On Families of Weakly Admissible Filtered φ-Modules and the Adjoint Quotient of GL_d

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Abstract. We study the relation of the notion of weak admissibility in families of filtered φ-modules, as considered in [He], with the adjoint quotient. We show that the weakly admissible subset is an open subvariety in the fibers over the adjoint quotient. Further we determine the image of the weakly admissible set in the adjoint quotient generalizing earlier work of Breuil and Schneider.

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1. Introduction

Filtered φ-modules appear in p-adic Hodge-theory as a category of linear algebra data describing *crystalline* representations of the absolute Galois group of a local p-adic field. More precisely, there is an equivalence of categories between crystalline representations and *weakly admissible* filtered φ-modules, see [CF]. Here weak admissibility is a semi-stability condition relating the slopes of the φ-linear endomorphism Φ with the filtration.
In our companion paper [He] we define and study arithmetic families of filtered φ-modules and crystalline representations. Our families are parameterized by rigid analytic spaces or adic spaces in the sense of Huber, see [Hu2] for example. We show that the condition of being weakly admissible is an open condition [He, Theorem 1.1] and that there is an open subset of the weakly admissible locus over which there exists a family of crystalline representations giving rise to the family of filtered φ-modules [He, Theorem 1.3].
In this paper we study the weakly admissible locus in more detail. In the setting of period domains in the sense of Rapoport and Zink [RZ], the weakly admissible locus is an admissible open subset of a flag variety. Contrarily, the

weakly admissible locus in our set up has an algebraic nature as soon as we fix the Frobenius Φ, or even the conjugacy class of its semi-simplification. Further we analyze the image of the weakly admissible locus in the adjoint quotient. The question whether there exists a weakly admissible filtration for a fixed conjugacy class of the semisimplification of the Frobenius already appears in work of Breuil and Schneider [BS] on the p-adic Langlands correspondence. Unlike the characterization in [BS], our characterization of the set of automorphisms Φ for which there exists a weakly admissible filtration is purely in terms of the adjoint quotient of GL_d.
Our main results are as follows: Fix a finite extension K of $\mathbb{Q}_p$ and write K_0 for the maximal unramified extension of $\mathbb{Q}_p$ inside K. Let $d > 0$ be an integer and denote by $A \subset \mathrm{GL}_d$ the diagonal torus. For a dominant cocharacter

$$\nu : \mathbb{G}_{m,\bar{\mathbb{Q}}_p} \longrightarrow (\mathrm{Res}_{K/\mathbb{Q}_p} A)_{\bar{\mathbb{Q}}_p}$$

we write Gr_ν for the partial flag variety of $\mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_d$ parametrizing flags of "type ν", see section 3.1 for the precise definition. This variety is defined over the reflex field E of ν. As in [He, 4.1] we denote by

$$\mathfrak{D}_\nu = \big((\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_d)_E \times \mathrm{Gr}_\nu\big)/(\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_d)_E$$

the stack of filtered φ-modules with filtration of "type ν" on the category of adic spaces locally of finite type. The action of $(\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_d)_E$ is explicitly described in (3.4). Let W denote the Weyl group of GL_d. We will define a morphism

$$\alpha : \mathfrak{D}_\nu \longrightarrow (A/W)^{\mathrm{ad}}$$

to the adification (see [Hu2, Remark 4.6 (i)]) of the adjoint quotient A/W and prove the following theorem.

Theorem 1.1. *Let $x \in (A/W)^{\mathrm{ad}}_E$ and form the 2-fiber product*

$$\begin{array}{ccc} \alpha^{-1}(x)^{\mathrm{wa}} & \longrightarrow & \mathfrak{D}_\nu^{\mathrm{wa}} \\ \big\downarrow & & \big\downarrow \\ x & \longrightarrow & (A/W)^{\mathrm{ad}}_E. \end{array}$$

Then there Artin stack in schemes $\mathfrak{A}$ over the field $k(x)$ such that

$$\alpha^{-1}(x)^{\mathrm{wa}} = \mathfrak{A}^{\mathrm{ad}}.$$

The stack $\mathfrak{A}$ is the stack quotient of a quasi-projective $k(x)$-variety.

Further we determine the image of the weakly admissible locus $\mathfrak{D}_\nu^{\mathrm{wa}}$ under the morphism α. The description of this image works in the category of analytic spaces in the sense of Berkovich.

Theorem 1.2. *Let ν be a dominant coweight as above. There is a dominant coweight $\mu(\nu)$ of GL_d associated to ν such that*

$$\alpha^{-1}(x)^{\mathrm{wa}} \neq \emptyset \Longleftrightarrow x \in (A/W)_{\leq \mu(\nu)}.$$

Here $(A/W)_{\leq\mu(\nu)}$ is a Newton-stratum in the sense of Kottwitz [Ko].

The coweight $\mu(\nu)$ which appears in the theorem is explicit and defined in Definition 5.4

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2. Filtered φ-modules

Throughout this section we denote by F a topological field containing $\mathbb{Q}_p$ with a continuous valuation $v_F : F \to \Gamma_F \cup \{0\}$ in the sense of [Hu1, 2, Definition] that is Γ_F is a totally ordered abelian group (written multiplicative) and

$$\begin{aligned} v_F(0) &= 0 \\ v_F(1) &= 1 \\ v_F(ab) &= v_F(a)v_F(b) \\ v_F(a+b) &\leq \max\{v_F(a), v_F(b)\}, \end{aligned}$$

where the order on Γ_F is extended to $\Gamma_F \cup \{0\}$ by $0 < \gamma$ for all $\gamma \in \Gamma_F$. We will introduce the notion of a filtered φ-module with coefficients in F and define weak admissibility for these objects.

Recall that K_0 is an unramified extension of $\mathbb{Q}_p$ with residue field k and write $f = [K_0 : \mathbb{Q}_p]$. We write φ for the lift of the absolute Frobenius to K_0.

2.1. φ-modules with coefficients. In this subsection we define and study what we call *isocrystals over k with coefficients in F*.

Definition 2.1. An *isocrystal over k with coefficients in F* is a free $F \otimes_{\mathbb{Q}_p} K_0$-module D of finite rank together with an automorphism $\Phi : D \to D$ that is semi-linear with respect to $\mathrm{id} \otimes \varphi : F \otimes_{\mathbb{Q}_p} K_0 \to F \otimes_{\mathbb{Q}_p} K_0$.

A morphism $f : (D, \Phi) \to (D', \Phi')$ is an $F \otimes_{\mathbb{Q}_p} K_0$-linear map $f : D \to D'$ such that

$$f \circ \Phi = \Phi' \circ f.$$

The category of isocrystals over k with coefficients in F is denoted by $\mathrm{Isoc}(k)_F$.

It is easy to see that $\mathrm{Isoc}(k)_F$ is an F-linear abelian $\otimes$-category with the obvious notions of direct sums and tensor products.

Remark 2.2. (i) Given an $F \otimes_{\mathbb{Q}_p} K_0$-module D of finite type, the existence of a semi-linear automorphism $\Phi : D \to D$ implies that D is free over $F \otimes_{\mathbb{Q}_p} K_0$. This fact will be used in the sequel.

(ii) In the classical setting an isocrystal over k is a finite-dimensional K_0-vector space with φ-linear automorphism Φ, i.e. an object in $\mathrm{Isoc}(k)_{\mathbb{Q}_p}$.

If F is finite over $\mathbb{Q}_p$, then an isocrystal over k with coefficients in F is the same as an object $(D, \Phi) \in \mathrm{Isoc}(k)_{\mathbb{Q}_p}$ together with a map

$$F \longrightarrow \mathrm{End}_\Phi(D),$$

where the subscript Φ on the right hand side indicates that the endomorphisms commute with Φ (compare [DOR, VIII, 5] for example). This is clearly equivalent to our definition.

Let F' be an extension of F with valuation $v_{F'}: F' \to \Gamma_{F'} \cup \{0\}$ extending the valuation v_F. The *extension of scalars* from F to F' is the functor

$$- \otimes_F F' : \mathrm{Isoc}(k)_F \longrightarrow \mathrm{Isoc}(k)_{F'} \tag{2.1}$$

that maps $(D, \Phi) \in \mathrm{Isoc}(k)_F$ to the object $(D \otimes_F F', \Phi \otimes \mathrm{id})$.
If F' is a finite extension of F, then we also define the *restriction of scalars*

$$\epsilon_{F'/F} : \mathrm{Isoc}(k)_{F'} \longrightarrow \mathrm{Isoc}(k)_F. \tag{2.2}$$

This functor maps $(D', \Phi') \in \mathrm{Isoc}(k)_{F'}$ to itself, forgetting the F'-action but keeping the F-action.
We write $\Gamma_F \otimes \mathbb{Q}$ for the localisation of the abelian group Γ_F. Then every element $\gamma' \in \Gamma_F \otimes \mathbb{Q}$ can be written as a single tensor $\gamma \otimes r$ and we extend the total order of Γ_F to $\Gamma_F \otimes \mathbb{Q}$ by

$$a \otimes \tfrac{1}{m} < b \otimes \tfrac{1}{n} \Leftrightarrow a^n < b^m.$$

DEFINITION 2.3. Let $(D, \Phi) \in \mathrm{Isoc}(k)_F$ and $d = \mathrm{rk}_{F \otimes_{\mathbb{Q}_p} K_0} D$. The map $\Phi^f : D \to D$ is an F-linear automorphism of the fd-dimensional F-vector space D.
(i) Define the *Newton slope* of (D, Φ) as

$$\lambda_N^{(F)}(D, \Phi) = v_F(\det_F \Phi^f) \otimes \tfrac{1}{f^2 d} \in \Gamma_F \otimes \mathbb{Q}.$$

Here $\det_F$ means that we take the determinant of an F-linear map on an F-vector space.
(ii) Let $\lambda \in \Gamma_F \otimes \mathbb{Q}$. An object $(D, \Phi) \in \mathrm{Isoc}(k)_F$ is called *purely of Newton-slope* λ if for all Φ-stable $F \otimes_{\mathbb{Q}_p} K_0$-submodules $D' \subset D$ we have $\lambda_N^{(F)}(D', \Phi|_{D'}) = \lambda$.

