τ for every $\tau \in I_F$ and such that, for every $\gamma \in \Gamma$, we have $f_i(\gamma(z)) = \det(\gamma)^{w_1} j(\gamma, z)^k f_i(z)$ where $k \in \mathbb{Z}[I_F]$ is $w_2 - w_1 + t$. The space $S_{w_1,w_2}(\Gamma_{K_1(\mathfrak{m})}(\mathfrak{c}_i); \mathbb{C})$ of Hilbert modular cuspforms is the subspace of $M_{w_1,w_2}(\Gamma_{K_1(\mathfrak{m})}(\mathfrak{c}_i); \mathbb{C})$, consisting of functions vanishing at all cusps.

LEMMA 2.6. *We have an isomorphism*:

$$S_{w_1,w_2}(K_1(\mathfrak{m}); \mathbb{C}) \cong \bigoplus_{1 \leq i \leq h_F} S_{w_1,w_2}(\Gamma_{K_1(\mathfrak{m})}(\mathfrak{c}_i); \mathbb{C}),\ \varphi \mapsto (f_i(z))_{1 \leq i \leq h_F}, \tag{7}$$

whose definition is explained in the proof below.

Proof. In order to explain the correspondence of the map (7), we recall the notion of Hilbert automorphic functions on $\mathrm{GL}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$ of weight (w_1, w_2), level $\Gamma_{K_1(\mathfrak{m})}(\mathfrak{c}_i)$ which are defined to be functions $\varphi_{\mathbb{R}}$ satisfying the following conditions:
(i) $\varphi_{\mathbb{R}}(\gamma g) = \varphi_{\mathbb{R}}(g)$ for all $\gamma \in \Gamma_{K_1(\mathfrak{m})}(\mathfrak{c}_i)$ and $g \in \mathrm{GL}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$.
(ii) $\varphi_{\mathbb{R}}(gu) = \det(u)^{w_1} \exp(-\sqrt{-1} \sum_{\tau \in I_F} k_\tau \theta_\tau) \varphi_{\mathbb{R}}(g)$, for all $u \in \mathrm{GL}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$ and $g \in \mathrm{GL}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$, where $\theta_\tau \in \mathbb{R}$ is given by $u_\tau \in \mathbb{R}^\times \cdot \left(\begin{smallmatrix} \cos(\theta_\tau) & -\sin(\theta_\tau) \\ \sin(\theta_\tau) & \cos(\theta_\tau) \end{smallmatrix}\right)$ and $k_\tau = w_{2,\tau} - w_{1,\tau} + 1$.
(iii) $z \mapsto \varphi_{\mathbb{R}}\left(\left(\begin{smallmatrix} 1 & \mathrm{Re}\, z \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \sqrt{\mathrm{Im}\, z} & 0 \\ 0 & \sqrt{\mathrm{Im}\, z}^{-1} \end{smallmatrix}\right)\right)$ is holomorphic at $z_\tau \in \mathfrak{H}$ for every $\tau \in I_F$.

Note that, since $\mathfrak{H}^{I_F}$ is isomorphic to $\mathrm{SL}_2(F \otimes_{\mathbb{Q}} \mathbb{R})/\mathrm{SO}_2(\mathbb{R})^d$, for modular forms $f_i(z)$ of weight (w_1, w_2) and level $\Gamma_{K_1(\mathfrak{m})}(\mathfrak{c}_i)$ in the sense of Definition 2.5, $(cz+d)^k f_i(z)$ corresponds naturally to a Hilbert automorphic function $\varphi_{\mathbb{R},i}$ on $\mathrm{GL}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$ of weight (w_1, w_2), level $\Gamma_{K_1(\mathfrak{m})}(\mathfrak{c}_i)$ in the above sense. In fact, this correspondence gives a bijection between two spaces.
On the other hand, let us consider an element φ of $M_{w_1,w_2}(K_1(\mathfrak{m}))$ as introduced in Definition 2.1 and Definition 2.2. For each $1 \leq i \leq h_F$, we denote by $\varphi_{\mathbb{R},i}$ the function $\varphi(\left(\begin{smallmatrix} t_i & 0 \\ 0 & 1 \end{smallmatrix}\right) g)$ restricted to $\mathrm{GL}_2(F \otimes_{\mathbb{Q}} \mathbb{R}) \subset \mathrm{GL}_2(\mathbb{A}_F^{\mathrm{fin}})$ where we choose $t_i \in \mathbb{A}_F^{\mathrm{fin}}$ such that $\mathfrak{c}_i\,\mathfrak{d} = (t_i)$. $\varphi_{\mathbb{R},i}$ is naturally a Hilbert automorphic function $\varphi_{\mathbb{R},i}$ on $\mathrm{GL}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$ of weight k, level $\Gamma_{K_1(\mathfrak{m})}(\mathfrak{c}_i)$ in the above sense. In fact, this correspondence gives a bijection between $M_{w_1,w_2}(K_1(\mathfrak{m}))$ and the direct sum for $1 \leq i \leq h_F$ of the spaces of Hilbert automorphic functions on $\mathrm{GL}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$ of weight k, level $\Gamma_{K_1(\mathfrak{m})}(\mathfrak{c}_i)$.
Now, the desired isomorphism is obtained by taking the composition of two correspondences explained above. □

Since f_i is invariant with respect to the action of the matrix $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ with $x \in (\mathfrak{c}_i\,\mathfrak{d})^{-1}$, f_i has a q-expansion of the form $\sum_{\xi \in (\mathfrak{c}_i)_+} a(f_i, \xi) q^\xi$, where $q^\xi = \exp(2\pi\sqrt{-1}\,\mathrm{Tr}_{F/\mathbb{Q}}(\xi z))$. Note the important relation:

$$a(f_i, \epsilon\xi) = \epsilon^{w_1} a(f_i, \xi)\ ,\ \text{for all } \epsilon \in \mathfrak{o}_+^\times\,. \tag{8}$$

We will describe the relation between Fourier coefficients and Hecke eigenvalues.

LEMMA 2.7. *For each $\mathfrak{c}_i$ and for an element $\xi \in (\mathfrak{c}_i)_+$, the quantity*

$$a(f_i, \xi)\xi^{-w_1}$$

depends only on the ideal $(\xi)\,\mathfrak{c}_i \subset \mathfrak{o}$.

DEFINITION 2.8. For an integral ideal $\mathfrak{a}$ of $\mathfrak{o}$, there is a unique i such that $\mathfrak{a}$ belongs to the same narrow ideal class as $\mathfrak{c}_i^{-1}$, that is $\mathfrak{a} = \xi\,\mathfrak{c}_i^{-1}$ with $\xi \in (\mathfrak{c}_i)_+$. We put then:

$$C(\varphi, \mathfrak{a}) := a(f_i, \xi)\xi^{-w_1}.$$

LEMMA 2.9. ([H1, Prop.4.1,Thm.5.2]) *Let $\varphi \in S_{w_1,w_2}(K_1(\mathfrak{m}))$ be an eigen cuspform. Then, for every integral ideal $\mathfrak{a}$ of $\mathfrak{o}$, the eigenvalue of $T_\mathfrak{a}$ on φ is $C(\varphi, \mathfrak{a})$.*

DEFINITION 2.10. We define the standard L-function of $\varphi \in S_{w_1,w_2}(K_1(\mathfrak{m}))$ to be:

$$L(\varphi, s) := \sum_{\mathfrak{a} \subset \mathfrak{o}} \frac{C(\varphi, \mathfrak{a})}{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{a})^s}.$$

Similarly, for any Hecke character ϕ of finite order, we define the standard L-function of $\varphi \in S_{w_1,w_2}(K_1(\mathfrak{m}))$ twisted by ϕ to be:

$$L(\varphi \otimes \phi, s) := \sum_{\mathfrak{a} \subset \mathfrak{o}} \frac{C(\varphi, \mathfrak{a})\phi(\mathfrak{a})}{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{a})^s}.$$

2.6. EICHLER-SHIMURA MAP. Let $\varphi \in S_{w_1,w_2}(K_1(\mathfrak{m}),\mathbb{C})$ be an eigen cuspform. Recall that, by (7), φ corresponds to $\sum_{1\le i\le h_F} f_i(z)$ where $f_i(z) \in S_{w_1,w_2}(\Gamma_{K_1(\mathfrak{m})}(\mathfrak{c}_i);\mathbb{C})$ for each i. Then, the integration of the vector-valued d-form

$$\sum_{1\le i\le h_F} f_i(z_\tau)\textstyle\prod_{\tau\in I_F}(X_\tau+z_\tau Y_\tau)^{w_{2,\tau}-w_{1,\tau}-1} \bigwedge_{\tau\in I_F} dz_\tau$$

on h_F-copies of $\mathfrak{H}^{I_F}$ defines an element of

$$H^d(Y_1(\mathfrak{m});\mathcal{L}(w_1,w_2;\mathbb{C})) \cong \bigoplus_{1\le i\le h_F} H^d(\Gamma_{K_1(\mathfrak{m})}(\mathfrak{c}_i)\backslash\mathfrak{H}^{I_F};\mathcal{L}(w_1,w_2;\mathbb{C})).$$

We call this the Eichler-Shimura class of φ and we denote it by $[\varphi] \in H^d(Y_1(\mathfrak{m});\mathcal{L}(w_1,w_2;\mathbb{C}))$. Let $H^d(Y_1(\mathfrak{m});\mathcal{L}(w_1,w_2;\mathbb{C}))[\lambda_\varphi]$ (resp. $H^d_c(Y_1(\mathfrak{m});\mathcal{L}(w_1,w_2;\mathbb{C}))[\lambda_\varphi]$) be the component on which the Hecke algebra for $M_{w_1,w_2}(K_1(\mathfrak{m});\mathbb{C})$ acts by the Hecke eigenvalues of φ. The class $[\varphi] \in H^d(Y_1(\mathfrak{m});\mathcal{L}(w_1,w_2;\mathbb{C}))$ naturally falls in $H^d(Y_1(\mathfrak{m});\mathcal{L}(w_1,w_2;\mathbb{C}))[\lambda_\varphi]$. By the isomorphism $H^d(Y_1(\mathfrak{m});\mathcal{L}(w_1,w_2;\mathbb{C}))[\lambda_\varphi] \cong H^d_c(Y_1(\mathfrak{m});\mathcal{L}(w_1,w_2;\mathbb{C}))[\lambda_\varphi]$, we also have the Eichler-Shimura class in $H^d_c(Y_1(\mathfrak{m});\mathcal{L}(w_1,w_2;\mathbb{C}))[\lambda_\varphi]$.

2.7. MELLIN TRANSFORM. In this section, we compute $L(\varphi,s)$ for an eigen cuspform $\varphi \in S_{w_1,w_2}(K_1(\mathfrak{m}))$ by using a Mellin transform.

$$\begin{aligned}
&\sum_{1\le i\le h_F} \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{c}_i)^s \int_{\sqrt{-1}(F\otimes\mathbb{R})^\times_+/\mathfrak{o}^\times_+} f_i(z) z^{s-t-w_1} \bigwedge_{\tau\in I_F} dz_\tau \\
&= \sum_{1\le i\le h_F} \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{c}_i)^s \int_{\sqrt{-1}(F\otimes\mathbb{R})^\times_+/\mathfrak{o}^\times_+} \sum_{\xi\in(\mathfrak{c}_i)_+} a(f_i,\xi)\exp(2\pi\sqrt{-1}\,\mathrm{Tr}_{F/\mathbb{Q}}(\xi z)) z^{s-t-w_1} \bigwedge_{\tau\in I_F} dz_\tau \\
&= \sum_{1\le i\le h_F} \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{c}_i)^s \times \\
&\quad\times \int_{\sqrt{-1}(F\otimes\mathbb{R})^\times_+/\mathfrak{o}^\times_+} \sum_{\xi\in(\mathfrak{c}_i)_+} a\frac{a(f_i,\xi)\xi^{w_1}}{\mathrm{N}_{F/\mathbb{Q}}(\xi)^s} \exp(2\pi\sqrt{-1}\,\mathrm{Tr}_{F/\mathbb{Q}}(\xi z))(\xi z)^{s-t-w_1} \bigwedge_{\tau\in I_F} d(\xi z_\tau)
\end{aligned}$$

By using the invariance of the integrant with respect to the action of units, the last term is equal to:

$$\sum_{1\le i\le h_F} \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{c}_i)^s \sum_{\xi\in(\mathfrak{c}_i)_+/\mathfrak{o}_+^\times} \frac{a(f_i,\xi)\xi^{w_1}}{\mathrm{N}_{F/\mathbb{Q}}(\xi)^s} \int_{\sqrt{-1}(F\otimes\mathbb{R})_+^\times} \exp(2\pi\sqrt{-1}\,\mathrm{Tr}_{F/\mathbb{Q}}(z)) z^{s-t-w_1} \bigwedge_{\tau\in I_F} dz_\tau$$

$$= \sum_{1\le i\le h_F} \sum_{\xi\in(\mathfrak{c}_i)_+/\mathfrak{o}_+^\times} \frac{a(f_i,\xi)\xi^{w_1}}{\mathrm{N}_{F/\mathbb{Q}}(\xi\,\mathfrak{c}_i^{-1})^s} \int_{\sqrt{-1}(F\otimes\mathbb{R})_+^\times} \exp(2\pi\sqrt{-1}\,\mathrm{Tr}_{F/\mathbb{Q}}(z)) z^{s-t-w_1} \bigwedge_{\tau\in I_F} dz_\tau$$

$$= \left(\sum_{\mathfrak{a}\subset\mathfrak{o}} \frac{C(\varphi,\mathfrak{a})}{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{a})^s}\right) \times \prod_{\tau\in I_F} \int_0^{\sqrt{-1}\infty} \exp(2\pi\sqrt{-1}y_\tau) y_\tau^{s-1-w_{1,\tau}} dy_\tau$$

$$= \left(\sum_{\mathfrak{a}\subset\mathfrak{o}} \frac{C(\varphi,\mathfrak{a})}{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{a})^s}\right) \times \prod_{\tau\in I_F} \int_0^{\infty} \exp(-2\pi y_\tau)(\sqrt{-1}y_\tau)^{s-1-w_{1,\tau}} d(\sqrt{-1}y_\tau)$$

$$= \left(\sum_{\mathfrak{a}\subset\mathfrak{o}} \frac{C(\varphi,\mathfrak{a})}{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{a})^s}\right) \times \prod_{\tau\in I_F} \frac{(\sqrt{-1})^{s-w_{1,\tau}}}{(2\pi)^{s-w_{1,\tau}}} \int_0^{\infty} \exp(-2\pi y_\tau)(2\pi y_\tau)^{s-1-w_{1,\tau}} d(2\pi y_\tau)$$

$$= L(\varphi,s) \prod_{\tau\in I_F} \frac{\Gamma(s-w_{1,\tau})}{(-2\pi\sqrt{-1})^{s-w_{1,\tau}}}.$$

3. MODULAR SYMBOL CYCLE

The image of $(F\otimes\mathbb{R})_+^\times/\,\mathfrak{o}_+^\times$ in the Hilbert modular variety Y_K of level K is a closed d-cycle, which is a generalization to the Hilbert modular case, of the classical modular symbol linking the cusps 0 and ∞ on a modular curve. Recall that the number of components of Y_K is $h_K = |F^\times\backslash\mathbb{A}_F^\times/\det(K)(F\otimes\mathbb{R})_+^\times|$. Thus Y_K is a quotient of $\mathfrak{H}^{I_F}\coprod\cdots\coprod\mathfrak{H}^{I_F}$ (h_K copies). For each i satisfying $1\le i\le h_K$, we fix an element $u_i\in F$ of the cusp $\mathbb{P}^1(F)$ of i-th component $\mathfrak{H}^{I_F}$. We define $H_{u_i}\subset\mathfrak{H}^{I_F}$ to be

$$H_{u_i} = \{u_i + \sqrt{-1}y \mid y\in(F\otimes\mathbb{R})_+^\times\}.$$

For each $\boldsymbol{u} = \{u_i\in F\}_{1\le i\le h_K}$, we denote by $H_{\boldsymbol{u}}\subset\mathfrak{H}^{I_F}\coprod\cdots\coprod\mathfrak{H}^{I_F}$ (h_K copies) the disjoint union $\coprod_{i=1}^{h_K} H_{u_i}$. Let $\mathfrak{m}$ be an integral ideal of F such that $\mathfrak{m}u_i$ is contained in $\mathfrak{o}$ for every i. Let E be a finite index subgroup of $\mathfrak{o}_+^\times$ consisting of totally positive units e congruent to 1 modulo $\mathfrak{m}$. For such a sufficiently small subgroup $E\subset\mathfrak{o}_+^\times$, we have:

$$G_{K,E,\boldsymbol{u}}:\ H_{\boldsymbol{u}}/E \longrightarrow Y_K.$$

Note that each component of $H_{\boldsymbol{u}}/E$ is a finite cover of $(F\otimes_{\mathbb{Q}}\mathbb{R})_+^\times/\,\mathfrak{o}_+^\times$ and it is homeomorphic to $\mathbb{R}\times(\mathbb{S}^1)^{d-1}$. In Section 2.4, we introduced a local system $\mathcal{L}(w_1,w_2;A)$ on Y_K for any subring A of $\mathbb{C}$ or $\overline{\mathbb{Q}}_p$ which contains the Galois closure of $\mathfrak{o}[\mathfrak{d}^{-1}\mathfrak{c}_i^{-1}]$ for each $1\le i\le h_K$. By definition, the pull-back of the local system $\mathcal{L}(w_1, w_2; A)$ to $H_{\boldsymbol{u}}/E$ is the etale space

$$(H_{\boldsymbol{u}} \times L(w_1, w_2; A))/E$$

where $e \in E$ acts on $H_{\boldsymbol{u}}$ by multiplication and acts on $L(w_1, w_2; A)$ via the matrix $\begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}$.

