The C^*-Algebra of an Affine Map on the 3-Torus

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Abstract. We study the C^*-algebra of an affine map on a compact abelian group and give necessary and sufficient conditions for strong transitivity when the group is a torus. The structure of the C^*-algebra is completely determined for all strongly transitive affine maps on a torus of dimension one, two or three.

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1 Introduction

The purpose with the present paper is to present a complete description and classification of the simple C^*-algebras which arise from the generalised crossed product construction of Renault, Deaconu and Anantharaman-Delaroche when it is applied to affine maps on tori of dimension ≤ 3. The paper is written in the conviction that progress on our understanding of the relationship between dynamical systems and operator algebras can benefit both areas and that it is improved by having rich classes of examples where the dynamical systems and the associated C^*-algebras are equally tractable. The affine maps of tori constitute a class of dynamical systems that are well studied and whose structures are relatively transparent when compared to other systems. As the present paper will demonstrate our knowledge of C^*-algebras is now comprehensive enough to allow a complete identification and classification of the corresponding C^*-algebras, provided the affine maps are strongly transitive and the dimension of the torus does not exceed 3.
There are many other compact abelian groups for which it would be desirable to have a better understanding of the C^*-algebras associated to affine maps which are local homeomorphisms. For this reason we maintain a high level of generality before we specialise to tori of low dimension. Specifically we first describe the general construction of Renault, Deaconu and Anantharaman-Delaroche

from [Re], [De] and [An], which produces a locally compact étale groupoid and hence a C^*-algebra out of a local homeomorphism. We develop a bit of the structure theory that we need when we specialise to affine maps. In particular, we show that the KK-equivalence class of the C^*-algebra is preserved by an appropriate notion of homotopy. It follows from this that the C^*-algebra of a locally homeomorphic and surjective affine map on a path-connected compact group is KK-equivalent, in a unit-preserving way, to the C^*-algebra of its linear part. This means that as far as the calculation of the K-groups is concerned it suffices to consider group endomorphisms. Furthermore, it follows from the classification theorem of Kirchberg and Phillips that the C^*-algebras of two strongly transitive affine maps on the same compact metrizable path-connected group are isomorphic provided they are purely infinite and the maps have the same linear parts. Thanks to a recent result from [CT] we know that the C^*-algebra of a locally injective surjection on a compact finite dimensional metric space is purely infinite if it is simple, provided only that the map is not injective. All in all this means that for the C^*-algebras of non-injective strongly transitive locally homeomorphic affine maps on a compact metrizable path-connected group it is not only the K-theory, but also the algebra itself which is completely determined by the K-theory of the C^*-algebra coming from the linear part of the map. Before we apply this to tori we first show that the C^*-algebra of a locally homeomorphic affine map on a compact abelian group is the universal C^*-algebra generated by a unitary representation of the dual group and an isometry subject to two relations, cf. Theorem 3.2. This result is motivated by a recent paper by Cuntz and Vershik ([CV]) where this is done for exact endomorphisms.

Turning the attention to tori we first give necessary and sufficient conditions for an affine surjection on a torus to be strongly transitive, cf. Theorem 4.3. It remains then to calculate the K-theory, and in order to obtain a complete calculation, covering all strongly transitive locally homeomorphic affine maps, we restrict to tori of dimension ≤ 3. When all the eigenvalues of the integer matrix which defines the endomorphism are strictly larger than one in absolute value the endomorphism is expanding, and in this case the calculation was performed in dimension 1 and 2 in [EHR]. The calculations in [EHR] were based on a six-terms exact sequence which in the commutative case is the same as the Deaconu-Muhly sequence from [DM] and [Th2]. We state here what this sequence becomes in the general two and three dimensional cases and complete in this way the calculation in these dimensions.

It remains then to handle the injective case which means that we must determine the C^*-algebras arising from minimal affine homeomorphisms. On the circle these are just the irrational rotation algebras and they are well understood. On the two-torus the minimal affine homeomorphisms are all conjugate to one of the Furstenberg transformations whose C^*-algebras have been characterised through the work of Lin and Phillips, [LP], [Ph2]. We show that both the methods and the results of Lin and Phillips carry over with little effort to the three-dimensional case. This part of the paper has some overlap with recent

work of Reihani, [Rei], where the K-theory of the C^*-algebras of Furstenberg transformations is studied.
Finally, we summarise our results in the three Sections 6.1, 6.2 and 6.3. They contain a description of the ordered K-theory groups, together with the position of the distinguished element of the K_0-group represented by the unit, for all the simple C^*-algebras one can obtain from the transformation groupoid of an affine local homeomorphism on a torus of dimension ≤ 3. This characterises these C^*-algebras since they are all classified by K-theory.

2 Algebras from local homeomorphisms

In this section we describe the construction of an étale groupoid and a C^*-algebra from a local homeomorphism. It was introduced in increasing generality by J. Renault [Re], V. Deaconu [De] and Anantharaman-Delaroche [An]. Although the focus in this paper is on cases where the space is compact it will be crucial to have access to statements and results from the locally compact case.

2.1 The definition

Let X be a second countable locally compact Hausdorff space and $\varphi : X \to X$ a local homeomorphism. Set

$$\Gamma_\varphi = \{(x,k,y) \in X \times \mathbb{Z} \times X :\ \exists n,m \in \mathbb{N},\ k = n-m,\ \varphi^n(x) = \varphi^m(y)\}.$$

This is a groupoid with the set of composable pairs being

$$\Gamma_\varphi^{(2)} \ = \ \{((x,k,y),(x',k',y')) \in \Gamma_\varphi \times \Gamma_\varphi :\ y = x'\}.$$

The multiplication and inversion are given by

$$(x,k,y)(y,k',y') = (x,k+k',y') \text{ and } (x,k,y)^{-1} = (y,-k,x).$$

Note that the unit space of Γ_φ can be identified with X via the map $x \mapsto (x,0,x)$. Under this identification the range map $r : \Gamma_\varphi \to X$ is the projection $r(x,k,y) = x$ and the source map the projection $s(x,k,y) = y$.
To turn Γ_φ into a locally compact topological groupoid, fix $k \in \mathbb{Z}$. For each $n \in \mathbb{N}$ such that $n + k \geq 0$, set

$$\Gamma_\varphi(k,n) = \left\{(x,l,y) \in X \times \mathbb{Z} \times X :\ l = k,\ \varphi^{k+n}(x) = \varphi^n(y)\right\}.$$

This is a closed subset of the topological product $X \times \mathbb{Z} \times X$ and hence a locally compact Hausdorff space in the relative topology. Since φ is locally injective, $\Gamma_\varphi(k,n)$ is an open subset of $\Gamma_\varphi(k,n+1)$ and hence the union

$$\Gamma_\varphi(k) = \bigcup_{n \geq -k} \Gamma_\varphi(k,n)$$

is a locally compact Hausdorff space in the inductive limit topology. The disjoint union

$$\Gamma_\varphi = \bigcup_{k\in\mathbb{Z}} \Gamma_\varphi(k)$$

is then a locally compact Hausdorff space in the topology where each $\Gamma_\varphi(k)$ is an open and closed set. In fact, as is easily verified, Γ_φ is a locally compact groupoid in the sense of [Re] and an étale groupoid, i.e. the range and source maps are local homeomorphisms.

To obtain a C^*-algebra, consider the space $C_c(\Gamma_\varphi)$ of continuous compactly supported functions on Γ_φ. They form a $*$-algebra with respect to the convolution-like product

$$fg(x,k,y) = \sum_{z,n+m=k} f(x,n,z)g(z,m,y)$$

and the involution

$$f^*(x,k,y) = \overline{f(y,-k,x)}.$$

To obtain a C^*-algebra, let $x \in X$ and consider the Hilbert space H_x of square summable functions on $s^{-1}(x) = \{(x',k,y') \in \Gamma_\varphi : \ y' = x\}$ which carries a representation π_x of the $*$-algebra $C_c(\Gamma_\varphi)$ defined such that

$$\left(\pi_x(f)\psi\right)(x',k,x) = \sum_{z,n+m=k} f(x',n,z)\psi(z,m,x)$$

when $\psi \in H_x$. One can then define a C^*-algebra $C_r^*(\Gamma_\varphi)$ as the completion of $C_c(\Gamma_\varphi)$ with respect to the norm

$$\|f\| = \sup_{x\in X} \|\pi_x(f)\|.$$

Since we assume that X is second countable it follows that $C_r^*(\Gamma_\varphi)$ is separable. It is this C^*-algebra we study in the present paper when φ is an affine map.

Note that the C^*-algebra can be constructed from any locally compact étale groupoid in the place of Γ_φ, see e.g. [Re], [An]. Note also that $C_r^*(\Gamma_\varphi)$ is nothing but the classical crossed product $C_0(X)\times_\varphi \mathbb{Z}$ when φ is a homeomorphism.

2.2 The structure

By construction $C_r^*(\Gamma_\varphi)$ carries an action β by the circle group $\mathbb{T}$ defined such that

$$\beta_\lambda(f)(x,k,y) = \lambda^k f(x,k,y)$$

when $f \in C_c(\Gamma_\varphi)$. This is the *gauge action* and it gives us an important tool for the study of the structure of $C_r^*(\Gamma_\varphi)$. To describe the fixed point algebra of the gauge action note that the canonical conditional expectation $P : C_r^*(\Gamma_\varphi) \to C_r^*(\Gamma_\varphi)^\beta$, given by

$$P(a) = \int_{\mathbb{T}} \beta_\lambda(a)\ d\lambda,$$

maps $C_c(\Gamma_\varphi)$ onto $C_c(\Gamma_\varphi(0))$. If we denote the open subgroupoid $\Gamma_\varphi(0)$ by R_φ, it follows that

$$C_r^*(\Gamma_\varphi)^\beta = C_r^*(R_\varphi).$$

To unravel the structure of $C_r^*(R_\varphi)$ and $C_r^*(\Gamma_\varphi)$, consider for $n \in \mathbb{N}$ the set

$$R(\varphi^n) = \{(x,y) \in X \times X : \ \varphi^n(x) = \varphi^n(y)\}.$$

Since φ is a local homeomorphism $R(\varphi^n)$ is a locally compact étale groupoid (an equivalence relation, in fact) in the relative topology inherited from $X \times X$, and we can consider its (reduced) groupoid C^*-algebra $C_r^*(R(\varphi^n))$. Now $R(\varphi^n)$ can be identified with an open subgroupoid of $R_\varphi \subseteq \Gamma_\varphi$ via the map $(x,y) \mapsto (x,0,y)$ and when we suppress this identification in the notation we have that

$$R_\varphi = \bigcup_n R(\varphi^n).$$

It follows that the embeddings $C_c(R(\varphi^n)) \subseteq C_c\left(R\left(\varphi^{n+1}\right)\right) \subseteq C_c(R_\varphi)$ extend to embeddings $C_r^*(R(\varphi^n)) \subseteq C_r^*\left(R\left(\varphi^{n+1}\right)\right) \subseteq C_r^*(R_\varphi)$, cf. e.g. Proposition 1.9 in [Ph3], and hence that

