The Classification of Real Purely Infinite Simple C*-Algebras

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Abstract. We classify real Kirchberg algebras using united K-theory. Precisely, let A and B be real simple separable nuclear purely infinite C*-algebras that satisfy the universal coefficient theorem such that $A_{\mathbb{C}}$ and $B_{\mathbb{C}}$ are also simple. In the stable case, A and B are isomorphic if and only if $K^{CRT}(A) \cong K^{CRT}(B)$. In the unital case, A and B are isomorphic if and only if $(K^{CRT}(A), [1_A]) \cong (K^{CRT}(B), [1_B])$. We also prove that the complexification of such a real C*-algebra is purely infinite, resolving a question left open from [43]. Thus the real C*-algebras classified here are exactly those real C*-algebras whose complexification falls under the classification result of Kirchberg [26] and Phillips [35]. As an application, we find all real forms of the complex Cuntz algebras $\mathcal{O}_n$ for $2 \leq n \leq \infty$.

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1. Introduction

One of the highlights of the classification theory of simple amenable C*-algebras is the classification of purely infinite nuclear simple C*-algebras, obtained by Kirchberg and Phillips in [26] and [35]. This classification theorem relies in an essential way on the Universal Coefficient Theorem established by Rosenberg and Schochet in [40], where it was observed that "For reasons pointed out already by Atiyah, there can be no good Künneth Theorem or Universal Coefficient Theorem for the KKO groups of real C*-algebras; this explains why we deal only with complex C*-algebras". Thus at the time of the Kirchberg and Phillips classification, the lack of a universal coefficient theorem was the primary barrier to extending the classification result to real C*-algebras.

However, in [8], a new invariant called united K-theory was introduced for real C*-algebras and in [9] a universal coefficient theorem was proven for real

C*-algebras using united K-theory. In the present paper, we take advantage of these developments to provide a complete classification of a class of real simple purely infinite C*-algebras in terms of united K-theory. The real C*-algebras that are classified are exactly those real C*-algebras for which the complexification is covered by the Kirchberg and Phillips theory. As an application of our classification we determine all the real forms of the complex Cuntz algebras $\mathcal{O}_n$ for $1 \leq n \leq \infty$: there are two such forms when n is odd and one when n is even or infinite.

The overall framework of the proof will be the same as that in the paper [35] and the underlying theory on which that paper was built. Furthermore, many of the proofs in the development leading to the main theorems of [35] carry over to the real case without significant change. In those cases, we will simply refer to the established proofs in the literature without reproducing them here. However there are many situations where the arguments in the real case require modification and we will then provide full proofs or full discussion of the necessary modifications.

In Section 2, we describe the invariant of united K-theory and summarize its key properties. In Section 3 we then establish real analogues of some of the fundamental properties of purely infinite algebras, in the course of which we resolve a problem left hanging in [43] and [13] by showing that the complexification of a purely infinite simple real C*-algebra is also purely infinite (using the original definition for simple algebras). Following the complex case, as developed in [38], we then establish (in Theorem 5.2) criteria for two unital homomorphisms from the real Cuntz algebra $\mathcal{O}_n^{\mathbb{R}}$ (n even) to be approximately unitarily equivalent. Modifications of the complex arguments are required to establish some of the preliminary results: in Section 4 we modify the required results about exponential rank, noting that the close link between self-adjoint and skew-adjoint elements is absent in a real C*-algebra, and in Section 5 we modify the result from [15] establishing the Rokhlin property of the Bernoulli shift on the CAR-algebra.

Our next step is to establish real analogues of Kirchberg's tensor product theorems and his embedding theorem. This is achieved in Section 6 by using the relevant complex results and the embedding of $\mathbb{C}$ into $M_2(\mathbb{R})$. In Sections 7, 8, 9 and 10, we closely follow [35] indicating how the results achieved for the complex case can be obtained in the real case. In particular, Section 7 contains a key result about uniqueness of homomorphisms from $\mathcal{O}_\infty^{\mathbb{R}}$ to a real purely infinite C*-algebras. Section 8 contains the theory of asymptotic morphisms in the context of real C*-algebras and Section 9 culminates in a theorem identifying KK-theory to a group of asymptotic unitary equivalence classes of asymptotic morphisms as in Section 4 of [35]. To accomplish this, we make use of the axiomatic characterization of KK-theory for real C*-algebras established in [12]. This development culminates in Section 10, which contains the statements and proofs of our classification theorems, and in Section 11, which uses these results to describe the real forms of Cuntz algebras. The notation we use in these sections closely follows that in [35].

We will use the notation $\mathcal{H}^{\mathbb{R}}$ for a real Hilbert space; and $\mathcal{B}(\mathcal{H}^{\mathbb{R}})$ and $\mathcal{K}^{\mathbb{R}}$ for the real C*-algebras of bounded and compact operators $\mathcal{H}^{\mathbb{R}}$. For the complex versions of these objects we will use $\mathcal{H}$, $\mathcal{B}(\mathcal{H})$, and $\mathcal{K}$. For a C*-algebra A, we will write $M_n(A)$ for the matrix algebra over A; and M_n will stand for $M_n(\mathbb{R})$. Following standard convention, we will use $\mathcal{O}_n$ for the complex Cuntz algebras and $\mathcal{O}_n^{\mathbb{R}}$ for the real versions. The complexification of a real C*-algebra A will be denoted by $A_{\mathbb{C}}$. We will use Φ throughout to denote the conjugate linear automorphism of $A_{\mathbb{C}}$ defined by $a + ib \mapsto a - ib$ (for $a, b \in A$). Note that A can be recovered from Φ as the fixed point set. Finally, a tensor product written as $A \otimes B$ will in most cases be the C*-algebra tensor product over $\mathbb{R}$, but should be understood to be a tensor product over $\mathbb{C}$ if both A and B are known to be complex C*-algebras. Recall that if A and B are real C*-algebras, then $(A \otimes B)_{\mathbb{C}} \cong A_{\mathbb{C}} \otimes B_{\mathbb{C}}$.