DEFINITION 3.1. A cohomological double-digit weight $(w_1, w_2) \in \mathbb{Z}[I_F] \times \mathbb{Z}[I_F]$ is said to be *critical* if $w_{1,\tau} < 0 \leq w_{2,\tau}$ holds for every $\tau \in I_F$.

For each critical (w_1, w_2), we consider the map:

$$L(w_1, w_2; A) \longrightarrow A \tag{9}$$

(10)

$$\sum a_w X^{w_2-w_1-t-w} Y^w \mapsto \text{the coefficient of } \begin{pmatrix} w_2 - w_1 - t \\ -w_1 - t \end{pmatrix} X^{w_2} Y^{-w_1-t},$$

where all the indeterminates are of multi-index and w runs through elements in $\mathbb{Z}[I_F]$ satisfying $0 \leq w \leq w_2 - w_1 - t$. Note that, by definition, a_w is divisible by $\begin{pmatrix} w_2 - w_1 - t \\ -w_1 - t \end{pmatrix}$. Hence, (9) induces a map of sheaves

$$\mathcal{L}(w_1, w_2; A) \longrightarrow A \tag{11}$$

on $H_{\boldsymbol{u}}/E$.

DEFINITION 3.2. Let A be a subring of $\mathbb{C}$ or $\overline{\mathbb{Q}}_p$ which contains the Galois closure of $\mathfrak{o}[\mathfrak{d}^{-1}\mathfrak{c}_i^{-1}]$ for each $1 \leq i \leq h_K$ and let (w_1, w_2) be a critical cohomological double-digit weight. For each $\boldsymbol{u} = \{u_i \in F\}_{1 \leq i \leq h_K}$, we have the Gysin map

$$H_c^d(Y_K; \mathcal{L}(w_1, w_2; A)) \longrightarrow H_c^d(H_{\boldsymbol{u}}/E; \mathcal{L}(w_1, w_2; A)) \tag{12}$$

for the map $G_{K,E,\boldsymbol{u}}$. We define the evaluation map

$$\mathrm{ev}_{K,\boldsymbol{u}}^{(w_1,w_2)}: \ H_c^d(Y_K; \mathcal{L}(w_1, w_2; A)) \longrightarrow A \otimes_{\mathbb{Z}} \mathbb{Q} \tag{13}$$

to be the composition of (12) and the map (14) defined as follows:
(14)

$$H_c^d(H_{\boldsymbol{u}}/E; \mathcal{L}(w_1, w_2; A)) \longrightarrow H_c^d(H_{\boldsymbol{u}}/E; A) \cong A^{\oplus h_K} \longrightarrow A \xrightarrow{\times \frac{1}{[\mathfrak{o}_+^\times : E]}} A \otimes_{\mathbb{Z}} \mathbb{Q}.$$

(the first arrow of (14) is induced by (11) and the second one is the trace map.)

REMARK 3.3. Later, in §5, we use the evaluation maps to define a certain measure on $\varprojlim_\beta \mathrm{Cl}_F^+(p^\beta)$. Since K varies in $K_1(\mathfrak{n}) \cap K_{11}(p^\alpha)$ where $p^\alpha \subset \mathfrak{o}$ becomes arbitrary small, it might seem that the denominator $\frac{1}{[\mathfrak{o}_+^\times : E]}$ is out of control. However, by distribution property with respect to α proved at Lemma 5.3 and Corollary 5.7, it will follow that the measure we obtain is integral.

We recall some basic facts and fix notations related to cusps and evaluation maps. For a multi-index notation $p^\beta = \mathfrak{p}_1^{\beta_1}\cdots\mathfrak{p}_s^{\beta_s}$, we have the extension as follows:

$$\mathbf{1} \longrightarrow (\mathfrak{o}/p^\beta)^\times/\mathfrak{o}_+^\times \longrightarrow \mathrm{Cl}_F^+(p^\beta) \longrightarrow \mathrm{Cl}_F^+ \longrightarrow \mathbf{1}. \tag{15}$$

When $K = K_1(\mathfrak{n}p^\alpha)$ for fixed tame conductor $\mathfrak{n}$, the number of components of Y_K is $h_F = |\mathrm{Cl}_F^+|$. We fix a splitting of the sequence (15) in a manner which is compatible with respect to β. Every element of $\boldsymbol{u} \in \mathrm{Cl}_F^+(p^\beta)$ is identified with $\{u_i \in (\mathfrak{o}/p^\beta)^\times/\mathfrak{o}_+^\times\}_{1\le i\le h_F}$. Here, we note that $p^{-\beta}/\mathfrak{o} \subset F$ and we fix representatives of $(\mathfrak{o}/p^\beta)^\times/\mathfrak{o}_+^\times$ in $p^{-\beta}/\mathfrak{o}$ in a manner which is compatible with respect to $\beta = (\beta_1, \cdots, \beta_s)$. Under this situation, we denote $\mathrm{ev}_{K,\boldsymbol{u}}^{(w_1,w_2)}$ with $K = K_1(\mathfrak{n}p^\alpha)$ and $\boldsymbol{u} \in \mathrm{Cl}_F^+(p^\beta)$ by $\mathrm{ev}_{\boldsymbol{u},\alpha,\beta}^{(w_1,w_2)}$.
In §5, we need a similar construction for the level $K_Z(p^\alpha) = ZK_1(\mathfrak{n}) \cap K_{11}(p^\alpha)$ for fixed tame conductor $\mathfrak{n}$ where Z is the p-Sylow subgroup of $(\mathfrak{o}\otimes\mathbb{Z}_p)^\times$ viewed as a subgroup of the center of $\mathrm{GL}_2(\mathbb{A}_F^{\mathrm{fin}})$ as introduced at the end of the introduction. The number of components $h_{K_Z(p^\alpha)} = |F^\times\backslash\mathbb{A}_F^\times/\det(K_Z(p^\alpha))(F\otimes\mathbb{R})_+^\times|$ of $Y_{K_Z(p^\alpha)}$ is $2^t h_F$ independently of α, where t satisfies $1 \le t \le s$. Since p is not equal to 2, we have

$$\mathbf{1} \longrightarrow ((\mathfrak{o}/p^\beta)^\times/\mathfrak{o}_+^\times)_p \longrightarrow \mathrm{Cl}_F^+(p^\beta)_p \longrightarrow (F^\times\backslash\mathbb{A}_F^\times/\det(K_Z(p^\alpha))(F\otimes\mathbb{R})_+^\times)_p \longrightarrow \mathbf{1}. \tag{16}$$

As in the previous case, we fix a splitting of the sequence (16) in a manner which is compatible with respect to $\beta = (\beta_1, \cdots, \beta_s)$. If we consider the situation where the coefficient ring is p-adic and the cohomology $H_c^d(Y_{K_Z(p^\alpha)}; \mathcal{L}(w_1, w_2; A))$ is localized at a maximal ideal of the p-adic Hecke algebra acting on the cohomology (we denote the maximal ideal by $\overline{\rho}$), we have the cusp associated to $\boldsymbol{u} = \{u_i \in ((\mathfrak{o}/p^\beta)^\times/\mathfrak{o}_+^\times)_p\} \in \mathrm{Cl}_F^+(p^\beta)_p$ and an evaluation map:

$$\mathrm{ev}_{K_Z(p^\alpha),\boldsymbol{u}}^{(w_1,w_2)} : H_c^d(Y_{K_Z(p^\alpha)}; \mathcal{L}(w_1, w_2; A))_{\overline{\rho}} \longrightarrow A\otimes_{\mathbb{Z}}\mathbb{Q}. \tag{17}$$

We also denote $\mathrm{ev}_{K_Z(p^\alpha),\boldsymbol{u}}^{(w_1,w_2)}$ for $\boldsymbol{u} \in \mathrm{Cl}_F^+(p^\beta)_p$ by $\mathrm{ev}_{\boldsymbol{u},\alpha,\beta}^{(w_1,w_2)}$

3.1. COHOMOLOGICAL INTERPRETATION OF SPECIAL VALUES. Now we consider the paring between the modular symbol cycle and the cohomology class of φ under Eichler-Shimura map and relate it to the special value with help of the subsection 2.7.
By the definition of Eichler-Shimura class, we have the following proposition:

THEOREM 3.4. *Let φ be an eigen cuspform of critical double-digit weight (w_1, w_2) and level $K_1(\mathfrak{n}p^\alpha)$. We denote by $[\varphi] \in H_c^d(Y_1(\mathfrak{n}p^\alpha); \mathcal{L}(w_1, w_2; \mathbb{C}))[\lambda_\varphi]$ the Eichler-Shimura class of a cuspform φ. Let $\beta = (\beta_1, \cdots, \beta_s)$ be a multi-index as above. Then, for each $\boldsymbol{u} = \{u_i \in (\mathfrak{o}/p^\beta)^\times/\mathfrak{o}_+^\times\} \in \mathrm{Cl}_F^+(p^\beta)$, we have*

$$\mathrm{ev}_{\boldsymbol{u},\alpha,\beta}^{(w_1,w_2)}([\varphi]) = \frac{1}{[\mathfrak{o}_+^\times : E]} \sum_{1\le i\le h_F} \int_{\sqrt{-1}(F\otimes\mathbb{R})_+^\times/E} f_i(z+u_i) z^{-t-w_1} \bigwedge_{\tau\in I_F} dz_\tau$$

via the correspondence from φ to $\{f_i(z)\}$ obtained in Lemma 2.6 where E is a sufficiently small subgroup of $\mathfrak{o}_+^\times$ with finite index introduced at the beginning of Section 3.

Let $\mathfrak{o}_\varphi$ be the ring of integers of the Hecke field which is obtained by adjoining the eigenvalues of Hecke operators to $\mathbb{Q}$ and let $\mathfrak{o}_{\varphi,(p)}$ be the localization of $\mathfrak{o}_\varphi$ at the prime ideal above p which is specified from the fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.

Definition 3.5. Let φ be an eigen cuspform of critical double-digit weight (w_1, w_2) and level $K_1(\mathfrak{n}p^\alpha)$. Let v be an $\mathfrak{o}_{\varphi,(p)}$-basis of the free $\mathfrak{o}_{\varphi,(p)}$-module of rank one obtained by $H_c^d(Y_1(\mathfrak{n}p^\alpha); \mathcal{L}(w_1, w_2; \mathfrak{o}_{\varphi,(p)}))[\lambda_\varphi, \epsilon]$ modulo $\mathfrak{o}_{\varphi,(p)}$-torsion part. By abuse of notation, we denote the image of the above v via the map

$$H_c^d(Y_1(\mathfrak{n}p^\alpha); \mathcal{L}(w_1, w_2; \mathfrak{o}_{\varphi,(p)}))[\lambda_\varphi, \epsilon]/(\mathfrak{o}_{\varphi,(p)}\text{-torsion}) \hookrightarrow \hookrightarrow H_c^d(Y_1(\mathfrak{n}p^\alpha); \mathcal{L}(w_1, w_2; \mathbb{C}))[\lambda_\varphi, \epsilon]$$

by the same letter v. We define the complex period $C^\epsilon_{\varphi,\infty,v} \in \mathbb{C}^\times$ to be the complex number satisfying $v = C^\epsilon_{\varphi,\infty,v} \cdot [\varphi]$.

Definition 3.6. Let ϕ be a finite order character of $\mathrm{Cl}_F^+(p^\beta)$. As in (15), $(\mathfrak{o}/p^\beta)^\times/\mathfrak{o}_+^\times$ is canonically identified with a subgroup of $\mathrm{Cl}_F^+(p^\beta)$. Thus we have a homomorphism:

$$(p^{-\beta}/\mathfrak{o})^\times \xrightarrow{\sim} (\mathfrak{o}/p^\beta)^\times \twoheadrightarrow (\mathfrak{o}/p^\beta)^\times/\mathfrak{o}_+^\times \hookrightarrow \mathrm{Cl}_F^+(p^\beta)$$

where the first term $(p^{-\beta}/\mathfrak{o})^\times$ is defined to be a subset of $p^{-\beta}/\mathfrak{o}$ consisting of elements whose annihirater is p^β. Then, the Gauss sum $G(\phi)$ for ϕ is defined as follows:

$$G(\phi) = \sum_{x \in (p^{-\beta}/\mathfrak{o})^\times} \phi(x)\exp(2\pi\sqrt{-1}\mathrm{Tr}_{F/\mathbb{Q}}(x)).$$

By a similar calculation as §2.7, we have

$$\frac{1}{[\mathfrak{o}_+^\times : E]} \sum_{1 \le i \le h_F} \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{c}_i)^s \sum_{u_j \in ((p^{-\beta}/\mathfrak{o})^\times/\mathfrak{o}_+^\times)_i} \phi_i^{-1}(u_j) \int_{\sqrt{-1}(F\otimes\mathbb{R})_+^\times/E} f_i(z+u_j) z^{s-t-w_1} \bigwedge_{\tau \in I_F} dz_\tau$$
$$= L(\varphi \otimes \phi, s) G(\phi^{-1}) \prod_{\tau \in I_F} \frac{\Gamma(s - w_{1,\tau})}{(-2\pi\sqrt{-1})^{s-w_{1,\tau}}},$$

for any finite order character ϕ of $\mathrm{Cl}_F^+(p^\beta)$, where $((p^{-\beta}/\mathfrak{o})^\times/\mathfrak{o}_+^\times)_i$ is the inverse image of each $\mathfrak{c}_i \in \mathrm{Cl}_F^+$ in (15) which is isomorphic to $(p^{-\beta}/\mathfrak{o})^\times/\mathfrak{o}_+^\times$ and ϕ_i is a map on each component $((p^{-\beta}/\mathfrak{o})^\times/\mathfrak{o}_+^\times)_i$ induced by ϕ. In the integral of the above equation, we choose a lift $u_i \in (p^{-\beta}/\mathfrak{o})^\times$ of $u_i \in (p^{-\beta}/\mathfrak{o})^\times/\mathfrak{o}_+^\times$ (we use the same symbol by abuse of notation). It is not difficult to see that the integral does not depend on the choice of such a lift. Using this, we have the following corollary:

COROLLARY 3.7. *Let $\varphi \in S_{w_1,w_2}(K_1(\mathfrak{n}\,p^\alpha))$ be a normalized eigen cusp form of weight (w_1, w_2) and of level $K_1(\mathfrak{n}\,p^\alpha)$. Let v be an $\mathfrak{o}_{\varphi,(p)}$-basis of $H^d_c(Y_1(\mathfrak{n}p^\alpha); \mathcal{L}(w_1, w_2; \mathfrak{o}_{\varphi,(p)}))[\lambda_\varphi, \epsilon]$ modulo $\mathfrak{o}_{\varphi,(p)}$-torsion part. Then we have*

$$\sum_{\boldsymbol{u}\in \mathrm{Cl}^+_F(p^\beta)} \phi^{-1}(\boldsymbol{u})\mathrm{ev}^{(w_1,w_2)}_{\boldsymbol{u},\alpha,\beta}(v) = G(\phi^{-1})\frac{L(\varphi\otimes\phi, 0)}{C^\epsilon_{\varphi,\infty,v}}.$$

for any Herke character of finite order, where ϵ is the equal to ϕ restricted to $(F\otimes\mathbb{R})^\times$ identifying $(F\otimes\mathbb{R})^\times/(F\otimes\mathbb{R})^\times_+$ with $\{\pm 1\}^{I_F}$.