LEMMA 2.4. *Let F' be an extension of F with valuation $v_{F'}$ extending v_F and $(D, \Phi) \in \mathrm{Isoc}(k)_F$. Then*

$$\lambda_N^{(F')}(D \otimes_F F', \Phi \otimes \mathrm{id}) = \lambda_N^{(F)}(D, \Phi).$$

If in addition F' is finite over F and $(D', \Phi') \in \mathrm{Isoc}(k)_{F'}$, then

$$\lambda_N^{(F)}(\epsilon_{F'/F}(D', \Phi')) = \lambda_N^{(F')}(D', \Phi').$$

Proof. These are straightforward computations. □

As the Newton slope is preserved under extension and restriction of scalars we will just write λ_N in the sequel.

Remark 2.5. Let $(D, \Phi) \in \mathrm{Isoc}(k)_{\mathbb{Q}_p}$ be an object of rank d and denote for the moment by v_p the usual p-adic valuation on $\mathbb{Q}_p$. Write $|-| = p^{-v_p(-)}$. Then the value group of the absolute value is $\Gamma_{\mathbb{Q}_p} = p^{\mathbb{Z}}$ and we identify $\Gamma_{\mathbb{Q}_p} \otimes \mathbb{Q}$ with the subgroup $p^{\mathbb{Q}}$ of $\mathbb{R} \backslash \{0\}$. Our definitions then imply

$$\lambda_N(D, \Phi) = p^{-\frac{1}{d} v_p(\det_{K_0} \Phi)}. \tag{2.3}$$

Here $v_p(\det_{K_0}\Phi)$ is the p-adic valuation of the determinant over K_0 of any matrix representing the semi-linear map Φ in some chosen basis. This matrix is well defined up to φ-conjugation and hence the valuation of the determinant is independent of choices. Note that (the negative of) the exponent in (2.3) is the usual Newton slope of the isocrystal (D,Φ) over k, compare [Zi] for example.

Proposition 2.6. *Let $(D,\Phi)\in \mathrm{Isoc}(k)_F$, then there exist unique elements $\lambda_1<\lambda_2<\cdots<\lambda_r\in\Gamma_F\otimes\mathbb{Q}$ and a unique decomposition*

$$D=D_1\oplus D_2\oplus\cdots\oplus D_r$$

of D into Φ-stable $F\otimes_{\mathbb{Q}_p}K_0$-submodules such that $(D_i,\Phi|_{D_i})$ is purely of Newton slope λ_i.

Proof. First we show the existence of such a decomposition. The uniqueness will then follow from Lemma 2.9 below.
Step 1: Assume first that there exists an embedding $\psi_0: K_0\hookrightarrow F$.
We obtain an isomorphism

$$F\otimes_{\mathbb{Q}_p}K_0 \xrightarrow{\cong} \textstyle\prod_{\psi:K_0\to F} F.$$

The endomorphism $\mathrm{id}\otimes\varphi$ on the left hand side translates to the shift of the factors on the right hand side. Further we obtain the corresponding decomposition

$$D=\prod_{\psi} V_\psi$$

into F-vector spaces V_ψ and F-linear isomorphisms

$$\Phi_\psi=\Phi|_{V_\psi}: V_\psi \xrightarrow{\cong} V_{\psi\circ\varphi}.$$

There is a bijection between the Φ-stable subspaces D' of D and the $\Phi^f|_{V_{\psi_0}}$-stable subspaces of V_{ψ_0} given by $D'\mapsto D'\cap V_{\psi_0}$.
Given $D'\subset D$ and $U=D'\cap V_{\psi_0}$ we have

$$\lambda_N(D',\Phi|_{D'})=v_F(\det_F\Phi^f|_U)\otimes\tfrac{1}{f\dim_F U}\in\Gamma_F\otimes\mathbb{Q}.$$

Hence the desired decomposition of D is induced by the decomposition of V_{ψ_0} into the maximal $\Phi^f|_{V_{\psi_0}}$-stable subspaces $U\subset V_{\psi_0}$ such that

$$v_F(\det_F\Phi^f|_{U'})\otimes\tfrac{1}{f\dim_F U'}=\lambda_i$$

for all Φ^f-stable subspaces $U'\subset U$.
Step 2: If there is no embedding ψ of K_0 into F, then we find a finite extension $F'=FK_0$ of F such that K_0 embeds into F'. We want to deduce the result from Step 1 by Galois descent. The extension F' is Galois over F, as K_0 is Galois over $\mathbb{Q}_p$. Further we extend the valuation from F to F' by setting $v_F(\mathcal{O}_{K_0}^\times)=\{1\}$, where $\mathcal{O}_{K_0}\subset K_0$ is the ring of integers.

Write (D', Φ') for the extension of scalars of $(D, \Phi) \in \mathrm{Isoc}(k)_F$ to $\mathrm{Isoc}(k)_{F'}$. Then there exists $\lambda_1 < \lambda_2 < \dots < \lambda_r \in \Gamma_F \otimes \mathbb{Q}$ and a decomposition

$$D' = D'_1 \oplus D'_2 \oplus \dots D'_r \tag{2.4}$$

such that the D'_i are Φ'-stable and $(D'_i, \Phi'|_{D'_i})$ is purely of slope λ_i. Now the action of the Galois group $\mathrm{Gal}(F'/F)$ preserves the valuation on F' and hence also the Newton slope of a Φ'-stable subobject of D'. It follows that $\mathrm{Gal}(F'/F)$ preserves the decomposition (2.4) and hence this decomposition descends to D. □

Remark 2.7. Proposition 2.6 replaces the slope decomposition in the classical context (c. f. [Zi, VI, 3] for example).

DEFINITION 2.8. Let $(D, \Phi) \in \mathrm{Isoc}(k)_F$ and denote by $D = \bigoplus D_i$ a decomposition of D into Φ-stable submodules purely of slope $\lambda_i \in \Gamma_F \otimes \mathbb{Q}$ as in Proposition 2.6. We will refer to this as the *slope decomposition.* Further, for $\lambda \in \Gamma_F \otimes \mathbb{Q}$ we define

$$D_\lambda = \begin{cases} D_i & , \ \lambda = \lambda_i \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 2.9. *Let $f : (D, \Phi) \to (D', \Phi')$ be a morphism in* $\mathrm{Isoc}(k)_F$. *Consider slope decompositions $D = \bigoplus D_i$ and $D' = \bigoplus D'_j$ as in Proposition* 2.6. *Then for all $\lambda \in \Gamma_F \otimes \mathbb{Q}$*

$$f(D_\lambda) \subset D'_\lambda.$$

Proof. This is an immediate consequence of $f \circ \Phi = \Phi' \circ f$. □

2.2. FILTERED ISOCRYSTALS WITH COEFFICIENTS. Recall that K is a totally ramified extension of K_0. We denote by $e = [K : K_0]$ the ramification index of K. In this section we define the basic object of our study.

DEFINITION 2.10. A *K-filtered isocrystal over k with coefficients in F* is a triple $(D, \Phi, \mathcal{F}^\bullet)$, where $(D, \Phi) \in \mathrm{Isoc}(k)_F$ and $\mathcal{F}^\bullet$ is a descending, separated and exhaustive $\mathbb{Z}$-filtration on $D_K = D \otimes_{K_0} K$ by (not necessarily free) $F \otimes_{\mathbb{Q}_p} K$-submodules.
A morphism

$$f : (D, \Phi, \mathcal{F}^\bullet) \longrightarrow (D', \Phi', \mathcal{F}'^\bullet)$$

is a morphism $f : (D, \Phi) \to (D', \Phi')$ in $\mathrm{Isoc}(k)_F$ such that $f \otimes \mathrm{id} : D_K \to D'_K$ respects the filtrations.
The category of K-filtered isocrystals over k with coefficients in F is denoted by $\mathrm{Fil}\,\mathrm{Isoc}(k)_F^K$.

It is easy to see that $\mathrm{Fil}\,\mathrm{Isoc}(k)_F^K$ is an F-linear $\otimes$-category. Further it has obvious notions of kernels, cokernels and exact sequences. For an extension F' of F we again have an *extension of scalars* like in (2.1),

$$- \otimes_F F' : \mathrm{Fil}\,\mathrm{Isoc}(k)_F^K \longrightarrow \mathrm{Fil}\,\mathrm{Isoc}(k)_{F'}^K.$$

If F' is finite over F, we also have a *restriction of scalars* like in (2.2),

$$\epsilon_{F'/F} : \mathrm{Fil}\,\mathrm{Isoc}(k)^K_{F'} \longrightarrow \mathrm{Fil}\,\mathrm{Isoc}(k)^K_F.$$

In the following we will often shorten our notation and just write D for an object $(D, \Phi, \mathcal{F}^\bullet) \in \mathrm{Fil}\,\mathrm{Isoc}(k)^K_F$.
We now want to develop a slope theory for filtered isocrystals and define weakly admissible objects.

DEFINITION 2.11. Let $(D, \Phi, \mathcal{F}^\bullet) \in \mathrm{Fil}\,\mathrm{Isoc}(k)^K_F$. We define

$$\begin{aligned}
\deg \mathcal{F}^\bullet &= \sum_{i\in\mathbb{Z}} \tfrac{1}{ef} i \dim_F \mathrm{gr}_i \mathcal{F}^\bullet \\
\deg_F(D) &= (v_F(\det_F \Phi^f) \otimes \tfrac{1}{f^2})^{-1}\ v_F(p)^{\deg(\mathcal{F}^\bullet)} \in \Gamma_F \otimes \mathbb{Q} \\
\mu_F(D) &= \deg_F(D)(1 \otimes \tfrac{1}{d}) \in \Gamma_F \otimes \mathbb{Q}.
\end{aligned}$$

We call $\mu_F(D)$ the *slope* of D.

