Cyclic Cohomology of Lie Algebras

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Abstract. We define and completely determine the category of Yetter-Drinfeld modules over Lie algebras. We prove a one to one correspondence between Yetter-Drinfeld modules over a Lie algebra and those over the universal enveloping algebra of the Lie algebra. We associate a mixed complex to a Lie algebra and a stable-Yetter-Drinfeld module over it. We show that the (truncated) Weil algebra, the Weil algebra with generalized coefficients defined by Alekseev-Meinrenken, and the perturbed Koszul complex introduced by Kumar-Vergne are examples of such a mixed complex.

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Contents

1 Introduction

One of the well-known complexes in mathematics is the Chevalley-Eilenberg complex of a Lie algebra $\mathfrak{g}$ with coefficients in a $\mathfrak{g}$-module V [2].

$$C^\bullet(\mathfrak{g},V): \qquad V \xrightarrow{d_{\rm CE}} V\otimes\mathfrak{g}^* \xrightarrow{d_{\rm CE}} V\otimes\wedge^2\mathfrak{g}^* \xrightarrow{d_{\rm CE}} \cdots \tag{1.1}$$

Through examples, we can see that when the coefficients space V is equipped with more structures, then the complex $(C^\bullet(\mathfrak{g},V),d_{\rm CE})$, together with another operator $d_{\rm K}: C^\bullet(\mathfrak{g},V)\to C^{\bullet-1}(\mathfrak{g},V)$, called Koszul boundary, turns into a mixed complex. That is $d_{\rm CE}+d_{\rm K}$ defines a coboundary on the total complex

$$W^\bullet = \bigoplus_{\bullet\geq p\geq 0} C^{2p-\bullet}(\mathfrak{g},V).$$

Among examples, one observes that,

- the well-known (truncated) Weil complex is achieved by $V := S(\mathfrak{g}^*)_{[2q]}$ the (truncated) polynomial algebra of $\mathfrak{g}$,
- the Weil algebra with generalized coefficients defined by Alekseev-Meinrenken in [1] is obtained by $V := \mathcal{E}'(\mathfrak{g}^*)$, the convolution algebra of compactly supported distributions on $\mathfrak{g}^*$,
- finally it was shown by Kumar-Vergne that if V is a module over the Weyl algebra $D(\mathfrak{g})$ then $(W^\bullet, d_{\rm CE}+d_{\rm K})$ is a complex which is called perturbed Koszul complex [14].

In this paper we prove that $(W^\bullet, d_{\rm CE}+d_{\rm K})$ is a complex if and only if V is a unimodular stable module over the Lie algebra $\widetilde{\mathfrak{g}}$, where $\widetilde{\mathfrak{g}} := \mathfrak{g}^*\rtimes\mathfrak{g}$ is the semidirect product Lie algebra $\mathfrak{g}^*$ and $\mathfrak{g}$. Here $\mathfrak{g}^* := \mathrm{Hom}(\mathfrak{g},\mathbb{C})$ is thought of as an abelian Lie algebra acted upon by the Lie algebra $\mathfrak{g}$ via the coadjoint representation.

Next, we show that any Yetter-Drinfeld module over the enveloping Hopf algebra $U(\mathfrak{g})$ yields a module over $\widetilde{\mathfrak{g}}$ and conversely any locally conilpotent module over $\widetilde{\mathfrak{g}}$ amounts to a Yetter-Drinfeld module over the Hopf algebra $U(\mathfrak{g})$. This correspondence is accompanied with a quasi-isomorphism which reduces to the antisymmetrization map if the module V is merely a $\mathfrak{g}$-module. The isomorphism generalizes the computation of the Hopf cyclic cohomology of $U(\mathfrak{g})$ in terms of the Lie algebra homology of $\mathfrak{g}$ carried out by Connes-Moscovici in [4].

Throughout the paper, $\mathfrak{g}$ denotes a finite dimensional Lie algebra over $\mathbb{C}$, the field of complex numbers. We denote by $X_1, \ldots, X_N$ and $\theta^1, \ldots, \theta^N$ a dual basis for $\mathfrak{g}$ and $\mathfrak{g}^*$ respectively, and by $\delta \in \mathfrak{g}^*$ the trace of the adjoint representation of $\mathfrak{g}$ on itself. All tensor products are over $\mathbb{C}$.

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2 The model complex for G-differential algebras

In this section we first recall G-differential algebras and their basic properties. Then we introduce our model complex which is the main motivation of this paper. The model complex includes as examples Weil algebra and their truncations, perturbed Koszul complex introduced by Kumar- Vergne in [14], and Weil algebra with generalized coefficients introduced by Alekseev-Meinrenken [1].

2.1 G-differential algebras

Let $\widehat{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a graded Lie algebra, where $\mathfrak{g}_{-1}$ and $\mathfrak{g}_0$ are N-dimensional vector spaces with bases $\iota_1, \cdots, \iota_N$, and $\mathcal{L}_1, \cdots, \mathcal{L}_N$ respectively, and $\mathfrak{g}_1$ is generated by d.

We let C^i_{jk} denote the structure constants of the Lie algebra $\mathfrak{g}_0$ and assume that the graded-bracket on $\widehat{g}$ is defined as follows.

$$[\iota_p, \iota_q] = 0, \tag{2.1}$$
$$[\mathcal{L}_p, \iota_q] = C^r_{pq}\iota_r, \tag{2.2}$$
$$[\mathcal{L}_p, \mathcal{L}_q] = C^r_{pq}\mathcal{L}_r, \tag{2.3}$$
$$[d, \iota_k] = \mathcal{L}_k, \tag{2.4}$$
$$[d, \mathcal{L}_k] = 0, \tag{2.5}$$
$$[d, d] = 0. \tag{2.6}$$

Now let G be a (connected) Lie group with Lie algebra $\mathfrak{g}$. We assume $\widehat{\mathfrak{g}}$ be as above with $\mathfrak{g}_0 \cong \mathfrak{g}$ as Lie algebras.

A graded algebra A is called a G-differential algebra if there exists a representation $\rho : G \to \mathrm{Aut}(A)$ of the group G and a graded Lie algebra homomorphism

$\hat{\rho} : \widehat{g} \to \mathrm{End}(A)$ compatible in the following way:

$$\left.\frac{d}{dt}\right|_{t=0} \rho(exp(tX)) = \hat{\rho}(X) \tag{2.7}$$

$$\rho(a)\hat{\rho}(X)\rho(a^{-1}) = \hat{\rho}(Ad_a X) \tag{2.8}$$

$$\rho(a)\iota_X \rho(a^{-1}) = \iota_{Ad_a X} \tag{2.9}$$

$$\rho(a)d\rho(a^{-1}) = d \tag{2.10}$$

for any $a \in G$ and any $X \in \mathfrak{g}$. For further discussion on G-differential algebras we refer the reader to [8, chapter 2] and [1].

The exterior algebra $\bigwedge \mathfrak{g}^*$ and the Weil algebra are examples of G-differential algebras.

Here we recall $W(\mathfrak{g})$, the Weil algebra of a finite dimensional Lie algebra $\mathfrak{g}$, by

$$W(\mathfrak{g}) = \bigwedge \mathfrak{g}^* \otimes S(\mathfrak{g}^*),$$

with the grading

$$W(\mathfrak{g}) = \bigoplus_{l \geq 0} W^l(\mathfrak{g}), \tag{2.11}$$

where

$$W^l(\mathfrak{g}) = \bigoplus_{p+2q=l} W^{p,q}, \qquad W^{p,q} := \wedge^p \mathfrak{g}^* \otimes S^q(\mathfrak{g}^*). \tag{2.12}$$

It is equipped with two degree $+1$ differentials as follows. The first one is

$$\begin{aligned} &d_{\mathrm{K}} : \wedge^p \mathfrak{g}^* \otimes S^q(\mathfrak{g}^*) \to \wedge^{p-1}\mathfrak{g}^* \otimes S^{q+1}(\mathfrak{g}^*) \\ &\varphi \otimes R \mapsto \sum_j \iota_{X_j}(\varphi) \otimes R\theta^j \end{aligned} \tag{2.13}$$

and it is called the Koszul coboundary. The second one is the Chevalley-Eilenberg coboundary (Lie algebra cohomology coboundary)

$$d_{\mathrm{CE}} : \wedge^p \mathfrak{g}^* \otimes S^q(\mathfrak{g}^*) \to \wedge^{p+1}\mathfrak{g}^* \otimes S^q(\mathfrak{g}^*). \tag{2.14}$$

Then $d_{\mathrm{CE}} + d_{\mathrm{K}} : W^l(\mathfrak{g}) \to W^{l+1}(\mathfrak{g})$ equips $W(\mathfrak{g})$ with a differential graded algebra structure. It is known that via coadjoint representation $W(\mathfrak{g})$ is a G-differential algebra.

A G-differential algebra is called locally free if there exists an element

$$\Theta = \sum_i X_i \otimes \theta^i \in (\mathfrak{g} \otimes A^{odd})^G$$

called the algebraic connection form.

We assume that $\Theta \in (\mathfrak{g} \otimes A^1)^G$, and we have

$$\iota_k(\Theta) = X_k, \quad \text{and} \quad \mathcal{L}_k(\theta^i) = -C^i_{kl}\theta^l.$$

2.2 The model complex

Let (A, Θ) be a locally free G-differential algebra with $\dim(G) = N$. We assume that V is a vector space with elements L_k and L^k in $\mathrm{End}(V)$, $1 \leq k \leq N$.

We consider the graded space $A \otimes V$ with the grading induced from that of A. Using all information of the G-differential algebra structure of A and the connection form $\Theta \in (\mathfrak{g} \otimes A^1)^G$, we introduce the map

$$D(x \otimes v) := d(x) \otimes v + \theta^k x \otimes L_k(v) + \iota_k(x) \otimes L^k(v) \tag{2.15}$$

as a sum of a degree $+1$ map and a degree -1 map.

Proposition 2.1. *Let (A, Θ) be a locally free G-differential algebra. Then the map*

$$d_{\mathrm{K}}(x \otimes v) = \iota_k(x) \otimes L^k(v) \tag{2.16}$$

is a differential, that is $d_{\mathrm{K}}^2 = 0$, if and only if V is a $\mathbb{C}^N$-module via $L^k s$, i.e.,

$$[L^p, L^q] = 0, \qquad 1 \leq p, q, \leq N.$$

Proof. Assume that $[L^j, L^i] = 0$. Then

$$d_{\mathrm{K}} \circ d_{\mathrm{K}}(x \otimes v) = \iota_l \iota_k(x) \otimes L^l L^k(v) = 0 \tag{2.17}$$

by the commutativity of L_ks and the anti-commutativity of ι_ks.

Conversely, if d_{K} has the property $d_{\mathrm{K}} \circ d_{\mathrm{K}} = 0$, then by using $\iota_k(\theta^j) = \delta_k^j$ we have

$$d_{\mathrm{K}} \circ d_{\mathrm{K}}(\theta^i \theta^j \otimes v) = d_{\mathrm{K}}(\theta^j \otimes L^i(v) - \theta^i \otimes L^j(v)) = 1 \otimes [L^j, L^i](v) = 0 \tag{2.18}$$

which implies $[L^j, L^i] = 0$. ☐

Definition 2.2. *[8]. For a commutative locally free G-differential algebra A, the element $\Omega = \sum_i \Omega^i \otimes X_i \in (A^2 \otimes \mathfrak{g})^G$, satisfying*

$$d(\theta^i) = -\frac{1}{2} C^i_{pq} \theta^p \theta^q + \Omega^i, \tag{2.19}$$

is called the curvature of the connection $\Theta = \sum_i \theta^i \otimes X_i$.

We call a commutative locally free G-differential algebra (A, Θ) flat if $\Omega = 0$, or equivalently

$$d(\theta^k) = -\frac{1}{2} C^k_{pq} \theta^p \theta^q. \tag{2.20}$$

Proposition 2.3. *Let (A, Θ) be a commutative locally free flat G-differential algebra. Then the map*

$$d_{\mathrm{CE}}(x \otimes v) = d(x) \otimes v + \theta^k x \otimes L_k(v) \tag{2.21}$$

is a differential, that is $d_{\mathrm{CE}}^2 = 0$, if and only if V is a $\mathfrak{g}$-module via L_k, that is $[L_t, L_l] = C^k_{tl} L_k$.

Proof. Using the commutativity of A we see that

$$\begin{aligned} d_{\mathrm{CE}} \circ d_{\mathrm{CE}}(1 \otimes v) &= \sum_k d_{\mathrm{CE}}(\theta^k \otimes L_k(v)) \\ &= \sum_k d(\theta^k) \otimes L_k(v) + \sum_{k,t} \theta^t \theta^k \otimes L_t L_k(v) \\ &= -\sum_{l,t} \frac{1}{2} C^k_{tl} \theta^t \theta^l \otimes L_k(v) + \sum_{t,l} \frac{1}{2} \theta^t \theta^l \otimes [L_t, L_l](v), \end{aligned} \tag{2.22}$$

which proves the claim. □

One notes that we use the commutativity of the differential graded algebra in the above proposition. It would be interesting if one finds an argument for a similar proposition to cover the noncommutative and twisted noncommutative Weil algebra as discussed in [1] and [3] respectively.