2. Preliminaries on United K-Theory

United K-theory was developed in the commutative context in [14] and subsequently extended to the context of real C*-algebras in [8]. United K-theory consists of the three separate K-theory modules as well as several natural transformations among them. In this section, we give the definition of united K-theory and summarize the features needed in this paper. Details are in [8], [9], [10].

Definition 2.1. Let A be a real C*-algebra. The *united K-theory* of A is given by

$$K^{CRT}(A) = \{KO_*(A), KU_*(A), KT_*(A), r, c, \varepsilon, \zeta, \psi_U, \psi_T, \gamma, \tau\} .$$

In this definition, $KO_*(A) = K_*(A)$ is the standard K-theory of a real C*-algebra, considered as a graded module over the ring $K_*(\mathbb{R})$. This means in particular that there are operations

$$\begin{aligned} \eta_O &\colon KO_n(A) \to KO_{n+1}(A) \\ \xi &\colon KO_n(A) \to KO_{n+4}(A) \\ \beta_O &\colon KO_n(A) \to KO_{n+8}(A) \end{aligned}$$

corresponding to multiplication by the elements of the same name in $KO_*(\mathbb{R})$. The operation β_O is the periodicity isomorphism of real K-theory.

The second item $KU_*(A) = K_*(A_{\mathbb{C}})$ is the K-theory of the complexification of A, having period 2. It is a module over $K_*(\mathbb{C})$, which is to say that that there is an isomorphism of period 2 and the two remaining groups are independent with no operations between them.

Finally, $KT_*(A)$ is the period 4 self-conjugate K-theory originally defined in the topological setting in [1]. In the non-commutative setting, it is more easily defined as $KT_*(A) = K_*(T \otimes A)$ in terms of the algebra $T = \{f \in C([0,1], \mathbb{C}) \mid$

$f(0) = \overline{f(1)}\}$ (see [8]). Self-conjugate K-theory is a module over the ring $K_*(T)$, giving operations

$$\begin{aligned} \eta_O &\colon KT_n(A) \to KT_{n+1}(A) \\ \omega &\colon KT_n(A) \to KT_{n+3}(A) \\ \beta_T &\colon KT_n(A) \to KT_{n+4}(A)\ . \end{aligned}$$

The rest of the information in united K-theory consists of operations

$$\begin{aligned} c_n &\colon KO_n(A) \longrightarrow KU_n(A) & r_n &\colon KU_n(A) \longrightarrow KO_n(A) \\ \varepsilon_n &\colon KO_n(A) \longrightarrow KT_n(A) & \zeta_n &\colon KT_n(A) \longrightarrow KU_n(A) \\ (\psi_U)_n &\colon KU_n(A) \longrightarrow KU_n(A) & (\psi_T)_n &\colon KT_n(A) \longrightarrow KT_n(A) \\ \gamma_n &\colon KU_n(A) \longrightarrow KT_{n-1}(A) & \tau_n &\colon KT_n(A) \longrightarrow KO_{n+1}(A) \end{aligned}$$

among the three K-theory modules.
For example, c is induced by the natural inclusion $A \to A_{\mathbb{C}}$; r by the inclusion $A_{\mathbb{C}} \to M_2(A)$; and ψ_U by the involution Φ on $A_{\mathbb{C}}$. These operations are known to satisfy the following relations (see Proposition 1.7 of [8]):

$$\begin{array}{rclrclrcl} rc &=& 2 & \psi_U\beta_U &=& -\beta_U\psi_U & \xi &=& r\beta_U^2 c \\ cr &=& 1+\psi_U & \psi_T\beta_T &=& \beta_T\psi_T & \omega &=& \beta_T\gamma\zeta \\ r &=& \tau\gamma & \varepsilon\beta_O &=& \beta_T^2\varepsilon & \beta_T\varepsilon\tau &=& \varepsilon\tau\beta_T + \eta_T\beta_T \\ c &=& \zeta\varepsilon & \zeta\beta_T &=& \beta_U^2\zeta & \varepsilon r\zeta &=& 1+\psi_T \\ (\psi_U)^2 &=& 1 & \gamma\beta_U^2 &=& \beta_T\gamma & \gamma c\tau &=& 1-\psi_T \\ (\psi_T)^2 &=& 1 & \tau\beta_T^2 &=& \beta_O\tau & \tau &=& -\tau\psi_T \\ \psi_T\varepsilon &=& \varepsilon & \gamma &=& \gamma\psi_U & \tau\beta_T\varepsilon &=& 0 \\ \zeta\gamma &=& 0 & \eta_O &=& \tau\varepsilon & \varepsilon\xi &=& 2\beta_T\varepsilon \\ \zeta &=& \psi_U\zeta & \eta_T &=& \gamma\beta_U\zeta & \xi\tau &=& 2\tau\beta_T\ . \end{array}$$