Remark 2.12. As in Lemma 2.4, one easily sees that the slope μ_F is preserved under extension and restriction of scalars. Hence we will just write μ in the sequel.

Now we have a Harder-Narasimhan formalism as in [DOR, Chapter 1]. The only difference is that our valuations are written multiplicatively, while in the usual theory they are written additively. We will only sketch the proofs and refer to [DOR] for the details.

LEMMA 2.13. *Let*

$$0 \longrightarrow D' \longrightarrow D \longrightarrow D'' \longrightarrow 0$$

be a short exact sequence in $\mathrm{Fil}\,\mathrm{Isoc}(k)^K_F$. *Then*

$$\deg_F(D) = \deg_F(D') \deg_F(D'').$$

Further

$$\max\{\mu(D'), \mu(D'')\} \geq \mu(D) \geq \min\{\mu(D'), \mu(D'')\}.$$

The sequence $\mu(D'), \mu(D), \mu(D'')$ *is either strictly increasing or strictly decreasing or stationary.*

Proof. The first assertion is obvious from the definitions and the second is a direct consequence. □

LEMMA 2.14. *Let* $f : D \to D'$ *be a morphism in* $\mathrm{Fil}\,\mathrm{Isoc}(k)^K_F$. *Then*

$$\deg_F(\mathrm{coim}\, f) \geq \deg_F(\mathrm{im}\, f).$$

Proof. Replacing D by $\mathrm{coim}\, f$ and D' by $\mathrm{im}\, f$, we may assume that f is an isomorphism in $\mathrm{Isoc}(k)_F$. Now the assertion follows easily from

$$(f \otimes \mathrm{id})(\mathcal{F}^i) \subset \mathcal{F}'^i.$$

□

DEFINITION 2.15. An object $(D, \Phi, \mathcal{F}^\bullet) \in \mathrm{Fil\,Isoc}(k)_F^K$ is called *semi-stable* if, for all Φ-stable subobjects $D' \subset D$, we have $\mu(D') \geq \mu(D)$. It is called *stable* if the inequality is strict for all proper subobjects. Finally D is called *weakly admissible* if it is semi-stable of slope 1.

Note that semi-stability is defined using "$\geq$" instead of "$\leq$" (as in [DOR]), since our valuations are written multiplicatively.

Remark 2.16. Let $(D, \Phi, \mathcal{F}^\bullet) \in \mathrm{Fil\,Isoc}(k)_{\mathbb{Q}_p}^K$. Using the notations of Remark 2.5, we find

$$\mu(D) = p^{\frac{1}{d}\left(v_p(\det_{K_0}\Phi) - \sum_i i \dim_K(\mathcal{F}^i/\mathcal{F}^{i+1})\right)}.$$

Hence we see that D is weakly admissible if and only if it is weakly admissible in the sense of [CF, 3.4].

PROPOSITION 2.17. *Let $D, D' \in \mathrm{Fil\,Isoc}(k)_F^K$ be semi-stable objects.*
(i) *If $\mu(D) < \mu(D')$, then* $\mathrm{Hom}(D, D') = 0$.
(ii) *If $\mu(D) = \mu(D') = \mu$, then for all $f \in \mathrm{Hom}(D, D')$ we have* $\mathrm{im}\, f \cong \mathrm{coim}\, f$ *and the objects* $\ker f$, $\mathrm{coker}\, f$ *and* $\mathrm{im}\, f$ *are semi-stable of slope μ.*

Proof. The proof is the same as in [DOR, Proposition 1.1.20] □

COROLLARY 2.18. *Let $\mu \in \Gamma \otimes \mathbb{Q}$, then the full subcategory of* $\mathrm{Fil\,Isoc}(k)_F^K$ *consisting of semi-stable objects of slope μ is an abelian, artinian and noetherian category which is stable under extensions. The simple objects are exactly the stable ones.*

Proof. The proof is the same as the proof of [DOR, Corollary 1.2.21]. □

The main result of this section is the existence of a Harder-Narasimhan filtration for the objects in $\mathrm{Fil\,Isoc}(k)_F^K$. The existence of this filtration will also imply that semi-stability (and hence weak admissibility) is preserved under extension and restriction of scalars.

PROPOSITION 2.19. *Let $D \in \mathrm{Fil\,Isoc}(k)_F^K$, then there exist unique elements $\mu_1 < \mu_2 < \cdots < \mu_r \in \Gamma_F \otimes \mathbb{Q}$ and a unique filtration*

$$0 = D_0 \subset D_1 \subset D_2 \subset \cdots \subset D_r = D$$

of D in $\mathrm{Fil\,Isoc}(k)_F^K$ *such that D_i/D_{i-1} is semi-stable of slope μ_i.*

Proof. The proof is similar to the proof of [DOR, Proposition 1.3.1 (a)].
First we prove the existence of the filtration. The uniqueness will then follow from Lemma 2.21 below.
By the existence of the slope decomposition in Proposition 2.6, the set

$$\{\mu(D') \mid D' \subset D \text{ stable under } \Phi\}$$

is finite. Hence there is a unique minimal element μ_1 and we claim that there is a maximal subobject $D_1 \subset D$ of slope μ_1 which then must be semi-stable. This follows, as the sum of two subobjects of slope μ_1 has again slope μ_1, by Lemma 2.13 and the minimality of μ_1.
Proceeding with D/D_1 the claim follows by induction. □

DEFINITION 2.20. Let $D \in \mathrm{Fil}\,\mathrm{Isoc}(k)_F^K$ and denote by

$$0 = D_0 \subset D_1 \subset D_2 \subset \cdots \subset D_r = D$$

a filtration as in Proposition 2.19. This filtration is called the *Harder-Narasimhan filtration* of D. For $\mu \in \Gamma_F \otimes \mathbb{Q}$ we define

$$D_{(\mu)} = \begin{cases} 0 & \text{if } \mu < \mu_1 \\ D_i & \text{if } \mu_i \leq \mu < \mu_{i+1} \\ D & \text{if } \mu \geq \mu_r. \end{cases}$$

LEMMA 2.21. *Let $f : D \to D'$ be a morphism in* $\mathrm{Fil}\,\mathrm{Isoc}(k)_F^K$ *and fix filtrations of D and D' as in Proposition* 2.19. *Let $\mu \in \Gamma_F \otimes \mathbb{Q}$, then* (*with the notation of Definition* 2.20)*:*

$$f(D_{(\mu)}) \subset D'_{(\mu)}.$$

Proof. The proof is the same as in [DOR, Proposition 1.3.1 (b)]. □

COROLLARY 2.22. *Let F' be an extension of F with valuation $v_{F'}$ extending v_F and $D \in$* $\mathrm{Fil}\,\mathrm{Isoc}(k)_F^K$. *Then D is semi-stable of slope μ, if and only if $D' = D \otimes_F F'$ is semi-stable of slope μ.*
If in addition F' is finite over F, then $D' \in$ $\mathrm{Fil}\,\mathrm{Isoc}(k)_{F'}^K$ *is semi-stable of slope μ, if and only if $\epsilon_{F'/F}(D') \in$* $\mathrm{Fil}\,\mathrm{Isoc}(k)_F^K$ *is semi-stable of slope μ.*

Proof. First it is clear that semi-stability of D' implies semi-stability of D, as every Φ-stable F-subspace of D defines a Φ-stable F'-subspace of D' which has the same slope.
Now assume that D is semi-stable. We may assume that F' is finitely generated over F, as every counterexample for the semi-stability condition is defined over a finitely generated extension. Then F' is an algebraic extension of a purely transcendental extension and we can treat both cases separately.
Assume first that F' is an algebraic extension of F. We may replace it by its Galois hull and denote by $G = \mathrm{Gal}(F'/F)$ the Galois group of F' over F. Then the action of G preserves the valuation on F'. We denote by

$$0 = D'_0 \subset D'_1 \subset D'_2 \subset \cdots \subset D'_r = D'$$

the Harder-Narasimhan filtration of D'. The action of G commutes with Φ and preserves the filtration $\mathcal{F}^\bullet \otimes_F F'$ of $D' \otimes_{K_0} K$. It follows that it preserves the slope of a Φ-stable subobject and hence preserves the Harder-Narasimhan filtration. It follows that the filtration descends to F and hence it can only have one step, as D is semi-stable.
Assume now that F' is purely transcendental over F. By the above discussion of algebraic extensions we may also assume that F is algebraically closed. Again we write $G = \mathrm{Aut}(F'/F)$ for the group of F-automorphisms of F'. As above we only need to check that G preserves the slope of a Φ-stable subobject of D'. Let $U \subset D'$ be such a Φ-stable subspace. Then U is a direct sum of

indecomposable Φ-modules U_i such that the isomorphism class of U_i is defined over F. This can be seen as follows: We decompose D as a product

$$D = V_1 \times \cdots \times V_f$$

of F-vector spaces such that Φ induces a linear map $\Phi_i : V_i \to V_{i+1}$, where $V_{f+1} := V_1$. We can choose bases of these vector spaces such that the matrix of Φ is represented by the tuple $(\mathrm{id}, \dots, \mathrm{id}, A)$ for some $A \in \mathrm{GL}_d(F)$ in Jordan canonical form. If we choose a similar canonical form for $\Phi|_U$, then it is clear that every Jordan-Block for $\Phi|_U$ is a sub-Jordan-block of A.
It follows that $\det_{F'}(\Phi^f|_U) \in F$ and hence the action of G preserves the slope of U, which yields the claim by the same descent argument as above.
Now assume that F' is finite over F and D' is a semi-stable object of $\mathrm{Fil}\,\mathrm{Isoc}(k)^K_{F'}$. Consider the Harder-Narasimhan filtration of $\epsilon_{F'/F}(D')$. By Lemma 2.21 the filtration steps are stable under the operation of F'. Hence the filtration can have only one step.
If conversely $\epsilon_{F'/F}(D')$ is known to be semi-stable, then every Φ-stable F'-subspace of D' *is* a Φ-stable F-subspace of $\epsilon_{F'/F}(D')$, and hence semi-stability of D' follows. $\square$

3. Families of filtered φ-modules

It is shown in [He, 4] that the stack of weakly admissible filtered φ-modules is an open substack of the stack of filtered φ-modules. We briefly recall this result before we study the weakly admissible locus in the fibers over the adjoint quotient. We write Rig_E for the category of rigid analytic spaces over a finite extension E of $\mathbb{Q}_p$ (see [BGR] and also [Bo] for an introduction to rigid geometry) and $\mathrm{Ad}_E^{\mathrm{lft}}$ for the category of adic spaces locally of finite type over E, see [Hu2].

