Extending Characters from Hall Subgroups

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Abstract. Suppose that G is a finite π-separable group. A classical result asserts that all irreducible characters of a Hall π-subgroup H of G extend to G if and only if H has a normal complement in G. Now, we fix a prime p and analyze when only the p'-degree irreducible characters of H extend to G.

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1. Introduction

We come back to an old result of C. Sah ([18]) that asserts that in a finite π-separable group, all irreducible complex characters of a Hall π-subgroup H of G extend to G if and only if G has a normal π-complement. Now, we fix a prime p and we wish to characterize when only the p'-degree characters of H extend to G.

Theorem A. *Let G be a finite π-separable group. Let H be a Hall π-subgroup of G, let K be a π-complement of G, and let p be a prime. Then every $\alpha \in \mathrm{Irr}(H)$ of p'-degree extends to G if and only if there is $P \in \mathrm{Syl}_p(H)$ such that $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(K)$.*

Of course, Theorem A is far more general than Sah's theorem, although we pay the price of using the Classification of Finite Simple Groups. This is not that surprising, however: in the case where H is normal in G, Theorem A is equivalent to proving a well-known consequence of the (yet unproven) McKay conjecture (see [17, Thm. C]) which therefore now becomes established.
As an easy consequence of our main result we obtain:

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Corollary B. *Suppose that A acts coprimely on a finite group G, and let $P \in \mathrm{Syl}_p(G)$ be A-invariant. Then all p'-degree irreducible characters of G are A-invariant if and only if $[\mathbf{N}_G(P), A] = 1$.*

The paper is split into two parts. In Section 2 we prove Theorem A and Corollary B modulo a statement (Theorem 2.2) on finite quasi-simple groups which is then shown in Section 3, using the classification, properties of algebraic groups and Deligne–Lusztig theory.

2. Proof of Theorem A

In our first result we use the Gajendragadkar's π-special characters (whose main properties can be found in [3]).

Lemma 2.1. *Suppose that G is π-separable and let H be a Hall π-subgroup of G. Let $L \lhd G$. Suppose that $\alpha \in \mathrm{Irr}(H)$ extends to G and is such that $H \cap L \subseteq \ker \alpha$. Then there is an extension $\beta \in \mathrm{Irr}(G)$ such that $L \subseteq \ker \beta$. In particular, if all p'-degree irreducible characters of H extend to G, then all p'-degree irreducible characters of HL/L extend to G/L.*

Proof. Let $\tilde{\alpha} \in \mathrm{Irr}(HL)$ be the unique irreducible character of HL that extends α and has L in its kernel. Notice that, by using the definition, $\tilde{\alpha}$ is π-special. By hypothesis, α extends to G. By [6, Thm. F], α has a π-special extension $\hat{\alpha}$ to G. Now by [3, Prop. (6.1)], we have that $\hat{\alpha}_{HL}$ is a π-special extension of α_H to HL. But, again by the uniqueness part in loc.cit., we have that $\hat{\alpha}_{HL} = \tilde{\alpha}$, and we are done. □

In order to prove Theorem A, we shall need the following non-trivial result whose proof (that uses the Classification of Finite Simple Groups) we defer until Section 3 below.

Theorem 2.2. *Let p be a prime. Suppose that $M \lhd G$ has p-power index, and has a normal Hall subgroup S such that M/S is not divisible by p. Assume that S is a non-abelian quasi-simple group of order divisible by p with $\mathbf{C}_G(S) = \mathbf{Z}(S)$ a p'-group. Let $P \in \mathrm{Syl}_p(G)$. Then the following are equivalent:*

(i) *all p'-degree P-invariant irreducible characters of S are M-invariant, and*
(ii) *there exists a complement K of S in M normalized by P and with $[\mathbf{N}_S(P), K] = 1$.*

We shall frequently write condition (ii) of the above theorem in the following more convenient form.

Lemma 2.3. *Let $S \lhd M$. Suppose that P, K are subgroups of M with $S \cap K = 1$. Then P normalizes K and $[\mathbf{N}_S(P), K] = 1$ if and only if $\mathbf{N}_{SP}(P) \subseteq \mathbf{N}_M(K)$.*

Proof. Since $P \subseteq \mathbf{N}_{SP}(P)$, by Dedekind's lemma we have that $\mathbf{N}_{SP}(P) = \mathbf{N}_S(P)P$. Thus, if P normalizes K and $[\mathbf{N}_S(P), K] = 1$ it is clear that $\mathbf{N}_{SP}(P) \subseteq \mathbf{N}_M(K)$. Conversely, we have that P and $\mathbf{N}_S(P)$ normalize K. Then, using that $S \lhd M$, we have that $[\mathbf{N}_S(P), K] \subseteq S \cap K = 1$. □

Also, the group theoretical conclusion in Theorem A has another convenient form.

LEMMA 2.4. *Suppose that G is a π-separable group with a Hall π-subgroup H and a Hall π-complement K. Let $P \in \mathrm{Syl}_p(H)$. If P normalizes K, then $\mathbf{N}_G(P) = \mathbf{N}_H(P)\mathbf{N}_K(P)$. In particular, $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(K)$ if and only if $\mathbf{N}_H(P) \subseteq \mathbf{N}_G(K)$.*

Proof. This is an application of Lemma (2.1) of [10]. □

LEMMA 2.5. *Suppose that P is a p-group acting on a group S that does not have a normal p-complement. Then there exists $1 \neq \chi \in \mathrm{Irr}(S)$ which is P-invariant of p'-degree.*

Proof. The semidirect product SP cannot have a normal p-complement. By Thompson's Theorem (Corollary (12.2) of [5]), there exists a non-linear $\gamma \in \mathrm{Irr}(SP)$ of degree not divisible by p. Now, $\gamma_S = \chi$ is irreducible (by Corollary (11.29) of [5]) and P-invariant. It is clear that χ is not trivial since otherwise γ would be a p'-degree character of a p-group, while γ is not linear. □

The following is one direction of Theorem A.

THEOREM 2.6. *Let G be a finite π-separable group. Let H be a Hall π-subgroup of G, let K be a π-complement of G, and let p be a prime. Suppose that every $\alpha \in \mathrm{Irr}(H)$ of p'-degree extends to G. Then there is $P \in \mathrm{Syl}_p(H)$ such that $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(K)$.*

Proof. If p does not divide $|H|$, then all irreducible characters of H extend to G, and G has a normal π-complement by Sah's Theorem (see [18, Thm. 5]). So we may assume that $p \in \pi$.

We prove that there exists $P \in \mathrm{Syl}_p(H)$ such that $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(K)$ by induction on $|G|$. Since H is a Hall subgroup of G, notice that every Sylow p-subgroup of H is a Sylow p-subgroup of G.

Let N be a fixed but arbitrary minimal normal subgroup of G.

Step 1. We can assume that there is $P \in \mathrm{Syl}_p(H)$ such that $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(K)N$. Also, $N \subseteq H$. In particular, $\mathbf{O}_{\pi'}(G) = 1$.

By Lemma 2.1, we know that all p'-degree irreducible characters of HN/N extend to G/N. Therefore, by induction and using that $\mathbf{N}_G(KN) = \mathbf{N}_G(K)N$, we conclude that there is $P \in \mathrm{Syl}_p(H)$ such that $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(K)N$. Now, since G is π-separable, we have that either N is a π-group or a π'-group. In the second case, $N \subseteq K$, and $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(K)$. So we will assume in the following that N is a π-group. Hence, $N \subseteq H$.

Step 2. It suffices to show that there is $m \in N$ such that $\mathbf{N}_{NP}(P^m)$ normalizes K.

Indeed, suppose that $\mathbf{N}_{NP}(P^m)$ normalizes K for some $m \in N$. Hence P^m, which is contained in NP, normalizes K. Then we have by Step 1 that

$\mathbf{N}_G(P^m) \subseteq \mathbf{N}_G(K)N = G_1$. Now, $P^m \in \mathrm{Syl}_p(G_1)$, and we may apply Lemma (2.1) of [10] to conclude that

$$\mathbf{N}_{G_1}(P^m) = \mathbf{N}_N(P^m)\mathbf{N}_{\mathbf{N}_G(K)}(P^m) \subseteq \mathbf{N}_G(K)\,.$$

However $\mathbf{N}_G(P^m) \subseteq G_1$, so $\mathbf{N}_{G_1}(P^m) = \mathbf{N}_G(P^m)$. Now, $P^m \subseteq PN \subseteq H$ is a Sylow p-subgroup of H, and we are done in this case.