Considering the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ as a commutative Lie algebra, we can define the Lie bracket on $\widetilde{\mathfrak{g}} := \mathfrak{g}^* \rtimes \mathfrak{g}$ by

$$\big[\alpha \oplus X \ , \ \beta \oplus Y\big] \ := \ \big(\mathcal{L}_X(\beta) - \mathcal{L}_Y(\alpha)\big) \oplus \big[X \ , \ Y\big]. \tag{2.23}$$

Accordingly, the next proposition determines the necessary and sufficient conditions on V for $(A \otimes V, d_{\mathrm{CE}} + d_{\mathrm{K}})$ to be a differential complex.

Proposition 2.4. *Let A be a commutative locally free flat G-differential algebra and V a $\mathfrak{g}$-module via L_ks and a $\mathbb{C}^N$-module via L^ks. Then, $(A \otimes V, D)$ is a differential complex if and only if V is*

$$\textit{unimodular stable} \qquad \sum_k L^k L_k = 0, \tag{2.24}$$

and

$$\widetilde{\mathfrak{g}}\textit{-module} \qquad [L^i, L_j] = \sum_k C^i_{jk} L^k. \tag{2.25}$$

Proof. Once we have $d^2_{\mathrm{CE}} = 0 = d_{\mathrm{K}}$, then $(A \otimes V, d_{\mathrm{CE}} + d_{\mathrm{K}})$ is a differential complex *i.e.,*

$$\begin{aligned} 0 &= d_{\mathrm{CE}} \circ d_{\mathrm{K}} + d_{\mathrm{K}} \circ d_{\mathrm{CE}} \\ &= \mathcal{L}_k \otimes L^k + \theta^k \iota_t \otimes [L_k, L^t] + \mathrm{Id} \otimes L^k L_k, \end{aligned} \tag{2.26}$$

if and only if (2.24) and (2.25) hold. □

3 Lie algebra cohomology and Perturbed Koszul complex

In this section we specialize the model complex $(A \otimes V, D)$ defined in (2.15) for $A = \bigwedge \mathfrak{g}^*$. We show that the perturbed Koszul complex defined in [14] is an example of the model complex. As another example of the model complex, we cover the Weil algebra with generalized coefficients introduced in [1].

3.1 Lie algebra cohomology

Let $\mathfrak{g}$ be a finite dimensional Lie algebra and V be a right $\mathfrak{g}$-module. Let also $\{\theta^i\}$ and $\{X_i\}$ be dual bases for $\mathfrak{g}^*$ and $\mathfrak{g}$. The Chevalley-Eilenberg complex $C(\mathfrak{g}, M)$ is defined by

$$V \xrightarrow{d_{\mathrm{CE}}} C^1(\mathfrak{g}, V) \xrightarrow{d_{\mathrm{CE}}} C^2(\mathfrak{g}, V) \xrightarrow{d_{\mathrm{CE}}} \cdots , \tag{3.1}$$

where $C^q(\mathfrak{g}, V) = \mathrm{Hom}(\wedge^q\mathfrak{g}, V)$ is the vector space of all alternating linear maps on $\mathfrak{g}^{\otimes q}$ with values in V. If $\alpha \in C^q(\mathfrak{g}, V)$, then

$$\begin{aligned} d_{\mathrm{CE}}(\alpha)(X_0, \ldots, X_q) = \sum_{i<j}(-1)^{i+j}\alpha([X_i, X_j], X_0 \ldots \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_q)+ \\ \sum_i(-1)^{i+1}\alpha(X_0, \ldots, \widehat{X}_i, \ldots X_q)X_i. \end{aligned} \tag{3.2}$$

Alternatively, we may identify $C^q(\mathfrak{g}, V)$ with $\wedge^q\mathfrak{g}^* \otimes V$ and the coboundary d_{CE} with the following one

$$\begin{aligned} &d_{\mathrm{CE}}(v) = -\theta^i \otimes v \cdot X_i, \\ &d_{\mathrm{CE}}(\beta \otimes v) = d_{\mathrm{dR}}(\beta) \otimes v - \theta^i \wedge \beta \otimes v \cdot X_i. \end{aligned} \tag{3.3}$$

where $d_{\mathrm{dR}} : \wedge^p\mathfrak{g}^* \to \wedge^{p+1}\mathfrak{g}^*$ is the de Rham derivation defined by $d_{\mathrm{dR}}(\theta^i) = \frac{-1}{2}C^i_{jk}\theta^j\theta^k$. We denote the cohomology of $(C^\bullet(\mathfrak{g}, V), d_{\mathrm{CE}})$ by $H^\bullet(\mathfrak{g}, V)$ and refer to it as the Lie algebra cohomology of $\mathfrak{g}$ with coefficients in V.

3.2 Perturbed Koszul complex

With the same assumptions on $\mathfrak{g}$ and V in the previous subsection, we specialize the model complex $A \otimes V$ defined in (2.15) for $A = \bigwedge \mathfrak{g}^*$. Indeed we have $W^n(\mathfrak{g}, V) := \wedge^n\mathfrak{g}^* \otimes V$, for $n \geq 0$, with differentials $d_{\mathrm{CE}} : W^n(\mathfrak{g}, V) \to W^{n+1}(\mathfrak{g}, V)$ defined in (3.3) and

$$\begin{aligned} &d_{\mathrm{K}} : W^n(\mathfrak{g}, V) \to W^{n-1}(\mathfrak{g}, V) \\ &\alpha \otimes v \mapsto \sum_i \iota_{X_i}(\alpha) \otimes v \lhd \theta^i. \end{aligned} \tag{3.4}$$

An immediate example is the (truncated) Weil algebra.

Example 3.1 (Weil algebra). Let $\mathfrak{g}$ be a (finite dimensional) Lie algebra and set $V = S(\mathfrak{g}^*)$ - the polynomial algebra on $\mathfrak{g}$. Then V is a unimodular stable right $\mathfrak{g}$-module via the (co)adjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$ and the initial multiplication of the symmetric algebra as the action of $\mathfrak{g}^*$.

Example 3.2 (Truncated Weil algebra). Let $V = S(\mathfrak{g}^*)_{[2n]}$ be the truncated polynomial algebra on $\mathfrak{g}$. With the same structure as it is defined in Example 3.1 one obtains the differential complex $W(\mathfrak{g}, S(\mathfrak{g}^*)_{[2n]})$.

To be able to interpret the coefficient space further, we introduce the crossed product algebra

$$\widetilde{D}(\mathfrak{g}) := S(\mathfrak{g}^*) \blacktriangleright\!\!\vartriangleleft U(\mathfrak{g}). \tag{3.5}$$

In the next proposition, by $\widetilde{\mathfrak{g}}$ we mean $\mathfrak{g}^* \blacktriangleright\!\!\vartriangleleft \mathfrak{g}$ with the Lie bracket defined in (2.23).

Proposition 3.3. *The algebras $\widetilde{D}(\mathfrak{g})$ and $U(\mathfrak{g}^* \blacktriangleright\!\!\vartriangleleft \mathfrak{g})$ are isomorphic.*

Proof. It is a simple case of [15, Theorem 7.2.3], that is

$$U(\mathfrak{g}^* \blacktriangleright\!\!\vartriangleleft \mathfrak{g}) = U(\mathfrak{g}^*) \blacktriangleright\!\!\vartriangleleft U(\mathfrak{g}) = S(\mathfrak{g}^*) \blacktriangleright\!\!\vartriangleleft U(\mathfrak{g}) = \widetilde{D}(\mathfrak{g}) \tag{3.6}$$

□

Next, we recall the compatibility for a module over a crossed product algebra, for a proof see [16, Lemma 3.6].

Lemma 3.4. *Let $\mathcal{H}$ be a Hopf algebra, and A an $\mathcal{H}$-module algebra. Then V is a right module on the crossed product algebra $A \blacktriangleright\!\!\vartriangleleft \mathcal{H}$ if and only if V is a right module on A and a right module on $\mathcal{H}$ such that*

$$(v \cdot h) \cdot a = (v \cdot (h_{(1)} \rhd a)) \cdot h_{(2)} \tag{3.7}$$

Finally we restate Proposition 2.4 in the case of $A = \bigwedge \mathfrak{g}^*$.

Proposition 3.5. *The graded space $(W^\bullet(\mathfrak{g}, V), d_{\mathrm{CE}} + d_{\mathrm{K}})$ is a complex if and only if V is a unimodular stable right $\widetilde{\mathfrak{g}}$-module.*

Example 3.6 (Weil algebra with generalized coefficients [1]). Let $\mathcal{E}'(\mathfrak{g}^*)$ be the convolution algebra of compactly supported distributions on $\mathfrak{g}^*$. The symmetric algebra $S(\mathfrak{g}^*)$ is canonically identified with the subalgebra of distributions supported at the origin. This immediately results with a natural $S(\mathfrak{g}^*)$-module structure on $\mathcal{E}'(\mathfrak{g}^*)$ via its own multiplication.

Regarding the coordinate functions μ_i, $1 \le i \le N$ as multiplication operators, we also have $[\mu_i, \theta^j] = \delta^i_j$.

The Lie derivative is described as follows.

$$\mathcal{L}_i = C^k_{ij}\theta^j\mu_k, \quad 1 \le i \le N. \tag{3.8}$$

Therefore,

$$\tau : \mathfrak{g} \to \mathrm{End}(\mathcal{E}'(\mathfrak{g}^*)), \quad X_i \mapsto C^k_{ji}\mu_k\theta^j = -L_{X_i} - \delta(X_i)I \tag{3.9}$$

is a map of Lie algebras, and hence equips $\mathcal{E}'(\mathfrak{g}^*)$ with a right $\mathfrak{g}$-module structure.

We first observe that

$$\sum_i (v \cdot X_i) \lhd \theta^i = C^k_{ji} v\mu_k\theta^j\theta^i = 0, \tag{3.10}$$

by the commutativity of $S(\mathfrak{g}^*)$ and the anti-commutativity of the lower indices of the structure coefficients.

Secondly we observe

$$\begin{aligned}&(v \cdot X_i) \lhd \theta^t = C^k_{ji} v\mu_k \theta^j \theta^t = \\ &C^k_{ji} v\theta^t \mu_k \theta^j + C^t_{ji} v\theta^j = (v \lhd \theta^t) \cdot X_i + v \lhd (X_i \rhd \theta^t),\end{aligned} \tag{3.11}$$

i.e., $\mathcal{E}'(\mathfrak{g}^*)$ is a right module over $S(\mathfrak{g}^*) \bowtie U(\mathfrak{g})$. Hence we have the complex $W(\mathfrak{g}, \mathcal{E}'(\mathfrak{g}^*))$.

Finally we remark that in [1] the authors consider compact groups and their Lie algebras, which are unimodular and hence $\delta = 0$. So, their and our actions of $\mathfrak{g}$ coincide.

3.3 Weyl algebra

Following [17] Appendix 1, let V be a (finite dimensional) vector space with dual V^*. Let $\mathscr{P}(V)$ be the algebra of all polynomials on V and $S(V)$ the symmetric algebra on V. Let us use the notation $D(V)$ for the algebra of differential operators on V with polynomial coefficients - the Weyl algebra on V. For any $v \in V$ we introduce the operator

$$\partial_v(f)(w) := \frac{d}{dt}\Big|_{t=0} f(w + tv). \tag{3.12}$$

As a result, we get an injective algebra map $v \mapsto \partial_v \in D(V)$. As a differential operator on V, ∂_v is identified with the derivative with respect to $v^* \in V^*$.

Using the bijective linear map $\mathscr{P}(V) \otimes S(V) \to D(V)$ defined as $f \otimes v \mapsto fv$, and the fact that $\mathscr{P}(V) \cong S(V^*)$, we conclude that $D(V) \cong S(V^*) \otimes S(V)$ as vector spaces.

Following [7], the standard representation of $D(V)$ is as follows. Let $\{{v_1}^*, \cdots, {v_n}^*\}$ be a basis of V^*. Then, forming $E = \mathbb{C}[{v_1}^*, \cdots, {v_n}^*]$, we consider the operators $P_i \in \mathrm{End}(E)$ as $\partial/\partial {v_i}^*$ and $Q^i \in \mathrm{End}(E)$ as multiplication by ${v_i}^*$. Then the relations are

$$\begin{aligned}&[P_i, Q^i] = I, \qquad [P_i, Q^j] = 0, \quad i \neq j \\ &[P_i, P_j] = 0, \qquad [Q^i, Q^j] = 0, \quad \forall\, i, j\end{aligned} \tag{3.13}$$

It is observed that if V is a module over $D(\mathfrak{g})$ then $(W^\bullet(\mathfrak{g}, V), d_{\mathrm{CE}} + d_{\mathrm{K}})$ is a differential complex [14]. We now briefly remark the relation of this result with our interpretation of the coefficient space (2.15). To this end, we first notice that if V is a right module over the Weyl algebra $D(\mathfrak{g})$, then it is module over the Lie algebra $\mathfrak{g}$ via the Lie algebra map

$$\tau : \mathfrak{g} \to D(\mathfrak{g}), \quad X_i \mapsto C^l_{ki} P_l Q^k. \tag{3.14}$$

Explicitly, we define the action of the Lie algebra as

$$v \cdot X_k = v\tau(X_k). \tag{3.15}$$

On the other hand, V is also a module over the symmetric algebra $S(\mathfrak{g}^*)$ via

$$v \lhd \theta^k = vQ^k. \tag{3.16}$$

Lemma 3.7. *Let V be a right module over $D(\mathfrak{g})$. Then V is unimodular stable.*

Proof. We immediately observe that

$$\sum_i (v \cdot X_i) \lhd \theta^i = \sum_i v \cdot (\tau(X_i)Q^i) = \sum_{i,l,k} v \cdot (C^k_{li} P_k Q^l Q^i) = 0 \tag{3.17}$$

by the commutativity of Qs and the anti-commutativity of the lower indices of the structure coefficients. ☐

Let us now introduce the map

$$\Phi : \widetilde{D}(\mathfrak{g}) \to D(\mathfrak{g}), \quad \theta^j \bowtie X_i \mapsto Q^j \tau(X_i) = C^l_{ki} Q^j P_l Q^k. \tag{3.18}$$

Lemma 3.8. *The map $\Phi : \widetilde{D}(\mathfrak{g}) \to D(\mathfrak{g})$ is well-defined.*

Proof. It is enough to prove $\Phi(X_i)\Phi(\theta^j) = \Phi(X_{i_{(1)}} \rhd \theta^j)\Phi(X_{i_{(2)}})$. To this, we observe

$$\begin{aligned} RHS &= \Phi(X_{i_{(1)}} \rhd \theta^j)\Phi(X_{i_{(2)}}) = -C^j_{ik} Q^k + C^l_{ki} Q^j P_l Q^k \\ &= C^l_{ki} P_l Q^j Q^k = C^l_{ki} P_l Q^k Q^j = \Phi(X_i)\Phi(\theta^j) = LHS. \end{aligned} \tag{3.19}$$

☐

Corollary 3.9. *If V is a right module over $D(\mathfrak{g})$, then V is a right module over $\widetilde{D}(\mathfrak{g}) = S(\mathfrak{g}^*) \bowtie U(\mathfrak{g})$.*

4 Lie algebra homology and Poincaré duality

In this section, for any Lie algebra $\mathfrak{g}$ and any stable $\widetilde{\mathfrak{g}}$-module V we define a complex dual to the model complex and we establish a Poincaré duality between these two complexes. The need for this new complex will be justified in the next sections.