3.1. Stacks of filtered φ-modules. Let d be a positive integer and ν an algebraic cocharacter

$$\nu : \bar{\mathbb{Q}}_p^\times \longrightarrow (\mathrm{Res}_{K/\mathbb{Q}_p} A_K)(\bar{\mathbb{Q}}_p), \tag{3.1}$$

where $A \subset \mathrm{GL}_d$ is the diagonal torus. We assume that this cocharacter is dominant with respect to the restriction B of the Borel subgroup of upper triangular matrices in $(\mathrm{GL}_d)_K$. We write $E \subset \bar{\mathbb{Q}}_p$ for the *reflex field* of ν, i.e. the field of definition of the cocharacter ν, see below for a precise characterization. Let Δ denote the set of simple roots (defined over $\bar{\mathbb{Q}}_p$) of $\mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_d$ with respect to B and denote by $\Delta_\nu \subset \Delta$ the set of all simple roots α such that $\langle \alpha, \nu\rangle = 0$. Here $\langle -, -\rangle$ is the canonical pairing between characters and cocharacters. We write P_ν for the parabolic subgroup of $(\mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_d)$ containing B and corresponding to $\Delta_\nu \subset \Delta$. This parabolic subgroup is defined over E, and the quotient by this parabolic is a projective E-variety

$$\mathrm{Gr}_{K,\nu} = (\mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_d)_E / P_\nu \tag{3.2}$$

representing the functor

$$S \mapsto \{\text{filtrations } \mathcal{F}^\bullet \text{ of } \mathcal{O}_S \otimes_{\mathbb{Q}_p} K^d \text{ of type } \nu\}$$

on the category of E-schemes. Here the filtrations are locally on S direct summands. Being of *type* ν means the following. Assume that the cocharacter

$$\nu : \bar{\mathbb{Q}}_p^\times \longrightarrow \prod_{\psi : K \to \bar{\mathbb{Q}}_p} \mathrm{GL}_d(\bar{\mathbb{Q}}_p)$$

is given by cocharacters

$$\nu_\psi : \lambda \mapsto \mathrm{diag}((\lambda^{i_1(\psi)})^{(m_1(\psi))}, \dots, (\lambda^{i_r(\psi)})^{(m_r(\psi))})$$

for some integers $i_j(\psi) \in \mathbb{Z}$, with $i_j(\psi) \neq i_{j'}(\psi)$ for $j \neq j'$, and multiplicities $m_j(\psi) > 0$. Then any point $\mathcal{F}^\bullet \in \mathrm{Gr}_{K,\nu}(\bar{\mathbb{Q}}_p)$ is a filtration $\prod_\psi \mathcal{F}^\bullet_\psi$ of $\prod_\psi \bar{\mathbb{Q}}_p^d$ such that

$$\dim_{\bar{\mathbb{Q}}_p} \mathrm{gr}_i(\mathcal{F}^\bullet_\psi) = \begin{cases} 0 & \text{if } i \notin \{i_1(\psi), \dots, i_r(\psi)\} \\ m_j(\psi) & \text{if } i = i_j(\psi). \end{cases}$$

In terms of the integers $i_j(\psi)$ and $m_j(\psi)$ the fact that ν is dominant means that $i_j(\psi) \geq i_{j+1}(\psi)$ for all j and ψ. The reflex field E of the character ν is characterized by the requirement

$$\mathrm{Gal}(\bar{\mathbb{Q}}_p/E) = \{\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \mid i_j(\psi) = i_j(\sigma\psi),\ m_j(\psi) = m_j(\sigma\psi)\}.$$

We denote by $\mathrm{Gr}^{\mathrm{rig}}_{K,\nu}$ resp. $\mathrm{Gr}^{\mathrm{ad}}_{K,\nu}$ the associated rigid space, resp. the associated adic space (cf. [BGR, 9.3.4] and [Hu2, Remark 4.6 (i)]).
Given ν as in (3.1) and denoting as before by E the reflex field of ν, we consider the following fpqc-stack $\mathfrak{D}_\nu$ on the category Rig_E (resp. on the category $\mathrm{Ad}^{\mathrm{lft}}_E$). For $X \in \mathrm{Rig}_E$ (resp. $\mathrm{Ad}^{\mathrm{lft}}_E$) the groupoid $\mathfrak{D}_\nu(X)$ consists of triples $(D, \Phi, \mathcal{F}^\bullet)$, where D is a coherent $\mathcal{O}_X \otimes_{\mathbb{Q}_p} K_0$-module which is locally on X free over $\mathcal{O}_X \otimes_{\mathbb{Q}_p} K_0$ and $\Phi : D \to D$ is an $\mathrm{id} \otimes \varphi$-linear automorphism. Finally $\mathcal{F}^\bullet$ is a filtration of $D_K = D \otimes_{K_0} K$ of type ν, i.e. after choosing fpqc-locally on X a basis of D, the filtration $\mathcal{F}^\bullet$ induces a map to $\mathrm{Gr}^{\mathrm{rig}}_{K,\nu}$ (resp. $\mathrm{Gr}^{\mathrm{ad}}_{K,\nu}$), compare also [PR, 5.a].

One easily sees that the stack $\mathfrak{D}_\nu$ is the stack quotient of the rigid space

$$X_\nu = (\mathrm{Res}_{K_0/\mathbb{Q}_p} \mathrm{GL}_d)^{\mathrm{rig}}_E \times \mathrm{Gr}^{\mathrm{rig}}_{K,\nu} \tag{3.3}$$

by the φ-conjugation action of $(\mathrm{Res}_{K_0/\mathbb{Q}_p} \mathrm{GL}_d)^{\mathrm{rig}}_E$ given by

$$(A, \mathcal{F}^\bullet) \cdot g = (g^{-1} A \varphi(g), g^{-1}\mathcal{F}^\bullet). \tag{3.4}$$

Here the canonical map $X_\nu \to \mathfrak{D}_\nu$ is given by

$$(A, \mathcal{F}^\bullet) \mapsto (\mathcal{O}_{X_\nu} \otimes_{\mathbb{Q}_p} K_0^d, A(\mathrm{id} \otimes \varphi), \mathcal{F}^\bullet).$$

3.2. THE WEAKLY ADMISSIBLE LOCUS. Fix a cocharacter ν with reflex field E as in the previous section. If $X \in \mathrm{Ad}_E^{\mathrm{lft}}$ and $x \in X$, then our definitions imply that, given $(D, \Phi, \mathcal{F}^\bullet) \in \mathfrak{D}_\nu(X)$, we have

$$(D \otimes k(x), \Phi \otimes \mathrm{id}, \mathcal{F}^\bullet \otimes k(x)) \in \mathrm{Fil}\,\mathrm{Isoc}(k)_{k(x)}^K$$

One of the main results of [He] is concerned with the structure of the weakly admissible locus in the stacks $\mathfrak{D}_\nu$ defined above.

THEOREM 3.1. *Let ν be a cocharacter as in* (3.1) *and X be an adic space locally of finite type over the reflex field of ν. If $(D, \Phi, \mathcal{F}^\bullet) \in \mathfrak{D}_\nu(X)$, then the weakly admissible locus*

$$X^{\mathrm{wa}} = \{x \in X \mid (D \otimes k(x), \Phi \otimes \mathrm{id}, \mathcal{F}^\bullet \otimes k(x)) \textit{ is weakly admissible}\}$$

is an open subset. Especially it has a canonical structure of an adic space.

Proof. This is [He, Theorem 4.1]. □

We can define a substack $\mathfrak{D}_\nu^{\mathrm{wa}} \subset \mathfrak{D}_\nu$ consisting of the weakly admissible filtered isocrystals. More precisely, for an adic space X the groupoid $\mathfrak{D}_\nu^{\mathrm{wa}}(X)$ consists of those triples $(D, \Phi, \mathcal{F}^\bullet)$ such that $(D \otimes k(x), \Phi \otimes \mathrm{id}, \mathcal{F}^\bullet \otimes k(x))$ is weakly admissible for all $x \in X$. Thanks to Corollary 2.22 it is clear that this is again an fpqc-stack. The following result is now an obvious consequence of Theorem 3.1.

COROLLARY 3.2. *The stack $\mathfrak{D}_\nu^{\mathrm{wa}}$ on the category of adic spaces locally of finite type over the reflex field of ν is an open substack of $\mathfrak{D}_\nu$.*

4. THE FIBERS OVER THE ADJOINT QUOTIENT

We now come to the main results of this paper. We want to link the weakly admissible locus in

$$(\mathrm{Res}_{K_0/\mathbb{Q}_p} \mathrm{GL}_d \times \mathrm{Gr}_{K,\nu})^{\mathrm{ad}}$$

as considered in the previous section to the adjoint quotient of the group GL_d. This relation was studied by Breuil and Schneider in [BS]. In this section we show that the fibers over the adjoint quotient are (base changes of) analytifications of schemes over $\mathbb{Q}_p$ and hence the period stacks considered here have a much more algebraic nature than the period spaces considered by Rapoport and Zink in [RZ]. In the next section we determine the image of the weakly admissible locus in the adjoint quotient and identify it with a closed Newton-stratum in the sense of Kottwitz [Ko].