$$C_r^*(R_\varphi) = \overline{\bigcup_n C_r^*(R(\varphi^n))}. \tag{2.1}$$

Lemma 2.1. *$C_r^*(R(\varphi))$ is Morita equivalent to $C_0(\varphi(X))$.*

Proof. Except for considerations regarding continuity the arguments are the same as in the proof of Theorem 3.2 in [Th1]. Let

$$G_\varphi = \{(y,x) \in X \times X : \ y = \varphi(x)\}$$

be the graph of φ. When $h \in C_c(G_\varphi)$ and $f \in C_c(R(\varphi))$ define $hf : G_\varphi \to \mathbb{C}$ such that

$$hf(y,x) = \sum_{z \in \varphi^{-1}(y)} h(y,z)f(z,x).$$

Then $hf \in C_c(G_\varphi)$ and we have turned $C_c(G_\varphi)$ into a right $C_c(R(\varphi))$-module. Similarly, when $g \in C_c(\varphi(X))$ we define $gh \in C_c(G_\varphi)$ such that

$$gh(y,x) = g(y)h(y,x),$$

so that $C_c(G_\varphi)$ is also a left $C_c(\varphi(X))$-module. Define a $C_c(R(\varphi))$-valued 'inner product' on $C_c(G_\varphi)$ such that

$$\langle h,k\rangle(x,y) = \overline{h(\varphi(x),x)}k(\varphi(y),y)$$

and a $C_c(\varphi(X))$-valued 'inner product' such that

$$(h,k)(y) = \sum_{z \in \varphi^{-1}(y)} h(y,z)\overline{k(y,z)}.$$

In this way $C_c(G_\varphi)$ becomes a $C_c(\varphi(X))$-$C_c(R(\varphi))$-pre-imprimitivity bimodule as defined by Raeburn and Williams in Definition 3.9 of [RW] and then Proposition 3.12 of [RW] shows that the completion of this bimodule is the required $C_0(\varphi(X))$-$C_r^*(R(\varphi))$-imprimitivity bimodule. $\square$

Let $\mathbb{K}$ denote the C^*-algebra of compact operators on a separable, infinite dimensional Hilbert space. By applying Lemma 2.1 to φ^n and combining with a well-known result of Brown, Green and Rieffel, [BGR], we conclude that

$$C_r^*\left(R\left(\varphi^n\right)\right) \otimes \mathbb{K} \cong C_0\left(\varphi^n(X)\right) \otimes \mathbb{K}.$$

In particular, it follows from (2.1) that $C_r^*(R_\varphi)$ is an inductive limit of C^*-algebras stably isomorphic to abelian C^*-algebras. When X is compact and φ is surjective it follows that $C_r^*\left(R_\varphi\right)$ is the inductive limit of a unital sequence of homogeneous C^*-algebras with spectrum X.
The next step will be to show that the gauge action is full.

Lemma 2.2. *Elements of the form fg^*, where $f,g \in C_c\left(\Gamma_\varphi(1)\right)$, span a dense subspace in $C_r^*\left(R_\varphi\right) = C_r^*\left(\Gamma_\varphi\right)^\beta$, and the same is true for the elements of the form hk^* where $h,k \in C_c\left(\Gamma_\varphi(-1)\right)$.*

Proof. For each $n \in \mathbb{N}$ set

$$R\left(\varphi^n\right) = \left\{(x,0,y) \in R_\varphi : \ \varphi^n(x) = \varphi^n(y)\right\}.$$

Let $F \in C_c\left(R\left(\varphi^n\right)\right),\ n \geq 2$. Using a partition of unity we can write F as a sum of functions in $C_c\left(R\left(\varphi^n\right)\right)$ each of which is supported in a subset of $R\left(\varphi^n\right)$ of the form $R\left(\varphi^n\right) \cap (U \times \{0\} \times V)$ where U, V are open subsets of X where φ^{n+1} is injective. We assume therefore that F is supported in $R\left(\varphi^n\right) \cap (U \times \{0\} \times V)$. Set $U_0 = r\left(R\left(\varphi^n\right) \cap (U \times \{0\} \times V)\right)$ and $V_0 = s\left(R\left(\varphi^n\right) \cap (U \times \{0\} \times V)\right)$, both open subsets of X. Set $K = r\left(\operatorname{supp} F\right)$, a compact subset of U_0. Let $h \in C_c(X)$ be such that $\operatorname{supp} h \subseteq U_0$ and $h(x) = 1$ for all $x \in K$. Set

$$A = \Gamma_\varphi(1,n) \cap \left(U_0 \times \{1\} \times \varphi(U_0)\right)$$

and

$$B = \Gamma_\varphi(-1,n) \cap \left(\varphi(U_0) \times \{-1\} \times V_0\right)$$

which are open in $\Gamma_\varphi(1)$ and $\Gamma_\varphi(-1)$, respectively. For every $(x,1,y) \in A$, set $f(x,1,y) = h(x)$ and note that f has compact support in A. When $(x,-1,y) \in B$ there is a unique element $x' \in U_0$ such that $\varphi(x') = x$ and $(x',0,y) \in R\left(\varphi^n\right)$. We can therefore define $g : B \to \mathbb{R}$ such that $g(x,-1,y) = F(x',0,y)$. Extending f and g to be zero outside A and B, respectively, we can consider them as elements of $C_c(\Gamma_\varphi)$. Then $f, g^* \in C_c\left(\Gamma_\varphi(1)\right)$. Since $fg = f\left(g^*\right)^* = F$ this completes the proof of the first assertion because $\bigcup_n C_c\left(R\left(\varphi^n\right)\right)$ is dense in $C_r^*\left(R_\varphi\right)$. The second assertion is proved in the same way. $\square$

Theorem 2.3. *There is an automorphism α on $C_r^*\left(R_\varphi\right) \otimes \mathbb{K}$ such that $C_r^*\left(\Gamma_\varphi\right) \otimes \mathbb{K}$ is $*$-isomorphic to the crossed product $\left(C_r^*\left(R_\varphi\right) \otimes \mathbb{K}\right) \rtimes_\alpha \mathbb{Z}$.*

Proof. It follows from Lemma 2.2 that Theorem 2 of [KT] applies to give an isomorphism

$$\left(C_r^*\left(\Gamma_\varphi\right) \rtimes_\beta \mathbb{T}\right) \otimes \mathbb{K} \;\cong\; C_r^*\left(R_\varphi\right) \otimes \mathbb{K}. \tag{2.2}$$

Let α_0 be the automorphism of $C_r^*\left(\Gamma_\varphi\right) \rtimes_\beta \mathbb{T}$ generating the action dual to β. Then $\left(C_r^*\left(\Gamma_\varphi\right) \rtimes_\beta \mathbb{T}\right) \rtimes_{\alpha_0} \mathbb{Z} \cong C_r^*\left(\Gamma_\varphi\right) \otimes \mathbb{K}$ by Takai duality, cf. e.g. Theorem 7.9.3 of [Pe]. Thus, when we let α be the automorphism of $C_r^*\left(R_\varphi\right) \otimes \mathbb{K}$ corresponding to $\alpha_0 \otimes \mathrm{id}_{\mathbb{K}}$ under the isomorphism (2.2) we deduce that $\left(C_r^*\left(R_\varphi\right) \otimes \mathbb{K}\right) \rtimes_\alpha \mathbb{Z} \cong C_r^*\left(\Gamma_\varphi\right) \otimes \mathbb{K}$. $\square$

When φ is proper and surjective, we can realise $C_r^*\left(\Gamma_\varphi\right)$ as a crossed product by an endomorphism via the procedure described in [De] and [An], and this can be used to give an alternative proof of Theorem 2.3. Without properness such an approach seems impossible.

Corollary 2.4. *a) $C_r^*\left(\Gamma_\varphi\right)$ is a separable nuclear C^*-algebra in the bootstrap category of Rosenberg and Schochet, [RS].*

b) Assume that φ is surjective and that $C_0(X)$ is KK-contractible. It follows that $C_r^*\left(\Gamma_\varphi\right)$ is KK-contractible.

Proof. a) is an immediate consequence of the preceding and b) follows from a) since Theorem 2.3 and the Pimsner-Voiculescu exact sequence, [PV], implies that the K-groups of $C_r^*\left(\Gamma_\varphi\right)$ are both zero when $C_0(X)$ is KK-contractible. $\square$

2.3 Homotopy of local homeomorphisms

Let Y be a compact metric space. A path $\sigma_t : Y \to Y, t \in [0,1]$, of surjective local homeomorphisms is called *a homotopy of local homeomorphisms* when the map $\Sigma : [0,1] \times Y \to [0,1] \times Y$ defined by

$$\Sigma(t,y) = (t, \sigma_t(y)) \tag{2.3}$$

is a local homeomorphism. We say then that $\{\sigma_t\}$ is a homotopy of local homeomorphism connecting σ_0 and σ_1, and that σ_0 and σ_1 are *homotopic as local homeomorphisms.*

Lemma 2.5. *Let $\sigma_0 : Y \to Y$ and $\sigma_1 : Y \to Y$ be surjective local homeomorphisms. Assume that σ_0 and σ_1 are homotopic as local homeomorphisms. Then there is a KK-equivalence $\lambda \in KK\left(C_r^*\left(\Gamma_{\sigma_0}\right), C_r^*\left(\Gamma_{\sigma_1}\right)\right)$ such that the induced isomorphism $\lambda_* : K_0\left(C_r^*\left(\Gamma_{\sigma_0}\right)\right) \to K_0\left(C_r^*\left(\Gamma_{\sigma_1}\right)\right)$ takes the element represented by the unit in $C_r^*\left(\Gamma_{\sigma_0}\right)$ to the one represented by the unit in $C_r^*\left(\Gamma_{\sigma_1}\right)$.*

Proof. Consider a homotopy $\{\sigma_t\}$ of local homeomorphisms connecting σ_0 to σ_1. Define $\Sigma : [0,1] \times Y \to [0,1] \times Y$ by (2.3) and observe that $\{0\} \times Y$ and $\{1\} \times Y$ are both closed totally Σ-invariant subsets of $[0,1] \times Y$. By Proposition

4.6 of [CT] we have therefore surjective $*$-homomorphisms $\pi_i : C_r^*(\Gamma_\Sigma) \to C_r^*(\Gamma_{\sigma_i})$ such that

$$\ker \pi_i \cong C_r^*\left(\Gamma_{\Sigma|_{Z_i}}\right),$$

where

$$Z_i = ([0,1] \setminus \{i\}) \times Y,$$

$i = 0, 1$. Since $C_0(Z_i)$ is a contractible C^*-algebra it follows from Corollary 2.4 that $C_r^*\left(\Gamma_{\Sigma|_{Z_i}}\right)$ is KK-contractible.