The following proposition is very important for the property of "invariance" of periods with respect to character twists:

PROPOSITION 3.8. *Let $\varphi \in S_{w_1,w_2}(K_1(\mathfrak{M}))$ be an eigen cuspform which is critical in the sense of Definition 3.1. Assume that φ is not a twist of another eigen cuspform by a Hecke character of a finite order. We take a twist $\varphi' = \varphi\otimes N^r_F\phi$ with the Norm character N_F and a Hecke character of finite order ϕ, which is of weight $(w_1 - rt, w_2 - rt)$. We assume that $(w_1 - rt, w_2 - rt)$ is also critical. Let v be an $\mathfrak{o}_{\varphi,(p)}$-basis of*

$$H^d_c(Y_1(\mathfrak{n}p^\alpha); \mathcal{L}(w_1, w_2; \mathfrak{o}_{\varphi,(p)}))[\lambda_\varphi, \epsilon]/(\mathfrak{o}_{\varphi,(p)}\text{-torsion})$$

and let v' be the twist $v[r]\otimes\phi$ which is an $\mathfrak{o}_{\varphi,(p)}$-basis of

$$H^d_c(Y_1(\mathfrak{n}p^\alpha); \mathcal{L}(w_1 - rt, w_2 - rt; \mathfrak{o}_{\varphi',(p)}[\phi]))[\lambda_{\varphi'}, \epsilon]/(\mathfrak{o}_{\varphi',(p)}\text{-torsion})$$

where $v[r]$ is a shift of weight of v by r.
Then we have:

$$C^\epsilon_{\varphi,\infty,v} = C^\epsilon_{\varphi',\infty,v'}\frac{G(\phi^{-1})}{(-2\pi\sqrt{-1})^{dr}}\prod_{\tau\in I_F}\frac{\Gamma(-w_{1,\tau}+r)}{\Gamma(-w_{1,\tau})}.$$

If there are not so much confusion, we sometimes omit the dependence on v of the period and denote it by $C_{\varphi,\infty}$. (see Remark 4.17 (2) for our idea for eliminating the dependence on the choice of v)

4. HIDA FAMILIES OF HILBERT MODULAR FORMS.

In the first half of this section we review Hida's theory of nearly ordinary Hilbert modular forms as developed in [H1] and [H2]. In particular, we recall Hida's control theorem for nearly ordinary p-adic Hecke algebras. Hida's proof relies, via the Jacquet-Langlands correspondence, on control theorems for cohomology groups coming from quaternion algebras over F which are totally definite, or indefinite but yield Shimura curves. The introduction of [H1] ends up with the hope that those results would be extended to *general quaternion algebras yielding varieties of higher dimension*. In the second half of this section we extend Hida's theory to the case of $M_2(F)$, corresponding to the d-dimensional Hilbert modular variety. Using the results of [Di1], we prove a control theorem 4.9 as well as a freeness result over the universal nearly ordinary p-adic Hecke algebra (see theorem 4.15).

The universal nearly ordinary p-adic Hecke algebra introduced by Hida is naturally an algebra over the Iwasawa algebra in $d+1+\delta_{F,p}$ variables ($\delta_{F,p}$ being the Leopoldt defect), and it is very important in our application in the following sections to separate the first d variables corresponding to the weight, from the other $1+\delta_{F,p}$ variables corresponding to twists by Hecke characters. We do that by a careful choice of level structure at primes dividing p.

Since we assumed that p is unramified in F, we denote by $\mathfrak{p}_1,\ldots,\mathfrak{p}_s$ the ideals of F above p. We fix an ideal $\mathfrak{n}$ of F prime to p.

4.1. VARIOUS LEVEL STRUCTURES. In Section 2.4, we introduced for $\alpha \geq 0$ the Hilbert modular varieties $Y_{11}(\mathfrak{n}p^\alpha)$, $Y_Z(p^\alpha)$ and $Y_1(p^\alpha)$ of level $K_1(\mathfrak{n})\cap K_{11}(p^\alpha)$, $K_Z(p^\alpha) = ZK_1(\mathfrak{n})\cap K_{11}(p^\alpha)$ and $K_1(\mathfrak{n}\,p^\alpha)$, as well as the sheaves $\mathcal{L}(w_1,w_2;A)$ on them. Here Z is a group of elements $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ with $a \in (\mathfrak{o}\otimes\mathbb{Z}_p)^\times$. Using the multi-index notation of 1.3, we have:

$$K_0(p^\alpha)/K_{11}(p^\alpha) \cong (\mathfrak{o}\,/p^\alpha)^\times \times (\mathfrak{o}\,/p^\alpha)^\times, \tag{18}$$

where the first factor is represented by the group of elements $\left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \middle|\, a \in (\mathfrak{o}\,/\prod \mathfrak{p}_i^{\alpha_i})^\times \right\}$ and the second factor is represented by the group of elements $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \middle|\, d \in (\mathfrak{o}\,/\prod \mathfrak{p}_i^{\alpha_i})^\times \right\}$. Hence, we have

$$K_0(p^\alpha)/K_{11}(p^\alpha)\,\mathfrak{o}^\times \cong \big((\mathfrak{o}\,/p^\alpha)^\times \times (\mathfrak{o}\,/p^\alpha)^\times\big)\,/\,\mathfrak{o}^\times \tag{19}$$

where we divide by the group generated by the elements $\left\{ \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \right\}$ where d runs through the image of $\mathfrak{o}^\times$ in $(\mathfrak{o}\,/p^\alpha)^\times$ which is also denoted by $\mathfrak{o}^\times$ by abuse of notation.

We introduce several tori as follows:

DEFINITION 4.1. Let us define T (resp. $T_{\rm n.o}$, T_Z, $T_{\rm ord}$) to be the p-Sylow subgroup of $\varprojlim_\alpha K_0(p^\alpha)/K_{11}(p^\alpha)$ (resp. $\varprojlim_\alpha K_0(p^\alpha)/K_{11}(p^\alpha)\,\mathfrak{o}^\times$, $\varprojlim_\alpha K_0(p^\alpha)/ZK_{11}(p^\alpha)$, $\varprojlim_\alpha K_0(p^\alpha)/K_1(p^\alpha)\,\mathfrak{o}^\times$).

By definition, we have the following relation:

$$\begin{array}{ccccc} & & T & & \\ & & \downarrow & & \\ T_{\rm ord} & \twoheadleftarrow & T_{\rm n.o} & \twoheadrightarrow & T_Z. \end{array} \tag{20}$$

DEFINITION 4.2. We define an Iwasawa algebra Λ (resp. $\Lambda_{\rm n.o}$, Λ_Z, $\Lambda_{\rm ord}$) to be $\mathcal{O}[[T]]$ (resp. $\mathcal{O}[[T_{\rm n.o}]]$, $\mathcal{O}[[T_Z]]$, $\mathcal{O}[[T_{\rm ord}]]$).

The algebra Λ (resp. Λ_Z) is isomorphic to $\mathcal{O}[[(1+p\,\mathbb{Z}_p)^{2d}]]$ (resp. $\mathcal{O}[[(1+p\,\mathbb{Z}_p)^d]]$). The algebra $\Lambda_{\rm n.o}$ (resp. $\Lambda_{\rm ord}$) is isomorphic to $\mathcal{O}[[(1+p\,\mathbb{Z}_p)^{d+1+\delta_{F,p}}]]$ (resp. $\mathcal{O}[[(1+p\,\mathbb{Z}_p)^{1+\delta_{F,p}}]]$), where $\delta_{F,p}$ is the Leopoldt defect for F and p which is expected to be zero by the Leopoldt conjecture.

We have the following relation:

$$
\begin{array}{ccccc}
 & & \Lambda & & \\
 & & \downarrow & & \\
\Lambda_{\mathrm{ord}} & \twoheadleftarrow & \Lambda_{\mathrm{n.o}} & \twoheadrightarrow & \Lambda_Z
\end{array}
\tag{21}
$$

where two vertical arrows are surjective. We remark that a $\Lambda_{\mathrm{n.o}}$-module (resp. Λ_Z-module, Λ_{ord}-module) M is naturally regarded as a Λ-module.

Let the module $\mathrm{H}^{\bullet}_{\mathrm{n.o}}(Y_{11}(\mathfrak{n}p^{\alpha}), \mathcal{L}(w_1, w_2; \mathcal{K}/\mathcal{O}))$ be the largest $\mathcal{O}$-direct summand of the module $\mathrm{H}^{\bullet}(Y_{11}(p^{\alpha}), \mathcal{L}(w_1, w_2; \mathcal{K}/\mathcal{O}))$ on which the Hecke operator $T_0(p)$ in Definition 2.2 is invertible. The Pontryagin dual $\mathcal{H}^{\bullet}_{\mathrm{n.o}}(w_1, w_2)$ of the limit $\varinjlim_{\alpha} \mathrm{H}^{\bullet}_{\mathrm{n.o}}(Y_{11}(\mathfrak{n}p^{\alpha}), \mathcal{L}(w_1, w_2; \mathcal{K}/\mathcal{O}))$ has a natural action of $T_{\mathrm{n.o}}$ and it is a module over the algebra $\Lambda_{\mathrm{n.o}}$.

Let the module $\mathrm{H}^{\bullet}_{\mathrm{n.o}}(Y_Z(\mathfrak{n}p^{\alpha}), \mathcal{L}(w_1, w_2; \mathcal{K}/\mathcal{O}))$ be the largest $\mathcal{O}$-direct summand of the module $\mathrm{H}^{\bullet}(Y_Z(\mathfrak{n}p^{\alpha}), \mathcal{L}(w_1, w_2; \mathcal{K}/\mathcal{O}))$ on which the Hecke operator $T_0(p)$ of Definition 2.2 is invertible. The Pontryagin dual $\mathcal{H}^{\bullet}_Z(w_1, w_2)$ of the limit $\varinjlim_{\alpha} \mathrm{H}^{\bullet}_{\mathrm{n.o}}(Y_Z(\mathfrak{n}p^{\alpha}), \mathcal{L}(w_1, w_2; \mathcal{K}/\mathcal{O}))$ has a natural action of T_Z and it is a module over the algebra Λ_Z.

Let the module $\mathrm{H}^{\bullet}_{\mathrm{ord}}(Y_1(\mathfrak{n}p^{\alpha}), \mathcal{L}(w_1, w_2; \mathcal{K}/\mathcal{O}))$ be the largest $\mathcal{O}$-direct summand of the module $\mathrm{H}^{\bullet}(Y_1(\mathfrak{n}p^{\alpha}), \mathcal{L}(w_1, w_2; \mathcal{K}/\mathcal{O}))$ on which the Hecke operator $T_0(p)$ of Definition 2.2 is invertible.

The Pontryagin dual $\mathcal{H}^{\bullet}_{\mathrm{ord}}(w_1, w_2)$ of the limit

$$\varinjlim_{\alpha} \mathrm{H}^{\bullet}_{\mathrm{ord}}(Y_1(\mathfrak{n}p^{\alpha}), \mathcal{L}(w_1, w_2; \mathcal{K}/\mathcal{O}))$$

has a natural action of T_{ord} and it is a module over $\mathcal{O}[[T_{\mathrm{ord}}]] \cong \mathcal{O}[[(1 + p\,\mathbb{Z}_p)^{1+\delta_{F,p}}]]$.

REMARK 4.3. Let $\overline{\rho}$ be a nearly ordinary residual Galois representation of G_F which is modular of level $K_1(\mathfrak{n}) \cap K_{11}(p^{\alpha})$ and nearly ordinary at p. For any cohomological weight $w = (w_1, w_2)$, we have

$$
\begin{aligned}
&\mathrm{H}^{\bullet}_{\mathrm{n.o}}(Y_{11}(\mathfrak{n}p^{\alpha}), \mathcal{L}(w_1, w_2; \mathcal{K}/\mathcal{O}))_{\overline{\rho}} \neq 0, \\
&\mathrm{H}^{\bullet}_{\mathrm{n.o}}(Y_Z(\mathfrak{n}p^{\alpha}), \mathcal{L}(w_1, w_2; \mathcal{K}/\mathcal{O}))_{\overline{\rho}} \neq 0,
\end{aligned}
$$

where $(\quad)_{\overline{\rho}}$ means the $\overline{\rho}$-part as explained at the equation (1). On the other hand, $\mathrm{H}^{\bullet}_{\mathrm{n.o}}(Y_1(\mathfrak{n}p^{\alpha}), \mathcal{L}(w_1, w_2; \mathcal{K}/\mathcal{O}))_{\overline{\rho}}$ can be zero for certain (w_1, w_2). However, there exists a double-digit weight (w_1, w_2) such that $\mathrm{H}^{\bullet}_{\mathrm{n.o}}(Y_1(\mathfrak{n}p^{\alpha}), \mathcal{L}(w_1, w_2; \mathcal{K}/\mathcal{O}))_{\overline{\rho}} \neq 0$.

We say that a residual representation $\overline{\rho}$ is realized in

$$\mathrm{H}^{\bullet}_{\mathrm{n.o}}(Y_1(\mathfrak{n}p^{\alpha}), \mathcal{L}(w_1, w_2; \mathcal{K}/\mathcal{O}))$$

when we have $\mathrm{H}^{\bullet}_{\mathrm{n.o}}(Y_1(\mathfrak{n}p^{\alpha}), \mathcal{L}(w_1, w_2; \mathcal{K}/\mathcal{O}))_{\overline{\rho}} \neq 0$. We recall the following lemma, which is straightforward from the construction of Hida family.

LEMMA 4.4. *Suppose that $\overline{\rho} : G_F \longrightarrow \overline{\mathbb{F}}_p$ is a nearly ordinary residual Galois representation of level $K_1(\mathfrak{n}p)$ and double-digit weight (a_1, a_2). Then $\overline{\rho}$*

is realized in the cohomology $\mathrm{H}^{\bullet}_{\mathrm{n.o}}(Y_1(\mathfrak{n}p^\alpha), \mathcal{L}(w_1, w_2; \mathcal{K}/\mathcal{O}))$ *if and only if the maps* $(\mathfrak{o}/p)^\times \longrightarrow (\mathfrak{o}/p)^\times,\ u \mapsto u^{w_1-a_1}$ *and* $(\mathfrak{o}/p)^\times \longrightarrow (\mathfrak{o}/p)^\times,\ u \mapsto u^{w_2-a_2}$ *annihilate the image of* $\mathfrak{o}_+^\times$ *in* $(\mathfrak{o}/p)^\times$.

When $\overline{\rho}$ is not realized in

$$\mathrm{H}^{\bullet}_{\mathrm{n.o}}(Y_1(\mathfrak{n}p^\alpha), \mathcal{L}(w_1, w_2; \mathcal{K}/\mathcal{O})),$$

$\mathrm{H}^{\bullet}_{\mathrm{n.o}}(Y_1(\mathfrak{n}p^\alpha), \mathcal{L}(w_1, w_2; \mathcal{K}/\mathcal{O}))_{\overline{\rho}}$ is defined to be zero. We denote by $\mathcal{H}^{\bullet}_{\mathrm{n.o}}(w_1, w_2)_{\overline{\rho}}$ the $\overline{\rho}$-part of $\mathcal{H}^{\bullet}_{\mathrm{n.o}}(w_1, w_2)$, which is defined to be the Pontrjagin dual of $\varinjlim_{\alpha} \mathrm{H}^{\bullet}_{\mathrm{n.o}}(Y_{11}(\mathfrak{n}p^\alpha), \mathcal{L}(w_1, w_2; \mathcal{K}/\mathcal{O}))_{\overline{\rho}}$.