First we need to recall some notations and facts about the adjoint quotient from [Ko]. We write $\mathrm{GL}_d = \mathrm{GL}(V)$ for the general linear group over $\mathbb{Q}_p$, where $V = \mathbb{Q}_p^d$, and $B \subset \mathrm{GL}_d$ for the Borel subgroup of upper triangular matrices. Further we denote by $A \subset B$ the diagonal torus and identify $X_*(A)$ and $X^*(A)$ with $\mathbb{Z}^d$ in the usual way, i.e. $(m_1, \dots, m_d) \in \mathbb{Z}^d$ defines

$$\big(t \mapsto \mathrm{diag}(t^{m_1}, \dots, t^{m_d})\big) \in X^*(A)$$
$$\text{resp. } \big(\mathrm{diag}(t_1, \dots, t_d) \mapsto t_1^{m_1} \dots t_d^{m_d}\big) \in X_*(A).$$

Let $\Delta = \{\alpha_1, \dots, \alpha_{d-1}\}$ be the simple roots defined by B, i.e. $\langle \alpha_i, \nu \rangle = \nu_i - \nu_{i+1}$ for all $\nu \in X_*(A)$. We also choose lifts

$$\omega_i = (1^{(i)}, 0^{(d-i)}) \in \mathbb{Z}^d = X^*(A)$$

of the dual basis $\varpi_1, \dots \varpi_{d-1} \in X^*(A \cap \mathrm{SL}_d)$ of the coroots. Finally $W = S_d$ denotes the Weyl group of (GL_d, A). There is a map

$$c : A \longrightarrow \mathbb{A}^{d-1} \times \mathbb{G}_m$$

which maps an element of A to the coefficients of its characteristic polynomial. This morphism identifies A/W with $\mathbb{A}^{d-1} \times \mathbb{G}_m$.
Now we will define a map

$$G = \mathrm{Res}_{K_0/\mathbb{Q}_p}(\mathrm{GL}_d)_{K_0} \longrightarrow A/W \tag{4.1}$$

that is invariant under φ-conjugation on the left side. Recall that we have identifications $\mathrm{GL}_d = \mathrm{GL}(V)$ and $(\mathrm{GL}_d)_{K_0} = \mathrm{GL}(V_0)$, where $V_0 = V \otimes_{\mathbb{Q}_p} K_0$. For an $\mathbb{Q}_p$-algebra R and $g \in G(R)$ we have the $R \otimes_{\mathbb{Q}_p} K_0$-linear automorphism $\Phi_g^f = (g(\mathrm{id} \otimes \varphi))^f$ of $R \otimes_{\mathbb{Q}_p} V_0$. Its characteristic polynomial is an element of $(R \otimes_{\mathbb{Q}_p} K_0)[T]$. Now this polynomial is invariant under $\mathrm{id} \otimes \varphi$ and hence it already lies in $R[T]$ which can be seen as follows: First we may assume that K_0 embeds into R, as $R = R' \cap (R \otimes_{\mathbb{Q}_p} K_0) \subset R' \otimes_{\mathbb{Q}_p} K_0$ for any extension R' of R. As in the proof of Proposition 2.6, we choose decompositions $R \otimes_{\mathbb{Q}_p} K_0 = \prod_{i=1}^f R$ and $D = \prod_{i=1}^f V_i$ such that Φ_g maps V_i to V_{i+1}, where $V_{f+1} = V_1$. Then Φ_g^f induces automorphisms Φ_i on V_i. It follows that

$$\mathrm{charpoly}(\Phi_g^f) = (\mathrm{charpoly}(\Phi_1), \dots, \mathrm{charpoly}(\Phi_f)).$$

However, $\Phi_{i+1} = \Phi|_{V_i} \circ \Phi_i \circ (\Phi|_{V_i})^{-1}$ and hence $\mathrm{charpoly}(\Phi_i) = \mathrm{charpoly}(\Phi_{i+1})$. We define the morphism in (4.1) by mapping $g \in G(R)$ to the coefficients of this polynomial. It is easy to check that this morphism is invariant under φ-conjugation on G and hence we get morphisms

$$\begin{tikzcd} G \times \mathrm{Gr}_{K,\nu} \arrow[r, "\widetilde{\alpha}"] \arrow[d] & (A/W)_E \arrow[d, "="] \\ \mathcal{D}_\nu \arrow[r, "\alpha"] & (A/W)_E, \end{tikzcd} \tag{4.2}$$

where $\mathcal{D}_\nu$ is the stack-quotient

$$\mathcal{D}_\nu = (G_E \times \mathrm{Gr}_{K,\nu})/G$$

on the category of E-schemes, where the action of G on $G_E \times \mathrm{Gr}_{K,\nu}$ is the same as in (3.4). Here ν is a cocharacter as in (3.1) and E is the reflex field of ν. We also write α and $\widetilde{\alpha}$ for the analytification of these morphisms.

THEOREM 4.1. *Let $x \in (A/W)_E^{\mathrm{ad}}$ and ν be a cocharacter as in* (3.1). *Then there exists a quasi-projective $k(x)$-scheme X which is an open subscheme of $\widetilde{\alpha}^{-1}(x)$ such that the weakly admissible locus in the fiber over x is given by*

$$\widetilde{\alpha}^{-1}(x)^{\mathrm{wa}} = X^{\mathrm{ad}}.$$

Proof. The proof will be similar to the proof of [He, Theorem 4.1].
Let $x = (c_1, \dots, c_d) \in k(x)^{d-1} \times k(x)^\times$ and let v_x denote the (multiplicative) valuation on $k(x)$. First note that

$$c_d = \det_{k(x)\otimes_{\mathbb{Q}_p} K_0}(\Phi^f) = \det_{k(x)}(\Phi^f)^{1/f}$$

and hence $\widetilde{\alpha}^{-1}(x)^{\mathrm{wa}} = \emptyset$ unless

$$v_x(c_d)^{1/f} = v_x(p)^{\sum_{j\in\mathbb{Z}} j \frac{1}{ef} \dim_{k(x)} \mathrm{gr}_j \mathcal{F}^\bullet},$$

where $\mathcal{F}^\bullet$ is the universal filtration on $\mathrm{Gr}_{K,\nu}$. In the following we will assume that this condition is satisfied. For $i \in \{0, \dots, d\}$, consider the following functor on the category of $\mathbb{Q}_p$-schemes,

$$S \longmapsto \left\{ \begin{array}{c} \mathcal{E} \subset \mathcal{O}_S \otimes_{\mathbb{Q}_p} V_0 \text{ locally free } \mathcal{O}_S \otimes_{\mathbb{Q}_p} K_0\text{-submodule} \\ \text{of rank } i \text{ that is locally on } S \text{ a direct summand} \end{array} \right\}.$$

Using the theory of Quot-schemes (see [FGA, Thm 3.1] for example) this functor is easily seen to be representable by a projective $\mathbb{Q}_p$-scheme $\mathrm{Gr}_{K_0,i}$.
We let $G = \mathrm{Res}_{K_0/\mathbb{Q}_p} \mathrm{GL}_d$ act on $\mathrm{Gr}_{K_0,i}$ in the following way: for a $\mathbb{Q}_p$-scheme S, let $A \in G(S)$ and $\mathcal{E} \in \mathrm{Gr}_{K_0,i}(S)$. We get a linear endomorphism A of $\mathcal{O}_S \otimes_{\mathbb{Q}_p} V_0$ and define the action of A on $\mathcal{E}$ by

$$A \cdot \mathcal{E} = A((\mathrm{id} \otimes \varphi)(\mathcal{E})).$$

Write

$$a : G \times \mathrm{Gr}_{K_0,i} \longrightarrow \mathrm{Gr}_{K_0,i}$$

for this action and consider the subscheme $Z_i \subset G \times \mathrm{Gr}_{K_0,i}$ defined by the following fiber product:

$$\begin{tikzcd} Z_i \arrow[r] \arrow[d] & G \times \mathrm{Gr}_{K_0,i} \arrow[d, "a\times \mathrm{id}"] \\ \mathrm{Gr}_{K_0,i} \arrow[r, "\Delta"] & \mathrm{Gr}_{K_0,i} \times \mathrm{Gr}_{K_0,i}\,. \end{tikzcd}$$

An S-valued point x of the scheme Z_i is a pair (g_x, U_x), where $g_x \in G(S)$ is a linear automorphism of $\mathcal{O}_S \otimes_{\mathbb{Q}_p} V_0$ and U_x is an $\mathcal{O}_S \otimes_{\mathbb{Q}_p} K_0$-submodule of rank i stable under $\Phi_x = g_x(\mathrm{id} \otimes \varphi)$. The scheme Z_i is projective over G via the first projection

$$\mathrm{pr}_i : Z_i \longrightarrow G.$$

Further we denote by $f_i \in \Gamma(Z_i, \mathcal{O}_{Z_i})$ the global section defined by

$$f_i(g_x, U_x) = \det((g_x(\mathrm{id} \otimes \varphi))^f|_{U_x})$$

(recall $f = [K_0 : \mathbb{Q}_p]$), where the determinant is the determinant as $\mathcal{O}_{Z_i}$-modules. We also write f_i for the global section on the associated adic space Z_i^{ad}.
We write $\mathcal{E}$ for the pullback of the universal bundle on Z_i to $Z_i \times \mathrm{Gr}_{K,\nu}$ and $\mathcal{F}^\bullet$ for the pullback of the universal filtration on $\mathrm{Gr}_{K,\nu}$. Then the fiber product

$$\mathcal{G}^\bullet = (\mathcal{E} \otimes_{K_0} K) \cap \mathcal{F}^\bullet$$

is a filtration of $\mathcal{E}\otimes_{K_0} K$ by coherent sheaves. By the semi-continuity theorem the function

$$h_i : x \longmapsto \sum_{j\in\mathbb{Z}} j\ \tfrac{1}{ef}\dim_{\kappa(x)}\mathrm{gr}_j\ \mathcal{G}^\bullet$$

is upper semi-continuous on $Z_i\times \mathrm{Gr}_{K,\nu}$ and hence so is

$$h_i^{\mathrm{ad}} : x \longmapsto \sum_{j\in\mathbb{Z}} j\ \tfrac{1}{ef}\dim_{k(x)}\mathrm{gr}_j\ (\mathcal{G}^\bullet)^{\mathrm{ad}}.$$

