Irreducible Modules over the Virasoro Algebra

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Received: April 11, 2011
Revised: September 22, 2011

Communicated by Edward Frenkel

Abstract. In this paper, we construct two different classes of Virasoro modules from twisting Harish-Chandra modules over the twisted Heisenberg-Virasoro algebra by an automorphism of the twisted Heisenberg-Virasoro algebra. Weight modules in the first class are some irreducible highest weight modules over the twisted Heisenberg-Virasoro algebra. The non-weight modules in the first class are irreducible Whittaker modules over the Virasoro algebra. We obtain concrete bases for all irreducible Whittaker modules (instead of a quotient of modules). This generalizes known results on Whittaker modules. The second class of modules are non-weight modules which are not Whittaker modules. We determine the irreducibility and isomorphism classes of these modules.

2010 Mathematics Subject Classification: 17B10, 17B20, 17B65, 17B66, 17B68
Keywords and Phrases: Virasoro algebra, Whittaker module, irreducible module.

1 Introduction

Throughout this paper, we will use $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$ to denote the sets of complex numbers, nonzero complex numbers, integers, nonnegative integers and positive integers respectively.
The theory of weight modules with finite-dimensional weight spaces over the Heisenberg algebra, the Virasoro algebra and the twisted Heisenberg-Virasoro algebra are fairly well developed. We refer the readers to [1], [4] [5], [11], [12], [13] and the references therein. For weight modules with infinite dimensional weight spaces, see [3], [7], [17]. Recently Whittaker modules over those algebras

were studied by many authors, see for example [2], [6], [10], [14], [16]. Besides Whittaker modules, some new non-weight modules over the Virasoro algebra were just constructed in [8].

We will use modules over the twisted Heisenberg-Virasoro algebra to study modules over the Virasoro algebra. Now we first recall the twisted Heisenberg-Virasoro algebra.

The twisted Heisenberg-Virasoro algebra $\mathbb{L}$ is the universal central extension of the Lie algebra $\{f(t)\frac{d}{dt} + g(t)|f, g \in \mathbb{C}[t, t^{-1}]\}$ of differential operators of order at most one on the Laurent polynomial algebra $\mathbb{C}[t, t^{-1}]$. More precisely, the twisted Heisenbeg-Virasoro algebra $\mathbb{L}$ is a Lie algebra over $\mathbb{C}$ with the basis

$$\{d_n, t^n, z_1, z_2, z_3|n \in \mathbb{Z}\}$$

and the Lie bracket given by

$$[d_n, d_m] = (m-n)d_{n+m} + \delta_{n,-m}\frac{n^3-n}{12}z_1, \tag{1.1}$$

$$[d_n, t^m] = mt^{m+n} + \delta_{n,-m}(n^2+n)z_2, \tag{1.2}$$

$$[t^n, t^m] = n\delta_{n,-m}z_3, \tag{1.3}$$

$$[\mathbb{L}, z_1] = [\mathbb{L}, z_2] = [\mathbb{L}, z_3] = 0. \tag{1.4}$$

The Lie algebra $\mathbb{L}$ has a Virasoro subalgebra Vir with basis $\{d_i, z_1|i \in \mathbb{Z}\}$, and a Heisenberg subalgebra H with basis $\{t^i, z_3|i \in \mathbb{Z}\}$.

Let σ be an endomorphism of $\mathbb{L}$, and V be any weight module of $\mathbb{L}$. We can make V into another $\mathbb{L}$-module, by defining the new action of $\mathbb{L}$ on V as

$$x \circ v = \sigma(x)v, \forall x \in \mathbb{L}, v \in V. \tag{1.5}$$

We will call the new module as *the twisted module of* V *by* σ, and denote it by V^σ.

To avoid any ambiguity, we will not omit the circ for the new action.

The module V^σ can be regarded as the Vir module by restriction to the Virasoro subalgebra. One important fact is that we can get a lot of new irreducible modules over Vir in this simple way, which include some new irreducible Whittaker modules. Since these modules are generally not weight modules, it is not trivial to determine isomorphism classes and irreducibility for these modules.

The paper is organized as follows. In section 2, we collect some known results for later use. In section 3, we construct our first class of Virasoro modules by twisting a highest weight $\mathbb{L}$ module (oscillator representation) with automorphisms of $\mathbb{L}$, then we obtain some new irreducible Whittaker modules $L_{\psi_m,\dot{z}_1}$, where $m > 0$, over the Virasoro algebra. This concrete realization allows us to give concrete bases for all irreducible Whittaker modules (not only as a quotient of modules). Our bases for irreducible Whittaker modules $L_{\psi_m,\dot{z}_1}$ with $m = 1$ generalize those results in [14] where it was required: $\psi_1(d_1)\psi_1(d_2) \neq 0$, and those results in [16] where an explicit formula for the Whittaker vector was give only for $\psi_1(d_1) \neq 0$ and $\psi_1(d_2) = 0$ in terms of Jack symmetric polynomial. In section 4, we construct our second class of Virasoro modules by twisting $\mathbb{L}$ modules of intermediate series with automorphisms of $\mathbb{L}$. Then we determine the isomorphism classes and irreducibility of these Virasoro modules.

2 Preliminaries

In this section, we collect some notations and known facts for later use. For details, we refer the readers to [9], [12], [15], and the references therein.