Let $\bullet$ denote the Kasparov product. Since we deal with separable nuclear C^*-algebras it follows from Theorem 19.5.7 of [Bl] that

$$KK\left(C_r^*(\Gamma_{\sigma_i}), C_r^*(\Gamma_\Sigma)\right) \xrightarrow{x \mapsto [\pi_i] \bullet x} KK\left(C_r^*(\Gamma_{\sigma_i}), C_r^*(\Gamma_{\sigma_i})\right) \tag{2.4}$$

and

$$KK\left(C_r^*(\Gamma_\Sigma), C_r^*(\Gamma_\Sigma)\right) \xrightarrow{x \mapsto [\pi_i] \bullet x} KK\left(C_r^*(\Gamma_\Sigma), C_r^*(\Gamma_{\sigma_i})\right) \tag{2.5}$$

are both isomorphisms because $\ker \pi_i$ is KK-contractible. It follows from the surjectivity of (2.4) that there is an element $[\pi_i]^{-1} \in KK\left(C_r^*(\Gamma_{\sigma_i}), C_r^*(\Gamma_\Sigma)\right)$ such that $[\pi_i] \bullet [\pi_i]^{-1} = \left[\mathrm{id}_{C_r^*(\Gamma_{\sigma_i})}\right]$. Then

$$[\pi_i] \bullet \left([\pi_i]^{-1} \bullet [\pi_i]\right) = \left([\pi_i] \bullet [\pi_i]^{-1}\right) \bullet [\pi_i] = [\pi_i]$$

by associativity of the Kasparov product so the injectivity of (2.5) implies that $[\pi_i]^{-1} \bullet [\pi_i] = \left[\mathrm{id}_{C_r^*(\Gamma_\Sigma)}\right]$, i.e. $[\pi_i]^{-1}$ is a KK-inverse of $[\pi_i]$. To finish the proof, set $\lambda = [\pi_1] \bullet [\pi_0]^{-1}$. $\square$

Recall that a continuous map $\psi : X \to X$ is *strongly transitive* when for every open non-empty subset V of X, there is an $N \in \mathbb{N}$ such that $\bigcup_{i=0}^N \psi^i(V) = X$. It was shown in [DS] that when $\phi : X \to X$ is a surjective local homeomorphism on a compact metric space X, the C^*-algebra $C_r^*(\Gamma_\phi)$ is simple if and only if X is not a finite set and ϕ is strongly transitive. In [CT] it was shown that the C^*-algebra of a non-injective and surjective strongly transitive local homeomorphism on a compact metric space of finite covering dimension is purely infinite. Combined with Lemma 2.5 this leads to the following.

Theorem 2.6. *Let X be a finite dimensional compact metric space and $\varphi : X \to X$, $\phi : X \to X$ two surjective local homeomorphisms, both non-injective and strongly transitive. Assume that ϕ and φ are homotopic as local homeomorphisms. It follows that $C_r^*(\Gamma_\phi) \cong C_r^*(\Gamma_\varphi)$.*

Proof. It follows from Corollary 6.6 of [CT] that the classification result of Kirchberg and Phillips applies, cf. Corollary 4.2.2 of [Ph1]. The conclusion follows therefore from Lemma 2.5. $\square$

2.4 Strong transitivity and exactness

Let X be a compact metric space which is not a finite set, and $\phi : X \to X$ a continuous map. Recall that ϕ is *exact* when for every open non-empty subset $V \subseteq X$ there is an $N \in \mathbb{N}$ such that $\phi^N(V) = X$. Thus exactness implies strong transitivity while the converse is generally not true. (For example an irrational rotation of the circle is strongly transitive but not exact.) It was pointed out in [DS] that a surjective local homeomorphism $\varphi : X \to X$ is exact if and only if $C_r^*(R_\varphi)$ is simple. Thus φ is exact if and only if $C_r^*(\Gamma_\varphi)$ and $C_r^*(R_\varphi)$ are both simple while φ is strongly transitive and not exact if and only $C_r^*(\Gamma_\varphi)$ is simple while $C_r^*(R_\varphi)$ is not.
With this section we want to point out that for locally injective and surjective endomorphisms of compact groups, strong transitivity is equivalent to exactness.

Lemma 2.7. *Let $\phi : X \to X$ be continuous, surjective and open. Then the following conditions are equivalent:*

i) ϕ is strongly transitive.

ii) $\bigcup_{n,m\in\mathbb{N}} \phi^{-m}(\phi^n(x))$ is dense in X for all $x \in X$.

iii) $\bigcup_{n\in\mathbb{N}} \phi^{-n}(x)$ is dense in X for all $x \in X$.

Proof. i) $\Rightarrow$ iii): If there is a point $x \in X$ such that

$$F = \overline{\bigcup_{n\in\mathbb{N}} \phi^{-n}(x)}$$

is not all of X, the set $U = X\backslash F$ is open, non-empty and satisfies that $x \notin \bigcup_n \phi^n(U)$, contradicting the strong transitivity of ϕ.
ii) $\Rightarrow$ i): Consider an open non-empty subset V of X. For every $x \in X$ there are $n, m \in \mathbb{N}$ such that $\phi^{-m}(\phi^n(x)) \cap V \neq \emptyset$, i.e. $x \in \phi^{-n}(\phi^m(V))$. Since ϕ is continuous and open, and X is compact, there is an $N \in \mathbb{N}$ such that $X = \bigcup_{i,j\leq N} \phi^{-i}\left(\phi^j(V)\right)$. Then $X = \phi^N(X) = \bigcup_{i=0}^{2N} \phi^i(V)$.
Since iii) $\Rightarrow$ ii) is trivial, the proof is complete. ☐

Proposition 2.8. *Let H be a compact group and $\alpha_0 : H \to H$ a continuous surjective group endomorphism with finite kernel. Then α_0 is exact if and only if α_0 is strongly transitive.*

Proof. Note that α_0 is open since its kernel is finite. Assume that α_0 is strongly transitive, and let $1 \in H$ be the neutral element. Consider an open non-empty subset $U \subseteq H$. Set

$$\Delta = \bigcup_n \ker \alpha_0^n = \bigcup_n \alpha_0^{-n}(1).$$

Then Δ is dense in H by Lemma 2.7. For every $x \in H$,

$$\bigcup_n \alpha_0^{-n}(\alpha_0^n(x)) = \{zx : \ z \in \Delta\},$$

and it follows that $\bigcup_n \alpha_0^{-n}\left(\alpha_0^n(x)\right)$ is dense in H for every $x \in H$. In particular, there is for every x an $n \in \mathbb{N}$ such that $x \in \alpha_0^{-n}\left(\alpha_0^n(U)\right)$. Since

$$\alpha_0^{-m}\left(\alpha_0^m(U)\right) \subseteq \alpha_0^{-m-1}\left(\alpha_0^{m+1}(U)\right)$$

for all m the compactness of H implies that $H = \alpha_0^{-N}\left(\alpha_0^N(U)\right)$ and therefore that $H = \alpha_0^N(U)$ for some N. □

3 The algebra of an affine map on a compact abelian group

Let H be a compact metrizable abelian group and let $G = \widehat{H}$ be its Pontryagin dual group. Let $\alpha : H \to H$ be a continuous affine map. That is, α is the composition of a continuous group endomorphism $\alpha_0 : H \to H$ and the translation by an element $h_0 \in H$, viz.

$$\alpha(h) = h_0\alpha_0(h).$$

We will refer to α_0 as *the linear part* of α. To ensure that the transformation groupoid of α is a well-behaved étale groupoid it is necessary to assume that α is a local homeomorphism.

Let $\langle\cdot, \cdot\rangle$ denote the duality between H and G. We can then define an endomorphism $\phi : G \to G$ such that

$$\langle\phi(g), h\rangle = \langle g, \alpha_0(h)\rangle . \tag{3.1}$$

Lemma 3.1. *The following conditions are equivalent.*

i) *α is a local homeomorphism.*

ii) *α_0 is a local homeomorphism.*

iii) $\ker\alpha_0$ *and* $\operatorname{coker}\alpha_0$ *are finite.*

iv) $\ker\phi$ *and* $\operatorname{coker}\phi$ *are finite.*

v) $\ker\alpha_0$ *and* $\ker\phi$ *are finite.*

Proof. Straightforward. □

Observe that when H is connected, $\operatorname{coker}\alpha_0$ is finite if and only if α_0 is surjective.

Assume that $\alpha : H \to H$ is an affine local homeomorphism. For each $g \in G$ we define a unitary U'_g in $C(H) \subseteq C_r^*(\Gamma_\alpha)$ in the usual way: $U'_g(x) = \langle g, x\rangle$. Then U' is a representation of G by unitaries in $C_r^*(\Gamma_\alpha)$. Set $N = \#\ker\alpha_0$ and define an isometry $V_\alpha \in C_c(\Gamma_\alpha)$ such that

$$V_\alpha(x,k,y) = \begin{cases} \frac{1}{\sqrt{N}} & \text{when } k = 1,\ y = \alpha(x) \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to check that $V_\alpha U'_g = \langle g, h_0\rangle\, U'_{\phi(g)} V_\alpha$ and that

$$\sum_{g \in G/\phi(G)} U'_g V_\alpha V^*_\alpha {U'_g}^* = 1.$$

It follows that we can consider the universal C^*-algebra $\mathcal{A}[\alpha]$ generated by unitaries $U_g, g \in G$, and an isometry S such that

$$U_g U_h = U_{g+h} \qquad SU_g = \langle g, h_0\rangle\, U_{\phi(g)} S \qquad \sum_{g \in G/\phi(G)} U_g S S^* U^*_g = 1. \tag{3.2}$$

Furthermore, there is a $*$-homomorphism $\nu : \mathcal{A}[\alpha] \to C^*_r(\Gamma_\alpha)$ such that $\nu(U_g) = U'_g$ and $\nu(S) = V_\alpha$. Note that the existence of ν implies that the canonical map $C(H) \to \mathcal{A}[\alpha]$ coming from the generators $U_g, g \in G$, is injective.