Step 3. If $\theta \in \mathrm{Irr}(N)$ of p'-degree extends to PN, then θ is K-invariant.

If $\eta \in \mathrm{Irr}(PN)$ is such an extension, then η^H has p'-degree. Hence there exists $\psi \in \mathrm{Irr}(H)$ over η of p'-degree. By hypothesis, we have that ψ extends to some $\chi \in \mathrm{Irr}(G)$. Let T be the stabilizer of θ in G, so that $PN \subseteq T$. If $\nu \in \mathrm{Irr}(T|\theta)$ is the Clifford correspondent of χ over θ, then $\nu^G = \chi$ has π-degree. It follows that T contains some π-complement of G. Thus K^g is contained in T for some $g \in G$. Now, we know that P normalizes KN by Step 1, and hence $|G : \mathbf{N}_G(K)N|$ is not divisible by p. Thus $|G : \mathbf{N}_G(K^g)N|$ is not divisible by p. By Corollary (1.2) of [19] applied in the group G/N with respect to the subgroup T/N, we have that $|T : \mathbf{N}_T(K^g)N|$ divides $|G : \mathbf{N}_G(K^g)N|$, and therefore $|T : \mathbf{N}_T(K^g)N|$ is not divisible by p either. It follows that there is some $R \in \mathrm{Syl}_p(T)$ normalizing K^gN. Hence $R \in \mathrm{Syl}_p(G)$. Now, R and P^g are Sylow p-subgroups of $\mathbf{N}_G(K^g)N$. Thus $P^{gmn_0} = R$ for some $m \in \mathbf{N}_G(K^g)$ and $n_0 \in N$. Also, $R = P^v$ for some $v \in T$ because R and P are Sylow p-subgroups of T. Now, write $m = x^g$ for some $x \in \mathbf{N}_G(K)$, so that $gm = xg$. We have that $P^{xgn_0} = R = P^v$ and thus $xgn_0v^{-1} \in \mathbf{N}_G(P) \subseteq \mathbf{N}_G(K)N$. Hence, $gn_0v^{-1} \in \mathbf{N}_G(K)N$, and $gn_0v^{-1} = wn$ for some $w \in \mathbf{N}_G(K)$ and $n \in N$. Finally, since $K^g \subseteq T$, we have that $K^{gn_0v^{-1}} \subseteq T$ (because $n_0, v \in T$). Thus $K^{wn} \subseteq T$ and $K = K^w \subseteq T^{n^{-1}} = T$, as claimed.

Step 4. We can assume that $NKP = G$. Thus $H = NP$ and $M = NK \triangleleft G$.

By Step 1, we have that $P \subseteq \mathbf{N}_G(KN) = \mathbf{N}_G(K)N$, so $G_0 = NKP$ is a subgroup of G and $NK \triangleleft G_0$. Write $H_0 = NP$ and $G_0 = NKP$, and notice that H_0 is a Hall π-subgroup of G_0. Also, K is a π-complement of G_0. Let $\eta \in \mathrm{Irr}(H_0)$ with p'-degree. Since $|H_0 : N|$ is a power of p, then, by Corollary (11.29) of [5], we have that $\theta = \eta_N \in \mathrm{Irr}(N)$ has p'-degree. Now θ has an extension to H_0. By Step 3, we conclude that θ is K-invariant. Now, $(|KN : N|, |N|) = 1$ and therefore θ has a canonical extension $\hat\theta$ to KN by Corollary (8.16) of [5], which is by uniqueness, therefore P-invariant. Hence, by Corollary (4.2) of [6], it follows that restriction defines a bijection

$$\mathrm{Irr}(G_0|\hat\theta) \to \mathrm{Irr}(H_0|\theta)\,.$$

We conclude that η extends to G_0. Now suppose that $G_0 < G$. Then, by induction, we conclude that there is $P_0 \in \mathrm{Syl}_p(H_0)$ such that $\mathbf{N}_{G_0}(P_0) \subseteq \mathbf{N}_{G_0}(K)$. Since $P \in \mathrm{Syl}_p(H_0)$, we have that $P_0 = P^n$ for some $n \in N$. Then $\mathbf{N}_{NP}(P^n) \subseteq \mathbf{N}_{G_0}(P_0) \subseteq \mathbf{N}_G(K)$, and we apply Step 2 in this case. Hence, we are reduced to the case where $G_0 = G$, $H = NP$ and $M = NK \triangleleft G$.

Step 5. We can assume that N is a direct product of non-abelian simple groups of order divisible by p which are transitively permuted by G. In particular,

$\mathbf{O}_{p'}(G) = \mathbf{O}_p(G) = 1$. Also, we can assume that every p'-degree P-invariant irreducible character of N is M-invariant.

Suppose first that N is a p'-group. Hence, P acts coprimely on NK, and because NK is π-separable, it follows (using, for instance, that the number of Hall ρ-subgroups of KN is not divisible by p, where ρ is the set of prime divisors of $|K|$) that P normalizes some K^n for some $n \in N$. Now every P-invariant character of N extends to PN (by Corollary (8.16) of [5]), and by Step 3 is K-invariant. Therefore, every P-invariant character of N is NK-invariant, and therefore K^n-invariant. Thus every irreducible P-invariant character of N is PK^n-invariant and by Lemma (2.2) of [16], we conclude that $\mathbf{C}_N(PK^n) = \mathbf{C}_N(P)$. Hence, $\mathbf{N}_N(P) = \mathbf{C}_N(P) \subseteq \mathbf{C}_N(K^n) \subseteq \mathbf{N}_G(K^n)$. We had that $P^{n^{-1}}$ normalizes K and now we have that $\mathbf{N}_N(P^{n^{-1}}) \subseteq \mathbf{N}_G(K)$. Thus $\mathbf{N}_{NP}(P^{n^{-1}}) = \mathbf{N}_N(P^{n^{-1}})P^{n^{-1}} \subseteq \mathbf{N}_G(K)$, and this case is complete by Step 2.

Suppose now that N is a p-group. Hence $N \subseteq P$ and $H = P$. In this case, the hypotheses tell us that every linear character of P extends to G. By Tate's Theorem (use, for instance, Theorem (6.31) of [5]), we conclude that G has a normal p-complement. Hence $K \triangleleft G$, and in this case the theorem is proved.

So we may assume that N is a direct product of non-abelian simple groups of order divisible by p, which are transitively permuted by G. In particular $\mathbf{O}^p(N) = N$, and it follows that every P-invariant p'-degree irreducible character of N extends to PN by Corollary (8.16) of [5]. Therefore by Step 3, every P-invariant p'-degree character of N is K-invariant, and hence M-invariant.

Step 6. We can assume that N is a minimal normal subgroup of NP. Hence $N = S^{g_1} \times \cdots \times S^{g_t}$, where $\{S^{g_1}, \ldots, S^{g_t}\}$ is the P-orbit of S, a non-abelian simple group of order divisible by p, $g_i \in P$, and $g_1 = 1$. Also, we can assume that $t > 1$.

We can write $N = U \times V$, where $U > 1$ and $V \geq 1$ are P-invariant, and U is the direct product of the P-orbit of a simple group S. That is, $U = S^{g_1} \times \cdots \times S^{g_t}$, where $\{S^{g_1}, \ldots, S^{g_t}\}$ is the P-orbit of S, $g_i \in P$, and $g_1 = 1$. By Lemma 2.5, let $1 \neq \eta \in \mathrm{Irr}(S)$ be $\mathbf{N}_P(S)$-invariant of p'-degree. Then it is straightforward to show that $\nu = \eta^{g_1} \times \cdots \times \eta^{g_t}$ is P-invariant. Now, let $\tau = \nu \times 1_V \in \mathrm{Irr}(N)$, which is P-invariant of p'-degree. Then τ is K-invariant by Step 5. Hence $\ker \tau < N$ is K-invariant, and therefore G-invariant. Since $V \subseteq \ker \tau$, we conclude that $V = 1$, because N is a minimal normal subgroup of G.

Suppose now that N is simple. Since $\mathbf{Z}(N) = 1$, we have that $\mathbf{C}_M(N)$ is a π'-group. Since by Step 1, we know that $\mathbf{O}_{\pi'}(G) = 1$, then we have that $\mathbf{C}_M(N) = 1$. Since G/M is a p-group, then we conclude that $\mathbf{C}_G(N)$ is a p-group. But we know that $\mathbf{O}_p(G) = 1$, and thus we conclude that $\mathbf{C}_G(N) = 1$. Then, and using Step 5, we are in the hypothesis of Theorem 2.2. We conclude by this theorem (and Lemma 2.3) that there is a complement K_1 of N in M such that $\mathbf{N}_{NP}(P) \subseteq \mathbf{N}_G(K_1)$. Now, $K_1 = K^n$ for some $n \in N$, and we have that $\mathbf{N}_{NP}(P^{n^{-1}}) \subseteq \mathbf{N}_G(K)$. Then we are done by Step 2.

