On the Ground State Energy of the Translation Invariant Pauli-Fierz Model. II.

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Abstract.

We determine the ground state energy of the translation invariant Pauli-Fierz model for an electron with spin, to subleading order $\mathcal{O}(\alpha^2)$ with respect to powers of the finestructure constant α and prove rigorous error bounds of order $\mathcal{O}(\alpha^3)$. A main objective of our argument is its brevity.

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1. Introduction

We continue the study of the translation invariant Pauli-Fierz model [2], describing a nonrelativistic free electron interacting with the quantized electromagnetic field. In contrast with [2], we study now electron with spin. We are interested in quantitative properties of the ground state energy (Theorem 2.1) and its associated eigenfunctions (Theorem 2.2). In particular, we determine the subleading terms of the ground state energy up to order α^2, where α denotes the finestructure constant, and rigorously bound the error by a term of order α^3. In comparison with [2], the ground state energy is an order of magnitude larger in powers of α, due to the presence of electron spin.

Following the technique developed in [2] (see also [4]), our method is based on perturbations around the true ground state of the translation invariant operator, together with a bound on the expected photon number for this ground state, obtained by Chen and Fröhlich [8]. In particular, an important ingredient of the proof is the improvement of photon number estimates for different parts of the ground state.

A well-known difficulty connected to this problem arises from the fact that the ground state energy is not an isolated eigenvalue of the Hamiltonian, and that the form factor in the interaction term of the Hamiltonian contains a critical frequency space singularity (the infrared problem of Quantum Electrodynamics (QED)).

Estimates on the ground state energy play an important role, for instance, in binding problems, e.g., the determination of the Hydrogen binding energy [3]. The systematic study of Pauli-Fierz Hamiltonian was initiated in [1]. The first estimate for the translation invariant operator for spinless electron was obtained by [12]. Later on in [6], the model for electron with spin was considered, and the bound was obtained up to the order α^2 with an error term of the order $\alpha^{\frac{5}{2}} \log \alpha$. Such estimates are not sufficient to compute the correction to the binding energy due to the interaction with the radiation field. In [2] a new effective method was developed to obtain the self energy in the spinless case up to the order α^3 with an error $\mathcal{O}(\alpha^4)$. This result was later improved in [5] with computing the term $\mathcal{O}(\alpha^4)$ with error term $\mathcal{O}(\alpha^5)$. These last two results [2, 5] were crucial for proving that the binding energy in the case of the Hydrogen atom with spinless electron contained an $\alpha^5 \log \alpha$ term and that this term comes from the ground state energy of the Hydrogen atom and does not exist in the translation invariant case [3].

In the work at hand, we are starting to implement the same program for the model of a Hydrogen atom with spin 1/2 electron interacting with the quantized radiation field. The first step of this program is, as in [2], computing the self-energy, for the electron with spin, up to the order $\mathcal{O}(\alpha^3)$.

The Pauli-Fierz Hamiltonian H for a free electron coupled to the quantized electromagnetic field is defined by

$$H = :\big(i\nabla_x \otimes I_f - \sqrt{\alpha} A(x)\big)^2: + \sqrt{\alpha}\sigma \cdot B(x) + I_{el} \otimes H_f. \tag{1}$$

where $: \ \cdots \ :$ denotes normal ordering, corresponding to the subtraction of a normal ordering constant proportional to α. The operator H acts on the Hilbert space $\mathcal{H} := \mathcal{H}_{el} \otimes \mathcal{F}$, where $\mathcal{H}_{el} = L^2(\mathbb{R}^3, \mathbb{C}^2)$, is the Hilbert space of one non-relativistic electron, $\mathbb{R}^3$ is the configuration space of the electron, and $\mathbb{C}^2$ accomodates its spin.

We describe the quantized electromagnetic field by use of the Coulomb gauge condition. Accordingly, the one-photon Hilbert space is given by $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$, where $\mathbb{R}^3$ denotes the photon momentum and $\mathbb{C}^2$ accounts for the two independent transversal polarizations of the photon. The photon Fock space is then defined by

$$\mathcal{F} = \bigoplus_{n\in\mathbb{N}} \mathcal{F}_s^{(n)},$$

where the n-photons space $\mathcal{F}_s^{(n)} = \bigotimes_s^n \big(L^2(\mathbb{R}^3) \otimes \mathbb{C}^2\big)$ is the symmetric tensor product of n copies of $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$.

We use units such that $\hbar = c = 1$, and where the mass of the electron equals $m = 1/2$. The electron charge is then given by $e = \sqrt{\alpha}$, with $\alpha \approx 1/137$ denoting the fine structure constant. As usual, we will consider α as a parameter.

The operator that couples an electron to the quantized vector potential is given by

$$A(x) = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \frac{\zeta(|k|)}{2\pi|k|^{1/2}} \varepsilon_\lambda(k) \Big[e^{ikx} \otimes a_\lambda(k) + e^{-ikx} \otimes a^*_\lambda(k) \Big] \mathrm{d}k \,,$$

where by the Coulomb gauge condition, $\mathrm{div} A = 0$. The operators a_λ, a^*_λ satisfy the usual commutation relations

$$[a_\nu(k), a^*_\lambda(k')] = \delta(k-k')\delta_{\lambda,\nu}, \quad [a_\nu(k), a_\lambda(k')] = 0,$$

and there exists a unique unit ray $\Omega_f \in \mathcal{F}$, the Fock vacuum, which satisfies $a_\lambda(k)\Omega_f = 0$ for all $k \in \mathbb{R}^3$ and $\lambda \in \{1,2\}$. The vectors $\varepsilon_\lambda(k) \in \mathbb{R}^3$ are the following two orthonormal polarization vectors perpendicular to k,

$$\varepsilon_1(k) = \frac{(k_2, -k_1, 0)}{\sqrt{k_1^2 + k_2^2}} \qquad \text{and} \qquad \varepsilon_2(k) = \frac{k}{|k|} \wedge \varepsilon_1(k).$$

The function $\zeta(|k|)$ describes the ultraviolet cutoff on the wavenumbers k. We assume ζ to be of class C^1, with compact support.

The operator that couples the electron to the magnetic field is

$$B(x) = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \frac{\zeta(|k|)}{2\pi|k|^{1/2}} k \times i\varepsilon_\lambda(k) \Big[e^{ikx} \otimes a_\lambda(k) - e^{-ikx} \otimes a^*_\lambda(k) \Big] \mathrm{d}k \,.$$

In Equation (1), $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is the 3-component vector of Pauli matrices. The photon field energy operator H_f is given by

$$H_f = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} |k| a^*_\lambda(k) a_\lambda(k) \mathrm{d}k.$$

For convenience, in the following, we shall denote

$$A(x) = A^-(x) + A^+(x) \quad \text{and} \quad B(x) = B^-(x) + B^+(x)$$

where

$$A^-(x) := \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \frac{\zeta(|k|)}{2\pi|k|^{1/2}} \varepsilon_\lambda(k) e^{ikx} \otimes a_\lambda(k) \mathrm{d}k \,,$$

$A^+(x) := (A^-(x))^*$,

$$B^-(x) := \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \frac{\zeta(|k|)}{2\pi|k|^{1/2}} k \times i\varepsilon_\lambda(k) e^{ikx} \otimes a_\lambda(k) \mathrm{d}k \,,$$

and $B^+(x) := (B^-(x))^*$.