Let $(r_1, r_2) \in \mathbb{Z}[I_F] \times \mathbb{Z}[I_F]$. For a Λ-module M, we denote by $\mathrm{tw}^{(r_1,r_2)}(M)$ the module M whose Λ-module structure is twisted by

$$\Lambda \to \Lambda\ ,\ [a, d] \mapsto a^{r_1} d^{r_2} [a, d], \tag{22}$$

where we identify $(\mathfrak{o} \otimes \mathbb{Z}_p)^\times \times (\mathfrak{o} \otimes \mathbb{Z}_p)^\times$ with T by

$$(a, d) \mapsto \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.$$

By Hida, $\mathcal{H}^{\bullet}_{\mathrm{n.o}}(w_1, w_2)_{\overline{\rho}}$ is independent of (w_1, w_2) in the following sense:

Theorem 4.5. [H2, (3.3)] *The* Λ*-modules* $\mathcal{H}^{\bullet}_{\mathrm{n.o}}(w_1, w_2)_{\overline{\rho}}$ *and* $\mathrm{tw}^{(w_1-w_1', w_2-w_2')}(\mathcal{H}^{\bullet}_{\mathrm{n.o}}(w_1', w_2')_{\overline{\rho}})$ *are isomorphic to each other for* $(w_1, w_2), (w_1', w_2') \in \mathbb{Z}[I_F] \times \mathbb{Z}[I_F]$ *cohomological double-digit weights* (*cf.* §*2.4*).

Corollary 4.6. *Let* $(w_1, w_2), (w_1', w_2') \in \mathbb{Z}[I_F] \times \mathbb{Z}[I_F]$ *be cohomological weights* (*cf.* §*2.4*).

(i) *The* $\Lambda_{\mathrm{n.o}}$*-module* $\mathcal{H}^{\bullet}_{\mathrm{n.o}}(w_1, w_2)_{\overline{\rho}}$ *is always isomorphic to* $\mathcal{H}^{\bullet}_{\mathrm{n.o}}(w_1', w_2')_{\overline{\rho}}$ *with* $\Lambda_{\mathrm{n.o}}$*-structure is twisted by the above twists.*

(ii) *The* Λ_{ord}*-module* $\mathcal{H}^{\bullet}_{\mathrm{ord}}(w_1, w_2)_{\overline{\rho}}$ *is isomorphic to*

$$\mathrm{tw}^{(w_1-w_1', w_2-w_2')}(\mathcal{H}^{\bullet}_{\mathrm{ord}}(w_1', w_2')_{\overline{\rho}})$$

with Λ_{ord}*-structure twisted by* (*22*) *if and only if we have* $w_1 = w_1'$ *and* $w_2 - w_2' \in \mathbb{Z}\,t$.

(iii) *The* Λ_Z*-module* $\mathcal{H}^{\bullet}_{Z}(w_1, w_2)_{\overline{\rho}}$ *is isomorphic to* $\mathcal{H}^{\bullet}_{Z}(w_1', w_2')_{\overline{\rho}}$ *with* Λ_Z*-structure is twisted by* (*22*) *if and only if* $w_1 + w_2 = w_1' + w_2'$.

Definition 4.7. Let (w_1, w_2) be a cohomological double-digit weight. We denote by $\mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}}(w_1, w_2)$ (resp. $\mathbb{T}^{Z}_{\overline{\rho}}(w_1, w_2)$, $\mathbb{T}^{\mathrm{ord}}_{\overline{\rho}}(w_1, w_2)$) the Hecke algebra which is defined to be the subalgebra in the $\Lambda_{\mathrm{n.o}}$-linear (resp. Λ_Z-linear, Λ_{ord}-linear) endomorphism ring of $\mathcal{H}^{d}_{\mathrm{n.o}}(w_1, w_2)_{\overline{\rho}}$ (resp. $\mathcal{H}^{d}_{Z}(w_1, w_2)_{\overline{\rho}}$, $\mathcal{H}^{d}_{\mathrm{ord}}(w_1, w_2)_{\overline{\rho}}$) generated by all Hecke operators.

Suppose that $\overline{\rho}$ is realized in $\mathrm{H}^{\bullet}_{\mathrm{n.o}}(Y_1(\mathfrak{n}p^\alpha), \mathcal{L}(w_1, w_2; \mathcal{K}/\mathcal{O}))$ and it is also realized in $\mathrm{H}^{\bullet}_{\mathrm{n.o}}(Y_1(\mathfrak{n}p^\alpha), \mathcal{L}(w_1', w_2'; \mathcal{K}/\mathcal{O}))$. By Corollary 4.6, $\mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}}(w_1, w_2)$ (resp. $\mathbb{T}^{Z}_{\overline{\rho}}(w_1, w_2)$, $\mathbb{T}^{\mathrm{ord}}_{\overline{\rho}}(w_1, w_2)$) is isomorphic to $\mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}}(w_1', w_2')$ (resp. $\mathbb{T}^{Z}_{\overline{\rho}}(w_1', w_2')$, $\mathbb{T}^{\mathrm{ord}}_{\overline{\rho}}(w_1', w_2')$) without any condition (resp. if $w_1 + w_2 = w_1' + w_2'$, if $w_1 = w_1'$ and $w_2 - w_2' \in \mathbb{Z}[t]$).

4.2. CONTROL THEOREM FOR NEARLY ORDINARY COHOMOLOGY. As explained in the introduction, the condition (Van$_{\overline{\rho}}$) is essential for the construction of our p-adic L-function. Before starting the construction, we will list some of known sufficient conditions which ensure the condition (Van$_{\overline{\rho}}$) stated at the beginning of §1.2.

REMARK 4.8. Let $\overline{\rho}$ be a residual Galois representation of G_F which is nearly ordinary at p and of level $K_1(\mathfrak{n}) \cap K_{11}(p)$.

(1) The paper [Di1] by Dimitrov shows that (Van$_{\overline{\rho}}$) holds if the conditions ($\star$) and ($\star\star$) below are true:

($\star$) $\overline{\rho}$ is realized in $H^d_{\mathrm{n.o}}(Y_{K_1(\mathfrak{n})}, \mathcal{L}(a_1, a_2, \mathcal{K}/\mathcal{O}))$ of double-digit weight $a = (a_1, a_2) \in \mathbb{Z}[I_F] \times \mathbb{Z}[I_F]$ satisfying

$$\sum_{\tau \in I_F} (a_{2,\tau} - a_{1,\tau}) < p - 1.$$

($\star\star$) The representation $\underset{\tau \in I_F}{\otimes} \overline{\rho}(\tau^{-1} \cdot \tau)$ of the absolute Galois group $G_{\widetilde{F}}$ is irreducible of order divisible by p, where $\widetilde{F}/\mathbb{Q}$ denotes the Galois closure of $F/\mathbb{Q}$ in $\overline{\mathbb{Q}}$.

We also remark that, if F is Galois over $\mathbb{Q}$ and if φ is a (parabolic) newform which is not a theta series, nor a twist of a base change, then for almost all ordinary primes p, $\overline{\rho} = \rho_\varphi \mod p$ satisfies ($\star$) and ($\star\star$).

(2) The paper [LS] by Lan and Suh shows that the condition (Van$_{\overline{\rho}}$) is true if the weight of $\overline{\rho}$ is regular.

The general formalism of control theorems established in [H1, §10] and [H4] and the condition (Van$_{\overline{\rho}}$) immediately implies the following theorem.

THEOREM 4.9. *Let $\overline{\rho}$ be a residual Galois representation of G_F which is nearly ordinary at p and of level $K_1(\mathfrak{n}) \cap K_{11}(p)$ satisfying the condition* (Van$_{\overline{\rho}}$) *stated at the beginning of §1.2. Then we have,*

(i) *$(\mathcal{H}^q_{\mathrm{n.o}})_{\overline{\rho}}$ is a free $\Lambda_{\mathrm{n.o}}$-module of finite rank and vanishes if $q \neq d$.*

(ii) *For any cohomological double-digit weight (w_1, w_2), we have the following isomorphism:*

$$\begin{aligned}
&\mathrm{tw}^{(w_1,w_2)}((\mathcal{H}^d_{\mathrm{n.o}})_{\overline{\rho}}) \underset{\Lambda_{\mathrm{n.o}}}{\otimes} \mathcal{O}\left[(K_0(p^\alpha)/K_{11}(p^\alpha)\,\mathfrak{o}^\times)_p\right] \\
&\cong (\mathcal{H}^d_{\mathrm{n.o}}(w_1, w_2))_{\overline{\rho}} \underset{\Lambda_{\mathrm{n.o}}}{\otimes} \mathcal{O}\left[(K_0(p^\alpha)/K_{11}(p^\alpha)\,\mathfrak{o}^\times)_p\right] \\
&\cong \mathrm{H}^d_{\mathrm{n.o}}(Y_{11}(\mathfrak{n}p^\alpha), \mathcal{L}(w_1, w_2; \mathcal{K}/\mathcal{O}))^{\mathrm{PD}}_{\overline{\rho}}.
\end{aligned}$$

COROLLARY 4.10. *Let $\overline{\rho}$ be a residual Galois representation of G_F which is nearly ordinary at p and of level $K_1(\mathfrak{n}) \cap K_{11}(p)$ satisfying the condition* (Van$_{\overline{\rho}}$).

(i) *$\mathcal{H}^q_{\mathrm{ord}}(w_1, w_2)_{\overline{\rho}}$ (resp. $\mathcal{H}^q_Z(w_1, w_2)_{\overline{\rho}}$) is a free Λ_{ord}-module (resp. Λ_Z-module) of finite rank and vanishes if $q \neq d$.*

(ii) *We have the following isomorphisms:*

$$\mathcal{H}^d_{\mathrm{ord}}(w_1,w_2)_{\overline{\rho}}\otimes_{\Lambda_{\mathrm{ord}}}\mathcal{O}\left[(K_0(p^\alpha)/K_1(p^\alpha)\,\mathfrak{o}^\times)_p\right]\cong \mathrm{H}^d_{\mathrm{n.o}}(Y_1(\mathfrak{n}p^\alpha),\mathcal{L}(w_1,w_2;\mathcal{K}/\mathcal{O}))^{\mathrm{PD}}_{\overline{\rho}}$$
$$\mathcal{H}^d_{Z}(w_1,w_2)_{\overline{\rho}}\otimes_{\Lambda_{Z}}\mathcal{O}\left[(K_0(p^\alpha)/ZK_{11}(p^\alpha))_p\right]\cong \mathrm{H}^d_{\mathrm{n.o}}(Y_Z(\mathfrak{n}p^\alpha),\mathcal{L}(w_1,w_2;\mathcal{K}/\mathcal{O}))^{\mathrm{PD}}_{\overline{\rho}}$$

Theorem 4.9 (ii) and Corollary 4.10 (ii) ensure the existence of Hida families in our setting (cf. Remark 1.5 for the relation between our construction and Hida's construction [H1], [H2], [H3]).
Now, we recall that we have another group of global units:

$$\left\{ \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} \middle| \, d\in\mathfrak{o}^\times \right\}. \tag{23}$$

We have the following definition by using the above group.

Definition 4.11. (1) Let $w=(w_1,w_2)$ be a cohomological pair and let $\alpha=(\alpha_1,\cdots,\alpha_s)$ be an element of $\mathbb{Z}^s$. We define an ideal $P^{\mathrm{n.o}}_{w;\alpha}\in\Lambda_{\mathrm{n.o}}$ (resp. $P^{Z}_{w;\alpha}\in\Lambda_{Z}$) to be the ideal such that

$$\mathrm{tw}^{-w_1,-w_2}(P^{\mathrm{n.o}}_{w;\alpha})=\mathrm{Ker}\left[\Lambda_{\mathrm{n.o}}\longrightarrow\mathcal{O}\left[(K_0(p^\alpha)/K_{11}(p^\alpha)\,\mathfrak{o}^\times)_p\right]\right]$$
$$\mathrm{tw}^{-w_1,-w_2}(P^{Z}_{w;\alpha})=\mathrm{Ker}\left[\Lambda_{Z}\longrightarrow\mathcal{O}\left[(K_0(p^\alpha)/ZK_{11}(p^\alpha))_p\right]\right]$$

(2) Let $\mathcal{O}\left[(K_0(p^\alpha)/K_{11}(p^\alpha)\,\mathfrak{o}^\times)_p\right]^{\mathrm{arith}}$ (resp. $\mathcal{O}\left[(K_0(p^\alpha)/ZK_{11}(p^\alpha))_p\right]^{\mathrm{arith}}$) the quotient ring of

$$\mathcal{O}\left[(K_0(p^\alpha)/K_{11}(p^\alpha)\,\mathfrak{o}^\times)_p\right] \text{ (resp. } \mathcal{O}\left[(K_0(p^\alpha)/ZK_{11}(p^\alpha))_p\right])$$

by the global units action of (23). We define an ideal $P^{\mathrm{n.o,arith}}_{w;\alpha}\in\Lambda_{\mathrm{n.o}}$ (resp. $P^{Z,\mathrm{arith}}_{w;\alpha}\in\Lambda_{Z}$) to be the ideal such that

$$\mathrm{tw}^{-w_1,-w_2}(P^{\mathrm{n.o,arith}}_{w;\alpha})=\mathrm{Ker}\left[\Lambda_{\mathrm{n.o}}\longrightarrow\mathcal{O}\left[(K_0(p^\alpha)/K_{11}(p^\alpha)\,\mathfrak{o}^\times)_p\right]^{\mathrm{arith}}\right]$$
$$\mathrm{tw}^{-w_1,-w_2}(P^{Z,\mathrm{arith}}_{w;\alpha})=\mathrm{Ker}\left[\Lambda_{Z}\longrightarrow\mathcal{O}\left[(K_0(p^\alpha)/ZK_{11}(p^\alpha))_p\right]^{\mathrm{arith}}\right]$$

Note that, $P^{\mathrm{n.o}}_{w;\alpha}$ and $P^{\mathrm{n.o,arith}}_{w;\alpha}$ are defined for any $w=(w_1,w_2)$, while $P^{Z}_{w;\alpha}$ and $P^{Z,\mathrm{arith}}_{w;\alpha}$ are defined only for $w=(w_1,w_2)$ such that $w_1+w_2=0$. By definition, we have $P^{\mathrm{n.o,arith}}_{w;\alpha}\supset P^{\mathrm{n.o}}_{w;\alpha}$ (resp. $P^{Z,\mathrm{arith}}_{w;\alpha}\supset P^{Z}_{w;\alpha}$).

4.3. Freeness result for nearly ordinary cohomology. Let us recall the following results:

Theorem 4.12. *Let $\overline{\rho}$ be a residual Galois representation of G_F which is nearly ordinary at p and of level $K_1(\mathfrak{n})\cap K_{11}(p)$ satisfying the condition* ($\mathrm{Van}_{\overline{\rho}}$). *Let us fix a cohomological double-digit weight $a=(a_1,a_2)\in\mathbb{Z}[I_F]\times\mathbb{Z}[I_F]$. Then, the Hecke algebra $\mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}}(a_1,a_2)$ (resp. $\mathbb{T}^{Z}_{\overline{\rho}}(a_1,a_2)$) is free over $\Lambda_{\mathrm{n.o}}$ (resp. Λ_Z) when it is realized in $\mathrm{H}^\bullet_{\mathrm{n.o}}(Y_1(\mathfrak{n}p^\alpha),\mathcal{L}(a_1,a_2;\mathcal{K}/\mathcal{O}))$ in the sense given before*

Lemma 4.4. For any cohomological double-digit weight $w = (w_1, w_2)$, we have exact control:

$$\mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}}(a_1, a_2) \otimes_{\Lambda_{\mathrm{n.o}}} \Lambda_{\mathrm{n.o}}/P^{\mathrm{n.o,arith}}_{w-a,\alpha} \cong (\mathfrak{h}^{\mathrm{n.o}}_{w;\alpha})_{\overline{\rho}},$$
$$\mathbb{T}^{Z}_{\overline{\rho}}(a_1, a_2) \otimes_{\Lambda_Z} (\Lambda_Z/P^{Z,\mathrm{arith}}_{w-a;\alpha}) \cong (\mathfrak{h}^{Z}_{w,\alpha})_{\overline{\rho}}$$

where $\mathfrak{h}^{\mathrm{n.o}}_{w,\alpha}$ (resp. $\mathfrak{h}^{Z}_{w,\alpha}$) is the Hecke algebra acting on $\mathrm{H}^d_{\mathrm{n.o}}(Y_{11}(p^\alpha), \mathcal{L}(w_1, w_2; \mathcal{K}/\mathcal{O}))^{\mathrm{PD}}_{\overline{\rho}}$ (resp. $\mathrm{H}^d_{\mathrm{n.o}}(Y_Z(\mathfrak{n}p^\alpha), \mathcal{L}(w_1, w_2; \mathcal{K}/\mathcal{O}))^{\mathrm{PD}}_{\overline{\rho}}$).