For $m\in\mathbb{Z}$ we write $Y_{i,m}\subset Z_i\times\mathrm{Gr}_{K,\nu}$ (resp. $Y_{i,m}^{\mathrm{ad}}\subset Z_i^{\mathrm{ad}}\times \mathrm{Gr}_{K,\nu}^{\mathrm{ad}}$) for the closed subscheme (resp. the closed adic subspace)

$$\begin{aligned} Y_{i,m} &= \{y\in Z_i\times \mathrm{Gr}_{K,\nu}\mid h_i(y)\geq m\},\\ Y_{i,m}^{\mathrm{ad}} &= \{y\in Z_i^{\mathrm{ad}}\times\mathrm{Gr}_{K,\nu}^{\mathrm{ad}}\mid h_i^{\mathrm{ad}}(y)\geq m\}.\end{aligned}$$

Then the definitions imply that

$$\begin{aligned} \mathrm{pr}_{i,m} &: Y_{i,m}\longrightarrow G\times\mathrm{Gr}_{K,\nu}\\ \mathrm{pr}_{i,m} &: Y_{i,m}^{\mathrm{ad}}\longrightarrow (G\times\mathrm{Gr}_{K,\nu})^{\mathrm{ad}}\end{aligned}$$

are proper morphism. Now

$$S_{i,m}=\{y=(g_y,U_y,\mathcal{F}_y^\bullet)\in Y_{i,m}\times_{(G\times\mathrm{Gr}_{K,\nu})}\widetilde{\alpha}^{-1}(x)\mid v_y(f_i(g_y,U_y))>v_y(p)^{f^2m}\}$$

is a union of connected components of

$$Y_{i,m}\times_{(G\times \mathrm{Gr}_{K,\nu})}\widetilde{\alpha}^{-1}(x),$$

which can be seen as follows: Let $\lambda_1,\dots,\lambda_d$ denote the zeros of the polynomial

$$X^d+c_1X^{d-1}+\dots+c_{d-1}X+c_d.$$

Then every possible value of the f_i is a product of some of the λ_i and hence f_i can take only finitely many values.
We conclude that the subset $\bigcup_{i,m}\mathrm{pr}_{i,m}(S_{i,m})$ is closed and claim that

$$\widetilde{\alpha}^{-1}(x)^{\mathrm{wa}}=\big(\widetilde{\alpha}^{-1}(x)\backslash\bigcup_{i,m}\mathrm{pr}_{i,m}(S_{i,m})\big)^{\mathrm{ad}}.$$

Indeed, let $z=(g_z,\mathcal{F}_z^\bullet)\in\widetilde{\alpha}^{-1}(x)\subset G^{\mathrm{ad}}\times\mathrm{Gr}_{K,\nu}^{\mathrm{ad}}$. Then the object

$$(k(z)\otimes V_0, g_z(\mathrm{id}\otimes\varphi),\mathcal{F}_z^\bullet)$$

is not weakly admissible if and only if there exists a $g_z(\mathrm{id}\otimes\varphi)$-stable subspace $U_z\subset k(z)\otimes V_0$ of some rank, violating the weak admissibility condition. This means $z\in\bigcup_{i,m}\mathrm{pr}_{i,m}(S_{i,m})^{\mathrm{ad}}$. Here we implicitly use that fact that weak admissibility is stable under extension of scalars (see Corollary 2.22).

□

Remark 4.2. In view of the period domains considered in [RZ] it can be surprising that this weakly admissible locus is indeed the adification of a scheme, not just an analytic space. The main reason is the following: In [RZ] the isocrystal is fixed and the counter examples one has to exclude for the weak admissibility condition are parametrized by the $\mathbb{Q}_p$-valued points of an algebraic variety. In our setting the isocrystal is not fixed and the Frobenius Φ may vary. Hence the set of counter examples is the algebraic variety itself rather than its $\mathbb{Q}_p$-valued points.

Example 4.3. This example illustrates the difference with period spaces in the sense of Rapoport-Zink. Let $K = \mathbb{Q}_p$ and $d = 2$. We consider the Frobenius $\Phi = \mathrm{diag}(p, p)$ and fix the filtration $\mathcal{F}^\bullet$ such that

$$\dim \mathcal{F}^i = \begin{cases} 2 & i \leq 0 \\ 1 & i = 1, 2 \\ 0 & i \geq 3. \end{cases}$$

In this case the flag variety $\mathrm{Gr}_{K,\nu}$ is the projective line $\mathbb{P}^1$ and the period space of [RZ] (or rather [DOR]) is the Drinfeld upper halfplane $\mathbb{P}^1 \backslash \mathbb{P}^1(\mathbb{Q}_p)$ which is not a scheme. On the other hand the weakly admissible set in the sense discussed above is obviously empty.

COROLLARY 4.4. *Let $x \in (A/W)^{\mathrm{ad}}_E$ and consider the 2-fiber product*

$$\begin{tikzcd} \alpha^{-1}(x)^{\mathrm{wa}} \arrow[r] \arrow[d] & \mathfrak{D}_\nu^{\mathrm{wa}} \arrow[d] \\ x \arrow[r] & (A/W)^{\mathrm{ad}}_E. \end{tikzcd}$$

Then there exists an Artin stack in schemes $\mathfrak{A}$ over the field $k(x)$ such that

$$\alpha^{-1}(x)^{\mathrm{wa}} \cong \mathfrak{A}^{\mathrm{ad}}$$

Proof. This is an immediate consequence of Theorem 4.1 □

We end the discussion of the fibers over the adjoint quotient by discussing three examples.

Example 4.5. Let $K = \mathbb{Q}_p$ and $d = 3$. We take $\Phi = \mathrm{diag}(1, p, p^2)$ and fix the type of the filtration $\mathcal{F}^\bullet$ such that

$$\dim \mathcal{F}^i = \begin{cases} 3 & i \leq 0 \\ 2 & i = 1 \\ 1 & i = 2 \\ 0 & i \geq 3. \end{cases}$$

We write $G = \mathrm{GL}_3$ and $B \subset G$ for the Borel subgroup of upper triangular matrices. Further $X = G/B$ is the full flag variety, and we are interested in the weakly admissible locus in X. One easily checks that

$$X^{\mathrm{wa}} = \{\mathcal{F}^\bullet \in X \mid \mathcal{F}^1 \cap V_1 = 0, \text{ and } \mathcal{F}^2 \not\subset V_{12}\},$$

where $0 \subset V_1 \subset V_{12} \subset \mathbb{Q}_p^3$ is the standard flag fixed by the Borel B. The subset X^{wa} is obviously stable under B and, in fact,

$$X^{\mathrm{wa}} = Bw_0B/B,$$

where w_0 is the longest Weyl group element. If x denotes the image of Φ in the adjoint quotient, then Φ is a representative of the unique φ-conjugacy class in G mapping to x and we further have $\alpha^{-1}(x) = A\backslash G/B$ and

$$\alpha^{-1}(x)^{\mathrm{wa}} = A\backslash Bw_0B/B.$$

Example 4.6. We use the same notations as in the example above, but this time $\Phi = \mathrm{diag}(1, 1, p^3)$. Then

$$X^{\mathrm{wa}} = \{\mathcal{F}^\bullet \in X \mid \mathcal{F}^1 \cap V_{12} = 0\}.$$

As $\dim V_{12} = \dim \mathcal{F}^1 = 2$ it follows that $X^{\mathrm{wa}} = \emptyset$. In this case there is a second φ-conjugacy class in G mapping to the same point in the adjoint quotient as Φ. A representative of this second φ-conjugacy class is given by

$$\Phi' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p^3 \end{pmatrix}.$$

Example 4.7. In this example let $d = 2$ and $\Phi = \mathrm{diag}(1, p)$. Let K be a ramified extension of $\mathbb{Q}_p$ of degree e and consider flags of the type $(1, \dots, 1)$, i.e. the cocharacter is defined over $\mathbb{Q}_p$, the only non-trivial filtration step is $\mathcal{F}^1 = (\mathcal{F}_i^1)_{i=1\dots,e}$ and the base change of the flag variety $X = \mathrm{Gr}_{K,\nu}$ to K is

$$X_K = \mathbb{P}^1_K \times \cdots \times \mathbb{P}^1_K.$$

The weakly admissible locus in X_K is given by

$$X_K^{\mathrm{wa}} = \{\mathcal{F}^\bullet = (\mathcal{F}^\bullet)_i \mid \mathcal{F}_i^1 \neq \infty \text{ for all } i \in \{1, \dots, e\}\}.$$

Let $G = \mathrm{Res}_{K/\mathbb{Q}_p} \mathrm{GL}_2$ and $B \subset G$ the Weil-restriction of the Borel subgroup of upper triangular matrices. Again we write w_0 for the longest Weyl group element of G. Then

$$X^{\mathrm{wa}} = Bw_0B/B \subset X = G/B.$$

If again x denotes the image of Φ in the adjoint quotient, then $\alpha^{-1}(x) = A\backslash X$ and

$$\alpha^{-1}(x)^{\mathrm{wa}} = A\backslash Bw_0B/B.$$

5. Newton strata and weak admissibility

The proof of Theorem 4.1 suggests that the weakly admissible locus in the fibers over a point in A/W does only depend on the valuation of the zeros of the characteristic polynomial associated to the points of the adjoint quotient. Hence we want to extend the result that the fibers over the adjoint quotient are nice spaces to the pre-image of the Newton strata in the adjoint quotient. Here we work in the category of analytic spaces in the sense of Berkovich (see [Be]), as it is not obvious how to generalize the notion of Newton strata (as defined in [Ko]) to adic spaces. Though the weakly admissible locus is not a Berkovich space in general [He, Example 4.4], we show that it becomes a Berkovich space, if we restrict ourselves to the pre-images of the Newton strata. Further we want to identify the image of the weakly admissible locus in the adjoint quotient with a (closed) Newton-stratum. As usual we will write $\mathscr{H}(x)$ for the residue field at a point x in an analytic space and $X^{\rm an}$ for the analytic space associated to a scheme X.