The system is translationally invariant, and H commutes with the operator of total momentum

$$P_{tot} = i\nabla_x \otimes I_f + I_{el} \otimes P_f,$$

where $i\nabla_x$ and $P_f = \sum_{\lambda=1,2} \int k a^*_\lambda(k) a_\lambda(k) \mathrm{d}k$ denote respectively the electron and the photon momentum operators.

Therefore, if $\mathcal{H}_p \cong \mathbb{C}^2 \otimes \mathcal{F}$ denotes the fibre Hilbert space corresponding to conserved total momentum p, for any fixed value p of the total momentum, the restriction of H to the fibre space $\mathcal{H}_p$ is given by (see e.g. [7])

$$H(p) =: (p - P_f - \sqrt{\alpha}A(0))^2 : +\sqrt{\alpha}\sigma \cdot B(0) + H_f. \tag{2}$$

Henceforth, we will write $A^\pm := A^\pm(0)$ and $B^\pm = B^\pm(0)$.
It is known that $\inf \mathrm{spec}(H) = \inf \mathrm{spec}(H(0))$ for small α [9]. Moreover, the ground state energy of the one electron self-energy operator with total momentum $p = 0$ is an eigenvalue of multiplicity two [7]. The case of spinless electron, without restriction on α, was investigated earlier in [10].
In the sequel, we shall study the operator $H(p = 0)$.

2. Statements of the main results

Consider

$$\Omega_0 := \lambda\Omega_f\,, \quad \text{with } \lambda \in \mathbb{C}^2 \text{ such that } |\lambda| = 1\,, \tag{3}$$

and define

$$\Gamma_1 := -(H_f + P_f^2)^{-1}\sigma.B^+\Omega_0 \tag{4}$$

$$\Gamma_2 = -(H_f + P_f^2)^{-1}\left[\sigma \cdot B^+\Gamma_1 + 2A^+ \cdot P_f\Gamma_1 + A^+ \cdot A^+\Omega_0\right]\,. \tag{5}$$

On $\mathbb{C}^2 \otimes \mathcal{F}$, we define the positive bilinear form

$$\langle v,\, w\rangle_* := \langle v,\, (H_f + P_f^2)w\rangle\,, \tag{6}$$

and its associated semi-norm $\|v\|_* = \langle v,\, v\rangle_*$.

Theorem 2.1 (Ground state energy of $H(0)$). *We have*

$$\begin{aligned} &\inf\ \mathrm{spec}(H(0)) = -\alpha\|\Gamma_1\|_*^2 \\ &+ \alpha^2\left(2\|A^-\Gamma_1\|^2 - \|\Gamma_2\|_*^2 + \|\Gamma_1\|_*^2\,\|\Gamma_1\|^2\right) + \mathcal{O}(\alpha^3)\,. \end{aligned} \tag{7}$$

The proof of the Theorem consists in proving an upper bound obtained with a trial state (see inequality (12) in Section 3), and a lower bound obtained by variational estimates (see (59) in Section 6).

Remark 2.1. *Recall that the ground state of $H(0)$ is twice degenerate* [7]. *The result in Theorem 2.1 does not depend on the choice of Ω_0.*
According to Lemma 7.2, the term of order α is nonzero.

In the remainder, we will need the following notations. For $n \in \mathbb{N}$, let P_n be the orthogonal projection onto the subspace $\mathbb{C}^2 \otimes \mathcal{F}_n$ of the space $\mathbb{C}^2 \otimes \mathcal{F}$, and $P_{\geq n}$ be the orthogonal projection onto the space $\mathbb{C}^2 \otimes \left(\bigoplus_{k\geq n} \mathcal{F}_k\right)$.
Let Φ_0 be a ground state of $H(0)$ with the condition $P_0\Phi_0 = \Omega_0$. Taking the $\langle\, .\,,\, .\,\rangle_*$-orthonormal projections of Φ_0 along the vectors Γ_1 and Γ_2, and denoting by R the component in the $\langle\, .\,,\, .\,\rangle_*$-orthogonal complement of their span, we get

$$\Phi_0 = \Omega_0 + \alpha^{\frac{1}{2}}\gamma_1\Gamma_1 + \alpha\gamma_2\Gamma_2 + R \tag{8}$$

where for $i = 1, 2$ we assume

$$\langle \Gamma_i,\, R\rangle_* = 0 \quad \text{and} \quad P_0 R = 0\,. \tag{9}$$

Theorem 2.2. *For Φ_0 defined by* (8) *and* (9)*, we have*

$$|\gamma_1 - 1| = \mathcal{O}(\alpha) \quad |\gamma_2 - 1| = \mathcal{O}(\alpha^{\frac{1}{2}}) \tag{10}$$

$$\|R\|_* = \mathcal{O}(\alpha^{\frac{3}{2}}) \quad \|R\| = \mathcal{O}(\alpha)\,. \tag{11}$$

The statement of this theorem follows immediately from the proof of Theorem 2.1.

3. Upper bound to the ground state energy

In this section, we prove the upper bound for the ground state energy

$$\begin{aligned} &\inf\ \mathrm{spec}(H(0)) \\ &\leq -\alpha\|\Gamma_1\|_*^2 + \alpha^2\left(2\|A^-\Gamma_1\|^2 - \|\Gamma_2\|_*^2 + \|\Gamma_1\|_*^2\,\|\Gamma_1\|^2\right) + \mathcal{O}(\alpha^3)\,. \end{aligned} \tag{12}$$

Let us define the following trial state

$$\Theta = \Omega_f + \sqrt{\alpha}\,\Gamma_1 + \alpha\,\Gamma_2\,.$$

Since

$$\begin{aligned} H(0) =& H_f + P_f^2 + 4\sqrt{\alpha}\mathcal{R}e P_f\cdot A^- + 2\alpha\mathcal{R}e A^+\cdot A^+ \\ &+ 2\alpha A^+\cdot A^- + 2\sqrt{\alpha}\mathcal{R}e\,\sigma\cdot B^- \end{aligned} \tag{13}$$

we obtain

$$\begin{aligned} &\langle H(0)\Theta,\,\Theta\rangle = \alpha\|\Gamma_1\|_*^2 + \alpha^2\|\Gamma_2\|_*^2 + 2\alpha^2\mathcal{R}e\langle A^+\cdot A^+\Omega_f,\,\Gamma_2\rangle \\ &+ 2\alpha\mathcal{R}e\langle\sigma\cdot B^-\Gamma_1,\,\Omega_f\rangle + 2\alpha^2\|A^-\Gamma_1\|^2 + 4\alpha^2\mathcal{R}e\langle P_f\cdot A^-\Gamma_2,\,\Gamma_1\rangle \\ &+ 2\alpha^2\mathcal{R}e\langle\sigma\cdot B^-\Gamma_2,\,\Gamma_1\rangle + 2\alpha^3\|A^-\Gamma_2\|^2 \\ &= -\alpha^2\|\Gamma_1\|_*^2 - \alpha^2\|\Gamma_2\|_*^2 + 2\alpha^2\|A^-\Gamma_1\|^2 + 2\alpha^3\|A^-\Gamma_2\|^2 \end{aligned} \tag{14}$$

where in the last equality, we used

$$\begin{aligned} &\alpha^2\mathcal{R}e\langle A^+\cdot A^+\Omega_f,\,\Gamma_2\rangle + 2\alpha^2\mathcal{R}e\langle P_f\cdot A^-\Gamma_2,\,\Gamma_1\rangle + \alpha^2\mathcal{R}e\langle\sigma\cdot B^-\Gamma_2,\,\Gamma_1\rangle \\ &= -\alpha^2\|\Gamma_2\|_*^2\,, \end{aligned}$$

and

$$2\alpha\mathcal{R}e\langle\sigma\cdot B^-\Gamma_1,\,\Omega_f\rangle = -2\alpha\|\Gamma_1\|_*^2\,.$$