Let us recall the definition of weight modules and highest weight modules over $\mathbb{L}$.

It is well-known that $\mathbb{L}$ has a natural $\mathbb{Z}$-gradation: deg d_n=deg t^n=n and deg $z_i = 0$ for $i = 1, 2, 3$. Set

$$\mathbb{L}_+ = \sum_{n>0}(\mathbb{C}d_n + \mathbb{C}t^n),\ \mathbb{L}_- = \sum_{n<0}(\mathbb{C}d_n + \mathbb{C}t^n),$$

and

$$\mathbb{L}_0 = \mathbb{C}d_0 + \mathbb{C}t^0 + \mathbb{C}z_1 + \mathbb{C}z_2 + \mathbb{C}z_3.$$

Then we have the triangular decomposition $\mathbb{L} = \mathbb{L}_+ \oplus \mathbb{L}_0 \oplus \mathbb{L}_-$.

For any $\mathbb{L}$-module V and $(\lambda, \lambda_H, c_1, c_2, c_3) \in \mathbb{C}^5$, set

$$V_{(\lambda,\lambda_H,c_1,c_2,c_3)} = \\ \big\{v \in V \bigm| d_0 v = \lambda v, t^0 v = \lambda_H v, \text{ and } z_i v = c_i v \text{ for } i = 1, 2, 3\big\},$$

which we generally call the weight space of V corresponding to the weight $(\lambda, \lambda_H, c_1, c_2, c_3) \in \mathbb{C}^5$. When t^0, z_1, z_2, z_3 act as scalars on the whole space V, we shall simply write V_λ instead of $V_{(\lambda,\lambda_H,c_1,c_2,c_3)}$.

An $\mathbb{L}$-module V is called a *weight module* if V is the sum of all its weight spaces.

A weight $\mathbb{L}$-module V is called *a highest weight module with highest weight* $(\lambda, \lambda_H, c_1, c_2, c_3) \in \mathbb{C}^5$, if there exists a nonzero weight vector $v \in V_{(\lambda,\lambda_H,c_1,c_2,c_3)}$ such that

1) V is generated by v as an $\mathbb{L}$-module;
2) $\mathbb{L}_+ v = 0$.

It is well known that, up to isomorphism, there exists a unique irreducible highest weight module $V(\lambda, \lambda_H, c_1, c_2, c_3)$ over $\mathbb{L}$ with the highest weight $(\lambda, \lambda_H, c_1, c_2, c_3) \in \mathbb{C}^5$.

For any $a, b \in \mathbb{C}$, we have the Vir module $A_{a,b}$, called the module of intermediate series, which has basis $\{t^k | k \in \mathbb{Z}\}$ such that z_1 acts trivially and

$$d_i t^k = (a + k + bi)t^{i+k}, \forall i, k \in \mathbb{Z}. \tag{2.1}$$

It is well-known that $A_{a,b}$ is irreducible if and only if $a \notin \mathbb{Z}$ or $b \notin \{0, 1\}$. We put $A'_{a,b} = A_{a,b}$ if $A_{a,b}$ is irreducible; otherwise $A'_{a,b}$ be the unique nontrivial irreducible sub-quotient of $A_{a,b}$.

Let us summarize some well-known results for the modules of intermediate series.

Theorem 1. *Let $a, b, a_1, b_1 \in \mathbb{C}$.*

(1) If $b \notin \{0,1\}$, then $A_{a,b} \cong A_{a_1,b_1}$ if and only if $b = b_1$ and $a - a_1 \in \mathbb{Z}$;

(2) If $b \in \{0,1\}$ and $a \notin \mathbb{Z}$, then we have $A_{a,b} \cong A_{a_1,b_1}$ if and only if $b_1 \in \{0,1\}$ and $a - a_1 \in \mathbb{Z}$;

(3) If $b \in \{0,1\}$ and $a \in \mathbb{Z}$, then we have $A_{a,b} \cong A_{a_1,b_1}$ if and only if $b_1 = b$ and $a_1 \in \mathbb{Z}$;

(4) If $a \in \mathbb{Z}$, then $A'(a,0) \cong A'(a,1) \cong A'(0,0)$.

We also need the following result from [15], and we will write it in a slightly different form for later use.
For any

$$\alpha = \sum_{i\in\mathbb{Z}} a_i t^i \in \mathbb{C}[t, t^{-1}], b \in \mathbb{C}, \tag{2.2}$$

we have the $\sigma = \sigma_{\alpha,b} \in \mathrm{Aut}(\mathbb{L})$ defined as

$$\begin{aligned}\sigma(d_n) = {} & d_n + t^n(\alpha + nb) - (n+1)a_{-n}z_2 \\ & -(\frac{\sum_i a_i a_{-n-i}}{2} + a_{-n}nb)z_3 + \delta_{n,0}b(z_2 + \frac{b}{2}z_3)\end{aligned} \tag{2.3}$$

$$\sigma(t^n) = t^n + \delta_{n,0}bz_3 - a_{-n}z_3, \sigma(z_1) = z_1 - 24bz_2 - 12b^2z_3, \tag{2.4}$$

$$\sigma(z_2) = z_2 + bz_3, \sigma(z_3) = z_3. \tag{2.5}$$

This can be verified directly, but one has to use the following formula

$$\sum_i (m+i)a_i a_{-m-n-i}z_3 = \frac{m-n}{2}\sum_i a_i a_{-m-n-i}z_3.$$

It is clear that

$$\sigma_{\alpha,b}\sigma_{\alpha_1,b_1} = \sigma_{\alpha+\alpha_1,b+b_1}, \ \forall\ \alpha, \alpha_1 \in \mathbb{C}[t, t^{-1}], b, b_1 \in \mathbb{C}. \tag{2.6}$$