Theorem 3.2. *Let H be a compact metrizable abelian group and $\alpha : H \to H$ an affine map. Assume that α is a local homeomorphism. Then $C^*_r(\Gamma_\alpha) \cong \mathcal{A}[\alpha]$.*

Proof. To construct the desired isomorphism we will show that the isomorphism $\rho : \mathcal{O}_E \to C^*_r(\Gamma_\alpha)$ from Proposition 3.2 in [Th2] factorises through ν, i.e. that there is a $*$-homomorphism $\mu : \mathcal{O}_E \to \mathcal{A}[\alpha]$ such that

$$\begin{tikzcd} \mathcal{O}_E \arrow[rr, "\rho"] \arrow[dr, "\mu"'] & & C^*_r(\Gamma_\alpha) \\ & \mathcal{A}[\alpha] \arrow[ur, "\nu"'] & \end{tikzcd} \tag{3.3}$$

commutes. Since ρ is an isomorphism this will complete the proof if we also show that μ is surjective. Let $g_i, i = 1, 2, \ldots, N$, be elements in G representing the distinct elements of $G/\phi(G)$. Notice that it follows from the third of the three relations in (3.2) that

$$S^* U^*_{g_i} U_{g_j} S = \begin{cases} 1 & \text{when } i = j \\ 0 & \text{when } i \neq j. \end{cases}$$

Combined with the second relation this implies that

$$S^* U_g S = \begin{cases} 0 & \text{when } g \notin \phi(G) \\ \overline{\langle k, h_0\rangle} U_k & \text{when } g = \phi(k),\ k \in G. \end{cases}$$

In particular, it follows that the closure of $C(H)S$ in $\mathcal{A}[\alpha]$ is a Hilbert $C(H)$-module with the 'inner product' $(a, b) = a^*b$. The existence of the $*$-homomorphism ν, or a simple direct calculation shows that

$$V^*_\alpha U'_g V_\alpha = \begin{cases} 0 & \text{when } g \notin \phi(G) \\ \overline{\langle k, h_0\rangle} U'_k & \text{when } g = \phi(k),\ k \in G. \end{cases}.$$

Since the C^*-correspondence E is the closure of $C(H)V_\alpha$ in $C_r^*(\Gamma_\alpha)$ it follows that we can define an isometry $t : E \to \mathcal{A}[\alpha]$ such that $t(fV_\alpha) = fS$ for all $f \in C(H)$. Together with the inclusion $\pi : C(H) \to \mathcal{A}[\alpha]$ this isometry t gives us a representation of the C^*-correspondence E in the sense of Katsura, cf. Definition 2.1 of [Ka]. To show that this representation is covariant in the sense of [Ka] it suffices by Proposition 3.3 in [Ka] to show, in Katsura's notation, that $C(H) \subseteq \psi_t(\mathcal{K}(E))$. This follows from the observation that

$$U_k = \sum_j U_k U_{g_j} S S^* U_{g_j}^* = \sum_j \psi_t\left(\theta_{U'_k U'_{g_j} V_\alpha,\ U'_{g_j} V_\alpha}\right)$$

for all k. Thus (π, t) is covariant and by Definition 3.5 in [Ka] there is therefore a $*$-homomorphism $\mu : \mathcal{O}_E \to \mathcal{A}[\alpha]$ whose range is generated by $\pi(C(H))$ and $t(E)$. But this is all of $\mathcal{A}[\alpha]$, i.e. μ is surjective.
It remains now only to show that $\rho = \nu \circ \mu$. To this end observe that the two $*$-homomorphisms agree on the canonical copies of $C(H)$ and E inside $\mathcal{O}_E$. As $\mathcal{O}_E$ is generated by these subsets the proof is complete. ☐

3.1 Affine maps with the same linear part

The following result follows immediately from Lemma 2.5 and Theorem 2.6.

Theorem 3.3. *Let H be a compact abelian path-connected second countable group and $\alpha_0 : H \to H$ a continuous group endomorphism with finite non-trivial kernel. Let $h_i \in H, i = 1, 2,$ and define $\alpha_i : H \to H$ such that $\alpha_i(h) = h_i\alpha_0(h)$.*
a) It follows that $C_r^*(\Gamma_{\alpha_1})$ and $C_r^*(\Gamma_{\alpha_2})$ are KK-equivalent.
b) Assume in addition that α_1 and α_2 are both strongly transitive, and that H is of finite covering dimension. It follows that

$$C_r^*(\Gamma_{\alpha_1}) \cong C_r^*(\Gamma_{\alpha_2}).$$

Corollary 3.4. *Let H be a compact abelian path-connected second countable group of finite covering dimension. Let $\alpha : H \to H$ be an affine map whose linear part α_0 is a continuous group endomorphism with finite non-trivial kernel. Assume that α is exact. Then α_0 is exact and*

$$C_r^*(\Gamma_\alpha) \cong C_r^*(\Gamma_{\alpha_0}).$$

Proof. It is easy to see that α is exact if and only of α_0 is. Apply then Theorem 3.3. ☐

Theorem 3.3 a) is generally not true when H is not connected, but it may be that b) of Theorem 3.3 and Corollary 3.4 remain true also when H is not connected; at least we do not know of a counterexample.

Example 3.5. Let A be a finite abelian group of order $N \geq 2$. On the infinite product $A^{\mathbb{N}}$ the shift α_0, given by $\alpha_0((a_n)) = (a_{n+1})$, is a surjective exact endomorphism with finite kernel. Fix an element $x = (x_n) \in A^{\mathbb{N}}$ and consider the affine map $\alpha : A^{\mathbb{N}} \to A^{\mathbb{N}}$ defined by

$$\alpha((a_n)) = (x_n a_{n+1}).$$

The dual group is $\oplus_{k\in\mathbb{N}} A$ and the dual endomorphism ϕ of α_0 is given by

$$\phi((a_n)) = (0, a_0, a_1, a_2, \dots)$$

Consider the C^*-algebra $\mathcal{A}[\alpha]$ generated by unitaries and an isometry satisfying (3.2). For $a \in A$, set $g_a = (a, 0, 0, 0, \dots)$. Then $V_a = U_{g_a} S,\ a \in A$, is a collection of isometries in $\mathcal{A}[\alpha]$ and $\sum_{a\in A} V_a V_a^* = 1$. Since

$$U_{g_b} = \sum_{a\in A} V_{b+a} V_a^*$$

and

$$U_{\phi^k(g_a)} V_b V_b^* \in \mathbb{C} V_b U_{\phi^{k-1}(g_a)} V_b^*$$

for all $k \geq 1$ and all a, b, we conclude that the V_a's generate $\mathcal{A}[\alpha]$. It follows that $\mathcal{A}[\alpha]$ is a copy of the Cuntz algebra $\mathcal{O}_N$. In particular, $\mathcal{A}[\alpha]$ is independent of the translation part of α.

4 Strongly transitive affine surjections on tori

A continuous map $T : \mathbb{T}^n \to \mathbb{T}^n$ on the n-torus is affine when it is the composition of a group endomorphism $\phi : \mathbb{T}^n \to \mathbb{T}^n$ and the translation by an element $\lambda \in \mathbb{T}^n$, i.e.

$$Tx = \lambda\phi(x).$$

Being a continuous group endomorphism ϕ is determined by an integer matrix $A = (a_{ij}) \in M_n(\mathbb{Z})$. Specifically, $\phi = \phi_A$ where ϕ_A is given by the formula

$$\phi_A\left(t_1, t_2, \dots, t_n\right) = \left(t_1^{a_{11}} t_2^{a_{12}} \cdots t_n^{a_{1n}}, t_1^{a_{21}} t_2^{a_{22}} \cdots t_n^{a_{2n}}, \cdots\cdots, t_1^{a_{n1}} t_2^{a_{n2}} \cdots t_n^{a_{nn}}\right) \tag{4.1}$$

for all $(t_1, t_2, \dots, t_n) \in \mathbb{T}^n$. It follows from Lemma 3.1 that T is a local homeomorphism if and only if it is surjective and finite-to-one. In fact, since surjectivity of ϕ_A is equivalent to non-singularity of A, and hence implies that T is finite-to-one we conclude that T is a local homeomorphism if and only if A is non-singular, i.e. $\operatorname{Det} A \neq 0$. We call A *the matrix of the linear part* of T.

In [Kr] Krzyzewski has given an algebraic characterisation of which surjective group endomorphisms of tori are strongly transitive and we will here use his results to obtain a similar characterisation of which affine surjections are strongly transitive. To formulate Krzyzewski's result recall that a non-constant polynomial

$$a_k x^k + a_{k-1} x^{k-1} + \cdots + a_0$$

is called *unimodular* when $a_i \in \mathbb{Z}$ for all i, $a_k = 1$, and $a_0 \in \{-1, 1\}$.

Theorem 4.1. *(Krzyzewski, [Kr]) Let $A \in M_n(\mathbb{Z})$ be non-singular, i.e.* $\operatorname{Det} A \neq 0$*, and let $f_A(x) = \operatorname{Det}(x1 - A)$ be the characteristic polynomial of A. The group endomorphism ϕ_A of $\mathbb{T}^n$ is strongly transitive if and only if no unimodular polynomial divides f_A.*

Lemma 4.2. *Let $\phi : \mathbb{T}^n \to \mathbb{T}^n$ be a surjective affine endomorphism and $A \in M_n(\mathbb{Z})$ the matrix of its linear part. Assume that 1 is not an eigenvalue of A. There is then a translation τ on $\mathbb{T}^n$ such that $\tau\phi\tau^{-1} = \phi_A$.*

Proof. Let $\lambda \in \mathbb{R}^n$ be a vector such that $\mathbb{R}^n \ni x \mapsto Ax + \lambda$ is a lift of ϕ, i.e. $\phi(p(x)) = p(Ax + \lambda)$ where $p : \mathbb{R}^n \to \mathbb{T}^n$ is the canonical surjection. Since $A - 1$ is surjective on $\mathbb{R}^n$ by assumption there is a vector $\mu \in \mathbb{R}^n$ such that $\lambda = (A - 1)\mu$. Define τ such that $\tau t = p(\mu)t$ and note that $\tau\phi\tau^{-1} = \phi_A$. □

Theorem 4.3. *Let $A \in M_n(\mathbb{Z})$ be an integral matrix with non-zero determinant and let $f_A(x) = \operatorname{Det}(x1 - A)$ be the characteristic polynomial of A. Write*

$$f_A(x) = (1 - x)^k g(x)$$

where $k \in \{0, 1, 2, \dots, n\}$ is the algebraic multiplicity of 1 as a root of f_A.

1) *If no unimodular polynomial divides f_A, every affine local homeomorphism of $\mathbb{T}^n$ with ϕ_A as linear part is exact and conjugate to ϕ_A.*

2) *If $k \geq 1$ but no unimodular polynomial divides g, let S be the set of elements $\mu \in \mathbb{T}^n$ with the property that the closed subgroup of $\mathbb{T}^n$ generated by μ and*
$$\left\{x^{-1}\phi_A(x) : \ x \in \mathbb{T}^n\right\} \tag{4.2}$$
is all of $\mathbb{T}^n$. Then S is a dense proper subset of $\mathbb{T}^n$ such that an affine map
$$Tx = \lambda\phi_A(x) \tag{4.3}$$
is strongly transitive if and only if $\lambda \in S$. In this case no affine local homeomorphism with ϕ_A as linear part is exact.