United K-theory takes values in the algebraic category of *CRT*-modules. A *CRT*-module consists of a triple (M^O, M^U, M^T) of graded modules, one over each of the rings $K_*(\mathbb{R})$, $K_*(\mathbb{C})$, and $K_*(T)$; as well as natural transformations $c, r, \varepsilon, \zeta, \psi_U, \psi_T, \gamma, \tau$ that satisfy the above relations.
For any real C*-algebra A, the *CRT*-module $K^{CRT}(A)$ is *acyclic*, which means that the sequences

$$\cdots \longrightarrow KO_n(A) \xrightarrow{\eta_O} KO_{n+1}(A) \xrightarrow{c} KU_{n+1}(A) \xrightarrow{r\beta_U^{-1}} KO_{n-1}(A) \longrightarrow \cdots$$

$$\cdots \longrightarrow KO_n(A) \xrightarrow{\eta_O^2} KO_{n+2}(A) \xrightarrow{\varepsilon} KT_{n+2}(A) \xrightarrow{\tau\beta_T^{-1}} KO_{n-1}(A) \longrightarrow \cdots$$

$$\cdots \longrightarrow KU_{n+1}(A) \xrightarrow{\gamma} KT_n(A) \xrightarrow{\zeta} KU_n(A) \xrightarrow{1-\psi_U} KU_n(A) \longrightarrow \cdots$$

are exact.
The important advantage of the full united K-theory over ordinary K-theory for a real C*-algebra A is that it yields both a Künneth formula (Theorem 4.2 of

[8]) and a univeral coefficient theorem (Theorem 1.1 of [9]). For later reference, we now state two results that follow from those fundamental theorems.

Proposition 2.2. *For any real C*-algebra A,*

(1) $K^{CRT}(\mathcal{O}_2^{\mathbb{R}} \otimes A) = 0$
(2) $K^{CRT}(\mathcal{O}_\infty^{\mathbb{R}} \otimes A) \cong K^{CRT}(A)$.

Proof. By Table IV of [8], we have $K^{CRT}(\mathcal{O}_2^{\mathbb{R}}) = 0$. Then (1) follows by the Künneth formula.
The unital inclusion $\mathbb{R} \to \mathcal{O}_\infty^{\mathbb{R}}$ induces an isomorphism on united K-theory. This follows from Theorem 4 of [10] and the fact that the unital inclusion $\mathbb{C} \to \mathcal{O}_\infty$ induces an isomorphism on (complex) K-theory. Thus, Theorem 3.5 of [8] gives $K^{CRT}(\mathcal{O}_\infty^{\mathbb{R}}) \otimes_{CRT} K^{CRT}(A) \cong K^{CRT}(A)$ and $\mathrm{Tor}(K^{CRT}(\mathcal{O}_\infty^{\mathbb{R}}), K^{CRT}(A)) = 0$. Then the isomorphism of (2) follows by the Main Theorem of [8]. □

Recall from [41] that the bootstrap class $\mathcal{N}$ is the smallest subcategory of complex, separable, nuclear C*-algebras that contains the separable type I C*-algebras; that is closed under the operations of taking inductive limits, stable isomorphisms, and crossed products by $\mathbb{Z}$ and $\mathbb{R}$; and that satisfies the two out of three rule for short exact sequences (i.e. if $0 \to A \to B \to C \to 0$ is exact and two of A, B, C are in $\mathcal{N}$, then the third is also in $\mathcal{N}$).

Proposition 2.3 (Corollary 4.11 of [9]). *Let A and B be real separable C*-algebras such that $A_{\mathbb{C}}$ and $B_{\mathbb{C}}$ are in $\mathcal{N}$. Then A and B are KK-equivalent if and only if $K^{CRT}(A) \cong K^{CRT}(B)$.*

This last result is the essential preliminary result for our classification of real purely infinite simple C*-algebras. We will also make use of Theorem 1 of [10], which states that every countable acyclic *CRT*-module can be realized as the united K-theory a real separable C*-algebra, indeed the C*-algebra can even be taken to be simple and purely infinite.
We now describe a simpler variation of united K-theory that, by results from [23], contains as much information as the full version of united K-theory.

Definition 2.4. Let A be a real C*-algebra. Then

$$K^{CR}(A) = \{KO_*(A), KU_*(A), r, c, \psi_U\}$$

For any real C*-algebra, $K^{CR}(A)$ is an acyclic *CR*-module, which means that the relations

$$rc = 2 \qquad \psi_U\beta_U = -\beta_U\psi_U \qquad \xi = r\beta_U^2 c$$
$$cr = 1 + \psi_U \qquad \psi_U^2 = 1 \qquad \psi_U c = c$$

are satisfied and that the sequence

$$\cdots \longrightarrow KO_n(A) \xrightarrow{\eta_O} KO_{n+1}(A) \xrightarrow{c} KU_{n+1}(A) \xrightarrow{r\beta_U^{-1}} KO_{n-1}(A) \longrightarrow \cdots$$

is exact.

Let Γ be the forgetful functor from the category *CRT*-modules to the category of *CR*-modules. It is immediate from Theorem 4.2.1 of [23] that Γ is injective (but not surjective) on the class of acyclic *CRT*-modules. Hence we have the following result.