4.1 Lie algebra homology

Let $\mathfrak{g}$ be a Lie algebra and V be a right $\mathfrak{g}$-module. We recall the Lie algebra homology complex $C_q(\mathfrak{g}, V) = \wedge^q \mathfrak{g} \otimes V$ by

$$\cdots \xrightarrow{\partial_{\rm CE}} C_2(\mathfrak{g}, V) \xrightarrow{\partial_{\rm CE}} C_1(\mathfrak{g}, V) \xrightarrow{\partial_{\rm CE}} V \tag{4.1}$$

where

$$\partial_{\mathrm{CE}}(X_0 \wedge \cdots \wedge X_{q-1} \otimes v) = \sum_i (-1)^i X_0 \wedge \cdots \wedge \widehat{X}_i \wedge \cdots \wedge X_{q-1} \otimes v \cdot X_i + \sum_{i<j} (-1)^{i+j} [X_i, X_j] \wedge X_0 \wedge \cdots \wedge \widehat{X}_i \wedge \cdots \wedge \widehat{X}_j \wedge \cdots \wedge X_{q-1} \otimes v \tag{4.2}$$

We call the homology of the complex $(C_\bullet(\mathfrak{g}, V), \partial_{\mathrm{CE}})$ the Lie algebra homology of $\mathfrak{g}$ with coefficients in V and denote it by $H_\bullet(\mathfrak{g}, V)$.

4.2 Poincaré duality

Let V to be a right $\mathfrak{g}$-module and right $S(\mathfrak{g}^*)$-module. We consider the graded vector space $C_n(\mathfrak{g}, V) := \wedge^n \mathfrak{g} \otimes V$ with two differentials $\partial_{\mathrm{CE}} : C_{n+1}(\mathfrak{g}, V) \to C_n(\mathfrak{g}, V)$ defined in (4.2), and

$$\partial_{\mathrm{K}} : C_n(\mathfrak{g}, V) \to C_{n+1}(\mathfrak{g}, V), \quad Y_1 \wedge \cdots \wedge Y_n \otimes v \mapsto \sum_i X_i \wedge Y_1 \wedge \cdots \wedge Y_n \otimes v \lhd \theta^i \tag{4.3}$$

Let us first justify that ∂_{K} is a differential.

Lemma 4.1. *We have $\partial_{\mathrm{K}} \circ \partial_{\mathrm{K}} = 0$.*

Proof. We observe that by the commutativity of $S(\mathfrak{g}^*)$ and the anti-commutativity of the wedge product we have

$$\begin{aligned} &\partial_{\mathrm{K}} \circ \partial_{\mathrm{K}}(Y_1 \wedge \cdots \wedge Y_n \otimes v) = \sum_i \partial_{\mathrm{K}}(X_i \wedge Y_1 \wedge \cdots \wedge Y_n \otimes v \lhd \theta^i) \\ &= \sum_{i,j} X_j \wedge X_i \wedge Y_1 \wedge \cdots \wedge Y_n \otimes v \lhd \theta^i\theta^j = 0. \end{aligned} \tag{4.4}$$

□

We say that a right $\widetilde{\mathfrak{g}}$-module V is stable if

$$\sum_i (v \lhd \theta^i) \cdot X_i = 0. \tag{4.5}$$

Proposition 4.2. *The space $(C_\bullet(\mathfrak{g}, V), \partial_{\mathrm{CE}} + \partial_{\mathrm{K}})$ is a differential complex if and only if V is stable right $\widetilde{\mathfrak{g}}$-module.*

Proof. On the one hand we have

$$\begin{aligned}
&\partial_{\mathrm{CE}}(\partial_{\mathrm{K}}(Y_0 \wedge \cdots \wedge Y_n \otimes v))\\
&= \sum_i \partial_{\mathrm{CE}}(X_i \wedge Y_0 \wedge \cdots \wedge Y_n \otimes v \lhd \theta^i)\\
&= \sum_i Y_0 \wedge \cdots \wedge Y_n \otimes (v \lhd \theta^i) \cdot X_i\\
&\quad + \sum_{i,j} (-1)^{j+1} X_i \wedge Y_0 \wedge \cdots \wedge \widehat{Y_j} \wedge \cdots \wedge Y_n \otimes (v \lhd \theta^i) \cdot Y_j\\
&\quad + \sum_{i,j} (-1)^{j+1} [X_i, Y_j] \wedge Y_0 \wedge \cdots \wedge \widehat{Y_j} \wedge \cdots \wedge Y_n \otimes v \lhd \theta^i\\
&\quad + \sum_{i,j} (-1)^{j+k} [Y_j, Y_k] \wedge X_i \wedge Y_0 \wedge \cdots \wedge \widehat{Y_j} \wedge \cdots \wedge \widehat{Y_k} \wedge \cdots \wedge Y_n \otimes v \lhd \theta^i,
\end{aligned} \tag{4.6}$$

and on the other hand

$$\begin{aligned}
&\partial_{\mathrm{K}}(\partial_{\mathrm{CE}}(Y_0 \wedge \cdots \wedge Y_n \otimes v))\\
&= \sum_j (-1)^j \partial_{\mathrm{K}}(Y_0 \wedge \cdots \wedge \widehat{Y_j} \wedge \cdots \wedge Y_n \otimes v \cdot Y_j)\\
&\quad + \sum_{j,k} (-1)^{j+k} \partial_{\mathrm{K}}([Y_j, Y_k] \wedge Y_0 \wedge \cdots \wedge \widehat{Y_j} \wedge \cdots \wedge \widehat{Y_k} \wedge \cdots \wedge Y_n \otimes v)\\
&= \sum_{i,j} (-1)^j X_i \wedge Y_0 \wedge \cdots \wedge \widehat{Y_j} \wedge \cdots \wedge Y_n \otimes (v \wedge Y_j) \lhd \theta^i\\
&\quad + \sum_{i,j,k} (-1)^{j+k+1} [Y_j, Y_k] \wedge X_i \wedge Y_0 \wedge \cdots \wedge \widehat{Y_j} \wedge \cdots \wedge \widehat{Y_k} \wedge \cdots \wedge Y_n \otimes v \lhd \theta^i.
\end{aligned} \tag{4.7}$$

Therefore, the complex is a mixed complex if and only if

$$\begin{aligned}
&(\partial_{\mathrm{CE}} \circ \partial_{\mathrm{K}} + \partial_{\mathrm{K}} \circ \partial_{\mathrm{CE}})(Y_0 \wedge \cdots \wedge Y_n \otimes v) =\\
&\sum_i Y_0 \wedge \cdots \wedge Y_n \otimes (v \lhd \theta^i) \cdot X_i +\\
&\sum_{i,j} (-1)^{j+1} X_i \wedge Y_0 \wedge \cdots \wedge \widehat{Y_j} \wedge \cdots \wedge Y_n \otimes [(v \lhd \theta^i) \cdot Y_j - (v \cdot Y_j) \lhd \theta^i] +\\
&\sum_{i,j} (-1)^{j+1} [X_i, Y_j] \wedge Y_0 \wedge \cdots \wedge \widehat{Y_j} \wedge \cdots \wedge Y_n \otimes v \lhd \theta^i = 0.
\end{aligned} \tag{4.8}$$

Now, if we assume that $(C^\bullet(\mathfrak{g}, V), \partial_{\mathrm{CE}} + \partial_{\mathrm{K}})$ is a differential complex, then we obtain the stability condition (4.5) evaluating (4.8) on $1 \otimes v$. Similarly we observe that V is a $\widetilde{\mathfrak{g}}$-module by evaluating (4.8) on $Y \otimes v$.

The converse argument is obvious. ∎

Proposition 4.3. *A vector space V is a unimodular stable right $\widetilde{\mathfrak{g}}$-module, if and only if $V\otimes\mathbb{C}_\delta$ is a stable right $\widetilde{\mathfrak{g}}$-module.*

Proof. Indeed, if V is unimodular stable right $\widetilde{\mathfrak{g}}$-module, that is $\sum_i (v\lhd X_i)\cdot\theta^i = 0$, for any $v\in V$, then

$$\begin{aligned}&\sum_i ((v\otimes 1_{\mathbb{C}})\cdot\theta^i)\lhd X^i = \sum_i (v\cdot\theta^i)\cdot X_i\otimes 1_{\mathbb{C}} + v\delta(X_i)\otimes 1_{\mathbb{C}}\\ &=\sum_i (v\cdot X_i)\lhd\theta^i\otimes 1_{\mathbb{C}} = 0,\end{aligned}\tag{4.9}$$

which proves that $V\otimes\mathbb{C}_\delta$ is stable. Similarly we observe that for $1\le i,j\le N$,

$$\begin{aligned}&((v\otimes 1_{\mathbb{C}})\cdot X_j)\lhd\theta^i = (v\cdot X_j\otimes 1_{\mathbb{C}} + v\delta(X_j)\otimes 1_{\mathbb{C}})\lhd\theta^i =\\ &((v\cdot X_j)\lhd\theta^i + v\delta(X_j)\lhd\theta^i)\otimes 1_{\mathbb{C}} =\\ &(v\lhd(X_j\rhd\theta^i) + (v\lhd\theta^i)\cdot X_j + v\delta(X_j)\lhd\theta^i)\otimes 1_{\mathbb{C}} =\\ &(v\otimes 1_{\mathbb{C}})\lhd(X_j\rhd\theta^i) + ((v\otimes 1_{\mathbb{C}})\lhd\theta^i)\cdot X_j\end{aligned}\tag{4.10}$$

i.e., $V\otimes\mathbb{C}_\delta$ is a right $\widetilde{\mathfrak{g}}$-module.

The converse argument is similar. ☐

Let us now briefly recall the Poincaré isomorphism by

$$\mathfrak{D}_P : \wedge^k\mathfrak{g}^*\to\wedge^{N-k}\mathfrak{g},\qquad \eta\mapsto\iota(\eta)\varpi,\tag{4.11}$$

where $\varpi = X_1\wedge\cdots\wedge X_N$ is the covolume element of $\mathfrak{g}$. By definition $\iota(\theta^i) : \wedge^\bullet\mathfrak{g}\to\wedge^{\bullet-1}\mathfrak{g}$ is given by

$$\langle\iota(\theta^i)\xi,\theta^{j_1}\wedge\cdots\wedge\theta^{j_{r-1}}\rangle := \langle\xi,\theta^i\wedge\theta^{j_1}\wedge\cdots\wedge\theta^{j_{r-1}}\rangle,\quad \xi\in\wedge^r\mathfrak{g}.\tag{4.12}$$

Finally, for $\eta = \theta^{i_1}\wedge\cdots\wedge\theta^{i_k}$, the interior multiplication $\iota(\eta) : \wedge^\bullet\mathfrak{g}\to\wedge^{\bullet-k}\mathfrak{g}$ is defined by

$$\iota(\eta) := \iota(\theta^{i_k})\circ\cdots\circ\iota(\theta^{i_1}).\tag{4.13}$$

Proposition 4.4. *Let V be a stable right $\widetilde{\mathfrak{g}}$-module. Then the Poincaré isomorphism induces a map of complexes between the complex $W(\mathfrak{g}, V\otimes\mathbb{C}_{-\delta})$ and the complex $C(\mathfrak{g}, V)$.*

Proof. Let us first introduce the notation $\widetilde{V} := V\otimes\mathbb{C}_{-\delta}$. We can identify $\widetilde{V}$ with V as a vector space, but with the right $\mathfrak{g}$-module structure deformed as $v\lhd X := v\cdot X - v\delta(X)$.