3) *If there is a unimodular polynomial which divides g, then no affine local homeomorphism of $\mathbb{T}^n$ with ϕ_A as linear part is strongly transitive.*

Proof. 1) In this case 1 is not an eigenvalue of A and hence every affine local homeomorphism with ϕ_A as linear part is conjugate to ϕ_A by Lemma 4.2. It follows from Theorem 4.1 and Proposition 2.8 that ϕ_A is exact.
2) Note that an affine map is exact if and only if its linear part is. Moreover it follows from Theorem 4.1 and Proposition 2.8 that ϕ_A is not exact; hence no affine map with ϕ_A as linear part is exact. This justifies the last assertion in 2) and shows that the set S is proper since it does not contain the neutral element. It is dense because it contains all topological group generators of $\mathbb{T}^n$. To prove the remaining assertions in 2), consider first the affine map (4.3) and assume

that it is strongly transitive. Let H be the closed subgroup of $\mathbb{T}^n$ generated by λ and the set (4.2). It is easy to check that $T^{-1}(H) \subseteq H$ and then condition iii) of Lemma 2.7 shows that $H = \mathbb{T}^n$, i.e. $\lambda \in S$. Conversely, assume that $\lambda \in S$. We use now Theorem III.12. on page 50 of [N] to get $W \in Gl_n(\mathbb{Z})$ and matrices $B_{11} \in M_{n-k}(\mathbb{Z})$, $B_{22} \in M_k(\mathbb{Z})$ and a $(n-k) \times k$ integral matrix B_{12} such that

$$WAW^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix} \tag{4.4}$$

and such that B_{22} is an upper triangular matrix of the form

$$B_{22} = \begin{pmatrix} 1 & b_{12} & b_{13} & \dots & b_{1k} \\ 0 & 1 & b_{23} & \dots & b_{2k} \\ 0 & 0 & 1 & \dots & b_{3k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

and g is the characteristic polynomial of B_{11}. By exchanging λ with $\phi_W(\lambda)$ and A with WAW^{-1} we may assume that A is the matrix on the right-hand side of (4.4). Let V and U be open non-empty subsets of $\mathbb{R}^{n-k}$ and $\mathbb{R}^k$, respectively. It follows from Theorem 4.1 and Proposition 2.8 that $\phi_{B_{11}}$ is exact which implies that there is a $N \in \mathbb{N}$ such that

$$B_{11}^j V + \mathbb{Z}^{n-k} = \mathbb{R}^{n-k} \tag{4.5}$$

for all $j \geq N$. Let $\alpha \in \mathbb{R}^{n-k}, \beta \in \mathbb{R}^k$ be vectors such that $p((\alpha,\beta)) = \lambda$ where $p : \mathbb{R}^n \to \mathbb{T}^n$ is the canonical surjection. Let $\rho \in \mathbb{T}^k$ be the image of β and note that the affine map $z \mapsto \rho\phi_{B_{22}}(z)$ is an affine homeomorphism of $\mathbb{T}^k$ which, thanks to the block diagonal form (4.4) is a factor of T. Since $\lambda \in S$ it follows that ρ and the set $\left\{z^{-1}\phi_{B_{22}}(z) : \ z \in \mathbb{T}^k\right\}$ generate $\mathbb{T}^k$ as a topological group. It follows then from Theorem 4 in [HP] that $x \mapsto \rho\phi_{B_{22}}(x)$ is a minimal homeomorphism of $\mathbb{T}^k$. There is therefore an $M \in \mathbb{N}$ such that

$$\bigcup_{j=N}^{N+M} \left(B_{22}^j U + \sum_{l=0}^{j-1} B_{22}^l \beta \right) + \mathbb{Z}^k = \mathbb{R}^k. \tag{4.6}$$

Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be the affine map $Lx = Ax + (\alpha,\beta)$. Thanks to the block form of A it follows from (4.5) and (4.6) that

$$\bigcup_{j=N}^{N+M} L^j(V \times U) + \mathbb{Z}^n = \mathbb{R}^n,$$

proving that $x \mapsto \lambda\phi_A(x)$ is strongly transitive.

3) It follows from Theorem III.12 on page 50 of [N] that there are $W_1 \in Gl_n(\mathbb{Z})$, matrices $A_{11} \in M_k(\mathbb{Z})$, $A_{22} \in M_{n-k}(\mathbb{Z})$ and a $k \times (n-k)$ integral matrix A_{12} such that

$$W_1 A W_1^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \tag{4.7}$$

and the characteristic polynomials of A_{11} and A_{22} are $(1-x)^k$ and g, respectively. We may therefore assume that A is the matrix on the right-hand side of (4.7). Let T be an affine local homeomorphism with ϕ_A as linear part. The projection $q : \mathbb{T}^n \to \mathbb{T}^{n-k}$ to the last coordinates gives then a factor map to an affine local homeomorphism $R : \mathbb{T}^{n-k} \to \mathbb{T}^{n-k}$ whose linear part is $\phi_{A_{22}}$. Since 1 is not an eigenvalue of A_{22} it follows from Lemma 4.2 that R is conjugate to $\phi_{A_{22}}$ which by Theorem 4.1 is not strongly transitive. Since strong transitivity is inherited by factors it follows that T is not strongly transitive either. □

4.1 Local homeomorphisms of the circle

In this section we want to point out that for the torus of lowest dimension the group endomorphisms comprise all strongly transitive non-invertible local homeomorphisms, up to conjugacy.

Let $\mathbb{T}$ be the unit circle in the complex plane. For any continuous map $\phi : \mathbb{T} \to \mathbb{T}$ there is a unique continuous map $g : [0,1] \to \mathbb{R}$ such that $g(0) \in [0,1[$ and $\phi\left(e^{2\pi i t}\right) = e^{2\pi i g(t)}$ for all $t \in [0,1]$. The value $g(1) - g(0) \in \mathbb{Z}$ is the *degree* of ϕ and we denote it by d_ϕ. For maps of positive degree the following can be deduced from the work of Boyland in [B].

Theorem 4.4. *Let $\phi : \mathbb{T} \to \mathbb{T}$ be a local homeomorphism such $|d_\phi| \geq 2$. Assume that ϕ is strongly transitive. Then ϕ is conjugate to the endomorphism $z \mapsto z^{d_\phi}$.*

Proof. The proof is essentially the same which is standard for expanding maps. Let $p : \mathbb{R} \to \mathbb{T}$ be the covering $p(t) = e^{2\pi i t}$ and choose a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $\phi \circ p = p \circ g$. Note that $g(x+1) = g(x) + d_\phi$ and that g is strictly increasing when $d_\phi \geq 2$ and strictly decreasing when $d_\phi \leq -2$. Let M denote the set of non-decreasing continuous functions $f : \mathbb{R} \to \mathbb{R}$ with the property that $f(t+1) = f(t) + 1$ for all t. Then M is a complete metric space in the metric D defined by

$$D(f_1, f_2) = \sup_{t \in \mathbb{R}} |f_1(t) - f_2(t)| .$$

Define $T_g : M \to M$ such that

$$T_g(f) = d_\phi^{-1} f \circ g.$$

Then $D\left(T_g(f_1), T_g(f_2)\right) \leq \frac{1}{|d_\phi|} D(f_1, f_2) \leq 2^{-1} D(f_1, f_2)$, and there is therefore a function $h \in M$ such that $T_g(h) = h$. Define $\alpha : \mathbb{T} \to \mathbb{T}$ such that $\alpha \circ p = p \circ h$ and observe that $\alpha \circ \phi = \beta_{d_\phi} \circ \alpha$, where $\beta_{d_\phi}(z) = z^{d_\phi}$. We claim that h is strictly increasing. Indeed, if not there is a non-empty open interval in $\mathbb{R}$ on which h is constant, and hence also a non-empty open interval $I \subseteq \mathbb{T}$ on which α is constant. Since ϕ is strongly transitive there is an $N \in \mathbb{N}$ such that $\mathbb{T} = \bigcup_{j=0}^{N} \phi^j(I)$. Since $\alpha \circ \phi^j = \beta_{d_\phi}^j \circ \alpha$ for all $j \in \mathbb{N}$ it follows that α is constant on $\phi^j(I)$ for all j, whence α is constant because $\mathbb{T}$ is connected. This is impossible since $h \in M$ and hence h is strictly increasing as claimed. It follows that α is a conjugacy. □

5 K-theory calculations

Let $T : \mathbb{T}^n \to \mathbb{T}^n$ be a continuous affine map which is also a local homeomorphism. It follows from Lemma 2.5 that the K-theory groups of $C_r^*(\Gamma_T)$ are unchanged when we replace T by its linear part. We focus therefore in this section on the calculation of the K-groups of $C_r^*(\Gamma_\phi)$ when ϕ is a group endomorphism of $\mathbb{T}^n$. The main tool will be the six-terms exact sequence from Theorem 3.7 in [EHR]. Note that it applies to our case since $C_r^*(\Gamma_\phi)$ is an Exel system in the sense of [EHR] by Theorem 9.1 of [EV], and that it is the same as the Deaconu-Muhly six-terms exact sequence considered in [Th2].

Let $\phi_A : \mathbb{T}^n \to \mathbb{T}^n$ be a group endomorphism given by the integral non-singular matrix A, i.e. ϕ_A is defined by (4.1). Set $D = |\mathrm{Det}\, A|$ and let $(\phi_A)_* : K_*(C(\mathbb{T}^n)) \to K_*(C(\mathbb{T}^n)), * = 0, 1$, be the homomorphism induced by the endomorphism of $C(\mathbb{T}^n)$ sending f to $f \circ \phi_A$ and let $\iota : C(\mathbb{T}^n) \to C_r^*(\Gamma_{\phi_A})$ be the canonical embedding. It follows from Lemma 4.3 in [EHR] that the six terms exact sequence from Theorem 3.7 of [EHR], or from Theorem 3.3 in [Th2], takes the form

$$\begin{tikzcd}
K_0(C(\mathbb{T}^n)) \arrow[r, "\mathrm{id} - D{(\phi_A)_0}^{-1}"] & K_0(C(\mathbb{T}^n)) \arrow[r, "\iota_*"] & K_0(C_r^*(\Gamma_{\phi_A})) \arrow[d] \\
K_1(C_r^*(\Gamma_{\phi_A})) \arrow[u] & K_1(C(\mathbb{T}^n)) \arrow[l, "\iota_*"] & K_1(C(\mathbb{T}^n)) \arrow[l, "\mathrm{id} - D{(\phi_A)_1}^{-1}"]
\end{tikzcd} \tag{5.1}$$

Consequently

$$K_0(C_r^*(\Gamma_{\phi_A})) \cong \operatorname{coker}\left(\mathrm{id} - D{(\phi_A)_0}^{-1}\right) \oplus \ker\left(\mathrm{id} - D{(\phi_A)_1}^{-1}\right) \tag{5.2}$$

and

$$K_1(C_r^*(\Gamma_{\phi_A})) \cong \operatorname{coker}\left(\mathrm{id} - D{(\phi_A)_1}^{-1}\right) \oplus \ker\left(\mathrm{id} - D{(\phi_A)_0}^{-1}\right). \tag{5.3}$$

For classification purposes it is important to keep track of the distinguished element of $K_0(C_r^*(\Gamma_{\phi_A}))$ represented by the unit in $C_r^*(\Gamma_{\phi_A})$. This is always quite easy because the unit of $C_r^*(\Gamma_{\phi_A})$ is the image of the unit in $C(\mathbb{T}^n)$ under the embedding ι. In particular, the unit represents always an element in the direct summand $\operatorname{coker}\left(\mathrm{id} - D{(\phi_A)_0}^{-1}\right)$.