Step 7. Conclusion.

As before, write $N = S_1 \times \cdots \times S_t$, where $S_i = S^{g_i}$, $g_i \in P$ and $S = S_1$. Also recall that $t > 1$ and therefore $\mathbf{N}_G(S) < G$.
By using Steps 5 and 6, we have that $G = P\mathbf{N}_G(S)$. Thus $|G : \mathbf{N}_G(S)|$ is a power of p. Now, since G/N has a normal p-complement M/N, and $N \subseteq \mathbf{N}_G(S)$, we conclude that $M \subseteq \mathbf{N}_G(S)$. In particular, $M \subseteq \mathbf{N}_G(S^{g_i})$ for all i.
Now, using that G/M is a p-group, let $G_2 \triangleleft G$ be containing $\mathbf{N}_G(S)$ with $|G_2| < |G|$. Notice that $\mathbf{N}_P(S)$ is a Sylow p-subgroup of $\mathbf{N}_G(S)$. Also $\mathbf{N}_P(S) \subseteq P_2 = P \cap G_2 \in \mathrm{Syl}_p(G_2)$. Now, let $\eta \in \mathrm{Irr}(NP_2)$ of p'-degree. We have that $\eta_N \in \mathrm{Irr}(N)$ is P_2-invariant of p'-degree. Write $\eta_N = \theta_1 \times \cdots \times \theta_t$, where $\theta_i \in \mathrm{Irr}(S_i)$. Then θ_1 is $\mathbf{N}_P(S)$-invariant, because we can write $N = S \times S'$, where S and S' are $\mathbf{N}_P(S)$-invariant. Now, let

$$\theta = \theta_1 \times (\theta_1)^{g_2} \times \cdots \times (\theta_1)^{g_t} ,$$

which is P-invariant. By Step 5, the character θ is M-invariant. In particular θ_1 is M-invariant. Since $\mathbf{N}_P(S^{g_i}) \subseteq P_2 \triangleleft P$, we can repeat the same argument with every S^{g_i} and every θ_i to conclude that η_N is M-invariant. By induction applied in the group G_2 with respect to the Hall π-subgroup NP_2 and Hall π-complement K, we conclude that there exists $P_3 \in \mathrm{Syl}_p(P_2N)$ such that $\mathbf{N}_{G_2}(P_3) \subseteq \mathbf{N}_G(K)$. Hence P_3 normalizes K and is such that $[\mathbf{N}_N(P_3), K] = 1$ by Lemma 2.3. Now, $P_3 \cap N \in \mathrm{Syl}_p(N)$, and also $[P_3 \cap N, K] = 1$. In particular, $P_3 \cap N \subseteq \mathbf{N}_N(K)$ and by the Frattini argument, we see that $|G : \mathbf{N}_G(K)|$ is not divisible by p.
Now, $P_3 \subseteq P_1 \in \mathrm{Syl}_p(\mathbf{N}_G(K))$, and we have that $P_1 \in \mathrm{Syl}_p(G)$. Recall that $G = MP = (KN)P$. Since $P \in \mathrm{Syl}_p(G)$, we may write $(P_1)^{kn^{-1}x} = P$ for some $k \in K$, $x \in P$ and $n \in N$. Hence $(P_1)^k = P^n$. Since P_1 normalizes K, we have that $(P_1)^k$ normalizes K and thus P^n normalizes K. Also, since $[\mathbf{N}_N(P_3), K] = 1$, we have that $[\mathbf{N}_N((P_3)^k), K] = 1$, where $(P_3)^k \subseteq P^n$.
Finally, let $Q = (P_3)^k \cap N = (P_3 \cap N)^k \in \mathrm{Syl}_p(N)$. Now, notice that, by elementary group theory, if R is any p-subgroup of G such that $R \cap N = Q$, then $\mathbf{N}_G(R) \subseteq \mathbf{N}_G(Q)$ and $\mathbf{N}_N(R)/Q = \mathbf{C}_{\mathbf{N}_N(Q)/Q}(R)$.
Since $(P_3)^k \cap N = Q \in \mathrm{Syl}_p(N)$, and $(P_3)^k \subseteq (P_1)^k$, we also have that $(P_1)^k \cap N = Q$. In particular,

$$\mathbf{N}_N((P_1)^k)/Q = \mathbf{C}_{\mathbf{N}_N(Q)/Q}((P_1)^k) \subseteq \mathbf{C}_{\mathbf{N}_N(Q)/Q}((P_3)^k) = \mathbf{N}_N((P_3)^k)/Q$$

and we conclude that $\mathbf{N}_N((P_1)^k) \subseteq \mathbf{N}_N((P_3)^k) \subseteq \mathbf{C}_G(K)$. Hence, we have found $(P_1)^k \in \mathrm{Syl}_p(G)$ such that $(P_1)^k$ normalizes K and $[\mathbf{N}_N((P_1)^k), K] = 1$. Hence

$$\mathbf{N}_{N(P_1)^k}((P_1)^k) \subseteq \mathbf{N}_G(K)$$

using Lemma 2.3. Since $(P_1)^k = P^n \in \mathrm{Syl}_p(H)$, we use Step 2 to finish the proof of the theorem. □

In order to prove the remaining direction of Theorem A, we need one more lemma.

Lemma 2.7. *Suppose that A acts as automorphisms on a finite group G, where G is a direct product of a set of subgroups $\mathcal{X}$, which are permuted by A. Let $B \subseteq A$, and suppose that B acts transitively on $\mathcal{X}$. Let $S \in \mathcal{X}$. Then $\mathbf{C}_G(A) = \mathbf{C}_G(B)$ if and only if $\mathbf{C}_S(\mathbf{N}_A(S)) = \mathbf{C}_S(\mathbf{N}_B(S))$.*

Proof. By [9, Lemma 2.2], we have that $\mathbf{C}_G(A) \cong \mathbf{C}_S(\mathbf{N}_A(S))$ and $\mathbf{C}_G(B) \cong \mathbf{C}_S(\mathbf{N}_B(S))$. Since $\mathbf{C}_G(A) \subseteq \mathbf{C}_G(B)$ and $\mathbf{C}_S(\mathbf{N}_A(S)) \subseteq \mathbf{C}_S(\mathbf{N}_B(S))$, the proof easily follows. $\square$

The following completes the proof of Theorem A.

Theorem 2.8. *Let G be a finite π-separable group. Let H be a Hall π-subgroup of G, let K be a π-complement of G, and let p be a prime. Suppose that there is $P \in \mathrm{Syl}_p(H)$ such that $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(K)$. Then every $\alpha \in \mathrm{Irr}(H)$ of p'-degree extends to G.*