3 Irreducible Whittaker modules over the Virasoro algebra

Let us recall the oscillator representation of the Heisenberg-Virasoro algebra $\mathbb{L}$ on the Fock space $B = \mathbb{C}[x_1, x_2, \ldots, x_n, \ldots]$. The action of $\mathbb{L}$ is defined as (see Prop.2.3, Lemma 2.2 in [9])

$$t^n = \frac{\partial}{\partial x_n}, t^{-n} = nx_n, \forall\, n \in \mathbb{N}, \tag{3.1}$$

$$t^0 = 0, z_3 = 1; z_2 = 0, z_1 = 1, \tag{3.2}$$

$$d_k = -\frac{1}{2}\sum_{i\in\mathbb{Z}} : t^{-i}\cdot t^{i+k} :, \forall\, k\in\mathbb{Z}. \tag{3.3}$$

Actually, B is isomorphic to the irreducible highest weight module $V(0,0,1,0,1)$ over $\mathbb{L}$ as in [12].
For any homogenous polynomial $u = x_{i_1}^{l_1}\dots x_{i_k}^{l_k}\in B$, define $\deg(u) = \sum_j i_j l_j$, and denote $B_i = \operatorname{span}\{u = x_{i_1}^{l_1}\dots x_{i_k}^{l_k}\in B \mid \deg(u) = i\}$ for all $i\in\mathbb{N}$. Then we have the weight space decomposition $B = \oplus_{i\in\mathbb{N}} B_i$, where B_i has the weight $-i$.
For any $0\neq f = \sum_{i=0}^n f_i$ with $f_i\in B_i$ and $f_n\neq 0$, denote $\deg(f) = n$, and $\mathrm{htm}(f) = f_n$.
When $\alpha\in\mathbb{C}[t]$, we have made the Fock space B into an irreducible highest weight module $B^{\sigma_{\alpha,b}}$. It is easy to verify (or from results in [1]) that: for any $b\in\mathbb{C}$, $\alpha = \sum_{i=0}^n a_i t^i\in\mathbb{C}[t]$, we have $B^{\sigma_{\alpha,b}}\cong V(\frac{b^2-a_0^2}{2}, b-a_0, 1-12b^2, b, 1)$.
From now on in this section we fix $b\in\mathbb{C}$ and $\alpha = \sum_{i=-m}^{m'} a_i t^i\in A\backslash\mathbb{C}[t]$ with $m > 0$ and $a_{-m}\neq 0$.

Lemma 2. *In $B^{\sigma_{\alpha,b}}$ we have*

$$d_n\circ 1 = 0, \forall\, n\geq 2m+1, \tag{3.4}$$

$$d_n\circ 1 = -\Big(\frac{\sum_i a_i a_{-n-i}}{2} + a_{-n} nb\Big), \forall\, m\leq n\leq 2m, \tag{3.5}$$

$$z_1\circ 1 = 1-12b^2, \tag{3.6}$$

$$\mathrm{htm}(d_n\circ(x_{i_1}^{k_1}\dots x_{i_l}^{k_l})) = (m-n)a_{-m}x_{m-n}x_{i_1}^{k_1}\dots x_{i_l}^{k_l}, \forall\, n < m. \tag{3.7}$$

$$\mathrm{htm}((d_{i_1}d_{i_2}\dots d_{i_k})\circ 1) = a x_{m-i_1}x_{m-i_2}\dots x_{m-i_k} \tag{3.8}$$

for all $i_1, i_2, \dots, i_k < m$, where $a = \prod_{j=1}^k (a_{-m}(m-i_j))\in\mathbb{C}^*$.

Proof. These follow from straightforward computations by using the formulas (2.3)-(2.5). □

From (3.8) we know the following

Lemma 3. *The following set*

$$\mathcal{B} = \{(d_{i_1}d_{i_2}\dots d_{i_k})\circ 1 \mid i_1\leq i_2\leq\dots\leq i_k < m\}$$

is a basis of $B^{\sigma_{\alpha,b}}$.

Theorem 4. *For any $\alpha\in A\backslash\mathbb{C}[t]$ and $b\in\mathbb{C}$, the module $B^{\sigma_{\alpha,b}}$ is irreducible over* Vir.

