Equivariant Cobordism of Schemes

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Abstract. Let k be a field of characteristic zero. For a linear algebraic group G over k acting on a scheme X, we define the equivariant algebraic cobordism of X and establish its basic properties. We explicitly describe the relation of equivariant cobordism with equivariant Chow groups, K-groups and complex cobordism.

We show that the rational equivariant cobordism of a G-scheme can be expressed as the Weyl group invariants of the equivariant cobordism for the action of a maximal torus of G. As applications, we show that the rational algebraic cobordism of the classifying space of a complex linear algebraic group is isomorphic to its complex cobordism.

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1. Introduction

Let k be a field of characteristic zero. Based on the construction of the motivic algebraic cobordism spectrum MGL by Voevodsky, Levine and Morel [31] gave a geometric construction of the algebraic cobordism and showed that this is a universal oriented Borel-Moore homology theory in the category of varieties over the field k. Their definition was extended by Deshpande [9] in the equivariant set-up that led to the notion of the equivariant cobordism of smooth varieties acted upon by linear algebraic groups. This in particular allowed one to define the algebraic cobordism of the classifying spaces analogous to their complex cobordism.
Apart from its many applications in the equivariant set up which are parallel to the ones in the non-equivariant world, an equivariant cohomology theory often leads to the description of the corresponding non-equivariant cohomology by mixing the geometry of the variety with the representation theory of the underlying groups.

Our aim in this first part of a series of papers is to develop the theory of equivariant cobordism in the category of all k-schemes with action of a linear algebraic group. We establish the fundamental properties of this theory and give applications. In the second part [20] of this series, we shall give many important applications of the results of this paper. Some further applications of the results of this paper to the computation of the non-equivariant cobordism rings appear in [21], [24] and [15]. We now describe some of the main results in of this paper.

Let G be a linear algebraic group over k. In this paper, a scheme will mean a quasi-projective k-scheme and all G-actions will be assumed to be linear. If X is a smooth scheme with a G-action, Deshpande defined the equivariant cobordism $\Omega^G_*(X)$ using the coniveau filtration on the Levine-Morel cobordism of certain smooth mixed spaces. This was based on the construction of the Chow groups of classifying spaces in [39] and the equivariant Chow groups in [10].

Using a *niveau* filtration on the algebraic cobordism, which is based on the analogous filtration on any Borel-Moore homology theory as described in [2, Section 3], we define the equivariant algebraic cobordism of any k-scheme with G-action in Section 4. This is defined by taking a projective limit over the quotients of the Levine-Morel cobordism of certain mixed spaces by various levels of the niveau filtration. In order to make sense of this construction, one needs to prove various properties of the above niveau filtration which is done in Section 3. These equivariant cobordism groups coincide with the one in [9] for smooth schemes. We also show in Section 5 how one can recover the formula for the cobordism group of certain classifying spaces directly from the above definition, by choosing suitable models for the underlying mixed spaces.

In Section 5, we establish the basic properties such as functoriality, homotopy invariance, exterior product, projection formula and existence of Chern classes for equivariant vector bundles in Theorem 5.2. Although we do not have the equivariant version of the localization sequence for the algebraic cobordism, we shall show that the restriction map induced by a G-equivariant open immersion is indeed surjective.

In Section 7, we show how the equivariant cobordism is related to other equivariant cohomology theories such as equivariant Chow groups, equivariant K-groups and equivariant complex cobordism. Using some properties of the niveau filtration and known relation between the non-equivariant cobordism and Chow groups, we deduce an explicit formula (*cf.* Proposition 7.2) which relates the equivariant cobordism and the equivariant Chow groups of k-schemes. Using this and the main results of [17], we give a formula in Theorem 7.4 which relates the equivariant cobordism with the equivariant K-theory of smooth schemes. We also construct a natural transformation from the algebraic to the equivariant version of the complex cobordism for schemes over the field of complex numbers.

Our next main result of this paper is Theorem 8.6, where we show that for a connected linear algebraic group G acting on a scheme X, there is a canonical isomorphism $\Omega_*^G(X) \xrightarrow{\cong} \left(\Omega_*^T(X)\right)^W$ with rational coefficients, where T is a split maximal torus of a Levi subgroup of G with Weyl group W. This is mainly achieved by the Morita isomorphism of Proposition 5.4 and a detour to the motivic cobordism MGL and its extension MGL' to singular schemes by Levine [28]. The use of MGL'-theory in our context is motivated by the recent comparison result of Levine [29] which shows that the Levine-Morel cobordism theory is a piece of the more general MGL'-theory.

As an easy consequence of Proposition 7.2, we recover Totaro's cycle class map (*cf.* [39])

$$CH^*(BG) \to MU^*(BG) \otimes_{\mathbb{L}} \mathbb{Z} \to H^*(BG, \mathbb{Z})$$

for a complex linear algebraic group G. It is conjectured that this map is an isomorphism of rings. This conjecture has been shown to be true by Totaro for some classical groups such as BGL_n, O_n, Sp_{2n} and SO_{2n+1}. Although, we can not say anything about this conjecture here, we do show as a consequence of Theorem 8.6 that the map $CH^*(BG) \to MU^*(BG) \otimes_{\mathbb{L}} \mathbb{Z}$ is indeed an isomorphism of rings with the rational coefficients (see Theorem 8.9 for the full statement). We do this by first showing that there is a natural ring homomorphism $\Omega^*(BG) \to MU^*(BG)$ (with integer coefficients) which lifts Totaro's map. We then show that this map is in fact an isomorphism with rational coefficients using Theorem 8.6.

We now make a remark on our definition and the notation for the equivariant cobordism groups. In many of the topology texts, the cobordism rings of classifying spaces are expressed as rings which are complete. For example, one often writes $MU^*(\mathbb{C}\mathrm{P}^\infty)$ as the formal power series $\mathbb{L}[[t]]$ instead of the graded power series ring. This does not allow one to write an expression of the cobordism in each degree. Since our interest is to give an expression of the cobordism groups in each component, we shall express the equivariant cobordism of a smooth scheme as a graded ring. This notation has been used earlier by other authors in the topological context (see [25, Section 2], [27]). We refer the reader to Subsection 6.1 for more about the comparison between the two notations.

We end this introduction with the following comment. One of the initial motivations for this article was to find a definition of the equivariant algebraic cobordism which has all the expected properties of an equivariant cohomology theory and which is simultaneously, simple enough to compute. Although much of this objective is achieved, the equivariant cobordism as considered here has two drawbacks. The first one is that there is not always a natural map from the algebraic to the complex equivariant cobordism for complex varieties with group action (*cf.* Proposition 7.5). A more serious problem is the lack of the localization sequence (*cf.* Proposition 5.3). One way to take care of these two problems is to consider Voevodsky's motivic cobordism MGL from the equivariant point of view. It turns out that although this approach does have certain advantages, it becomes computationally much harder. So the challenge is to study and analyze the situations when these two approaches yield the same answer so that one can use either of the two, depending on what one would like to prove. These questions will be studied in further detail in [23].

2. Recollection of algebraic cobordism

In this section, we briefly recall the definition of algebraic cobordism of Levine-Morel. We also recall the other definition of this object as given by Levine-Pandharipande. Since we shall be concerned with the study of schemes with group actions and the associated quotient schemes, and since such quotients often require the original scheme to be quasi-projective, we shall assume throughout this paper that all schemes over k are quasi-projective.

Notations. We shall denote the category of quasi-projective k-schemes by $\mathcal{V}_k$. By a scheme, we shall mean an object of $\mathcal{V}_k$. The category of smooth quasi-projective schemes will be denoted by $\mathcal{V}^S_k$. If G is a linear algebraic group over k, we shall denote the category of quasi-projective k-schemes with a G-action and G-equivariant maps by $\mathcal{V}_G$. The associated category of smooth G-schemes will be denoted by $\mathcal{V}^S_G$. All G-actions in this paper will be assumed to be linear. Recall that this means that all G-schemes are assumed to admit G-equivariant ample line bundles. This assumption is always satisfied for normal schemes (*cf.* [36, Theorem 2.5], [37, 5.7]).

2.1. Algebraic cobordism. Before we define the algebraic cobordism, we recall the Lazard ring $\mathbb{L}$. It is a polynomial ring over $\mathbb{Z}$ on infinite but countably many variables and is given by the quotient of the polynomial ring $\mathbb{Z}[A_{ij} | (i,j) \in \mathbb{N}^2]$ by the relations, which uniquely define the universal formal group law $F_{\mathbb{L}}$ of rank one on $\mathbb{L}$. This formal group law is given by the power series

$$F_{\mathbb{L}}(u,v) = u + v + \sum_{i,j \ge 1} a_{ij} u^i v^j,$$

where a_{ij} is the equivalence class of A_{ij} in the ring $\mathbb{L}$. The Lazard ring is graded by setting the degree of a_{ij} to be $1-i-j$. In particular, one has $\mathbb{L}_0 = \mathbb{Z}, \mathbb{L}_{-1} = \mathbb{Z}a_{11}$ and $\mathbb{L}_i = 0$ for $i \ge 1$, that is, $\mathbb{L}$ is non-positively graded. We shall write $\mathbb{L}_*$ for the graded ring such that $\mathbb{L}_{*,i} = \mathbb{L}_{-i}$ for $i \in \mathbb{Z}$. We now define the algebraic cobordism of Levine and Morel [31].

Let X be an equi-dimensional k-scheme. A cobordism cycle over X is a family $\alpha = [Y \xrightarrow{f} X, L_1, \cdots, L_r]$, where Y is a smooth scheme, the map f is projective, and L_i's are line bundles on Y. Here, one allows the set of line bundles to be empty. The degree of such a cobordism cycle is defined to be $\deg(\alpha) = \dim_k(Y) - r$ and its codimension is defined to be $\dim(X) - \deg(\alpha)$. Let $\mathcal{Z}^*(X)$ be the free abelian group generated by the cobordism cycles of the above type. Note that this group is graded by the codimension of the cycles. In particular, for $j \in \mathbb{Z}$, $\mathcal{Z}^j(X)$ is the free abelian group on cobordism cycles $\alpha = [Y \xrightarrow{f} X, L_1, \cdots, L_r]$, where Y is smooth and irreducible and codimension of α is j. We impose several relations on $\mathcal{Z}^*(X)$ in order to define the algebraic cobordism group. The first among these is the so called *dimension axiom*: let $\mathcal{R}^*_{\dim}(X)$ be the graded subgroup of $\mathcal{Z}^*(X)$ generated by the cobordism cycles $\alpha = [Y \xrightarrow{f} X, L_1, \cdots, L_r]$ such that $\dim_k Y < r$. Let

$$\mathcal{Z}^*_{\dim}(X) = \frac{\mathcal{Z}^*(X)}{\mathcal{R}^*_{\dim}(X)}.$$

For a line bundle L on X and cobordism cycle α as above, we define the Chern class operator on $\mathcal{Z}^*_{\dim}(X)$ by letting $c_1(L)(\alpha) = [Y \xrightarrow{f} X, L_1, \cdots, L_r, f^*(L)]$. Next, we impose the so called *section axiom*. Let $\mathcal{R}^*_{\rm sec}(X)$ be the graded subgroup of $\mathcal{Z}^*_{\dim}(X)$ generated by cobordism cycles of the form $[Y \to X, L] - [Z \to X]$, where $Y \xrightarrow{s} L$ is a section of the line bundle L on Y which is transverse to the zero-section, and $Z \hookrightarrow Y$ is the closed subvariety of Y defined by the zeros of s. The transversality of s ensures that Z is a smooth variety. In particular, $[Z \to X]$ is a well-defined cobordism cycle on X. Define

$$\underline{\Omega}^*(X) = \frac{\mathcal{Z}^*_{\dim}(X)}{\mathcal{R}^*_{\rm sec}(X)}.$$

The assignment $X \mapsto \underline{\Omega}^*(X)$ is called the pre-cobordism theory.
Finally, we impose the *formal group law* on the cobordism using the following relation. For X as above, let $\mathcal{R}^*_{\rm FGL}(X) \subset \mathbb{L}\otimes_{\mathbb{Z}}\underline{\Omega}^*(X)$ be the graded $\mathbb{L}$-submodule generated by elements of the form

$$\left\{F_{\mathbb{L}}\left(c_1(L), c_1(M)\right)(x) - c_1(L \otimes M)(x) | x \in \underline{\Omega}^*(X), L, M \in \mathrm{Pic}(X)\right\}.$$

We define the *algebraic cobordism* group of X by

$$\Omega^*(X) = \frac{\mathbb{L}\otimes_{\mathbb{Z}}\underline{\Omega}^*(X)}{\mathcal{R}^*_{\rm FGL}(X)}. \tag{2.1}$$

If X is not necessarily equi-dimensional, we define $\mathcal{Z}_*(X)$ to be same as $\mathcal{Z}^*(X)$ except that $\mathcal{Z}_*(X)$ is now graded by the degree of the cobordism cycles. In particular, $\mathcal{Z}_i(X)$ is the free abelian group on cobordism cycles $[Y \xrightarrow{f} X, L_1, \cdots, L_r]$ such that f is projective and Y is smooth and irreducible such that $\dim(Y) - r = i$. One then defines $\Omega_*(X)$ to be the quotient of $\mathbb{L}_* \otimes_{\mathbb{Z}} \underline{\Omega}_*(X)$ in the same way as above. Note that for X equi-dimensional of dimension d and $i \in \mathbb{Z}$, one has $\Omega^i(X) \cong \Omega_{d-i}(X)$.
Observe that $\Omega^*(X)$ is a graded $\mathbb{L}$-module such that $\Omega^j(X) = 0$ for $j > \dim(X)$ and $\Omega^j(X)$ can be non-zero for any given $-\infty < j \le \dim(X)$. Similarly, $\Omega_*(X)$ is a graded $\mathbb{L}_*$-module which has no component in the negative degrees and it can be non-zero in arbitrarily large positive degree.
The following is the main result of Levine and Morel from which most of their other results on algebraic cobordism are deduced. We refer to *loc. cit.* for more properties.

Theorem 2.1. *The functor $X \mapsto \Omega_*(X)$ is the universal Borel-Moore homology on the category $\mathcal{V}_k$. In other words, it is universal among the homology theories on $\mathcal{V}_k$ which have functorial push-forward for projective morphism, pull-back for smooth morphism (any morphism of smooth schemes), Chern classes for line bundles, and which satisfy Projective bundle formula, homotopy invariance,* *the above dimension, section and formal group law axioms. Moreover, for a k-scheme X and closed subscheme Z of X with open complement U, there is a localization exact sequence*

$$\Omega_*(Z) \to \Omega_*(X) \to \Omega_*(U) \to 0.$$

It was also shown in *loc. cit.* that the natural composite map

$$\Phi : \mathbb{L} \to \mathbb{L} \otimes_{\mathbb{Z}} \underline{\Omega}^*(k) \twoheadrightarrow \Omega^*(k)$$
$$a \mapsto [a]$$

is an isomorphism of commutative graded rings.
As an immediate corollary of Theorem 2.1, we see that for a smooth k-scheme X and an embedding $\sigma : k \to \mathbb{C}$, there is a natural morphism of graded rings

$$\Phi^{top}_X : \Omega^*(X) \to MU^{2*}(X_\sigma(\mathbb{C})), \tag{2.2}$$

where $MU^*(X_\sigma(\mathbb{C}))$ is the complex cobordism ring of the complex manifold $X_\sigma(\mathbb{C})$ given by the complex points of $X \times_k \mathbb{C}$. This map is an isomorphism for $X = \mathrm{Spec}(k)$. In particular, there are isomorphisms of graded rings

$$\mathbb{L} \xrightarrow{\cong} \Omega^*(k) \xrightarrow{\cong} MU^{2*} \xrightarrow{\cong} MU^*, \tag{2.3}$$

where MU^* is the complex cobordism ring of a point. As a corollary, we see that for any field extension $k \hookrightarrow K$, the natural map $\Omega^*(k) \to \Omega^*(K)$ is an isomorphism.