Proposition 2.5. *Let A and B be real C*-algebras. Then $K^{CRT}(A) \cong K^{CRT}(B)$ if and only if $K^{CR}(A) \cong K^{CR}(B)$.*

Note, however, that the results of [10] do not extend to *CR*-modules. Not every countable acyclic *CR*-module can be realized as $K^{CR}(A)$ for a real C*-algebra A.

3. Preliminaries on Real Simple Purely Infinite C*-Algebras

In this section, we provide some preliminaries on simple and purely infinite C*-algebras, including a theorem characterizing simple purely infinite real C*-algebras in terms of their complexification. One direction of this characterization was achieved in [43] and [13].

Let A be a real unital C*-algebra, let $\mathcal{U}(A)$ denote the group of unitary elements in A, and let $\mathcal{U}_0(A)$ denote the connected component of the identity in $\mathcal{U}(A)$. Note that if u is a unitary in a real C*-algebra, then the spectrum $\sigma(u) \subseteq \mathbb{T}$ satisfies $\overline{\sigma(u)} = \sigma(u)$ and the real C*-algebra generated by u is isomorphic to the algebra of complex-valued continuous functions f on $\sigma(u)$ that satisfy $\overline{f(z)} = f(\overline{z})$. (If a is an element of A, then by definition the spectrum $\sigma(a)$ is found by passing to $A_{\mathbb{C}}$.)

We begin by making an explicit mention of a fairly well-known result about real simple C*-algebras.

Definition 3.1. A real C*-algebra A is *c-simple* if $A_{\mathbb{C}}$ is simple.

Proposition 3.2. *A simple real C*-algebra A is either c-simple or is isomorphic to a simple complex C*-algebra.*

Proof. Let I be a proper ideal in $A_{\mathbb{C}}$. Then $J = A \cap I \cap \Phi(I) = 0$ and so $I \cap \Phi(I) = 0$. Furthermore, $I + \Phi(I) = A_{\mathbb{C}}$. It then follows that the map $x \mapsto x + \Phi(x)$ is an isomorphism from the complex C*-algebra I onto A. $\square$

As the structure of simple complex C*-algebras is comparatively well-understood, our primary interest lies in c-simple C*-algebras.

As in the complex case, we will use the tilde $\sim$ to denote the relation of Murray-von Neumann equivalence of projections. A projection is said to be infinite if it is Murray-von Neumann equivalent to a proper subprojection of itself. The following definition of purely infinite is from [43]. Bearing in mind subsequent developments, such as [27] and [28], a different definition should be made in the non-simple case. However the focus in this paper is on simple algebras, for which the definition below is appropriate.

Definition 3.3. Let A be a real simple C*-algebra.

(1) A subalgebra B is a *regular hereditary* subalgebra of A if there is an element $x \in A_+$ such that $B = \overline{xAx}$.
(2) A is *purely infinite* if every regular hereditary subalgebra of A contains an infinite projection.

Proposition 3.4. *Let A be a separable simple purely infinite real C^*-algebra. Then either A is unital or there is a real unital simple purely infinite C^*-algebra A_0 such that $A \cong \mathcal{K}^{\mathbb{R}} \otimes A_0$.*

Proof. As in Section 27.5 of [2]. □

Proposition 3.5. *Let A be a simple purely infinite C^*-algebra and let p be a projection in A. Then pAp and A are stably isomorphic.*

Proof. In the complex case, this result follows from Corollary 2.6 of [16]. The proof of that result and the proofs of the preliminary lemmas of Section 2 of [16] work the same in the real case. □

For the rest of this section, f_ε will denote the real-valued function such that $f_\varepsilon(t) = 0$ for $t \leq \varepsilon/2$, $f_\varepsilon(t) = 1$ for $t \geq \varepsilon$, and $f_\varepsilon(t)$ is linear on $[\varepsilon/2, \varepsilon]$.

Lemma 3.6. *For any real C^*-algebra A, the following are equivalent.*
(1) *For any non-zero $a, b \in A$ there exist $x, y \in A$ with $a = xby$.*
(2) *For any non-zero positive $a, b \in A$ there exists $x \in A$ with $a = xbx^*$.*

Proof. (2) $\Rightarrow$ (1). Let $0 \neq a, b \in A$. As in the complex case, described in 1.4.5 of [33], there exists $u \in A$ with $a = u(a^*a)^{1/4}$. Let $x \in A$ with $(a^*a)^{1/4} = xbb^*x^*$ and observe that $a = (ux)b(b^*x^*)$.
(1) $\Rightarrow$ (2). This uses the argument for the complex case, from Lemma 1.7 and Proposition 1.10 of [18]. If $a, b \in A$ are positive and non-zero and ε is chosen so that $f_\varepsilon(b) \neq 0$ then $a = (zz^*zk)b(zz^*zk)^*$, where x, y are chosen so that $a^{1/6} = xf_\varepsilon(b)y$, $k \geq 0$ is chosen so that $f_{\varepsilon/2}(b) = kbk$ and $z = x(f_\varepsilon(b)yy^*f_\varepsilon(b))^{1/2}$. □

Lemma 3.7. *Let A be a real C^*-algebra such that for all non-zero elements a, b there exist x, y with $a = xby$. Suppose that A contains a non-zero projection and let c be a non-zero positive element such that $\overline{cAc} \neq A$. Then $\overline{cAc}$ contains an infinite projection.*