Proof. Recall that we have assumed that $\alpha = \sum_{i=-m}^{m'} a_i t^i \in A\backslash\mathbb{C}[t]$ with $m > 0$ and $a_{-m} \neq 0$. Let V be a nonzero Vir submodule of $B^{\sigma_{\alpha,b}}$, and $0 \neq f \in V$ with lowest degree. Suppose that $f \notin \mathbb{C}$, and $\deg(f) = n$. Say $f_n = \mathrm{htm}(f)$ with $\frac{\partial}{\partial x_{i_1}}(f_n) \neq 0$. Then we have

$$g = (d_{i_1+m} + t^{i_1+m}(\alpha + (i_1+m)b) \cdot f = d_{i_1+m} \circ f + af \in V,$$

where $a = \frac{\sum_i a_i a_{-m-i_1-i}}{2} \in \mathbb{C}$. It is straightforward to compute that

$$\mathrm{htm}(g) = \mathrm{htm}(a_{-m}t^{i_1} \cdot f) = a_{-m}\frac{\partial}{\partial x_{i_1}}(f_n) \neq 0.$$

And $\deg(g) = n - i_1 < n$, which contradicts the choice of f. So $1 \in V$. Thus $V = B^{\sigma_{\alpha,b}}$ by Lemma 3. Therefore $B^{\sigma_{\alpha,b}}$ is irreducible as a Vir module. □

For any $m \in \mathbb{N}$, denote $\mathrm{Vir}_{\geq m} = \oplus_{i\geq m}\mathbb{C}d_i$. Let $\psi_m : \mathrm{Vir}_{\geq m} \to \mathbb{C}$ be any nonzero homomorphism of Lie algebras and $\dot{z_1} \in \mathbb{C}$. Defined the one dimensional $\mathrm{Vir}_{\geq m} + \mathbb{C}z_1$ module $\mathbb{C}v$ by $d_i v = \psi_m(d_i)v$ and $z_1 v = \dot{z_1}v$. Then we have the induced Vir module

$$L_{\psi_m,\dot{z_1}} = \mathrm{Ind}_{U(\mathrm{Vir}_{\geq m}+\mathbb{C}z_1)}^{U(\mathrm{Vir})}\mathbb{C}v.$$

Note that $L_{\psi_m,\dot{z_1}}$ is a Whittaker module with respect to the Whittaker pair $(\mathrm{Vir}, \mathrm{Vir}_{\geq m} + \mathbb{C}z_1)$, and $w = 1 \otimes v$ be a cyclic Whittaker vector in the sense of [2].

From the PBW theorem, $L_{\psi_m,\dot{z_1}}$ has a basis

$$\{(d_{i_1}d_{i_2}\dots d_{i_k})w | i_1 \leq i_2 \leq \dots \leq i_k < m\}. \tag{3.9}$$

For $m = 1$, the irreducibility of $L_{\psi_1,\dot{z_1}}$ with $\psi_1(d_1)\psi_1(d_2) \neq 0$ was studied in [14] (see Proposition 4.8 and 6.1 in [14]), and an explicit formula for the Whittaker vector was give in [16] for $\psi_1(d_1) \neq 0$ and $\psi_1(d_2) = 0$ in terms of Jack symmetric polynomial.

Theorem 5. *Let $b \in \mathbb{C}$ and $\alpha = \sum_{i=-m}^{m'} a_i t^i \in A$ with $a_{-m} \neq 0$ and $m > 0$. Then $B^{\sigma_{\alpha,b}} \cong L_{\psi_m,1-12b^2}$ with $\psi_m(d_n) = -(\frac{\sum_i a_i a_{-n-i}}{2} + a_{-n}nb)$, for all $n \geq m$.*

Proof. It follows by Lemma 2, Lemma 3 and (3.9). □

Theorem 6. *Suppose $\dot{z_1} \in \mathbb{C}$ and $\psi_m(d_{2m}) \neq 0$. Then the Whittaker module $L_{\psi_m,\dot{z_1}}$ over* Vir *is irreducible.*

Proof. Denote $\psi_m(d_n) = \dot{d_n}$ for $n = m, \dots, 2m$. It is easy to see that there exist $b \in \mathbb{C}$ and some $\alpha(t) = \sum_{i=-m}^{0} a_i t^i \in \mathbb{C}[t^{-1}]$ with $a_{-m} \neq 0$ satisfying the following equations:

$$\begin{cases} \dot{z}_1 = 1 - 12b^2, \\ \dot{d}_{2m} = -\dfrac{a_{-m}^2}{2}, \\ \dot{d}_{2m-1} = -(a_{-m}a_{-m+1} + a_{-2m+1}(2m-1)b), \\ \vdots \\ \dot{d}_n = -(\dfrac{\sum_i a_i a_{-n-i}}{2} + a_{-n}nb), \\ \vdots \\ \dot{d}_m = -(\dfrac{\sum_i a_i a_{-m-i}}{2} + a_{-m}mb). \end{cases}$$

From Theorem 5 we know that $L_{\psi_m,\dot{z}_1} \cong B^{\sigma_{\alpha,b}}$. Using Theorem 4 we see that $L_{\psi_m,\dot{z}_1}$ is irreducible over Vir. □

Now we can give the main result in this section.

Theorem 7. *Suppose that $\dot{z_1} \in \mathbb{C}$, $m \in \mathbb{N}$, and $\psi_m : \mathrm{Vir}_{\geq m} \to \mathbb{C}$ is a Lie algebra homomorphism. Then the Whittaker module $L_{\psi_m,\dot{z_1}}$ over* Vir *is irreducible if and only if $\psi_m(d_{2m}) \neq 0$ or $\psi_m(d_{2m-1}) \neq 0$.*

