Change of Selmer Group for Big Galois Representations and Application to Normalization

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Abstract. The goal of this note is to prove, under some assumptions, a formula relating the Selmer groups of isogenous Galois representations. Local and global Euler-Poincaré characteristic formulas are key tools in the proof. With additional hypotheses, we use the isogeny formula to study how the formation of Selmer groups interacts with normalization of the coefficient ring and discuss how a main conjecture for a big Galois representation over a non-normal ring follows from a corresponding conjecture over the normalization.

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1 Introduction

1.1. Set $G_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ and suppose given a continuous Galois representation

$$\rho : G_{\mathbf{Q}} \longrightarrow \mathrm{Aut}_R(T),$$

where R is a ring finite and free over the power series ring $\mathcal{O}[\![X_1, \ldots, X_n]\!]$, with $\mathcal{O}$ the integer ring of a p-adic field, and T is a finitely-generated R-module. One can, under suitable hypotheses, attach a *Selmer group* $\mathrm{Sel}(\rho)$ to such ρ. This Selmer group is a finitely-generated R-module which is canonically defined in terms of the Galois cohomology of ρ.
The basic question we investigate below is the following. Given representations ρ_1 and ρ_2 as above on R-modules T_1 and T_2 which are *isogenous*, i.e., such that there is an $R[G_{\mathbf{Q}}]$-linear homomorphism $T_1 \to T_2$ with R-torsion cokernel, how are the Selmer groups $\mathrm{Sel}(\rho_1)$ and $\mathrm{Sel}(\rho_2)$ related? We prove the following

formula relating the support divisors of $\mathrm{Sel}(\rho_1)$ and $\mathrm{Sel}(\rho_2)$ in terms of local and global invariants of the quotient $Q = T_2/\phi(T_1)$ (see Theorem ?? for the precise statement).

Theorem. *If T_1 and T_2 satisfy certain natural hypotheses (cf. ??), then*

$$\operatorname{div}\bigl(\mathrm{Sel}(\rho_1)\bigr) - \operatorname{div}\bigl(\mathrm{Sel}(\rho_2)\bigr) = \\ = \sum_{v \text{ real}} \operatorname{div}(Q_{K_v}) - (r_1 + r_2)\operatorname{div}(Q) + \sum_{v|p} [K_v : \mathbf{Q}_p] \operatorname{div}(\mathrm{F}_v^+ Q).$$

1.2. Our main motivation (and a key example of this type of representation) comes from Hida theory. Let f be a p-ordinary cuspidal newform. By work of Hida [?], such f belongs to a p-adic family $\mathcal{F}$ of newforms, which can be viewed as a formal power series with coefficients in a ring R finite and free over $\mathcal{O}[\![X]\!]$, where $\mathcal{O}$ is a suitable finite extension of $\mathbf{Z}_p$. The specializations of $\mathcal{F}$ at appropriate values of T are power series expansions of classical p-stabilized newforms of varying weight, level, and character. One can attach a Galois representation $\rho_{\mathcal{F}}$ to $\mathcal{F}$ on a rank 2 module T over the ring R interpolating the p-adic Galois representation attached to the classical newforms arising as specializations of $\mathcal{F}$. Many of the hypotheses imposed in ?? are automatically satisfied by these representations.

1.3. An early investigation of how isogenies affect Iwasawa invariants was undertaken by Schneider [?], who gave a formula relating the μ-invariants for Selmer groups of isogenous abelian varieties over $\mathbf{Z}_p$-extensions of number fields. This formula was generalized by Perrin-Riou [?] to more general p-adic representations. More recently, Ochiai [?] has given a similar formula for invariants of big Galois representations with coefficients in a power series ring $\mathbf{Z}_p[\![T_1, \dots, T_n]\!]$. Our isogeny formula is a generalization of Ochiai's and has a similar proof, which, in particular, depends on Euler-Poincaré characteristic formulas and Poitou-Tate duality.

1.4. In Theorem ??, we prove somewhat general Euler-Poincaré characteristic formulas for big Galois representations. For $p > 2$, the theorem can be deduced from the corresponding statements in Nekovář [?, 4.6.9 and 7.8.6] (which exclude the case of $p = 2$). Our main result, the isogeny formula of Theorem ??, follows from a series of computations involving these. Fortunately, many of the needed computations are contained in Greenberg's series of papers [?, ?, ?]. In a certain sense, therefore, this note may be viewed as an addition to that series. Some of the results contained here can also be found in the second author's thesis [?, Ch. 1].

1.5. Under an additional "p-criticality" assumption on the representation T (cf. ??), we show in §?? that the corresponding normalized representation $\widetilde{T}$

obtained by extending scalars to the normalization $\widetilde{R}$ of R gives Selmer groups which, when considered as R-modules, have the same divisor on $\operatorname{Spec} R$. Using this fact and some elementary commutative algebra, we discuss how a main conjecture for the representation $\widetilde{T}$ implies a corresponding main conjecture for T. Thus, under our admittedly somewhat strict hypotheses, main conjectures, roughly speaking, commute with normalization. This result should not be surprising to the experts; its study was suggested by Greenberg [?, §1].

1.6. We remark here on some of the hypotheses we impose, some of which could be considered rather strong. The conditions (??)–(??) and the p-criticality hypothesis imposed in §?? are somewhat standard and are known to hold for many of the representations arising "in nature" from the study of Hida families as discussed briefly above, with the possible exception of (??), which has nonetheless been extensively studied. There are two additional, less standard, hypotheses we employ.

The first of these is that the Galois modules we consider are assumed to be *free* over the coefficient ring. There are two places where we make serious use of this hypothesis. The first is in the application of a result of Greenberg [?, Lemma 2.2.6] on vanishing of Galois invariants. We feel that this result is probably true for even torsion-free modules. The second is in the proof of Theorem ??, where we make use of the following property of free modules M over a ring R with module-finite normalization $\widetilde{R}$: the divisor (on $\operatorname{Spec} R$) associated to the torsion R-module $(M \otimes_R \widetilde{R})/M$ is $\operatorname{rank}_R M$ times the divisor associated to $\widetilde{R}/R$. It is unclear to us whether there is a weaker hypothesis on R-modules which guarantees this to hold.