Proof. The argument from $(vii) \Rightarrow (i)$ of Theorem 2.2 of [31] applies to the real case to show that for any non-trivial projection p and positive element x there is a Murray-von Neumann equivalence between p and a subprojection of x. We will repeatedly use this fact.
In the unital case, this shows that the unit 1 is Murray-van Neumann equivalent to a projection of $\overline{cAc}$, which is necessarily infinite.
Now suppose that A has no unit but has a non-zero projection p. Applying the fact above to a non-zero positive element d in $(1-p)A(1-p)$ gives a projection q such that $p \sim q$ and $p \perp q$. Now apply the fact again using the projection $p + q$ and the positive element p to show that $p + q$ is infinite. Finally, apply the same fact using the projection $p + q$ and the positive element c to show that $p + q$ is Murray-von Neumann equivalent to a projection in $\overline{cAc}$. □

Lemma 3.8. *Let A be a real simple C^*-algebra. Then the following are equivalent:*

(1) *A is purely infinite,*
(2) *A is not isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ and for each pair of non-zero elements $a, b \in A$ there exist $x, y \in A$ such that $a = xby$,*
(3) *A is not isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ and for each pair of non-zero positive elements $a, b \in A$ there exists $x \in A$ such that $a = xbx^*$.*

Furthermore, if these conditions are satisfied, then for all $\varepsilon > 0$ the element x in (3) can be chosen to satisfy $\|x\| \leq (\|a\|/\|b\|)^{1/2} + \varepsilon$.

Proof. As the result is well-known in the complex case, we may assume by Theorem 3.1 that A is c-simple. By Lemma 3.6, (2) and (3) are equivalent.
For (1) $\Rightarrow$ (2), let a, b be non-zero elements of A, identified with $e_{11}(\mathcal{K}^{\mathbb{R}} \otimes A)e_{11}$. We are assuming $A_{\mathbb{C}}$ is simple, so Theorem 2.4 of [17] applied to the unital algebra pAp implies that $\mathcal{K} \otimes pA_{\mathbb{C}}p$ is algebraically simple. Then by Proposition 3.5, $\mathcal{K} \otimes A_{\mathbb{C}}$ is algebraically simple, whence $\mathcal{K}^{\mathbb{R}} \otimes A$ is. The argument from $(ii) \Rightarrow (xi)$ of Theorem 2.2 of [31], then produces $x, y \in \mathcal{K}^{\mathbb{R}} \otimes A$ with $a = xby$, so $a = (e_{11}xe_{11})b(e_{11}ye_{11})$.
For (2) $\Rightarrow$ (1), we use a simplified argument based on the proof of Theorem 1.2 of [31]. Note first that if a nonzero projection can be found in A then Lemma 3.7 gives the result. (In particular, this takes care of the unital case.) Let a, d be non-zero positive elements of A with $da = ad = a$ (for a positive element x with norm 1 take $a = f_{1/2}(x)$ and $d = f_{1/4}(x)$). Then let $s, t \in A$ with $d = sat$ and let $y = (as^*sa)^{1/2}t$. An easy argument shows that $|y||y^*| = |y^*|$ hence $f_{1/2}(|y|)f_{1/8}(|y^*|) = f_{1/8}(|y^*|)$. Unless $f_{1/4}(|y|)$ is a projection, Lemma 4.2 of [7] gives a scaling element $t \in A$. In this case, $p_n = f_n + f_n^{1/2}tf_n^{1/2} + f_n^{1/2}t^*f_n^{1/2}$ (where $f_n = t^n(t^*)^n - t^{n+1}(t^*)^{n+1}$ for $n \geq 2$) is a projection by Theorem 3.1 of [7], .
The final condition holds as in Lemma 2.4 of [28]. □

Theorem 3.9. *A real c-simple C^*-algebra A is purely infinite if and only if $A_{\mathbb{C}}$ is purely infinite.*

Proof. From Theorem 3.3 of [43] we know that A is purely infinite if $A_{\mathbb{C}}$ is.
For the converse, suppose A is purely infinite, let ω be a free ultrafilter on $\mathbb{N}$ and let A_ω be the corresponding ultrapower algebra, defined in Definition 6.2.2 of [39]. Note that the proofs of Proposition 6.2.6 of [39] and the preliminary Lemma 6.2.3 carry over directly to the real case (using Lemma 3.8). Therefore A_ω is simple and purely infinite. Suppose that D is a dimension function, as defined in Definition I.1.2 of [5], on the complexification $(A_\omega)_{\mathbb{C}} \cong (A_{\mathbb{C}})_\omega$. For each positive non-zero a, b in A_ω there exist $x, y \in A_\omega$ with $b = xax^*$ and $a = yby^*$ so $D(a) = D(b)$. For any infinite projection $p \in A_\omega$, there exists a projection $q < p$ with $D(p) = D(q) + D(p - q) = D(p) + D(p)$, so $D(a) = D(p) = 0$ for each positive $a \in A_\omega$. Then for each positive $a \in (A_\omega)_{\mathbb{C}}$, we have $0 \leq D(a) \leq D(a + \Phi(a)) = 0$. So there is no dimension function on $(A_\omega)_{\mathbb{C}}$ and therefore, by Theorem II.2.2 of [5], no 2-quasitrace. Therefore $A_{\mathbb{C}}$

is weakly purely infinite by Theorem 4.8 of [28]. By Corollary 4.16 of [28] it is therefore purely infinite. □

Corollary 3.10. (1) *If A and B are stably isomorphic real C*-algebras, and if A is purely infinite and c-simple then so is B.*

(2) *Any inductive limit of real purely infinite c-simple C*-algebras is again purely infinite and c-simple.*

(3) *If A and B are purely infinite and c-simple, then so is $A \otimes_{min} B$.*

Proof. These results follow immediately from Theorem 3.9 and the same results in the complex case (see Proposition 4.1.8 of [39]). □

We now work toward showing that the K_0 and K_1 groups of a real purely infinite algebra can be described in a similar way to the complex case. The next two lemmas provide the required modification of Lemma 1.7 of [19].