In general, for n large, it is not easy to give explicit descriptions of the maps $(\phi_A)_0$ and $(\phi_A)_1$. When $n \le 3$ the calculations are not too complicated and can be based either on the realization of $K_*(C(\mathbb{T}^n))$ as an exterior algebra, cf. [EHR], or more simply on a repeated use of the Künneth theorem, [S]. In the following sections we combine the resulting six-terms exact sequences with Theorem 4.3 to obtain a complete list of the simple C^*-algebras which arise from strongly transitive affine maps on a torus $\mathbb{T}^n$ when $n \le 3$.

5.1 The circle

As pointed out in [EHR] the calculation of the K-theory groups of the C^*-algebra of an endomorphism of the circle of positive degree has been carried out by several mathematicians, and [EHR] contains the calculation for endomorphisms of negative degree.
Let $a \in \mathbb{Z}\backslash\{0\}$ and define $\phi_a : \mathbb{T} \to \mathbb{T}$ such that $\phi_a(t) = t^a$. Then

1) $K_0\left(C_r^*\left(\Gamma_{\phi_a}\right)\right) = \mathbb{Z}_{a-1} \oplus \mathbb{Z}$ and $K_1\left(C_r^*\left(\Gamma_{\phi_a}\right)\right) = \mathbb{Z}$ when $a \geq 2$,

2) $K_0\left(C_r^*\left(\Gamma_{\phi_1}\right)\right) = \mathbb{Z}^2 = K_1\left(C_r^*\left(\Gamma_{\phi_1}\right)\right)$,

3) $K_0\left(C_r^*\left(\Gamma_{\phi_{-1}}\right)\right) = \mathbb{Z}$ and $K_1\left(C_r^*\left(\Gamma_{\phi_{-1}}\right)\right) = \mathbb{Z} \oplus \mathbb{Z}_2$ and

4) $K_0\left(C_r^*\left(\Gamma_{\phi_a}\right)\right) = \mathbb{Z}_{|a|-1}$ and $K_1\left(C_r^*\left(\Gamma_{\phi_a}\right)\right) = \mathbb{Z}_2$ when $a \leq -2$.

5.2 The two-torus

In this case the diagram (5.1) takes the form

$$\begin{tikzcd}
\mathbb{Z}^2 \arrow[r, "\begin{pmatrix} 1-|\mathrm{Det}\,A| & 0 \\ 0 & 1-\epsilon(A) \end{pmatrix}"] & \mathbb{Z}^2 \arrow[r] & K_0\left(C_r^*\left(\Gamma_{\phi_A}\right)\right) \arrow[d] \\
K_1\left(C_r^*\left(\Gamma_{\phi_A}\right)\right) \arrow[u] & \mathbb{Z}^2 \arrow[l] & \mathbb{Z}^2 \arrow[l, "1-\epsilon(A)A"']
\end{tikzcd}$$

where $\epsilon(A)$ denotes the sign of $\mathrm{Det}\,A$. This gives us the following conclusions.

1) $K_0\left(C_r^*\left(\Gamma_{\phi_A}\right)\right) \cong \mathbb{Z} \oplus \mathbb{Z}_{\mathrm{Det}\,A-1} \oplus \ker(1-A)$ with the unit $[1] \in K_0\left(C_r^*\left(\Gamma_{\phi_A}\right)\right)$ represented by $1 \in \mathbb{Z}_{\mathrm{Det}\,A-1}$ and $K_1\left(C_r^*\left(\Gamma_{\phi_A}\right)\right) \cong \mathbb{Z} \oplus \mathrm{coker}(1-A)$, when $\mathrm{Det}\,A \geq 2$.

2) $K_0\left(C_r^*\left(\Gamma_{\phi_A}\right)\right) \cong \mathbb{Z}^2 \oplus \ker(1-A)$ with the unit $[1] \in K_0\left(C_r^*\left(\Gamma_{\phi_A}\right)\right)$ represented by $(1,0) \in \mathbb{Z}^2$ and $K_1\left(C_r^*\left(\Gamma_{\phi_A}\right)\right) \cong \mathbb{Z}^2 \oplus \mathrm{coker}(1-A)$, when $\mathrm{Det}\,A = 1$.

3) $K_0\left(C_r^*\left(\Gamma_{\phi_A}\right)\right) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \ker(1+A)$ with the unit $[1] \in K_0\left(C_r^*\left(\Gamma_{\phi_A}\right)\right)$ represented by $1 \in \mathbb{Z}$ and $K_1\left(C_r^*\left(\Gamma_{\phi_A}\right)\right) \cong \mathbb{Z} \oplus \mathrm{coker}(1+A)$, when $\mathrm{Det}\,A = -1$.

4) $K_0\left(C_r^*\left(\Gamma_{\phi_A}\right)\right) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{|\mathrm{Det}\,A|-1} \oplus \ker(1+A)$ with the unit $[1] \in K_0\left(C_r^*\left(\Gamma_{\phi_A}\right)\right)$ represented by $1 \in \mathbb{Z}_{|\mathrm{Det}\,A|-1}$ and $K_1\left(C_r^*\left(\Gamma_{\phi_A}\right)\right) \cong \mathrm{coker}(1+A)$, when $\mathrm{Det}\,A \leq -2$.

When we specialise to the cases where all eigenvalues of A have modulus greater than 1 we recover Corollary 4.12 from [EHR].

5.3 The three-dimensional torus

In this case the exact sequence (5.1) takes the form

$$\begin{tikzcd}
\mathbb{Z}^4 \arrow[rr, "\begin{pmatrix} 1-|\mathrm{Det}\,A| & 0 \\ 0 & 1-\epsilon(A)A \end{pmatrix}"] & & \mathbb{Z}^4 \arrow[r] & K_0\left(C_r^*\left(\Gamma_{\phi_A}\right)\right) \arrow[d] \\
K_1\left(C_r^*\left(\Gamma_{\phi_A}\right)\right) \arrow[u] & & \mathbb{Z}^4 \arrow[ll] & \mathbb{Z}^4 \arrow[l, "\begin{pmatrix} 1-\epsilon(A) & 0 \\ 0 & 1-\epsilon(A)\,\mathrm{cof}(A) \end{pmatrix}"]
\end{tikzcd}$$

where $\mathrm{cof}(A) = \mathrm{Det}\,A\left(A^{-1}\right)^t$ is the cofactor matrix of A. It follows that

1) $K_0\left(C_r^*\left(\Gamma_{\phi_A}\right)\right) \cong \mathbb{Z} \oplus \ker(1-\mathrm{cof}(A)) \oplus \mathbb{Z}_{\mathrm{Det}\,A-1} \oplus \mathrm{coker}(1-A)$ with the unit $[1] \in K_0\left(C_r^*\left(\Gamma_{\phi_A}\right)\right)$ represented by $1 \in \mathbb{Z}_{\mathrm{Det}\,A-1}$ and $K_1\left(C_r^*\left(\Gamma_{\phi_A}\right)\right) \cong \mathbb{Z} \oplus \ker(1-A) \oplus \mathrm{coker}\left(1-\mathrm{cof}(A)\right)$, when $\mathrm{Det}\,A \geq 2$.

2) $K_0\left(C_r^*\left(\Gamma_{\phi_A}\right)\right) \cong K_1\left(C_r^*\left(\Gamma_{\phi_A}\right)\right) \cong \mathbb{Z}^2 \oplus \ker(1-A) \oplus \mathrm{coker}(1-A)$ with the unit $[1] \in K_0\left(C_r^*\left(\Gamma_{\phi_A}\right)\right)$ represented by $(1,0) \in \mathbb{Z}^2$, when $\mathrm{Det}\,A = 1$.

3) $K_0\left(C_r^*\left(\Gamma_{\phi_A}\right)\right) \cong \mathbb{Z} \oplus \ker(1-A) \oplus \mathrm{coker}(1+A)$ with the unit $[1] \in K_0\left(C_r^*\left(\Gamma_{\phi_A}\right)\right)$ represented by $1 \in \mathbb{Z}$ and $K_1\left(C_r^*\left(\Gamma_{\phi_A}\right)\right) \cong \mathbb{Z} \oplus \ker(1+A) \oplus \mathrm{coker}\left(1-A\right) \oplus \mathbb{Z}_2$, when $\mathrm{Det}\,A = -1$.

4) $K_0\left(C_r^*\left(\Gamma_{\phi_A}\right)\right) \cong \ker(1+\mathrm{cof}(A)) \oplus \mathbb{Z}_{|\,\mathrm{Det}\,A|-1} \oplus \mathrm{coker}(1+A)$ with the unit $[1] \in K_0\left(C_r^*\left(\Gamma_{\phi_A}\right)\right)$ represented by $1 \in \mathbb{Z}_{|\,\mathrm{Det}\,A|-1}$ and $K_1\left(C_r^*\left(\Gamma_{\phi_A}\right)\right) \cong \ker(1+A) \oplus \mathrm{coker}(1+\mathrm{cof}(A)) \oplus \mathbb{Z}_2$, when $\mathrm{Det} \leq -2$.

6 The C^*-algebras of strongly transitive affine surjections on an n-torus, $n \leq 3$

6.1 The C^*-algebra of a strongly transitive local homeomorphism on the circle

A continuous affine map T of the circle has the form

$$Tt = e^{2\pi i\alpha}t^a$$

for some $\alpha \in \mathbb{R}$ and some $a \in \mathbb{Z}$. By combining the results of Section 4 with the K-theory calculations listed in Section 5.1 we obtain the following conclusion.

A) When $a \geq 2$ the C^*-algebra $C_r^*\left(\Gamma_T\right)$ is the same for all $\alpha \in \mathbb{R}$, it is purely infinite and simple with K-theory groups

$$K_0\left(C_r^*(\Gamma_T)\right) \cong \mathbb{Z} \oplus \mathbb{Z}_{a-1}, \quad K_1\left(C_r^*(\Gamma_T)\right) \cong \mathbb{Z}.$$

The unit of $C_r^*(\Gamma_T)$ corresponds to $1 \in \mathbb{Z}_{a-1} \subseteq K_0\left(C_r^*(\Gamma_T)\right)$.

B) When $a \leq -2$ the C^*-algebra $C_r^*(\Gamma_T)$ is the same for all $\alpha \in \mathbb{R}$, it is purely infinite and simple with K-theory groups

$$K_0\left(C_r^*(\Gamma_T)\right) \cong \mathbb{Z}_{|a|-1}, \quad K_1\left(C_r^*(\Gamma_T)\right) \cong \mathbb{Z}_2.$$

The unit of $C_r^*(\Gamma_T)$ corresponds to $1 \in \mathbb{Z}_{|a|-1} \subseteq K_0\left(C_r^*(\Gamma_T)\right)$.