The identity $\|\Theta\|^2 = 1 + \alpha\|\Gamma_1\|^2 + \alpha^2\|\Gamma_2\|^2$ together with (14) yields

$$\begin{aligned} &\inf\mathrm{spec}(H(0)) \leq \frac{\langle\, H(0)\Theta,\,\Theta\,\rangle}{\|\Theta\|^2} \\ &= -\alpha^2\|\Gamma_1\|_*^2 - \alpha^2\|\Gamma_2\|_*^2 + 2\alpha^2\|A^-\Gamma_1\|^2 + \alpha^2\|\Gamma_1\|_*^2\|\Gamma_1\|^2 + \mathcal{O}(\alpha^3)\,. \end{aligned}$$

which concludes the proof of the upper bound (12).

4. A priori estimates

Let Φ_0 denote the ground state of $H(0)$ with the normalization condition $P_0\Phi_0 = \Omega_0$, where Ω_0 is defined by (3).
For Γ_1 defined by (4), we decompose Φ_0 as

$$\Phi_0 = \Omega_0 + (\gamma_1\Gamma_1 + R_1) + P_{\geq 2}\Phi_0 \quad \text{with} \quad \langle\Gamma_1,\, R_1\rangle_* = 0\,, \gamma_1 \in \mathbb{C}\,. \tag{15}$$

Proposition 4.1. *The following estimate holds*

$$\|\Phi_0\|_* = \mathcal{O}(\alpha)\,. \tag{16}$$

Proof.

$$\begin{aligned}
&\langle H(0)\Phi_0,\, \Phi_0\rangle = \langle(H_f + P_f^2)\Phi_0,\, \Phi_0\rangle + 4\sqrt{\alpha}\mathcal{R}e\langle P_f\cdot A^-\Phi_0\,\Phi_0\rangle \\
&+ 2\sqrt{\alpha}\mathcal{R}e\langle\sigma\cdot B^-\Phi_0,\, \Phi_0\rangle + 2\alpha\mathcal{R}e\langle A^-\cdot A^-\Phi_0\,\Phi_0\rangle + 2\alpha\|A^-\Phi_0\|^2 \\
&\geq \langle(H_f + P_f^2)\Phi_0,\, \Phi_0\rangle - c\sqrt{\alpha}\|H_f^{\frac12}\Phi_0\|\,\|P_f\Phi_0\| \\
&- c\sqrt{\alpha}\|H_f^{\frac12}\Phi_0\|\,\|\Phi_0\| - c\alpha\|H_f^{\frac12}\Phi_0\|\,(\|H_f^{\frac12}\psi\| + \|\psi\|) \\
&\geq \langle(H_f + P_f^2)\Phi_0,\, \Phi_0\rangle - (c\sqrt{\alpha} + \frac14)\|H_f^{\frac12}\Phi_0\|^2 \\
&- c\sqrt{\alpha}\|P_f\Phi_0\|^2 - c\alpha\|\Phi_0\|^2 \\
&\geq \frac12\langle(H_f + P_f^2)\Phi_0,\, \Phi_0\rangle - c\alpha = \frac12\|\Phi_0\|_*^2 - c\alpha
\end{aligned} \tag{17}$$

using in the second inequality of (17) that for all $\psi \in \mathbb{C}^2\otimes\mathcal{F}$ we have $\|A^-\psi\| \leq c\|H_f^{\frac12}\psi\|$, $\|B^-\psi\| \leq c\|H_f^{\frac12}\psi\|$ and $\|A^+\psi\| \leq c(\|H_f^{\frac12}\psi\| + \|\psi\|)$ (see e.g. [11, Lemma A4]). The proof of (16) follows from (17) and the fact that $\langle H(0)\Phi_0,\, \Phi_0\rangle \leq \langle H(0)\Omega_0,\, \Omega_0\rangle = 0$. □

Proposition 4.2. *There exists $c > 0$ such that for all $\phi \in \mathbb{C}^2\otimes\mathcal{F}$,*

$$\langle H(0)\phi,\, \phi\rangle - \frac12\|\phi\|_*^2 \geq -c\alpha\|\phi\|^2\,. \tag{18}$$

Proof. The proof is done by repeating all steps in (17), replacing Φ_0 by ϕ. □

The next result is a consequence of an a priori photon number bound for the ground state obtained in [8, Proposition 5.1], whose statement is given in Lemma 7.3 for total momentum $p = 0$.

Proposition 4.3. *The following holds*

$$\|P_{\geq 1}\Phi_0\|^2 = \mathcal{O}(\alpha)\,. \tag{19}$$

Proof. Applying Lemma 7.3, we obtain

$$\begin{aligned}
\|P_{\geq 1}\Phi_0\|^2 \leq \langle P_{\geq 1}\Phi_0,\, N_f\, P_{\geq 1}\Phi_0\rangle &= \sum_{\lambda=1,2}\int \|a_\lambda(k)\Phi_0\|^2\mathrm{d}k \\
&\leq c\int \frac{\alpha}{|k|^2}\zeta(|k|)^2\mathrm{d}k \leq c\alpha\,,
\end{aligned}$$

where N_f is the photon number operator. □

Corollary 4.1. *There exists* $\alpha_0 > 0$ *such that* γ_1 *is uniformly bounded in* $\alpha \in [0, \alpha_0]$.

Proof. Since $\langle R_1, \Gamma_1\rangle_* = 0$, then from Proposition 4.1, we have that there exists α_0 and c such that for all α smaller than α_0,

$$\alpha\langle (H_f + P_f^2)\gamma_1\Gamma_1, \Gamma_1\rangle \le c\alpha \tag{20}$$

which implies $|\gamma_1|^2 \le c\,(\langle (H_f + P_f^2)\Gamma_1, \Gamma_1\rangle)^{-1}$. We conclude the proof by applying Lemma 7.2. □

Corollary 4.2. *We have*

$$\|R_1\|_*^2 = \mathcal{O}(\alpha) \quad \textit{and} \quad \|R_1\|^2 = \mathcal{O}(\alpha)\,. \tag{21}$$

Proof. Applying Proposition 4.1 and Corollary 4.1 gives

$$\|R_1\|_* \le \|\Pi_1\Phi_0\|_* + \sqrt{\alpha}\,|\gamma_1|\,\|\Gamma_1\|_* \le c\sqrt{\alpha}\,.$$

Similarly, applying Proposition 4.3 and Corollary 4.1 we get

$$\|R_1\| \le \|\Pi_1\Phi_0\| + \sqrt{\alpha}\,|\gamma_1|\,\|\Gamma_1\| \le c\sqrt{\alpha}\,. \tag{22}$$