2.2. Cobordism via double point degeneration. To enforce the formal group law on the algebraic cobordism in order to make it an oriented cohomology theory on the category of smooth varieties, Levine and Morel artificially imposed this condition by tensoring their pre-cobordism theory with the Lazard ring. Although they were able to show that the resulting map $\mathcal{Z}_*(X) \to \Omega_*(X)$ is still surjective, they were unable to describe the explicit geometric relations in $\mathcal{Z}_*(X)$ that define $\Omega_*(X)$. This was subsequently accomplished by Levine-Pandharipande [32]. We conclude our introduction to the algebraic cobordism by briefly discussing the construction of Levine-Pandharipande. For $n \geq 1$, let $\square^n$ denote the space $(\mathbb{P}^1_k - \{1\})^n$.

Definition 2.2. A morphism $Y \xrightarrow{\pi} \square^1$ is called a *double point degeneration*, if Y is a smooth scheme and $\pi^{-1}(0)$ is scheme-theoretically given as the union $A \cup B$, where A and B are smooth divisors on Y which intersect transversely. The intersection $D = A \cap B$ is called the double point locus of π. Here, A, B and D are allowed to be disconnected or, even empty.

For a double point degeneration as above, notice that the scheme D is also smooth and $\mathcal{O}_D(A+B)$ is trivial. In particular, one sees that $N_{A/D} \otimes_D N_{B/D} \cong \mathcal{O}_D$. This is turn implies that the projective bundles $\mathbb{P}(\mathcal{O}_D \oplus N_{A/D}) \to D$ and $\mathbb{P}(\mathcal{O}_D \oplus N_{B/D}) \to D$ are isomorphic, where $N_{A/D}$ and $N_{B/D}$ are the normal bundles of D in A and B respectively. Let $\mathbb{P}(\pi) \to D$ denote any of these two projective bundles.

Let X be a k-scheme and let $Y \xrightarrow{f} X \times \square^1$ be a projective morphism from a smooth scheme Y. Assume that the composite map $\pi : Y \to X \times \square^1 \to \square^1$ is a double point degeneration such that $Y_\infty = \pi^{-1}(\infty)$ is smooth. We define the cobordism cycle on X associated to the morphism f to be the cycle

$$C(f) = [Y_\infty \to X] - [A \to X] - [B \to X] + [\mathbb{P}(\pi) \to X]. \tag{2.4}$$

Let $\mathcal{M}_*(X)$ be the free abelian group on the isomorphism classes of the morphisms $[Y \xrightarrow{f} X]$, where Y is smooth and irreducible and f is projective. Then $\mathcal{M}_*(X)$ is a graded abelian group, where the grading is by the dimension of Y. Let $\mathcal{R}_*(X)$ be the subgroup of $\mathcal{M}_*(X)$ generated by all cobordism cycles $C(f)$, where $C(f)$ is as in (2.4). Note that $\mathcal{R}_*(X)$ is a graded subgroup of $\mathcal{M}_*(X)$. Define

$$\omega_*(X) = \frac{\mathcal{M}_*(X)}{\mathcal{R}_*(X)}. \tag{2.5}$$

Theorem 2.3 ([32]). *There is a canonical isomorphism*

$$\omega_*(X) \xrightarrow{\cong} \Omega_*(X) \tag{2.6}$$

of oriented Borel-Moore homology theories on $\mathcal{V}$.

3. Niveau filtration on algebraic cobordism

In this section, we introduce the niveau filtration on the algebraic cobordism which plays an important role in the definition of the equivariant algebraic cobordism. Our main result here is a refined localization sequence for the cobordism which preserves the niveau filtration. This new localization sequence will have interesting consequences in the study of the equivariant cobordism.

Let X be a k-scheme of dimension d. For $j \in \mathbb{Z}$, let Z_j be the set of all closed subschemes $Z \subset X$ such that $\dim_k(Z) \le j$ (we assume $\dim(\emptyset) = -\infty$). The set Z_j is then ordered by the inclusion. For $i \ge 0$, we define

$$\Omega_i(Z_j) = \varinjlim_{Z \in Z_j} \Omega_i(Z) \quad \text{and} \quad \text{put}$$

$$\Omega_*(Z_j) = \bigoplus_{i \ge 0} \Omega_i(Z_j).$$

It is immediate that $\Omega_*(Z_j)$ is a graded $\mathbb{L}_*$-module and there is a graded $\mathbb{L}_*$-linear map $\Omega_*(Z_j) \to \Omega_*(X)$.

Following [2, Section 3], we let Z_j/Z_{j-1} denote the ordered set of pairs $(Z, Z') \in Z_j \times Z_{j-1}$ such that $Z' \subset Z$ with the ordering

$$(Z, Z') \ge (Z_1, Z'_1) \quad \text{if} \quad Z_1 \subseteq Z \quad \text{and} \quad Z'_1 \subseteq Z'.$$

If $(Z, Z') \ge (Z_1, Z'_1)$, then the functoriality of the push-forward maps and the localization sequence yield a map $\Omega_i(Z_1 - Z'_1) \to \Omega_i(Z - Z')$ (*cf.* (3.1)). We let

$$\Omega_i\left(Z_j/Z_{j-1}(X)\right) := \varinjlim_{(Z,Z') \in Z_j/Z_{j-1}} \Omega_i(Z - Z').$$

Lemma 3.1. *For $f : X' \to X$ projective, the push-forward map $\Omega_*(X') \xrightarrow{f_*} \Omega_*(X)$ induces a push-forward map $\Omega_*\left(Z_j/Z_{j-1}(X')\right) \to \Omega_*\left(Z_j/Z_{j-1}(X)\right)$.*

Proof. Let $(Z, Z') \in Z_j/Z_{j-1}(X')$. Then $(W, W') = (\mathrm{Im}(Z), \mathrm{Im}(Z')) \in Z_j/Z_{j-1}(X)$. It suffices now to show that f_* induces a natural map $\Omega_*(Z - Z') \to \Omega_*(W - W')$. However, this follows directly from the localization exact sequences

$$\begin{tikzcd} \Omega_*(Z') \arrow[r] \arrow[d, "f_*"'] & \Omega_*(Z) \arrow[r] \arrow[d, "f_*"] & \Omega_*(Z - Z') \arrow[r] \arrow[d, dotted] & 0 \\ \Omega_*(W') \arrow[r] & \Omega_*(W) \arrow[r] & \Omega_*(W - W') \arrow[r] & 0 \end{tikzcd} \tag{3.1}$$

and the fact that the square on the left is commutative. □

For $x \in Z_j$, let

$$\widetilde{\Omega_*(k(x))} = \varinjlim_{U \subseteq \overline{\{x\}}} \Omega_*(U), \tag{3.2}$$

where the limit is taken over all non-empty open subsets of $\overline{\{x\}}$. Taking the limit over the localization sequences

$$\Omega_*(Z') \to \Omega_*(Z) \to \Omega_*(Z - Z') \to 0$$

for $(Z, Z') \in Z_j/Z_{j-1}$, one now gets an exact sequence

$$\Omega_*(Z_{j-1}) \to \Omega_*(Z_j) \to \bigoplus_{x \in (Z_j - Z_{j-1})} \widetilde{\Omega_*(k(x))} \to 0. \tag{3.3}$$

Definition 3.2. We define $F_j\Omega_*(X)$ to be the image of the natural $\mathbb{L}_*$-linear map $\Omega_*(Z_j) \to \Omega_*(X)$. In other words, $F_j\Omega_*(X)$ is the image of all $\Omega_*(W) \to \Omega_*(X)$, where $W \to X$ is a projective map such that $\dim(\mathrm{Image}(W)) \le j$. Using the localization sequence, this is same as saying that $F_j\Omega_*(X)$ is the set of all elements $s \in \Omega_*(X)$ such that $i^*(s) = 0$ for some open subset $i : U \hookrightarrow X$, whose complement has dimension at most j.

One checks at once that there is a canonical *niveau filtration*

$$0 = F_{-1}\Omega_*(X) \subseteq F_0\Omega_*(X) \subseteq \cdots \subseteq F_{d-1}\Omega_*(X) \subseteq F_d\Omega_*(X) = \Omega_*(X). \tag{3.4}$$

Lemma 3.3. *If $f : X' \to X$ is a projective morphism, then $f_*\left(F_j\Omega_*(X')\right) \subseteq F_j\Omega_*(X)$. If $g : X' \to X$ is a smooth morphism of relative dimension r, then $g^*\left(F_j\Omega_*(X)\right) \subseteq F_{j+r}\Omega_*(X')$.*

Proof. The first assertion is obvious from the definition. In fact, the push-forward map preserves the niveau filtration at the level of the free abelian groups of cobordism cycles. The second assertion also follows immediately using the fact that for a cobordism cycle $[Y \to X]$, one has $g^*([Y \to X]) =$

$[Y \times_X X' \to X']$. This in turn implies that $g^* \circ f_* = f'_* \circ g'^*$ for a Cartesian square

$$\begin{array}{ccc} W' & \xrightarrow{f'} & X' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ W & \xrightarrow[f]{} & X \end{array}$$

such that f is projective and g is smooth. ☐

PROPOSITION 3.4. *Let X be a k-scheme and let Z be a closed subscheme of X with the complement U. Then for every $j \geq \dim(Z)$, there is an exact sequence*

$$\Omega_*(Z) \to F_j\Omega_*(X) \to F_j\Omega_*(U) \to 0.$$

Proof. Since $j \geq \dim(Z)$, we see that the image of the map $\Omega_*(Z) \to \Omega_*(X)$ lies in $F_j\Omega_*(X)$. Using the localization sequence of the algebraic cobordism (*cf.* Theorem 2.1), we only need to show that the map $F_j\Omega_*(X) \to F_j\Omega_*(U)$ is surjective.

Let $F_j\mathcal{Z}_*(X)$ be the free abelian group on cobordism cycles $[Y \xrightarrow{f} X]$ such that Y is irreducible and $\dim(f(Y)) \leq j$. Note that $f(Y)$ is a closed and irreducible subscheme of X since Y is irreducible and f is projective. It is then clear that $F_j\mathcal{Z}_*(X) \subset \mathcal{Z}_*(X)$ and $F_j\mathcal{Z}_*(X) \twoheadrightarrow F_j\Omega_*(X)$.

Let $[Y \xrightarrow{f} U]$ be a cobordism cycle on U such that Y is smooth and irreducible, f is projective and $\dim(f(Y)) \leq j$. We have a factorization $Y \hookrightarrow \mathbb{P}^n_k \times U \to U$ where the first inclusion is a closed immersion. Let $\overline{Y}$ denote a resolution of singularities of the closure of Y in $\mathbb{P}^n_k \times X$ and let $\overline{Y} \xrightarrow{\bar{f}} X$ be the projection map. It is then easy to verify that $[\overline{Y} \xrightarrow{\bar{f}} X]$ is a cobordism cycle on X which restricts to $[Y \xrightarrow{f} U]$ in $\Omega_*(U)$ and $\dim(\bar{f}(\overline{Y})) = \dim(f(Y)) \leq j$. This proves the required surjection. ☐

THEOREM 3.5. *Let X be a k-scheme and let Z be a closed subscheme of X with the complement U. Then for every $j \in \mathbb{Z}$, there is an exact sequence*

$$\frac{\Omega_*(Z)}{F_j\Omega_*(Z)} \to \frac{\Omega_*(X)}{F_j\Omega_*(X)} \to \frac{\Omega_*(U)}{F_j\Omega_*(U)} \to 0. \tag{3.5}$$

Proof. Let $f : U \to X$ and $g : Z \to X$ be the inclusion maps. The surjectivity of the second map in (3.5) follows from the localization sequence of algebraic cobordism (*cf.* Theorem 2.1). It suffices thus to show that

$$f^{*(-1)}\left(F_j\Omega_*(U)\right) \subseteq \mathrm{Image}\left(\Omega_*(Z) \oplus F_j\Omega_*(X) \to \Omega_*(X)\right) \tag{3.6}$$

in order to prove the theorem.

So let $\alpha \in \Omega_*(X)$ be such that $\beta = f^*(\alpha) \in F_j\Omega_*(U)$. We can find a closed subscheme $q : W \hookrightarrow U$ of dimension at most j and a cobordism cycle $\beta' \in \Omega_*(W)$ such that $\beta = q_*(\beta')$. Let Y be the closure of W in X and let $W \xrightarrow{f'} Y \xrightarrow{p} X$ be the open and the closed immersions.

Using Theorem 2.1 (the localization sequence), we can find $\alpha' \in \Omega_*(Y)$ such that $\beta' = f'^*(\alpha')$. We conclude from this that $f^*(\alpha - p_*(\alpha')) = 0$ in $\Omega_*(U)$. Using Theorem 2.1 (the localization sequence) again, we see that $\alpha = g_*(\gamma) + p_*(\alpha')$ for some $\gamma \in \Omega_*(Z)$. Since $\dim(Y) = \dim(W) \le j$, it also follows that $p_*(\alpha') \in F_j\Omega_*(X)$. This proves (3.6) and hence the theorem. □

The following is an immediate consequence of Theorem 3.5.

Corollary 3.6. *Let X be a k-scheme. Then for any $j \ge 0$ and any closed subscheme $Z \subset X$ of dimension at most j, the natural map $\Omega_*(X) \to \Omega_*(X-Z)$ induces an isomorphism*

$$\frac{\Omega_*(X)}{F_j\Omega_*(X)} \xrightarrow{\cong} \frac{\Omega_*(X-Z)}{F_j\Omega_*(X-Z)}.$$

Lemma 3.7. *For a k-scheme X and $i \ge 0$, the natural map $\Omega_i(X) \to \mathrm{CH}_i(X)$ has the factorization*

$$\Omega_i(X) \to \frac{\Omega_i(X)}{F_{i-1}\Omega_i(X)} \to \mathrm{CH}_i(X).$$

Proof. By Theorem 2.3, $\Omega_*(X)$ is generated by the cobordism cycles $[Y \to X]$, where Y is smooth and f is projective. It follows from the definition of the niveau filtration that $F_j\Omega_*(X)$ is generated by the cobordism cycles of the form $i_*([Y \to Z])$, where $Z \xhookrightarrow{\phi} X$ is a closed subscheme of X of dimension at most j. Since $\Omega_* \to \mathrm{CH}_*$ is a natural transformation of oriented Borel-Moore homology theories, we get a commutative diagram

$$\begin{tikzcd}
\Omega_i(Z) \arrow[r] \arrow[d, "\phi_*"'] & \mathrm{CH}_i(Z) \arrow[d, "\phi_*"] \\
\Omega_i(X) \arrow[r] & \mathrm{CH}_i(X).
\end{tikzcd}$$

The lemma now follows from the fact that $\mathrm{CH}_i(Z) = 0$ if $j \le i-1$. □

Lemma 3.8. *For any $s \in F_j\Omega_*(X)$, there are elements $a_i \in \mathbb{L}_*$ and $s_i \in \Omega_{\le j}(X)$ such that $s = \sum_i a_i s_i$.*

Proof. It is a simple variant of the generalized degree formula [30, Theorem 4.7]. We can assume that s is a homogeneous element of $\Omega_*(X)$. We have seen in the proof of Proposition 3.4 that $F_j\mathcal{Z}_*(X)$ a free abelian subgroup of $\mathcal{Z}_*(X)$ such that $F_j\mathcal{Z}_*(X) \twoheadrightarrow F_j\Omega_*(X)$. We can thus assume that $s = [Y \xrightarrow{f} X]$, where Y is smooth and irreducible and f is projective such that $\dim(f(Y)) \le j$.
Let $\iota : f(Y) = W \hookrightarrow X$ denote the inclusion of the closed subset and let U be the complement of W in X. Then the image of s dies in $\Omega_*(U)$ under the restriction map. It follows from the localization sequence of the algebraic cobordism (*cf.* Theorem 2.1) and [30, Theorem 4.7] that we can write $s = \iota_*\left(a[\widetilde{W} \to W] + \sum_i a_i[\widetilde{Z_i} \to Z_i]\right)$, where $\widetilde{W} \to W$ is a resolution of singularities of W, $\widetilde{Z_i} \to Z_i$ is a resolution of singularities of a closed subscheme Z_i

of W of dimension strictly less than that of W and $a, a_i \in \mathbb{L}_*$. It follows from this expression that $s = \sum_i a_i s_i$ such that $s_i \in \Omega_{\le j}(X)$ and $a_i \in \mathbb{L}_*$ $\square$

Proposition 3.9. *Let $E \xrightarrow{f} X$ be a vector bundle of rank r. Then the pull-back map $f^* : \Omega_*(X) \to \Omega_*(E)$ induces an isomorphism*

$$F_j\Omega_*(X) \xrightarrow{\cong} F_{j+r}\Omega_*(E)$$

for all $j \in \mathbb{Z}$. In particular, $F_{<r}\Omega_*(E) = 0$.