Proof. Again, we can assume that $p \in \pi$. We argue by double induction, first on $|G : \mathbf{O}_\pi(G)|$, and second on $|G|$. If $U \leq G$ and $N \triangleleft G$, then notice that $|U : \mathbf{O}_\pi(U)| \leq |G : \mathbf{O}_\pi(G)|$, and that $|G/N : \mathbf{O}_\pi(G/N)| \leq |G : \mathbf{O}_\pi(G)|$.
By hypothesis, we have that $K \triangleleft KP \leq G$. Also, if $KP \leq U < G$, by induction, we have that the theorem is valid for U with respect to any Hall subgroup of U containing P. Let $\alpha \in \mathrm{Irr}(H)$ be of p'-degree. We want to show that α extends to G. If $N \triangleleft G$, then we have that $\mathbf{N}_{G/N}(PN/N) \subseteq \mathbf{N}_{G/N}(KN/N)$. Therefore, if $1 < N$ is a π'-group, we easily see that $|G/N : \mathbf{O}_\pi(G/N)| < |G : \mathbf{O}_\pi(G)|$, and we deduce that $\hat{\alpha} \in \mathrm{Irr}(HN/N)$ (the unique extension of α to HN having N in its kernel) extends to G/N, by induction. Hence, we may assume that $\mathbf{O}_{\pi'}(G) = 1$.
Now, let $N = \mathbf{O}_\pi(G) \subseteq H$. Suppose that $Z \subseteq N$ is normal in G. Since α has p'-degree, there exists $\hat{\eta} \in \mathrm{Irr}(ZP)$ of p'-degree under α. Hence $\eta = \hat{\eta}_Z \in \mathrm{Irr}(Z)$, because ZP/Z is a p-group. Now, if $ZPK < G$, by induction, the group ZPK with Hall subgroup ZP and complement K satisfies the hypothesis of the theorem. We conclude that $\hat{\eta}$ extends to ZPK, and therefore that η is K-invariant. Hence $PK \subseteq T = I_G(\eta)$, the stabilizer of η in G. We have that $HT = G$ because $K \subseteq T$. Let $\nu \in \mathrm{Irr}(T \cap H)$ be the Clifford correspondent of α over η. If $T < G$, by induction, we conclude that ν has an extension $\hat{\nu} \in \mathrm{Irr}(T)$. Then $(\hat{\nu}^G)_H = \nu^H = \alpha$ (using Mackey), and we are done in this case. Hence, we conclude that whenever $Z \subseteq N$ is normal in G then either $ZKP = G$ or η is G-invariant.
In the latter case, where η is G-invariant, we use the theory of character triple isomorphisms, as developed in [7]. Since η extends to ZP, then by using Theorem 5.2 of [7] and its proof, we can find a character triple (G^*, Z^*, η^*) isomorphic to (G, Z, η), where Z^* is a p'-group, and also a π-group. Now, $(PZ)^* = P^* \times Z^*$ for a unique Sylow p-subgroup P^* of G^*. Also, H^* is a Hall π-subgroup of G^* and, using the Schur–Zassenhaus theorem, we have that $(KZ)^* = K^* \times Z^*$ for a unique subgroup K^* of G^*, which turns out to be a Hall π-complement of G^*. Also, using that $\mathbf{N}_G(PZ)^* = \mathbf{N}_{G^*}(P^* \times Z^*) = \mathbf{N}_{G^*}(P^*)$ and that $\mathbf{N}_{G^*}(K^* \times Z^*) = \mathbf{N}_{G^*}(K^*)$, we see that the hypotheses of

the theorem are satisfied in G^*. Furthermore, $N^* = \mathbf{O}_\pi(G^*)$, and therefore $|G : \mathbf{O}_\pi(G)| = |G^* : \mathbf{O}_\pi(G^*)|$.

Now let $\theta \in \mathrm{Irr}(N)$ be an irreducible constituent of α_N. Suppose that θ is G-invariant. Then, using the notation of the previous paragraph with $Z = N$, we have that $N^* = \mathbf{O}_\pi(G^*)$ is central in G^*. By the Hall–Higman's 1.2.3 Lemma, it follows that $V = \mathbf{O}_{\pi'}(G^*) > 1$. Then $|G^*/V : \mathbf{O}_\pi(G^*/V)| < |G^* : N^*| = |G : \mathbf{O}_\pi(G)|$, and by induction, and arguing as in the first paragraph of the proof, we conclude that α^* extends to G^*, and therefore that α extends to G. Hence, by the previous paragraph, we may assume that $NPK = G$. Thus $H = NP$.

Suppose now that θ is K-invariant. In this case, θ has a canonical extension ρ to $M = NK \triangleleft G$, using that $(|M : N|, |N|) = 1$. Also ρ is P-invariant by uniqueness. Also, in this case, we know by Corollary (4.2) of [6] that restriction defines a bijection $\mathrm{Irr}(G|\rho) \to \mathrm{Irr}(NP|\theta)$, and we conclude that α extends to G. Hence, it is enough to show that θ is K-invariant.

Now, let N/Z be a chief factor of G. Since $ZKP < G$, we conclude by the second paragraph of the proof that α_Z has a G-invariant irreducible constituent. Hence, by using again character triple isomorphisms, it is no loss to assume that Z is a central p'-subgroup of G.

In our present situation, and using Lemma 2.3, notice that our hypotheses now are that P normalizes K (that is, $M \triangleleft G$) and that $[\mathbf{N}_N(P), K] = 1$. If N/Z is a p'-group, then $\mathbf{N}_N(P) = \mathbf{C}_N(P)$, and thus $\mathbf{C}_N(KP) = \mathbf{C}_N(P)$. In this case, and using that θ is P-invariant, θ is K-invariant by Lemma (2.2) of [16]. If N/Z is a p-group, since P normalizes K, we have that $[\mathbf{O}_p(G), K] = 1$, and in this case we have that θ is K-invariant too (since $N \subseteq \mathbf{O}_p(G) \times Z$).

Hence we may assume that N/Z is the direct product $S_1/Z \times \ldots \times S_t/Z$ of non-abelian simple groups of order divisible by p which are transitively permuted by G. In particular $\mathbf{O}_p(G) = 1$. Since $\mathbf{O}_{\pi'}(G) = 1$, we easily deduce that $\mathbf{C}_M(N) = \mathbf{Z}(N) = Z$. Now, $\mathbf{C}_H(N)$ is a Hall π-subgroup of $\mathbf{C}_G(N)$, and $\mathbf{C}_K(N)$ is a Hall π-complement of $\mathbf{C}_G(N)$. Hence $\mathbf{C}_G(N) = \mathbf{C}_H(N)\mathbf{C}_K(N)$. Since $\mathbf{C}_K(N) = Z$, we see that $\mathbf{C}_G(N) \subseteq H = NP$. In particular, $\mathbf{C}_G(N)/Z$ is a p-group. Since $\mathbf{O}_p(G) = 1$, we see that $\mathbf{C}_G(N) = Z$. Now, since N/Z is a direct product of non-abelian simple groups, we have that $N'Z = N$. Since Z is a p'-group, then N/N' is a p'-group, and $N \cap P = N' \cap P$. Suppose that N is not perfect. If $N'P = NP$, then $N = N \cap N'P = N'(N \cap P) \subseteq N'$. Thus $N'P < P$. By Corollary (4.2) of [6], restriction then defines a bijection $\mathrm{Irr}(NP|\lambda) \to \mathrm{Irr}(N'P|\lambda_{Z\cap N'})$, and also $\mathrm{Irr}(N|\lambda) \to \mathrm{Irr}(N'|\lambda_{Z\cap N'})$. By the inductive hypothesis applied in $N'PK$, we conclude that $\alpha_{NP'}$ extends to $N'PK$. Thus $\alpha_{N'}$ is K-invariant. By uniqueness in the restriction map, we deduce that $\alpha_N = \theta$ is also K-invariant, and we are done in this case too. Thus we may assume that N is perfect.

If $t = 1$, that is, if N/Z is simple, then we may apply Theorem 2.2 to conclude that θ is K-invariant. So we may assume that $t > 1$. Hence $\mathbf{N}_G(S_1) < G$.

Let $Q = P \cap N \in \mathrm{Syl}_p(N)$, and let $Q_i = Q \cap S_i = P \cap S_i \in \mathrm{Syl}_p(S_i)$. Since $[\mathbf{N}_N(P), K] = 1$, we have that $[Q, K] = 1$. Now, let $1 \neq x \in Q_i$, and let $k \in K$. Then $x = x^k \in S_i \cap (S_i)^k$. If $(S_i)^k \neq S_i$, then $(S_i)^k \cap S_i = Z$, a p'-group, and this is not possible. We conclude that $K \subseteq \mathbf{N}_G(S_i)$ for all i. Since $S_i \lhd N$, then $M \subseteq \mathbf{N}_G(S_i)$ and we conclude that all the S_i's are P-conjugate, and $P\mathbf{N}_G(S_i) = G$.

Now, using that Z is a p'-group, we have that $Q = Q_1 \times \cdots \times Q_t$. Write $S_i = S^{g_i}$ for some $g_i \in P$, where $S = S_1$. Let $P_0 = \mathbf{N}_P(S) \in \mathrm{Syl}_p(\mathbf{N}_G(S))$. Since $N \subseteq \mathbf{N}_G(S)$, we have that $P_0 \cap N \in \mathrm{Syl}_p(N)$ and $P_0 \cap S \in \mathrm{Syl}_p(S)$. Necessarily, $P_0 \cap N = Q$ and $P_0 \cap S = Q_1$. Also, since P normalizes K, then P_0 normalizes KS. Thus $KS \lhd KSP_0 = G_0 < G$.

We wish to apply the inductive hypothesis in G_0, where here $H_0 = SP_0$ is a Hall π-subgroup of G_0, and K is a π-complement of G_0. Notice that $P_0 \in \mathrm{Syl}_p(SP_0)$, since $P_0 \cap S \in \mathrm{Syl}_p(S)$ and $|SP_0 : P_0| = |S : S \cap P_0|$.