Proof. "$\Rightarrow$": Suppose that $\psi_m(d_{2m}) = 0$, and $\psi_m(d_{2m-1}) = 0$. Then it is staightforward to see that $L_{\psi_m,\dot{z_1}}$ has a proper submodule generated by $d_{m-1}\cdot 1$.
"$\Leftarrow$" From Theorem 8, we need only to consider the case where $\psi_m(d_{2m}) = 0$ and $\psi_m(d_{2m-1}) \neq 0$.
Let $\mathrm{Vir}[\frac{1}{2}\mathbb{Z}]$ be the Virasoro algebra with the basis $\{d_k, z \,|\, k \in \frac{1}{2}\mathbb{Z}\}$ and subject to the relations:

$$[d_n, d_m] = (m-n)d_{n+m} + \delta_{n,-m}\frac{n^3-n}{12}z_1,\ \forall\ m,n \in \frac{1}{2}\mathbb{Z}.$$

Then $\mathrm{Vir}[\mathbb{Z}]$ is a subalgebra of $\mathrm{Vir}[\frac{1}{2}\mathbb{Z}]$. Now we define $\psi_{m-1/2}$ on $\mathrm{Vir}[\frac{1}{2}\mathbb{Z}]$ as

$$\psi_{m-1/2}(d_k) = \psi_m(d_k),\ \psi_{m-1/2}(d_{k-1/2}) = 0, \forall\, k \in \mathbb{N}.$$

From Theorem 8 we know that the Whittaker module $L_{\psi_{m-1/2},\dot{z_1}}$ with respect to the Whittaker pair $(\mathrm{Vir}[\frac{1}{2}\mathbb{Z}], \mathrm{Vir}_{\geq m-1/2} + \mathbb{C}z_1)$ is irreducible with a basis:

$$(d_{k_1}d_{k_2}\cdots d_{k_r})(d_{p_1}d_{p_2}\cdots d_{p_t})1$$

where $k_1, k_2, \cdots, k_r \in 1/2 + \mathbb{Z}$ with $k_1 \leq k_2 \leq \cdots \leq k_r \leq m-1$, and $p_1, p_2, \cdots, p_t \in \mathbb{Z}$ with $p_1 \leq p_2 \leq \cdots \leq p_t \leq m-1$. Clearly,

$$W = \mathrm{span}\{d_{p_1}d_{p_2}\cdots d_{p_t}1 \,|\, p_1, p_2, \cdots, p_t \in \mathbb{Z} \text{ with } p_1 \leq p_2 \leq \cdots \leq p_t \leq m-1\}$$

is a $\mathrm{Vir}[\mathbb{Z}]$-module which is a Whittaker module isomorphic to $L_{\psi_m,\dot{z_1}}$ with respect to the Whittaker pair $(\mathrm{Vir}[\mathbb{Z}], \mathrm{Vir}_{\geq m} + \mathbb{C}z_1)$.

We want to show that W is irreducible as a Vir[$\mathbb{Z}$]-module. To the contrary, we assume that W is not irreducible. Take a nonzero proper submodule V of W. Let V' be the span of the following subspaces

$$(d_{k_1}d_{k_2}\cdots d_{k_r})V; k_1, k_2, \cdots, k_r \in 1/2+\mathbb{Z} \text{ with } k_1 \leq k_2 \leq \cdots \leq k_r \leq m-1.$$

Then V' is a proper subspace of $L_{\psi_{m-1/2},\dot{z_1}}$. Using PBW Theorem one can easily show that V' is a submodules of $L_{\psi_{m-1/2},\dot{z_1}}$ over Vir[$\frac{1}{2}\mathbb{Z}$], which is a contradiction. Thus W is irreducible as a Vir[$\mathbb{Z}$]-module. Therefore the Whittaker module $L_{\psi_m,\dot{z_1}}$ is irreducible over Vir[$\mathbb{Z}$] if $\psi_m(d_{2m-1})\neq 0$. □

We like to mention that the results in [14] and [16] are the special case of the above theorem with $m=1$ and $\psi_1(d_1)\psi_1(d_2)\neq 0$ or $\psi_1(d_1)\neq 0$.

4 Irreducible modules over Vir with $z_1 = 0$

We can make $A=\mathbb{C}[t,t^{-1}]$ into an $\mathbb{L}$-module by defining $d_i = t^{i+1}\frac{d}{dt}$, t^i acting as multiplication by t^i, and z_1, z_2, z_3 acting as zero, i.e.,

$$d_i\cdot t^j = jt^{i+j}, t^i\cdot t^j = t^{i+j}, z_k\cdot A = 0, \forall\, k=1,2,3.$$

The module on A is isomorphic to $V(0,0;1)$ defined on Page 187 of [12] which is an irreducible module over $\mathbb{L}$. We will use this module instead of the most general case $V(a,b;F)$, where $a,b,F\in\mathbb{C}$, since we will essentially obtain isomorphic Virasoro modules.
For any $\alpha\in A, b\in\mathbb{C}$, we have the $\mathbb{L}$-module $A_{\alpha,b}=A^{\sigma_{\alpha,b}}$. The action of $\mathbb{L}$ on $A_{\alpha,b}$ is

$$d_n\circ t^i = (\alpha+i+nb)t^{n+i}, \forall\, i,n\in\mathbb{Z}, \tag{4.1}$$

$$t^j\circ t^i = t^{i+j}, z_k\circ A = 0, \forall\, i,j\in\mathbb{Z}, k=1,2,3. \tag{4.2}$$

In this section, the irreducibility and the isomorphism classes of such modules are completely determined.
Note that if $\alpha\in\mathbb{C}$, then $A_{\alpha,b}$ is simply a weight module of intermediate series in [12].