Remark 3.10. The reader should be warned that the map f^* shifts the degree of the grading by r.

Proof. Using Lemma 3.8, this can be proved in the same way as [9, Lemma 3.3], where a similar result is proven for smooth varieties and coniveau filtration. We sketch the proof in the singular case.

The homotopy invariance of the algebraic cobordism tells us that the natural map $\Omega_*(X) \xrightarrow{f^*} \Omega_*(E)$ is an isomorphism. So we only need to show that this map is surjective at each level of the niveau filtration. So let $e \in F_{j+r}\Omega_*(E)$. We can assume that $e \in \Omega_i(E)$ is a homogeneous element.

By Lemma 3.8, we can write $e = \sum_p a_p s_p$, where each s_p is a homogeneous element of $\Omega_{\le j+r}(E)$ and $a_p \in \mathbb{L}_*$. Since f^* is an isomorphism of graded abelian groups which shifts the degree by r, we can write $s_p = f^*(x_p)$ such that $x_p \in \Omega_{\le j}(X)$. Letting $x = \sum_p a_p x_p$, we see that $x \in F_j\Omega_*(X)$ and $s = f^*(x)$. This proves the proposition. $\square$

4. Equivariant algebraic cobordism

In this text, G will denote a linear algebraic group of dimension g over k. All representations of G will be finite dimensional. The definition of equivariant cobordism needs one to consider certain kind of mixed spaces which in general may not be a scheme even if the original space is a scheme. The following well known (*cf.* [10, Proposition 23]) lemma shows that this problem does not occur in our context and all the mixed spaces in this paper are schemes with ample line bundles.

Lemma 4.1. *Let H be a linear algebraic group acting freely and linearly on a k-scheme U such that the quotient U/H exists as a quasi-projective variety. Let X be a k-scheme with a linear action of H. Then the mixed quotient $X \stackrel{H}{\times} U$ exists for the diagonal action of H on $X \times U$ and is quasi-projective. Moreover, this quotient is smooth if both U and X are so. In particular, if H is a closed subgroup of a linear algebraic group G and X is a k-scheme with a linear action of H, then the quotient $G \stackrel{H}{\times} X$ is a quasi-projective scheme.*

Proof. It is already shown in [10, Proposition 23] using [12, Proposition 7.1] that the quotient $X \stackrel{H}{\times} U$ is a scheme. Moreover, as U/H is quasi-projective, [12, Proposition 7.1] in fact shows that $X \stackrel{H}{\times} U$ is also quasi-projective. The similar conclusion about $G \overset{H}{\times} X$ follows from the first case by taking $U = G$ and by observing that G/H is a smooth quasi-projective scheme (*cf.* [3, Theorem 6.8]). The assertion about the smoothness is clear since $X \times U \to X \overset{H}{\times} U$ is a principal H-bundle. $\square$

For any integer $j \geq 0$, let V_j be an l-dimensional representation of G and let U_j be a G-invariant open subset of V_j such that the codimension of the complement $(V_j - U_j)$ in V_j is at least j and G acts freely on U_j such that the quotient U_j/G is a quasi-projective scheme. Such a pair (V_j, U_j) will be called a *good* pair for the G-action corresponding to j (*cf.* [17, Section 2]). It is easy to see that a good pair always exists (*cf.* [10, Lemma 9]). Let X_G denote the mixed quotient $X \overset{G}{\times} U_j$ of the product $X \times U_j$ by the diagonal action of G, which is free.
Let X be a k-scheme of dimension d with a G-action. Fix $j \geq 0$ and let (V_j, U_j) be an l-dimensional good pair corresponding to j. For $i \in \mathbb{Z}$, set

$$\Omega^G_i(X)_j = \frac{\Omega_{i+l-g}\left(X \overset{G}{\times} U_j\right)}{F_{d+l-g-j}\Omega_{i+l-g}\left(X \overset{G}{\times} U_j\right)}. \tag{4.1}$$

Lemma 4.2. *For a fixed $j \geq 0$, the group $\Omega^G_i(X)_j$ is independent (in a canonical way) of the choice of the good pair (V_j, U_j).*

Proof. Let (V_j, U_j) and (V'_j, U'_j) be two good pairs of dimensions and l and l' respectively corresponding to j. Using the results of Section 3, one can follow the proof of the similar result for the equivariant Chow groups in [10, Proposition 1] to construct a canonical isomorphism

$$\alpha_{vv'} : \frac{\Omega_{i+l-g}\left(X \overset{G}{\times} U_j\right)}{F_{d+l-g-j}\Omega_{i+l-g}\left(X \overset{G}{\times} U_j\right)} \xrightarrow{\cong} \frac{\Omega_{i+l'-g}\left(X \overset{G}{\times} U'_j\right)}{F_{d+l'-g-j}\Omega_{i+l'-g}\left(X \overset{G}{\times} U'_j\right)}$$

as follows.
We let $V = V_j \oplus V'_j$ and $U = (U_j \oplus V'_j) \cup (V_j \oplus U'_j)$. Let G act diagonally on V. Then it is easy to see that the complement of the open subset $X \overset{G}{\times} (U_j \oplus V'_j)$ in $X \overset{G}{\times} U$ has dimension at most $d + l + l' - g - j$. Hence by Corollary 3.6, the map

$$\frac{\Omega_{i+l+l'-g}\left(X \overset{G}{\times} U\right)}{F_{d+l+l'-g-j}\Omega_{i+l+l'-g}\left(X \overset{G}{\times} U\right)} \xrightarrow{\iota^*_v} \frac{\Omega_{i+l+l'-g}\left(X \overset{G}{\times} (U_j \oplus V'_j)\right)}{F_{d+l+l'-g-j}\Omega_{i+l+l'-g}\left(X \overset{G}{\times} (U_j \oplus V'_j)\right)} \tag{4.2}$$

is an isomorphism. On the other hand, the map $X \overset{G}{\times} (U_j \oplus V'_j) \xrightarrow{\phi_v} X \overset{G}{\times} U_j$ is a vector bundle of rank l' and hence by Proposition 3.9, the map
(4.3)

$$\frac{\Omega_{i+l-g}\left(X \overset{G}{\times} U_j\right)}{F_{d+l-g-j}\Omega_{i+l-g}\left(X \overset{G}{\times} U_j\right)} \xrightarrow{\phi_v^*} \frac{\Omega_{i+l+l'-g}\left(X \overset{G}{\times} (U_j \oplus V'_j)\right)}{F_{d+l+l'-g-j}\Omega_{i+l+l'-g}\left(X \overset{G}{\times} (U_j \oplus V'_j)\right)}$$

is also an isomorphism. Combining the above two isomorphisms, we get the canonical isomorphism

$$(\iota_v^*)^{-1} \circ \phi_v^* : \frac{\Omega_{i+l-g}\left(X \overset{G}{\times} U_j\right)}{F_{d+l-g-j}\Omega_{i+l-g}\left(X \overset{G}{\times} U_j\right)} \xrightarrow{\cong} \frac{\Omega_{i+l+l'-g}\left(X \overset{G}{\times} U\right)}{F_{d+l+l'-g-j}\Omega_{i+l+l'-g}\left(X \overset{G}{\times} U\right)}.$$

In the same way, we also get an isomorphism

$$(\phi_{v'}^*)^{-1} \circ \iota_{v'}^* : \frac{\Omega_{i+l+l'-g}\left(X \overset{G}{\times} U\right)}{F_{d+l+l'-g-j}\Omega_{i+l+l'-g}\left(X \overset{G}{\times} U\right)} \xrightarrow{\cong}$$

$$\xrightarrow{\cong} \frac{\Omega_{i+l'-g}\left(X \overset{G}{\times} U'_j\right)}{F_{d+l'-g-j}\Omega_{i+l'-g}\left(X \overset{G}{\times} U'_j\right)}.$$

The composite $\alpha_{vv'} = ((\phi_{v'}^*)^{-1} \circ \iota_{v'}^*) \circ ((\iota_v^*)^{-1} \circ \phi_v^*)$ is the desired canonical isomorphism (see the proof of [39, Theorem 1.1]). $\square$

Lemma 4.3. *For $j' \geq j \geq 0$, there is a natural surjective map $\Omega_i^G(X)_{j'} \twoheadrightarrow \Omega_i^G(X)_j$.*

Proof. Choose a good pair $(V_{j'}, U_{j'})$ for j'. Then it is clearly a good pair for j too. Moreover, there is a natural surjection

$$\frac{\Omega_{i+l-g}\left(X \overset{G}{\times} U_{j'}\right)}{F_{d+l-g-j'}\Omega_{i+l-g}\left(X \overset{G}{\times} U_{j'}\right)} \twoheadrightarrow \frac{\Omega_{i+l-g}\left(X \overset{G}{\times} U_{j'}\right)}{F_{d+l-g-j}\Omega_{i+l-g}\left(X \overset{G}{\times} U_{j'}\right)}.$$

On the other hand, the left and the right terms are $\Omega_i^G(X)_{j'}$ and $\Omega_i^G(X)_j$ respectively by Lemma 4.2. $\square$

Definition 4.4. Let X be a k-scheme of dimension d with a G-action. For any $i \in \mathbb{Z}$, we define the *equivariant algebraic cobordism* of X to be

$$\Omega_i^G(X) = \varprojlim_j \Omega_i^G(X)_j.$$

The reader should note from the above definition that unlike the ordinary cobordism, the equivariant algebraic cobordism $\Omega^G_i(X)$ can be non-zero for any $i \in \mathbb{Z}$. We set

$$\Omega^G_*(X) = \bigoplus_{i\in\mathbb{Z}} \Omega^G_i(X).$$

If X is an equi-dimensional k-scheme with G-action, we let $\Omega^i_G(X) = \Omega^G_{d-i}(X)$ and $\Omega^*_G(X) = \underset{i\in\mathbb{Z}}{\oplus} \Omega^i_G(X)$. We shall denote the equivariant cobordism $\Omega^*_G(k)$ of the ground field by $S(G)$. This is also called the algebraic cobordism of the classifying space of G and often written as $\Omega^*(BG)$.

Remark 4.5. If G is the trivial group, we can take the good pair (V_j, V_j) for every j where V_j is any l-dimensional k-vector space. In that case, we get $\Omega_{i+l}(X \stackrel{G}{\times} V_j) \cong \Omega_{i+l}(X \times V_j)$ which is isomorphic to $\Omega_i(X)$ by the homotopy invariance of the non-equivariant cobordism. Moreover, $F_{d+l-j}\Omega_{i+l}(X \times V_j)$ is isomorphic to $F_{d-j}\Omega_i(X)$ by Proposition 3.9 and this last term is zero for all large j. In particular, we see from (4.1) and the definition of the equivariant cobordism that there is a canonical isomorphism $\Omega^G_*(X) \xrightarrow{\cong} \Omega_*(X)$.

Remark 4.6. Let X be a G-scheme and let H be a closed normal subgroup of G with quotient W. If (V_j, U_j) is a good pair for the G-action for any $j \geq 0$, then W naturally acts on the mixed quotient $X_j = X \stackrel{H}{\times} U_j$ and hence it acts on $\Omega_*(X_j)$. Since W acts on X_j by automorphisms, it keeps the niveau filtration invariant. In particular, it acts on $\Omega^H_*(X)_j$. It is clear that if (V'_j, U'_j) is another good pair, then the isomorphisms in (4.2) and (4.3) are W-equivariant. In other words, the W-action on $\Omega_*(X_j)$ does not depend on the choice of good pairs. Furthermore, for $j' \geq j$, we can choose a good pair for j' and that makes the maps in the inverse system $\{\Omega_*(X_j)\}_{j\geq 0}$ W-equivariant. We conclude that W acts on the equivariant cobordism $\Omega^H_*(X)$. One example of such a situation is where H is a maximal torus in a linear algebraic group and G is its normalizer. The quotient W is then the Weyl group. In that case, $\Omega^H_*(X)$ becomes a $\mathbb{Z}[W]$-module.

Remark 4.7. It is easy to check from the above definition of the niveau filtration that if X is a smooth and irreducible k-scheme of dimension d, then $F_j\Omega_i(X) = F^{d-j}\Omega^{d-i}(X)$, where $F^\bullet\Omega^*(X)$ is the coniveau filtration used in [9]. Furthermore, one also checks in this case that if G acts on X, then

$$\Omega^i_G(X) = \varprojlim_j \frac{\Omega^i\left(X \stackrel{G}{\times} U_j\right)}{F^j\Omega^i\left(X \stackrel{G}{\times} U_j\right)}, \tag{4.4}$$

where (V_j, U_j) is a good pair corresponding to any $j \geq 0$. Thus the above definition 4.4 of the equivariant cobordism coincides with that of [9] for smooth schemes.

Remark 4.8. As is evident from the above definition (see Example 6.6), the equivariant cobordism $\Omega^G_i(X)$ can not in general be computed in terms of the algebraic cobordism of one single mixed space. This makes these groups more complicated to compute than the equivariant Chow groups, which can be computed in terms of a single mixed space. This also motivates one to ask if the equivariant cobordism can be defined in such a way that they can be calculated using one single mixed space in a given degree. It follows however from Lemmas 4.2 and 4.3 that for a given i and j, each component of the projective system $\left\{\Omega^G_i(X)_j\right\}_{j\ge 0}$ can be computed using a single mixed space.

4.1. Change of groups. If $H \subset G$ is a closed subgroup of dimension h, then any l-dimensional good pair (V_j, U_j) for G-action is also a good pair for the induced H-action. Moreover, for any $X \in \mathcal{V}_G$ of dimension d, $X \overset{H}{\times} U_j \to X \overset{G}{\times} U_j$ is an étale locally trivial G/H-fibration and hence a smooth map (*cf.* [3, Theorem 6.8]) of relative dimension $g-h$. This induces the inverse system of pull-back maps

$$\Omega^G_i(X)_j = \frac{\Omega_{i+l-g}\left(X \overset{G}{\times} U_j\right)}{F_{d+l-g-j}\Omega_{i+l-g}\left(X \overset{G}{\times} U_j\right)} \to h$$

$$\to \frac{\Omega_{i+l-h}\left(X \overset{H}{\times} U_j\right)}{F_{d+l-h-j}\Omega_{i+l-h}\left(X \overset{H}{\times} U_j\right)} = \Omega^H_i(X)_j$$

and hence a natural restriction map

$$r^G_{H,X} : \Omega^G_*(X) \to \Omega^H_*(X). \tag{4.5}$$

Taking $H = \{1\}$ and using Remark 4.5, we get the *forgetful* map

$$r^G_X : \Omega^G_*(X) \to \Omega_*(X) \tag{4.6}$$

from the equivariant to the non-equivariant cobordism. Since $r^G_{H,X}$ is obtained as a pull-back under the smooth map, it commutes with any projective push-forward and smooth pull-back (*cf.* Theorem 5.2). We remark here that although the definition of $r^G_{H,X}$ uses a good pair, it is easy to see as in Lemma 4.2 that it is independent of the choice of such good pairs.

4.2. Fundamental class of cobordism cycles. Let $X \in \mathcal{V}_G$ and let $Y \xrightarrow{f} X$ be a morphism in $\mathcal{V}_G$ such that Y is smooth of dimension d and f is projective. For any $j \ge 0$ and any l-dimensional good pair (V_j, U_j), $[Y_G \xrightarrow{f_G} X_G]$ is an ordinary cobordism cycle of dimension $d + l - g$ by Lemma 5.1 and hence defines an element $\alpha_j \in \Omega^G_d(X)_j$. Moreover, it is evident that the image of $\alpha_{j'}$ is α_j for $j' \ge j$. Hence we get a unique element $\alpha \in \Omega^G_d(X)$, called the *G-equivariant fundamental class* of the cobordism cycle $[Y \xrightarrow{f} X]$. We also see from this more generally that if $[Y \xrightarrow{f} X, L_1, \cdots, L_r]$ is as above with each L_i a G-equivariant line bundle on Y, then this defines a unique class in $\Omega^G_{d-r}(X)$. It is interesting question to ask under what conditions on the group G, the equivariant cobordism group $\Omega^G_*(X)$ is generated by the fundamental classes of G-equivariant cobordism cycles on X. It turns out that this question indeed has a positive answer if G is a split torus by [20, Theorem 4.11].