The second is the condition (??) on the rank of compact Selmer groups, which is necessary in order to conclude the surjectivity of a certain localization map. It is a difficult and interesting question whether this condition holds for representations arising from Hida theory and is not true in general (cf. [?, §4.9] or [?, §7(d)] for an example).

2 Notation

2.1. Fix a prime p. Let R be a complete Noetherian local domain with maximal ideal $\mathfrak{m}$ and assume that R is finite and free over $\mathbf{Z}_p[\![T_1, \ldots, T_n]\!]$. If V is a finite-dimensional vector space over the fraction field $\operatorname{Frac} R$ of R, then we call an R-submodule $T \subseteq V$ an *R-lattice* in V if T is a finitely-generated R-module and $T \otimes_R \operatorname{Frac} R = V$ (where, here and subsequently, "=" means "canonically isomorphic").

Let $K/\mathbf{Q}$ be a finite extension. Fix a finite set Σ of primes of K containing the archimedean primes and the primes lying over p. Denote by K_Σ the maximal extension of K unramified outside Σ and set $G_\Sigma = \operatorname{Gal}(K_\Sigma/K)$. Our main objects of study in what follows are Galois representations

$$\rho : G_\Sigma \to \mathrm{GL}_n(\operatorname{Frac} R)$$

which are *continuous* in the sense that the representation space $V = V_\rho$ of ρ admits a G_Σ-stable R-lattice T such that the induced representation, which by abuse of notation we still denote by $\rho : G_\Sigma \to \mathrm{Aut}_R(T)$, is continuous for the Krull topology on G_Σ and the topology on $\mathrm{Aut}_R(T)$ induced by the topology on R.

In what follows, we shall be studying *free* lattices, i.e., R-submodules of $(\mathrm{Frac}\, R)^{\oplus n}$ of rank n which are free R-modules. Without additional assumptions on R, it may not be the case that any continuous R-linear representation of $\mathrm{Gal}(K_\Sigma/K)$ admits a $\mathrm{Gal}(K_\Sigma/K)$-stable lattice which is free as an R-module.

2.2. If M is any $\mathbf{Z}_p$-module, denote by $M^\vee$ the *Pontryagin dual* of M, i.e., $M^\vee = \mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Q}_p/\mathbf{Z}_p)$. Note that R is a compact $\mathbf{Z}_p$-module, so its dual $R^\vee$ is discrete; we endow $R^\vee$ with the trivial Galois action. If M is a cofinitely-generated, discrete R-module, then, by, e.g., Greenberg [?, Prop. 3.2], the (continuous) Galois cohomology groups $\mathrm{H}^i(K_\Sigma/K, M)$ are likewise cofinitely-generated R-modules.

2.3 Ordinary data. For notational convenience, we now define a notion of *ordinary datum* over R. Such a datum X consists of a pair (T, F), where T is a finitely-generated *free* R-module with continuous G_Σ-action and F consists of G_{K_v}-submodules $\mathrm{F}_v^+ T \subseteq T$, one for each prime v of K lying over p, such that $\mathrm{F}_v^+ T$ and $\mathrm{F}_v^- T = T/\mathrm{F}_v^+ T$ are free R-modules. We refer to the chain $T \supseteq \mathrm{F}_v^+ T \supseteq 0$ as the *local filtration* on T at v given by F.

Given ordinary data $X_1 = (T_1, \mathrm{F}_1)$ and $X_2 = (T_2, \mathrm{F}_2)$, we define a *homomorphism* $\phi : X_1 \to X_2$ to be an $R[G_\Sigma]$-linear homomorphism $\phi : T_1 \to T_2$ which is compatible with the filtrations in the sense that $\phi(\mathrm{F}_v^+ T_1) \subseteq \mathrm{F}_v^+ T_2$ for all $v \mid p$.

2.4. We now define discrete modules associated to a datum $X = (T, \mathrm{F})$. Denote by $W^* = W^*_X$ the discrete Galois module $W^* = \mathrm{Hom}_R\big(T, R^\vee(1)\big)$ dual to T. Thus, $W^* \cong T^\vee$ as an R-module. (Note that we do not define here a compact module T^* with Galois action the Tate dual of that on T.) The filtrations F on T induce filtrations $W^* \supseteq \mathrm{F}_v^+ W^* \supseteq 0$ for $v \mid p$ via $\mathrm{F}_v^+ W^* = \mathrm{Hom}_R\big(\mathrm{F}_v^- T, R^\vee(1)\big)$.

2.5 Local conditions. A set Δ of *local conditions* for an $R[G_\Sigma]$-module M is a choice of submodule $\mathrm{H}^1_{\mathrm{f}}(K_v, M) \subseteq \mathrm{H}^1(K_v, M)$ for each $v \in \Sigma$. Given an ordinary datum $X = (T, \mathrm{F})$, we define, following Greenberg [?, §4], the *Greenberg* local conditions for W^* as follows: if $v \nmid p$, then set

$$\mathrm{H}^1_{\mathrm{f}}(K_v, W^*) = \mathrm{H}^1_{\mathrm{ur}}(K_v, W^*) = \ker\big(\mathrm{H}^1(K_v, W^*) \xrightarrow{\mathrm{res}} \mathrm{H}^1(I_v, W^*)\big),$$

where $I_v \subseteq G_{K_v}$ is the inertia group. For $v \mid p$, set

$$\mathrm{H}^1_{\mathrm{f}}(K_v, W^*) = \ker\big(\mathrm{H}^1(K_v, W^*) \longrightarrow \mathrm{H}^1(I_v, \mathrm{F}_v^- W^*)\big),$$

where the homomorphism on the right is induced by the quotient $W^* \to \mathrm{F}_v^- W^*$ and restriction to I_v.
Let $X = (T, \mathrm{F})$ be an ordinary datum. Recall that Tate local duality gives a perfect pairing

$$\mathrm{H}^1(K_v, T) \times \mathrm{H}^1(K_v, W^*) \longrightarrow R^\vee.$$

We define the Greenberg local conditions $\mathrm{H}^1_{\mathrm{f}}(K_v, T)$ for T as the orthogonal complements of the Greenberg local conditions $\mathrm{H}^1_{\mathrm{f}}(K_v, W^*)$ for W^* under this pairing.