C) When $a = \pm 1$, T is a homeomorphism and $C_r^*\left(\Gamma_T\right)$ is finite. When $a = -1$, T is not strongly transitive and $C_r^*\left(\Gamma_T\right)$ is not simple. When $a = 1$, T is strongly transitive if and only if α is not rational. When $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $C_r^*\left(\Gamma_T\right)$ is an irrational rotation algebra and its structure is well-known. See [EE].

It is well-known that two irrational rotation algebras are isomorphic if and only if the two irrational rotations are conjugate. Now combine this with the observation that the degree a of T can be read off from the K-theory groups of $C_r^*(\Gamma_T)$, and the well-known fact that a minimal homeomorphism of the circle is conjugate to an irrational rotation. Combining with Theorem 4.4 we obtain then the following result regarding strongly transitive local homeomorphisms of the circle.

Proposition 6.1. *Two strongly transitive local homeomorphisms φ and ψ of the circle are conjugate if and only if the associated C^*-algebras $C_r^*\left(\Gamma_\varphi\right)$ and $C_r^*\left(\Gamma_\psi\right)$ are isomorphic.*

6.2 The C^*-algebra of a strongly transitive affine surjection on the two-torus

An affine local homeomorphism of $\mathbb{T}^2$ has the form

$$Tx = \lambda \phi_A(x) \tag{6.1}$$

for some $A \in M_2(\mathbb{Z})$ with $\operatorname{Det} A \neq 0$ and some $\lambda \in \mathbb{T}^2$.
Most of the following results summarise the results of Theorem 4.3, Corollary 3.4 and Section 5.2, but the case $\operatorname{Det} A = 1$ uses the calculation of N.C. Phillips from Example 4.9 of [Ph2] and the classification results of Lin and Phillips from [LP].
Assuming that neither 1 nor -1 is an eigenvalue of A we have the following:

A) When $\operatorname{Det} A \geq 2$ the C^*-algebra $C_r^*(\Gamma_T)$ is isomorphic to $C_r^*\left(\Gamma_{\phi_A}\right)$, it is purely infinite and simple with K-theory groups

$$K_0\left(C_r^*(\Gamma_T)\right) \cong \mathbb{Z} \oplus \mathbb{Z}_{\operatorname{Det} A-1}, \quad K_1\left(C_r^*(\Gamma_T)\right) \cong \mathbb{Z} \oplus \operatorname{coker}(1-A).$$

The unit of $C_r^*(\Gamma_T)$ corresponds to $1 \in \mathbb{Z}_{\operatorname{Det} A-1} \subseteq K_0\left(C_r^*(\Gamma_T)\right)$.

B) When $\operatorname{Det} A \in \{-1, 1\}$, T is not strongly transitive and $C_r^*(\Gamma_T)$ is not simple.

C) When $\operatorname{Det} A \leq -2$ the C^*-algebra $C_r^*(\Gamma_T)$ is isomorphic to $C_r^*\left(\Gamma_{\phi_A}\right)$, it is purely infinite and simple with K-theory groups

$$K_0\left(C_r^*(\Gamma_T)\right) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{|\operatorname{Det} A|-1}, \quad K_1\left(C_r^*(\Gamma_T)\right) \cong \operatorname{coker}(1+A).$$

The unit of $C_r^*(\Gamma_T)$ corresponds to $1 \in \mathbb{Z}_{|\operatorname{Det} A|-1} \subseteq K_0\left(C_r^*(\Gamma_T)\right)$.

Note that it follows from Theorem 4.3 that T is not strongly transitive and $C_r^*\left(\Gamma_T\right)$ not simple when -1 is an eigenvalue while 1 is not.

Assuming that 1 is an eigenvalue of A we have the following:

D) When $\operatorname{Det} A \geq 2$ the set of λ's for which T is strongly transitive is the dense proper subset of $\mathbb{T}^2$ consisting of the elements $\lambda \in \mathbb{T}^2$ with the property that the closed group generated by λ and the set $\left\{x^{-1}\phi_A(x) : \ x \in \mathbb{T}^2\right\}$ is all of $\mathbb{T}^2$. The corresponding C^*-algebras $C_r^*(\Gamma_T)$ are the same simple and purely infinite C^*-algebra with K-theory groups

$$K_0\left(C_r^*(\Gamma_T)\right) \cong \mathbb{Z}^2 \oplus \mathbb{Z}_{\operatorname{Det} A-1}, \quad K_1\left(C_r^*(\Gamma_T)\right) \cong \mathbb{Z} \oplus \operatorname{coker}(1-A).$$

The unit of $C_r^*(\Gamma_T)$ corresponds to $1 \in \mathbb{Z}_{\operatorname{Det} A-1} \subseteq K_0\left(C_r^*(\Gamma_T)\right)$.

E) When $\operatorname{Det} A = 1$, the set of λ's for which T is strongly transitive is a dense proper subset of $\mathbb{T}^2$. For each such λ the C^*-algebra $C_r^*\left(\Gamma_T\right)$ is a simple unital AH-algebra with no dimension growth, a unique trace state and real rank zero, cf. Example 5.6 of [LP]. The ordered K-theory groups depend on λ and are calculated in Example 4.9 of [Ph2].

F) When $\operatorname{Det} A = -1$, T is not strongly transitive and $C_r^*\left(\Gamma_T\right)$ is not simple.

G) When $\operatorname{Det} A \leq -2$ the set of λ's for which T is strongly transitive is the dense proper subset of $\mathbb{T}^2$ which consists of the elements $\lambda \in \mathbb{T}^2$ with the property that the closed group generated by λ and the set $\left\{x^{-1}\phi_A(x) : \ x \in \mathbb{T}^2\right\}$ is all of $\mathbb{T}^2$. The corresponding C^*-algebras $C_r^*(\Gamma_T)$ are the same simple and purely infinite C^*-algebra with K-theory groups

$$K_0\left(C_r^*(\Gamma_T)\right) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{|\operatorname{Det} A|-1}, \quad K_1\left(C_r^*(\Gamma_T)\right) \cong \operatorname{coker}(1+A).$$

The unit of $C_r^*(\Gamma_T)$ corresponds to $1 \in \mathbb{Z}_{|\operatorname{Det} A|-1} \subseteq K_0\left(C_r^*(\Gamma_T)\right)$.

6.3 The C^*-algebra of a strongly transitive affine surjection on the three-dimensional torus

We consider now an affine map $T : \mathbb{T}^3 \to \mathbb{T}^3$ of the form

$$Tx = \lambda\phi_A(x) \tag{6.2}$$

where $A \in M_3(\mathbb{Z})$ and $\operatorname{Det} A \neq 0$.

Assume first that none of the numbers $1, -1, \operatorname{Det} A$ and $-\operatorname{Det} A$ are an eigenvalue of A.

A) When $\operatorname{Det} A \geq 2$ the C^*-algebra $C_r^*(\Gamma_T)$ is isomorphic to $C_r^*(\Gamma_{\phi_A})$, it is purely infinite and simple with K-theory groups

$$K_0(C_r^*(\Gamma_T)) \cong \mathbb{Z} \oplus \mathbb{Z}_{\operatorname{Det} A-1} \oplus \operatorname{coker}(1-A),$$
$$K_1(C_r^*(\Gamma_T)) \cong \mathbb{Z} \oplus \operatorname{coker}(1-\operatorname{cof}(A)).$$

The unit of $C_r^*(\Gamma_T)$ corresponds to $1 \in \mathbb{Z}_{\operatorname{Det} A-1} \subseteq K_0(C_r^*(\Gamma_T))$.

B) When $\operatorname{Det} A \in \{-1, 1\}$, T is not strongly transitive and $C_r^*(\Gamma_T)$ is not simple.

C) When $\operatorname{Det} A \leq -2$ the C^*-algebra $C_r^*(\Gamma_T)$ is isomorphic to $C_r^*(\Gamma_{\phi_A})$, it is purely infinite and simple with K-theory groups

$$K_0(C_r^*(\Gamma_T)) \cong \mathbb{Z}_{|\operatorname{Det} A|-1} \oplus \operatorname{coker}(1+A),$$
$$K_1(C_r^*(\Gamma_T)) \cong \mathbb{Z}_2 \oplus \operatorname{coker}(1+\operatorname{cof}(A))$$

The unit of $C_r^*(\Gamma_T)$ corresponds to $1 \in \mathbb{Z}_{|\operatorname{Det} A|-1} \subseteq K_0(C_r^*(\Gamma_T))$.

It follows from Theorem 4.3 that T is not strongly transitive when one of the numbers -1, $\operatorname{Det} A$ and $-\operatorname{Det} A$ is an eigenvalue of A, but 1 is not, and when both 1 and -1 are eigenvalues of A. It remains therefore only to consider the case when 1 is an eigenvalue, but -1 is not.

Assume that 1 is an eigenvalue of A and that -1 is not.

D) When $\operatorname{Det} A \geq 2$ the set of λ's for which T is strongly transitive is the dense proper subset of $\mathbb{T}^3$ which consists of $\lambda \in \mathbb{T}^3$ with the property that the closed group generated by λ and the set $\left\{x^{-1}\phi_A(x) : \ x \in \mathbb{T}^3\right\}$ is all of $\mathbb{T}^3$. The corresponding C^*-algebras $C_r^*(\Gamma_T)$ are all the same simple and purely infinite C^*-algebra with K-theory groups

$$K_0(C_r^*(\Gamma_T)) \cong \mathbb{Z} \oplus \ker(1-\operatorname{cof}(A)) \oplus \mathbb{Z}_{\operatorname{Det} A-1} \oplus \operatorname{coker}(1-A),$$
$$K_1(C_r^*(\Gamma_T)) \cong \mathbb{Z} \oplus \ker(1-A) \oplus \operatorname{coker}(1-\operatorname{cof}(A)).$$

The unit of $C_r^*(\Gamma_T)$ corresponds to $1 \in \mathbb{Z}_{\operatorname{Det} A-1} \subseteq K_0(C_r^*(\Gamma_T))$.

E) When $\operatorname{Det} A = 1$, T is not strongly transitive unless 1 is the only eigenvalue of A in which case the set of λ's for which T is strongly transitive is the dense proper subset of $\mathbb{T}^3$ which consists of the λ's in $\mathbb{T}^3$ with the property that the closed group generated by λ and the set $\left\{x^{-1}\phi_A(x) : \ x \in \mathbb{T}^3\right\}$ is all of $\mathbb{T}^3$. With λ in this set, the C^*-algebra $C_r^*(\Gamma_T)$ is a unital simple AH-algebra with no dimension growth, a unique trace state and real rank zero. The ordered K-theory of $C_r^*(\Gamma_T)$ depends on λ and is calculated in Subsection 6.4 below.

F) When $\operatorname{Det} A = -1$, T is not strongly transitive and $C_r^*(\Gamma_T)$ is not simple.