5. Some properties of equivariant cobordism

In this section, we establish some basic properties of equivariant algebraic cobordism that are analogous to the non-equivariant case. We begin with the following elementary result. This will be used in the sequel for the morphisms between mixed quotients.

LEMMA 5.1. *Let $f : X \to Y$ be a projective G-equivariant map in $\mathcal{V}_G$ with free G-actions such that Y/G is quasi-projective. Then $X/G \in \mathcal{V}_k$ and the induced map $\overline{f} : X/G \to Y/G$ of quotients is projective.*

Proof. It follows from our assumption and [12, Proposition 7.1] that X/G exists and that $\overline{f} : X' = X/G \to Y/G = Y'$ is a morphism in $\mathcal{V}_k$. Furthermore, the square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X' \arrow[d, "\overline{f}"] \\ Y \arrow[r] & Y' \end{tikzcd}$$

is Cartesian. Since both the horizontal maps are the principal G-bundles, they are smooth and surjective. Since proper maps have smooth descent (in fact fpqc descent), we see that $\overline{f} : X' \to Y'$ is proper. Since these schemes are quasi-projective, we leave it as an exercise to show that $\overline{f}$ is also quasi-projective and hence must be projective. $\square$

THEOREM 5.2. *The equivariant algebraic cobordism satisfies the following properties.*

(i) Functoriality : The assignment $X \mapsto \Omega^G_*(X)$ is covariant for projective maps and contravariant for smooth maps in $\mathcal{V}_G$. It is also contravariant for l.c.i. morphisms in $\mathcal{V}_G$. Moreover, for a fiber diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

in $\mathcal{V}_G$ with f projective and g smooth, one has $g^* \circ f_* = {f'}_* \circ {g'}^* : \Omega^G_*(X) \to \Omega^G_*(Y')$.

(ii) Homotopy : If $f : E \to X$ is a G-equivariant vector bundle, then $f^* : \Omega^G_*(X) \xrightarrow{\cong} \Omega^G_*(E)$.

(iii) Chern classes : For any G-equivariant vector bundle $E \xrightarrow{f} X$ of rank r, there are equivariant Chern class operators $c^G_m(E) : \Omega^G_*(X) \to \Omega^G_{*-m}(X)$ for $0 \le m \le r$ with $c^G_0(E) = 1$. These Chern classes have same functoriality properties as in the non-equivariant case. Moreover, they satisfy the Whitney sum formula.
(iv) Free action : If G acts freely on X with quotient Y, then $\Omega^G_*(X) \xrightarrow{\cong} \Omega_*(Y)$.
(v) Exterior Product : There is a natural product map

$$\Omega^G_i(X) \otimes_{\mathbb{Z}} \Omega^G_{i'}(X') \to \Omega^G_{i+i'}(X \times X').$$

In particular, $\Omega^G_*(k)$ is a graded algebra and $\Omega^G_*(X)$ is a graded $\Omega^G_*(k)$-module for every $X \in \mathcal{V}_G$.
(vi) Projection formula : For a projective map $f : X' \to X$ in $\mathcal{V}^S_G$, one has for $x \in \Omega^G_*(X)$ and $x' \in \Omega^G_*(X')$, the formula : $f_*(x' \cdot f^*(x)) = f_*(x') \cdot x$.

Proof. Assume that the dimensions of X and Y are m and n respectively and let $d = m-n$ be the relative dimension of a projective G-equivariant morphism $f : X \to Y$. For a fixed $j \ge 0$, let (V_j, U_j) be an l-dimensional good pair for j. Since f is projective, Lemma 5.1 implies that $\overline{f} : X_G \to Y_G$ is projective and hence by Theorem 2.1 and Lemma 3.3, there is a push-forward map

$$\frac{\Omega_{i+l-g}(X_G)}{F_{m+l-g-j}\Omega_{i+l-g}(X_G)} \to \frac{\Omega_{i+l-g}(Y_G)}{F_{m+l-g-j}\Omega_{i+l-g}(Y_G)} = \frac{\Omega_{i+l-g}(Y_G)}{F_{n+l-g-(j-d)}\Omega_{i+l-g}(Y_G)}.$$

In particular, we get a compatible system of maps

$$\Omega^G_i(X)_{j+d} \to \Omega^G_i(Y)_j.$$

Taking the inverse limits, one gets the desired push-forward map $\Omega^G_i(X) \xrightarrow{f_*} \Omega^G_i(Y)$.
If f is smooth of relative dimension d, then $\overline{f} : X_G \to Y_G$ is also smooth of same relative dimension. Hence, we get a compatible system of pull-back maps $\Omega^G_i(Y)_j \xrightarrow{\overline{f}^*} \Omega^G_{i+d}(X)_j$. Taking the inverse limit, we get the desired pull-back map of the equivariant algebraic cobordism groups. If f is a l.c.i. morphism of G-schemes, the same proof applies using the existence of similar map in the non-equivariant case. The required commutativity of the pull-back and push-forward maps follows exactly in the same way from the corresponding result for the non-equivariant cobordism groups.
To prove the homotopy property, let $E \xrightarrow{f} X$ be a G-equivariant vector bundle of rank r. For any $j \ge 0$, let (V_j, U_j) be a good pair for j. Then the map of mixed quotients $E_G \to X_G$ is a vector bundle of rank r (cf. [10, Lemma 1]). Hence by Proposition 3.9, the pull-back map $\Omega^G_i(X)_j \to \Omega^G_{i+r}(E)_j$ is an isomorphism. If $j' \ge j$, then we can choose a common good pair for both j and j'. Hence, we have a pull-back map of the inverse systems

$\{\Omega^G_i(X)_j\} \to \{\Omega^G_{i+r}(E)_j\}$ which is an isomorphism at each level. Hence $f^* : \Omega^G_i(X) \to \Omega^G_{i+r}(E)$ is an isomorphism.

To define the Chern classes of an equivariant vector bundle E of rank r, we choose an l-dimensional good pair (V_j, U_j) and consider the vector bundle $E_G \to X_G$ as above and let $c^G_{m,j} : \Omega_{i+l-g}(X_G) \to \Omega_{i+l-g-m}(X_G)$ be the non-equivariant Chern class as in [31, 4.1.7]. For a closed subscheme $Z \overset{\iota}{\hookrightarrow} X_G$, the projection formula for the non-equivariant cobordism

$$c^G_{m,j}(E_G) \circ \iota_* = \iota_* \circ \left(c^G_{m,j}\left(\iota^*(E_G)\right)\right) \tag{5.1}$$

implies that $c^G_{m,j}(E_G)$ descends to maps $c^G_{m,j} : \Omega^G_i(X)_j \to \Omega^G_{i-m}(X)_j$.

One shows as in Lemma 4.2 that this is independent of the choice of the good pairs. Furthermore, choosing a common good pair for $j' \geq j$, we see that $c^G_{m,j}$ actually defines a map of the inverse systems. Taking the inverse limit, we get the Chern classes $c^G_m(E) : \Omega^G_i(X) \to \Omega^G_{i-m}(X)$ for $0 \leq m \leq r$ with $c^G_0(E) = 1$. The functoriality and the Whitney sum formula for the equivariant Chern classes are easily proved along the above lines using the analogous properties of the non-equivariant Chern classes.

The statement about the free action follows from [9, Lemma 7.2] and Remark 4.5.

We now show the existence of the exterior product of the equivariant cobordism which requires some work. Let d and d' be the dimensions of X and X' respectively. We first define maps

$$\Omega^G_i(X)_j \otimes \Omega^G_{i'}(X')_j \to \Omega^G_{i+i'}(X \times X')_j \quad \text{for } j \geq 0. \tag{5.2}$$

Let (V_j, U_j) be an l-dimensional good pair for j and let $\alpha = [Y \xrightarrow{f} X_G]$ and $\alpha' = [Y' \xrightarrow{f'} X'_G]$ be the cobordism cycles on X_G and X'_G respectively. Using the fact that $X \times U_j \to X_G$ and $X' \times U_j \to X'_G$ are principal G-bundles, we get the unique cobordism cycles $[\widetilde{Y} \to X \times U_j]$ and $[\widetilde{Y'} \to X' \times U_j]$ whose G-quotients are the above chosen cycles. We define $\alpha \star \alpha' = [\widetilde{Y} \overset{G}{\times} \widetilde{Y'} \to (X \times X')_G]$. Note that $(V_j \times V_j, U_j \times U_j)$ is a good pair for j of dimension $2l$ and $(X \times X')_G$ is the quotient of $X \times X' \times U_j \times U_j$ for the free diagonal action of G and $\alpha \star \alpha'$ is a well defined cobordism cycle by Lemma 5.1.

Suppose now that $W \xrightarrow{p} X_G \times \square^1$ is a projective morphism from a smooth scheme W such that the composite map $\pi : W \to X_G \times \square^1 \to \square^1$ is a double point degeneration with $W_\infty = \pi^{-1}(\infty)$ smooth. Letting G act trivially on $\square^1$, this gives a unique G-equivariant double point degeneration $\widetilde{W} \xrightarrow{\widetilde{p}} X \times U_j \times \square^1$ of G-schemes. This implies in particular that $\widetilde{W} \times \widetilde{Y'} \xrightarrow{\widetilde{p} \times \widetilde{f'}} X \times X' \times U_j \times U_j \times \square^1$ is also a G-equivariant double point degeneration whose quotient for the free G-action gives a double point degeneration $\widetilde{W} \overset{G}{\times} \widetilde{Y'} \xrightarrow{q} (X \times X')_G \times \square^1$. Moreover, it is easy to see from this that $C(p) \star \alpha' = C(q)$ (cf. (2.4)). Reversing the roles of X and X' and using (2.4) and Theorem 2.3, we get the maps

$$\Omega_{i+l-g}(X_G) \otimes \Omega_{i'+l-g}(X'_G) \to \Omega_{i+i'-2l-g}\left((X \times X')_G\right).$$

It is also clear from the definition of $\alpha \star \alpha'$ and the niveau filtration that

$$\begin{aligned}&\{F_{d+l-g-j}\Omega_{i+l-g}(X_G) \otimes \Omega_{i'+l-g}(X'_G)\} + \\ &\qquad + \{\Omega_{i+l-g}(X_G) \otimes F_{d'+l-g-j}\Omega_{i'+l-g}(X'_G)\} \to \\ &\qquad\qquad \to F_{d+d'+2l-g-j}\Omega_{i+i'-2l-g-j}\left((X \times X')_G\right).\end{aligned}$$

This defines the maps as in (5.2). One can now show as in Lemma 4.2 that these maps are independent of the choice of the good pairs. We get the desired exterior product as the composite map

$$\begin{aligned}\Omega^G_i(X) \otimes_{\mathbb{Z}} \Omega^G_{i'}(X') &= \varprojlim_j \Omega^G_i(X)_j \otimes_{\mathbb{Z}} \varprojlim_j \Omega^G_{i'}(X')_j \\ &\to \varprojlim_j \left(\Omega^G_i(X)_j \otimes_{\mathbb{Z}} \Omega^G_{i'}(X')_j\right) \qquad (5.3)\\ &\to \varprojlim_j \Omega^G_{i+i'}(X \times X')_j = \Omega^G_{i+i'}(X \times X'). \qquad (5.4)\end{aligned}$$

Finally for X smooth, we get the product structure on $\Omega^*_G(X)$ via the composite $\Omega^*_G(X) \otimes_{\mathbb{Z}} \Omega^*_G(X) \to \Omega^*_G(X \times X) \xrightarrow{\Delta^*_X} \Omega^*_G(X)$. The projection formula can now be proven by using the non-equivariant version of such a formula (*cf.* [31, 5.1.4]) at each level of the projective system $\{\Omega^i_G(X)_j\}$ and then taking the inverse limit. $\square$

We now turn our attention to the question of the localization sequence in equivariant cobordism. In the topological context, Buhstaber-Miscenko [6, 7] defined the topological K-theory of an infinite CW-complex as the projective limit of the K-theory of finite skeleta. They suggested that this theory might not have the Gysin exact sequence. In [26], Landweber showed that such a phenomenon for the K-theory is also reflected in the complex cobordism. This makes us believe that one should not expect the full localization sequence for the equivariant algebraic cobordism considered here. On the positive side however, we can prove the following weaker result.

Proposition 5.3. *Let X be a G-scheme of dimension d and let $f : U \hookrightarrow X$ be a G-invariant open subscheme. Then the restriction map $f^* : \Omega^G_*(X) \to \Omega^G_*(U)$ is surjective.*

Proof. Let Z be the complement of U in X with the reduced induced closed subscheme structure and let $g : Z \hookrightarrow X$ be the inclusion map.
We fix integers $i \in \mathbb{Z}$ and $j \geq 0$ and choose a good pair (V_j, U_j) of dimension l for j. Then we see that Z is a G-invariant closed subscheme of X and $Z_G \subseteq X_G$ is a closed subscheme with the complement U_G. Hence by applying Theorem 3.5

at the appropriate levels of the niveau filtration and taking the quotients, we get an exact sequence

$$\frac{\Omega_{i+l-g}(Z_G)}{F_{d+l-g-j}\Omega_{i+l-g}(Z_G)} \to \frac{\Omega_{i+l-g}(X_G)}{F_{d+l-g-j}\Omega_{i+l-g}(X_G)} \to \frac{\Omega_{i+l-g}(U_G)}{F_{d+l-g-j}\Omega_{i+l-g}(U_G)} \to 0.$$

If $d' = \dim(Z)$, then $F_{d'+l-g-j}\Omega_{i+l-g}(Z_G) \subseteq F_{d+l-g-j}\Omega_{i+l-g}(Z_G)$ and hence by Lemma 4.2, we get an exact sequence of inverse systems

$$\Omega_i^G(Z)_j \xrightarrow{\phi_j^i} \Omega_i^G(X)_j \to \Omega_i^G(U)_j \to 0. \tag{5.5}$$

Setting M_j^i and N_j^i to be the kernel and the image of the map ϕ_j^i respectively, (5.5) can be split into the short exact sequences of inverse systems

$$0 \to M_j^i \to \Omega_i^G(Z)_j \to N_j^i \to 0; \tag{5.6}$$

$$0 \to N_j^i \to \Omega_i^G(X)_j \to \Omega_i^G(U)_j \to 0. \tag{5.7}$$

It follows from Lemma 4.3 that $\{\Omega_i^G(Z)_j\}_{j\ge 0}$ is an inverse system of surjective maps and hence so is $\{N_j^i\}_{j\ge 0}$. In particular, it satisfies the Mittag-Leffler (ML) condition. As a consequence, we get an exact sequence of inverse limits

$$\varprojlim_j N_j^i \to \Omega_i^G(X) \to \Omega_i^G(U) \to 0$$

and this proves the proposition. □

Proposition 5.4 (Morita Isomorphism). *Let $H \subset G$ be a closed subgroup and let $X \in \mathcal{V}_H$. Then there is a canonical isomorphism*

$$\Omega_*^G\left(G \overset{H}{\times} X\right) \xrightarrow{\cong} \Omega_*^H(X). \tag{5.8}$$

Proof. Define an action of $H \times G$ on $G \times X$ by

$$(h,g)\cdot(g',x) = \left(gg'h^{-1}, hx\right),$$

and an action of $H \times G$ on X by $(h,g)\cdot x = hx$. Then the projection map $G \times X \xrightarrow{p} X$ is $(H \times G)$-equivariant which is a G-torsor. Hence by [9, Lemma 7.2], the natural map $\Omega_*^H(X) \xrightarrow{p^*} \Omega_*^{H\times G}(G \times X)$ is an isomorphism. On the other hand, the projection map $G \times X \to G \overset{H}{\times} X$ is $(H \times G)$-equivariant which is an H-torsor. Hence we get an isomorphism $\Omega_*^G\left(G \overset{H}{\times} X\right) \xrightarrow{\cong} \Omega_*^{H\times G}(G \times X)$. The proposition follows by combining these two isomorphisms. □

6. Computations

Let X be a k-scheme of dimension d with a G-action. We have seen above that unlike the situation of Chow groups, the cobordism group $\Omega_{i+l-g}\left(X \overset{G}{\times} U_j\right)$ is not independent of the choice of the l-dimensional good pair (V_j, U_j) even if j is large enough. This anomaly is rectified by considering the quotients of the cobordism groups of the good pairs by the niveau filtration. Our main result in this section is to show that if we suitably choose a sequence of good pairs $\{(V_j, U_j)\}_{j\geq 0}$, then the above equivariant cobordism group can be computed without taking quotients by the niveau filtration. This reduction is often very helpful in computing the equivariant cobordism groups.