5.1. NEWTON STRATA. We first need to recall more notations from [Ko]. We write $\mathfrak{a} = X_*(A) \otimes_{\mathbb{Z}} \mathbb{R}$, and $\mathfrak{a}_{\rm dom} \subset \mathfrak{a}$ for the subset of dominant elements, i.e the elements $\mu \in \mathfrak{a}$ such that $\langle \alpha_i, \mu\rangle \geq 0$ for all $i \in \{1, \dots, d-1\}$. For $c = (c_1, \dots, c_d) \in (A/W)^{\rm an}$ we write

$$d_c = (-v_c(c_1), \dots, -v_c(c_d)) \in \widetilde{\mathbb{R}}^{d-1} \times \mathbb{R}, \tag{5.1}$$

where v_c denotes the (additive) valuation on $\mathscr{H}(c)$ normalized by $v_c(p) = 1$, and $\widetilde{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$. Note that there is a sign in (5.1), as Kottwitz uses a different sign convention. For $a \in A^{\rm an}$ define $\nu_a \in \mathfrak{a}$ by requiring

$$\langle \lambda, \nu_a\rangle = -v_a(\lambda(a))$$

for all $\lambda \in X^*(A)$, where we write v_a for the (additive) valuation on $\mathscr{H}(a)$. By [Ko, Proposition 1.4.1] there is a continuous map $r : \mathfrak{a} \to \mathfrak{a}_{\rm dom}$ mapping $x \in \mathfrak{a}$ to the dominant element with the smallest distance to x, and this map extends in a continuous way to $\widetilde{\mathbb{R}}^{d-1} \times \mathbb{R}$. Here $\mathfrak{a} \subset \widetilde{\mathbb{R}}^{d-1} \times \mathbb{R}$ via the chosen identification $X_*(A) = \mathbb{Z}^d$. Then we find that $r(d_{c(a)})$ is the unique dominant element in the W-orbit of ν_a. This follows from [Ko, Theorem 1.5.1] for all $a \in A(\bar{\mathbb{Q}}_p)$ and, for an arbitrary point, from the fact that $A(\bar{\mathbb{Q}}_p)$ is dense in $A^{\rm an}$ and the continuity of the construction. Note that $r(d_c) = (-v_c(\lambda_1) \geq \dots \geq -v_c(\lambda_d))$ if the λ_i are the roots of the characteristic polynomial associated to c and Kottwitz shows that this does only depend on d_c and not on c itself.

DEFINITION 5.1. For $\mu \in \mathfrak{a}_{\rm dom}$ we define

$$\begin{aligned}(A/W)_\mu &= \{c \in A/W \mid r(d_c) = \mu\}\\ (A/W)_{\leq\mu} &= \{c \in A/W \mid r(d_c) \leq \mu\}.\end{aligned}$$

Here " $\leq$ " is the usual dominance order on dominant coweights. We will call the first of these subsets the *Newton stratum* defined by μ and the second the *closed Newton stratum* defined by μ.

We need another description of these sets to identify them as analytic subspaces of the adjoint quotient.

PROPOSITION 5.2. *Let $\mu \in \mathfrak{a}_{\rm dom}$ and $I_\mu = \{i \in \{1, \dots, d-1\} \mid \langle \alpha_i, \mu\rangle = 0\}$. Then*

$$(A/W)_\mu = \left\{ c = (c_1 \dots, c_d) \in (\mathbb{A}^{d-1} \times \mathbb{G}_m)^{\rm an} \;\middle|\; \begin{array}{c} v_c(c_i) \geq -\langle \omega_i, \mu\rangle \ , i \in I_\mu \\ v_c(c_i) = -\langle \omega_i, \mu\rangle \ , i \notin I_\mu \end{array} \right\}$$

$$(A/W)_{\leq\mu} = \left\{ c = (c_1 \dots, c_d) \in (\mathbb{A}^{d-1} \times \mathbb{G}_m)^{\rm an} \;\middle|\; \begin{array}{c} v_c(c_i) \geq -\langle \omega_i, \mu\rangle \ , i \neq d \\ v_c(c_d) = -\langle \omega_d, \mu\rangle. \end{array} \right\}$$

Proof. For all points in $(A/W)^{\rm an}(\bar{\mathbb{Q}}_p)$ this follows from [Ko, Theorem 1.5.2]. Again the proposition follows from continuity, and the fact that the points in $(A/W)^{\rm an}(\bar{\mathbb{Q}}_p)$ are dense in $(A/W)^{\rm an}$. $\square$

The category of (strict) analytic spaces is a full subcategory of the category of adic spaces locally of finite type, see [Hu3, 8.3] and the references cited there. Hence we can restrict the stacks $\mathfrak{D}_\nu$ and $\mathfrak{D}_\nu^{\mathrm{wa}}$ to the category of analytic spaces. We write again $\mathfrak{D}_\nu$ and $\mathfrak{D}_\nu^{\mathrm{wa}}$ for these restrictions. Further we write $\widetilde{\alpha}^{\mathrm{an}}$ (resp. α^{an}) for the analytifications of the morphisms defined in (4.2).

THEOREM 5.3. *Let ν be a cocharacter as in* (3.1) *and $\mu \in \mathfrak{a}_{\mathrm{dom}}$. Then the weakly admissible locus in the inverse image $(\widetilde{\alpha}^{\mathrm{an}})^{-1}((A/W)^{\mathrm{an}}_\mu)$ is an analytic space.*

Proof. The proof is almost identical with the proof of Theorem 4.1. If we replace Z_i by

$$\tilde{Z}_i = (Z_i)^{\mathrm{an}} \times_{(G\times \mathrm{Gr}_{K,\nu})^{\mathrm{an}}} (\widetilde{\alpha}^{\mathrm{an}})^{-1}\big((A/W)^{\mathrm{an}}_\mu\big)$$

and $Y_{i,m}$ by

$$\tilde{Y}_{i,m} = Y_{i,m} \times_{(G\times \mathrm{Gr}_{K,\nu})^{\mathrm{an}}} (\widetilde{\alpha}^{\mathrm{an}})^{-1}\big((A/W)^{\mathrm{an}}_\mu\big).$$

Here the functions f_i are not locally constant on $\tilde{Z}_i$ and $\tilde{Y}_{i,m}$, but their valuations (or absolute values) are. As in the proof of Theorem 4.1[1] it follows that

$$\tilde{S}_{i,m} = \{y = (g_y, U_y, \mathcal{F}^\bullet_y) \in \tilde{Y}_{i,m} \mid |f_i(g_y, U_y)|_y > |p|^{f^2 m}\}$$

is a union of connected components of $\tilde{Y}_{i,m}$ and we use the properness of the projection $\mathrm{pr}_{i,m} : \tilde{Y}_{i,m} \to (\widetilde{\alpha}^{\mathrm{an}})^{-1}\big((A/W)^{\mathrm{an}}_\mu\big)$ to conclude that

$$(\widetilde{\alpha}^{\mathrm{an}})^{-1}\big((A/W)^{\mathrm{an}}_\mu\big)^{\mathrm{wa}} = (\widetilde{\alpha}^{\mathrm{an}})^{-1}\big((A/W)^{\mathrm{an}}_\mu\big) \setminus \bigcup \mathrm{pr}_{i,m}(\tilde{S}_{i,m})$$

is an open subspace. □

5.2. THE IMAGE OF THE WEAKLY ADMISSIBLE LOCUS. In this section we determine the image of the weakly admissible locus under the map defined in (4.1). In the case of a regular cocharacter ν it was shown by Breuil and Schneider that the set of points $a \in A$ such that there exists a weakly admissible filtered φ-module $(D, \Phi, \mathcal{F}^\bullet)$ with $(\Phi^f)^{\mathrm{ss}} = a$ is an affinoid domain, see [BS, Proposition 3.2]. Here we extend this result to the general case and give a description of this image purely in terms of the adjoint quotient A/W. The difference with the description in [BS] is that we do not need to fix an order of the eigenvalues corresponding to the order of their valuations. We fix a coweight ν as in (3.1). This coweight determines the jumps of the filtration $\mathcal{F}^\bullet$ on $\mathrm{Gr}_{K,\nu}$. After passing to $\bar{\mathbb{Q}}_p$ the filtration is given by

$$\mathcal{F}^\bullet = \prod_\psi \mathcal{F}^\bullet_\psi$$

where the product runs over all embeddings $\psi : K \hookrightarrow \bar{\mathbb{Q}}_p$.
We write $\{x_{\psi,1} > x_{\psi,2} > \dots > x_{\psi,r}\}$ for the jumps of the filtration $\mathcal{F}^\bullet_\psi$, i.e.

$$\mathrm{gr}_i\, \mathcal{F}^\bullet_\psi \neq 0 \Leftrightarrow i \in \{x_{\psi,1}, \dots, x_{\psi,r}\}.$$

[1]Note that in this case this does not show that the weakly admissible locus is a scheme, as our base is not a point but the Newton-stratum $(A/W)^{\mathrm{an}}_\mu$ which is not a scheme.