Lemma 3.11. *Let A be a real c-simple purely infinite unital C*-algebra and let $u \in \mathcal{U}(A)$ and let $\lambda \in \sigma(u)$. For any $\varepsilon > 0$ there exists $v \in \mathcal{U}(A)$ such that $\|u - v\| < \varepsilon$ and*

(1) *if $\lambda = \lambda^*$ then $v = v_0 + \lambda p$ where p is a non-zero projection in A and $v_0 \in \mathcal{U}(p^\perp A p^\perp)$.*

(2) *if $\lambda \neq \lambda^*$ then $v = v_0 + \lambda p_1 + \lambda^* p_2$ where p_1 and p_2 are orthogonal non-zero orthogonal projections in $A_\mathbb{C}$ satisfying $\Phi(p_1) = p_2$ and $v_0 \in \mathcal{U}((p_1 + p_2)^\perp A (p_1 + p_2)^\perp)$.*

Proof. First assume that $\lambda = \lambda^*$. Let h be a positive function on $\sigma(u)$ such that $\text{supp}(h) \subset N_{\varepsilon_0}(\lambda)$ and $h(z^*) = h(z)$ for all $z \in \sigma(u)$. Then $h(u) \in A$ and let p be a non-zero projection in $\overline{h(u)Ah(u)}$. As in the proof of Lemma 1.7 of [19], we have $\|u - (p^\perp u p^\perp + \lambda p)\| \leq 3\varepsilon_0$. Then the polar decomposition of $(p^\perp u p^\perp + \lambda p)$ yields a unitary v of the required form that, if ε_0 is sufficiently small, will satisfy $\|u - v\| < \varepsilon$.

Now assume $\lambda \neq \lambda^*$. Choose ε_0 small enough so that $N_{\varepsilon_0}(\lambda) \cap N_{\varepsilon_0}(\lambda^*) = \emptyset$. Let h_1 be a positive function on $\sigma(u)$ such that $\text{supp}(h_1) \subset N_{\varepsilon_0}(\lambda)$. By Theorem 3.9, $A_\mathbb{C}$ is purely infinite so there is a non-zero projection p_1 in $B = \overline{h_1(u) A_\mathbb{C} h_1(u)}$.

Define $p_2 = \Phi(p_1) \in \Phi(B)$ and $p = p_1 + p_2$. Now $\Phi(h_1(u)) = h_2(u)$ where h_2 is the continuous function on $\sigma(u)$ defined by $h_2(z) = h_1(z^*)$. Since $\text{supp}(h_2) \subset N_{\varepsilon_0}(\lambda^*)$, we have $h_1(u)h_2(u) = 0$. Thus p_1 and p_2 are orthogonal projections and $p \in A$.

As in Lemma 1.7 of [19], we have $\|up_1 - \lambda p_1\| \leq \varepsilon_0$ and $\|up_2 - \lambda^* p_2\| \leq \varepsilon_0$ from which it follows that $\|u - (p^\perp u p^\perp + \lambda p_1 + \lambda^* p_2)\| \leq 8\varepsilon_0$. The required unitary v is obtained by taking the polar decomposition of $p^\perp u p^\perp + \lambda p_1 + \lambda^* p_2$ in A. □

Lemma 3.12. *Let A and u be as above. Then there is a projection p in A and a unitary v in $\mathcal{U}(p^\perp A p^\perp)$ such that $u \sim v + p$.*

Proof. If $1 \in \sigma(u)$ then using Lemma 3.11, approximate u by an element of the form $v + p$. If the approximation is close enough, then the two unitaries will be in the same path component.

If $\lambda \in \sigma(u)$ where $\lambda \neq \lambda^*$, use Lemma 3.11 to approximate u by $v + \lambda p_1 + \lambda^* p_2$. Then we can easily find a path from $\lambda p_1 + \lambda^* p_2$ to $p_1 + p_2$ in $(p_1+p_2)A(p_1+p_2)$.

The only possibility left is $u = -1$. In that case, find two orthogonal projections q_1 and q_2 and a partial isometry s such that $ss^* = q_1$ and $s^*s = q_2$. Let $p = q_1 + q_2$. The projection p can be rotated to $-p$ within the 2×2 matrix algebra generated by q_1, q_2 and s. Hence the unitary $-1 = -(p^\perp) + -p$ can be connected to the unitary $-(p^\perp) + p$. □

Proposition 3.13. *Let A be a c-simple purely infinite real C*-algebra. Then*

1. $K_0(A) = \{[p] \mid p$ *is a non-zero projection in* $A\}$
2. $K_1(A) = \mathcal{U}(A)/\mathcal{U}_0(A)$ *(for A unital).*

Proof. In the complex case, these results are proven in Section 1 of [19]. The proofs of those results as well as the proofs of the preliminary lemmas carry over to the real case, with two modifications. The first is to the proof of Lemma 1.7 of [19], which we already addressed with the proof of Lemma 3.12 above.