□

5. Lower bound up to the order α

In the present section, we derive a sharp lower bound for the ground state energy $\langle H(0)\Phi_0, \Phi_0\rangle/\|\Phi_0\|^2$, up to the order α, with rest of order α^2. The proof also implies improved estimates on γ_1, $\|R_1\|_*$ and $\|P_{\ge 2}\Phi_0\|_*$. These results are stated as follows

Proposition 5.1. *The following holds*

$$\inf \operatorname{spec}(H(0)) = -\alpha\|\Gamma_1\|_*^2 + \mathcal{O}(\alpha^2) \tag{23}$$

$$\|R_1\|_* = \mathcal{O}(\alpha) \tag{24}$$

$$\|P_{\ge 2}\Phi_0\|_* = \mathcal{O}(\alpha) \tag{25}$$

$$\gamma_1 = 1 + \mathcal{O}(\sqrt{\alpha}), \quad \mathcal{I}m\gamma_1 = \mathcal{O}(\sqrt{\alpha}) \tag{26}$$

Proof. Using the decomposition (15) for Φ_0, and the identity (13) for $H(0)$, we get

$$\begin{aligned}
&\langle H(0)\Phi_0, \Phi_0\rangle \\
&= \alpha|\gamma_1|^2\|\Gamma_1\|_*^2 + \|R_1\|_*^2 + \|P_{\ge 2}\Phi_0\|_*^2 + 4\sqrt{\alpha}\mathcal{R}e\langle P_f\cdot A^- P_{\ge 2}\Phi_0, P_{\ge 1}\Phi_0\rangle \\
&+ 2\sqrt{\alpha}\mathcal{R}e\langle \sigma\cdot B^-(\sqrt{\alpha}\gamma_1\Gamma_1 + R_1 + P_{\ge 2}\Phi_0), \Phi_0\rangle + 2\alpha\|A^-\Phi_0\|^2 \\
&+ 2\alpha\mathcal{R}e\langle A^-\cdot A^- P_{n\ge 2}\Phi_0, \Phi_0\rangle\,.
\end{aligned} \tag{27}$$

Applying [11, Lemma A4], Corollary 4.1 and Corollary 4.2, the fourth term in the right hand side of (27) is estimated as

$$\begin{aligned}
&4\sqrt{\alpha}\mathcal{R}e\langle P_f\cdot A^- P_{\ge 2}\Phi_0, P_{\ge 1}\Phi_0\rangle \\
&\ge -\epsilon\|H_f^{\frac{1}{2}} P_{\ge 2}\Phi_0\|^2 - c(\epsilon)\alpha\|P_f P_1\Phi_0\|^2 - c(\epsilon)\alpha\|P_f P_{\ge 2}\Phi_0\|^2 \\
&\ge -\epsilon\|P_{\ge 2}\Phi_0\|_*^2 - c\alpha^2\,.
\end{aligned} \tag{28}$$

In (28), as well as in the sequel, we shall omit the ϵ-dependence of the constants c since ϵ will eventually be given a fixed value independent of α.
Similarly, using in addition $\langle \sigma \cdot B^- \Gamma_1,\, \Omega_0 \rangle = -\|\Gamma_1\|_*^2$ and $\langle \sigma \cdot B^- R_1,\, \Omega_0 \rangle = \langle R_1,\, \Gamma_1 \rangle_* = 0$, we estimate the fifth term in the right hand side of (27) as

$$
\begin{aligned}
& 2\sqrt{\alpha}\mathcal{R}e\langle \sigma \cdot B^-(\sqrt{\alpha}\gamma_1\Gamma_1 + R_1 + P_{\geq 2}\Phi_0),\, \Phi_0\rangle \\
& = 2\alpha\mathcal{R}e\langle \sigma \cdot B^- \gamma_1\Gamma_1,\, \Omega_0\rangle + 2\sqrt{\alpha}\mathcal{R}e\langle \sigma \cdot B^- R_1,\, \Omega_0\rangle \\
& + 2\sqrt{\alpha}\mathcal{R}e\langle \sigma \cdot B^- P_{\geq 2}\Phi_0,\, P_{\geq 1}\Phi_0\rangle \\
& \geq -2\alpha\mathcal{R}e\gamma_1\|\Gamma_1\|_*^2 - \epsilon\|H_f^{\frac{1}{2}}P_{n\geq 2}\Phi_0\|^2 - c\alpha\|P_{\geq 1}\Phi_0\|^2 \\
& \geq -2\alpha\mathcal{R}e\gamma_1\|\Gamma_1\|_*^2 - \epsilon\|P_{\geq 2}\Phi_0\|_*^2 - c\alpha^2
\end{aligned}
\tag{29}
$$

The sixth term in the right hand side of (27) is nonnegative, and with similar arguments as above, the seventh is bounded by

$$
\begin{aligned}
& 2\alpha\mathcal{R}e\langle A^- \cdot A^- P_{\geq 2}\Phi_0,\, \Phi_0\rangle \geq -c\alpha\|A^- P_{\geq 2}\Phi_0\|\,\|A^+\Phi_0\| \\
& \geq -\epsilon\|P_{\geq 2}\Phi_0\|_*^2 - c\alpha^2\,.
\end{aligned}
\tag{30}
$$

Collecting (27)-(30) gives

$$
\begin{aligned}
& \langle H(0)\Phi_0,\, \Phi_0\rangle \\
& \geq \frac{1}{2}\|P_{\geq 2}\Phi_0\|_*^2 + \alpha|1-\gamma_1|^2\|\Gamma_1\|_*^2 - \alpha\|\Gamma_1\|_*^2 + \|R_1\|_*^2 - c\alpha^2
\end{aligned}
\tag{31}
$$

To prove (23) we first note that from the decomposition (15) of Φ_0 and Proposition 4.3, we have $\|\Phi_0\|^2 = 1 + \mathcal{O}(\alpha)$. Therefore, (23) is a consequence of this equality, the upper bound (12), and the lower bound (31).
The estimates (24)-(26) are direct consequences of (31) and the fact that $\langle H(0)\Phi_0,\, \Phi_0\rangle \leq 0$. $\square$

6. Lower bound up to the order α^2

Equipped with the estimates of Sections 4 and 5, we are now ready to establish a lower bound up to the order α^2, with error term of the order α^3 for the ground state energy.
For Γ_1 defined by (4) and for γ_1 given by the decomposition (15) of Φ_0, we define

$$
\Gamma_2^{(\gamma_1)} = -(H_f + P_f^2)^{-1}\left(\gamma_1\sigma \cdot B^+\Gamma_1 + 2\gamma_1 A^+ \cdot P_f\Gamma_1 + A^+ \cdot A^+\Omega_0\right)\,,
\tag{32}
$$

and we define γ_2 and R_2 (depending on γ_1) by

$$
P_2\Phi_0 = \alpha\gamma_2\Gamma_2^{(\gamma_1)} + R_2 \quad \text{and} \quad \langle R_2, \Gamma_2^{(\gamma_1)}\rangle_* = 0\,.
\tag{33}
$$

Thus, we have

$$
\Phi_0 = \Omega_0 + \sqrt{\alpha}\gamma_1\Gamma_1 + R_1 + \alpha\gamma_2\Gamma_2^{(\gamma_1)} + R_2 + P_{\geq 3}\Phi_0\,.
\tag{34}
$$

6.1. Preliminary estimates. Before estimating the ground state energy, we need to prove some estimates for the vectors occurring in the decomposition (34) for Φ_0.