Theorem 6.1. *Let $\{(V_j, U_j)\}_{j\geq 0}$ be a sequence of l_j-dimensional good pairs such that*
(i) $V_{j+1} = V_j \oplus W_j$ as representations of G with $\dim(W_j) > 0$ and
(ii) $U_j \oplus W_j \subsetneq U_{j+1}$ as G-invariant open subsets.
(iii) $\operatorname{codim}_{V_{j+1}}(V_{j+1} \setminus U_{j+1}) > \operatorname{codim}_{V_j}(V_j \setminus U_j)$.
Then for any scheme X as above and any $i \in \mathbb{Z}$, we have

$$\varprojlim_j \Omega_{i+l_j-g}\left(X \overset{G}{\times} U_j\right) \xrightarrow{\cong} \Omega_i^G(X).$$

Moreover, such a sequence $\{(V_j, U_j)\}_{j\geq 0}$ of good pairs always exists.

Proof. Let $\{(V_j, U_j)\}_{j\geq 0}$ be a sequence of good pairs as in the theorem. We have natural maps

$$\begin{aligned}(6.1)\qquad \Omega_{i+l_{j+1}-g}\left(X \overset{G}{\times} U_{j+1}\right) &\twoheadrightarrow \\ &\twoheadrightarrow \Omega_{i+l_{j+1}-g}\left(X \overset{G}{\times} (U_j \oplus W_j)\right) \xleftarrow{\cong} \Omega_{i+l_j-g}\left(X \overset{G}{\times} U_j\right),\end{aligned}$$

where the first map is the restriction to an open subset and the second is the pull-back via a vector bundle. Taking the quotients by the niveau filtrations, we get natural maps (*cf.* proof of Lemma 4.2)
(6.2)

$$\begin{array}{ccc}
\dfrac{\Omega_{i+l_{j+1}-g}\left(X \overset{G}{\times} U_{j+1}\right)}{F_{d+l_{j+1}-g-j-1}\Omega_{i+l_{j+1}-g}\left(X \overset{G}{\times} U_{j+1}\right)} & \longrightarrow & \dfrac{\Omega_{i+l_{j+1}-g}\left(X \overset{G}{\times} (U_j \oplus W_j)\right)}{F_{d+l_{j+1}-g-j-1}\Omega_{i+l_{j+1}-g}\left(X \overset{G}{\times} (U_j \oplus W_j)\right)} \\
 & & \Big\uparrow \cong \\
\dfrac{\Omega_{i+l_j-g}\left(X \overset{G}{\times} U_j\right)}{F_{d+l_j-g-j}\Omega_{i+l_j-g}\left(X \overset{G}{\times} U_j\right)} & \twoheadleftarrow & \dfrac{\Omega_{i+l_j-g}\left(X \overset{G}{\times} U_j\right)}{F_{d+l_j-g-j-1}\Omega_{i+l_j-g}\left(X \overset{G}{\times} U_j\right)}
\end{array}$$

where the right vertical arrow is an isomorphism by Proposition 3.9. Setting $X_j = X \overset{G}{\times} U_j$, we get natural maps

$$(6.3)\qquad \begin{array}{ccc}
\Omega_{i+l_{j+1}-g}(X_{j+1}) & \xrightarrow{\nu_j^{j+1}} & \Omega_{i+l_j-g}(X_j) \\
\Big\downarrow & & \Big\downarrow \\
\dfrac{\Omega_{i+l_{j+1}-g}(X_{j+1})}{F_{d+l_{j+1}-g-j-1}\Omega_{i+l_{j+1}-g}(X_{j+1})} & \to & \dfrac{\Omega_{i+l_j-g}(X_j)}{F_{d+l_j-g-j}\Omega_{i+l_j-g}(X_j)}.
\end{array}$$

Since (V_j, U_j) is a good pair for each j, we see that $\frac{\Omega_{i+l_j-g}(X_j)}{F_{d+l_j-g-j}\Omega_{i+l_j-g}(X_j)} \cong \Omega_i^G(X)_j$. Hence, we only have to show that the map

$$\varprojlim_j \Omega_{i+l_j-g}(X_j) \to \varprojlim_j \frac{\Omega_{i+l_j-g}(X_j)}{F_{d+l_j-g-j}\Omega_{i+l_j-g}(X_j)} \tag{6.4}$$

is an isomorphism in order to prove the theorem.
To prove (6.4), we only need to show that for any given $j \geq 0$, the map $\Omega_{i+l_{j'}-g}(X_{j'}) \xrightarrow{\nu_j^{j'}} \Omega_{i+l_j-g}(X_j)$ factors through

$$\frac{\Omega_{i+l_{j'}-g}(X_{j'})}{F_{d+l_{j'}-g-j'}\Omega_{i+l_{j'}-g}(X_{j'})} \to \Omega_{i+l_j-g}(X_j) \quad \text{for all } j' \gg j. \tag{6.5}$$

However, it follows from (6.2) that $\nu_j^{j'}$ induces the map

$$\frac{\Omega_{i+l_{j'}-g}(X_{j'})}{F_{d+l_{j'}-g-j'}\Omega_{i+l_{j'}-g}(X_{j'})} \to \frac{\Omega_{i+l_j-g}(X_j)}{F_{d+l_j-g-j'}\Omega_{i+l_j-g}(X_j)}.$$

On the other hand $F_{d+l_j-g-j'}\Omega_{i+l_j-g}(X_j)$ vanishes for $j' \gg j$. This proves (6.5) and hence (6.4).
Finally, it follows easily from the proof of Lemma 4.2 (see also [39, Remark 1.4]) that a sequence of good pairs as in Theorem 6.1 always exists. ☐

As a simple corollary of Theorem 6.1, we get the following localization sequence for the equivariant cobordism in a special case.

Corollary 6.2. *Let X be a G-scheme and let $Z \subseteq X$ be a G-invariant closed subscheme with the complement U. Assume that there is a G-equivariant projective morphism $p : X \to Y$ whose restriction to Z is an isomorphism. Then there is a short exact sequence*

$$0 \to \Omega_*^G(Z) \to \Omega_*^G(X) \to \Omega_*^G(U) \to 0. \tag{6.6}$$

Proof. Let $\{(V_j, U_j)\}_{j\geq 0}$ be a sequence of good pairs as in Theorem 6.1. The localization sequence for the ordinary algebraic cobordism yields for any $i \in \mathbb{Z}$, an exact sequence of inverse systems

$$\Omega_{i+l_j-g}\left(Z \overset{G}{\times} U_j\right) \to \Omega_{i+l_j-g}\left(X \overset{G}{\times} U_j\right) \to \Omega_{i+l_j-g}\left(U \overset{G}{\times} U_j\right) \to 0. \tag{6.7}$$

It follows from Lemma 5.1 that there are projective morphisms $Z \overset{G}{\times} U_j \xrightarrow{f_j} X \overset{G}{\times} U_j \xrightarrow{p_j} Y \overset{G}{\times} U_j$ such that $p_j \circ f_j$ is an isomorphism. This implies that the map ${p_j}_* \circ {f_j}_*$ is an isomorphism. In other words, (6.8) is in fact a short exact sequence of inverse systems
(6.8)

$$0 \to \Omega_{i+l_j-g}\left(Z \overset{G}{\times} U_j\right) \to \Omega_{i+l_j-g}\left(X \overset{G}{\times} U_j\right) \to \Omega_{i+l_j-g}\left(U \overset{G}{\times} U_j\right) \to 0.$$

We have moreover seen in (6.1) that $\{\Omega_{i+l_j-g}(Z \stackrel{G}{\times} U_j)\}_{j\geq 0}$ is an inverse system of surjective maps. It follows that (6.8) remains short exact after taking limit, which proves (6.6). $\square$

Another consequence of Theorem 6.1 is that for a linear algebraic group G acting on a scheme X of dimension d, the forgetful map $r^G_X : \Omega^G_*(X) \to \Omega_*(X)$ (*cf.* (4.6)) can be easily shown to be analogous to the one used in [10, Subsection 2.2] for the Chow groups. This interpretation of the forgetful map has some interesting applications in the computation of the non-equivariant cobordism using the equivariant techniques (*cf.* [20], [21]).

So let $\{(V_j, U_j)\}_{j\geq 0}$ be a sequence of good pairs as in Theorem 6.1. We choose a k-rational point $x \in U_0$ and let x_j be its image in U_j/G under the natural map $U_0 \to U_0/G \to U_j/G$. Setting $X_j = X \stackrel{G}{\times} U_j$, this yields a commutative diagram

(6.9)
$$\begin{tikzcd} X \times U_j \arrow[r, "p_j"] \arrow[d, "\pi_j"'] & X_j \arrow[r, "\phi_j"] \arrow[d, "\psi_j"'] & X_{j+1} \arrow[d, "\psi_{j+1}"] \\ U_j \arrow[r] & U_j/G \arrow[r] & U_{j+1}/G \end{tikzcd}$$

such that the left square is Cartesian and

(6.10)
$$X \cong \pi_j^{-1}(x) \xrightarrow{\cong} \psi_j^{-1}(x_j) \xrightarrow{\cong} \psi_{j+1}^{-1}(x_{j+1}).$$

Let $\nu_j : \psi_j^{-1}(x_j) \hookrightarrow X_j$ be the closed embedding. Notice that since U_j/G is smooth and ψ_j is flat, it follows that ν_j is a regular closed embedding (hence an l.c.i. morphism). Using the identification in (6.10), we get maps $\nu_j^* : \Omega_*(X_j) \to \Omega_*(X)$ such that $\nu_j^* \circ \phi_j^* = \nu_{j+1}^*$. Taking the limit over $j \geq 0$, this yields for any $i \in \mathbb{Z}$, a restriction map

(6.11)
$$\widetilde{r}^G_X : \Omega^G_i(X) = \varprojlim_{j\geq 0} \Omega^G_i(X)_j \to \Omega_i(X).$$

Corollary 6.3. *The maps* r^G_X, $\widetilde{r}^G_X : \Omega^G_i(X) \to \Omega_i(X)$ *coincide.*

Proof. Using the construction of the map r^G_X in (4.6) and the diagram (6.9), it suffices to show that for any $i \in \mathbb{Z}$, the natural maps

$$\frac{\Omega_i(X)}{F_{d-j}\Omega_i(X)} \leftarrow \frac{\Omega_i(X \times V_j)}{F_{d+l_j-j}\Omega_i(X \times V_j)} \to \frac{\Omega_i(X \times U_j)}{F_{d+l_j-j}\Omega_i(X \times U_j)}$$

are isomorphisms for all $j \gg 0$. Here, the first map is the restriction induced by the section corresponding to the rational point $x_j \in U_j/G$, and the second map is the restriction to an open subset. The existence of the first map follows from Proposition 3.9. The assertion that these two maps are isomorphisms follows immediately from Corollary 3.6 and Proposition 3.9. $\square$

Remark 6.4. It follows from Corollary 6.3 that the map $\widetilde{r}^G_X$ does not depend on the choice of the k-rational point $x \in U_0$.

6.1. Graded vs. completed cobordism rings. Another consequence of Theorem 6.1 is that it allows us to explain the relation between the graded (or noncomplete) and the nongraded (or complete) versions of the equivariant cobordism rings of smooth G-schemes. If $\{(V_j, U_j)\}_{j\ge 0}$ is a sequence of l_j-dimensional good pairs as in Theorem 6.1, then it can be easily checked from the proof of this theorem that the expression

$$\widehat{\Omega^*_G(X)} = \varprojlim_j \Omega^*\left(X \stackrel{G}{\times} U_j\right) \tag{6.12}$$

is well-defined and there is a natural map $\iota_X : \Omega^*_G(X) \to \widehat{\Omega^*_G(X)}$.
Furthermore, the surjectivity of the map $\Omega^*\left(X \stackrel{G}{\times} U_{j+1}\right) \to \Omega^*\left(X \stackrel{G}{\times} U_j\right)$ (*cf.* (6.1)) implies that ι_X identifies $\widehat{\Omega^*_G(X)}$ as the completion of $\Omega^*_G(X)$ with respect to the linear topology given by the decreasing filtration

$$F^j\Omega^*_G(X) = \mathrm{Ker}\left(\Omega^*_G(X) \twoheadrightarrow \Omega^*\left(X \stackrel{G}{\times} U_j\right)\right).$$

6.2. Formal group law in equivariant cobordism. Let G be a linear algebraic group over k acting on a scheme X of dimension d. We have seen before that the equivariant line bundles on X give rise to the equivariant Chern class operators on $\Omega^G_*(X)$. Below, we write down an expression for the equivariant Chern class of the tensor product of two such line bundles.
Let $\{(V_j, U_j)\}_{j\ge 0}$ be a sequence of l_j-dimensional good pairs as in Theorem 6.1. Letting $X_j = X \stackrel{G}{\times} U_j$, we see that for every $j \ge 0$, $\Omega_*(X_j) = \bigoplus_{i\in\mathbb{Z}} \Omega_{i+l_j-g}(X_j)$ is an $\mathbb{L}$-module and for $j' \ge j$, there is a natural surjection $\Omega_*(X_{j'}) \twoheadrightarrow \Omega_*(X_j)$ of $\mathbb{L}$-modules.
Given G-equivariant line bundles L, M on X, we get line bundles L_j, M_j on X_j, where $L_j = L \stackrel{G}{\times} U_j$ for $j \ge 0$. The formal group law of the non-equivariant cobordism yields

$$c_1((L\otimes M)_j) = \\ = c_1(L_j\otimes M_j) = c_1(L_j) + c_1(M_j) + \sum_{i,i'\ge 1} a_{i,i'}\left(c_1(L_j)\right)^i \circ \left(c_1(M_j)\right)^{i'}.$$

Note that if $(x_j) \in \Omega^G_i(X)$, then the evaluation of the operator $c^G_1(L)(x_j)$ at any level $j \ge 0$ is a finite sum above.
Taking the limit over $j \ge 0$ and noting that the sum (and the product) in the equivariant cobordism groups are obtained by taking the limit of the sums (and the products) at each level of the inverse system, we get the same formal group law for the equivariant Chern classes:

$$c^G_1(L\otimes M) = c^G_1(L) + c^G_1(M) + \sum_{i,i'\ge 1} a_{i,i'}\left(c^G_1(L)\right)^i \circ \left(c^G_1(M)\right)^{i'}. \tag{6.13}$$

Note that the coefficients $a_{i,i'}$ are homogeneous elements of $\mathbb{L}$ and can be considered as elements of $S(G)$ under the natural inclusion of graded rings $\mathbb{L} \hookrightarrow S(G)$. One should also observe that unlike the case of ordinary cobordism, the evaluation of the above sum on any given equivariant cobordism cycle may no longer be finite. In other words, the equivariant Chern classes are not in general locally nilpotent.

6.3. Cobordism ring of classifying spaces. Let R be a Noetherian ring and let $A = \underset{j \in \mathbb{Z}}{\oplus} A_j$ be a $\mathbb{Z}$-graded R-algebra with $R \subseteq A_0$. Recall that the *graded power series ring* $S^{(n)} = \underset{i \in \mathbb{Z}}{\oplus} S_i$ is a graded ring such that S_i is the set of formal power series of the form $f(\mathbf{t}) = \sum\limits_{m(\mathbf{t}) \in \mathcal{C}} a_{m(\mathbf{t})} m(\mathbf{t})$ such that $a_{m(\mathbf{t})}$ is a homogeneous element of A of degree $|a_{m(\mathbf{t})}|$and $|a_{m(\mathbf{t})}| + |m(\mathbf{t})| = i$. Here, $\mathcal{C}$ is the set of all monomials in $\mathbf{t} = (t_1, \cdots, t_n)$ and $|m(\mathbf{t})| = i_1 + \cdots + i_n$ if $m(\mathbf{t}) = t_1^{i_1} \cdots t_n^{i_n}$. We call $|m(\mathbf{t})|$ to be the degree of the monomial $m(\mathbf{t})$.
We shall write the above graded power series ring as $A[[\mathbf{t}]]_{\mathrm{gr}}$ to distinguish it from the usual formal power series ring $A[[\mathbf{t}]]$. Notice that if A is only non-negatively graded, then $S^{(n)}$ is nothing but the standard polynomial ring $A[t_1, \cdots, t_n]$ over A. It is also easy to see that $S^{(n)}$ is indeed a graded ring which is a subring of the formal power series ring $A[[t_1, \cdots, t_n]]$. The following result summarizes some basic properties of these rings. The proof is straightforward and is left as an exercise.