We prove the commutativity of the (co)boundaries via the (inverse) Poincaré isomorphism, *i.e.,*

$$\begin{aligned}&\mathfrak{D}_P^{-1} : \wedge^p\mathfrak{g}\otimes V\to\wedge^{N-p}\mathfrak{g}^*\otimes\widetilde{V}\\ &\xi\otimes v\mapsto\mathfrak{D}_P^{-1}(\xi\otimes v),\end{aligned}\tag{4.14}$$

where for an arbitrary $\eta \in \wedge^{N-p}\mathfrak{g}$

$$\langle \eta, \mathfrak{D}_P^{-1}(\xi \otimes v)\rangle := \langle \eta\xi, \omega^*\rangle v. \tag{4.15}$$

Here, $\omega^* \in \wedge^N \mathfrak{g}^*$ is the volume form.

The commutativity of the diagram

$$\begin{tikzcd}
\wedge^p \mathfrak{g} \otimes V \arrow[r, "\partial_{\mathrm{CE}}"] \arrow[d, "\mathfrak{D}_P^{-1}"'] & \wedge^{p-1}\mathfrak{g} \otimes V \arrow[d, "\mathfrak{D}_P^{-1}"] \\
\wedge^{N-p}\mathfrak{g}^* \otimes \widetilde{V} \arrow[r, "d_{\mathrm{CE}}"'] & \wedge^{N-p+1}\mathfrak{g}^* \otimes \widetilde{V}
\end{tikzcd}$$

follows from the Poincaré duality in Lie algebra homology - cohomology, [13, Chapter VI, Section 3]. For the commutativity of the diagram

$$\begin{tikzcd}
\wedge^p \mathfrak{g} \otimes V \arrow[r, "\partial_{\mathrm{K}}"] \arrow[d, "\mathfrak{D}_P^{-1}"'] & \wedge^{p+1}\mathfrak{g} \otimes V \arrow[d, "\mathfrak{D}_P^{-1}"] \\
\wedge^{N-p}\mathfrak{g}^* \otimes \widetilde{V} \arrow[r, "d_{\mathrm{K}}"'] & \wedge^{N-p-1}\mathfrak{g}^* \otimes \widetilde{V}
\end{tikzcd}$$

we take an arbitrary $\xi \in \wedge^p\mathfrak{g}$, $\eta \in \wedge^{N-p-1}\mathfrak{g}$ and $v \in V$. Then

$$\begin{aligned}
&\mathfrak{D}_P^{-1}(\partial_{\mathrm{K}}(\xi \otimes v))(\eta) = \langle \eta X_i \xi, \omega^*\rangle v \lhd \theta^i = \\
&(-1)^{N-p-1}\langle X_i\eta\xi, \omega^*\rangle v \lhd \theta^i = (-1)^{N-p-1} d_{\mathrm{K}}(\mathfrak{D}_P^{-1}(\xi \otimes v))(\eta).
\end{aligned} \tag{4.16}$$

$\square$

5 Lie algebra coaction and SAYD coefficients

In this section we identify the coefficients we discussed in the previous sections of this paper with stable-anti-Yetter-Drinfeld module over the universal enveloping algebra of the Lie algebra in question. To this end, we introduce the notion of comodule over a Lie algebra.

5.1 SAYD modules and cyclic cohomology of Hopf algebras

Let $\mathcal{H}$ be a Hopf algebra. By definition, a character $\delta : \mathcal{H} \to \mathbb{C}$ is an algebra map. A group-like $\sigma \in \mathcal{H}$ is the dual object of the character, *i.e.,* $\Delta(\sigma) = \sigma \otimes \sigma$. The pair (δ, σ) is called a modular pair in involution [6] if

$$\delta(\sigma) = 1, \quad \text{and} \quad S_\delta^2 = Ad_\sigma, \tag{5.1}$$

where $Ad_\sigma(h) = \sigma h \sigma^{-1}$ and S_δ is defined by

$$S_\delta(h) = \delta(h_{(1)})S(h_{(2)}). \tag{5.2}$$

We recall from [10] the definition of a right-left stable-anti-Yetter-Drinfeld module over a Hopf algebra $\mathcal{H}$. Let V be a right module and left comodule over a Hopf algebra $\mathcal{H}$. We say that it is stable-anti-Yetter-Drinfeld (SAYD) module over $\mathcal{H}$ if

$$\blacktriangledown(v\cdot h) = S(h_{(3)})v_{\langle -1\rangle}h_{(1)}\otimes v_{\langle 0\rangle}\cdot h_{(2)}, \qquad v_{\langle 0\rangle}\cdot v_{\langle -1\rangle} = v, \tag{5.3}$$

for any $v\in V$ and $h\in\mathcal{H}$. It is shown in [10] that any MPI defines a one dimensional SAYD module and all one dimensional SAYD modules come this way.

Let V be a right-left SAYD module over a Hopf algebra $\mathcal{H}$. Let

$$C^q(\mathcal{H},V) := V\otimes\mathcal{H}^{\otimes q}, \quad q\geq 0. \tag{5.4}$$

We recall

$$\begin{aligned}
&\text{face operators} \quad \partial_i : C^q(\mathcal{H},V)\to C^{q+1}(\mathcal{H},V), && 0\leq i\leq q+1\\
&\text{degeneracy operators} \quad \sigma_j : C^q(\mathcal{H},V)\to C^{q-1}(\mathcal{H},V), && 0\leq j\leq q-1\\
&\text{cyclic operators} \quad \tau : C^q(\mathcal{H},V)\to C^q(\mathcal{H},V), &&
\end{aligned}$$

by

$$\begin{aligned}
&\partial_0(v\otimes h^1\otimes\ldots\otimes h^q) = v\otimes 1\otimes h^1\otimes\ldots\otimes h^q,\\
&\partial_i(v\otimes h^1\otimes\ldots\otimes h^q) = v\otimes h^1\otimes\ldots\otimes h^i{}_{(1)}\otimes h^i{}_{(2)}\otimes\ldots\otimes h^q,\\
&\partial_{q+1}(v\otimes h^1\otimes\ldots\otimes h^q) = v_{\langle 0\rangle}\otimes h^1\otimes\ldots\otimes h^q\otimes v_{\langle -1\rangle},\\
&\sigma_j(v\otimes h^1\otimes\ldots\otimes h^q) = (v\otimes h^1\otimes\ldots\otimes\varepsilon(h^{j+1})\otimes\ldots\otimes h^q),\\
&\tau(v\otimes h^1\otimes\ldots\otimes h^q) = v_{\langle 0\rangle}h^1{}_{(1)}\otimes S(h^1{}_{(2)})\cdot(h^2\otimes\ldots\otimes h^q\otimes v_{\langle -1\rangle}),
\end{aligned} \tag{5.5}$$

where $\mathcal{H}$ acts on $\mathcal{H}^{\otimes q}$ diagonally.

The graded module $C^\bullet(\mathcal{H},V)$ endowed with the above operators is then a cocyclic module [9], which means that ∂_i, σ_j and τ satisfy the following identities

$$\begin{aligned}
&\partial_j\partial_i = \partial_i\partial_{j-1}, && \text{if} \quad i<j,\\
&\sigma_j\sigma_i = \sigma_i\sigma_{j+1}, && \text{if} \quad i\leq j,\\
&\sigma_j\partial_i = \begin{cases}\partial_i\sigma_{j-1}, & \text{if} \quad i<j\\ \mathrm{Id} & \text{if} \quad i=j \text{ or } i=j+1\\ \partial_{i-1}\sigma_j & \text{if} \quad i>j+1,\end{cases}\\
&\tau\partial_i = \partial_{i-1}\tau, && 1\leq i\leq q\\
&\tau\partial_0 = \partial_{q+1}, \qquad \tau\sigma_i = \sigma_{i-1}\tau, && 1\leq i\leq q\\
&\tau\sigma_0 = \sigma_n\tau^2, \qquad \tau^{q+1} = \mathrm{Id}.
\end{aligned} \tag{5.6}$$

We use the face operators to define the Hochschild coboundary

$$b : C^q(\mathcal{H}, V) \to C^{q+1}(\mathcal{H}, V), \qquad b := \sum_{i=0}^{q+1} (-1)^i \partial_i \tag{5.7}$$

It is known that $b^2 = 0$. As a result, one obtains the Hochschild complex of the coalgebra $\mathcal{H}$ with coefficients in the bicomodule V. Here, the right comodule defined trivially. The cohomology of $(C^\bullet(\mathcal{H}, V), b)$ is denoted by $H^\bullet_{\rm coalg}(\mathcal{H}, V)$.

We use the rest of the operators to define the Connes boundary operator,

$$B : C^q(\mathcal{H}, V) \to C^{q-1}(\mathcal{H}, V), \qquad B := \left(\sum_{i=0}^{q} (-1)^{qi} \tau^i \right) \sigma_{q-1} \tau. \tag{5.8}$$

It is shown in [5] that for any cocyclic module we have $b^2 = B^2 = (b+B)^2 = 0$. As a result, we define the cyclic cohomology of $\mathcal{H}$ with coefficients in the SAYD module V, which is denoted by $HC^\bullet(\mathcal{H}, V)$, as the total cohomology of the bicomplex

$$C^{p,q}(\mathcal{H}, V) = \begin{cases} V \otimes \mathcal{H}^{\otimes q-p}, & \text{if} \quad 0 \le p \le q, \\ \\ \quad 0, & \quad \text{otherwise.} \end{cases} \tag{5.9}$$

We can also define the periodic cyclic cohomology of $\mathcal{H}$ with coefficients in V, which is denoted by $HP^*(\mathcal{H}, V)$, as the total cohomology of direct sum total of the bicomplex

$$C^{p,q}(\mathcal{H}, V) = \begin{cases} V \otimes \mathcal{H}^{\otimes q-p}, & \text{if} \quad p \le q, \\ \\ \quad 0, & \quad \text{otherwise.} \end{cases} \tag{5.10}$$

It can be seen that the periodic cyclic complex and hence the cohomology is $\mathbb{Z}_2$ graded.

5.2 SAYD modules over Lie algebras

We need to define the notion of comodule over a Lie algebra $\mathfrak{g}$ to be able to make a passage from the stable $\widetilde{\mathfrak{g}}$-modules we already defined in the previous sections to SAYD modules over the universal enveloping algebra $U(\mathfrak{g})$.

Definition 5.1. *We say a vector space V is a left comodule over the Lie algebra $\mathfrak{g}$ if there is a map $\blacktriangledown_{\mathfrak{g}} : V \to \mathfrak{g} \otimes V$ such that*

$$v_{[-2]} \wedge v_{[-1]} \otimes v_{[0]} = 0, \tag{5.11}$$

where $\blacktriangledown_{\mathfrak{g}}(v) = v_{[-1]} \otimes v_{[0]}$, and

$$v_{[-2]} \otimes v_{[-1]} \otimes v_{[0]} = v_{[-1]} \otimes (v_{[0]})_{[-1]} \otimes (v_{[0]})_{[0]}.$$

Proposition 5.2. *Let $\mathfrak{g}$ be a Lie algebra and V be a vector space. Then, V is a right $S(\mathfrak{g}^*)$-module if and only if it is a left $\mathfrak{g}$-comodule.*

Proof. Assume that V is a right module over the symmetric algebra $S(\mathfrak{g}^*)$. Then for any $v \in V$ there is an element $v_{[-1]} \otimes v_{[0]} \in \mathfrak{g}^{**} \otimes V \cong \mathfrak{g} \otimes V$ such that for any $\theta \in \mathfrak{g}^*$

$$v \lhd \theta = v_{[-1]}(\theta) v_{[0]} = \theta(v_{[-1]}) v_{[0]}. \tag{5.12}$$

Hence define the linear map $\blacktriangledown_{\mathfrak{g}} : V \to \mathfrak{g} \otimes V$ by

$$v \mapsto v_{[-1]} \otimes v_{[0]}. \tag{5.13}$$

The compatibility needed for V to be a right module over $S(\mathfrak{g}^*)$,which is $(v \lhd \theta) \lhd \eta - (v \lhd \eta) \lhd \theta = 0$ translates directly into

$$\alpha(v_{[-2]} \wedge v_{[-1]}) \otimes v_{[0]} = (v_{[-2]} \otimes v_{[-1]} - v_{[-1]} \otimes v_{[-2]}) \otimes v_{[0]} = 0, \tag{5.14}$$

where $\alpha : \wedge^2 \mathfrak{g} \to U(\mathfrak{g})^{\otimes 2}$ is the anti-symmetrization map. Since the anti-symmetrization is injective, we have

$$v_{[-2]} \wedge v_{[-1]} \otimes v_{[0]} = 0. \tag{5.15}$$

Hence, V is a left $\mathfrak{g}$-comodule.

Conversely, assume that V is a left $\mathfrak{g}$-comodule via the map $\blacktriangledown_{\mathfrak{g}} : V \to \mathfrak{g} \otimes V$ defined by $v \mapsto v_{[-1]} \otimes v_{[0]}$. We define the right action

$$V \otimes S(\mathfrak{g}^*) \to V, \quad v \otimes \theta \mapsto v \lhd \theta := \theta(v_{[-1]}) v_{[0]}, \tag{5.16}$$

for any $\theta \in \mathfrak{g}^*$ and any $v \in V$. Thus,

$$(v \lhd \theta) \lhd \eta - (v \lhd \eta) \lhd \theta = (v_{[-2]} \otimes v_{[-1]} - v_{[-1]} \otimes v_{[-2]})(\theta \otimes \eta) \otimes v_{[0]} = 0, \tag{5.17}$$

proving that V is a right module over $S(\mathfrak{g}^*)$. $\square$

Having understood the relation between the left $\mathfrak{g}$-coaction and right $S(\mathfrak{g}^*)$-action, it is natural to investigate the relation with left $U(\mathfrak{g})$-coaction.