Proposition 6.1. *There exits $\alpha_1 > 0$ and $c > 0$ such that for all $\alpha \in (0, \alpha_1)$ and all $\gamma_1 \in (\frac{1}{2}, \frac{3}{2})$, we have*

$$|\gamma_2| < c\,.$$

Proof. From (25) of Proposition 5.1 we have

$$\|P_2\Phi_0\|_* = \alpha^2|\gamma_2|^2\|\Gamma_2^{(\gamma_1)}\|_*^2 + \|R_2\|_*^2 < c\alpha^2\,. \tag{35}$$

Together with Lemma 7.1, this yields the result. □

Proposition 6.2. *There exists $\alpha_2 > 0$ and $c > 0$ such that for all $\alpha \in (0, \alpha_2)$, for all $\epsilon > 0$, and for all $\gamma_1 \in (\frac{1}{2}, \frac{3}{2})$*

$$\begin{aligned} \|R_1\|^2 &\le c\,\epsilon^{-1}\alpha^2 + \epsilon\alpha^{-1}\|H_f^{\frac{1}{2}}R_1\|^2\,, \\ \|R_2\|^2 &\le c\,\epsilon^{-1}\alpha^2 + \epsilon\alpha^{-1}\|H_f^{\frac{1}{2}}R_2\|^2\,, \\ \|P_{\ge 3}\Phi_0\|^2 &\le c\,\epsilon^{-1}\alpha^2 + \epsilon\alpha^{-1}\|H_f^{\frac{1}{2}}P_{\ge 3}\Phi_0\|^2\,. \end{aligned} \tag{36}$$

Proof. We have, applying lemma 7.3

$$\|a_\lambda(k)R_2\| \le \|a_\lambda(k)P_2\Phi_0\| + \alpha\,|\gamma_2|\,\|a_\lambda(k)\Gamma_2^{(\gamma_1)}\| \le c\sqrt{\alpha}/|k|\,. \tag{37}$$

Therefore, given $\epsilon > 0$, we have

$$\begin{aligned} \|R_2\|^2 &\le \sum_{\lambda=1,2}\int \|a_\lambda(k)R_2\|^2 \mathrm{d}k \\ &= \sum_{\lambda=1,2}\int_{\epsilon|k|<\alpha} \|a_\lambda(k)R_2\|^2 \mathrm{d}k + \sum_{\lambda=1,2}\int_{\epsilon|k|\ge\alpha} \|a_\lambda(k)R_2\|^2 \mathrm{d}k \\ &\le \int_{\epsilon|k|<\alpha} \left|\frac{c\sqrt{\alpha}}{|k|}\right|^2 \mathrm{d}k + \int_{\epsilon|k|\ge\alpha} \frac{\epsilon|k|}{\alpha}\|a_\lambda(k)R_2\|^2 \mathrm{d}k \\ &\le c^2\epsilon^{-1}\alpha^2 + \epsilon\alpha^{-1}\|H_f^{\frac{1}{2}}R_2\|^2\,. \end{aligned}$$

The bounds for $\|R_1\|$ and $\|P_{\ge 3}\Phi_0\|$ can be derived similarly. □

To estimate $\langle H(0)\Phi_0,\, \Phi_0\rangle$ we use the above decomposition (34) of Φ_0 and the identity (13).

6.2. Terms involving $P_1\Phi_0$ but not $P_{\ge 2}\Phi_0$. Denoting by (I) the terms in $\langle H(0)\Phi_0,\, \Phi_0\rangle$ involving $P_1\Phi_0$ but not $P_{\ge 2}\Phi_0$ we have

$$\begin{aligned} (I) =& \alpha|\gamma_1|^2\,\|\Gamma_1\|_*^2 + \|R_1\|_*^2 + 2\sqrt{\alpha}\mathcal{R}e\langle\sigma\cdot B^-(\sqrt{\alpha}\gamma_1\Gamma_1 + R_1),\, \Omega_0\rangle \\ &+ 2\alpha\|A^-(\sqrt{\alpha}\gamma_1\Gamma_1 + R_1)\|^2\,. \end{aligned} \tag{38}$$

Using the definition of Γ_1 and $\langle \Gamma_1,\, R_1\rangle_* = 0$, we get that the third term in the right hand side equals $-2\alpha\mathcal{R}e\gamma_1\|\Gamma_1\|_*^2$. Using the bound (26) on γ_1, we get that the fourth term in the right hand side of (38) is estimated as

$$\begin{aligned}&2\alpha^2|\gamma_1|^2\|A^-\Gamma_1\|^2 + 4\alpha^{\frac32}\mathcal{R}e\langle A^-\gamma_1\Gamma_1,\, A^-R_1\rangle + 2\alpha\|A^-R_1\|^2\\ &\geq 2\alpha^2|\gamma_1|^2\|A^-R_1\|^2 - \epsilon\|H_f^{\frac12}R_1\|^2 + \mathcal{O}(\alpha^3)\,.\end{aligned}\tag{39}$$

The above thus implies

$$\begin{aligned}(I) \geq\ & \alpha\|\Gamma_1\|_*^2(|\gamma_1|^2 - 2\mathcal{R}e\gamma_1) + (1-\epsilon)\|R_1\|_*^2\\ &+ 2\alpha^2|\gamma_1|^2\|A^-\Gamma_1\|^2 + \mathcal{O}(\alpha^3)\\ =& -\alpha^2\|\Gamma_1\|_*^2 + \alpha|1-\gamma_1|^2\|\Gamma_1\|_*^2 + 2\alpha^2(|\gamma_1|^2-1)\|A^-\Gamma_1\|^2\\ &+ 2\alpha^2\|A^-\Gamma_1\|^2 + (1-\epsilon)\|R_1\|_*^2 + \mathcal{O}(\alpha^3)\end{aligned}$$

This yields

$$\begin{aligned}(I) \geq\ & -\alpha^2\|\Gamma_1\|_*^2 + 2\alpha^2\|A^-\Gamma_1\|^2 + \frac12\alpha|1-\gamma_1|^2\|\Gamma_1\|_*^2\\ &+ \frac12\alpha|1-\gamma_1|^2\|\Gamma_1\|_*^2 - c\alpha^2\left||\gamma_1|^2-1\right|\|\Gamma_1\|_*^2 + (1-\epsilon)\|R_1\|_*^2 + \mathcal{O}(\alpha^3)\\ &\geq -\alpha^2\|\Gamma_1\|_*^2 + 2\alpha^2\|A^-\Gamma_1\|^2 + \frac12\alpha|1-\gamma_1|^2\|\Gamma_1\|_*^2\\ &+\alpha\|\Gamma_1\|_*^2\left(\frac12(1-\mathcal{R}e\gamma_1)^2 + \frac12(\mathcal{I}m\gamma_1)^2 - c\alpha\,|\mathcal{R}e\gamma_1 - 1| - c\alpha(\mathcal{I}m\gamma_1)^2\right)\\ &+ (1-\epsilon)\|R_1\|_*^2 + \mathcal{O}(\alpha^3)\\ &\geq -\alpha^2\|\Gamma_1\|_*^2 + 2\alpha^2\|A^-\Gamma_1\|^2 + \frac12\alpha|1-\gamma_1|^2\|\Gamma_1\|_*^2\\ &+ (1-\epsilon)\|R_1\|_*^2 + \mathcal{O}(\alpha^3)\,,\end{aligned}\tag{40}$$

using $\frac12(1-\mathcal{R}e\gamma_1)^2 - c\alpha\,|\mathcal{R}e\gamma_1 - 1| \geq -c'\alpha^2$ and $\frac12(\mathcal{I}m\gamma_1)^2 - c\alpha\mathcal{I}m\gamma_1 \geq -c'\alpha^2$.