Proof. By Theorem 4.9 (ii) and by the assumption ($\mathrm{Van}_{\overline{\rho}}$), we have

$$\mathcal{H}^d_{\mathrm{n.o}}(a_1, a_2)_{\overline{\rho}} \otimes \Lambda_{\mathrm{n.o}}/P^{\mathrm{n.o,arith}}_{w-a;\alpha} \cong (\mathrm{H}^d(Y_{K_1(\mathfrak{n})\cap K_{11}(p^\alpha)}, \mathcal{L}(w_1, w_2; \mathcal{K}/\mathcal{O}))_{\overline{\rho}})^{\mathrm{PD}}$$

at each cohomological double-digit weight $w = (w_1, w_2)$, By specializing at $P^{\mathrm{n.o,arith}}_{w-a;\alpha}$, we identify

$$\mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}}(a_1, a_2) \otimes \Lambda_{\mathrm{n.o}}/P^{\mathrm{n.o,arith}}_{w-a;\alpha} \subset \mathrm{End}_{\mathcal{O}}\big(\mathcal{H}^d_{\mathrm{n.o}}(a_1, a_2)_{\overline{\rho}} \otimes \Lambda_{\mathrm{n.o}}/P^{\mathrm{n.o,arith}}_{w-a;\alpha}\big)$$

with $\mathfrak{h}^{\mathrm{n.o}}_{w;\alpha}$. The statement for $\mathfrak{h}^{Z}_{w;\alpha}$ is done in the same way. Hence, we omit the proof for $\mathfrak{h}^{Z}_{w;\alpha}$. □

DEFINITION 4.13. Let $w = (w_1, w_2)$ be a cohomological weight and let α be an element of $\mathbb{N}^s$. A ring homomorphism $\kappa : \Lambda^{\mathrm{n.o}} \longrightarrow \overline{\mathbb{Q}}_p$ is called an arithmetic specialization of weight w and level p^α if the kernel of κ contains $P^{\mathrm{n.o,arith}}_{w;\alpha}$. For any finite $\Lambda^{\mathrm{n.o}}$-algebra R, a ring homomorphism $\kappa : R \longrightarrow \overline{\mathbb{Q}}_p$ is called an arithmetic specialization of weight w and level p^α if $\Lambda^{\mathrm{n.o}} \longrightarrow R \xrightarrow{\kappa} \overline{\mathbb{Q}}_p$ is an arithmetic specialization of weight w and level p^α in the above sense. The arithmetic specialization of Λ^Z or a finite Λ^Z-algebra is defined similarly.

REMARK 4.14. By Theorem 4.12, we have a one-to-one correspondence under the condition ($\mathrm{Van}_{\overline{\rho}}$):

$$\left\{\begin{array}{c}\text{normalized Hilbert modular eigen cuspforms } \varphi \\ \text{of double-digit weight } (w_1, w_2), \text{ level } K_1(\mathfrak{n}\, p^\alpha) \\ \text{which are nearly ordinary at } p \\ \text{and whose residual Galois representations} \\ \text{are isomorphic to } \overline{\rho}\end{array}\right\}$$
$$\xrightarrow{\sim} \left\{\begin{array}{c}\text{arithmetic specializations} \\ \kappa : \mathbb{T}_{\overline{\rho}} \longrightarrow \overline{\mathbb{Q}}_p \\ \text{which have weight } (w_1, w_2) \text{ and level } p^\alpha\end{array}\right\}$$

We denote by $\kappa_\varphi : \mathbb{T}_{\overline{\rho}} \longrightarrow \overline{\mathbb{Q}}_p$ the arithmetic specialization associated to an eigen cuspform φ via the above correspondence.

As is explained after Definition 4.7, $\mathbb{T}^{Z}_{\overline{\rho}}(w_1, w_2)$ depends only on $w_1 + w_2$. Hence, we denote $\mathbb{T}^{Z}_{\overline{\rho}}(w_1, w_2)$ by $\mathbb{T}^{Z, w_1+w_2}_{\overline{\rho}}$. Note that, once we fix a branch $\mathcal{R}$ of $\mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}}(w_1, w_2)$, a branch $\mathcal{R}^{Z, w_1+w_2} := \mathcal{R} \otimes_{\mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}}(w_1, w_2)} \mathbb{T}^{Z, w_1+w_2}_{\overline{\rho}}$ of $\mathbb{T}^{Z, w_1+w_2}_{\overline{\rho}}$ is induced for each cohomological double-digit weight (w_1, w_2). We will present the following refinement of Theorem 4.9 which takes into counts the action of the whole Hecke algebra.

Theorem 4.15. *Let $\overline{\rho}$ be a residual Galois representation of G_F which is nearly ordinary at p and of level $K_1(\mathfrak{n}) \cap K_{11}(p)$ satisfying the condition* ($\mathrm{Van}_{\overline{\rho}}$)*. Let $w = (w_1, w_2)$ be a cohomological double-digit weight and let $\mathcal{R}$ be a branch of $\mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}}(w_1, w_2)$. Then,*

(1) *The $\mathcal{R}$-module $\mathcal{M}(w_1, w_2)_{\mathcal{R}} := \mathcal{H}^d_{\mathrm{n.o}}(w_1, w_2)_{\overline{\rho}} \otimes_{\mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}}} \mathcal{R}$ is free of rank one over $\mathcal{R}$. Similarly, The $\mathcal{R}^{Z,w_1+w_2}$-module $\mathcal{M}^Z(w_1, w_2)_{\mathcal{R}^{Z,w_1+w_2}} := \mathcal{M}(w_1, w_2)_{\mathcal{R}} \otimes_{\mathcal{R}} \mathcal{R}^{Z,w_1+w_2}$ is free of rank 2^d over $\mathcal{R}^{Z,w_1+w_2}$.*

(2) *For any character ϵ of the $\{\pm 1\}^{I_F}$, $\mathcal{M}(w_1, w_2)_{\mathcal{R}}[\epsilon]$ (resp. $\mathcal{M}^Z(w_1, w_2)_{\mathcal{R}^{Z,w_1+w_2}}[\epsilon]$) is free of rank one over $\mathcal{R}$ (resp. $\mathcal{R}^{Z,w_1+w_2}$).*

Proof. We will write the proof only for $\mathcal{M}(w_1, w_2)_{\mathcal{R}}$ and $\mathcal{M}(w_1, w_2)_{\mathcal{R}}[\epsilon]$. We can apply exactly the same proof for $\mathcal{M}^Z(w_1, w_2)_{\mathcal{R}^{Z,w_1+w_2}}$ and $\mathcal{M}^Z(w_1, w_2)_{\mathcal{R}^{Z,w_1+w_2}}[\epsilon]$.

Since p is not equal to 2 and the group $\{\pm 1\}^{I_F}$ is of 2-power order, we have a direct sum decomposition as $\mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}}(w_1, w_2)$-module with respect to the action of the group $\{\pm 1\}^{I_F}$ on the Betti cohomology as follows:

$$\mathcal{H}^d_{\mathrm{n.o}}(w_1, w_2)_{\overline{\rho}} = \bigoplus_{\epsilon} \mathcal{H}^d_{\mathrm{n.o}}(w_1, w_2)_{\overline{\rho}}[\epsilon].$$

This induces the direct sum decomposition as $\mathcal{R}$-module

$$\mathcal{M}(w_1, w_2)_{\mathcal{R}} = \bigoplus_{\epsilon} \mathcal{M}(w_1, w_2)_{\mathcal{R}}[\epsilon].$$

Hence, the statement (2) implies the statement (1). We will prove the statement (2) in the rest of the proof. By Theorem 4.9 (ii) and by the condition ($\mathrm{Van}_{\overline{\rho}}$), we have

$$\begin{aligned}\mathcal{M}(w_1, w_2)_{\mathcal{R}}[\epsilon] \otimes_{\mathcal{R}} \mathcal{R}/\mathfrak{M} &\cong \mathcal{H}^d_{\mathrm{n.o}}(w_1, w_2)_{\overline{\rho}}[\epsilon] \otimes_{\mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}}} \mathcal{R}/\mathfrak{M} \\ &\cong (\mathrm{H}^d(Y_{K_1(\mathfrak{n}) \cap K_0(p)}, \mathcal{L}(w_1, w_2; \mathcal{R}/\mathfrak{M})[\epsilon])^{\mathrm{PD}} \\ &\cong (\mathrm{H}^d(Y_{K_1(\mathfrak{n})}, \mathcal{L}(w_1, w_2; \mathcal{R}/\mathfrak{M}))[\epsilon])^{\mathrm{PD}}\end{aligned}$$

where the second isomorphism holds thanks to the ordinarity. Note that the condition ($\mathrm{Van}_{\overline{\rho}}$) easily implies that $\mathrm{H}^d(Y_{K_1(\mathfrak{n})}, \mathcal{L}(w_1, w_2; \mathcal{R}/\mathfrak{M}))[\epsilon])$ is of rank one over $\mathcal{R}/\mathfrak{M}$. By Nakayama's lemma, $\mathcal{M}(w_1, w_2)_{\mathcal{R}}[\epsilon]$ is a cyclic $\mathcal{R}$-module. Thus, $\mathcal{M}(w_1, w_2)_{\mathcal{R}}[\epsilon]$ is isomorphic to $\mathcal{R}/I$ for some ideal I of $\mathcal{R}$. Since $\mathcal{R}$ is an integral domain which is integral over $\Lambda_{\mathrm{n.o}}$, we have $I \neq 0$ if and only if $I \cap \Lambda_{\mathrm{n.o}} \neq 0$. Since $\mathcal{M}(w_1, w_2)_{\mathcal{R}}[\epsilon]$ is a torsion-free $\Lambda_{\mathrm{n.o}}$-module by Theorem 4.12, this implies $I = 0$. Thus $\mathcal{M}(w_1, w_2)_{\mathcal{R}}[\epsilon]$ is a free $\mathcal{R}$-module of rank one. □

4.4. p-adic periods. Let us consider the homomorphism

$$T^{\mathrm{n.o}} \twoheadrightarrow ((\mathfrak{o} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times / \overline{\mathfrak{o}^\times})_p. \tag{24}$$

induced from the map $T = ((\mathfrak{o}\otimes_{\mathbb{Z}}\mathbb{Z}_p)^\times/\overline{\mathfrak{o}^\times})_p \times ((\mathfrak{o}\otimes_{\mathbb{Z}}\mathbb{Z}_p)^\times/\overline{\mathfrak{o}^\times})_p \twoheadrightarrow ((\mathfrak{o}\otimes_{\mathbb{Z}}\mathbb{Z}_p)^\times/\overline{\mathfrak{o}^\times})_p$, $(a,d) \mapsto ad^{-1}$. Note that the kernel of (24) is equal to Z.

By identifying $((\mathfrak{o}\otimes_{\mathbb{Z}}\mathbb{Z}_p)^\times/\overline{\mathfrak{o}^\times})_p$ canonically with $\mathrm{Gal}(F_{\{p\}}/F)$ where $F_{\{p\}}$ is the maximal abelian pro-p extension of F unramified outside primes above p, we have the isomorphism

$$\Lambda \cong \Lambda_Z\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[\mathrm{Gal}(F_{\{p\}}/F)]]. \tag{25}$$

The isomorphism (25) induces the following isomorphism for any cohomological double-digit weight $(w_1,w_2)\in\mathbb{Z}[I_F]\times\mathbb{Z}[I_F]$:

$$\mathcal{H}^d_{\mathrm{n.o}}(w_1,w_2)_{\overline{\rho}} \cong \mathcal{H}^d_Z(w_1,w_2)_{\overline{\rho}}\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[\mathrm{Gal}(F_{\{p\}}/F)]]. \tag{26}$$

Let $w=(w_1,w_2)\in\mathbb{Z}[I_F]\times\mathbb{Z}[I_F]$ be a cohomological double-digit weight. Since $w_1+w_2\in t\mathbb{Z}$, there exists an integer $r=r(w)$ such that $w_1+w_2=rt$, we denote by $w^Z=(w_1^Z,w_2^Z)\in\mathbb{Z}[I_F]\times\mathbb{Z}[I_F]$ by

$$w_i^Z = \begin{cases} w_i-(\frac{r}{2}+1)t & \text{if } r \text{ is even,}\\ w_i-\frac{(r+1)}{2}t & \text{if } r \text{ is odd,}\end{cases}$$

for $i=1,2$.

We also have the identification:

$$\mathcal{H}^d_{\mathrm{n.o}}(w_1,w_2)_{\overline{\rho}} \cong \mathcal{H}^d_{\mathrm{n.o}}(w_1^Z,w_2^Z)_{\overline{\rho}}. \tag{27}$$

induced by the twist by $\chi_{\mathrm{cyc}}^{\frac{r(w)}{2}+1}$ (resp. $\chi_{\mathrm{cyc}}^{\frac{r(w)+1}{2}}$) when $r(w)$ is even (reps. odd). The isomorphisms (26) and (27) induce the following isomorphism:

$$\mathcal{H}^d_{\mathrm{n.o}}(w_1,w_2)_{\overline{\rho}} \cong \mathcal{H}^d_Z(w_1^Z,w_2^Z)_{\overline{\rho}}\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[\mathrm{Gal}(F_{\{p\}}/F)]]. \tag{28}$$

Note that $w_1^Z+w_2^Z$ is equal to $-2t$ (resp. $-t$) when $r(w)$ is even (resp. odd). For a branch $\mathcal{R}$ of $\mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}}\subset\mathrm{End}_{\Lambda_{\mathrm{n.o}}}(\mathcal{H}^d_{\mathrm{n.o}}(w_1,w_2)_{\overline{\rho}})$, the identification (28) induces the following isomorphism:

$$\mathcal{R}\cong\begin{cases}\mathcal{R}^{Z,-2t}\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[\mathrm{Gal}(F_{\{p\}}/F)]] & \text{when } r(w) \text{ is even,}\\ \mathcal{R}^{Z,-t}\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[\mathrm{Gal}(F_{\{p\}}/F)]] & \text{when } r(w) \text{ is odd,}\end{cases} \tag{29}$$

where $\mathcal{R}^{Z,-2t}$ (resp. $\mathcal{R}^{Z,-t}$) is a branch of $\mathbb{T}^{Z,-2t}_{\overline{\rho}}$ (resp. $\mathbb{T}^{Z,-t}_{\overline{\rho}}$) when $r(w)$ is even (resp. odd).

Thus, we have the following one-to-one map:

$$\left\{\begin{matrix}\text{arithmetic specializations } \kappa \text{ of weight } w=(w_1,w_2)\\ \text{and level } K_1(\mathfrak{n}p^\alpha) \text{ on } \mathcal{R}\end{matrix}\right\} \tag{30}$$

$$\longrightarrow\left\{\begin{matrix}\text{a pair } (\kappa^Z,\eta_\kappa) \text{ which consists of}\\ \text{arithmetic specializations } \kappa^Z \text{ on } \mathcal{R}^{Z,-2t} \text{ (resp. } \mathcal{R}^{Z,-t})\\ \text{of weight } w^Z=(w_1^Z,w_2^Z)\\ \eta_\kappa:\mathrm{Gal}(F_{\{p\}}/F)\longrightarrow\overline{\mathbb{Q}}_p^\times \text{ such that } \eta_\kappa/\chi_{\mathrm{cyc}}^{\frac{r(w)}{2}+1}\\ \text{(resp. } \eta_\kappa/\chi_{\mathrm{cyc}}^{\frac{r(w)+1}{2}}) \text{ is of finite order}\end{matrix}\right\},$$

when $r(w)$ is even (resp. odd).