By Lemma 2.4, we need to check that $\mathbf{N}_{SP_0}(P_0)$ normalizes K. By Lemma 2.3, we need to check that P_0 normalizes K and $[\mathbf{N}_S(P_0), K] = 1$. Since P normalizes K, we only need to check that $[\mathbf{N}_S(P_0), K] = 1$. By hypothesis, we know that $[\mathbf{N}_N(P), K] = 1$.

Now, $P_0 \cap \mathbf{N}_S(Q_1) = Q_1$ and therefore $\mathbf{N}_{G_0}(P_0) \subseteq \mathbf{N}_{G_0}(Q_1)$. We easily conclude that

$$\mathbf{N}_S(P_0)/Q_1 = \mathbf{C}_{\mathbf{N}_S(Q_1)/Q_1}(P_0)\,.$$

By the same argument,

$$\mathbf{N}_N(P)/Q = \mathbf{C}_{\mathbf{N}_N(Q)/Q}(P)\,.$$

Since $[\mathbf{N}_N(P), K] = 1$, then $\mathbf{C}_{\mathbf{N}_N(Q)/Q}(P) = \mathbf{C}_{\mathbf{N}_N(Q)/Q}(PK)$.

Now, we have that $\mathbf{N}_N(Q)/Q$ is KP-isomorphic to the direct product of $\mathbf{N}_{S_i}(Q_i)/Q_i$ and that these factors are transitively permuted by P. By Lemma 2.7, we know that

$$\mathbf{C}_{\mathbf{N}_N(Q)/Q}(KP) = \mathbf{C}_{\mathbf{N}_N(Q)/Q}(P)$$

if and only if

$$\mathbf{C}_{\mathbf{N}_S(Q_1)/Q_1}(KP_0) = \mathbf{C}_{\mathbf{N}_S(Q_1)/Q_1}(P_0) = \mathbf{N}_S(P_0)/Q_1\,.$$

We conclude that K acts trivially on $\mathbf{N}_S(P_0)/Q_1$. Since $[Q_1, K] = 1$, by coprime action we have that $[K, \mathbf{N}_S(P_0)] = 1$, as desired.

Hence, we can apply the inductive hypothesis in G_0. Now, by using the notation of central products used in Section 5.1 of [8], we can write $\theta = \theta_1 \cdot \ldots \cdot \theta_t$, where $\theta_i \in \mathrm{Irr}(S_i)$. Since θ is P-invariant, then we conclude that θ_1 is P_0-invariant. Since the determinantal order and the degree of θ_1 are coprime to p, we see that θ_1 extends to a p'-degree character of SP_0. By induction, this character extends to G_0, and we conclude that θ_1 is K-invariant. The same argument applies to every θ_i, and we conclude that θ is K-invariant. This finishes the proof of the theorem. □

The proof of Corollary B now is immediate:

Proof of Corollary B. Let $\Gamma = GA$ be the semidirect product. Then Γ has a Hall π-subgroup G and a π-complement A. By coprime action, G has an A-invariant Sylow p-subgroup P, and all of them are $\mathbf{C}_G(A)$-conjugate. Also $\mathbf{N}_\Gamma(P) = \mathbf{N}_G(P)A$ and $\mathbf{N}_\Gamma(A) = \mathbf{C}_G(A) \times A$. Suppose that $[\mathbf{N}_G(P), A] = 1$.

Then $\mathbf{N}_\Gamma(P) \subseteq \mathbf{N}_\Gamma(A)$ and by Theorem A, we have that all irreducible p'-degree characters of G extend to Γ and are, therefore, A-invariant.
Conversely, if all p'-degree irreducible characters of G are A-invariant, then all of them extend to Γ since $(|\Gamma : G|, |G|) = 1$. Hence, by Theorem A, there is a Sylow p-subgroup P_1 of G such that $\mathbf{N}_\Gamma(P_1) \subseteq \mathbf{N}_\Gamma(A)$. Hence $[\mathbf{N}_G(P_1), A] = 1$. In particular, P_1 is A-invariant, and we conclude that $(P_1)^c = P$ for some $c \in \mathbf{C}_G(A)$. Then $[\mathbf{N}_G(P), A] = 1$, and the proof is complete. □

3. Proof of Theorem 2.2

The aim of this section is the proof of Theorem 2.2 which we restate as follows:

Theorem 3.1. *Let S be quasi-simple, normal in a group X with $S/Z(S) \leq X/Z(S) \leq \mathrm{Aut}(S/Z(S))$, and let p be a prime dividing $|S|$, with $|Z(S)|$ prime to p. Let M/S be normal in X/S of order prime to $|S|$, such that X/M is a p-group. Let $P \in \mathrm{Syl}_p(X)$. Then the following are equivalent:*

(i) *all P-invariant characters in $\mathrm{Irr}_{p'}(S)$ are M-invariant;*
(ii) *there exists a complement K of S in M normalized by P with $[N_S(P), K] = 1$.*

The statement is trivially true when $M = S$, so we may assume that the quasi-simple group S has outer automorphisms of order prime to $|S|$, which by the classification forces S to be of Lie type and M/S to consist of field automorphisms:

Proposition 3.2. *Let S be finite quasi-simple and $\sigma \in \mathrm{Aut}(S)$ with $\gcd(o(\sigma), |S|) = 1$. Then S is of Lie type and σ is a field automorphism. In particular,* $\mathrm{Out}(S)$ *has a cyclic central π'-Hall subgroup, where $\pi = \pi(|S|)$.*

Proof. According to [4, Cor. 5.1.4], any automorphism of a finite quasi-simple group S is induced by an automorphism of the simple quotient $S/Z(S)$, so it suffices to deal with the case that S is simple. Since the outer automorphism group of finite simple groups not of Lie type has order a power of 2, clearly S must be of Lie type. In this case, $\mathrm{Out}(S)$ is described in [4, §2.5]: it is an extension of the normal subgroup D of diagonal automorphisms (which is cyclic or a Klein four group) by the commuting product of the cyclic group of field automorphisms with the group of graph automorphisms (the latter being a subgroup of $\mathfrak{S}_3$). Now the order of D is only divisible by prime divisors of the order of the Weyl group of S. But by Remark 3.3 below, since $o(\sigma)$ is prime to $|S|$, it is only divisible by primes larger than those occurring in $|D|$, so σ can only act trivially on D. This shows the claim. □

Remark 3.3. In the preceding proof, we used the following observation: if W is a Weyl group, and p a prime divisor of its order, then all primes dividing $p-1$ also divide $|W|$. This can easily be checked by inspection. It also follows from the rationality of Weyl groups.
In fact, even more is true: all primes smaller than p divide $|W|$; no a priori proof of this is known to the authors.

3.1. The general setup. We fix the following notation throughout this section. Let $\mathbf{G}$ be a simple linear algebraic group of simply connected type over the algebraic closure of a finite field of characteristic r, and $\mathbf{T} \leq \mathbf{G}$ a maximal torus. Then for any graph automorphism of the Dynkin diagram of $\mathbf{G}$ and any integral power $q = r^a$ there exists a Steinberg endomorphism $F_a : \mathbf{G} \to \mathbf{G}$ such that F_a acts as $q\phi$ on the character group $X(\mathbf{T})$ of $\mathbf{T}$, with ϕ of finite order inducing the given graph automorphism on the Weyl group. Similarly, if $\mathbf{G}$ is of type B_2, G_2 or F_4, and $r = 2, 3$, respectively 2, then for any odd power $q = \sqrt{r}^a$ of $\sqrt{r}$ there exists a Steinberg endomorphism $F_a : \mathbf{G} \to \mathbf{G}$ such that F_a acts as $q\phi$ on $X(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{R}$ and ϕ induces the non-trivial graph automorphism of the Coxeter diagram of $\mathbf{G}$ (see [15, Thm. 22.5], for example).
Now for a Steinberg endomorphism $F = F_a$ as above, let $G = \mathbf{G}^F$ be the corresponding finite group of fixed points. Then, with a finite number of exceptions, G is a quasi-simple group of Lie type, and all finite simple groups of Lie type arise by this setup as $G/Z(G)$, except for ${}^2F_4(2)'$. Since the latter group has no coprime automorphisms, this exception is of no importance for our question.
So we may and will now assume that $G = \mathbf{G}^F$ is perfect. Then, except for a finite number of cases, G is the universal Schur cover of the simple group $G/Z(G)$. Since none of the simple groups of Lie type with exceptional Schur multiplier have coprime automorphisms (see [4, §6.1]), we may assume that $G/Z(G)$ is none of these. Thus, the quasi-simple group S from the statement of Theorem 3.1 can be realized as a central quotient $S = G/Z$ of a group G as above, for some $Z \leq Z(G)$, and in particular all irreducible characters of S occur among the characters of G.
According to Proposition 3.2 we will have to consider field automorphisms of S. Now any coprime field automorphism γ of G, and hence of S, is induced by a Steinberg endomorphism of $\mathbf{G}$ as follows. For fixed r and $\mathbf{G}$, we may assume the F_a to be chosen compatibly such that $F_{an} = F_a^n$ for any n coprime to $o(\phi)$. In this setting, if γ is a coprime field automorphism of G of order f, then in particular, f is prime to the order of ϕ, and thus there exists a Steinberg endomorphism $F_c : \mathbf{G} \to \mathbf{G}$ such that $F = F_{cf} = F_c^f$ and $\langle\gamma\rangle = \langle F_c|_G\rangle$ in $\mathrm{Out}(G)$. The centralizer of γ in G is then just the fixed point group $\mathbf{G}^{F_c}$ under F_c, a finite group of Lie type of the same type as G.