Let $\blacktriangledown : V \to U(\mathfrak{g}) \otimes V$ be a left $U(\mathfrak{g})$-comodule structure on the linear space V. Then composing via the canonical projection $\pi : U(\mathfrak{g}) \to \mathfrak{g}$, we get a linear map $\blacktriangledown_{\mathfrak{g}} : V \to \mathfrak{g} \otimes V$.

$$\begin{tikzcd} V \arrow[r, "\blacktriangledown"] \arrow[dr, "\blacktriangledown_{\mathfrak{g}}"'] & U(\mathfrak{g}) \otimes V \arrow[d, "\pi \otimes id"] \\ & \mathfrak{g} \otimes V \end{tikzcd}$$

Lemma 5.3. *If $\blacktriangledown : V \to U(\mathfrak{g}) \otimes V$ is a coaction, then so is $\blacktriangledown_{\mathfrak{g}} : V \to \mathfrak{g} \otimes V$.*

Proof. If we write $\blacktriangledown(v) = v_{\overline{[-1]}} \otimes v_{\overline{[0]}}$ then

$$\begin{aligned} v_{[-2]} \wedge v_{[-1]} \otimes v_{[0]} = \pi(v_{\overline{[-2]}}) \wedge \pi(v_{\overline{[-1]}}) \otimes v_{\overline{[0]}} = \\ \pi(v_{\overline{[-1]}^{(1)}}) \wedge \pi(v_{\overline{[-1]}^{(2)}}) \otimes v_{\overline{[0]}} = 0 \end{aligned} \tag{5.18}$$

by the cocommutativity of $U(\mathfrak{g})$. □

For the reverse process which is to obtain a $U(\mathfrak{g})$-comodule out of a $\mathfrak{g}$-comodule, we will need the following concept.

DEFINITION 5.4. *Let V be a $\mathfrak{g}$-comodule via $\blacktriangledown_{\mathfrak{g}} : V \to \mathfrak{g} \otimes V$. Then we call the coaction locally conilpotent if it is conilpotent on any one dimensional subspace. In other words, $\blacktriangledown_{\mathfrak{g}} : V \to \mathfrak{g} \otimes V$ is locally conilpotent if and only if for any $v \in V$ there exists $n \in \mathbb{N}$ such that $\blacktriangledown_{\mathfrak{g}}^n(v) = 0$.*

EXAMPLE 5.5. If V is an SAYD module on $U(\mathfrak{g})$, then by [11, Lemma 6.2] we have the filtration $V = \cup_{p \in \mathbb{Z}} F_p V$ defined as $F_0 V = V^{coU(\mathfrak{g})}$ and inductively

$$F_{p+1}V/F_pV = (V/F_pV)^{coU(\mathfrak{g})} \tag{5.19}$$

Then the induced $\mathfrak{g}$-comodule V is locally conilpotent.

EXAMPLE 5.6. Let $\mathfrak{g}$ be a Lie algebra and $S(\mathfrak{g}^*)$ be the symmetric algebra on $\mathfrak{g}^*$. For $V = S(\mathfrak{g}^*)$, consider the coaction

$$S(\mathfrak{g}^*) \to \mathfrak{g} \otimes S(\mathfrak{g}^*), \quad \alpha \mapsto X_i \otimes \alpha\theta^i, \tag{5.20}$$

called the Koszul coaction. The corresponding $S(\mathfrak{g}^*)$-action on V coincides with the multiplication of $S(\mathfrak{g}^*)$. Therefore, the Koszul coaction is not locally conilpotent.

One notes that the Koszul coaction is locally conilpotent on any truncation of the symmetric algebra.

Let $\{U_k(\mathfrak{g})\}_{k\geq 0}$ be the canonical filtration of $U(\mathfrak{g})$, *i.e.,*

$$U_0(\mathfrak{g}) = \mathbb{C} \cdot 1, \quad U_1(\mathfrak{g}) = \mathbb{C} \cdot 1 \oplus \mathfrak{g}, \quad U_p(\mathfrak{g}) \cdot U_q(\mathfrak{g}) \subseteq U_{p+q}(\mathfrak{g}) \tag{5.21}$$

Let us call an element in $U(\mathfrak{g})$ as symmetric homogeneous of degree k if it is the canonical image of a symmetric homogeneous tensor of degree k over $\mathfrak{g}$. Let $U^k(\mathfrak{g})$ be the set of all symmetric elements of degree n in $U(\mathfrak{g})$.

We recall from [7, Proposition 2.4.4] that

$$U_k(\mathfrak{g}) = U_{k-1}(\mathfrak{g}) \oplus U^k(\mathfrak{g}). \tag{5.22}$$

In other words, there is a (canonical) projection

$$\begin{aligned} &\theta_k : U_k(\mathfrak{g}) \to U^k(\mathfrak{g}) \cong U_k(\mathfrak{g})/U_{k-1}(\mathfrak{g}) \\ &X_1 \cdots X_k \mapsto \sum_{\sigma \in S_k} X_{\sigma(1)} \cdots X_{\sigma(k)}. \end{aligned} \tag{5.23}$$

So, fixing an ordered basis of the Lie algebra $\mathfrak{g}$, we can say that the above map is bijective on the PBW-basis elements.

Let us consider the unique derivation of $U(\mathfrak{g})$ extending the adjoint action of the Lie algebra $\mathfrak{g}$ on itself, and call it $\mathrm{ad}(X) : U(\mathfrak{g}) \to U(\mathfrak{g})$ for any $X \in \mathfrak{g}$. By [7, Proposition 2.4.9], $\mathrm{ad}(X)(U^k(\mathfrak{g})) \subseteq U^k(\mathfrak{g})$ and $\mathrm{ad}(X)(U_k(\mathfrak{g})) \subseteq U_k(\mathfrak{g})$. So by applying $\mathrm{ad}(X)$ to both sides of (5.22), we observe that the preimage of $\mathrm{ad}(Y)(\sum_{\sigma \in S_k} X_{\sigma(1)} \cdots X_{\sigma(k)})$ is $\mathrm{ad}(Y)(X_1 \cdots X_k)$.

Proposition 5.7. *For a locally conilpotent $\mathfrak{g}$-comodule V, the linear map*

$$\begin{aligned} &\blacktriangledown : V \to U(\mathfrak{g}) \otimes V \\ &v \mapsto 1 \otimes v + \sum_{k \geq 1} \theta_k^{-1}(v_{[-k]} \cdots v_{[-1]}) \otimes v_{[0]} \end{aligned} \tag{5.24}$$

defines a $U(\mathfrak{g})$-comodule structure.

Proof. For an arbitrary basis element $v^i \in V$, let us write

$$v^i{}_{[-1]} \otimes v^i{}_{[0]} = \alpha_k^{ij} X_j \otimes v^k \tag{5.25}$$

where $\alpha_k^{ij} \in \mathbb{C}$. Then, by the coaction compatibility $v_{[-2]} \wedge v_{[-1]} \otimes v_{[0]} = 0$ we have

$$v^i{}_{[-2]} \otimes v^i{}_{[-1]} \otimes v^i{}_{[0]} = \sum_{j_1, j_2} \alpha_{l_2}^{ij_1j_2} X_{j_1} \otimes X_{j_2} \otimes v^{l_2}, \tag{5.26}$$

such that $\alpha_{l_2}^{ij_1j_2} := \alpha_{l_1}^{ij_1} \alpha_{l_2}^{l_1j_2}$ and $\alpha_{l_2}^{ij_1j_2} = \alpha_{l_2}^{ij_2j_1}$.

We have

$$\blacktriangledown(v^i) = 1 \otimes v^i + \sum_{k \geq 1} \sum_{j_1 \leq \cdots \leq j_k} \alpha_{l_k}^{ij_1 \cdots j_k} X_{j_1} \cdots X_{j_k} \otimes v^{l_k}, \tag{5.27}$$

because for $k \geq 1$

$$v^i{}_{[-k]} \otimes \cdots \otimes v^i{}_{[-1]} \otimes v^i{}_{[0]} = \sum_{j_1, \cdots, j_k} \alpha_{l_k}^{ij_1 \cdots j_k} X_{j_1} \otimes \cdots \otimes X_{j_k} \otimes v^{l_k}, \tag{5.28}$$

where $\alpha_{l_k}^{ij_1 \cdots j_k} := \alpha_{l_1}^{ij_1} \cdots \alpha_{l_k}^{l_{k-1}j_k}$, and for any $\sigma \in S_k$ we have

$$\alpha_{l_k}^{ij_1 \cdots j_k} = \alpha_{l_k}^{ij_{\sigma(1)} \cdots j_{\sigma(k)}}. \tag{5.29}$$

At this point, the counitality is immediate,

$$(\varepsilon \otimes id) \circ \blacktriangledown(v^i) = v^i. \tag{5.30}$$

On the other hand, to prove the coassociativity we first observe that

$$
\begin{aligned}
&(id\otimes \blacktriangledown)\circ\blacktriangledown(v^i)=1\otimes\blacktriangledown(v^i)+\sum_{k\geq 1}\sum_{j_1\leq\cdots\leq j_k}\alpha^{ij_1\cdots j_k}_{l_k}X_{j_1}\cdots X_{j_k}\otimes\blacktriangledown(v^{l_k})\\
&=1\otimes 1\otimes v^i+\sum_{k\geq 1}\sum_{j_1\leq\cdots\leq j_k}\alpha^{ij_1\cdots j_k}_{l_k}1\otimes X_{j_1}\cdots X_{j_k}\otimes v^{l_k}+\\
&\sum_{k\geq 1}\sum_{j_1\leq\cdots\leq j_k}\alpha^{ij_1\cdots j_k}_{l_k}X_{j_1}\cdots X_{j_k}\otimes 1\otimes v^{l_k}+\\
&\sum_{k\geq 1}\sum_{j_1\leq\cdots\leq j_k}\alpha^{ij_1\cdots j_k}_{l_k}X_{j_1}\cdots X_{j_k}\otimes(\sum_{t\geq 1}\sum_{r_1\leq\cdots\leq r_t}\alpha^{l_kr_1\cdots r_t}_{s_t}X_{r_1}\cdots X_{r_t}\otimes v^{s_t}),
\end{aligned}
\tag{5.31}
$$

where $\alpha^{l_kr_1\cdots r_t}_{s_t}:=\alpha^{l_kr_1}_{s_1}\cdots\alpha^{s_{t-1}r_t}_{s_t}$. Then we notice that

$$
\begin{aligned}
&\Delta(\sum_{k\geq 1}\sum_{j_1\leq\cdots\leq j_k}\alpha^{ij_1\cdots j_k}_{l_k}X_{j_1}\cdots X_{j_k})\otimes v^{l_k}\\
&=\sum_{k\geq 1}\sum_{j_1\leq\cdots\leq j_k}\alpha^{ij_1\cdots j_k}_{l_k}1\otimes X_{j_1}\cdots X_{j_k}\otimes v^{l_k}\\
&\quad+\sum_{k\geq 1}\sum_{j_1\leq\cdots\leq j_k}\alpha^{ij_1\cdots j_k}_{l_k}X_{j_1}\cdots X_{j_k}\otimes 1\otimes v^{l_k}\\
&\quad+\sum_{k\geq 2}\sum_{j_1\leq\cdots\leq r_1\leq\cdots\leq r_p\leq\cdots\leq j_k}\alpha^{ij_1\cdots j_k}_{l_k}X_{r_1}\cdots X_{r_p}\otimes X_{j_1}\cdots\widehat{X}_{r_1}\cdots\widehat{X}_{r_p}\cdots X_{j_k}\otimes v^{l_k}\\
&=\sum_{k\geq 1}\sum_{j_1\leq\cdots\leq j_k}\alpha^{ij_1\cdots j_k}_{l_k}1\otimes X_{j_1}\cdots X_{j_k}\otimes v^{l_k}\\
&\quad+\sum_{k\geq 1}\sum_{j_1\leq\cdots\leq j_k}\alpha^{ij_1\cdots j_k}_{l_k}X_{j_1}\cdots X_{j_k}\otimes 1\otimes v^{l_k}\\
&\quad+\sum_{p\geq 1}\sum_{k-p\geq 1}\\
&\qquad\sum_{q_1\leq\cdots\leq q_{k-p}}\sum_{r_1\leq\cdots\leq r_p}\alpha^{ir_1\cdots r_p}_{l_p}\alpha^{l_pq_1\cdots q_{k-p}}_{l_k}X_{r_1}\cdots X_{r_p}\otimes X_{q_1}\cdots X_{q_{k-p}}\otimes v^{l_k},
\end{aligned}
\tag{5.32}
$$

where for the last equality we write the complement of $r_1\leq\cdots\leq r_p$ in $j_1\leq\cdots\leq j_k$ as $q_1\leq\cdots\leq q_{k-p}$. Then (5.29) implies that

$$
\alpha^{ij_1\cdots j_k}_{l_k}=\alpha^{ir_1\cdots r_pq_1\cdots q_{k-p}}_{l_k}=\alpha^{ir_1\cdots r_p}_{l_p}\alpha^{l_pq_1\cdots q_{k-p}}_{l_k}. \tag{5.33}
$$

As a result,

$$
(id\otimes\blacktriangledown)\circ\blacktriangledown(v^i)=(\Delta\otimes id)\circ\blacktriangledown(v^i). \tag{5.34}
$$

This is the coassociativity and the proof is now complete. $\square$

Let us denote by ${}^{\mathfrak{g}}\mathrm{conil}\mathcal{M}$ the subcategory of locally conilpotent left $\mathfrak{g}$-comodules of the category of left $\mathfrak{g}$-comodules ${}^{\mathfrak{g}}\mathcal{M}$ with colinear maps.