Let us fix an $\mathcal{R}^{Z,-2t}$-basis element (resp. $\mathcal{R}^{Z,-t}$-basis element) $b^{\mathrm{even},\epsilon} \in \mathcal{M}^Z(-2t,0)_{\mathcal{R}^{Z,-2t}}[\epsilon]$ (resp. $b^{\mathrm{odd},\epsilon} \in \mathcal{M}^Z(-t,0)_{\mathcal{R}^{Z,-t}}[\epsilon]$).
For any arithmetic specialization κ on $\mathcal{R}$ of cohomological double-digit weight $w=(w_1,w_2)$ with w_1+w_2, we have

$$\begin{aligned}(31)\quad & \kappa^Z(\mathcal{M}(w_1^Z,w_2^Z)_{\mathcal{R}^{Z,w_1^Z+w_2^Z}}[\epsilon]) \\ &\cong \left(\mathrm{H}^d_{\mathrm{n.o}}(Y_Z(\mathfrak{n}p^\alpha),\mathcal{L}(w_1^Z,w_2^Z;\mathcal{K}/\mathcal{O}))^{\mathrm{PD}}_{\overline{\rho}}[\epsilon]\right)\otimes_{(\mathfrak{h}^Z_{w^Z;\alpha})_{\overline{\rho}}}\kappa(\mathcal{R}) \\ &\cong \left(\mathrm{H}^d_{\mathrm{n.o}}(Y_1(\mathfrak{n}p^\alpha),\mathcal{L}(w_1^Z,w_2^Z;\mathcal{K}/\mathcal{O}))^{\mathrm{PD}}_{\overline{\rho}}[\epsilon]\right)\otimes_{(\mathfrak{h}^{\mathrm{n.o}}_{w^Z;\alpha})_{\overline{\rho}}}\kappa(\mathcal{R})\end{aligned}$$

Hence, if we denote by $b^{\mathrm{even},\epsilon}_{\kappa^Z}$ (resp. $b^{\mathrm{odd},\epsilon}_{\kappa^Z}$) the image of $b^{\mathrm{even},\epsilon}$ (resp. $b^{\mathrm{odd},\epsilon}$) via the specialization κ^Z, we have the canonical isomorphism

$$(32)\qquad \left(\mathrm{H}^d_{\mathrm{n.o}}(Y_1(\mathfrak{n}p^\alpha),\mathcal{L}(w_1,w_2;\mathcal{K}/\mathcal{O}))^{\mathrm{PD}}_{\overline{\rho}}[\epsilon]\right)\otimes_{(\mathfrak{h}^{\mathrm{n.o}}_{w;\alpha})_{\overline{\rho}}}\kappa(\mathcal{R})\cong b^{\epsilon}_{\kappa^Z}\cdot\kappa(\mathcal{R})$$

Definition 4.16. Assume the condition ($\mathrm{Van}_{\overline{\rho}}$) for $\overline{\rho}$ associated to the fixed branch $\mathcal{R}$. Let $\kappa=\kappa_\varphi$ be an arithmetic specialization on $\mathcal{R}$ which corresponds to a nearly ordinary eigenform φ of level $K_1(\mathfrak{n}p^\alpha)$ and of weight $w=(w_1,w_2)$. Denote by $\mathfrak{o}_\varphi$ the ring of integers of the Hecke field for φ and let be the localization at the prime over p induced from the fixed embedding $\mathfrak{o}_\varphi\hookrightarrow\overline{\mathbb{Z}}_p$.

$$\begin{aligned}&\mathrm{H}^d_{\mathrm{n.o}}(Y_1(\mathfrak{n}p^\alpha),\mathcal{L}(w_1,w_2;\mathfrak{o}_{\varphi,(p)}))_{\overline{\rho}}[\epsilon] \\ &\longrightarrow \left(\mathrm{H}^d_{\mathrm{n.o}}(Y_1(\mathfrak{n}p^\alpha),\mathcal{L}(w_1,w_2;\mathcal{O}))_{\overline{\rho}}[\epsilon]\right)\otimes_{(\mathfrak{h}^{\mathrm{n.o}}_{w^Z;\alpha})_{\overline{\rho}}}\kappa(\mathcal{R}) \\ &\longrightarrow \left(\mathrm{H}^d_{\mathrm{n.o}}(Y_1(\mathfrak{n}p^\alpha),\mathcal{L}(w_1^Z,w_2^Z;\mathcal{O}))_{\overline{\rho}}[\epsilon]\right)\otimes_{(\mathfrak{h}^{\mathrm{n.o}}_{w^Z;\alpha})_{\overline{\rho}}}\kappa(\mathcal{R})\end{aligned}$$

where $\mathfrak{o}_{\varphi,(p)}$ is the discrete valuation ring defined before Definition 3.5 and $\mathcal{O}$ contains the completion of $\mathfrak{o}_{\varphi,(p)}$. The image of the module $\mathrm{H}^d_{\mathrm{n.o}}(Y_1(\mathfrak{n}p^\alpha),\mathcal{L}(w_1,w_2;\mathfrak{o}_{\varphi,(p)}))_{\overline{\rho}}[\epsilon]$ gives an $\mathfrak{o}_{\varphi,(p)}$-rational structure on the image $\mathrm{H}^d_{\mathrm{n.o}}(Y_1(\mathfrak{n}p^\alpha),\mathcal{L}(w_1^Z,w_2^Z;\mathcal{O}))_{\overline{\rho}}[\epsilon]\otimes_{(\mathfrak{h}^{\mathrm{n.o}}_{w^Z;\alpha})_{\overline{\rho}}}\kappa(\mathcal{R})$ which is a free $\kappa(\mathcal{R})$-module of rank one. Since $\mathfrak{o}_{\varphi,(p)}$ is principal, the image of the above composite map is free of rank one over $\mathfrak{o}_{\varphi,(p)}$. Let v be an $\mathfrak{o}_{\varphi,(p)}$-basis of the image. On the other hand, we have the isomorphism

$$\begin{aligned}&\left(\mathrm{H}^d_{\mathrm{n.o}}(Y_1(\mathfrak{n}p^\alpha),\mathcal{L}(w_1^Z,w_2^Z;\mathcal{O}))_{\overline{\rho}}[\epsilon]\right)\otimes_{(\mathfrak{h}^{\mathrm{n.o}}_{w^Z;\alpha})_{\overline{\rho}}}\kappa(\mathcal{R}) \\ &\cong \left(\mathrm{H}^d_{\mathrm{n.o}}(Y_1(\mathfrak{n}p^\alpha),\mathcal{L}(w_1^Z,w_2^Z;\mathcal{K}/\mathcal{O}))^{\vee}_{\overline{\rho}}[\epsilon]\right)\otimes_{(\mathfrak{h}^{\mathrm{n.o}}_{w^Z;\alpha})_{\overline{\rho}}}\kappa(\mathcal{R})\cong b^{\epsilon}_{\kappa^Z}\cdot\kappa(\mathcal{R})\end{aligned}$$

via the Poincare duality theorem. We define the p-adic period $C^{\epsilon}_{\varphi,p}=C^{\epsilon}_{\varphi,p,v}$ in $\kappa(\mathcal{R})^\times$ by $b^{\epsilon}_{\kappa^Z}=C^{\epsilon}_{\varphi,p,v}\cdot v$

Remark 4.17. (1) A p-adic period does not depend on the twist by a finite character. That is, if an arithmetic specialization κ on $\mathcal{R}$ corresponds to a modular form φ and if an arithmetic specialization κ' on $\mathcal{R}$ corresponds to a modular form $\varphi'=\varphi\otimes\phi$ with a finite character ϕ of $\mathrm{Gal}(F_\infty/F)$, we have $C^{\epsilon}_{\varphi,p,v}=C^{\epsilon}_{\varphi',p,v}$.

(2) Note that the p-adic period $C^{\epsilon}_{\varphi,p,v}$ as well as the complex period $C^{\epsilon}_{\varphi,\infty,v}$ depend on the choice of v and they are well-defined only up to multiplication of an element in $(\mathfrak{o}_{\varphi,(p)})^{\times}$. Nevertheless, for two different choices of v, the p-adic and complex periods are incremented by the same amount. Hence, the pair $(C^{\epsilon}_{\varphi,p}, C^{\epsilon}_{\varphi,\infty})$ (or the ratio "$C^{\epsilon}_{\varphi,p}/C^{\epsilon}_{\varphi,\infty}$") is independent of the choice of v. By this reason, later in the interpolation of the p-adic L-function, we will denote the periods by $C^{\epsilon}_{\varphi,p}$ and $C^{\epsilon}_{\varphi,\infty}$ without specifying the choice of v.

5. p-ADIC L-FUNCTION

5.1. STATEMENT OF THE MAIN THEOREM (THEOREM A). Under the one-to-one correspondence between arithmetic specializations of $\mathcal{R}$ and modular forms φ on the Hida family over $\mathcal{R}$, we sometimes denote an arithmetic specialization κ of $\mathcal{R}$ by κ_{φ} to specify the corresponding modular form φ (cf. Remark 4.14). As is explained after Definition 4.7, $\mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}}(w_1, w_2)$ is independent of the choice of a cohomological weight (w_1, w_2). Hence, we denote $\mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}}(w_1, w_2)$ by $\mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}}$. We sometimes denote the complex and p-adic periods $C^{\epsilon}_{\varphi,\infty,v}$ and $C^{\epsilon}_{\varphi,p,v}$ by $C_{\varphi,\infty,v}$ and $C_{\varphi,p,v}$ when ϵ is the trivial character $\mathbf{1}$. We also denote $b^{\mathrm{even},\epsilon}$ (resp. $b^{\mathrm{odd},\epsilon}$) by b^{even} (resp. b^{odd}) when ϵ is the trivial character $\mathbf{1}$.
The theorem below is the main theorem of our paper which we denoted by Theorem A in the introduction.

THEOREM 5.1. *Let $\overline{\rho}$ be a residual Galois representation of G_F which is nearly ordinary at p and of level $K_1(\mathfrak{n}) \cap K_{11}(p)$ satisfying the conditions* ($\mathrm{Van}_{\overline{\rho}}$) *and* ($\mathrm{Ir}_{\overline{\rho}}$) *stated at the beginning of §1.2. Let $\mathcal{R}$ be a branch of $\mathbb{T}^{\mathrm{n.o}}_{\overline{\rho}}$ and let us fix an $\mathcal{R}^{Z,-2t}$-basis (resp. $\mathcal{R}^{Z,-t}$-basis) element b^{even} (resp. b^{odd}) of $\mathcal{M}^{Z}(-2t,0)_{\mathcal{R}^{Z,-2t}}[\mathbf{1}]$ (resp. $\mathcal{M}^{Z}(-t,0)_{\mathcal{R}^{Z,-t}}[\mathbf{1}]$) (see Theorem 4.15). Assume that $\mathcal{R}^{Z,-2t}$ (resp. $\mathcal{R}^{Z,-2t}$) is a Gorenstein algebra.*
Then, there exists a p-adic L-function $L_p(\mathcal{R}; b^{\mathrm{even}}) \in \mathcal{R}$ (resp. $L_p(\mathcal{R}; b^{\mathrm{odd}}) \in \mathcal{R}$) which satisfies the interpolation property

$$\frac{\kappa_{\varphi}(L_p(\mathcal{R}; b^{\mathrm{even}}))}{C_{\varphi,p}} = \prod_{\mathfrak{p}\,|(p)} A_{\mathfrak{p}}(\varphi) \frac{L(\varphi, 0)}{C_{\varphi,\infty}}$$

(*resp. the same interpolation for* $L_p(\mathcal{R}; b^{\mathrm{odd}})$),

for every p-stabilized nearly ordinary eigen cuspform $\varphi \in S_{w_1,w_2}(K_1(\mathfrak{n}p^*))$ of critical cohomological weight $w = (w_1, w_2)$ on $\mathcal{R}$ such that $w_1 + w_2 = rt$ for an even (resp. odd) integer r where $C^{\epsilon}_{\varphi,p}$ is a p-adic period (cf. Definition 4.16) and $A_{\mathfrak{p}}(\varphi)$ is defined as follows:

$$A_{\mathfrak{p}}(\varphi) = \begin{cases} \left(1 - \dfrac{1}{N_F(\mathfrak{p}) a_{\mathfrak{p}}(\varphi)}\right) & \text{if } \mathfrak{p} \nmid \mathrm{Cond}(\phi_0 \omega^{-j}), \\ \left(\dfrac{1}{N_F(\mathfrak{p}) a_{\mathfrak{p}}(\varphi^0)}\right)^{\mathrm{ord}_{\mathfrak{p}}\ \mathrm{Cond}(\phi_0\omega^{-j})} & \text{if } \mathfrak{p}\,|\mathrm{Cond}(\phi_0 \omega^{-j}). \end{cases}$$

where φ^0 (resp. ϕ_0) is a unique ordinary form of weight (w_1, w_2) (resp. a unique finite order character of $\mathrm{Gal}(\widetilde{F}_\infty/F)$) which is not a twist of another nearly ordinary form of weight $(k-2, j-1)$ by a finite order Hecke character of F with p-primary conductor such that $\varphi = \varphi^0 \otimes \phi_0\omega^{-j}$. The number $N_F(\mathfrak{p})$ is the absolute norm of the prime ideal $\mathfrak{p}$.

5.2. Construction of $L_p(\mathcal{R}; b^{\mathrm{even}})$ and $L_p(\mathcal{R}; b^{\mathrm{odd}})$. The p-adic L-function $L_p(\mathcal{R}; b^{\mathrm{even}})$ (resp. $L_p(\mathcal{R}; b^{\mathrm{odd}})$) will be constructed as an element of $\mathcal{R} \cong \mathcal{R}^{Z,-2t}[[\mathrm{Gal}(F_{\{p\}}/F)]]$ (resp. $\mathcal{R} \cong \mathcal{R}^{Z,-t}[[\mathrm{Gal}(F_{\{p\}}/F)]]$) interpolating special values of L-function of Hilbert modular forms twisted by Hecke characters. Since the construction of $L_p(\mathcal{R}; b^{\mathrm{odd}})$ is parallel to that of $L_p(\mathcal{R}; b^{\mathrm{even}})$, we will explain mainly the construction of $L_p(\mathcal{R}; b^{\mathrm{even}})$. Note that, for a fixed cohomological weight $w^Z = (w_1^Z, w_2^Z) \in \mathbb{Z}[I_F] \times \mathbb{Z}[I_F]$ with $w_1^Z + w_2^Z = -2t$, we have:

$$\mathcal{R}^{Z,-2t}[[\mathrm{Gal}(F_{\{p\}}/F)]] = \varprojlim_{\alpha,\beta} \mathcal{R}^Z_{w^Z;\alpha}[\mathrm{Cl}_F^+(p^\beta)_p],$$

where $\mathcal{R}^Z_{w^Z,\alpha}$ is a local component of the full Hecke algebra $\mathfrak{h}_\alpha^{Z,(w_1^Z,w_2^Z)}$ acting on the cohomology $\mathrm{H}^d(Y_Z(\mathfrak{n}p^\alpha), \mathcal{L}(w_1^Z, w_2^Z; \mathcal{K}/\mathcal{O}))$ generated by T_λ with λ prime to $p\,\mathfrak{n}$, U_λ with $\lambda|\,\mathfrak{n}$, and U_{ϖ_i} for ϖ_i running over all uniformizers of $\mathfrak{p}_i$. Recall that, it is an immediate consequence of Corollary 4.6 that $\varprojlim_\alpha \mathcal{R}^Z_{w^Z;\alpha}$ depends only on $w_1^Z + w_2^Z$. Hence the notation $\mathcal{R}^{Z,-2t}$ is justified (cf. notations given after Remark 4.14).

Therefore, in order to construct $L_p(\mathcal{R}; b^{\mathrm{even}})$, it is enough to construct an inverse system

$$\left\{x_{\alpha,\beta} = x_{\alpha,\beta}^{(w_1^Z,w_2^Z)}(b^{\mathrm{even}}) \in \mathcal{R}^Z_{w^Z,\alpha}[\mathrm{Cl}_F^+(p^\beta)_p]\right\}_{\alpha,\beta}$$

with respect to multi-indexes α and β for each $w^Z = (w_1^Z, w_2^Z)$ with $w_1^Z + w_2^Z = -2t$ in $\mathbb{Z}[I_F]$. For $w^Z = (w_1^Z, w_2^Z)$ with $w_1^Z + w_2^Z = -2t$, we will define $L_p^{(w_1^Z,w_2^Z)}(\mathcal{R}; b^{\mathrm{even}}) \in \mathcal{R}^{Z,-2t}[[\mathrm{Gal}(F_{\{p\}}/F)]]$ to be $\varprojlim_{\alpha,\beta} x_{\alpha,\beta}^{(w_1^Z,w_2^Z)}(b^{\mathrm{even}})$ and we denote $L_p^{(-2t,0)}(\mathcal{R}; b^{\mathrm{even}})$ by $L_p(\mathcal{R}; b^{\mathrm{even}})$. Finally, by help of the canonical identification $\mathcal{R} = \mathcal{R}^{Z,-2t}[[\mathrm{Gal}(F_{\{p\}}/F)]]$, we regard $L_p(\mathcal{R}; b^{\mathrm{even}})$ as an element of $\mathcal{R}$.