G) When $\mathrm{Det}\, A \leq -2$ the set of λ's for which T is strongly transitive is the dense proper subset of $\mathbb{T}^3$ which consists of $\lambda \in \mathbb{T}^3$ with the property that the closed group generated by λ and the set $\left\{x^{-1}\phi_A(x) : \ x \in \mathbb{T}^3\right\}$ is all of $\mathbb{T}^3$. The corresponding C^*-algebras $C_r^*(\Gamma_T)$ are all the same simple and purely infinite C^*-algebra with K-theory groups

$$K_0\left(C_r^*(\Gamma_T)\right) \cong \ker(1 + \mathrm{cof}(A)) \oplus \mathbb{Z}_{|\,\mathrm{Det}\, A|-1} \oplus \mathrm{coker}(1 + A),$$
$$K_1\left(C_r^*(\Gamma_T)\right) \cong \mathbb{Z}_2 \oplus \mathrm{coker}(1 + \mathrm{cof}(A))$$

The unit of $C_r^*(\Gamma_T)$ corresponds to $1 \in \mathbb{Z}_{|\,\mathrm{Det}\, A|-1} \subseteq K_0\left(C_r^*(\Gamma_T)\right)$.

6.4 Minimal affine homeomorphisms of the three-dimensional torus and their C^*-algebras

In this section we justify the statements made under E) in the preceding section concerning the C^*-algebras of a minimal affine homeomorphism of $\mathbb{T}^3$. As we shall show the conclusion concerning the structure of the algebras will follow from results of Lin and Phillips from [LP] once we have calculated the K-groups and the action of the traces on K_0. To do this we use the method of Phillips from Example 4.9 of [Ph2]. First of all we note that a minimal affine homeomorphism of a torus is uniquely ergodic with the Haar measure as the unique invariant Borel probability measure. This follows from Theorem 4 of [Pa] and it means that there is only a single trace state to consider.

Let $A \in M_3(\mathbb{Z})$ be a matrix for which 1 is the only eigenvalue and let $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{T}^3$ be an element such that $Tx = \lambda\phi_A(x)$ is minimal. Since 1 is the only eigenvalue of A minimality of T is equivalent to the condition that the closed group generated by λ and the set $\left\{x^{-1}\phi_A(x) : \ x \in \mathbb{T}^3\right\}$ is all of $\mathbb{T}^3$. This follows from Theorem 4.3, but in the present case this is actually a result of Hoare and Parry, cf. Theorem 4 in [HP].

It follows from Theorem III.12 on page 50 of [N] that there an element $W \in Gl_3(\mathbb{Z})$ and integers $a, b, c \in \mathbb{Z}$ such that

$$WAW^{-1} = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

We will therefore assume that A is equal to the matrix on the right-hand side. Let $\alpha_T : C(\mathbb{T}^3) \to C(\mathbb{T}^3)$ be the automorphism $\alpha_T(f) = f \circ T$ so that $C_r^*(\Gamma_T)$ is isomorphic to the crossed product $C(\mathbb{T}^3) \times_{\alpha_T} \mathbb{Z}$, cf. Proposition 1.8 of [Ph3]. Let τ be the trace state of $C(\mathbb{T}^3) \times_{\alpha_T} \mathbb{Z}$ induced by the Haar-measure of $\mathbb{T}^3$. Thanks to the unique ergodicity of T this is the only trace state of $C(\mathbb{T}^3) \times_{\alpha_T} \mathbb{Z}$. We aim to calculate the map

$$\tau_* : K_0\left(C(\mathbb{T}^3) \times_{\alpha_T} \mathbb{Z}\right) \to \mathbb{R}.$$

¿From the six-terms exact sequence of Pimsner and Voiculescu, [PV], we consider the piece

$$\begin{tikzcd} K_0(C(\mathbb{T}^3)) \arrow[r, "\mathrm{id} - {(\alpha_T)_*}^{-1}"] & K_0(C(\mathbb{T}^3)) \arrow[r, "i_*"] & K_0(C^*_r(\Gamma_T)) \arrow[d, "\partial"] \\ & K_1\left(C(\mathbb{T}^3)\right) & K_1\left(C(\mathbb{T}^3)\right) \arrow[l, "\mathrm{id} - {(\alpha_T)_*}^{-1}"] \end{tikzcd} \tag{6.3}$$

which gives an isomorphism

$$K_0(C^*_r(\Gamma_T)) \cong \operatorname{coker}\left(\mathrm{id} - {(\alpha_T)_*}^{-1}\right) \oplus \ker\left(\mathrm{id} - {(\alpha_T)_*}^{-1}\right). \tag{6.4}$$

¿From the calculations in Section 5.3 we deduce that $\operatorname{coker}\left(\mathrm{id} - {(\alpha_T)_*}^{-1}\right) \cong \mathbb{Z} \oplus \operatorname{coker}(1 - A)$ and

$$\ker\left(\mathrm{id} - {(\alpha_T)_*}^{-1}\right) \cong \mathbb{Z} \oplus \ker(1 - (A^t)^{-1}). \tag{6.5}$$

On $K_0(C(\mathbb{T}^3))$ all traces induce the map which takes a projection to its rank and it follows therefore that $\tau_* : \mathbb{Z} \oplus \operatorname{coker}(1 - A) \to \mathbb{R}$ is the map which picks up the first coordinate from $\mathbb{Z}$ and annihilates $\operatorname{coker}(1 - A)$.

The first direct summand $\mathbb{Z}$ in $\ker\left(\mathrm{id} - {(\alpha_T)_*}^{-1}\right)$ is generated, as a subgroup of $K_1\left(C(\mathbb{T}^3)\right)$ by the image u of $[z]\otimes[z]\otimes[z] \in K_1(C(\mathbb{T}))\otimes K_1(C(\mathbb{T}))\otimes K_1(C(\mathbb{T}))$ under the composed map

$$K_1(C(\mathbb{T}))\otimes K_1(C(\mathbb{T}))\otimes K_1(C(\mathbb{T})) \to K_1(C(\mathbb{T}))\otimes K_0\left(C(\mathbb{T}^2)\right) \to K_1\left(C(\mathbb{T}^3)\right)$$

coming from two applications of the Künneth theorem. (Here z denotes the identity function on $\mathbb{T}$, considered as a unitary in $C(\mathbb{T})$.) It follows from Theorem V. 12 and Theorem VI. 11 in [Ex] that τ_* takes this generator u, considered as an element of $K_0(C^*_r(\Gamma_T))$, to an integer $k \in \mathbb{Z}$. By exchanging $u - k[1]$ for u we can therefore assume that τ_* annihilates the first $\mathbb{Z}$-summand from $\ker\left(\mathrm{id} - {(\alpha_T)_*}^{-1}\right)$.

To get a picture of how τ_* acts on $\ker(1 - (A^t)^{-1})$ observe that

$$\ker(1 - (A^t)^{-1}) = \left\{(x_1, x_2, x_3) \in \mathbb{Z}^3 : \ ax_1 = cx_2 + (b - ac)x_1 = 0\right\}.$$

The identification of $\ker(1-(A^t)^{-1})$ with a subgroup of $K_1(C(\mathbb{T}^3))$ coming from (6.5) will be suppressed in the following; it is given by the map $(x_1, x_2, x_3) \mapsto \sum_{i=1}^3 x_i[u_i]$ where $u_i \in C(\mathbb{T}^3)$ is the unitary $u_i(z_1, z_2, z_3) = z_i$. Fix a group embedding $\Phi_0 : \ker(1 - (A^t)^{-1}) \to K_0(C^*_r(\Gamma_T))$ such that $\partial \circ \Phi_0 = \mathrm{id}$. Write $\lambda_j = e^{2\pi i \alpha_j}$ for some $\alpha_j \in \mathbb{R}, j = 1, 2, 3$. It follows then from Theorem IX. 11 of [Ex] that for any element $\xi = (x_1, x_2, x_3)$ of $\ker(1 - (A^t)^{-1})$ there is an integer k_ξ such that

$$\tau_* \circ \Phi_0(\xi) = x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 + k_\xi.$$

We can therefore change Φ_0 on group generators to obtain another embedding $\Phi : \ker(1-(A^t)^{-1}) \to K_0(C_r^*(\Gamma_T))$ such that $\Phi_0(\xi) - \Phi(\xi) \in \mathbb{Z}[1]$ for all $\xi \in \ker(1-(A^t)^{-1})$ and

$$\tau_* \circ \Phi(x_1, x_2, x_3) = \sum_{j=1}^{3} x_j \alpha_j$$

for all $x_1, x_2, x_3 \in \ker(1-(A^t)^{-1})$. It follows that there is an isomorphism

$$\Psi : \mathbb{Z}^2 \oplus \ker(1-(A^t)^{-1}) \oplus \operatorname{coker}(1-A) \to K_0(C_r^*(\Gamma_T))$$

such that $\tau_* \circ \Psi(x, y, u, v) = x + \eta(u)$, where $x, y \in \mathbb{Z},\ u \in \ker(1-(A^t)^{-1}),\ v \in \operatorname{coker}(1-A)$, and $\eta : \ker(1-(A^t)^{-1}) \to \mathbb{R}$ is given by

$$\eta(x_1, x_2, x_3) = x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3.$$

In all cases $\alpha_3 \in \tau_*(K_0(C_r^*(\Gamma_T)))$. Since λ and $\left\{x^{-1}\phi_A(x) :\ x \in \mathbb{T}^3\right\}$ must generate $\mathbb{T}^3$ in order for T to be minimal we see that α_3 must be irrational. It follows therefore from Corollary 5.3 of [LP] that $C_r^*(\Gamma_T)$ is a unital simple AH-algebra with no dimension growth, a unique trace state and real rank zero, as claimed in E) of Section 6.3. Furthermore, it follows from Theorem 4.5 in [Ph2] that the positive semi-group of $K_0(C_r^*(\Gamma_T))$ under the isomorphism Ψ becomes the set

$$\{0\} \cup \left\{(x, y, u, v) \in \mathbb{Z}^2 \oplus \ker(1-(A^t)^{-1}) \oplus \operatorname{coker}(1-A) :\ x + \eta(u) > 0\right\}.$$

In this way we have obtained a complete description of $K_0(C_r^*(\Gamma_T))$ as a partially ordered group. In this picture the order unit coming from the unit in $C_r^*(\Gamma_T)$ is $(1, 0, 0, 0)$.

Remark 6.2. When only one of the λ_i's are different from 1 the preceding description of the order on K_0 can be obtained from Theorem 7.2 of [Rei]. On the other hand it follows from the calculation above that in our setting the range of the trace on K_0 can have rank 3 and 4 which is not possible when the C^*-algebra comes from a Furstenberg transformation.

The difficulties in extending our approach to tori of higher dimensions come not only from the increasing complexity of the K-theory calculations, but arises also from the algebraic conditions for strong transitivity described in Theorem 4.3. Already in dimension 4 they become quite complicated, cf. Corollary 3 in [Kr], and can no longer be described in a simple way in terms of the eigenvalues for the matrix of the linear part.

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