Secondly, in the proof of Lemma 1.1 of [19] the author uses an element of the form

$$\widetilde{w} = w + w^* + (1 - w^*w - ww^*), \qquad (\text{where } w^2 = 0)$$

that is a unitary lying in the finite dimensional C*-algebra generated by w. In the complex case it follows that $\widetilde{w} \in \mathcal{U}_0(A)$, whereas in the real case unitary groups of finite dimensional C*-algebras are not connected in general.

However, if instead we take $\widetilde{w} = w - w^* + (1 - w^*w - ww^*)$ then $\widetilde{w}$ is in the connected component of the identity, as it corresponds to a matrix of the form $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The proof of Lemma 1.1 of [19] can be completed without change using this alternative $\widetilde{w}$. □

We note that part (1) of Proposition 3.13 appeared as Proposition 11 of [10].

4. Exponential Rank

Definition 4.1. An element a in a real C*-algebra A is *skew-adjoint* if $a^* = -a$. The set of skew-adjoint elements is denoted by A_{sk}.

If a is skew-adjoint, then $\sigma(a) = -\sigma(a) \subseteq i\mathbb{R}$ and the real unital C*-algebra generated by a is isomorphic to

$$\{f \in C(\sigma(a), \mathbb{C}) \mid f(it)^* = f(-it)\}\,.$$

Furthermore, if a is a skew-adjoint element in a real unital C*-algebra A, then $\exp(a)$ is a unitary in A.

Lemma 4.2. *Let A and B be unital real C*-algebras.*

1. $\mathcal{U}_0(A) = \{\prod_{i=1}^n \exp(k_i) \mid k_i \in A_{sk}, n \in \mathbb{N}\}$.
2. *If $\alpha\colon A \to B$ is unital and surjective, then $\alpha(\mathcal{U}_0(A)) = \mathcal{U}_0(B)$.*

Proof. Suppose first that u is a unitary element in A with $\|u-1\| < 2$. Then $-1 \notin \sigma(u)$. We define a continuous function $f\colon \mathbb{T}\setminus\{0\} \to i(-\pi,\pi)$ by $f(\exp(it)) = it$ for $t \in (-\pi,\pi)$. Then $f(u)$ is in the real C*-algebra generated by u, is skew-adjoint, and satisfies $\exp(f(u)) = u$.
More, generally, if $u \in \mathcal{U}_0(A)$ then there exists a chain
$u = u_0, u_1, u_2, \ldots, u_n = 1$ with $\|u_{i-1} - u_i\| < 2$ for all $i \in \{1,2,\ldots,n\}$. Then applying the previous paragraph we have $u_{i-1}u_i^* = \exp(k_i)$ for all i with $1 \le i \le n$. Then $u = \prod_{i=1}^n \exp(k_i)$.
Conversely, if $\{k_i\}_{i=1}^n$ is any collection of skew-adjoint elements, then $u(t) = \prod_{i=1}^n \exp(tk_i)$ for $0 \le t \le 1$ is a continuous path of unitaries from 1_A to $\prod_{i=1}^n \exp(k_i)$. This proves (1).
For (2), the inclusion $\alpha(\mathcal{U}_0(A)) \subseteq \mathcal{U}_0(B)$ is immediate. Let $u \in \mathcal{U}_0(B)$. Then $u = \prod_{i=1}^n \exp(k_i)$ for some skew-adjoint elements $k_i \in B$. Let $l_i \in A$ be elements such that $\alpha(l_i) = k_i$. We may assume that l_i is skew-adjoint for all i, by replacing with $\frac{1}{2}(l_i - l_i^*)$ if necessary. Then $u = \alpha\left(\prod_{i=1}^n \exp(l_i)\right)$. □

Let $\mathcal{E} = \{\exp(k) \mid k \in A_{sk}\}$ and let $\mathcal{E}^n$ be the set of all products of at most n elements of $\mathcal{E}$. Thus $\mathcal{U}_0(A) = \cup_{n=1}^\infty \mathcal{E}^n$. The argument in the proof above also implies that the set $\mathcal{E}^{n+1}$ contains the topological closure of $\mathcal{E}^n$ so that we have the an increasing sequence

$$\mathcal{E} \subseteq \overline{\mathcal{E}} \subseteq \mathcal{E}^2 \subseteq \overline{(\mathcal{E})^2} \subseteq \cdots \subseteq (\mathcal{E})^n \subseteq \overline{(\mathcal{E}^n)} \subseteq (\mathcal{E})^{n+1} \subseteq \ldots$$

similar to that in [37], motivating the following definition.
Definition 4.3.