2.6. Given a set of local conditions Δ on an $R[G_\Sigma]$-module M, set

$$\mathrm{H}^1_{\mathrm{s}}(K_v, M) = \mathrm{H}^1(K_v, M)/\mathrm{H}^1_{\mathrm{f}}(K_v, M)$$

and define the *Selmer group* over K attached to M and Δ by

$$\mathrm{Sel}_\Delta(M) = \ker\Big(\mathrm{H}^1(G_\Sigma, M) \longrightarrow \bigoplus_{v \in \Sigma} \mathrm{H}^1_{\mathrm{s}}(K_v, M)\Big),$$

where the homomorphism on the right-hand side is induced by the obvious local-to-global map. If $M = W^*$ or T and Δ is the set of Greenberg local conditions for M, then we omit the Δ and denote the corresponding Selmer group by $\mathrm{Sel}(M)$.
For a G_Σ-module M and $i \geq 0$, we further define *Shafarevich-Tate groups*

$$^{i}(M) = \ker\Big(\mathrm{H}^i(K, M) \longrightarrow \bigoplus_{v \in \Sigma} \mathrm{H}^i(K_v, M)\Big).$$

Thus, $^{1}(M) = \mathrm{Sel}_\Delta(M)$ for Δ the set of local conditions defined by setting $\mathrm{H}^1_{\mathrm{f}}(K_v, M) = 0$ for all $v \in \Sigma$.
The representations arising in the case of Hida families come equipped with additional structure that allows other natural definitions of local conditions (e.g., the so-called *Bloch-Kato* local conditions) which in general give rise to Selmer groups different from those discussed above. Ochiai has studied the relationship between these Selmer groups, cf. [?, §3].

2.7. If M is a finitely-generated R-module and $\mathfrak{p} \subseteq R$ is a prime ideal, then we denote the $\mathfrak{p}$-*length* of M by

$$\mathrm{lgth}_{\mathfrak{p}} M = \mathrm{lgth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}},$$

which is finite if M is a torsion R-module. A simple argument shows that

$$\mathrm{lgth}_{\mathfrak{p}} M = \sum_{j=0}^{\infty} \mathrm{rank}_{R/\mathfrak{p}}\, \mathfrak{p}^j M/\mathfrak{p}^{j+1} M. \tag{2.7.1}$$

A finitely-generated R-module M is said to be *pseudo-null* if $\mathrm{lgth}_{\mathfrak{p}} M = 0$ for every height 1 prime $\mathfrak{p} \subseteq R$. Equivalently, M is pseudo-null if the set $\mathrm{Ass}_R(M)$

of associated primes of M contains only primes of height 2 or greater. If M is cofinitely-generated, we say M is *copseudo-null* if $M^\vee$ is pseudo-null. If R has dimension 2 and finite residue field, then a finitely-generated, resp. cofinitely-generated, R-module is pseudo-null, resp. copseudo-null, if and only if it contains only finitely many elements.

2.8 Conditions on X. Fix an ordinary datum $X = (T, \mathrm{F})$. Below, we often subject X to the following conditions.

(2.8.1) $T(-1)_{G_\Sigma}$ is a pseudo-null R-module.

(2.8.2) For each prime $v \mid p$ of K, $(\mathrm{F}_v^- T)(I_v) = 0$, $(\mathrm{F}_v^+ T(-1))(K_v) = 0$, $(\mathrm{F}_v^+ T)(K_v) = 0$, and $(\mathrm{F}_v^- W^*)(K_v)$ is copseudo-null over R.

(2.8.3) $\mathrm{Sel}(W^*)$ is a cotorsion R-module.

(2.8.4) No subquotient of $W^*[\mathfrak{m}]$ is isomorphic to μ_p as a G_K-module.

(2.8.5) For all $v \in \Sigma$ with $v \nmid p\infty$, $T(K_v) = 0$ and $W^*(K_v)$ is copseudo-null over R.

(2.8.6) $\mathrm{rank}_R \mathrm{Sel}(T) = 0$.

Note that $T(-1)_{G_\Sigma} \cong W^*(K)^\vee$, so (??) is equivalent to the statement that $W^*(K)$ is copseudo-null. A similar remark applies to (??) and the modules $(\mathrm{F}_v^- W^*)(K_v)^\vee \cong \mathrm{F}_v^+ T(-1)_{G_{K_v}}$. Condition (??) implies $T(K) = 0$. Indeed, $(T/\mathfrak{m}T) = \mathrm{Hom}_{\mathbf{Z}_p}(W^*, \mu_{p^\infty}) = 0$ under this assumption, so $T(K)/\mathfrak{m}T(K) = 0$ as well. As mentioned in the introduction, one cannot expect (??) to hold in general. As discussed, e.g., in [?, §4.9] or [?, §7(d)], there are interesting representations arising from Hida theory for which it should hold and for which it should not hold. In the context of those examples, though not generally, (??) and (??) should be equivalent.

3 Duality formulas

3.1. This section is devoted to the proof of various duality results for Selmer groups. The first several subsections (up to ??) are devoted to the proof of the following theorem, the global (??) and local (??) Euler-Poincaré characteristic formulas. This theorem can be deduced from Nekovář [?, 4.6.9 and 7.8.6], at least in the case $p > 2$.