6.3. TERMS INVOLVING $P_2\Phi_0$ BUT NOT $P_{\geq 3}\Phi_0$. Denoting by (II) the terms in $\langle H(0)\Phi_0,\, \Phi_0\rangle$ involving $P_2\Phi_0$ but not $P_{\geq3}\Phi_0$ we have

$$\begin{aligned}(II) =\ & \alpha^2|\gamma_2|^2\|\Gamma_2^{(\gamma_1)}\|_*^2 + \|R_2\|_*^2\\ &+ 4\alpha^2\mathcal{R}e\gamma_2\overline{\gamma_1}\langle P_f\cdot A^-\Gamma_2^{(\gamma_1)},\, \Gamma_1\rangle + 4\alpha^{\frac32}\mathcal{R}e\gamma_2\langle P_f\cdot A^-\Gamma_2^{(\gamma_1)},\, R_1\rangle\\ &+ 4\alpha\mathcal{R}e\overline{\gamma_1}\langle P_f\cdot A^-R_2,\, \Gamma_1\rangle + 4\sqrt{\alpha}\mathcal{R}e\langle P_f\cdot A^-R_2,\, R_1\rangle\\ &+ 2\alpha^2\mathcal{R}e\gamma_2\overline{\gamma_1}\langle \sigma\cdot B^-\Gamma_2^{(\gamma_1)},\, \Gamma_1\rangle + 2\alpha^{\frac32}\mathcal{R}e\gamma_2\langle\sigma\cdot B^-\Gamma_2^{(\gamma_1)},\, R_1\rangle\\ &+ 2\alpha\mathcal{R}e\overline{\gamma_1}\langle\sigma\cdot B^-R_2,\, \Gamma_1\rangle + 2\sqrt{\alpha}\mathcal{R}e\langle\sigma\cdot B^-R_2,\, R_1\rangle\\ &+ 2\alpha^2\mathcal{R}e\gamma_2\langle A^-\cdot A^-\Gamma_2^{(\gamma_1)}, \Omega_0\rangle + 2\alpha\mathcal{R}e\langle A^-\cdot A^-R_2,\, \Omega_0\rangle\\ &+ 2\alpha^3|\gamma_2|^2\|A^-\Gamma_2^{(\gamma_1)}\|^2 + 2\alpha\|A^-R_2\| + 4\alpha^2\mathcal{R}e\gamma_2\langle A^+\cdot A^-\Gamma_2^{(\gamma_1)},\, R_2\rangle\,.\end{aligned}\tag{41}$$

The sum of the fifth, the ninth and the twelfth terms in the right hand side of (41) is equal to $-2\alpha\langle R_2,\, \Gamma_2^{(\gamma_1)}\rangle_* = 0$; the sum of the third, seventh and eleventh terms is equal to $-2\alpha^2\gamma_2\|\Gamma_2^{(\gamma_1)}\|_*^2$; the sum of the fourth and the sixth terms is bounded below by

$$\begin{aligned}-c\sqrt{\alpha}\|H_f^{\frac{1}{2}}P_2\Phi_0\|\,\|H_f^{\frac{1}{2}}R_1\| &\geq -\epsilon\|R_1\|_*^2 - c(\epsilon)\alpha\|P_2\Phi_0\|_*^2\\ &\geq -\epsilon\|R_1\|_*^2 - c\alpha^3\,,\end{aligned}$$

according to (25) of Proposition 5.1. Applying Proposition 6.2, yields

$$\begin{aligned}2\sqrt{\alpha}\mathcal{R}e\langle\sigma\cdot B^-R_2,\, R_1\rangle &\geq -\epsilon\|R_2\|_*^2 - c\alpha\|R_1\|^2\\ &\geq -\epsilon\|R_2\|_*^2 - \epsilon\|R_1\|_*^2 - c\alpha^3\,.\end{aligned}$$

The term $2\alpha^{\frac{3}{2}}\mathcal{R}e\gamma_2\langle\sigma\cdot B^-\Gamma_2^{(\gamma_1)},\, R_1\rangle$ in (41) is estimated by

$$\begin{aligned}2\alpha^{\frac{3}{2}}\mathcal{R}e\gamma_2\langle\sigma\cdot B^-\Gamma_2^{(\gamma_1)},\, R_1\rangle &= 2\alpha^{\frac{3}{2}}\mathcal{R}e\gamma_2\langle H_f^{-\frac{1}{2}}\sigma\cdot B^-\Gamma_2^{(\gamma_1)},\, H_f^{\frac{1}{2}}R_1\rangle\\ &\geq -\epsilon\|R_1\|_*^2 - c\alpha^3\,,\end{aligned} \tag{42}$$

since the norm of $H_f^{-\frac{1}{2}}\sigma\cdot B^-\Gamma_2^{(\gamma_1)}$ is uniformly bounded in $\gamma_1\in(\frac{1}{2},\,\frac{3}{2})$. Finally we have

$$4\alpha^2\mathcal{R}e\gamma_2\langle A^+\cdot A^-\Gamma_2^{(\gamma_1)},\, R_2\rangle \geq -\epsilon\|R_2\|_*^2 - c\alpha^4\,. \tag{43}$$

Collecting all the above estimates in (41) thus gives

$$\begin{aligned}(II) &\geq \alpha^2\|\Gamma_2^{(\gamma_1)}\|_*^2(|\gamma_2|^2 - 2\gamma_2) + (1-\epsilon)\|R_2\|_*^2 - \epsilon\|R_1\|_*^2 - c\alpha^3\\ &\geq -\alpha^2\|\Gamma_2^{(\gamma_1)}\|_*^2 + (1-\epsilon)\|R_2\|_*^2 - \epsilon\|R_1\|_*^2 - c\alpha^3\,.\end{aligned} \tag{44}$$

6.4. Remaining terms. We collect in (III) all the terms in $\langle H(0)\Phi_0,\,\Phi_0\rangle$ that have not been treated in subsections 6.2 and 6.3. This yields

$$\begin{aligned}(III) &= \langle H(0)P_{\geq 3}\Phi_0,\,\Phi_0\rangle\\ &= \langle H(0)P_{\geq 3}\Phi_0,\, P_{\geq 3}\Phi_0\rangle + \langle H(0)P_{\geq 3}\Phi_0,\,(1-P_{\geq 3})\Phi_0\rangle\end{aligned} \tag{45}$$

Applying Proposition 6.2, the first term in the right hand side of (45) is bounded below using the following estimate

$$\begin{aligned}\langle H(0)P_{\geq 3}\Phi_0,\, P_{\geq 3}\Phi_0\rangle - \frac{1}{2}\|P_{\geq 3}\Phi_0\|_*^2 &\geq -c\alpha\|P_{\geq 3}\Phi_0\|^2\\ &\geq -c\epsilon^{-1}\alpha^3 - \epsilon\|P_{\geq 3}\Phi_0\|_*^2\,.\end{aligned} \tag{46}$$