3.2. Action on $\mathrm{Irr}(G)$. In order to determine the action of coprime automorphisms on $\mathrm{Irr}(G)$, we first recall Lusztig's parametrization. For this, let $\mathbf{G}^*$ be a group in duality with $\mathbf{G}$, with compatible Steinberg endomorphisms $F_a^* : \mathbf{G}^* \to \mathbf{G}^*$, $a \geq 1$ (respectively a odd, when G is a Suzuki or Ree group). We'll write $G^* := {\mathbf{G}^*}^{F^*}$ for the group of fixed points of $F^* = F_a^*$. Then by the results of Lusztig, there is a partition

$$\mathrm{Irr}(G) = \bigsqcup_{s \in G^*/\sim} \mathcal{E}(G, s)$$

into Lusztig series $\mathcal{E}(G, s)$, indexed by semisimple elements s of the dual group G^* modulo conjugation. Now recall that $S = G/Z$ for some subgroup $Z \leq$

$Z(G)$. Let $T \leq G$ be a maximal torus, with dual torus $T^* \leq G^*$. So any $s \in T^*$ is a linear character of T, and $\chi \in \mathcal{E}(G,s)$ is trivial on $Z \leq T$ if and only if $Z \leq \ker(s)$ (see [13, Lemma 2.2]). This defines a subgroup of T^* of index $|Z|$; we denote by S^* the normal subgroup of G^* generated by all these subgroups, so $G^*/S^* \cong Z$ (and hence $[G^*,G^*] \leq S^*$). Thus, for $s \in G^*$ semisimple, the characters in $\mathcal{E}(G,s)$ have Z in their center, so descend to $S = G/Z$, if and only if $s \in S^*$.

$$\begin{tikzcd} G^*_{\rm ss} \arrow[r, dotted, "\text{param. irred. of}"] & G \arrow[d, ->>] \\ S^*_{\rm ss} \arrow[u, hook] \arrow[r, dotted, "\text{param. irred. of}"] & S \end{tikzcd}$$

Now any field automorphism of G permutes the various Lusztig series $\mathcal{E}(G,s)$. More precisely, if γ is induced by F_c as described above, then by Lusztig $\mathcal{E}(G,s)$ is γ-stable if and only if the G^*-class of s is F_c^*-stable. We need the following:

Lemma 3.4. *In the above situation, assume that $\mathcal{E}(G,s)$ is γ-stable. Then γ fixes each orbit in $\mathcal{E}(G,s)$ under diagonal automorphisms.*

Proof. Let $\mathbf{H} = C_{\mathbf{G}^*}(s)$. Lusztig [12, Prop. 5.1] proves the existence of a surjection

$$\psi : \mathcal{E}(G,s) \to \mathcal{E}(\mathbf{H}^{\circ F},1) \pmod{\mathbf{H}^F/\mathbf{H}^{\circ F}}$$

with fibres the orbits under the action of diagonal automorphisms, such that multiplicities of $\rho \in \mathcal{E}(G,s)$ in Deligne–Lusztig characters are determined by those of elements of $\psi(\rho)$. We claim that ψ can be chosen γ-equivariant. The result then follows since field automorphisms act trivially on $\mathcal{E}(\mathbf{H}^{\circ F},1)$ (see e.g. [2, Prop. 6.6]).

Now unipotent characters are uniquely determined by their multiplicities in Deligne–Lusztig characters, and thus the claim follows, unless H has a component of exceptional type (see [2, Prop. 6.3]). But in the latter case, characters with same multiplicities in all Deligne–Lusztig characters have distinct eigenvalues of Frobenius attached, and since γ commutes with F, these are respected by γ. $\square$

In our situation, as remarked in the proof of Proposition 3.2, any prime divisor of $o(\gamma)$ is larger than the order of the group of diagonal automorphisms. Thus γ must in fact fix all elements in $\mathcal{E}(G,s)$. We hence get the following characterization of γ-stable characters of S (recall that all irreducible characters of S occur among those of G):

Proposition 3.5. *Let γ be a coprime automorphism of S, where $S = G/Z$ with $G = \mathbf{G}^F$, $Z \leq Z(G)$ as above. Then $\chi \in \mathrm{Irr}(S)$ is γ-stable if and only if $\chi \in \mathcal{E}(G,s)$ for some F_c^*-stable semisimple element $s \in S^*$, where γ is induced by F_c.*

Proof. This follows from the above, using the fact that any F_c^*-stable conjugacy class of the connected group $\mathbf{G}$ contains an F_c^*-stable (and hence $F^* = F_c^{*f}$-stable) element (see e.g., [15, Thm. 21.11]). □

3.3. Defining characteristic. We now first dispose of an easy case; observe that in defining characteristic, all semisimple characters lie in $\mathrm{Irr}_{p'}(G)$, by Lusztig's Jordan decomposition of characters (but note that when r is bad $\mathrm{Irr}_{p'}(G)$ may contain non-semisimple characters).

Proposition 3.6. *Let S be quasi-simple of Lie type, γ a coprime automorphism of S. Let $p = r$, the defining characteristic of S. Let A be a p-group of automorphisms of S. Then γ does not centralize a Sylow p-subgroup of S, and moreover not all A-invariant elements of $\mathrm{Irr}_{p'}(G)$ are γ-invariant. In particular, Theorem 3.1 holds in this situation.*

Proof. Let f denote the order of γ. As above, we may choose $F, F_c : \mathbf{G} \to \mathbf{G}$ such that S is a central quotient of $G = \mathbf{G}^F$, $F = F_c^f$ and γ is induced by $F_c|_G$. Then $G_0 := \mathbf{G}^{F_c}$ is a group of the same type as G, but over a subfield $\mathbb{F}_{q_0}$ of $\mathbb{F}_q$, of index $[\mathbb{F}_q : \mathbb{F}_{q_0}] = f$. The order formula for groups of Lie type (see e.g., [15, Table 24.1]) then shows that the p-parts of G and G_0 differ. Since $Z(G)$ has order prime to p, the same holds for the p-parts of S and $C_S(\gamma)$. In particular, γ does not centralize a Sylow p-subgroup of S, whence (ii) of Theorem 3.1 does not hold.
Since diagonal automorphisms have order prime to the defining characteristic, A consists of graph and field automorphisms only. Thus, A induces a group of automorphisms A^* on the dual G^* of G, and we write $H := C_{G^*}(A^*)$ for its fixed point group. Now let $s \in H \cap [G^*, G^*]$ be semisimple, not centralized by γ^* (which exists by Zsigmondy's theorem). Then by Proposition 3.5 the Lusztig series $\mathcal{E}(G, s)$ consists of characters with $Z(G)$ in their kernel, invariant under A, but not invariant under γ. Moreover, the semisimple character in $\mathcal{E}(G, s)$ is of p'-degree. This shows that (i) of Theorem 3.1 is not satisfied either, thus completing the proof. □

3.4. Sylow subgroups. We now turn to the non-defining primes. There we require the following crucial result from [13, Prop. 7.3]:

Proposition 3.7. *Let G be as above, $p \neq r$ and $\chi \in \mathrm{Irr}_{p'}(G)$. Then there is some semisimple $s \in G^*$ centralizing a Sylow p-subgroup of G^* with $\chi \in \mathcal{E}(G, s)$. Conversely, if $s \in G^*$ centralizes a Sylow p-subgroup, then there is some $\chi \in \mathcal{E}(G, s)$ of p-height 0.*

We next consider the following generic case, which occurs for all large enough primes p:

Proposition 3.8. *Let $S = G/Z$ be quasi-simple of Lie type, with $G = \mathbf{G}^F$, $Z \leq Z(G)$ as above. Let $p \neq r$ and P a Sylow p-subgroup of G, P^* a Sylow p-subgroup of G^*.*

(a) *Then P is contained in a proper F-stable Levi subgroup of $\mathbf{G}$ if and only if P^* is contained in a proper F^*-stable Levi subgroup of $\mathbf{G}^*$.*

(b) *Assume that the condition in (a) is satisfied, and that moreover S has no diagonal or graph automorphism of order p. Then neither (i) nor (ii) in Theorem 3.1 holds.*

Proof. Let $\mathbf{L}$ be a proper F-stable Levi subgroup of $\mathbf{G}$ containing P. If $\mathbf{L}^*$ denotes an F^*-stable Levi subgroup of $\mathbf{G}^*$ dual to $\mathbf{L}$, then $|G^*| = |G|$ and $|{\mathbf{L}^*}^{F^*}| = |\mathbf{L}^F|$, so $\mathbf{L}^*$ contains a Sylow p-subgroup P^* of G^*. Since we may exchange the roles of $\mathbf{G}$, $\mathbf{G}^*$ in this argument, we obtain (a).

Now assume that P is contained in the F-stable proper Levi subgroup $\mathbf{L}$. Then $\mathbf{T} := Z(\mathbf{L})$ is an F-stable torus of dimension at least 1 (see [15, Prop. 12.6]). Thus, for a coprime automorphism γ of S, induced by F_c, $\mathbf{T}^F \setminus \mathbf{T}^{F_c}$ is non-empty. Let $\kappa : G \to G/Z = S$ denote the canonical epimorphism. Since $\kappa(\mathbf{T}^F) \le C_S(\kappa(P)) \le N_S(\kappa(P))$, this shows that $N_S(\kappa(P))$ is not centralized by γ.

On the other hand, the non-trivial torus $\mathbf{T}^* := Z(\mathbf{L}^*)$ lies in $C_{\mathbf{G}^*}(P^*)$. It follows that there exists a semisimple $s \in {\mathbf{T}^*}^{F^*} \setminus {\mathbf{T}^*}^{F_c^*} \subseteq C_{G^*}(P^*)$ which hence is not F_c^*-invariant. Then, by Proposition 3.5, the elements of $\mathcal{E}(G,s)$ are not γ-invariant. But $\mathcal{E}(G,s)$ contains characters of height 0 by Proposition 3.7 above.

Furthermore, by assumption the Sylow p-subgroup B of $\mathrm{Out}(S)$ consists of field automorphisms, hence is cyclic. Thus there exists $F_b : \mathbf{G} \to \mathbf{G}$ such that $\beta := F_b|_G$ generates B modulo inner automorphisms. Then we may argue as before with semisimple elements in $\mathbf{T}^{F_b} \setminus \mathbf{T}^{\langle F_b, F_c\rangle}$, respectively $s \in {\mathbf{T}^*}^{F_b^*} \setminus {\mathbf{T}^*}^{\langle F_b^*, F_c^*\rangle}$ to see that neither (i) nor (ii) of Theorem 3.1 are satisfied. $\square$

3.5. Bad primes and torsion primes. It remains to deal with the primes p not satisfying the condition in Proposition 3.8(a). As announced before, these are small:

Proposition 3.9. *Let $\mathbf{H}$ be a simple algebraic group in characteristic r with Steinberg endomorphism $F : \mathbf{H} \to \mathbf{H}$. Let $p \ne r$ be a prime and P a Sylow p-subgroup of $H := \mathbf{H}^F$. If P is not contained in any proper F-stable Levi subgroup of $\mathbf{H}$, then every semisimple element centralizing P is quasi-isolated, and in particular, p is a bad prime or torsion prime for $\mathbf{H}$.*

Proof. Clearly we have that $P \ne 1$. Let $g \in C_H(P)$ be semisimple, so $P \le C_{\mathbf{H}}(g)$. Now the connected group $C^\circ_{\mathbf{H}}(g)$ acts by conjugation on the set Ω of proper Levi subgroups of $\mathbf{H}$ containing $C_{\mathbf{H}}(g)$, and any orbit therein is F-stable; hence by [15, Thm. 21.11], if Ω is non-empty then it contains an F-stable element $\mathbf{L}$, and then $P \le C_{\mathbf{H}}(g) \le \mathbf{L}$, which is not the case. So $C_{\mathbf{H}}(g)$ is not contained in any proper Levi subgroup of $\mathbf{H}$, whence, by definition, g is quasi-isolated in $\mathbf{H}$ (see [15, Exmp. 14.4(2)]).

Apply this to $1 \ne g \in Z(P)$. If p is not a torsion prime for $\mathbf{H}$, the index $|C_{\mathbf{H}}(g) : C^\circ_{\mathbf{H}}(g)|$ is prime to p (see [15, Prop. 14.20]), so $P \le C^\circ_{\mathbf{H}}(g)$. By assumption, this does not lie in any proper Levi subgroup of $\mathbf{H}$, so by definition g is isolated. Now by a result of Deriziotis (see [15, Rem. 14.5]) and the algorithm of Borel and de Siebenthal (see [15, Thm. 13.12]), this implies that all prime divisors of $o(g)$ are bad for $\mathbf{H}$. □

We need the following two elementary results, where we write Φ_m for the mth cyclotomic polynomial:

Lemma 3.10. *Let $q \geq 1$, p a prime not dividing q, and e the multiplicative order of q modulo p. Then $p|\Phi_d(q)$ if and only if $d = ep^i$ for some $i \geq 0$.*

See e.g. [13, Lemma 5.2] for a proof.

Lemma 3.11. *Let $m, f \geq 1$. Then*

$$\Phi_m(X^f) = \prod_d \Phi_d(X)$$

where the product runs over all divisors d of mf which do not divide $m'f$ for any $m' < m$. In particular, when $\gcd(m, f) = 1$ *then $\Phi_m(X^f) = \prod_{d|f} \Phi_{md}(X)$.*

Proof. The roots of $\Phi_m(X)$ are the primitive mth roots of unity. Thus the roots of $\Phi_m(X^f)$ are mfth roots of unity whose fth power is a primitive mth root of unity, hence whose order is not a divisor of $m'f$, for any $m' < m$. □

Proposition 3.12. *Let $\mathbf{G}$ be connected reductive with Steinberg endomorphisms $F_c, F = F_c^f$ such that* $\gcd(f, |\mathbf{G}^F|) = 1$, *$\mathbf{H} \leq \mathbf{G}$ a connected reductive F_c-stable subgroup, p a prime dividing the order of the Weyl group of $\mathbf{G}$, $p \neq r$. Then $\mathbf{H}^{F_c}$ contains a Sylow p-subgroup of $\mathbf{H}^F$.*

Proof. The group $H_0 = \mathbf{H}^{F_c}$ is a group of the same type as $H = \mathbf{H}^F$. Thus there are non-negative integers $a(d)$ such that $|H_0|_{r'} = \prod_d \Phi_d(q_0)^{a(d)}$ and $|H|_{r'} = \prod_d \Phi_d(q)^{a(d)}$, with $q_0^f = q$, and d divides the order of the Weyl group W of $\mathbf{G}$ whenever $a(d) > 0$ (see [15, Cor. 24.6 and 24.7]). Since f is coprime to $|H|$, it's coprime to $p - 1$ by Remark 3.3, so q, q_0 have the same order e modulo p. Now, if p divides $\Phi_d(q) = \Phi_d(q_0^f)$, then $d = ep^i$ for some $i \geq 0$ by Lemma 3.10, and hence $\gcd(f, d) = 1$. Then the only factor of $\Phi_d(q_0^f)$ (in the factorization given by Lemma 3.11), which is divisible by p, is $\Phi_d(q_0)$ (again using Lemma 3.10). Thus the p-parts of $\Phi_d(q)$ and $\Phi_d(q_0)$ coincide, and hence $|H_0|_p = |H|_p$. □

Corollary 3.13. *Let S be quasi-simple of Lie type as above, γ a coprime automorphism and p a prime dividing the order of the Weyl group of S. Then the fixed group $C_S(\gamma)$ contains a Sylow p-subgroup of S.*

Proof. Write $S = G/Z$ with $G = \mathbf{G}^F$, $Z \leq Z(G)$. Now γ is induced by some Steinberg endomorphism F_c with $F_c^f = F$, where $f = o(\gamma)$ is coprime to $|S|$. Then $|G_0|_p = |G|_p$ by the preceding result, with $G_0 = \mathbf{G}^{F_c}$. Since the subgroup $Z \leq Z(G)$ has order only divisible by prime divisors of the order of the Weyl group, it is fixed by F_c (again by Remark 3.3) and hence $G_0/(Z \cap G_0) = G_0/Z \leq C_S(\gamma)$, which shows that $|S|_p = |C_S(\gamma)|_p$. □

After these preparations we can return to the proof of Theorem 3.1.