3.2 Theorem. *Suppose K has r_1 real places and r_2 conjugate pairs of complex places. For any cofinitely-generated cotorsion R-module D and height 1 prime $\mathfrak{p} \subseteq R$,*

$$\sum_{i=0}^{2} (-1)^i \operatorname{lgth}_{\mathfrak{p}} \mathrm{H}^i(G_\Sigma, D)^\vee = \sum_{v \text{ real}} \operatorname{lgth}_{\mathfrak{p}} D(K_v)^\vee - (r_1 + r_2) \operatorname{lgth}_{\mathfrak{p}} D^\vee \quad (3.2.1)$$

and, for every non-archimedean prime v of K,

$$\sum_{i=0}^{2}(-1)^i \operatorname{lgth}_{\mathfrak{p}} \mathrm{H}^i(K_v, D)^\vee = \begin{cases} 0 & v \nmid p \\ -[K_v : \mathbf{Q}_p] \operatorname{lgth}_{\mathfrak{p}} D^\vee & v \mid p \end{cases} \tag{3.2.2}$$

3.3. Define, for any cofinitely-generated cotorsion R-module D,

$$\delta_\Sigma(D) = \sum_{i=0}^{2} \operatorname{lgth}_{\mathfrak{p}}(-1)^i \mathrm{H}^i(G_\Sigma, D)^\vee - \sum_{v \text{ real}} \operatorname{lgth}_{\mathfrak{p}} D(K_v)^\vee + (r_1 + r_2) D^\vee.$$

Similarly, for v a non-archimedean prime of K, if $v \nmid p$, then set

$$\delta_v(D) = \sum_{i=0}^{2}(-1)^i \operatorname{lgth}_{\mathfrak{p}} \mathrm{H}^i(K_v, D)^\vee,$$

and if $v \mid p$, then set

$$\delta_v(D) = \sum_{i=0}^{2}(-1)^i \operatorname{lgth}_{\mathfrak{p}} \mathrm{H}^i(K_v, D)^\vee + [K_v : \mathbf{Q}_p] \operatorname{lgth}_{\mathfrak{p}} D^\vee.$$

Thus, $\delta_*(D)$ is the difference between the right-hand side and left-hand side of (??) (if $* = \Sigma$) or of (??) (if $*$ is a prime of K), and we need to show that $\delta_*(D) = 0$.

3.4. The proof of Theorem ?? proceeds by induction on $\operatorname{lgth}_{\mathfrak{p}} D^\vee$ and dévissage. The base case is the following.

Lemma. *If* $\operatorname{lgth}_{\mathfrak{p}} \mathfrak{p} D^\vee = 0$*, then* $\delta_*(D) = 0$ *for* $* = \Sigma$ *or a prime of* K*.*

Proof. Consider the short exact sequence

$$0 \longrightarrow D[\mathfrak{p}] \longrightarrow D \longrightarrow D/D[\mathfrak{p}] \longrightarrow 0.$$

By hypothesis, we have $\operatorname{lgth}_{\mathfrak{p}}(D/D[\mathfrak{p}])^\vee = 0$, so $\mathfrak{p} \notin \operatorname{Supp}(D/D[\mathfrak{p}])^\vee$. As $\operatorname{Supp} M^\vee \supseteq \operatorname{Supp} \mathrm{H}^i(G, M)^\vee$ for any $i \geq 0$, any cofinitely-generated R-module M, and $G = G_\Sigma$ or G_{K_v}, we see from the definition of δ_* that $\delta_*(D) = \delta_*(D[\mathfrak{p}])$. We may therefore assume without loss of generality that $D[\mathfrak{p}] = D$. Under this assumption, D and all the $\mathrm{H}^i(G, D)$ are cofinitely-generated $R/\mathfrak{p}$-modules, so $\operatorname{lgth}_{\mathfrak{p}} D^\vee = \operatorname{corank}_{R/\mathfrak{p}} D$ and $\operatorname{lgth}_{\mathfrak{p}} \mathrm{H}^i(G, D)^\vee = \operatorname{corank}_{R/\mathfrak{p}} \mathrm{H}^i(G, D)^\vee$ by (??). Rephrased in this way via coranks, the statement of the lemma becomes the same as [?, Prop. 4.1]. □

3.5 Lemma (Dévissage). *For any short exact sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of cofinitely-generated R-modules with G-action for $G = G_{K_v}$, resp. $G = G_\Sigma$, we have $\delta_*(A) - \delta_*(B) + \delta_*(C) = 0$ for $* = \Sigma$ or a prime of K.

Proof. As $\operatorname{lgth}_{\mathfrak{p}}$ is additive in exact sequences, we may ignore the terms in the definition of $\delta_*(D)$ which are multiples of $\operatorname{lgth}_{\mathfrak{p}} D^\vee$. The lemma is slightly more difficult when $p = 2$ and $* = \Sigma$, so let us first assume either $p > 2$ or $* \neq \Sigma$. If v is an archimedean prime of K, then, as $p > 2$, $\mathrm{H}^i(K_v, D) = 0$ for $i > 1$ and $D = A$, B, or C. The result then follows from the long exact cohomology sequence and the fact that G_Σ and G_{K_v} have p-cohomological dimension 2 under our assumptions.

Now suppose $p = 2$ and $* = \Sigma$. By the long exact G_Σ-cohomology sequence, we have

$$\sum_{i=0}^{2} \operatorname{lgth}_{\mathfrak{p}} \mathrm{H}^i(G_\Sigma, A) - \sum_{i=0}^{2} \operatorname{lgth}_{\mathfrak{p}} \mathrm{H}^i(G_\Sigma, B) + \sum_{i=0}^{2} \operatorname{lgth}_{\mathfrak{p}} \mathrm{H}^i(G_\Sigma, C) = \\ = \operatorname{lgth}_{\mathfrak{p}} \ker[\mathrm{H}^3(G_\Sigma, A) \to \mathrm{H}^3(G_\Sigma, B)]. \quad (3.5.1)$$

Recall ([?, Thm. 4.10], e.g.) that for any discrete ind-finite $R[G_\Sigma]$-module D, the natural map

$$\mathrm{H}^q(G_\Sigma, D) \longrightarrow \prod_{v \text{ real}} \mathrm{H}^q(K_v, D)$$

given by the product of restrictions to decomposition groups at real places is an isomorphism for $q \geq 3$, so the right-hand side of (??) is equal to

$$\operatorname{lgth}_{\mathfrak{p}} \prod_{v \text{ real}} \ker[\mathrm{H}^3(K_v, A) \to \mathrm{H}^3(K_v, B)].$$