The second term in the right hand side of (45) is

$$\begin{aligned}&\langle H(0)P_{\geq 3}\Phi_0,\,(1-P_{\geq 3})\Phi_0\rangle = 2\sqrt{\alpha}\mathcal{R}e\langle\sigma\cdot B^-P_3\Phi_0, P_2\Phi_0\rangle\\ &+ 4\sqrt{\alpha}\mathcal{R}e\langle P_f\cdot A^-P_3\Phi_0, P_2\Phi_0\rangle + 2\alpha\mathcal{R}e\langle A^-\cdot A^-P_3\Phi_0, P_1\Phi_0\rangle\\ &+ 2\alpha\mathcal{R}e\langle A^-\cdot A^-P_4\Phi_0,\, P_2\Phi_0\rangle\end{aligned} \tag{47}$$

We have

$$\begin{aligned}&2\alpha\mathcal{R}e\langle A^-\cdot A^-P_3\Phi_0,\, P_1\Phi_0\rangle \geq -\epsilon\|H_f^{\frac{1}{2}}P_3\Phi_0\|_*^2 - c\alpha^2\|A^+P_1\Phi_0\|^2\\ &\geq -\epsilon\|H_f^{\frac{1}{2}}P_3\Phi_0\|_*^2 - c\alpha^2\|P_1\Phi_0\|^2 \geq -\epsilon\|H_f^{\frac{1}{2}}P_3\Phi_0\|_*^2 - c\alpha^3\,,\end{aligned} \tag{48}$$

where in the last inequality, we used Proposition 4.3.

Similarly, we get

$$2\alpha\mathcal{R}e\langle A^- \cdot A^- P_4\Phi_0,\, P_2\Phi_0\rangle \geq -\epsilon\|H_f^{\frac{1}{2}} P_4\Phi_0\|_*^2 - c\alpha^3 \tag{49}$$

and

$$\begin{aligned}
&2\alpha\mathcal{R}e\langle \sigma \cdot B^- P_3\Phi_0,\, P_2\Phi_0\rangle \geq -\epsilon\|H_f^{\frac{1}{2}} P_3\Phi_0\|_*^2 - c\alpha\|P_2\Phi_0\|^2\\
&\geq -\epsilon\|H_f^{\frac{1}{2}} P_3\Phi_0\|_*^2 - c\alpha^3|\gamma_2|^2\|\Gamma_2^{(\gamma_1)}\|^2 - c\alpha\|R_2\|^2\\
&\geq -\epsilon\|H_f^{\frac{1}{2}} P_3\Phi_0\|_*^2 - c\alpha^3 - \epsilon\|H_f^{\frac{1}{2}} R_2\|^2\,,
\end{aligned} \tag{50}$$

where in the last inequality we applied Propositions 6.1 and 6.2.
The last term we have to estimate in (47) is

$$4\sqrt{\alpha}\mathcal{R}e\langle P_f \cdot A^- P_3\Phi_0,\, P_2\Phi_0\rangle \geq -c\sqrt{\alpha}\left(\|H_f^{\frac{1}{2}} P_3\Phi_0\|^2 + \|H_f^{\frac{1}{2}} P_2\Phi_0\|^2\right)\,. \tag{51}$$

Collecting (45)-(51) yields

$$(III) \geq \frac{1}{4}\|P_{\geq 3}\Phi_0\|_*^2 - \epsilon\|P_2\Phi_0\|_*^2 - c\alpha^3\,. \tag{52}$$

6.5. PROOF OF THE LOWER BOUND. The estimates (40), (44) and (52) for (I), (II) and (III) give

$$\begin{aligned}
\langle H(0)\Phi_0,\, \Phi_0\rangle \geq -\alpha\|\Gamma_1\|_*^2 + \frac{1}{2}\alpha|1-\gamma_1|^2\|\Gamma_1\|_*^2 + 2\alpha^2\|A^-\Gamma_1\|^2\\
- \alpha^2\|\Gamma_2^{(\gamma_1)}\|_*^2 + \frac{1}{4}\left(\|R_1\|_*^2 + \|R_2\|_*^2 + \|P_{\geq 3}\Phi_0\|_*^2\right) - c\alpha^3.
\end{aligned} \tag{53}$$

Next, we replace $\Gamma_2^{(\gamma_1)}$ by Γ_2 in the above expression and estimate the difference. For that sake, we estimate

$$\begin{aligned}
&\left|\|\Gamma_2^{(\gamma_1)}\|^2 - \|\Gamma_2\|^2\right| \leq c\left|\|\Gamma_2^{(\gamma_1)}\| - \|\Gamma_2\|\right| \leq c\|\Gamma_2^{(\gamma_1)} - \Gamma_1\|\\
&\leq c|\gamma - 1|\left\|\Gamma_2 + (H_f + P_f^2)^{-1}A^+ \cdot A^+\Omega_0\right\| \leq c|\gamma - 1|\,.
\end{aligned}$$

where in the first inequality we applied Lemma 7.1.
Applying this inequality, and Corollary 4.1, we thus can estimate in (53) the following two terms

$$\frac{1}{2}|1-\gamma_1|^2\alpha\|\Gamma_1\|_*^2 - \alpha^2\|\Gamma_2^{(\gamma_1)}\|_*^2 \geq -\alpha^2\|\Gamma_2\|_*^2 - c\alpha^3\,. \tag{54}$$

Thus, (54) and (53) give

$$\begin{aligned}
\langle H(0)\Phi_0,\, \Phi_0\rangle \geq &-\alpha\|\Gamma_1\|_*^2 + 2\alpha^2\|A^-\Gamma_1\|^2 - \alpha^2\|\Gamma_2\|_*^2\\
&+ \frac{1}{4}\left(\|R_1\|_*^2 + \|R_2\|_*^2 + \|P_{\geq 3}\Phi_0\|_*^2\right) - c\alpha^3\,.
\end{aligned} \tag{55}$$

Eventually, we compute

$$\frac{\langle H(0)\Phi_0,\, \Phi_0\rangle}{\|\Phi_0\|^2} = \frac{\langle H(0)\Phi_0,\, \Phi_0\rangle}{1 + \|P_1\Phi_0\|^2 + \|P_{\geq 2}\Phi_0\|^2} \geq \frac{\langle H(0)\Phi_0,\, \Phi_0\rangle}{1 + \|P_1\Phi_0\|^2} \tag{56}$$

since $\langle H(0)\Phi_0,\, \Phi_0\rangle \le 0$. Now using from proposition 4.3 that $\|P_{\ge 1}\Phi_0\|^2 = \mathcal{O}(\alpha)$, we obtain, applying (55) and (56)

$$\frac{\langle H(0)\Phi_0,\, \Phi_0\rangle}{\|\Phi_0\|^2} = \langle H(0)\Phi_0,\, \Phi_0\rangle + \|P_1\Phi_0\|^2\alpha\|\Gamma_1\|_*^2 + \mathcal{O}(\alpha^3)\,. \tag{57}$$