The construction of $x_{\alpha,\beta}^{(w_1^Z,w_2^Z)}(b^{\mathrm{even}})$ will occupy the next paragraph 5.3 and the desired interpolation property will be proved in the paragraph 5.4.

5.3. Proof of the distribution property of Theorem A. Recall that, for every $\boldsymbol{u} \in \mathrm{Cl}_F^+(p^\beta)_p$, we constructed:

$$\mathrm{ev}_{\boldsymbol{u},\alpha,\beta}^{(w_1^Z,w_2^Z)} \in \mathrm{Hom}_{\mathcal{O}}\big(H^d(Y_Z(\mathfrak{n}p^\alpha); \mathcal{L}(w_1^Z, w_2^Z; \mathcal{O}))_{\overline{\rho}}[\mathbf{1}], \mathcal{K}\big)$$

after the equation (17) of Section 3. Note that we can omit the compact support, since the localization at $\overline{\rho}$ kills the boundary cohomology.

The $\mathcal{R}^{Z,-2t}$-basis $\mathrm{tw}^{(w_1^Z,w_2^Z)}(b^{\mathrm{even}}) \in \mathcal{M}^Z(w_1^Z, w_2^Z)_{\mathcal{R}^{Z,-2t}}[\mathbf{1}]$ from Paragraph 4.4 specializes to an $\mathcal{R}^Z_{w^Z,\alpha}$-basis b_α^{even} of $H_c^d(Y_Z(\mathfrak{n}p^\alpha); \mathcal{L}(w_1^Z, w_2^Z; \mathcal{O}))_{\overline{\rho}}[\mathbf{1}]$. Now, let

us consider the composite map:
(33)
$$\mathrm{Hom}_{\mathcal{O}}(H^d(Y_Z(\mathfrak{n}p^\alpha);\mathcal{L}(w_1^Z,w_2^Z;\mathcal{O}))_{\overline{\mathfrak{p}}}[\mathbf{1}],\mathcal{K}) \twoheadrightarrow \mathrm{Hom}_{\mathcal{O}}(\mathcal{R}^Z_{w^Z,\alpha},\mathcal{K}) \overset{\sim}{\to} \mathcal{R}^Z_{w^Z,\alpha}\otimes_{\mathcal{O}}\mathcal{K},$$
where the first morphism of (33) is induced by $\mathcal{R}^Z_{w^Z,\alpha}$-basis b^{even}_α.

DEFINITION 5.2. We denote by $\mathrm{ev}^{(w_1^Z,w_2^Z)}_{\boldsymbol{u},\alpha,\beta}(b^{\mathrm{even}}_\alpha) \in \mathcal{R}^Z_{w^Z,\alpha}\otimes_{\mathcal{O}}\mathcal{K}$ the image of $\mathrm{ev}^{(w_1^Z,w_2^Z)}_{\boldsymbol{u},\alpha,\beta}$ by the map (33). Then, for each $\alpha=(\alpha_1,\cdots,\alpha_s),\beta=(\beta_1,\cdots,\beta_s)\in\mathbb{N}^s$ with $\alpha\le\beta$ and with $\beta_j>0$ for all j, we define $x^{(w_1^Z,w_2^Z)}_{\alpha,\beta}(b^{\mathrm{even}})\in(\mathcal{R}^Z_{w^Z,\alpha}\otimes_{\mathcal{O}}\mathcal{K})\left[\mathrm{Cl}^+_F(p^\beta)_p\right]$ to be:
$$x^{(w_1^Z,w_2^Z)}_{\alpha,\beta}(b^{\mathrm{even}}) = U_0(p)^{-\beta}\cdot\left(\sum_{\boldsymbol{u}\in\mathrm{Cl}^+_F(p^\beta)_p}\mathrm{ev}^{(w_1^Z,w_2^Z)}_{\boldsymbol{u},\alpha,\beta}(b^{\mathrm{even}}_\alpha)\cdot[\boldsymbol{u}]\right).$$
Here, $U_0(p)^\beta$ is the modified Hecke operator $\prod_{1\le i\le s}(U_{0,\mathfrak{p}_i})^{\beta_i}$ associated to fixed uniformizers $\varpi_{\mathfrak{p}_i}$ of $\mathfrak{p}_i$ (see 2.3 for the definition).

By the definition of ordinary forms in 2.3, the operator $U_0(p)$ defined as in Definition 5.2 is a unit in the nearly ordinary Hecke algebra $\mathcal{R}^{Z,-2t}$ and its quotients $\mathcal{R}^Z_{w^Z,\alpha}$.
We will show that the set $\left\{x^{(w_1^Z,w_2^Z)}_{\alpha,\beta}(b^{\mathrm{even}})\in(\mathcal{R}^Z_{w^Z,\alpha}\otimes_{\mathcal{O}}\mathcal{K})\left[\mathrm{Cl}^+_F(p^\beta)_p\right]\right\}_{\alpha,\beta}$ forms a projective system, by checking the compatibility with respect to α and β. Thus, our construction via the inverse limit of $x^{(w_1^Z,w_2^Z)}_{\alpha,\beta}(b^{\mathrm{even}})$ for the p-adic L-function $L_p(\mathcal{R};b^{\mathrm{even}})$ is an analogue of Iwasawa's construction via Stickelberger elements for Kubota-Leopoldt p-adic L-function.
The compatibility with respect to α is not so hard.

LEMMA 5.3. *For each β and for each pair α,α' satisfying $\alpha\le\alpha'$, $x^{(w_1^Z,w_2^Z)}_{\alpha',\beta}(b^{\mathrm{even}})$ is mapped to $x^{(w_1^Z,w_2^Z)}_{\alpha,\beta}(b^{\mathrm{even}})$ via the natural map $(\mathcal{R}^Z_{w^Z,\alpha'}\otimes_{\mathcal{O}}\mathcal{K})[\mathrm{Cl}^+_F(p^\beta)_p]\twoheadrightarrow(\mathcal{R}^Z_{w^Z,\alpha}\otimes_{\mathcal{O}}\mathcal{K})[\mathrm{Cl}^+_F(p^\beta)_p]$.*

Proof. Note that $\{b^{\mathrm{even}}_\alpha\}_{\alpha\ge0}$ is a compatible system by definition. Hence, it follows that $x^{(w_1^Z,w_2^Z)}_{\alpha,\beta}(b^{\mathrm{even}})$ is the image of $x^{(w_1^Z,w_2^Z)}_{\alpha',\beta}(b^{\mathrm{even}})$ by the natural homomorphism
$$(\mathcal{R}^Z_{w^Z,\alpha'}\otimes_{\mathcal{O}}\mathcal{K})\left[\mathrm{Cl}^+_F(p^\beta)_p\right]\twoheadrightarrow(\mathcal{R}^Z_{w^Z,\alpha}\otimes_{\mathcal{O}}\mathcal{K})\left[\mathrm{Cl}^+_F(p^\beta)_p\right].$$
Note also that the image of $U_0(p)^\beta$ in $\mathcal{R}^Z_{w^Z,\alpha}\otimes_{\mathcal{O}}\mathcal{K}$ is compatible with respect to α once β is fixed. Thus, we have the compatibility of $x^{(w_1^Z,w_2^Z)}_{\alpha,\beta}(b^{\mathrm{even}})$ and $x^{(w_1^Z,w_2^Z)}_{\alpha',\beta}(b^{\mathrm{even}})$. $\square$

The compatibility with respect to β which is more delicate than that of α is proved below.

Lemma 5.4. *For each α and for each pair β, β' satisfying $\beta \leq \beta'$, $x^{(w_1^Z, w_2^Z)}_{\alpha,\beta'}(b^{\mathrm{even}})$ is mapped to $x^{(w_1^Z, w_2^Z)}_{\alpha,\beta}(b^{\mathrm{even}})$ via the natural map $(\mathcal{R}^Z_{w^Z,\alpha} \otimes_{\mathcal{O}} \mathcal{K})[\mathrm{Cl}^+_F(p^{\beta'})_p] \twoheadrightarrow (\mathcal{R}^Z_{w^Z,\alpha} \otimes_{\mathcal{O}} \mathcal{K})[\mathrm{Cl}^+_F(p^{\beta})_p]$.*

Proof. It suffices to show the equality:

$$U_0(p)^{\beta'-\beta} \mathrm{ev}^{(w_1^Z, w_2^Z)}_{\boldsymbol{u},\alpha,\beta} = \sum_{\boldsymbol{u'}} \mathrm{ev}^{(w_1^Z, w_2^Z)}_{\boldsymbol{u'},\alpha,\beta'}, \tag{34}$$

for any $\boldsymbol{u} \in \mathrm{Cl}^+_F(p^\beta)_p$ in the space $\mathrm{Hom}_{\mathcal{O}}(H^d(Y_Z(\mathfrak{n}p^\alpha); \mathcal{L}(w_1^Z, w_2^Z; \mathcal{O}))_{\overline{\rho}}[\mathbf{1}], \mathcal{O})$, where $\boldsymbol{u'}$ runs through elements in $\mathrm{Cl}^+_F(p^{\beta'})_p$ which are mapped to $\boldsymbol{u}$.
By an inductive argument, the proof of (34) is reduced to the case of $\beta \leq \beta'$ where there exists j such that $\beta' = (\beta_1, \cdots, \beta_{j-1}, \beta_j + 1, \beta_{j+1}, \cdots, \beta_s)$. Then, the equation (34) is reduced to

$$U_{0,\varpi_j} \mathrm{ev}^{(w_1^Z, w_2^Z)}_{\boldsymbol{u},\alpha,\beta} = \sum_{\boldsymbol{u'}} \mathrm{ev}^{(w_1^Z, w_2^Z)}_{\boldsymbol{u'},\alpha,\beta'}, \tag{35}$$

where the sum runs through archimedean embeddings of F with which ϖ_j-adic topology is compatible with the induced p-adic embedding. For any $\alpha \geq 0$, we have the coset decomposition of the double coset $K_Z(p^\alpha) \begin{pmatrix} \varpi_{\mathfrak{p}_j} & 0 \\ 0 & 1 \end{pmatrix} K_Z(p^\alpha)$ which induces the operator $U_{\mathfrak{p}_j}$ as follows:

$$K_Z(p^\alpha) \begin{pmatrix} \varpi_{\mathfrak{p}_j} & 0 \\ 0 & 1 \end{pmatrix} K_Z(p^\alpha) = \coprod_a \begin{pmatrix} \varpi_{\mathfrak{p}_j} & a \\ 0 & 1 \end{pmatrix} K_Z(p^\alpha). \tag{36}$$

where a in the right-hand side runs through a set of representatives of $\mathfrak{o}^\times_+$-orbits in $\mathfrak{o}/(\varpi_{\mathfrak{p}_j}) = \mathfrak{o}/\mathfrak{p}_j$. By the sequence (16), $\boldsymbol{u}$ is identified with a set of certain numbers of elements $\{u_i\}$ where each u_i is an $(\mathfrak{o}^\times_+)_p$-orbits in $\mathfrak{o}/p^\beta = p^{-\beta}/\mathfrak{o}$. Since we have $p^{-\beta}\varpi_{\mathfrak{p}_j}^{-1} = p^{-\beta'}$, $\dfrac{a+u_i}{\varpi_{\mathfrak{p}_j}}$ runs through the set of representatives of a fixed $u_i \in p^{-\beta}/\mathfrak{o}$ in $p^{-\beta'}/\mathfrak{o}$ when a varies the representatives of the elements which appeared in (36). Here, when we talk about these representatives, we consider $p^{-\beta'}/\mathfrak{o} \twoheadrightarrow p^{-\beta}/\mathfrak{o}$ induced from $\mathfrak{o}/p^{\beta'} \twoheadrightarrow \mathfrak{o}/p^\beta$ by fixed identifications $p^{-\beta}/\mathfrak{o} = \mathfrak{o}/p^\beta$ and $p^{-\beta'}/\mathfrak{o} = \mathfrak{o}/p^{\beta'}$. This completes the proof. □

If there is an index j with $\beta_j = 0$, the element $x^{(w_1^Z, w_2^Z)}_{\alpha,\beta})(b^{\mathrm{even}}_{\kappa^Z})$ is not defined at Definition 5.2. We extend the definition to general β as follows.

Definition 5.5. Let S be the subset of $\{1, \ldots, s\}$ such that $\beta_j = 0$ for $j \in S$ and $\beta_j > 0$ for $j \notin S$. Then, we define $x^{(w_1^Z, w_2^Z)}_{\alpha,\beta}(b^{\mathrm{even}}) \in (\mathcal{R}^Z_{w^Z,\alpha} \otimes_{\mathcal{O}} \mathcal{K}) \left[\mathrm{Cl}^+_F(p^\beta)_p\right]$ to be:

$$x^{(w_1^Z, w_2^Z)}_{\alpha,\beta}(b^{\mathrm{even}}) =$$

$$= \prod_{j \in S} \left(1 - \frac{1}{N_F(\mathfrak{p}_j) U_{0,\mathfrak{p}_j}}\right) \cdot U_0(p)^{-\beta} \cdot \left(\sum_{\boldsymbol{u} \in \mathrm{Cl}^+_F(p^\beta)_p} \mathrm{ev}^{(w_1^Z, w_2^Z)}_{\boldsymbol{u},\alpha,\beta}(b^{\mathrm{even}}_\alpha) \cdot [\boldsymbol{u}] \right).$$

With this extended definition, we have the following proposition:

PROPOSITION 5.6. *For each pair α, α' satisfying $\alpha \leq \alpha'$ and for each pair β, β' satisfying $\beta \leq \beta'$, $x^{(w_1^Z, w_2^Z)}_{\alpha', \beta'}(b^{\mathrm{even}})$ is mapped to $x^{(w_1^Z, w_2^Z)}_{\alpha, \beta}(b^{\mathrm{even}})$ via the natural map $(\mathcal{R}^Z_{w^Z, \alpha'} \otimes_{\mathcal{O}} \mathcal{K})[\mathrm{Cl}^+_F(p^{\beta'})_p] \twoheadrightarrow (\mathcal{R}^Z_{w^Z, \alpha} \otimes_{\mathcal{O}} \mathcal{K})[\mathrm{Cl}^+_F(p^{\beta})_p]$.*

Proof. Thanks to Lemma 5.3 and Lemma 5.4, the desired compatibility holds unless there is an index j with $\beta_j = 0$. Hence, we fix α and we assume that there is j such that $\beta_j = 0$. Let $\beta' = (\beta'_1, \cdots, \beta'_s)$ be the multi-index such that $\beta'_j = 1$ and $\beta'_k = \beta_k$ for $k \neq j$. It suffices to prove that $x^{(w_1^Z, w_2^Z)}_{\alpha, \beta'}(b^{\mathrm{even}})$ is mapped to $x^{(w_1^Z, w_2^Z)}_{\alpha, \beta}(b^{\mathrm{even}})$ via the natural map $(\mathcal{R}^Z_{w^Z, \alpha} \otimes_{\mathcal{O}} \mathcal{K})[\mathrm{Cl}^+_F(p^{\beta'})_p] \twoheadrightarrow (\mathcal{R}^Z_{w^Z, \alpha} \otimes_{\mathcal{O}} \mathcal{K})[\mathrm{Cl}^+_F(p^{\beta})_p]$. Let us calculate the ratio:

$$\sum_{\widetilde{\boldsymbol{u}}}{}' \mathrm{ev}^{(w_1^Z, w_2^Z)}_{\widetilde{\boldsymbol{u}}, \alpha, \beta'}(b^{\mathrm{even}}_{\alpha}) \Big/ \sum_{\widetilde{\boldsymbol{u}}} \mathrm{ev}^{(w_1^Z, w_2^Z)}_{\widetilde{\boldsymbol{u}}, \alpha, \beta'}(b^{\mathrm{even}}_{\alpha}) \tag{37}$$

where $\widetilde{\boldsymbol{u}}$ runs through the element of $\mathfrak{o}^\times_+$-orbits of $\prod_{k \neq j}(\mathfrak{o}/\mathfrak{p}_k^{\beta_k})^\times \times \mathfrak{o}/\mathfrak{p}_j$ which are lifts of $\boldsymbol{u}$ in the summation of the denominator, and $\widetilde{\boldsymbol{u}}$ runs through the element of $\mathfrak{o}^\times_+$-orbits of $\prod_{k \neq j}(\mathfrak{o}/\mathfrak{p}_k^{\beta_k})^\times \times (\mathfrak{o}/\mathfrak{p}_j)^\times$ which are lifts of $\boldsymbol{u}$ in the summation of the numerator. The ratio is equal to the value of $1 - \dfrac{1}{N_F(\mathfrak{p}_j)U_{0, \mathfrak{p}_j}}$ acting on $\mathrm{ev}^{(w_1^Z, w_2^Z)}_{\widetilde{\boldsymbol{u}}, \alpha, \beta'}(b^{\mathrm{even}}_{\alpha})$. By modifying the definition counting this factor, we have the desired compatibility, which completes the proof. □

Finally, We have the following corollary.