Lemma 6.5. *(i) There are inclusions of rings $A[t_1, \cdots, t_n] \subset S^{(n)} \subset A[[t_1, \cdots, t_n]]$, where the first is an inclusion of graded rings.*
(ii) These inclusions are analytic isomorphisms with respect to the $\mathbf{t}$-adic topology. In particular, the induced maps of the associated graded rings

$$A[t_1, \cdots, t_n] \to \mathrm{Gr}_{(\mathbf{t})} S^n \to \mathrm{Gr}_{(\mathbf{t})} A[[t_1, \cdots, t_n]]$$

are isomorphisms.
(iii) $S^{(n-1)}[[t_n]]_{\mathrm{gr}} \xrightarrow{\cong} S^{(n)}$.
(iv) $\frac{S^{(n)}}{(t_{i_1}, \cdots, t_{i_r})} \xrightarrow{\cong} S^{(n-r)}$ *for any $n \geq r \geq 1$, where $S^{(0)} = A$.*
(v) The sequence $\{t_1, \cdots, t_n\}$ is a regular sequence in $S^{(n)}$.
(vi) If $A = R[x_1, x_2, \cdots]$ is a polynomial ring with $|x_i| < 0$ and $\lim\limits_{i \to \infty} |x_i| = -\infty$, then $S^{(n)} \xrightarrow{\cong} \varprojlim\limits_{i} R[x_1, \cdots, x_i][[\mathbf{t}]]_{\mathrm{gr}}$.

Examples 6.6. In the following examples, we compute $\Omega^*(BG) = \Omega^*_G(k)$ for some classical groups G over k. These computations follow directly from the definition of equivariant cobordism and suitable choices of good pairs.
We first consider the case when $G = \mathbb{G}_m$ is the multiplicative group. For any $j \geq 1$, we choose the good pair (V_j, U_j), where V_j is the j-dimensional representation of $\mathbb{G}_m$ with all weights -1 and U_j is the complement of the origin. We see then that $U_j/\mathbb{G}_m \cong \mathbb{P}^{j-1}_k$. Let ζ be the class of $c_1(\mathcal{O}(-1))(1) \in \Omega^1(\mathbb{P}^{j-1}_k)$. The projective bundle formula for the ordinary algebraic cobordism

implies that $(\Omega^i_G)_j = \bigoplus_{0 \le p \le j-1} \mathbb{L}^{i-p}\zeta^p$. Taking the inverse limit over $j \ge 1$, we find from this that for $i \in \mathbb{Z}$,

$$\Omega^i_{\mathbb{G}_m}(k) = \prod_{p \ge 0} \mathbb{L}^{i-p}\zeta^p.$$

It particular, if $x = \sum_{j=1}^{n} x_{i_j}$ is a sum of homogeneous elements of $\Omega^*(B\mathbb{G}_m)$, then we get a natural map

$$\Omega^*(B\mathbb{G}_m) \to \mathbb{L}[[t]]_{\mathrm{gr}} \tag{6.14}$$

$$x = \left(x_{i_1} = \prod a^{i_1}_p \zeta^p, \cdots, x_{i_n} = \prod a^{i_n}_p \zeta^p\right) \mapsto \sum_{p \ge 0} \left(\sum_{1 \le j \le n} a^{i_j}_p \right) t^p,$$

which is an isomorphism of graded $\mathbb{L}$-algebras. Observe that $\Omega^*\widehat{(B\mathbb{G}_m)}$ (*cf.* (6.12)) is the formal power series ring $\mathbb{L}[[t]]$.

For a general split torus T of rank n, we choose a basis $\{\chi_1, \cdots, \chi_n\}$ of the character group $\widehat{T}$. This is equivalent to a decomposition $T = T_1 \times \cdots \times T_n$ with each T_i isomorphic to $\mathbb{G}_m$ and χ_i is a generator of $\widehat{T_i}$. Let L_χ be the one-dimensional representation of T, where T acts via χ. For any $j \ge 1$, we take the good pair (V_j, U_j) such that $V_j = \prod_{i=1}^{n} L^{\oplus j}_{\chi_i}$, $U_j = \prod_{i=1}^{n} \left(L^{\oplus j}_{\chi_i} \setminus \{0\}\right)$ and T acts on V_j by $(t_1, \cdots, t_n)(x_1, \cdots, x_n) = (\chi_1(t_1)(x_1), \cdots, \chi_n(t_n)(x_n))$. It is then easy to see that $U_j/T \cong X_1 \times \cdots \times X_n$ with each X_i isomorphic to $\mathbb{P}^{j-1}_k$. Moreover, the T-line bundle L_{χ_i} gives the line bundle $L_{\chi_i} \overset{T_i}{\times} \left(L^{\oplus j}_{\chi_i} \setminus \{0\}\right) \to X_i$ which is $\mathcal{O}(\pm 1)$. Letting ζ_i be the first Chern class of this line bundle, the projective bundle formula for the non-equivariant cobordism shows that

$$\Omega^i_T(k) = \prod_{p_1, \cdots, p_n \ge 0} \mathbb{L}^{i - (\sum_{i=1}^{n} p_i)} \zeta_1^{p_1} \cdots \zeta_n^{p_n},$$

which is isomorphic to the set of formal power series in $\{\zeta_1, \cdots, \zeta_n\}$ of degree i with coefficients in $\mathbb{L}$. It particular, one concludes as in the rank one case above that

Proposition 6.7. *Let $\{\chi_1, \cdots, \chi_n\}$ be a chosen basis of the character group of a split torus T of rank n. The assignment $t_i \mapsto c^T_1(L_{\chi_i})$ yields a graded $\mathbb{L}$-algebra isomorphism*

$$\mathbb{L}[[t_1, \cdots, t_n]]_{\mathrm{gr}} \to \Omega^*(BT).$$

For $G = GL_n$, we can take a good pair for j to be (V_j, U_j), where V_j is the vector space of $n \times p$ matrices with $p > n$ with GL_n acting by left multiplication, and U_j is the open subset of matrices of maximal rank. Then the mixed quotient is the Grassmannian $Gr(n, p)$. We can now calculate the cobordism ring of $Gr(n, p)$ using the projective bundle formula (by standard stratification technique) and then we can use the similar calculations as above to get a natural isomorphism

$$\Omega^*(BGL_n) \to \mathbb{L}[[\gamma_1, \cdots, \gamma_n]]_{\mathrm{gr}} \tag{6.15}$$

of graded $\mathbb{L}$-algebras, where γ_i's are the elementary symmetric polynomials in $t_1, \cdots, t_n$ that occur in (6.14).
Another way to obtain the isomorphism (6.15) is to observe that the Weyl group of GL_n is the permutation group S_n and $t_{GL_n} = 1$, where t_G denotes the torsion index of a connected reductive group G. It follows from [15, Theorem 3.7] that we can assume the base field to be the field of complex numbers. Subsequently, it follows from [15, Proposition 4.8] that the natural map $\Omega^*(BGL_n) \to (\Omega^*(BT))^{S_n} = \mathbb{L}[[\gamma_1, \cdots, \gamma_n]]_{\mathrm{gr}}$ is an isomorphism. Using the same argument, on obtains an isomorphism $\Omega^*(BSL_n) \xrightarrow{\cong} \mathbb{L}[[\gamma_2, \cdots, \gamma_n]]_{\mathrm{gr}}$.

Remark 6.8. The cobordism rings of BGL_n and BSL_n have also been written down by Deshpande in [9, Section 4]. His expressions depend on the assumption that these groups are isomorphic to the complex cobordism. As the reader will find in Subsection 7.3, there may not in general exist a map from the algebraic to the complex equivariant cobordism although such a map does exist for a classifying space BG (*cf.* Corollary 7.7). Moreover, it is not clear when such a map is an isomorphism. We refer the reader to [15, Theorem 3.7] for a result in this direction.

7. Comparison with other equivariant cohomology theories

In this paper, we fix the following notation for the tensor product while dealing with inverse systems of modules over a commutative ring. Let A be a commutative ring with unit and let $\{L_n\}$ and $\{M_n\}$ be two inverse systems of A-modules with inverse limits L and M respectively. Following [38], one defines the *topological tensor product* of L and M by

$$L \widehat{\otimes}_A M := \varprojlim_n (L_n \otimes_A M_n). \tag{7.1}$$

In particular, if D is an integral domain with quotient field F and if $\{A_n\}$ is an inverse system of D-modules with inverse limit A, one has $A \widehat{\otimes}_D F = \varprojlim_n (A_n \otimes_D F)$. The examples $\widehat{\mathbb{Z}_{(p)}} = \varprojlim_n \mathbb{Z}/p^n$ and $\mathbb{Z}[[x]] \otimes_{\mathbb{Z}} \mathbb{Q} \to \varprojlim_n \frac{\mathbb{Z}[x]}{(x^n)} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[[x]]$ show that the map $A \otimes_D F \to A \widehat{\otimes}_D F$ is in general neither injective nor surjective. We shall denote $A \widehat{\otimes}_D F$ in the sequel by A_F to simplify the notations.
If R is a $\mathbb{Z}$-graded ring and if M and N are two R-graded modules, then recall that $M \otimes_R N$ is also a graded R-module given by the quotient of $M \otimes_{R_0} N$ by the graded submodule generated by the homogeneous elements of the type $ax \otimes y - x \otimes ay$ where a, x and y are the homogeneous elements of R, M and

N respectively. If all the graded pieces M_i and N_i are the limits of inverse systems $\{M_i^\lambda\}$ and $\{N_i^\lambda\}$ of R_0-modules, we define the *graded topological tensor product* as $M \widehat{\otimes}_R N = \bigoplus_{i \in \mathbb{Z}} \left(M \widehat{\otimes}_R N\right)_i$, where

$$\left(M \widehat{\otimes}_R N\right)_i = \varprojlim_{\lambda} \left(\bigoplus_{j+j'=i} \frac{M_j^\lambda \otimes_{R_0} N_{j'}^\lambda}{(ax \otimes y - x \otimes ay)} \right). \tag{7.2}$$

Notice that this reduces to the ordinary tensor product of graded R-modules if the underlying inverse systems are trivial.

7.1. Comparison with equivariant Chow groups. Let X be a k-scheme of dimension d with a G-action. It was shown by Levine and Morel [31] that there is a natural map $\Omega_*(X) \to \mathrm{CH}_*(X)$ of graded abelian groups which is a ring homomorphism if X is smooth. Moreover, this map induces a graded isomorphism

$$\Omega_*(X) \otimes_{\mathbb{L}} \mathbb{Z} \xrightarrow{\cong} \mathrm{CH}_*(X). \tag{7.3}$$

Recall from [39] and [10] that the equivariant Chow groups of X are defined as $\mathrm{CH}_i^G(X) = \mathrm{CH}_{i+l-g}\left(X \stackrel{G}{\times} U\right)$, where (V, U) is an l-dimensional good pair corresponding to $d-i+1$. It is known that $\mathrm{CH}_i^G(X)$ is well-defined and can be non-zero for any $-\infty < i \le d$. We set $\mathrm{CH}_*^G(X) = \bigoplus_i \mathrm{CH}_i^G(X)$. If X is equi-dimensional, we let $\mathrm{CH}^i_G(X) = \mathrm{CH}^G_{d-i}(X)$ and set $\mathrm{CH}^*_G(X) = \bigoplus_{i \ge 0} \mathrm{CH}^i_G(X)$. Notice that in this case, $\mathrm{CH}^i_G(X)$ is same as $\mathrm{CH}^i\left(X \stackrel{G}{\times} U\right)$, where (V, U) is an l-dimensional good pair corresponding to $i+1$.

If we fix $i \in \mathbb{Z}$ and choose an l-dimensional good pair (V_j, U_j) corresponding to $j \ge \max(0, d-i+1)$, the universality of the algebraic cobordism gives a unique map $\Omega_{i+l-g}\left(X \stackrel{G}{\times} U_j\right) \to \mathrm{CH}_{i+l-g}\left(X \stackrel{G}{\times} U_j\right)$. By Lemma 3.7, this map factors through

$$\frac{\Omega_{i+l-g}\left(X \stackrel{G}{\times} U_j\right)}{F_{i+l-g-1}\Omega_{i+l-g}\left(X \stackrel{G}{\times} U_j\right)} \to \mathrm{CH}_{i+l-g}\left(X \stackrel{G}{\times} U_j\right). \tag{7.4}$$

Since $j \ge d-i+1$ by the choice, we have $d+l-g-j \le i+l-g-1$ and hence we get the map

(7.5)

$$\Omega_i^G(X)_j = \frac{\Omega_{i+l-g}\left(X \stackrel{G}{\times} U_j\right)}{F_{d+l-g-j}\Omega_{i+l-g}\left(X \stackrel{G}{\times} U_j\right)} \to \mathrm{CH}_{i+l-g}\left(X \stackrel{G}{\times} U_j\right) = \mathrm{CH}_i^G(X).$$

It is easily shown using the proof of Lemma 4.2 that this map is independent of the choice of the good pair (V_j, U_j). Taking the inverse limit over $j \geq 0$, we get a natural map $\Omega_i^G(X) \to \mathrm{CH}_i^G(X)$ and hence a map of graded abelian groups

$$\Phi_X : \Omega_*^G(X) \to \mathrm{CH}_*^G(X) \tag{7.6}$$

which is in fact a map of graded $\mathbb{L}$-modules. Notice that the right side of (7.5) does not depend on j as long as $j \gg 0$. If X is equi-dimensional, we write the above map cohomologically as $\Omega_G^*(X) \to \mathrm{CH}_G^*(X)$.

Example 7.1. Let T be a split torus of rank n over k. It follows from Proposition 6.7 that $\Omega^*(BT)$ is isomorphic to the *graded* power series ring $\mathbb{L}[[t_1, \cdots, t_n]]_{\mathrm{gr}}$. One knows that $\mathrm{CH}^*(BT)$ is isomorphic to the polynomial ring $\mathbb{Z}[t_1, \cdots, t_n]$ (*cf.* [10, 3.2]). And the map $\Phi_k : \Omega^*(BT) \to \mathrm{CH}^*(BT)$ in this case is the obvious map $\mathbb{L}[[t_1, \cdots, t_n]]_{\mathrm{gr}} \to \mathbb{Z}[t_1, \cdots, t_n]$ obtained by killing the ideal $\mathbb{L}^{<0}$.

Proposition 7.2. *The map Φ_X induces an isomorphism of graded $\mathbb{L}$-modules*

$$\Phi_X : \Omega_*^G(X) \widehat{\otimes}_{\mathbb{L}} \mathbb{Z} \xrightarrow{\cong} \mathrm{CH}_*^G(X).$$

Proof. Let $\{(V_j, U_j)\}_{j \geq 0}$ be a sequence of l_j-dimensional good pairs as in Theorem 6.1. It follows from (7.3) that for any $i \in \mathbb{Z}$, there is a short exact sequence

$$0 \to \left(\mathbb{L}^{<0}\Omega_*(X_j) \cap \Omega_{i+l_j-g}(X_j)\right) \to \Omega_{i+l_j-g}(X_j) \to \mathrm{CH}_{i+l_j-g}(X_j) \to 0. \tag{7.7}$$

By comparing this exact sequence for $j' \geq j \geq 0$, using the surjection $\Omega_{i+l_{j+1}-g}(X_{j+1}) \to \Omega_{i+l_j-g}(X_j)$ as in (6.1) and using the localization sequences for the cobordism and Chow groups, we find that the map

$$\left(\mathbb{L}^{<0}\Omega_*(X_{j+1}) \cap \Omega_{i+l_{j+1}-g}(X_{j+1})\right) \to \left(\mathbb{L}^{<0}\Omega_*(X_j) \cap \Omega_{i+l_j-g}(X_j)\right)$$

is surjective for each $j \geq 0$. Taking the limit in (7.7) and using Theorem 6.1, we get a short exact sequence

$$0 \to \varprojlim_{j \geq 0} \left(\mathbb{L}^{<0}\Omega_*(X_j) \cap \Omega_{i+l_j-g}(X_j)\right) \to \Omega_i^G(X) \to \mathrm{CH}_i^G(X) \to 0.$$

Observe here that the inverse system $\{\mathrm{CH}_{i+l-g}(X_j)\}_{j \geq 0}$ is eventually constant with $\mathrm{CH}_i^G(X)$ as its limit. We now take the direct sum over $i \in \mathbb{Z}$ to get the desired isomorphism $\Omega_*^G(X) \widehat{\otimes}_{\mathbb{L}} \mathbb{Z} \xrightarrow{\cong} \mathrm{CH}_*^G(X)$. □

Let $C(G) = \mathrm{CH}_G^*(k)$ denote the equivariant Chow ring of the field k. The following is the equivariant analogue of (7.3).