Proposition 3.14. *Let $S = G/Z$ be quasi-simple of Lie type, with $G = \mathbf{G}^F$ and $Z \leq Z(G)$ as above, $p \neq r$ a prime such that no proper F-stable Levi subgroup of $\mathbf{G}$ contains a Sylow p-subgroup of G. Then (i) in Theorem 3.1 holds.*

Proof. Let $\chi \in \mathrm{Irr}_{p'}(S)$, so $\chi \in \mathcal{E}(G, s)$ for some semisimple element $s \in G^*$ centralizing a Sylow p-subgroup P^* of S^*, by Proposition 3.7. Now by Proposition 3.8, P^* is not contained in a proper F^*-stable Levi subgroup of G^*, so by Proposition 3.9, applied to $\mathbf{H} = \mathbf{G}^*$, we conclude that all semisimple elements in $C_{G^*}(P^*)$ are quasi-isolated, their order is only divisible by torsion primes and bad primes, so in particular only by divisors of the order of the Weyl group of $\mathbf{G}^*$. Let $\mathbf{T}^*$ be an F^*-stable torus of $\mathbf{G}^*$ containing s. Then Proposition 3.12 applied with $\mathbf{H} = \mathbf{T}^*$ shows that $s \in {\mathbf{T}^*}^{F_c^*}$. But then χ is F_c-stable by Proposition 3.5. □

For p odd let $e_p(q)$ denote the order of q modulo p, respectively the order of q modulo 4 when $p = 2$.

Proposition 3.15. *Let $S = G/Z$ be quasi-simple of Lie type, with $G = \mathbf{G}^F$ and $Z \leq Z(G)$ as above, $p \neq r$ a prime such that no proper F-stable Levi subgroup of $\mathbf{G}$ contains a Sylow p-subgroup of G. Then (ii) in Theorem 3.1 holds.*

Proof. By Proposition 3.9 our assumptions imply that p is a torsion or bad prime for S. By [13, Thms. 5.14, 5.19, 8.4] the normalizer of a Sylow p-subgroup S_p of S is contained in the normalizer of a Φ_e-torus $\mathbf{T}_e$, where $e = e_p(q)$, unless $S = \mathrm{SL}_3(q), \mathrm{SU}_3(q), G_2(q), {}^2F_4(q^2)$ with $p = 3$ or $S = \mathrm{Sp}_{2n}(q), {}^2G_2(q^2)$ with $p = 2$.
Now first assume that S is of exceptional type. Since a Sylow 2-subgroup of ${}^{(2)}E_6(q)$ (with q odd) is contained in a Levi subgroup of type ${}^{(2)}D_5(q)$, and a Sylow 3-subgroup of $E_7(q)$ (with $3 \not| q$) is contained in a Levi subgroup of type $E_6(q)$ or ${}^2E_6(q)$, these situations do not arise here. In the remaining cases, which are collected in Table 1, the Sylow normalizers may easily be worked out explicitly inside $N_S(\mathbf{T}_e)$, respectively they are already given in [14, Sect. 3 and 4] (see also [11, Sect. 4] for the case $p = 2$). In particular they depend only on the integer a, which is the same for S and for the centralizer of any coprime automorphism by Corollary 3.13.
Now consider the case where S is of classical type. If S is not of type A, then $p = 2$ is the only torsion prime, and from the above mentioned result on $N_S(S_p)$ (or from [1, Thm. 4]) it follows that a Sylow 2-subgroup S_2 of S is self-normalizing, or an extension of S_2 of degree 3^t where t is the number of summands in the 2-adic expansion of n for $S = \mathrm{Sp}_{2n}(q)$ with $q \equiv \pm 3 \pmod 8$. In particular, it's the same in S and in $C_S(\gamma)$.
Finally, when S is linear the possible torsion primes are divisors of $\gcd(n, q-1)$. For $\mathrm{SL}_3(q)$ with $p = 3$ and $q \equiv 4, 7 \pmod 9$ the Sylow p-normalizers is isomorphic to $3^{1+2}.Q_8$, independently of q (see [14, Sect. 3.1]), whence the

Table 1. Normalizers for Sylow subgroups of exceptional groups not contained in proper Levi subgroups

G	p	$N(S_p)$	
${}^2G_2(3^{2f+1})$	2	$2^3.7.3$	
$G_2(q)$	2	$(2^a)^2.2^2$	
	3	$(3^a)^2.W(G_2)$	$q \equiv 1, 8 \pmod 9$
		$3^{1+2}.Q_8.2$	$q \equiv 2, 4, 5, 7 \pmod 9$
${}^3D_4(q)$	2	$(2^a)^2.2^2$	
	3	$(3^a \times 3^{a+1}).W(G_2)$	
${}^2F_4(2^{2f+1})$	3	$(3^a)^2.W(G_2)$	$2^{2f+1} \equiv 8 \pmod 9$
		$3^{1+2}.Q_8.2$	$2^{2f+1} \equiv 2, 5 \pmod 9$
$F_4(q)$	2	$(2^a)^4.S_2(W(F_4))$	
	3	$(3^a)^4.3^2.2^3$	
$E_6(q)$	3	$(3^a)^6.S_3(W(E_6)).2^2$	$q \equiv 1 \pmod 3$
		$(3^a)^4.3^2.2^3$	$q \equiv 2 \pmod 3$
${}^2E_6(q)$	3	$(3^a)^6.S_3(W(E_6)).2^2$	$q \equiv 2 \pmod 3$
		$(3^a)^4.3^2.2^3$	$q \equiv 1 \pmod 3$
$E_7(q)$	2	$(2^a)^7.S_2(W(E_7))$	
$E_8(q)$	2	$(2^a)^8.S_2(W(E_8))$	
	3	$(3^a)^8.S_3(W(E_8)).2^4$	
	5	$(5^a)^8.S_5(W(E_8)).[2^6]$	$q \equiv \pm 1 \pmod 5$
		$(5^a)^4.5.4^2$	$q \equiv \pm 2 \pmod 5$

Here, $p^a = |\Phi_e(q)|_p$ with $e = e_p(q)$, respectively $3^a = |\Phi_4(2^{2f+1})|_3$ for the Ree groups, and $S_p(H)$ denotes a Sylow p-subgroup of H. Moreover, p^a stands for a cyclic group of that order, $[p^k]$ for an unspecified group of order p^k, H^k for a direct product of k copies of H.

claim holds. Exclude this case. Let $n = \sum_i p^{k_i}$ be the p-adic expansion of n (thus, any p^{k_i} occurs at most $p-1$ times). If this has at least two summands, then a proper Levi subgroup $\mathrm{GL}_{n_1}(q) \circ \mathrm{GL}_{n_2}(q)$, with $n_1 = p^{k_1}$, $n_2 = n - n_1$, contains a Sylow p-subgroup of S. Thus we may assume that $n = p^k$ is a p-power. For $p = 2$ the Sylow 2-subgroups are self-normalizing by [1, Thm. 4]. For odd p an easy matrix calculation shows that the normalizer of S_p (modulo the center) is an extension of a homocyclic group $(p^a)^{n-1}$ by the normalizer in $\mathfrak{S}_n$ of one of its Sylow p-subgroups, where $p^a = |q-1|_p$. So again, it is the same in S as in the centralizer of any coprime automorphism. The case of unitary groups is entirely similar. $\square$

The proof of Theorem 3.1 is now complete by Propositions 3.6, 3.8, 3.14 and 3.15. In the course of the proof we have established the following: if S is of Lie type and p not the defining characteristic, then (i) and (ii) in Theorem 3.1 are equivalent to

(iii) a Sylow p-subgroup of S is not contained in any proper Levi subgroup of S.

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