Assigning a $\mathfrak{g}$-comodule $\blacktriangledown_{\mathfrak{g}} : V \to \mathfrak{g}\otimes V$ to a $U(\mathfrak{g})$-comodule $\blacktriangledown : V \to U(\mathfrak{g})\otimes V$ determines a functor

$$ {}^{U(\mathfrak{g})}\mathcal{M} \xrightarrow{P} {}^{\mathfrak{g}}\mathrm{conil}\mathcal{M} \tag{5.35} $$

Similarly, constructing a $U(\mathfrak{g})$-comodule from a $\mathfrak{g}$-comodule determines a functor

$$ {}^{\mathfrak{g}}\mathrm{conil}\mathcal{M} \xrightarrow{E} {}^{U(\mathfrak{g})}\mathcal{M} \tag{5.36} $$

As a result, we can express the following proposition.

Proposition 5.8. *The categories* ${}^{\mathrm{U}(\mathfrak{g})}\mathcal{M}$ *and* ${}^{\mathfrak{g}}\mathrm{conil}\mathcal{M}$ *are isomorphic.*

Proof. We show that the functors

$$ {}^{U(\mathfrak{g})}\mathcal{M} \underset{E}{\overset{P}{\rightleftarrows}} {}^{\mathfrak{g}}\mathrm{conil}\mathcal{M} $$

are inverses to each other.

If $\blacktriangledown_{\mathfrak{g}} : V \to \mathfrak{g} \otimes V$ is a locally conilpotent $\mathfrak{g}$-comodule and $\blacktriangledown : V \to U(\mathfrak{g}) \otimes V$ the corresponding $U(\mathfrak{g})$-comodule, by the very definition the $\mathfrak{g}$-comodule corresponding to $\blacktriangledown : V \to U(\mathfrak{g}) \otimes V$ is exactly $\blacktriangledown_{\mathfrak{g}} : V \to \mathfrak{g} \otimes V$. This proves that

$$ P \circ E \;=\; \mathrm{Id}_{{}^{\mathfrak{g}}\mathrm{conil}\mathcal{M}} \,. \tag{5.37} $$

Conversely, let us start with a $U(\mathfrak{g})$-comodule $\blacktriangledown : V \to U(\mathfrak{g}) \otimes V$ and write the coaction by using the PBW-basis of $U(\mathfrak{g})$ as follows

$$ {v^i}_{(-1)} \otimes {v^i}_{(0)} = 1 \otimes v^i + \sum_{k\geq 1} \sum_{j_1 \leq \cdots \leq j_k} \gamma_{l_k}^{ij_1\cdots j_k} X_{j_1} \cdots X_{j_k} \otimes v^{l_k}. \tag{5.38} $$

So, the corresponding $\mathfrak{g}$-comodule $\blacktriangledown_{\mathfrak{g}} : V \to \mathfrak{g} \otimes V$ is given as follows

$$ {v^i}_{[-1]} \otimes {v^i}_{[0]} = \pi({v^i}_{(-1)}) \otimes {v^i}_{(0)} = \sum_j \gamma_k^{ij} X_j \otimes v^k. \tag{5.39} $$

Finally, the $U(\mathfrak{g})$-coaction corresponding to this $\mathfrak{g}$-coaction is defined on $v^i \in V$ as

$$ v^i \mapsto 1 \otimes v + \sum_{k\geq 1} \sum_{j_1 \leq \cdots \leq j_k} \gamma_{l_1}^{ij_1} \gamma_{l_2}^{l_1 j_2} \cdots \gamma_{l_k}^{l_{k-1} j_k} X_{j_1} \cdots X_{j_k} \otimes v^{l_k} \tag{5.40} $$

Therefore, we can recover $U(\mathfrak{g})$-coaction we started with if and only if

$$ \gamma_{l_k}^{ij_1\cdots j_k} = \gamma_{l_1}^{ij_1} \gamma_{l_2}^{l_1 j_2} \cdots \gamma_{l_k}^{l_{k-1} j_k}, \quad \forall k \geq 1. \tag{5.41} $$

The equation (5.41) is a consequence of the coassociativity $\blacktriangledown$. Indeed, applying the coassociativity as

$$(\Delta^{k-1} \otimes id) \circ \blacktriangledown = \blacktriangledown^k \tag{5.42}$$

and comparing the coefficients of $X_{j_1} \otimes \cdots \otimes X_{j_k}$ we conclude (5.41) for any $k \geq 1$. Hence, we proved

$$E \circ P = \mathrm{Id}_{U(\mathfrak{g})\mathcal{M}}\,. \tag{5.43}$$

The equation (5.41) implies that if $\blacktriangledown : V \to U(\mathfrak{g}) \otimes V$ is a left coaction, then its associated $\mathfrak{g}$-coaction $\blacktriangledown_{\mathfrak{g}} : V \to \mathfrak{g} \otimes V$ is locally conilpotent. □

For a $\mathfrak{g}$-coaction

$$v \mapsto v_{[-1]} \otimes v_{[0]}\,, \tag{5.44}$$

the associated $U(\mathfrak{g})$-coaction is denoted by

$$v \mapsto v_{\overline{[-1]}} \otimes v_{\overline{[0]}}\,. \tag{5.45}$$

Definition 5.9. *Let V be a right module and left comodule over a Lie algebra $\mathfrak{g}$. We call V a right-left AYD over $\mathfrak{g}$ if*

$$\blacktriangledown_{\mathfrak{g}}(v \cdot X) = v_{[-1]} \otimes v_{[0]} \cdot X + [v_{[-1]}, X] \otimes v_{[0]}\,. \tag{5.46}$$

Moreover, V is called stable if

$$v_{[0]} \cdot v_{[-1]} = 0. \tag{5.47}$$

Proposition 5.10. *Let $\blacktriangledown_{\mathfrak{g}} : V \to \mathfrak{g} \otimes V$ be a locally conilpotent $\mathfrak{g}$-comodule and $\blacktriangledown : V \to U(\mathfrak{g}) \otimes V$ the corresponding $U(\mathfrak{g})$-comodule structure. Then, V is a right-left AYD over $\mathfrak{g}$ if and only if it is a right-left AYD over $U(\mathfrak{g})$.*

Proof. Let us first assume V to be a right-left AYD module over $\mathfrak{g}$. For $X \in \mathfrak{g}$ and an element $v \in V$, AYD compatibility implies that

$$\begin{aligned}&(v \cdot X)_{[-k]} \otimes \cdots \otimes (v \cdot X)_{[-1]} \otimes (v \cdot X)_{[0]} = v_{[-k]} \otimes \cdots \otimes v_{[-1]} \otimes v_{[0]} \cdot X \\ &\quad + [v_{[-k]}, X] \otimes \cdots \otimes v_{[-1]} \otimes v_{[0]} + v_{[-k]} \otimes \cdots \otimes [v_{[-1]}, X] \otimes v_{[0]}\,.\end{aligned} \tag{5.48}$$

Multiplying in $U(\mathfrak{g})$, we get

$$\begin{aligned}&(v \cdot X)_{[-k]} \cdots (v \cdot X)_{[-1]} \otimes (v \cdot X)_{[0]} = \\ &v_{[-k]} \cdots v_{[-1]} \otimes v_{[0]} \cdot X - ad(X)(v_{[-k]} \cdots v_{[-1]}) \otimes v_{[0]}\,.\end{aligned} \tag{5.49}$$

So, for the extension $\blacktriangledown : V \to U(\mathfrak{g}) \otimes V$ we have

$$\begin{aligned}&(v \cdot X)_{\overline{[-1]}} \otimes (v \cdot X)_{\overline{[0]}} = 1 \otimes v \cdot X + \sum_{k \geq 1} \theta_k^{-1}((v \cdot X)_{[-k]} \cdots (v \cdot X)_{[-1]}) \otimes (v \cdot X)_{[0]} \\ &= 1 \otimes v \cdot X + \sum_{k \geq 1} \theta_k^{-1}(v_{[-k]} \cdots v_{[-1]}) \otimes v_{[0]} \cdot X - \sum_{k \geq 1} \theta_k^{-1}(ad(X)(v_{[-k]} \cdots v_{[-1]})) \otimes v_{[0]} \\ &= v_{\overline{[-1]}} \otimes v_{\overline{[0]}} \cdot X - \sum_{k \geq 1} ad(X)(\theta_k^{-1}(v_{[-k]} \cdots v_{[-1]})) \otimes v_{[0]} \\ &= v_{\overline{[-1]}} \otimes v_{\overline{[0]}} \cdot X - ad(X)(v_{\overline{[-1]}}) \otimes v_{\overline{[0]}} = S(X_{(3)})v_{\overline{[-1]}}X_{(1)} \otimes v_{\overline{[0]}} \cdot X_{(2)}\,.\end{aligned} \tag{5.50}$$

Here on the third equality we used the fact that the operator ad commute with θ_k, and on the fourth equality we used

$$\sum_{k\geq 1} ad(X)(\theta_k^{-1}(v_{[-k]}\cdots v_{[-1]}))\otimes v_{[0]} =$$
$$\sum_{k\geq 1} ad(X)(\theta_k^{-1}(v_{[-k]}\cdots v_{[-1]}))\otimes v_{[0]} + ad(X)(1)\otimes v = ad(X)(v_{\overline{[-1]}})\otimes v_{\overline{[0]}}. \tag{5.51}$$

By using the fact that AYD condition is multiplicative, we conclude that $\blacktriangledown : M \to U(\mathfrak{g})\otimes M$ satisfies the AYD condition on $U(\mathfrak{g})$.

Conversely assume that V is a right-left AYD over $U(\mathfrak{g})$. We first observe that

$$(\Delta\otimes id)\circ\Delta(X) = X\otimes 1\otimes 1 + 1\otimes X\otimes 1 + 1\otimes 1\otimes X \tag{5.52}$$

Accordingly,

$$\begin{aligned}\blacktriangledown(v\cdot X) &= v_{\overline{[-1]}}X\otimes v_{\overline{[0]}} + v_{\overline{[-1]}}\otimes v_{\overline{[0]}}\cdot X - Xv_{\overline{[-1]}}\otimes v_{\overline{[0]}}\\ &= -ad(X)(v_{\overline{[-1]}})\otimes v_{\overline{[0]}} + v_{\overline{[-1]}}\otimes v_{\overline{[0]}}\cdot X\end{aligned} \tag{5.53}$$

It is known that the projection map $\pi : U(\mathfrak{g})\to\mathfrak{g}$ commutes with the adjoint representation. So

$$\begin{aligned}\blacktriangledown_{\mathfrak{g}}(v\cdot X) &= -\pi(ad(X)(v_{\overline{[-1]}}))\otimes v_{\overline{[0]}} + \pi(v_{\overline{[-1]}})\otimes v_{\overline{[0]}}\cdot X\\ &= -ad(X)\pi(v_{\overline{[-1]}})\otimes v_{\overline{[0]}} + \pi(v_{\overline{[-1]}})\otimes v_{\overline{[0]}}\cdot X\\ &= [v_{[-1]}, X]\otimes v_{[0]} + v_{[-1]}\otimes v_{[0]}\cdot X.\end{aligned} \tag{5.54}$$

That is, V is a right-left AYD over $\mathfrak{g}$. ☐

Lemma 5.11. *Let $\blacktriangledown_{\mathfrak{g}} : V\to\mathfrak{g}\otimes V$ be a locally conilpotent $\mathfrak{g}$-comodule and $\blacktriangledown : V\to U(\mathfrak{g})\otimes V$ be the corresponding $U(\mathfrak{g})$-comodule structure. If V is stable over $\mathfrak{g}$, then it is stable over $U(\mathfrak{g})$.*

Proof. Writing the $\mathfrak{g}$-coaction in terms of basis elements as in (5.25), the stability reads

$$v^i{}_{[0]}v^i{}_{[-1]} = \alpha_k^{ij}v^k\cdot X_j = 0,\quad \forall i \tag{5.55}$$

Therefore, for the corresponding $U(\mathfrak{g})$-coaction we have

$$\begin{aligned}&\sum_{j_1\leq\cdots\leq j_k}\alpha_{l_1}^{ij_1}\cdots\alpha_{l_k}^{l_{k-1}j_k}v^{l_k}\cdot(X_{j_1}\cdots X_{j_k}) =\\ &\sum_{j_1\leq\cdots\leq j_{k-1}}\alpha_{l_1}^{ij_1}\cdots\alpha_{l_{k-1}}^{l_{k-2}j_{k-1}}(\sum_{j_k}\alpha_{l_k}^{l_{k-1}j_k}v^{l_k}\cdot X_{j_1})\cdot(X_{j_2}\cdots X_{j_k}) =\\ &\sum_{j_2,\cdots,j_k}\alpha_{l_1}^{ij_k}\cdots\alpha_{l_{k-1}}^{l_{k-2}j_{k-1}}(\sum_{j_1}\alpha_{l_k}^{l_{k-1}j_1}v^{l_k}\cdot X_{j_1})\cdot(X_{j_2}\cdots X_{j_k}),\end{aligned} \tag{5.56}$$

where on the second equality we used (5.29). This immediately implies that

$$v^i_{\underline{[0]}} \cdot v^i_{\underline{[-1]}} = v^i. \tag{5.57}$$

That is, the stability over $U(\mathfrak{g})$. $\square$

However, the converse is not true.