COROLLARY 5.7. *Assume the condition $(\mathrm{Ir}_{\overline{\rho}})$. For each $\alpha = (\alpha_1, \cdots, \alpha_s), \beta = (\beta_1, \cdots, \beta_s) \in \mathbb{N}^s$ with $\alpha \leq \beta$, the element $x^{(w_1^Z, w_2^Z)}_{\alpha, \beta}(b^{\mathrm{even}}) \in (\mathcal{R}^Z_{w^Z, \alpha} \otimes_{\mathcal{O}} \mathcal{K}) \left[\mathrm{Cl}^+_F(p^{\beta})_p\right]$ introduced at Definition 5.5 lies in the subalgebra $\mathcal{R}^Z_{w^Z, \alpha} \left[\mathrm{Cl}^+_F(p^{\beta})_p\right]$.*

Proof. Let us consider the case of $\alpha = 0$. In this case, since we consider the set of all $\boldsymbol{u}$ in $\mathrm{Cl}^+_F(p^{\beta})_p$, $(\coprod_{\boldsymbol{u}} H_{\boldsymbol{u}})/\mathfrak{o}^\times_+$ is well-defined. (Note that $\begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}$ is contained in $K_1(\mathfrak{n}) \subset K_{11}(p^0) = K_1(\mathfrak{n})$ for any unit $e \in \mathfrak{o}^\times_+$.)

We recall that the Betti cohomology with compact support $H^d_c(Y_{K_1(\mathfrak{n})}; \mathcal{L}(w_1, w_2; \mathcal{K}))$ is isomorphic to the étale cohomology with compact support $H^d_{\text{ét},c}((Y_{K_1(\mathfrak{n})})_{\overline{\mathbb{Q}}}; \mathcal{L}(w_1, w_2; \mathcal{K}))$ where $(Y_{K_1(\mathfrak{n})})_{\overline{\mathbb{Q}}}$ is a model of $Y_{K_1(\mathfrak{n})}$ over $\overline{\mathbb{Q}}$. By the fact that $Y_{K_1(\mathfrak{n})}$ has a model over $\mathbb{Q}$, the étale cohomology $H^d_{\text{ét},c}((Y_{K_1(\mathfrak{n})})_{\overline{\mathbb{Q}}}; \mathcal{L}(w_1, w_2; \mathcal{K}))$ has a natural Galois action of $G_{\mathbb{Q}}$. By [BL], for each eigen cusp form φ in $\overline{\rho}$-component, it is known that we have

$$H^d_{\text{ét},c}((Y_{K_1(\mathfrak{n})})_{\overline{\mathbb{Q}}}; \mathcal{L}(w_1, w_2; \mathcal{K}))[\lambda_\varphi] \cong (\rho_\varphi)^{\otimes [F:\mathbb{Q}]}$$

as a representation of G_F where ρ_φ is a Galois representation of φ on $\mathcal{K}$ which lifts our $\overline{\rho}$. We denote by A the Pontrjagin dual of

the image of the integral cohomology $H^d_{\text{ét},c}((Y_{K_1(\mathfrak{n})})_{\overline{\mathbb{Q}}};\mathcal{L}(w_1,w_2;\mathcal{O}))[\lambda_\varphi]$, in $H^d_{\text{ét},c}((Y_{K_1(\mathfrak{n})})_{\overline{\mathbb{Q}}};\mathcal{L}(w_1,w_2;\mathcal{K}))[\lambda_\varphi]$. The representation A is an $\mathcal{O}$-module isomorphic to $(\mathcal{K}/\mathcal{O})^{\oplus 2^d}$ with continuous $G_{\mathbb{Q}}$-action and $A[\varpi]$ is isomorphic to the dual of $\overline{\rho}^{\otimes[F:\mathbb{Q}]}$ as a representation of G_F for a uniformizer ϖ of $\mathcal{O}$.
Thus, the element

$$x^{(w_1^Z,w_2^Z)}_{0,\beta}(b^{\text{even}}) = \prod_{1\le j\le s}\left(1-\frac{1}{N_F(\mathfrak{p}_j)U_{0,\mathfrak{p}_j}}\right)\cdot \mathrm{ev}^{(w_1^Z,w_2^Z)}_{\boldsymbol{u},0,\beta}(b_0^{\text{even}})\cdot[\boldsymbol{u}]\,.$$

lies in $\mathcal{R}^Z_{w^Z,0}\left[\mathrm{Cl}^+_F(p^\beta)_p\right]$ since we can choose a finite-index subgroup E of $\mathfrak{o}_+^\times$ whose index $[\mathfrak{o}_+^\times:E]$ causes the denominator to be $E=\mathfrak{o}_+^\times$ (cf. Definition 3.2) and since the assumption that $\mathrm{ev}^{(w_1^Z,w_2^Z)}_{\boldsymbol{u},0,\beta}(b_0^{\text{even}})$ is not integral would induce a G_F-stable line in $A[\varpi]$, which contradicts to the result of [BL] and our condotion ($\mathrm{Ir}_{\overline{\rho}}$).
By Lemma 5.3 and by the fact that $\mathcal{R}^Z_{w^Z,\alpha}$ is the preimage of $\mathcal{R}^Z_{w^Z,0}\subset\mathcal{R}^Z_{w^Z,0}\otimes_{\mathcal{O}}\mathcal{K}$ under the natural projection map $\mathcal{R}^Z_{w^Z,\alpha}\otimes_{\mathcal{O}}\mathcal{K}\longrightarrow\mathcal{R}^Z_{w^Z,0}\otimes_{\mathcal{O}}\mathcal{K}$, we obtain the desired integrality for any α. $\square$

The $\mathcal{R}^{Z,-2t}$-basis $\mathrm{tw}^{(w_1^Z,w_2^Z)}(b^{\text{even}})\in\mathcal{M}^Z(w_1^Z,w_2^Z)_{\mathcal{R}^{Z,-2t}}[\mathbf{1}]$ from Paragraph 4.4 specializes to an $\mathcal{R}^Z_{w^Z,\alpha}$-basis b^{even}_α of $H^d_c(Y_Z(\mathfrak{n}p^\alpha);\mathcal{L}(w_1^Z,w_2^Z;\mathcal{O}))_{\overline{\rho}}[\mathbf{1}]$. Moreover, there exists an $\mathcal{O}$-algebra isomorphism $\mathrm{Hom}_{\mathcal{O}}(\mathcal{R}^Z_{w^Z,\alpha},\mathcal{O})\cong\mathcal{R}^Z_{w^Z,\alpha}$ since $\mathcal{R}^Z_{w^Z,\alpha}$ is Gorenstein by assumption. Now, let us consider the composite map:
(38)

$$\mathrm{Hom}_{\mathcal{O}}(H^d(Y_Z(\mathfrak{n}p^\alpha);\mathcal{L}(w_1^Z,w_2^Z;\mathcal{O}))_{\overline{\rho}}[\mathbf{1}],\mathcal{O})\twoheadrightarrow\mathrm{Hom}_{\mathcal{O}}(\mathcal{R}^Z_{w^Z,\alpha},\mathcal{O})\xrightarrow{\sim}\mathcal{R}^Z_{w^Z,\alpha},$$

where the first isomorphism of (38) is induced by $\mathcal{R}^Z_{w^Z,\alpha}$-basis b^{even}_α and the second isomorphism of (38) is the one induced by the Gorenstein property.

5.4. Proof of the interpolation property of Theorem A. We will evaluate $L_p(\mathcal{R};b^{\text{even}})$ at an arithmetic specialization $\kappa=\kappa_\varphi$ of $\mathcal{R}$ corresponding to a cuspform $\varphi\in S_{w_1,w_2}(K_1(\mathfrak{n}p^\alpha))$. Let r and η be as introduced in 4.4.
We have the following commutative diagram with a unique arithmetic specialization κ^Z_φ on $\mathcal{R}$:

$$\begin{tikzcd}
\mathcal{R} \arrow[r, "\kappa_\varphi"] \arrow[d] & \overline{\mathbb{Q}}_p \arrow[d, equal] \\
\mathcal{R}^{Z,-2t}[[\mathrm{Gal}(F_{\{p\}}/F)]] \arrow[r, "\eta_\kappa\circ\kappa^Z"'] & \overline{\mathbb{Q}}_p.
\end{tikzcd}$$

Let φ^Z be the eigen cuspform corresponding to the arithmetic specialization κ^Z_φ and let (w_1^0,w_2^0) be the critical pair which is the largest among the shifts (w_1^Z+nt,w_2^Z+nt) by integer multiples nt of t which are critical. We denote by φ^0 the unique twist of φ^Z of weight (w_1^0,w_2^0) which is not a twist of another nearly ordinary form of smaller level by a finite order Hecke character of F. (Since it will be a long and tedious description, we do not repeat what the twist by a finite order character of a modular form or the corresponding arithmetic specialization means.) Note that there exists a unique integer r_0 (resp. r_Z) and a unique finite order character ϕ_Z (resp. ϕ_0) of $\mathrm{Cl}_F^+(p^\infty)$ such that the form the newform associated with $\varphi \otimes N_F^{-r_0}\phi_0^{-1}$ (resp. $\varphi^Z \otimes N_F^{-r_Z}\phi_Z^{-1}$) is equal to φ^0. By definition, we have:

$$\kappa_\varphi(L_p(\mathcal{R}; b^{\mathrm{even}})) = (\eta_\kappa \circ \kappa_\varphi^Z)(L_p^{(-2t,0)}(\mathcal{R}; b^{\mathrm{even}})).$$

The element $\kappa_\varphi^Z(L_p^{(-2t,0)}(\mathcal{R}; b^{\mathrm{even}}))$ is an element in $\kappa(\mathcal{R})[[\mathrm{Gal}(F_{\{p\}}/F)]] \subset \kappa(\mathcal{R})[[\mathrm{Gal}(F_{\{p\}}/F)]]$. Recall that η_κ is equal to $\chi_{\mathrm{cyc}}^{r_Z+1}\phi_Z$ for some finite Hecke character ϕ_Z.
By construction, $\kappa_\varphi^Z(L_p^{(-2t,0)}(\mathcal{R}; b^{\mathrm{even}})) \otimes \phi_Z^{-1}$ is equal to

$$\left(\varprojlim_\beta \kappa_\varphi^Z(x_{\alpha,\beta}^{(w_1^Z,w_2^Z)})(b_{\kappa^Z}^{\mathrm{even}})\right)\otimes\phi_Z^{-1} = \varprojlim_\beta(\kappa_\varphi^Z\otimes\phi_Z^{-1})(x_{\alpha,\beta}^{(w_1^Z,w_2^Z)}\otimes\phi_Z^{-1})(b_{\kappa^Z}^{\mathrm{even}}\otimes\phi_Z^{-1}).$$

Let v^0 be a basis of a free $\mathfrak{o}_{\varphi^0,(p)}$-module $H_c^d(Y_1(\mathfrak{n}\,p^\alpha); \mathcal{L}(w_1^0, w_2^0; \mathfrak{o}_{\varphi^0,(p)}))[\lambda_\varphi, \epsilon]$ of rank one. Then, by Definition 5.2, we have:

$$\begin{aligned}
&\varprojlim_\beta(\kappa_\varphi^Z \otimes \phi_Z^{-1})(x_{\alpha,\beta}^{(w_1^Z,w_2^Z)} \otimes \phi_Z^{-1})(b_{\kappa^Z}^{\mathrm{even}} \otimes \phi_Z^{-1}) \\
&= \varprojlim_\beta \left(\sum_{\boldsymbol{u}\in \mathrm{Cl}_F^+(p^\beta)_p} U_0(p)^{-\beta}\mathrm{ev}_{\boldsymbol{u},\alpha,\beta}^{(w_1^Z,w_2^Z)}(b_{\kappa^Z}^{\mathrm{even}} \otimes \phi_Z^{-1})[\boldsymbol{u}]\right) \\
&= C_{\varphi,p,v^0} \cdot \varprojlim_\beta \left(\sum_{\boldsymbol{u}\in \mathrm{Cl}_F^+(p^\beta)_p} U_0(p)^{-\beta}\mathrm{ev}_{\boldsymbol{u},\alpha,\beta}^{(w_1^Z,w_2^Z)}(v^0[r_Z])[\boldsymbol{u}]\right).
\end{aligned}$$

The evaluation of $\varprojlim_\beta \sum_{\boldsymbol{u}\in \mathrm{Cl}_F^+(p^\beta)_p} U_0(p)^{-\beta}\mathrm{ev}_{\boldsymbol{u},\alpha,\beta}^{(w_1^Z,w_2^Z)}(v^0[r_Z])[\boldsymbol{u}]$ at $\chi_{\mathrm{cyc}}^{r_0-r_Z}\phi_0$ is equal to the evaluation of $\varprojlim_\beta \sum_{\boldsymbol{u}\in \mathrm{Cl}_F^+(p^\beta)_p} U_0(p)^{-\beta}\mathrm{ev}_{\boldsymbol{u},\alpha,\beta}^{(w_1,w_2)}(v^0[r_0])[\boldsymbol{u}]$ at ϕ_0 by the standard argument between Tate twists of critical values. Let $\prod_{j=1}^s \mathfrak{p}_j^{\beta_j}$ be the conductor of ϕ_0. If we have $\beta_j > 0$ for all j, we simply evaluate $U_0(p)^{-\beta} \cdot \left(\sum_{\boldsymbol{u}\in \mathrm{Cl}_F^+(p^\beta)_p} \mathrm{ev}_{\boldsymbol{u},\alpha,\beta}^{(w_1^Z,w_2^Z)}(v^0[r_0])\,[\boldsymbol{u}]\right)$. via the character ϕ_0, whose value is equal to:

$$\prod_{j=1}^{s}\left(\frac{1}{N_F(\mathfrak{p})a_{\mathfrak{p}}(\varphi^0)}\right)^{\beta_j}\left(\sum_{\boldsymbol{u}\in \mathrm{Cl}_F^+(p^\beta)_p}\phi_0^{-1}(\boldsymbol{u})\mathrm{ev}_{\boldsymbol{u},\alpha,\beta}^{(w_1,w_2)}(v^0[r_0])\right)$$
$$=\prod_{j=1}^{s}\left(\frac{1}{N_F(\mathfrak{p})a_{\mathfrak{p}}(\varphi^0)}\right)^{\beta_j}G(\phi_0^{-1})\frac{L(\varphi^0\otimes N_F^{r_0},0)}{C_{\varphi,\infty,v^0[r_0]}}$$
$$=\prod_{j=1}^{s}\left(\frac{1}{N_F(\mathfrak{p})a_{\mathfrak{p}}(\varphi^0)}\right)^{\beta_j}\frac{L(\varphi^0,0)}{C_{\varphi,\infty,v}},$$

where the first equality follows from Corollary 3.7 and the second equality follows from Proposition 3.8. This completes the proof of the interpolation property for $L_p(\mathcal{R};b^{\mathrm{even}})$ for κ_φ with $a_{\mathfrak{p}_j}(\varphi)=0$ for all j.
If there are j with $a_{\mathfrak{p}_j}(\varphi)\neq 0$ or equivalently $\beta_j=0$ for some component of the conductor $\prod_{j=1}^s \mathfrak{p}_j^{\beta_j}$ of ϕ_0, the interpolation is clear from the extendes definition of the element $x_{\alpha,\beta}^{(w_1^Z,w_2^Z)})(b^\epsilon_{\kappa^Z})$ at Definition 5.5 (see also Proposition 5.6) which complete the proof for $L_p(\mathcal{R};b^{\mathrm{even}})$.
The proof for $L_p(\mathcal{R};b^{\mathrm{odd}})$ is done in the same way except that we replace the role of $\mathcal{M}^Z(-2t,0)[\mathbf{1}]$ by that of $\mathcal{M}^Z(-t,0)[\mathbf{1}]$.

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