Since

$$\begin{aligned} &\|P_1\Phi_0\|^2\alpha\|\Gamma_1\|_*^2 \\ &= \alpha^2\|\Gamma_1\|^2\|\Gamma_1\|_*^2 + \alpha\|R_1\|^2\|\Gamma_1\|_*^2 + 2\alpha^{\frac32}\|\Gamma_1\|_*^2\mathcal{R}e\langle H_f^{-\frac12}\Gamma_1,\, H_f^{\frac12}R_1\rangle \\ &\ge \alpha^2\|\Gamma_1\|^2\|\Gamma_1\|_*^2 - c\alpha^{\frac32}\|\Gamma_1\|_*^2\|H_f^{-\frac12}\Gamma_1\|\,\|R_1\|_* \\ &\ge \alpha^2\|\Gamma_1\|^2\|\Gamma_1\|_*^2 - c\alpha^3 - \epsilon\|R_1\|_*^2\,, \end{aligned} \tag{58}$$

we obtain, together with (55) and (57)

$$\begin{aligned} &\frac{\langle H(0)\Phi_0,\, \Phi_0\rangle}{\|\Phi_0\|^2} \\ &\ge -\alpha\|\Gamma_1\|_*^2 + 2\alpha^2\|A^-\Gamma_1\|^2 - \alpha^2\|\Gamma_2\|_*^2 + \alpha^2\|\Gamma_1\|^2\|\Gamma_1\|_*^2 + \mathcal{O}(\alpha^3)\,. \end{aligned} \tag{59}$$

7. Appendix

Lemma 7.1. *There exists $\delta_1 > 0$, $\delta_2 > 0$, and $\alpha_0 > 0$ such that for all $\gamma_1 \in (\frac12,\, \frac32)$ and all $\alpha \in (0, \alpha_0)$, $\|\Gamma_2^{(\gamma_1)}\|_* \in (\delta_1,\, \delta_2)$.*

Proof. For the sake of simplicity, we shall fix here $\Omega_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\Omega_f$. The statement of the Lemma remains true for all Ω_0 defined as in (3).
We have

$$\begin{aligned} \Gamma_2^{(\gamma_1)} &= \gamma_1(H_f + P_f^2)^{-1}\Big[\sigma\cdot B^+(H_f+P_f^2)^{-1}\sigma\cdot B^+ \\ &\quad + 2A^+\cdot P_f(H_f+P_f^2)^{-1}\sigma\cdot B^+\Big]\Omega_0 - (H_f+P_f^2)^{-1}A^+\cdot A^+\Omega_0\,. \end{aligned} \tag{60}$$

In order to prove the bound below, it is sufficient to show that there exists a region $J \subset \mathbb{R}^3\times\mathbb{R}^3$ with strictly positive Lebesgue measure, and $\delta > 0$ such that for all α small enough independent of δ , for all $\gamma \in (\frac12,\, \frac32)$ and all $(k,\, k') \in J$, and for given $\lambda,\, \mu \in \{1,\, 2\}$, we have $|\Gamma_2^{(\gamma_1)}(k, \lambda;\, k', \mu)| > \delta$. For that sake, we shall prove that the third vector in the right hand side of (60) has a stronger singularity at the origin than the fist two vectors.
For all λ, μ in $\{1,\, 2\}$, we have, for all k and k' in the region $S_1 := \{(k,\, k') \mid \frac{|k'|}{2} \le |k| \le 2|k'|\}$

$$\begin{aligned} &\left|\left(\sigma\cdot B^+(H_f+P_f^2)^{-1}\Omega_0\right)\,(k, \lambda;\, k', \mu)\right| \\ &= \frac{1}{\sqrt2}\left|\sigma\cdot\frac{ik'\wedge\epsilon_\mu(k')\,\zeta(|k|)}{2\pi|k'|^{\frac12}}\,\frac{1}{|k|+|k|^2}\,\sigma\cdot\frac{ik\wedge\epsilon_\mu(k)}{2\pi|k|^{\frac12}} + \text{symmetric}\right| \\ &\le c\left(\frac{|k'|^{\frac12}}{|k|^{\frac12}} + \frac{|k|^{\frac12}}{|k'|^{\frac12}}\right) \le c\,, \end{aligned} \tag{61}$$

where the symmetric expression is with respect to $(k,\,\lambda)$ and $(k',\,\mu)$, and the constants c are independent of the variables and of the parameters α and γ_1. On the other hand, we have

$$
\begin{aligned}
&\left|\left(A^+\cdot P_f(H_f+P_f^2)^{-1}\sigma\cdot B^+\Omega_0\right)(k,\lambda;\,k',\mu)\right| \\
&= \frac{1}{\sqrt{2}}\left|\frac{\epsilon_\mu(k')\cdot k'}{2\pi|k'|^{\frac12}}\,\frac{1}{|k|+|k|^2}\,\sigma\cdot\frac{ik\wedge\epsilon_\lambda(k)}{2\pi|k|^{\frac12}}+\text{ symmetric }\right|. \qquad (62)
\end{aligned}
$$

Picking $S_2 = \{(k,\,k')\mid \frac{k_2'k_2+k_1'k_1}{\sqrt{k_1^2+k_2^2}\sqrt{{k_1'}^2+{k_2'}^2}}\}$, where For $k=(k_1,\,k_2,\,k_3)$ and $k'=(k_1',\,k_2',\,k_3')$, we obtain, for $\lambda=\mu=1$ and for k and k' in S_2,

$$
\begin{aligned}
&\left|\left(A^+\cdot P_f(H_f+P_f^2)^{-1}\sigma\cdot B^+\Omega_0\right)(k,1;\,k',1)\right| \\
&\geq c\left|\frac{1}{|k'|^{\frac12}|k|^{\frac12}}\,\frac{k_2'k_2+k_1'k_1}{\sqrt{k_1^2+k_2^2}\sqrt{{k_1'}^2+{k_2'}^2}}\right| \geq c\frac{1}{|k'|^{\frac12}|k|^{\frac12}}\,, \qquad (63)
\end{aligned}
$$

where again the constants are independent of k, k', γ_1 and α. Therefore, for any $\delta>0$ there exists $\epsilon>0$ such that for $S_3(\epsilon)=\{(k,\,k')\mid |k|\leq\epsilon,\,|k'|\leq\epsilon\}$, we have, for all α small enough and all $\gamma_1\in(\frac12,\,\frac32)$, that for all $(k,\,k')\in S_1\cap S_2\cap S_3(\epsilon)$, which is of positive Lebesgue measure, $|\Gamma_2^{(\gamma_1)}(k,1;\,k',1)|>\delta$. This concludes the proof of the existence of the uniform lower bound δ_1 for $\|\Gamma_2^{(\gamma_1)}\|_*$.
The proof of the upper bound for $\|\Gamma_2^{(\gamma_1)}\|_*$ is straightforward. □

Lemma 7.2. *We have*

$$
\|H_f^{\frac12}\Gamma_1\|<\infty\,,\quad 0<\|\Gamma_1\|_* \quad \textit{and} \quad 0<\|\Gamma_1\|\,.
$$

Proof. Straightforward computations. □

We end this appendix by recalling a useful result due to Chen and Fröhlich [8], which we reproduce below in the case of total momentum $p=0$, which is the case of interest for us.

Lemma 7.3. [8, Proposition 5.1] *For any normalized state ψ in the eigenspace associated to the ground state energy of $H(0)$, there exists $c>0$ such that*

$$
\|a_\lambda(k)\psi\|\leq\frac{c\sqrt{\alpha}}{|k|}\zeta(|k|)\,.
$$

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