Lemma 8. *(1) Let V be an irreducible $\mathbb{L}$ module, then $V^{\sigma_{\alpha,b}}$ is irreducible for any $\alpha\in A, b\in\mathbb{C}$.*

(2) $A_{\alpha,b}$ is irreducible as an $\mathbb{L}$ module for any $\alpha\in A, b\in\mathbb{C}$.

Proof. The statements in this Lemma are obvious. □

Lemma 9. *For any $k\in\mathbb{Z}$, let*

$$w_k = -\frac{1}{2}d_{k-1}d_1 - \frac{1}{2}d_{k+1}d_{-1} + d_kd_0 \in U(\text{Vir}). \tag{4.3}$$

Then $w_k\circ g = b(b-1)t^kg$, for all $k\in\mathbb{Z}$ and $g\in A_{\alpha,b}$.

Proof. For any $k \in \mathbb{Z}$ and $t^j \in A_{\alpha,b}$, we compute

$$(d_{k-i}d_i) \circ t^j = d_{k-i} \circ (\alpha + j + ib)t^{i+j} = (\alpha + j + ib)(\alpha + i + j + (k-i)b)t^{k+j}.$$

Taking $i = -1, 0, 1$ respectively, we get

$$\begin{aligned} w_k \circ t^j &= [-\frac{1}{2}(\alpha + j + b)(\alpha + 1 + j + (k-1)b) \\ &\qquad -\frac{1}{2}(\alpha + j - b)(\alpha - 1 + j + (k+1)b) + (\alpha + j)(\alpha + j + kb)]t^{k+j} \\ &= b(b-1)t^{k+j}. \end{aligned}$$

Thus $w_k \circ g = b(b-1)t^k g$ for all $k \in \mathbb{Z}$ and $g \in A_{\alpha,b}$. ☐

Corollary 10. *Suppose that $b \notin \{0, 1\}$, $0 \neq g \in A_{\alpha,b}$. Then $\mathrm{span}_{\mathbb{C}}\{w_i \circ g | i \in \mathbb{Z}\} = A_{\alpha,b}$ if and only if $g = ct^i$ for some $c \in \mathbb{C}^*$ and $i \in \mathbb{Z}$.*

Proof. From Lemma 9, $\mathrm{span}_{\mathbb{C}}\{w_i \circ g | i \in \mathbb{Z}\} = Ag$. It is clear that $Ag = A$ if and only if $g = ct^i$ for some $c \in \mathbb{C}^*$ and $i \in \mathbb{Z}$. ☐

Lemma 11. *If $\alpha \in A \setminus \mathbb{C}$, then $E_\alpha = \mathrm{span}_{\mathbb{C}}\{\alpha t^i + it^i | i \in \mathbb{Z}\} \neq A$.*

Proof. For any $0 \neq f = \sum_{i=s}^{r} b_i t^i \in A$ with $b_s, b_r \neq 0$, define $\deg(f) = (s, r)$ and $l(f) = r - s$. Suppose that

$$\alpha = \sum_{i=m}^{n} a_i t^i \in A \setminus \mathbb{C}, \quad \text{with } \deg \alpha = (m, n).$$

Case 1. $m < 0 < n$.
It is easy to see that $l(f) \geq n - m$ for all $f \in E_\alpha$, hence $E_\alpha \neq A$ in this case.
Case 2. $m \geq 0$ or $n \leq 0$.
Without loss of generality, we may assume that $m \geq 0$. If $m = 0$ then $n > 0$ since $\alpha \notin \mathbb{C}$; if $m > 0$ then $n \geq m > 0$. If $a_0 \notin \mathbb{Z}$ or $m > 0$, then it is easy to check that $l(f) \geq 1$ for all $0 \neq f \in E_\alpha$. So $E_\alpha \neq A$. If $a_0 \in \mathbb{Z}$ and $m = 0$, then $n > 0$ and $a_0 \neq 0$. It is not hard to see that $t^{-a_0} \notin E_\alpha$. ☐

Theorem 12. *Let $\alpha \in A, b \in \mathbb{C}$.*

(1) If $b \notin \{0, 1\}$, then $A_{\alpha,b}$ is irreducible as a Vir *module with action defined as in (4.1).*

(2) The Vir *module $A_{\alpha,1}$ is irreducible if and only if $\alpha \in \mathbb{C} \setminus \mathbb{Z}$. If $\alpha \notin \mathbb{C} \setminus \mathbb{Z}$, then*

$$d_0 \circ A_{\alpha,1} = \oplus_{i \in \mathbb{Z}} \mathbb{C}(\alpha + i)t^i$$

is the unique irreducible Vir *submodule of $A_{\alpha,b}$, and* Vir *acts on $A_{\alpha,1}/(d_0 \circ A_{\alpha,1})$ as zero.*

(3) The Vir *module $A_{\alpha,0}$ is irreducible if and only if $\alpha \notin \mathbb{Z}$.*

(4) We have $d_0 \circ A_{\alpha,1} \cong A_{\alpha,0}$ *as* Vir*-module if* $\alpha \notin \mathbb{Z}$.

Proof. (1). Suppose that $b \neq 0, 1$. Let M be a nonzero Vir submodule of $A_{\alpha,b}$. Then by lemma 9, we have $t^i \cdot M \subset M$ for all $i \in \mathbb{Z}$. Hence M is also an $\mathbb{L}$ submodule of $A_{\alpha,b}$. Thus $M = A_{\alpha,b}$ by lemma 8. So $A_{\alpha,b}$ is irreducible as Vir module in this case.