As G_{K_v} is cyclic of order 2 for v a real place, the cohomology groups $\mathrm{H}^i(K_v, D)$ are periodic of period 2 for $i > 0$, and all have equal $\mathfrak{p}$-length (cf. [?, Prop. 4.18]). This implies that

$$\operatorname{lgth}_{\mathfrak{p}} \ker[\mathrm{H}^3(K_v, A) \to \mathrm{H}^3(K_v, B)] = \operatorname{lgth}_{\mathfrak{p}} \ker[\mathrm{H}^1(K_v, A) \to \mathrm{H}^1(K_v, B)]$$

for real v, which, by the long exact G_{K_v}-cohomology sequences, shows that the right hand side of (??) is equal to

$$\sum_{v \text{ real}} \big(\operatorname{lgth}_{\mathfrak{p}} A(K_v) - \operatorname{lgth}_{\mathfrak{p}} B(K_v) + \operatorname{lgth}_{\mathfrak{p}} C(K_v)\big),$$

which proves the lemma for $p = 2$ and $* = \Sigma$. □

3.6 Proof of Theorem ??. The statement is true when $\operatorname{lgth}_{\mathfrak{p}} D^\vee = 0$ by Lemma ??, so assume $\operatorname{lgth}_{\mathfrak{p}} D^\vee > 0$. Consider the short exact sequence

$$0 \longrightarrow \mathfrak{p}D^\vee \longrightarrow D^\vee \longrightarrow D^\vee/\mathfrak{p}D^\vee \longrightarrow 0.$$

Lemma ?? implies the result if $\operatorname{lgth}_{\mathfrak{p}} \mathfrak{p}D^\vee = 0$. Similarly, if $\operatorname{lgth}_{\mathfrak{p}} D^\vee/\mathfrak{p}D^\vee = 0$, then $\operatorname{lgth}_{\mathfrak{p}} D^\vee = 0$ by Nakayama's Lemma, so *a fortiori* $\operatorname{lgth}_{\mathfrak{p}} \mathfrak{p}D^\vee = 0$ and we are again done by Lemma ??. We may therefore assume that both $\operatorname{lgth}_{\mathfrak{p}} \mathfrak{p}D^\vee$ and $\operatorname{lgth}_{\mathfrak{p}} D^\vee/\mathfrak{p}D^\vee$ are positive and thus less than $\operatorname{lgth}_{\mathfrak{p}} D^\vee$. The theorem then follows from dévissage (Lemma ??) and induction. □

3.7 Theorem (Poitou-Tate global duality). *There is a perfect pairing*

$$ {}^1(W^*) \times {}^2(T) \longrightarrow \mathbf{Q}_p/\mathbf{Z}_p $$

and a 9*-term exact sequence*

$$
\begin{aligned}
0 \longrightarrow \mathrm{H}^0(G_\Sigma, W^*) \longrightarrow \bigoplus_{v\in\Sigma} \mathrm{H}^0(K_v, W^*) \longrightarrow \mathrm{H}^2(G_\Sigma, T)^\vee \longrightarrow \\
\longrightarrow \mathrm{H}^1(G_\Sigma, W^*) \longrightarrow \bigoplus_{v\in\Sigma} \mathrm{H}^1(K_v, W^*) \longrightarrow \mathrm{H}^1(G_\Sigma, T)^\vee \longrightarrow \\
\longrightarrow \mathrm{H}^2(G_\Sigma, W^*) \longrightarrow \bigoplus_{v\in\Sigma} \mathrm{H}^2(K_v, W^*) \longrightarrow \mathrm{H}^0(G_\Sigma, T)^\vee \longrightarrow 0
\end{aligned}
$$

Proof. For all n, $R/\mathfrak{m}^n$ is finite. Note that $W^* = \varinjlim W^*[\mathfrak{m}^n]$, and $T = \varprojlim T/\mathfrak{m}^n T$. As $W^*[\mathfrak{m}^n] \cong \mathrm{Hom}_{\mathbf{Z}_p}\big(T/\mathfrak{m}^n T, \mathbf{Q}_p/\mathbf{Z}_p(1)\big)$, the theorem follows from the version for finite modules (see [?, Thm. I.4.10], for example) by taking limits. $\square$

3.8. For a G_Σ-module M with local filtrations at each $v \mid p$, e.g., for M arising from an ordinary datum, we define semi-local cohomology groups by

$$ \mathrm{H}^i_{\mathrm{loc}}(M) = \bigoplus_{v|p} \mathrm{H}^i(K_v, \mathrm{F}^-_v\, M) \oplus \bigoplus_{\substack{v\in\Sigma \\ v\nmid p\infty}} \mathrm{H}^i(K_v, M). $$

Additionally, let

$$ \mathrm{loc}^i_M : \mathrm{H}^i(G_\Sigma, M) \to \mathrm{H}^i_{\mathrm{loc}}(M) $$

be the natural localization map.

3.9 Lemma. *If X satisfies* (??) *and* (??)*, then the natural homomorphism*

$$ \mathrm{H}^1(G_\Sigma, W^*) \longrightarrow \bigoplus_{v\in\Sigma} \mathrm{H}^1_{\mathrm{s}}(K_v, W^*) $$

is surjective.