Example 5.12. It is known that $U(\mathfrak{g})$, as a left $U(\mathfrak{g})$-comodule via $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ and a right $\mathfrak{g}$-module via $ad : U(\mathfrak{g}) \otimes \mathfrak{g} \to U(\mathfrak{g})$ is stable. However, the associated $\mathfrak{g}$-comodule, is no longer stable. Indeed, for $u = X_1X_2X_3 \in U(\mathfrak{g})$, we have

$$u_{[-1]} \otimes u_{[0]} = X_1 \otimes X_2X_3 + X_2 \otimes X_1X_3 + X_3 \otimes X_1X_2 \tag{5.58}$$

Then,

$$u_{[0]} \cdot u_{[-1]} = [[X_1, X_2], X_3] + [[X_2, X_1], X_3] + [[X_1, X_3], X_2] = [[X_1, X_3], X_2] \tag{5.59}$$

which is not necessarily zero.

The following is the main result of this section.

Proposition 5.13. *Let V be a vector space, and $\mathfrak{g}$ be a Lie algebra. Then, V is a stable right $\widetilde{\mathfrak{g}}$-module if and only if it is a right-left SAYD module over $\mathfrak{g}$.*

Proof. Let us first assume that V is a stable right $\widetilde{\mathfrak{g}}$-module. Since V is a right $S(\mathfrak{g}^*)$-module it is a left $\mathfrak{g}$-comodule by Proposition 5.2. Accordingly

$$\begin{aligned}
&[v_{[-1]}, X_j] \otimes v_{[0]} + v_{[-1]} \otimes v_{[0]} \cdot X_j = \\
&[X_l, X_j]\theta^l(v_{[-1]}) \otimes v_{[0]} + X_t\theta^t(v_{[-1]}) \otimes v_{[0]} \cdot X_j = \\
&X_tC^t_{lj}\theta^l(v_{[-1]}) \otimes v_{[0]} + X_t\theta^t(v_{[-1]}) \otimes v_{[0]} \cdot X_j = \\
&X_t \otimes [v \lhd (X_j \rhd \theta^t) + (v \lhd \theta^t) \cdot X_j] = \\
&X_t \otimes (v \cdot X_j) \lhd \theta^t = X_t\theta^t((v \cdot X_j)_{[-1]}) \otimes (v \cdot X_j)_{[0]} = \\
&(v \cdot X_j)_{[-1]} \otimes (v \cdot X_j)_{[0]}
\end{aligned} \tag{5.60}$$

This proves that V is a right-left AYD module over $\mathfrak{g}$. On the other hand, for any $v \in V$,

$$v_{[0]} \cdot v_{[-1]} = \sum_i v_{[0]} \cdot X_i\theta^i(v_{[-1]}) = \sum_i (v \lhd \theta^i) \cdot X_i = 0 \tag{5.61}$$

Hence, V is stable too. As a result, V is SAYD over $\mathfrak{g}$.

Conversely, assume that V is a right-left SAYD module over $\mathfrak{g}$. So V is a right module over $S(\mathfrak{g}^*)$ and a right module over $\mathfrak{g}$. In addition we see that

$$\begin{aligned}
&v \lhd (X_j \rhd \theta^i) + (v \lhd \theta^i) \cdot X_j = C^i_{kj} v \lhd \theta^k + (v \lhd \theta^i) \cdot X_j = \\
&C^i_{kj}\theta^k(v_{[-1]})v_{[0]} + \theta^i(v_{[-1]})v_{[0]} \cdot X_j = \\
&\theta^i([v_{[-1]}, X_j])v_{[0]} + \theta^i(v_{[-1]})v_{[0]} \cdot X_j = \\
&(\theta^i \otimes id)([v_{[-1]}, X_j] \otimes v_{[0]} + v_{[-1]} \otimes v_{[0]} \cdot X_j) = \\
&\theta^t((v \cdot X_j)_{[-1]})(v \cdot X_j)_{[0]} = (v \cdot X_j) \lhd \theta^i.
\end{aligned} \tag{5.62}$$

Thus, V is a right $\widetilde{\mathfrak{g}}$-module. Finally, we prove the stability by

$$\sum_i (v \lhd \theta^i) \cdot X_i = \sum_i v_{[0]} \cdot X_i \theta^i(v_{[-1]}) = v_{[0]} \cdot v_{[-1]} = 0. \tag{5.63}$$

$\square$

Corollary 5.14. *Any right module over the Weyl algebra $D(\mathfrak{g})$ is a right-left SAYD module over the Lie algebra $\mathfrak{g}$.*

Finally, we state an analogous of Lemma 2.3 [10] to show that the category of ${}^{\mathfrak{g}}\mathcal{AYD}_{\mathfrak{g}}$ is monoidal.

Proposition 5.15. *Let M and N be two right-left AYD modules over $\mathfrak{g}$. Then $M \otimes N$ is also a right-left AYD over $\mathfrak{g}$ via the coaction*

$$\blacktriangledown_{\mathfrak{g}} : M \otimes N \to \mathfrak{g} \otimes M \otimes N, \quad m \otimes n \mapsto m_{[-1]} \otimes m_{[0]} \otimes n + n_{[-1]} \otimes m \otimes n_{[0]} \tag{5.64}$$

and the action

$$M \otimes N \otimes \mathfrak{g} \to M \otimes N, \quad (m \otimes n) \cdot X = m \cdot X \otimes n + m \otimes n \cdot X \tag{5.65}$$

Proof. We verify that

$$\begin{aligned}
&[(m \otimes n)_{[-1]}, X] \otimes (m \otimes n)_{[0]} + (m \otimes n)_{[-1]} \otimes (m \otimes n)_{[0]} \cdot X = \\
&[m_{[-1]}, X] \otimes m_{[0]} \otimes n + [n_{[-1]}, X] \otimes m \otimes n_{[0]} + \\
&m_{[-1]} \otimes (m_{[0]} \otimes n) \cdot X + n_{[-1]} \otimes (m \otimes n_{[0]}) \cdot X = \\
&(m \cdot X)_{[-1]} \otimes (m \cdot X)_{[0]} \otimes n + n_{[-1]} \otimes m \cdot X \otimes n_{[0]} + \\
&m_{[-1]} \otimes m_{[0]} \otimes n \cdot X + (n \cdot X)_{[-1]} \otimes m \otimes (n \cdot X)_{[0]} = \\
&\blacktriangledown_{\mathfrak{g}}(m \cdot X \otimes n + m \otimes n \cdot X) = \blacktriangledown_{\mathfrak{g}}((m \otimes n) \cdot X).
\end{aligned} \tag{5.66}$$

$\square$

5.3 Examples

This subsection is devoted to examples to illustrate the notion of SAYD module over a Lie algebra. We consider the representations and corepresentations of a

Lie algebra $\mathfrak{g}$ on a finite dimensional vector space V in terms of matrices. We then investigate the SAYD condition as a relation between these matrices and the Lie algebra structure of $\mathfrak{g}$.

Let also V be a n dimensional $\mathfrak{g}$-module with a basis $\{v^1, \cdots, v^n\}$. We express the module structure as

$$m^i \cdot X_j = \beta^i_{jk} m^k, \quad \beta^i_{jk} \in \mathbb{C}. \tag{5.67}$$

In this way, for any basis element $X_j \in \mathfrak{g}$ we obtain a matrix $B_j \in M_n(\mathbb{C})$ such that

$$(B_j)^i_k := \beta^i_{jk}. \tag{5.68}$$

Let $\blacktriangledown_{\mathfrak{g}} : V \to \mathfrak{g} \otimes V$ be a coaction. We write the coaction as

$$\blacktriangledown_{\mathfrak{g}}(v^i) = \alpha^{ij}_k X_j \otimes v^k, \quad \alpha^{ij}_k \in \mathbb{C}. \tag{5.69}$$

This way we get a matrix $A^j \in M_n(\mathbb{C})$ for any basis element $X_j \in \mathfrak{g}$ such that

$$(A^j)^i_k := \alpha^{ij}_k. \tag{5.70}$$

Lemma 5.16. *Linear map $\blacktriangledown_{\mathfrak{g}} : M \to \mathfrak{g} \otimes M$ forms a right $\mathfrak{g}$-comodule if and only if*

$$A^{j_1} \cdot A^{j_2} = A^{j_2} \cdot A^{j_1}. \tag{5.71}$$

Proof. It is just the translation of the coaction compatibility $v^i{}_{[-2]} \wedge v^i{}_{[-1]} \otimes v^i{}_{[0]} = 0$ in terms of the matrices A^i. □

Lemma 5.17. *Right $\mathfrak{g}$-module left $\mathfrak{g}$-comodule V is stable if and only if*

$$\sum_j A^j \cdot B_j = 0. \tag{5.72}$$

Proof. By the definition of the stability,

$$v^i{}_{[0]} \cdot v^i{}_{[-1]} = \alpha^{ij}_k v^k \cdot X_j = \alpha^{ij}_k \beta^k_{jl} v^l = 0 \tag{5.73}$$

Therefore,

$$\alpha^{ij}_k \beta^k_{jl} = (A^j)^i_k (B_j)^k_l = (A^j \cdot B_j)^i_l = 0. \tag{5.74}$$

□

We proceed to express the AYD condition.

Lemma 5.18. *The $\mathfrak{g}$-module-comodule V is a right-left AYD if and only if*

$$[B_q, A^j] = \sum_s A^s C^j_{sq}. \tag{5.75}$$

Proof. We first observe

$$\begin{aligned}\blacktriangledown_{\mathfrak{g}}(v^p\cdot X_q)&=\blacktriangledown_{\mathfrak{g}}(\beta^p_{qk}v^k)=\beta^p_{qk}\alpha^{kj}_l X_j\otimes v^l\\&=(B_q)^p_k(A^j)^k_l X_j\otimes v^l=(B_q\cdot A^j)^p_l X_j\otimes v^l.\end{aligned}\tag{5.76}$$

On the other hand, writing $\blacktriangledown_{\mathfrak{g}}(v^p)=\alpha^{pj}_l X_j\otimes v^l$,

$$\begin{aligned}&[v^p{}_{[-1]},X_q]\otimes v^p{}_{[0]}+v^p{}_{[-1]}\otimes v^p{}_{[0]}\cdot X_q=\alpha^{ps}_l[X_s,X_q]\otimes v^l+\alpha^{pj}_t X_j\otimes v^t\cdot X_q\\&=\alpha^{ps}_l C^j_{sq}X_j\otimes v^l+\alpha^{pj}_t\beta^t_{ql}X_j\otimes v^l=(\alpha^{ps}_l C^j_{sq}+(A^j\cdot B_q)^p_l)X_j\otimes v^l.\end{aligned}\tag{5.77}$$

□

Next, considering the Lie algebra $\mathfrak{sl}(2)$, we determine the SAYD modules over simple $\mathfrak{sl}(2)$-modules.

Example 5.19. Let $V=<\{v^1,v^2\}>$ be a two dimensional simple $\mathfrak{sl}(2)$-module. Then, by [12], the representation

$$\rho:\mathfrak{sl}(2)\to\mathfrak{gl}(V)\tag{5.78}$$

is the inclusion $\rho:\mathfrak{sl}(2)\hookrightarrow\mathfrak{gl}(2)$. Therefore, we have

$$B_1=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad B_2=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad B_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.\tag{5.79}$$

We want to find

$$A^1=\begin{pmatrix}x^1_1&x^1_2\\x^2_1&x^2_2\end{pmatrix},\qquad A^2=\begin{pmatrix}y^1_1&y^1_2\\y^2_1&y^2_2\end{pmatrix},\qquad A^3=\begin{pmatrix}z^1_1&z^1_2\\z^2_1&z^2_2\end{pmatrix},\tag{5.80}$$

such that together with the $\mathfrak{g}$-coaction $\blacktriangledown_{\mathfrak{sl}(2)}:V\to\mathfrak{sl}(2)\otimes V$, defined as $v^i\mapsto(A^j)^i_kX_j\otimes v^k$, V becomes a right-left SAYD over $\mathfrak{sl}(2)$. We first express the stability condition. To this end,

$$A^1\cdot B_1=\begin{pmatrix}x^1_2&0\\x^2_2&0\end{pmatrix},\qquad A^2\cdot B_2=\begin{pmatrix}0&y^1_1\\0&y^2_1\end{pmatrix},\qquad A^3\cdot B_3=\begin{pmatrix}z^1_1&-z^1_2\\z^2_1&-z^2_2\end{pmatrix},\tag{5.81}$$

and hence, the stability is

$$\sum_j A^j\cdot B_j=\begin{pmatrix}x^1_2+z^1_1&y^1_1-z^1_2\\x^2_2+z^2_1&y^2_1-z^2_2\end{pmatrix}=0.\tag{5.82}$$

Next, we consider the AYD condition

$$[B_q,A^j]=\sum_s A^sC^j_{sq}.\tag{5.83}$$

For $j = 1 = q$,

$$A^1 = \begin{pmatrix} x_1^1 & 0 \\ x_1^2 & x_2^2 \end{pmatrix}, \qquad A^2 = \begin{pmatrix} 0 & y_2^1 \\ 0 & y_2^2 \end{pmatrix}, \qquad A^3 = \begin{pmatrix} 0 & 0 \\ z_1^2 & 0 \end{pmatrix}. \tag{5.84}$$

Similarly, for $q = 2$ and $j = 1$, we arrive

$$A^1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A^2 = \begin{pmatrix} 0 & y_2^1 \\ 0 & y_2^2 \end{pmatrix}, \qquad A^3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.85}$$

Finally, for $j = 1$ and $q = 2$ we conclude

$$A^1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A^3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.86}$$

Thus, the only $\mathfrak{sl}(2)$-comodule structure that makes a 2-dimensional simple $\mathfrak{sl}(2)$-module V to be a right-left SAYD over $\mathfrak{sl}(2)$ is the trivial comodule structure.