(2). Note that $d_j \circ g = d_0 \circ (t^j g)$ for all $g \in A_{\alpha,1}, i \in \mathbb{Z}$. So $d_0 \circ A_{\alpha,1}$ is a Vir submodule of $A_{\alpha,1}$, and Vir acts trivially on $A_{\alpha,1}/(d_0 \circ A_{\alpha,1})$. For any nonzero Vir submodule M of $A_{\alpha,1}$. Let $AM = \text{span}\{t^i g | i \in \mathbb{Z}, g \in M\}$. Since $d_0 \circ t^i g = d_i \circ g \in M$ for all $g \in M$, then

$$d_0 \circ AM \subset M. \tag{4.4}$$

Noting that $d_j \circ (t^i g) = d_{i+j} \circ g$ we see that AM is a Vir submodule. Then AM is also an $\mathbb{L}$ module, and $AM = A$ by lemma 8. Combining with (4.4), we have $d_0 \circ A_{\alpha,1} \subset M$, i.e., $d_0 \circ A_{\alpha,1}$ is the unique minimum nonzero submodule of $A_{\alpha,1}$. Note that $d_0 \circ A_{\alpha,1} = \text{span}_{\mathbb{C}}\{(\alpha + i)t^i | i \in \mathbb{Z}\}$. Using Lemma 11 we see that $A_{\alpha,1}$ is not irreducible if $\alpha \notin \mathbb{C}$. For $\alpha \in \mathbb{C}$ it is well-known that $A_{\alpha,1}$ is not irreducible if and only if $\alpha \in \mathbb{Z}$. This proves (2).

(3) and (4). It is straightforward to check that $\eta : A_{\alpha,0} \to d_0 \circ A_{\alpha,1}$ with $\eta(g) = d_0 \circ g$ is a Vir-module epimorphism. From (2), we know that $A_{\alpha,0}$ is irreducible if and only if η is injective. Note that if $d_0 \circ g = 0$ for some $0 \neq g \in A_{\alpha,1}$, then $t\frac{d}{dt}(g) + \alpha g = 0$. By comparing the terms of highest and lowest degree in t respectively, we have $\alpha \in \mathbb{C}$. Using well-known results on $A_{\alpha,0}$ for $\alpha \in \mathbb{C}$, we see that the statements in (3) and (4) are true. So we have proved the theorem. □

LEMMA 13. *If* $\alpha_1 - \alpha_2 \in \mathbb{Z}$, *then* $A_{\alpha_1,b} \cong A_{\alpha_2,b}$ *as* Vir*-modules.*

Proof. Suppose that $\alpha_1 = \alpha_2 + k$ for some $k \in \mathbb{Z}$. Then it is straightforward to check that $\eta : A_{\alpha_1,b} \to A_{\alpha_2,b}$ with $\eta(t^i) = t^{i+k}$ is a Vir-module isomorphism. □

LEMMA 14. *For any* $\alpha \notin \mathbb{Z}$ *there exist finitely many nonzero* $v_\alpha^{(i)} \in U(\text{Vir})$ *for* $i \in \mathbb{Z}$ *such that for any* $k \in \mathbb{Z}$, *the element*

$$u_{\alpha,k} = \sum_i d_{i+k} v_\alpha^{(i)} \in U(\text{Vir}) \tag{4.5}$$

satisfies that $u_{\alpha,k} \circ g = t^k u_{\alpha,0} \circ g$ *in* $A_{\alpha,0}$ *for all* $g \in A_{\alpha,0}$.

Proof. Note that $A_{\alpha,0}$ is irreducible as a Vir module. Since $d_0 \circ 1 = \alpha \neq 0$, there exists $u_\alpha \in U(\text{Vir})$ such that $u_\alpha \circ d_0 \circ 1 = 1$. Let $u_{\alpha,0} = u_\alpha \cdot d_0 \in U(\text{Vir})$. Then we can write $u_{\alpha,0}$ as in (4.5) with $k = 0$, and we have the definition for $u_{\alpha,k}$. Note that $d_{k+i} \circ g = t^k(d_i \circ g)$ for all $g \in A_{\alpha,0}$ and $k \in \mathbb{Z}$. Then we can easily verify that $u_{\alpha,k} \circ g = t^k u_{\alpha,0} \circ g$ in $A_{\alpha,0}$ for all $g \in A_{\alpha,0}$. □

Theorem 15. *Let* $\alpha_1, \alpha_2 \in A, b_1, b_2 \in \mathbb{C}$.

(1) If $b_1 \notin \{0, 1\}$, *then* $A_{\alpha_1,b_1} \cong A_{\alpha_2,b_2}$ *as Vir modules if and only if* $\alpha_1 - \alpha_2 \in \mathbb{Z}$ *and* $b_1 = b_2$.

(2) If $b \in \{0, 1\}$, *then* $A_{\alpha_1,b} \cong A_{\alpha_2,b}$ *as Vir modules if and only if* $\alpha_1 - \alpha_2 \in \mathbb{Z}$.

(3) $A_{\alpha_1,0} \cong A_{\alpha_2,1}$ *as Vir modules if and only if* $\alpha_1 - \alpha_2 \in \mathbb{Z}$ *and* $\alpha_1 \in \mathbb{C}\backslash\mathbb{Z}$.