Further denote by $n_{\psi,i}$ the rank of $\mathcal{F}_\psi^{x_{\psi,i}}$ and write

$$m_{\psi,j}(i) = \max(0, n_{\psi,j} + i - d).$$

This is the minimal dimension of the intersection of $\mathcal{F}_\psi^{x_{\psi,j}}$ with a subspace of dimension i. For $i \in \{0, \dots, d\}$ define

$$l_i = \sum_\psi \tfrac{1}{ef} \left(\sum_{j=1}^{r-1} (x_{\psi,j} - x_{\psi,j+1}) m_{\psi,j}(i) + x_{\psi,r} m_{\psi,r}(i) \right). \tag{5.2}$$

DEFINITION 5.4. For a cocharacter ν and $i \in \{1, \dots, d\}$ define l_i as in (5.2). Define a rational dominant coweight $\mu(\nu) \in \mathfrak{a}_{\mathrm{dom}}$ by requiring that

$$\tfrac{1}{f}\langle \omega_i, \mu(\nu)\rangle = -l_i \text{ for all } i \in \{1, \dots, d\}.$$

The following result generalizes [BS, Proposition 3.2].

THEOREM 5.5. *Let ν be a cocharacter as in* (3.1) *and define $\mu(\nu)$ as in Definition* 5.4. *Let $x \in (A/W)^{\mathrm{an}}$, then $(\widetilde{\alpha}^{\mathrm{an}})^{-1}(x)^{\mathrm{wa}} \neq \emptyset$ if and only if $x \in (A/W)^{\mathrm{an}}_{\leq \mu(\nu)}$.*

Proof. Let $c = (c_1, \dots, c_d) \in (A/W)^{\mathrm{an}}_{\leq\mu(\nu)}$ and denote by $\lambda_1, \dots, \lambda_t$ the roots of

$$X^d + c_1 X^{d-1} + \dots + c_{d-1}X + c_d$$

with multiplicities m_i in some finite extension L of $\mathscr{H}(c)$ containing K_0. Let $D = L \otimes_{\mathbb{Q}_p} V_0 \cong \prod_{i=1}^d L^d$ and

$$g = (\mathrm{id}, \dots, \mathrm{id}, A) \in \prod_{i=1}^f \mathrm{GL}_d(L) \cong G(L),$$

where A is a matrix consisting of t Jordan blocks of size m_i with diagonal entries λ_i. Now the pair $(D, \Phi) = (D, g(\mathrm{id} \otimes \varphi)) \in \mathrm{Isoc}(k)_L$ has the property that there are only finitely many Φ-stable subobjects $D' \subset D$. If $D' \subset D$ is a rank i subobject then

$$t_N(D') := \tfrac{1}{f^2} v_p(\det \Phi^f|_{D'}) = \tfrac{1}{f} \sum_{j=1}^t m'_j v_p(\lambda_j)$$

for some multiplicities m'_j, where we write v_p for the additive valuation. Write $a = (\lambda_1^{(m_1)}, \dots, \lambda_t^{(m_t)}) \in A^{\mathrm{an}}$, then $c(a) = c$ and $r(d_{c(a)}) \leq \mu(\nu)$ by assumption. It follows that

$$\begin{aligned} t_N(D') &= \tfrac{1}{f} v_p(w\omega_i(a)) & &= -\tfrac{1}{f}\langle w\omega_i, \nu_a\rangle \\ &= -\tfrac{1}{f}\langle w'\omega_i, r(d_{c(a)})\rangle & &\geq -\tfrac{1}{f}\langle \omega_i, r(d_{c(a)})\rangle \\ &\geq -\tfrac{1}{f}\langle \omega_i, \mu(\nu)\rangle & &= l_i. \end{aligned}$$

for some $w, w' \in W$. Now for all Φ-stable $D' \subset D$ consider the open subset

$$U_{D'} \subset \mathrm{Gr}_{K,\nu} \otimes_{\mathbb{Q}_p} K_0$$

of all filtrations $\mathcal{F}^\bullet$ such that $\dim(\mathcal{F}_\psi^{x_{\psi,j}} \cap D'_K) = \max(0, n_{\psi,j} + i - d)$ for all embeddings ψ. This is open as the right hand side is the minimal possible dimension of such an intersection. Since $\mathrm{Gr}_{K,\nu}$ is geometrically irreducible we find that the intersection $\bigcap_{D' \subset D} U_{D'}$ is non-empty and hence there exists an F-valued point $\mathcal{F}^\bullet$ in this intersection, where F is some extension of L. Now we have $(D \otimes_{K_0} F, \Phi \otimes \mathrm{id}, \mathcal{F}^\bullet) \in \mathrm{Fil}\,\mathrm{Isoc}(k)_F^K$ and this object is weakly admissible since for all Φ-stable $D' \subset D$ we have

$$\deg(D') = l_i - t_N(D') \leq 0$$

where i is the rank of the subobject D' (and here we write the degree additively). Further, by the definition of g, we find that g maps to c under the map $\widetilde{\alpha}$.

Conversely assume that $c \in (A/W)^{\mathrm{an}}$ such that $\emptyset \neq \widetilde{\alpha}^{-1}(c)$. Let $(D, \Phi, \mathcal{F}^\bullet)$ be an F-valued point of this fiber for some field F containing K_0. Then D decomposes into $D_1 \times \cdots \times D_f$ and we denote by $\mu_1, \dots, \mu_t$ the distinct eigenvalues of $\Phi^f|_{D_1}$ and by d_i their multiplicities (as zeros of the characteristic polynomial). We write $(\lambda_1, \dots, \lambda_d) = (\mu_1^{(d_1)}, \dots, \mu_t^{(d_t)})$. Then $c = c(\lambda_1, \dots \lambda_d)$ and we claim that

$$\tfrac{1}{f} \sum_{j \in I} v_p(\lambda_j) \geq l_i$$

for all $I \subset \{1, \dots, d\}$ with $\sharp I = i$. This claim clearly implies $c \in (A/W)^{\mathrm{an}}_{\leq \mu(\nu)}$. Let $I \subset \{1, \dots, d\}$ and write $(\lambda_j)_{j \in I} = ({\lambda'_1}^{(m_1)}, \dots, {\lambda'_t}^{(m_t)})$, where we assume that the λ'_j are pairwise distinct. Then $\sum_{j=1}^t m_j = i = \sharp I$. Using the Jordan canonical form on easily sees that there exists a subobject $D' \subset D$ such that

$$(\Phi^f|_{D' \cap D_1})^{\mathrm{ss}} = \mathrm{diag}({\lambda'_1}^{(m_1)}, \dots {\lambda'_t}^{(m_t)})$$

and hence

$$\sum_{j \geq 1} m_j \tfrac{1}{f} v_p(\lambda'_j) \geq \sum_{j \in \mathbb{Z}} j \tfrac{1}{ef} \dim \mathrm{gr}_j\, D'_K \geq l_i,$$

which yields the claim. $\square$

We end by giving two examples of closed Newton strata in the adjoint quotient.

Example 5.6. Let $K = \mathbb{Q}_p$ and $d = 3$. We fix the cocharacter ν as in Example 4.5 and Example 4.6, i.e.

$$\dim \mathcal{F}^i = \begin{cases} 3 & i \leq 0 \\ 2 & i = 1 \\ 1 & i = 2 \\ 0 & i \geq 3. \end{cases}$$

One easily checks that $l_1 = 0$, $l_2 = 1$ and $l_3 = 3$, i.e.

$$\mu(\nu) : t \longmapsto \mathrm{diag}(1, t^{-1}, t^{-2}).$$

The image of the weakly admissible locus in the adjoint quotient is given by

$$(A/W)^{\mathrm{an}}_{\leq\mu(\nu)} = \left\{ c=(c_1,c_2,c_3)\in\mathbb{A}^2\times\mathbb{G}_m \;\middle|\; \begin{array}{l} v_c(c_1)\geq 0,\\ v_c(c_2)\geq 1,\\ v_c(c_3)=3. \end{array} \right\}$$

If $a=(a_1,a_2,a_3)\in A$ with $v_a(a_1)\leq v_a(a_2)\leq v_a(a_3)$, then [BS, Proposition 3.2] says that there exists a weakly admissible filtered φ-module $(D,\Phi,\mathcal{F}^\bullet)$ with filtration of type ν such that $\Phi^{\mathrm{ss}}=a$ if and only if

$$\begin{cases} 0\leq v_a(a_1)\\ 1\leq v_a(a_1)+v_a(a_2)\\ 3=v_a(a_1)+v_a(a_2)+v_a(a_3). \end{cases}$$

This is clearly equivalent to our condition in the adjoint quotient. This result also explains Example 4.5 and Example 4.6.

Example 5.7. Again we let $K=\mathbb{Q}_p$ and $d=3$. Fix a cocharacter ν such that

$$\dim\mathcal{F}^i = \begin{cases} 3 & i\leq 0\\ 2 & i=1\\ 0 & i\geq 2. \end{cases}$$

One easily checks that $l_1=0$, $l_2=1$ and $l_3=2$, and the image of the weakly admissible locus is

$$(A/W)^{\mathrm{an}}_{\leq\mu(\nu)} = \left\{ c=(c_1,c_2,c_3)\in\mathbb{A}^2\times\mathbb{G}_m \;\middle|\; \begin{array}{l} v_c(c_1)\geq 0,\\ v_c(c_2)\geq 1,\\ v_c(c_3)=2. \end{array} \right\}$$

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