Proof. Consider the exact sequence arising from local duality and the definitions of the various groups involved:

$$ \mathrm{H}^1(G_\Sigma, W^*) \longrightarrow \bigoplus_{v\in\Sigma} \mathrm{H}^1_{\mathrm{s}}(K_v, W^*) \longrightarrow \mathrm{Sel}(T)^\vee \longrightarrow {}^1(T)^\vee \longrightarrow 0. $$

Under (??), [?, Prop. 2.2.1] states that $\mathrm{H}^1(G_\Sigma, T)$ is Λ-torsion-free, whence R-torsion-free, so the same is true of $\mathrm{Sel}(T)$. The lemma then follows from the assumption (??). $\square$

3.10 Lemma. *If X satisfies (??) and (??), then $\mathrm{Sel}(W^*)/\ker \mathrm{loc}^1_{W^*}$ is a copseudo-null R-module. In particular,*

$$\mathrm{lgth}_{\mathfrak{p}} \mathrm{Sel}(W^*)^\vee = \mathrm{lgth}_{\mathfrak{p}} \ker(\mathrm{loc}^1_{W^*})^\vee$$

for every height 1 prime $\mathfrak{p} \subseteq R$. If X further satisfies (??) and (??), then $\mathrm{coker}\, \mathrm{loc}^1_{W^*}$ is a copseudo-null R-module.

Proof. The inflation-restriction sequence for $I_v \subseteq G_{K_v}$ implies that the quotient $\mathrm{Sel}(W^*)/\ker \mathrm{loc}^1_{W^*}$ injects into

$$\bigoplus_{v|p} \mathrm{H}^1\big(K_v^{\mathrm{ur}}/K_v, \mathrm{F}_v^- \, W^*(I_v)\big) \oplus \bigoplus_{\substack{v\in\Sigma \\ v\nmid p\infty}} \mathrm{H}^1\big(K_v^{\mathrm{ur}}/K_v, W^*(I_v)\big), \tag{3.10.1}$$

where K_v^{ur} is the maximal unramified extension of K_v. The lemma thus follows from the assumptions (??) and (??), which state that $\mathrm{F}_v^- \, W^*(I_v)$ and $W^*(I_v)$ are copseudo-null R-modules.

In case X also satisfies (??) and (??), then Lemma ?? gives that the homomorphism

$$\mathrm{H}^1(G_\Sigma, W^*) \longrightarrow \bigoplus_{v\in\Sigma} \mathrm{H}^1_{\mathrm{s}}(K_v, W^*)$$

defining $\mathrm{Sel}(W^*)$ is surjective. The module (??) above is the kernel of the quotient map

$$\mathrm{H}^1_{\mathrm{loc}}(W^*) \longrightarrow \bigoplus_{v\in\Sigma} \mathrm{H}^1_{\mathrm{s}}(K_v, W^*),$$

so the final statement in the lemma follows from the fact that (??) is copseudo-null. ☐

3.11 Lemma. *If X satisfies (??), (??), and (??), then the R-modules $\mathrm{H}^2_{\mathrm{loc}}(W^*)$ and $\mathrm{H}^2(G_\Sigma, W^*)$ are trivial.*

Proof. By local Tate duality, we have

$$\mathrm{H}^2(K_v, \mathrm{F}_v^- \, W^*) \cong (\mathrm{F}_v^+ \, T)(K_v)$$

for $v \mid p$, and

$$\mathrm{H}^2(K_v, W^*) \cong T(K_v)$$

for $v \in \Sigma$, $v \nmid p\infty$. Both of these are trivial by (??) and (??), respectively, so that $\mathrm{H}^2_{\mathrm{loc}}(K, W^*) = 0$.

We first show ${}^2(W^*) = 0$. By [?, Prop. 6.6], ${}^2(W^*)$ is coreflexive, so it suffices to show that $\mathrm{corank}_R \, {}^2(W^*) = 0$. By [?, Prop. 4.4], ${}^2(W^*)$ has the same R-corank as ${}^1(W^*)$. On the other hand, ${}^1(W^*) \subseteq \mathrm{Sel}(W^*)$, which is assumed R-cotorsion by (??). By definition of , we have an exact sequence

$$0 \longrightarrow {}^2(W^*) \longrightarrow \mathrm{H}^2(G_\Sigma, W^*) \longrightarrow \bigoplus_{v\in\Sigma} \mathrm{H}^2(K_v, W^*).$$

We have just seen that ${}^2(W^*) = 0$, and $\mathrm{H}^2(K_v, W^*) = 0$ for $v \in \Sigma$ by (??), (??), and local duality. ☐

4 Isogenies

4.1. If T_1 and T_2 are finitely generated R-modules and $\phi : X_1 \to X_2$ is a homomorphism with torsion kernel and cokernel, then we say that ϕ is an *isogeny* or that T_1 and T_2 are *isogenous* if we do not wish to make the homomorphism explicit. Note that an isogeny of torsion-free R-modules is necessarily injective. Similarly, if W_1 and W_2 are cofinitely-generated R-modules, then we say that a homomorphism $\psi : W_1 \to W_2$ is an isogeny (and that W_1 and W_2 are isogenous) if its Pontryagin dual $\psi^\vee : W_2^\vee \to W_1^\vee$ is an isogeny. If $X_i = (T_i, \mathrm{F}_i)$, $i = 1, 2$, are ordinary data, then we say that $\phi : X_1 \to X_2$ is an isogeny and hence that the X_i are isogenous if $\phi : T_1 \to T_2$ is an isogeny. A homomorphism of ordinary data $\phi : X_1 \to X_2$ is an isogeny if and only if the induced homomorphism $W^*_{X_2} \to W^*_{X_1}$ is an isogeny. Isogeny is an equivalence relation on the categories of finitely-generated R-modules, cofinitely-generated R-modules, and ordinary data, cf. [?, §2].

4.2. For the remainder of the section, fix ordinary data $X_1 = (T_1, \mathrm{F}_1)$ and $X_2 = (T_2, \mathrm{F}_2)$ and an isogeny $\phi : X_1 \to X_2$. Our goal is to use Theorem ?? to prove a formula (Theorem ??) relating the $\mathfrak{p}$-lengths of Selmer groups for X_1 and X_2 in terms of various Galois invariants of the quotient module $T_2/\phi(T_1)$, or, more precisely, its dual $C = C_\phi = \ker[W_2^* \to W_1^*]$. The key tools we need are the global Euler-Poincaré characteristic formulas above and Poitou-Tate duality, Theorem ??. The formula can be thought of as a reorganization of the information provided by Poitou-Tate duality under the assumptions (??)–(??).