Proof. From Theorem 1, Theorem 12, and Lemma 13, we see that all the sufficient conditions in (1)-(3) are satisfied. So we need only to prove the necessity of the conditions in (1)-(3).

(1). From Lemma 9, we have $b_1(b_1-1) = b_2(b_2-1) \neq 0$. So $b_2 \notin \{0, 1\}$. Suppose that $\sigma : A_{\alpha_1,b_1} \to A_{\alpha_2,b_2}$ is a Vir module isomorphism. By Lemma 9 again, we know that $\{w_k \circ 1 \,|\, k \in \mathbb{Z}\}$ is a basis for A_{α_1,b_1}. Then $\{w_k \circ \sigma(1) \,|\, k \in \mathbb{Z}\}$ is a basis for A_{α_2,b_2}. By Corollary 10, we know that $\sigma(1) = ct^{i_0}$ for some $c \in \mathbb{C}^*$ and $i_0 \in \mathbb{Z}$. Computing $\sigma(d_j \circ 1) = d_j \circ \sigma(1)$, we have $(i_0 + \alpha_2 - \alpha_1) + (b_2 - b_1)j = 0$, for all $j \in \mathbb{Z}$. Thus $\alpha_1 = \alpha_2 + i_0$ and $b_1 = b_2$.

(2). Case 1. $b = 0$.
If $\alpha_1 \in \mathbb{Z}$, then we have $\alpha_2 \in \mathbb{Z}$ by Theorem 12 (3).
Now suppose that $\alpha_1, \alpha_2 \notin \mathbb{Z}$, and that $\sigma : A_{\alpha_1,0} \to A_{\alpha_2,0}$ is a Vir-module isomorphism. Take $u_{\alpha_1,k}$ as in Lemma 14. Then $u_{\alpha_1,k} \circ 1 = t^k$, for all $k \in \mathbb{Z}$. Note that

$$\begin{aligned} \sigma(t^k) &= \sigma(u_{\alpha_1,k} \circ 1) = u_{\alpha_1,k} \circ \sigma(1) \\ &= t^k(u_{\alpha_1,0} \circ \sigma(1)) = t^k \sigma(u_{\alpha_1,0} \circ 1) = t^k \sigma(1) \in A_{\alpha_2,0}. \end{aligned}$$

Then

$$\begin{aligned} A_{\alpha_2,0} = \sigma(A_{\alpha_1,0}) &= \mathrm{span}_{\mathbb{C}}\{\sigma(u_{\alpha_1,k} \circ 1) | k \in \mathbb{Z}\} \\ &= \mathrm{span}_{\mathbb{C}}\{t^k \sigma(1) | k \in \mathbb{Z}\} = A\sigma(1). \end{aligned}$$

So $\sigma(1) = ct^{i_0}$ for some $c \in \mathbb{C}^*, i_0 \in \mathbb{Z}$, and

$$\sigma(t^i) = ct^{i+i_0}, \ \forall\, i \in \mathbb{Z}.$$

By a similar computation as in the last step in (1), we deduce that $\alpha_1 = \alpha_2 + i_0$ and the result follows in this case.
Case 2. $b = 1$.
Suppose that $A_{\alpha_1,1} \cong A_{\alpha_2,1}$ as Vir modules. If $\alpha_1 \in \mathbb{Z}$, then $A_{\alpha_1,1}$ is a weight module with respect to d_0. This forces $A_{\alpha_2,1}$ to be a weight module with respect to d_0. So $\alpha_2 \in \mathbb{C}$. Theorem 14 (2) ensures that $\alpha_1 - \alpha_2 \in \mathbb{Z}$.
Now suppose that $\alpha_1, \alpha_2 \notin \mathbb{Z}$. From Theorem 14 (4) we know that $A_{\alpha_1,0} \cong d_0 \circ A_{\alpha_1,1} \cong d_0 \circ A_{\alpha_2,1} \cong A_{\alpha_2,0}$ as irreducible Vir modules. Thus $\alpha_1 - \alpha_2 \in \mathbb{Z}$ by Case 1.

(3). If $\alpha_2 \in \mathbb{C}$, then $A_{\alpha_2,1}$ is a weight module with respect to d_0. This forces $A_{\alpha_1,0}$ to be a weight module with respect to d_0. So $\alpha_1 \in \mathbb{C}$. From Theorem 1 we know that $\alpha_1 - \alpha_2 \in \mathbb{Z}$ and $\alpha_1 \in \mathbb{C}\backslash\mathbb{Z}$.

Now suppose that $\alpha_2 \notin \mathbb{C}$, then $A_{\alpha_2,1}$ is a non-weight module with respect to d_0. This forces $A_{\alpha_1,0}$ to be a non-weight module with respect to d_0. So $\alpha_1 \notin \mathbb{C}$. From Theorem 12 (2) and (4) we know that $A_{\alpha_1,0}$ is irreducible while $A_{\alpha_2,1}$ is not irreducible. So $A_{\alpha_1,0}$ and $A_{\alpha_2,1}$ cannot be isomorphic in this case. $\square$

Acknowledgement. We are grateful to the referee for giving nice suggestions. X.G. is partially supported by NSF of China (11101380 and 11026155), R.L. is is partially supported by NSF of China, and K.Z. is partially supported by NSERC and NSF of China.

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