Corollary 7.3. *For a k-scheme X with a G-action, the natural map*

$$\Omega_*^G(X) \otimes_{S(G)} C(G) \to \mathrm{CH}_*^G(X)$$

is an isomorphism of $C(G)$-modules. This is a ring isomorphism if X is smooth.

Proof. It is clear that the above map is a ring homomorphism if X is smooth. So we only need to prove the first assertion. But this follows directly from the isomorphisms $\Omega^G_*(X)\otimes_{S(G)}C(G) \cong \Omega^G_*(X)\otimes_{S(G)}\left(S(G)\widehat{\otimes}_{\mathbb{L}}\mathbb{Z}\right) \cong \Omega^G_*(X)\widehat{\otimes}_{\mathbb{L}}\mathbb{Z}$ using Proposition 7.2. $\square$

7.2. Comparison with equivariant K-theory. It was shown by Levine and Morel in [30, Corollary 11.11] that the universal property of the algebraic cobordism implies that there is a canonical isomorphism of oriented cohomology theories

$$\Omega^*(X) \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}] \xrightarrow{\cong} K_0(X)[\beta, \beta^{-1}] \tag{7.8}$$

in the category of smooth k-schemes. This was later generalized to a complete algebraic analogue of the Conner-Floyd isomorphism

$$MGL^* \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}] \xrightarrow{\cong} K_*(X)[\beta, \beta^{-1}]$$

between the motivic cobordism and algebraic K-theory by Panin, Pimenov and Röndigs [33]. Since the equivariant cobordism is a Borel style cohomology theory, one can not expect an equivariant version of the isomorphism (7.8) even with the rational coefficients. However, we show here that the equivariant Conner-Floyd isomorphism holds after we base change the above by the completion of the representation ring of G with respect to the ideal of virtual representations of rank zero. In fact, it can be shown easily that such a base change is the minimal requirement. In Theorem 7.4, all cohomology groups are considered with rational coefficients (*cf.* Section 8).

For a linear algebraic group G, let $R(G)$ denote the representation ring of G. Let I denote the ideal of of virtual representations of rank zero in $R(G)$ and let $\widehat{R(G)}$ denote the associated completion of $R(G)$. Let $\widehat{C(G)}$ denote the completion of $C(G)$ with respect to the augmentation ideal of algebraic cycles of positive codimensions. For a scheme X with G-action, let $K^G_0(X)$ denote the Grothendieck group of G-equivariant vector bundles on X. By [11, Theorem 4.1], there is a natural ring isomorphism $\widehat{R(G)} \xrightarrow{\cong} \widehat{C(G)}$ given by the equivariant Chern character. We identify these two rings via this isomorphism. In particular, the maps $S(G) \twoheadrightarrow C(G) \to \widehat{C(G)}$ yield a ring homomorphism $S(G) \to \widehat{R(G)}$.

Theorem 7.4. *Let X be a smooth scheme with a G-action. Then, with rational coefficients, there is a natural isomorphism of rings*

$$\Psi_X : \Omega^*_G(X)\otimes_{S(G)}\widehat{R(G)} \xrightarrow{\cong} K^G_0(X) \otimes_{R(G)} \widehat{R(G)}.$$

Proof. By [17, Theorem 1.2], there is a Chern character isomorphism $K^G_0(X) \otimes_{R(G)} \widehat{R(G)} \xrightarrow{\cong} \mathrm{CH}^*(X) \otimes_{C(G)} \widehat{C(G)}$ of cohomology rings. Thus, we only need to show that the map $\Omega^*_G(X)\otimes_{S(G)}\widehat{C(G)} \to \mathrm{CH}^*(X) \otimes_{C(G)} \widehat{C(G)}$ is an isomorphism. However, we have

$$\begin{aligned}\Omega^*_G(X)\otimes_{S(G)}\widehat{C(G)} &\cong \left(\Omega^*_G(X)\otimes_{S(G)}C(G)\right)\otimes_{C(G)}\widehat{C(G)}\\ &\cong \mathrm{CH}^*_G(X)\otimes_{C(G)}\widehat{C(G)},\end{aligned}$$

where the last isomorphism follows from Corollary 7.3. This finishes the proof. □

7.3. COMPARISON WITH COMPLEX COBORDISM. Let G be a complex Lie group acting on a finite CW-complex X. We define the *equivariant complex cobordism* ring of X as

$$MU^*_G(X) := MU^*\left(X \overset{G}{\times} EG\right) \tag{7.9}$$

where $EG \to BG$ is universal principal G-bundle over the classifying space BG of G. If $E'G \to B'G$ is another such bundle, then the projection $(X \times EG \times E'G)/G \to X \overset{G}{\times} EG$ is a fibration with contractible fiber. In particular, $MU^*_G(X)$ is well-defined. Moreover, if G acts freely on X with quotient X/G, then the map $X \overset{G}{\times} EG \to X/G$ is a fibration with contractible fiber EG and hence we get $MU^*_G(X) \cong MU^*(X/G)$.

For a linear algebraic group G over $\mathbb{C}$ acting on a $\mathbb{C}$-scheme X, let $H^*_G(X,A)$ denote the (equivariant) cohomology of the complex analytic space $X(\mathbb{C})$ with coefficients in the ring A.

PROPOSITION 7.5. *Assume that $X \in \mathcal{V}^S_G$ is such that $H^*_G(X,\mathbb{Z})$ is torsion-free. Then there is a natural homomorphism of graded rings*

$$\rho^G_X:\ \Omega^*_G(X) \to MU^{2*}_G(X).$$

Proof. If $\{(V_j,U_j)\}$ is a sequence of good pairs as in Theorem 6.1, then the universality of the Levine-Morel cobordism gives a natural $\mathbb{L}$-algebra map of inverse systems

$$\Omega^i\left(X \overset{G}{\times} U_j\right) \to MU^{2i}\left(X \overset{G}{\times} U_j\right)$$

which after taking limits yields the map

$$\Omega^i_G(X) = \varprojlim_{j\ge 0} \Omega^i\left(X \overset{G}{\times} U_j\right) \to \varprojlim_{j\ge 0} MU^{2i}\left(X \overset{G}{\times} U_j\right). \tag{7.10}$$

On the other hand, it follows from [15, Lemma 3.2] (see also [39, Theorem 2.1]) that there is a Milnor exact sequence

(7.11)

$$0 \to {\varprojlim_{j\ge 0}}^1 MU^{2i-1}\left(X \overset{G}{\times} U_j\right) \to MU^{2i}\left(X \overset{G}{\times} EG\right) \to \varprojlim_{j\ge 0} MU^{2i}\left(X \overset{G}{\times} U_j\right) \to 0.$$

Moreover, it follows from our assumption and [26, Corollary 1] that the first term of this exact sequence vanishes. This yields the natural map $\rho^G_X : \Omega^i_G(X) \to MU^{2i}_G(X)$. □

It follows from the proof of [26, Corollary 1] that the first term in (7.11) always vanishes if we work over the rationals. We can thus imitate the proof of Proposition 7.5 to see that there is a natural map $\Omega^*_G(X)_{\mathbb{Q}} \to MU^{2*}_G(X)_{\mathbb{Q}}$. Combining this with Proposition 7.2 (with rational coefficients), one concludes the following.

Corollary 7.6. *For any $X \in \mathcal{V}^S_G$, there is a natural map of graded $\mathbb{L}_{\mathbb{Q}}$-algebras*

$$\rho^G_X : \ \Omega^*_G(X)_{\mathbb{Q}} \to MU^{2*}_G(X)_{\mathbb{Q}}.$$

In particular, there is a natural ring homomorphism

$$\overline{\rho}^G_X : \ \mathrm{CH}^*_G(X)_{\mathbb{Q}} \to MU^{2*}_G(X)_{\mathbb{Q}} \widehat{\otimes}_{\mathbb{L}_{\mathbb{Q}}} \mathbb{Q}$$

which factors the cycle class map $\mathrm{CH}^*_G(X) \to H^{2*}_G(X, \mathbb{Q})$.

Corollary 7.7. *There is a natural morphism $\Omega^*(BG) \to MU^{2*}(BG)$ of graded $\mathbb{L}$-algebras. In particular, there is a natural ring homomorphism $\mathrm{CH}^*(BG) \to MU^{2*}(BG)\widehat{\otimes}_{\mathbb{L}}\mathbb{Z}$ which factors the cycle class map $\mathrm{CH}^*(BG) \to H^{2*}(BG, \mathbb{Z})$.*

Proof. The first assertion follows immediately from (7.10), (7.11) and [26, Theorem 1] using the fact that BG is homotopy equivalent to the classifying space of its maximal compact subgroup. The second assertion follows from the first and Proposition 7.2 using the identification $\mathbb{L} \xrightarrow{\cong} MU^*$. □

Remark 7.8. The map $\mathrm{CH}^*(BG) \to MU^{2*}(BG)\widehat{\otimes}_{\mathbb{L}}\mathbb{Z}$ has also been constructed by Totaro [39] by a different method.

We shall study the above realization maps in more detail in the next section.

8. Reduction of arbitrary groups to tori

The main result of this section is to show that with the rational coefficients, the equivariant cobordism of schemes with an action of a connected linear algebraic group can be written in terms of the Weyl group invariants of the equivariant cobordism for the action of the maximal torus. This reduces the problems about the equivariant cobordism to the case where the underlying group is a torus. We draw some consequences of this for the cycle class map from the rational Chow groups to the complex cobordism groups of classifying spaces. We first prove some reduction results about the equivariant cobordism which reflect the relations between the G-equivariant cobordism and the equivariant cobordism for actions of subgroups of G. The results of this section are used in [19] and [21] to compute the non-equivariant cobordism ring of flag varieties and flag bundles.

Proposition 8.1. *Let G be a connected reductive group over k. Let B be a Borel subgroup of G containing a maximal torus T over k. Then for any $X \in \mathcal{V}_G$, the restriction map*

$$\Omega^B_*(X) \xrightarrow{r^B_{T,X}} \Omega^T_*(X) \tag{8.1}$$

is an isomorphism.

Proof. By Proposition 5.4, we only need to show that

$$\Omega_*^B\left(B \overset{T}{\times} X\right) \cong \Omega_*^B(X). \tag{8.2}$$

By [8, XXII, 5.9.5], there exists a characteristic filtration $B^u = U_0 \supseteq U_1 \supseteq \cdots \supseteq U_n = \{1\}$ of the unipotent radical B^u of B such that U_{i-1}/U_i is a vector group, each U_i is normal in B and $TU_i = T \ltimes U_i$. Moreover, this filtration also implies that for each i, the natural map $B/TU_i \to B/TU_{i-1}$ is a torsor under the vector bundle $U_{i-1}/U_i \times B/TU_{i-1}$ on B/TU_{i-1}. Hence, the homotopy invariance (*cf.* Theorem 5.2) gives an isomorphism

$$\Omega_*^B(B/TU_{i-1} \times X) \xrightarrow{\cong} \Omega_*^B(B/TU_i \times X).$$

Composing these isomorphisms successively for $i = 1, \cdots, n$, we get

$$\Omega_*^B(X) \xrightarrow{\cong} \Omega_*^B(B/T \times X).$$

The canonical isomorphism of B-varieties $B \overset{T}{\times} X \cong B/T \times X$ and Proposition 5.4 together now prove (8.2) and hence (8.1). □

Proposition 8.2. *Let H be a possibly non-reductive group over k. Let $H = L \ltimes H^u$ be the Levi decomposition of H (which exists since k is of characteristic zero). Then the restriction map*

$$\Omega_*^H(X) \xrightarrow{r_{L,X}^H} \Omega_*^L(X) \tag{8.3}$$

is an isomorphism.

Proof. Since the ground field is of characteristic zero, the unipotent radical H^u of H is split over k. Now the proof is exactly same as the proof of Proposition 8.1, where we just have to replace B and T by H and L respectively. □

Notation: All results in the rest of this section will be proven with the rational coefficients. In order to simplify our notations, an abelian group A from now on will actually mean the $\mathbb{Q}$-vector space $A \otimes_{\mathbb{Z}} \mathbb{Q}$, and an inverse limit of abelian groups will mean the limit of the associated $\mathbb{Q}$-vector spaces. In particular, all cohomology groups will be considered with the rational coefficients and $\Omega_i^G(X)$ will mean

$$\Omega_i^G(X) := \varprojlim_j \left(\Omega_i^G(X)_j \otimes_{\mathbb{Z}} \mathbb{Q}\right).$$

Notice that this is same as $\Omega_i^G(X)\widehat{\otimes}_{\mathbb{Z}}\mathbb{Q}$ in our earlier notation.

8.1. The motivic cobordism theory. Before we prove our main results of this section, we recall the theory of motivic algebraic cobordism $MGL_{*,*}$ introduced by Voevodsky in [41]. This is a bi-graded ring cohomology theory in the category of smooth schemes over k. Levine has recently shown in [28] that $MGL_{*,*}$ extends uniquely to a bi-graded oriented Borel-Moore homology theory $MGL'_{*,*}$ on the category of all schemes over k. This homology theory has exterior products, homotopy invariance, localization exact sequence and Mayer-Vietoris among other properties (*cf.* [*loc. cit.*, Section 3]). Moreover, the universality of Levine-Morel cobordism theory implies that there is a unique map

$$\vartheta : \Omega_* \to MGL'_{2*,*}$$

of oriented Borel-Moore homology theories. Our motivation for studying the motivic cobordism theory in this text comes from the following result of Levine.

Theorem 8.3 ([29]). *For any $X \in \mathcal{V}_k$, the map ϑ_X is an isomorphism.*

We recall from [28] that for a smooth k-scheme X, there is a Hopkins-Morel spectral sequence

$$E_2^{p,q}(n) = \mathrm{CH}^{n-q}(X, 2n-p-q) \otimes \mathbb{L}^q \Rightarrow MGL^{p+q,n}(X) \tag{8.4}$$

which is an algebraic analogue of the Atiyah-Hirzebruch spectral sequence in complex cobordism.
If X is possibly singular, we embed it as a closed subscheme of a smooth scheme M. Then, the functoriality of the above spectral sequence with respect to an open immersion yields a spectral sequence

$$E_2^{p,q}(n) = \mathrm{CH}^{n-q}_X(M, 2n-p-q) \otimes \mathbb{L}^q \Rightarrow MGL^{p+q,n}_X(M)$$

of cohomology with support. Since the higher Chow groups and the motivic cobordism groups of M with support in X are canonically isomorphic to the higher Chow groups and the Borel-Moore motivic cobordism groups of X (*cf.* [1], [29, Section 3]), the above spectral sequence is identified with

$$E^2_{p,q}(n) = \mathrm{CH}_n(X, p) \otimes \mathbb{L}^q \Rightarrow MGL'_{2n+2q-p,n+q}(X). \tag{8.5}$$

Now, suppose that a finite group G acts on X. By embedding X equivariantly in a smooth G-scheme M, the formula

$$MGL'_{p,q}(X) := \mathrm{Hom}_{\mathcal{SH}(k)}\left(\Sigma^\infty_T M/(M-X), \Sigma^{p',q'} MGL\right)$$

(where $p' = 2\dim(M) - p, q' = \dim(M) - q$) shows that G acts naturally on $MGL'_{p,q}(X)$. It also acts on the higher Chow groups $\mathrm{CH}_p(X,q)$ likewise, where we just have to replace MGL in (8.5) by the Eilenberg-MacLane spectrum $H\mathbb{Z}$. Recall furthermore that MGL and $H\mathbb{Z}$ are ring spectra and the spectral sequence (8.4) (and hence (8.5)) is obtained by first showing that there is a natural quotient morphism of ring spectra $MGL \to H\mathbb{Z}$. In particular, for any G-scheme X, (8.5) is a spectral sequence of $\mathbb{Z}[G]$-modules. The reader will notice that the rational coefficients have not been used so far and Theorem 8.3 as well as the spectral sequence (8.5) hold integrally.