4.3 Proposition. *If X satisfies (??)–(??), then for all height 1 primes $\mathfrak{p} \subseteq R$,*

$$\mathrm{lgth}_\mathfrak{p} \operatorname{Sel}(W_1^*)^\vee - \mathrm{lgth}_\mathfrak{p} \operatorname{Sel}(W_2^*)^\vee = \\ = \sum_{i=1}^{2} (-1)^i \big(\mathrm{lgth}_\mathfrak{p} \mathrm{H}^i(G_\Sigma, C)^\vee - \mathrm{lgth}_\mathfrak{p} \mathrm{H}^i_{\mathrm{loc}}(C)^\vee\big)$$

Proof. The commutative diagram

$$\begin{tikzcd} \mathrm{H}^1(G_\Sigma, C) \arrow[r, "\alpha"] \arrow[d] & \mathrm{H}^1(G_\Sigma, W_1^*) \arrow[r] \arrow[d, "\mathrm{loc}^1_{W_1^*}"'] & \mathrm{H}^1(G_\Sigma, W_2^*) \arrow[r, "\gamma"] \arrow[d, "\mathrm{loc}^1_{W_2^*}"'] & \mathrm{H}^2(G_\Sigma, C) \arrow[d] \\ \mathrm{H}^1_{\mathrm{loc}}(C) \arrow[r, "\beta"] & \mathrm{H}^1_{\mathrm{loc}}(W_1^*) \arrow[r] & \mathrm{H}^1_{\mathrm{loc}}(W_2^*) \arrow[r, "\delta"] & \mathrm{H}^2_{\mathrm{loc}}(C) \end{tikzcd}$$

has exact rows. Assumptions (??), resp. (??) and (??), imply that $\ker \alpha$, resp. $\ker \beta$, is copseudo-null over R. Likewise, γ and δ have trivial cokernel by Lemma ??. By Lemma ??, $\mathrm{loc}^1_{W_1^*}$ and $\mathrm{loc}^1_{W_2^*}$ have copseudo-null cokernels. Examining the $\mathfrak{p}$-lengths in the above diagram for a height 1 prime $\mathfrak{p} \subseteq R$

therefore gives

$$\operatorname{lgth}_{\mathfrak p}\ker(\operatorname{loc}^1_{W_1^*})^\vee - \operatorname{lgth}_{\mathfrak p}\ker(\operatorname{loc}^1_{W_2^*})^\vee = \\ = \sum_{i=1}^{2}(-1)^i\big(\operatorname{lgth}_{\mathfrak p}\mathrm{H}^i(G_\Sigma, C)^\vee - \operatorname{lgth}_{\mathfrak p}\mathrm{H}^1_{\mathrm{loc}}(C)^\vee\big),$$

which implies the proposition by the first statement of Lemma ??. □

4.4 Theorem. *If X_1 and X_2 satisfy (??)–(??), then, for every height 1 prime $\mathfrak p \subseteq R$,*

$$\operatorname{lgth}_{\mathfrak p}\operatorname{Sel}(W_1^*)^\vee - \operatorname{lgth}_{\mathfrak p}\operatorname{Sel}(W_2^*)^\vee = \\ = \sum_{v \text{ real}}\operatorname{lgth}_{\mathfrak p} C(K_v)^\vee - (r_1+r_2)\operatorname{lgth}_{\mathfrak p} C^\vee + \sum_{v\mid p}[K_v:\mathbf{Q}_p]\operatorname{lgth}_{\mathfrak p}(\mathrm{F}_v^- C)^\vee.$$

Proof. By (??), $\mathrm{H}^0(G_\Sigma, W_1^*)$ is copseudo-null, so $\mathrm{H}^0(G_\Sigma, C)$ and $\mathrm{H}^1_{\mathrm{loc}}(W_1^*)$ are also copseudo-null. The theorem thus follows immediately from Proposition ?? and the global Euler-Poincaré characteristic formula, Theorem ??. □

5 Application to normalization

5.1. We now apply the main result of §?? to study how Selmer groups behave with respect to normalization. Assume R is reduced and let $\widetilde R$ be the integral closure of R in its total ring of fractions. A well-known result of Nagata [?, Thm. 7] states that $\widetilde R$ is a finite R-module. If X is an ordinary datum over R, then set $\widetilde X = (\widetilde T, \widetilde{\mathrm F})$, where $\widetilde T = T\otimes_R \widetilde R$ and $\widetilde{\mathrm F}_v^+ T = (F_v^+ T)\otimes_R \widetilde R$. Since $\widetilde R$ is finite over R, we may view $\widetilde X$ as an ordinary datum over R or over $\widetilde R$, and the natural inclusion $T \to \widetilde T$ is an isogeny of ordinary data over R.

5.2. Fix an ordinary datum X over R. For $\Phi = \operatorname{Frac} R$ the fraction field of R, define $V = T\otimes_R \Phi$, so V is a finite-dimensional Φ-vector space with a Φ-linear action of G_Σ. The filtrations F induce filtrations $V \supseteq \mathrm{F}_v^+ V \supseteq 0$ for $v \mid p$. Define

$$\alpha(X) = \dim_\Phi(\operatorname{res}_{K/\mathbf Q} V)^+ = \sum_{v\mid\infty}\dim_\Phi V(K_v).$$

For $v \mid p$, define $\varepsilon_v(X) = \dim_K \mathrm{F}_v^+ V$. We say that X is *p-critical* if $\alpha(X) = \sum_{v\mid p}\varepsilon_v(x)$. For p-critical data, we have the following theorem regarding normalization.