Example 5.20. We investigate all possible coactions that make the truncated symmetric algebra $S(\mathfrak{sl}(2)^*)_{[2]}$ an SAYD module over $\mathfrak{sl}(2)$.

A vector space basis of $S(\mathfrak{sl}(2)^*)_{[2]}$ is $\{1 = \theta^0, \theta^1, \theta^2, \theta^3\}$ and the Koszul coaction is

$$\begin{aligned} & S(\mathfrak{sl}(2)^*)_{[2]} \to \mathfrak{sl}(2) \otimes S(\mathfrak{sl}(2)^*)_{[2]} \\ & \theta^0 \mapsto X_1 \otimes \theta^1 + X_2 \otimes \theta^2 + X_3 \otimes \theta^3 \\ & \theta^i \mapsto 0, \qquad i = 1, 2, 3 \end{aligned} \tag{5.87}$$

We first determine the right $\mathfrak{sl}(2)$ action to find the matrices B_1, B_2, B_3. We have

$$\theta^i \lhd X_j(X_q) = \theta^i \cdot X_j(X_q) = \theta^i([X_j, X_q]). \tag{5.88}$$

Therefore,

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 \end{pmatrix}, B_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.89}$$

Let $A^1 = (x_k^i), A^2 = (y_k^i), A^3 = (z_k^i)$ represent the $\mathfrak{g}$-coaction on V. According to the above expression of B_1, B_2, B_3, the stability is

$$\sum_j A^j \cdot B_j = \begin{pmatrix} 0 & y_3^0 + 2z_1^0 & x_3^0 - 2z_2^0 & -2x_1^0 + 2y_2^0 \\ 0 & y_3^1 + 2z_1^1 & x_3^1 - 2z_2^1 & -2x_1^1 + 2y_2^1 \\ 0 & y_3^2 + 2z_1^2 & x_3^2 - 2z_2^2 & -2x_1^2 + 2y_2^2 \\ 0 & y_3^3 + 2z_1^3 & x_3^3 - 2z_2^3 & -2x_1^3 + 2y_2^3 \end{pmatrix} = 0. \tag{5.90}$$

As before, we make the following observations. First,

$$[B_1, A^1] = \begin{pmatrix} 0 & 0 & -x^0_3 & 2x^0_1 \\ -2x^3_0 & -2x^3_1 & -2x^3_2 - x^1_3 & -2x^3_3 + 2x^1_1 \\ 0 & 0 & -x^2_3 & 2x^2_1 \\ x^2_0 & x^2_1 & x^2_2 - x^3_3 & x^2_3 - 2x^3_1 \end{pmatrix} = 2A^3 \tag{5.91}$$

and next

$$[B_2, A^1] = \begin{pmatrix} 0 & x^0_3 & 0 & -2x^0_2 \\ 0 & x^1_3 & 0 & -2x^1_2 \\ 2x^3_0 & 2x^3_1 + x^2_3 & 2x^3_2 & 2x^3_3 - 2x^2_2 \\ -x^1_0 & -x^1_1 + x^3_3 & -x^1_2 & -x^1_3 - 2x^3_2 \end{pmatrix} = 0 \tag{5.92}$$

Finally,

$$[B_3, A^1] = \begin{pmatrix} 0 & -2x^0_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2x^2_0 & -4x^2_1 & 0 & -2x^2_3 \\ 0 & -2x^3_1 & 0 & 0 \end{pmatrix} = -2A^1. \tag{5.93}$$

Hence, together with the stability one gets

$$A^1 = \begin{pmatrix} 0 & x^0_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ x^2_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.94}$$

and

$$[B_1, A^1] = \begin{pmatrix} 0 & 0 & 0 & 2x^0_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ x^2_0 & 0 & 0 & 0 \end{pmatrix} = 2A^3. \tag{5.95}$$

Similarly one computes

$$[B_1, A^2] = \begin{pmatrix} 0 & 0 & 0 & 2y^0_1 \\ -2y^3_0 & -2y^3_1 & 0 & 2y^1_1 \\ 0 & 0 & 0 & 2y^2_1 \\ y^2_0 & y^2_1 & 0 & 2y^3_1 \end{pmatrix} = 0, \tag{5.96}$$

as well as

$$[B_2, A^2] = \begin{pmatrix} 0 & 0 & 0 & -2y^0_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -y^1_0 & & 0 & 0 \end{pmatrix} = -2A^3, \tag{5.97}$$

and $[B_3, A^2] = 2A^2$. We conclude that

$$A^1 = \begin{pmatrix} 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 \\ d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; A^2 = \begin{pmatrix} 0 & 0 & c & 0 \\ d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; A^3 = \begin{pmatrix} 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2}d & 0 & 0 & 0 \end{pmatrix}. \tag{5.98}$$

One notes that $c=1, d=0$ recovers the Koszul coaction, but obviously it is not the only choice.

6 Cyclic cohomology of Lie algebras

In this section we show that for V, a SAYD module over a Lie algebra $\mathfrak{g}$, the (periodic) cyclic cohomology of $\mathfrak{g}$ with coefficients in V and the (periodic) cyclic cohomology of the enveloping Hopf algebra $U(\mathfrak{g})$ with coefficient in the corresponding SAYD over $U(\mathfrak{g})$ are isomorphic.

As a result of Proposition 4.2 and Proposition 5.13, we have the following definition.

Definition 6.1. *Let $\mathfrak{g}$ be a Lie algebra and V be a right-left SAYD module over $\mathfrak{g}$. We call the cohomology of the total complex of $(C_\bullet(\mathfrak{g},V), \partial_{\rm CE}+b_{\rm K})$ the cyclic cohomology of the Lie algebra $\mathfrak{g}$ with coefficients in the SAYD module V, and denote it by $HC^\bullet(\mathfrak{g},V)$. Similarly we denote its periodic cyclic cohomology by $HP^\bullet(\mathfrak{g},V)$.*

Our main result in this section is an analogous of Proposition 7 of [4].

Theorem 6.2. *Let $\mathfrak{g}$ be a Lie algebra and V be a SAYD module over the Lie algebra $\mathfrak{g}$. Then the periodic cyclic homology of $\mathfrak{g}$ with coefficients in V is the same as the periodic cyclic cohomology of $U(\mathfrak{g})$ with coefficients in the corresponding SAYD module V over $U(\mathfrak{g})$. In short,*

$$HP^\bullet(\mathfrak{g},V) \cong HP^\bullet(U(\mathfrak{g}),V). \tag{6.1}$$

Proof. The total coboundary of $C(\mathfrak{g},V)$ is $\partial_{\rm CE}+\partial_{\rm K}$ while the total coboundary of the complex $C(U(\mathfrak{g}),V)$ computing the cyclic cohomology of $U(\mathfrak{g})$ is $B+b$.

Next, we compare the E_1 terms of the spectral sequences of the total complexes corresponding to the filtration on the complexes which is induced by the filtration on V via [11, Lemma 6.2]. To this end, we first show that the coboundaries respect this filtration.

As it is indicated in the proof of [11, Lemma 6.2], each F_pV is a submodule of V. Thus, the Lie algebra homology boundary $\partial_{\rm CE}$ respects the filtration. As for $\partial_{\rm K}$, we notice for $v\in F_pV$

$$\partial_{\rm K}(X_1\wedge\cdots\wedge X_n\otimes v) = v_{[-1]}\wedge X_1\wedge\cdots\wedge X_n\otimes v_{[0]} \tag{6.2}$$

Since

$$\blacktriangledown(v) = v_{\overline{[-1]}}\otimes v_{\overline{[0]}} = 1\otimes v + v_{[-1]}\otimes v_{[0]} + \sum_{k\geq 2}\theta_k^{-1}(v_{[-k]}\cdots v_{[-1]})\otimes v_{[0]} \tag{6.3}$$

we observe that $v_{[-1]}\wedge X_1\wedge\cdots\wedge X_n\otimes v_{[0]} \in \wedge^{n+1}\mathfrak{g}\otimes F_{p-1}V$. Since $F_{p-1}V\subseteq F_pV$, we conclude that $\partial_{\rm K}$ respects the filtration.

Since the Hochschild coboundary $b : V \otimes U(\mathfrak{g})^{\otimes n} \to V \otimes U(\mathfrak{g})^{\otimes n+1}$ is the alternating sum of cofaces δ_i, it suffices to check each δ_i preserve the filtration, which is obvious for all cofaces except possibly the last one. However, for the last coface, we take $v \in F_pV$ and write

$$v_{\overline{[-1]}} \otimes v_{\overline{[0]}} = 1 \otimes v + v_{\langle -1\rangle} \otimes v_{\langle 0\rangle}, \qquad v_{\langle -1\rangle} \otimes v_{\langle 0\rangle} \in \mathfrak{g} \otimes F_{p-1}V. \tag{6.4}$$

We have

$$\delta_n(v \otimes u^1 \otimes \cdots \otimes u^n) = v_{\overline{[0]}} \otimes u^1 \otimes \cdots \otimes u^n \otimes v_{\overline{[-1]}} \in F_pV \otimes U(\mathfrak{g})^{\otimes\, n+1}. \tag{6.5}$$

Hence, we can say that b respects the filtration.

For the cyclic operator, the result again follows from the fact that F_p is a $\mathfrak{g}$-module. Indeed, for $v \in F_pV$

$$\tau_n(v\otimes u^1\otimes\cdots\otimes u^n) = v_{\overline{[0]}} \cdot u^1{}_{(1)} \otimes S(u^1{}_{(2)})\cdot(u^2\otimes\cdots\otimes u^n\otimes v_{\overline{[-1]}}) \in F_pV\otimes U(\mathfrak{g})^{\otimes\, n} \tag{6.6}$$

Finally we consider the extra degeneracy operator

$$\sigma_{-1}(v\otimes u^1\otimes\cdots\otimes u^n) = v\cdot u^1{}_{(1)}\otimes S(u^1{}_{(2)})\cdot(u^2\otimes\cdots\otimes u^n) \in F_pV\otimes U(\mathfrak{g})^{\otimes\, n} \tag{6.7}$$

which preserves the filtration again by using the fact that F_p is $\mathfrak{g}$-module and the coaction preserve the filtration. As a result now, we can say that the Connes' boundary B respects the filtration.

Now, the E_1-term of the spectral sequence associated to the filtration $(F_pV)_{p\geq 0}$ computing the periodic cyclic cohomology of the Lie algebra $\mathfrak{g}$ is known to be of the form

$$E_1^{j,\,i}(\mathfrak{g}) = H^{i+j}(F_{j+1}C(\mathfrak{g},V)/F_jC(\mathfrak{g},V), [\partial_{\rm CE} + \partial_{\rm K}]) \tag{6.8}$$

where, $[\partial_{\rm CE}+\partial_{\rm K}]$ is the induced coboundary operator on the quotient complex. By the obvious identification

$$F_{j+1}C(\mathfrak{g},V)/F_jC(\mathfrak{g},V) \cong C(\mathfrak{g}, F_{j+1}V/F_jV) = C(\mathfrak{g},(V/F_jV)^{co\mathfrak{g}}), \tag{6.9}$$

we observe that

$$E_1^{j,\,i}(\mathfrak{g}) = H^{i+j}(C(\mathfrak{g},(V/F_jV)^{coU(\mathfrak{g})}), [\partial_{\rm CE}]), \tag{6.10}$$

for $\partial_{\rm K}(F_{j+1}C(\mathfrak{g},V)) \subseteq F_jC(\mathfrak{g},V)$.

Similarly,

$$E_1^{j,\,i}(U(\mathfrak{g})) = H^{i+j}(C(U(\mathfrak{g}),(V/F_jV)^{coU(\mathfrak{g})}), [b+B]). \tag{6.11}$$

Finally, considering

$$E_1^{j,\,i}(\mathfrak{g}) = H^{i+j}(C(\mathfrak{g},(V/F_jV)^{co\mathfrak{g}}), [0] + [\partial_{\rm CE}]) \tag{6.12}$$

i.e., as a bicomplex with degree $+1$ differential is zero, the anti-symmetrization map $\alpha : C(\mathfrak{g},(V/F_jV)^{co\mathfrak{g}}) \to C(U(\mathfrak{g}),(V/F_jV)^{coU(\mathfrak{g})})$ induces a quasi-isomorphism $[\alpha] : E_1^{j,\,i}(\mathfrak{g}) \to E_1^{j,\,i}(U(\mathfrak{g}))$, $\forall i,j$ by Proposition 7 in [4]. ☐

Remark 6.3. *In case the $\mathfrak{g}$-module V has a trivial $\mathfrak{g}$-comodule structure, the coboundary $\partial_{\mathrm{K}} = 0$ and*

$$HP^{\bullet}(\mathfrak{g}, V) = \bigoplus_{n=\bullet \bmod 2} H_n(\mathfrak{g}, V). \tag{6.13}$$

In this case, the above theorem becomes [4, Proposition 7].

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