We shall now use rational coefficients everywhere and draw some consequences of Theorem 8.3 and (8.5). Since the functor of taking "G-invariants" is exact on the category of $\mathbb{Q}[G]$-modules, the spectral sequence (8.5) over the rationals, yields the spectral sequence of G-invariants

$$E'^{2}_{p,q}(n) = \left(\mathrm{CH}_n(X,p)\right)^G \otimes \mathbb{L}^q \Rightarrow \left(MGL'_{2n+2q-p,n+q}(X)\right)^G. \tag{8.6}$$

Recall that a connected and reductive group G over k is said to be *split*, if it contains a split maximal torus T over k such that G is given by a root datum relative to T. One knows that every connected and reductive group containing a split maximal torus is split (*cf.* [8, Chapter XXII, Proposition 2.1]). In such a case, the normalizer N of T in G and all its connected components are defined over k and the quotient N/T is the Weyl group W of the corresponding root datum. As an application of the spectral sequences (8.5) and (8.6), we get the following.

Lemma 8.4. *Let G be a connected reductive group with split maximal torus T and the associated Weyl group W. Let G act freely on a scheme X. Then, with rational coefficients, the pull-back map $\Omega_*(X/G) \to \left(\Omega_*(X/T)\right)^W$ (up to a shift) is an isomorphism.*

Proof. It follows from [17, Corollary 3.9] (see also [22, Corollary 8.9]) that in the case under consideration, the natural map $\mathrm{CH}_*(X/G,p) \to \left(\mathrm{CH}_*(X/T,p)\right)^W$ is an isomorphism for all $p \geq 0$. We can thus apply the spectral sequences (8.5) and (8.6) to conclude that the map $MGL'_{*,*}(X/G) \to \left(MGL'_{*,*}(X/T)\right)^W$ is an isomorphism. The lemma now follows from this isomorphism and Theorem 8.3. $\square$

Remark 8.5. The proof of Lemma 8.4 is based on the existence of the Atiyah-Hirzebruch spectral sequence in the motivic cobordism. There is no published proof of the existence of this spectral sequence, though it has been presented by the authors during various seminars. We can give another proof of the above lemma without using the spectral sequence as follows.

It was proven by Levine and Morel in [31, Theorems 4.1.28, 4.5.1] that there is a morphism of oriented Borel-Moore homology theories $\Omega_* \to \mathrm{CH}_*[\mathbf{t}]^{(\mathbf{t})}$ which is an isomorphism with rational coefficients. Here, $\mathrm{CH}_*[\mathbf{t}]$ is the polynomial module $\mathrm{CH}_*[t_1, t_2, \cdots]$ in infinitely many variables with $\deg(t_i) = i$. Recall also that $\mathrm{CH}_*[\mathbf{t}]^{(\mathbf{t})}$ is same as $\mathrm{CH}_*[\mathbf{t}]$ as $\mathrm{CH}_*(k)$-module.

Applying the above isomorphism to X/T and X/G and using the isomorphism $\mathrm{CH}_*(X/G) \xrightarrow{\cong} \left(\mathrm{CH}_*(X/T)\right)^W$ (*cf.* [10, Proposition 9]), we immediately get that the map $\Omega_*(X/G) \to \left(\Omega_*(X/T)\right)^W$ is an isomorphism.

Recall from Remark 4.6 that if a connected reductive group G acts on a scheme X, then the Weyl group W acts on $\Omega^T_*(X)$ where T is a maximal torus of G.

Theorem 8.6. [1] *Let G be a connected linear algebraic group and let L be a Levi subgroup of G with a split maximal torus T. Let W denote the Weyl group of L with respect to T. Then for any $X \in \mathcal{V}_G$, the natural map*

$$\Omega^G_*(X) \to \left(\Omega^T_*(X)\right)^W \tag{8.7}$$

is an isomorphism with rational coefficients.

Proof. By Proposition 8.2, we can assume that $G = L$ and hence G is a connected reductive group with split maximal torus T.
We choose a sequence of l_j-dimensional good pairs $\{(V_j, U_j)\}$ as in Theorem 6.1 for the G-action. Then, this is also a sequence of good pairs for the action of T. Setting $X^j_H = \{X \overset{H}{\times} U_j\}$ for any closed subgroup $H \subseteq G$, we see that $\{X^j_T\}$ is a sequence of W-schemes, each term of which has a free W-action. It follows from Lemma 8.4 (or Remark 8.5) that the smooth pull-back map

$$\Omega_{i+l_j-g}\left(X^j_G\right) \to \left(\Omega_{i+l_j-n}\left(X^j_T\right)\right)^W \tag{8.8}$$

is an isomorphism, where $\dim(G) = g$ and $\dim(T) = n$.
Since the action of W on the inverse system $\left\{\Omega_{i+l_j-n}\left(X^j_T\right)\right\}_j$ induces the similar action on the inverse limit and since the inverse limit commutes with taking the W-invariants, we get

$$\varprojlim_j \Omega_{i+l_j-g}\left(X^j_G\right) \xrightarrow{\cong} \left(\varprojlim_j \Omega_{i+l_j-n}\left(X^j_T\right)\right)^W. \tag{8.9}$$

Since the left and the right terms are same as $\Omega^G_i(X)$ and $\left(\Omega^T_i(X)\right)^W$ respectively by choice of our good pairs and Theorem 6.1, we conclude that $\Omega^G_i(X) \xrightarrow{\cong} \left(\Omega^T_i(X)\right)^W$. This completes the proof of the theorem. $\square$

Corollary 8.7. *Let $X \in \mathcal{V}_G$ be as in Theorem 8.6. Then the restriction map*

$$\Omega^G_*(X)_{\mathbb{Q}} \xrightarrow{r^G_{T,X}} \Omega^T_*(X)_{\mathbb{Q}} \tag{8.10}$$

is a split monomorphism which is natural for the morphisms in $\mathcal{V}_G$. In particular, if H is any closed subgroup of G, then there is a split injective map

$$\Omega^H_*(X)_{\mathbb{Q}} \xrightarrow{r^G_{T,X}} \Omega^T_*\left(G \overset{H}{\times} X\right)_{\mathbb{Q}}. \tag{8.11}$$

Proof. The first statement follows directly from Theorem 8.6, where the splitting is given by the trace map. The second statement follows from the first and Proposition 5.4. $\square$

[1] It has been shown recently in [15, Proposition 4.8] that the map $S(G) \to S(T)^W$ is an isomorphism over $\mathbb{Z}[t_G^{-1}]$, where t_G is the torsion index of G.

Before we apply Theorem 8.6 to study the rational cobordism rings of classifying spaces, we need the following topological analogue, which is much simpler to prove. Recall from (7.9) that if G is a complex Lie group and X is a finite CW-complex with a G-action, then its *equivariant complex cobordism* is defined as

$$MU^*_G(X) := MU^*\left(X \overset{G}{\times} EG\right). \tag{8.12}$$

Theorem 8.8. *Let G be a complex Lie group with a maximal torus T and Weyl group W. Then for any X as above, the natural map*

$$MU^*_G(X) \to \left(MU^*_T(X)\right)^W \tag{8.13}$$

is an isomorphism with rational coefficients.

Proof. As in the proof of Theorem 8.6, we can reduce to the case when G is reductive. It follows from the above definition of the equivariant complex cobordism and the similar definition of the equivariant singular cohomology of X, plus the Atiyah-Hirzebruch spectral sequence in topology that there is a spectral sequence

$$E_2^{p,q} = H^p_G(X, \mathbb{Q}) \otimes_{\mathbb{Q}} MU^q \Rightarrow MU^{p+q}_G(X). \tag{8.14}$$

Since the Atiyah-Hirzebruch spectral sequence degenerates rationally, we see that the above spectral sequence degenerates too. Since one knows that $H^*_G(X) \cong \left(H^*_T(X)\right)^W$ (*cf.* [4, Proposition 1]), the corresponding result for the cobordism follows. $\square$

Theorem 8.9. *For a connected linear algebraic group G over $\mathbb{C}$, the degree doubling map $\rho^G : \Omega^*(BG) \to MU^*(BG)$ (cf. Corollary 7.7) of $\mathbb{L}$-algebras, is an isomorphism with rational coefficients. In particular, the natural map of $\mathbb{Q}$-algebras*

$$\mathrm{CH}^*(BG)_{\mathbb{Q}} \xrightarrow{\overline{\rho}^G} MU^*(BG)\widehat{\otimes}_{\mathbb{L}}\mathbb{Q}$$

is an isomorphism.

Proof. To prove the first isomorphism, we can use Theorems 8.6 and 8.8 to reduce to the case of a torus. But this case is already known even with the integer coefficients (*cf.* (6.14) and [39]). The second isomorphism follows from the first and Proposition 7.2. $\square$

Remark 8.10. The map $\overline{\rho}^G : \mathrm{CH}^*(BG) \to MU^*(BG)\widehat{\otimes}_{\mathbb{L}}\mathbb{Z}$ was found by Totaro in [39] even before Levine and Morel discovered their algebraic cobordism. It was conjectured that the map $\overline{\rho}^G$ should be an isomorphism with the integer coefficients for a connected complex algebraic group G. Totaro modified this conjecture to an expectation that $\overline{\rho}^G$ should be an isomorphism after localization at a prime p such that $MU^*(BG)_{(p)}$ is concentrated in even degree. The above theorem proves the isomorphism in general with the rational coefficients. We also remark that the map $MU^*(BG)\widehat{\otimes}_{\mathbb{L}}\mathbb{Q} \to H^*(BG, \mathbb{Q})$ is an isomorphism (*cf.* [38]). The above result then shows that the cycle class map for the classifying space is an isomorphism with the rational coefficients. One wonders if the techniques of this paper could be applied to the algebraic version of the Brown-Peterson cobordism theory to prove the Totaro's modified conjecture. We do not know the answer yet.

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References

[1] S. Bloch, *Algebraic cycles and higher K-theory*, Advances in Math., 61, (1986), 267-304.

[2] S. Bloch, A. Ogus, *Gersten's conjecture and the homology of schemes*, Ann. Sci. École Norm. Sup., (4), 7, (1975), 181-201

[3] A. Borel, *Linear Algebraic groups*, Second edition, GTM 8, Springer-Verlag, (1991).

[4] M. Brion, *Equivariant cohomology and equivariant intersection theory*, Representation theory and algebraic geometry, NATO ASI series, C514, Kluwer Academic Publishers, (1997).

[5] M. Brion, *Equivariant Chow groups for torus actions*, Transform. Groups, 2, (1997), no. 3, 225-267.

[6] V. Buhstaber, A. Miscenko, *K-theory on the category of infinite cell complexes*, (Russian), Izv. Akad. Nauk SSSR Ser. Mat., 32, (1968), 560V604.

[7] V. Buhstaber, A. Miscenko, *Elements of infinite filtration in KK-theory*, (Russian), Dokl. Akad. Nauk SSSR, 178, (1968), 1234V1237.

[8] M. Demazure, A. Grothendieck, *Schémas en Groupes*, Lecture Notes in Math., 153, (1970), Springer-Verlag.

[9] D. Deshpande, *Algebraic Cobordism of Classifying Spaces*, MathArxiv, mathAG/0907.4437, (2009).

[10] D. Edidin, W. Graham, *Equivariant intersection theory*, Invent. Math., 131, (1998), 595-634.

[11] D. Edidin, W. Graham, *Riemann-Roch for equivariant Chow groups*, Duke Math. J., 102, (2000), no. 3, 567-594.

[12] J. Fogarty, F. Kirwan, D. Mumford, *Geometric Invariant Theory*, 3rd Edition, Springer-Verlag, (1994).

[13] W. Fulton, *Intersection Theory*, 2nd edition, Springer-Verlag, (1998).

[14] R. Joshua, *Algebraic K-theory and higher Chow groups of linear varieties*, Math. Proc. Cambridge Philos. Soc., 130, (2001), no. 1, 37-60.

[15] V. Kiritchenko, A. Krishna, *Equivariant cobordism of flag varieties and of symmetric varieties*, MathArxiv, mathAG/1104.1089, (2011).

[16] B. Kock, *Chow motifs and higher Chow theory of G/P*, Manuscripta Math., 70, (1991), no. 4, 363-372.

[17] A. Krishna, *Riemann-Roch for equivariant K-theory*, MathArxiv, mathAG/0906.1696, (2009).

[18] A. Krishna, *Equivariant K-theory and higher Chow groups of smooth varieties*, MathArxiv, mathAG/0906.3109, (2009).

[19] A. Krishna, *Equivariant cobordism of schemes*, MathArxiv, mathAG/1006.3176, (2010).
[20] A. Krishna, *Equivariant cobordism of schemes with torus action*, MathArxiv, mathAG/1010.6182, (2010).
[21] A. Krishna, *Cobordism of flag bundles*, MathArxiv, mathAG/1007.1083, (2010).
[22] A. Krishna, *Higher Chow groups of varieties with group actions*, Submitted to journal, (2011).
[23] A. Krishna, *The equivariant motivic cobordism*, In preparation, (2011).
[24] A. Krishna, V. Uma, *Cobordism rings of toric varieties*, MathArxiv, mathAG/1011.0573, (2010).
[25] P. Landweber, *Coherence, flatness and cobordism of classifying spaces*, Proc. Adv. study Inst. Alg. top., II, Aarhus, (1970), 256-269.
[26] P. Landweber, *Elements of infinite filtration in complex cobordism*, Math. Scand., 30, (1972), 223-226.
[27] P. Landweber, *Unique factorization in graded power series rings*, Proc. of the AMS., 42, (1974), 73-76.
[28] M. Levine, *Oriented cohomology, Borel-Moore homology, and algebraic cobordism*, Special volume in honor of Melvin Hochster, Michigan Math. J., 57, (2008).
[29] M. Levine, *Comparison of cobordism theories*, J. Algebra, 322, (2009), no. 9, 3291-3317.
[30] M. Levine, F. Morel, *Algebraic cobordism I*, Preprint, www.math.uiuc.edu/K-theory/0547, (2002).
[31] M. Levine, F. Morel, *Algebraic cobordism*, Springer Monographs in Mathematics, Springer, Berlin, (2007).
[32] M. Levine, R. Pandharipande, *Algebraic cobordism revisited*, Invent. Math., 176, (2009), no. 1, 63-130.
[33] I. Panin, K. Pimenov, O. Röndings, *On the relation of Voevodsky's algebraic cobordism to Quillen's K-theory*, Invent. Math., 175, (2009), no. 2, 435-451.
[34] D. Quillen, *Elementary proofs of some results of cobordism theory using Steenrod operations*, Advances in Math., 7, (1971), 29-56.
[35] T. Springer, *Linear algebraic groups*, Second edition, Progress in Math., 9, (1998), Birkhauser.
[36] H. Sumihiro, *Equivariant completion II*, J. Math. Kyoto, 15, (1975), 573-605.
[37] R. Thomason, *Equivariant algebraic vs. topological K-homology Atiyah-Segal-style*, Duke Math. J., 56, (1988), 589-636.
[38] B. Totaro, *Torsion algebraic cycles and complex cobordism*, J. Amer. Math. Soc., 10, (1997), no. 2, 467-493.
[39] B. Totaro, *The Chow ring of a classifying space*, Algebraic K-theory (Seattle, WA, 1997), Proc. Sympos. Pure Math., Amer. Math. Soc., 67, (1999), 249-281.

[40] G. Vezzosi, A. Vistoli, *Higher algebraic K-theory for actions of diagonalizable groups*, Invent. Math., 153, (2003), no. 1, 1-44.
[41] V. Voevodsky, $\mathbb{A}^1$*-homotopy theory*, Proceedings of the International Congress of Mathematicians, 1, (Berlin, 1998), Doc. Math., (1998), 579-604.

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