5.3 Theorem. *Let $\mathfrak p \subseteq R$ be a height 1 prime. If $p = 2$, then assume that $T(K_v)$ is a summand of T for each real place v of K. If X and $\widetilde X$, both viewed as ordinary data over R, are p-critical and satisfy (??)–(??), then*

$$\operatorname{lgth}_{\mathfrak p}\operatorname{Sel}(W^*)^\vee = \operatorname{lgth}_{\mathfrak p}\operatorname{Sel}(\widetilde{W^*})^\vee.$$

Proof. Let $C = \ker[\widetilde{W}^* \to W^*]$, the map being induced by the inclusion $T \hookrightarrow \widetilde{T}$. By (??), (??) and (??), we have that $\mathrm{H}^0(G_\Sigma, C)$ and $\mathrm{H}^0_{\mathrm{loc}}(C)$ are copseudo-null over R, so it suffices by Proposition ?? to show that

$$\sum_{i=0}^{2}(-1)^i \operatorname{lgth}_{\mathfrak{p}} \mathrm{H}^i(G_\Sigma, C)^\vee = \sum_{i=0}^{2}(-1)^i \mathrm{H}^i_{\mathrm{loc}}(C)^\vee \tag{5.3.1}$$

for all $\mathfrak{p} \subseteq R$ of height 1. The global Euler-Poincaré formula (??) gives the left-hand side of (??) as

$$\sum_{v \text{ real}} \operatorname{lgth}_{\mathfrak{p}} C(K_v)^\vee - (r_1 + r_2) \operatorname{lgth}_{\mathfrak{p}} C^\vee$$

and the local formula (??) gives the right-hand side as

$$\sum_{v|p} -[K_v : \mathbf{Q}_p] \operatorname{lgth}_{\mathfrak{p}} (\mathrm{F}_v^- \, C)^\vee.$$

Adding $[K : \mathbf{Q}] \operatorname{lgth}_{\mathfrak{p}} C^\vee$ to these yields

$$\sum_{v \text{ real}} \operatorname{lgth}_{\mathfrak{p}} C(K_v)^\vee + r_2 \operatorname{lgth}_{\mathfrak{p}} C^\vee$$

and

$$\sum_{v|p} [K_v : \mathbf{Q}_p] \operatorname{lgth}_{\mathfrak{p}} (\mathrm{F}_v^+ \, C)^\vee.$$

By freeness of T, $\operatorname{lgth}_{\mathfrak{p}} C^\vee = \operatorname{rank}_R T \operatorname{lgth}_{\mathfrak{p}} \widetilde{R}/R$, and similarly for $(\mathrm{F}_v^\pm \, C)^\vee$ and $C(K_v)$. The theorem thus follows from the p-criticality assumption on T. $\square$

5.4 Lemma. *Let M and N be torsion $\widetilde{R}$-modules and fix a height 1 prime $\mathfrak{q} \subseteq R$. Then $\operatorname{lgth}_{\mathfrak{p}} M = \operatorname{lgth}_{\mathfrak{p}} N$ for all height 1 primes $\mathfrak{p} \subseteq \widetilde{R}$ such that $\mathfrak{p} \mid \mathfrak{q}$ if and only if $\operatorname{lgth}_{\mathfrak{q}} M = \operatorname{lgth}_{\mathfrak{q}} N$.*

Proof. The content of the lemma is that the $\mathfrak{q}$-length of a torsion $\widetilde{R}$-module M (viewed as R-module) is determined by its $\mathfrak{p}$-lengths (viewed as $\widetilde{R}$-module) for $\mathfrak{p} \subseteq \widetilde{R}$ lying over $\mathfrak{q}$, and conversely. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the primes of $\widetilde{R}$ lying over $\mathfrak{q} \subseteq R$ and set $S = (R - \mathfrak{q}) \subseteq \widetilde{R}$. First consider a chain

$$M_{\mathfrak{q}} = M_{\mathfrak{q},0} \supseteq M_{\mathfrak{q},1} \supseteq \cdots \supseteq M_{\mathfrak{q},k}$$

computing the length of $M_{\mathfrak{q}}$ as $S^{-1}\widetilde{R}$-module. Each successive quotient in this chain is isomorphic to $\widetilde{R}/\mathfrak{p}_i$ for some i. Localization of this chain to $\widetilde{R}_{\mathfrak{p}_i}$ therefore computes $\operatorname{lgth}_{\mathfrak{p}_i} M$ after removing repeated submodules and we see that $\operatorname{lgth}_{\mathfrak{q}} M$ determines and is determined by these lengths and $\operatorname{lgth}_{\mathfrak{q}} \widetilde{R}/\mathfrak{p}_i$. $\square$

5.5. In the below corollary to Theorem ??, we say two finitely-generated R-modules M and N *have the same divisor* if $\operatorname{lgth}_{\mathfrak{p}} M = \operatorname{lgth}_{\mathfrak{q}} N$ for all height 1 primes $\mathfrak{p} \subseteq R$. Similarly, we say a finitely-generated R-module *has the same divisor* as an element $L \in R$ if M and R/L have the same divisor.

Corollary. *Let $0 \neq L \in R$ and let $\widetilde{L}$ be the image of L in $\widetilde{R}$. Using notation and assumptions as in Theorem ??, with the exception that we now view $\widetilde{X}$ as an ordinary datum over $\widetilde{R}$,* $\operatorname{Sel}(W^*)^\vee$ *has the same divisor as L if and only if* $\operatorname{Sel}(\widetilde{W}^*)^\vee$ *has the same divisor as $\widetilde{L}$.*

Proof. Viewing $\widetilde{R}$ as a rank 1 R-module, we use the formula [?, Lemma 11.7] to see that, for every height 1 prime $\mathfrak{q} \subseteq R$,

$$\operatorname{lgth}_{\mathfrak{q}} \widetilde{R}/(\widetilde{L}) = \operatorname{lgth}_{\mathfrak{q}} R/(L),$$

so the result follows by combining Theorem ?? and Lemma ??. □

5.6. Corollary ?? states, roughly speaking, that, under some assumptions, the formation of the divisor of the Selmer group of an ordinary datum commutes with normalization. In a situation where there is a p-adic L-function belonging to R associated with the ordinary datum X, the corollary provides some flexibility in proving a main conjecture for X, in that such a conjecture can be proved equivalently before or after normalization.

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