(1) The *exponential rank* of A, written $\mathrm{cer}(A)$, is equal to the integer n if $\mathcal{E}^n$ is the smallest set in this sequence to be equal to $\mathcal{U}_0(A)$ and is equal to the symbol $n+\varepsilon$ if $\overline{\mathcal{E}^n}$ is the smallest set to be equal to $\mathcal{U}_0(A)$. If $\mathcal{E}^n \neq \mathcal{U}_0(A)$ for all n then $\mathrm{cer}(A) = \infty$.
(2) The *exponential length* of A, written $\mathrm{cel}(A)$, is equal to the smallest number $0 < \mathrm{cel}(A) \le \infty$ such that every unitary u in $\mathcal{U}_0(A)$ can be written in the form

$$u = \exp(k_1)\exp(k_2)\cdots\exp(k_n)$$

where $k_i \in A_{sk}$ and

$$\|k_1\| + \|k_2\| + \cdots + \|k_n\| \le \mathrm{cel}(A)\ .$$

With these definitions, the proofs of Section 2 of [37] can be applied with minimal modification to prove the following results.

Lemma 4.4. *Let A be a real unital C*-algebra and let n be a positive integer.*

(1) *If $\mathrm{cel}(A) < n\pi$ then $\mathrm{cer}(A) \le n$.*
(2) *If $\mathrm{cel}(A) \le n\pi$ then $\mathrm{cer}(A) \le n+\varepsilon$.*

Lemma 4.5. *Let A be a real unital C*-algebra. If every unitary $u \in \mathcal{U}_0(A)$ can be connected to the identity by a rectifiable path of length no more than M, then* $\mathrm{cel}(A) \le M$.

Definition 4.6. A real C*-algebra A has *real skew rank zero* if the elements of A_{sk} with finite spectrum are dense in A_{sk}.

In the case of a complex C*-algebra A there is a bicontinuous bijection $A_{sa} \to A_{sk}$ given by multiplication by i, showing that A has skew rank zero if and only it has real rank zero. However, in the case of real C*-algebras things are more subtle. For example the condition of being skew-rank zero is not equivalent (in the unital case) to the condition that the invertible elements of A_{sk} are dense. Indeed, all finite dimensional real C*-algebras have real skew rank zero, but the invertibles of $(M_n)_{sk}$ are dense only if n is even.

Proposition 4.7. *Let A be a real unital c-simple purely infinite C*-algebra satisfying $[1] \in 2K_0(A)$. Then the invertibles of A_{sk} are dense in A_{sk} and A has real skew rank zero.*

Proof. Let A be a real purely infinite C*-algebra such that $[1] \in 2K_0(A)$. Let $a \in A_{sk}$ and let $\varepsilon > 0$ be given. Define functions $g\colon i\mathbb{R} \to \mathbb{R}$ and $f\colon i\mathbb{R} \to i\mathbb{R}$ by

$$g(it) = \max\{\varepsilon - |t|, 0\} \qquad \text{and} \qquad f(it) = \begin{cases} i(t+\varepsilon) & t \leq -\varepsilon \\ 0 & |t| < \varepsilon \\ i(t-\varepsilon) & t \geq \varepsilon\ . \end{cases}$$

Then $g(a) \in A_+$ and $f(a) \in A_{sk}$.
Since A is purely infinite, there is a projection $p \in \overline{g(a)Ag(a)}$ with $2[p] = [1] \in K_0(A)$. Then $[1-p] = [p]$ so there is a partial isometry s such that $s^*s = 1-p$ and $ss^* = p$. Since $f(a)g(a) = 0$ we have $f(a) = (1-p)f(a)(1-p)$.
Let $b = f(a) + \varepsilon(s - s^*)$. In matrix form under the decomposition indicated by the projection sum $1 = (1-p) + p$ we have

$$b = \begin{pmatrix} f(a) & -\varepsilon \\ \varepsilon & 0 \end{pmatrix}$$

whence b is invertible. This proves the first statement.
For the second statement, again let $a \in A_{sk}$ and let $\varepsilon > 0$ be given. By the first part of the theorem, we may assume that a is invertible, hence $\sigma(a) \subset i\mathbb{R} \setminus \{0\}$. Write $a = a_1 + a_2$ where the elements $a_i \in A_{\mathbb{C}}$ satisfy $\sigma(a_1) \subset i(0,\infty)$ and $\sigma(a_2) \subset i(-\infty, 0)$. Note also that $\Phi(a_1) = a_2$.
Since $A_{\mathbb{C}}$ is simple and purely infinite it has real rank zero, so there exists $b_1 \in (A_{\mathbb{C}})_{sk}$ such that $\sigma(b_1)$ is a finite subset of $i\mathbb{R}^+$ and $\|a_1 - b_1\| < \varepsilon/2$. Let $b_2 = \Phi(b_1)$ and let $b = b_1 + b_2$. Then b is a skew-adjoint element of A with finite spectrum and $\|a - b\| < \varepsilon$. □

Lemma 4.8. *Let A be a real c-simple unital C*-algebra such that $[1] \in 2K_0(A)$. Let $u \in \mathcal{U}(A)$ be a unitary such that $\sigma(u) \neq S^1$. Then for every $\varepsilon > 0$ there is a unitary v with finite spectrum such that $\|u - v\| < \varepsilon$.*

Proof. If $-1 \notin \sigma(u)$, then there is a continuous function $f\colon \sigma(u) \to i[-\pi, \pi]$ that is a right inverse to the function $it \mapsto \exp(it)$ and that satisfies $f(z^*) = f(z)^*$. Then $f(u) \in A_{sk}$ can be approximated within δ by a skew adjoint

element b with finite spectrum by Proposition 4.7. For an appropriate choice of δ, this implies that $\exp(b) \in \mathcal{U}(A)$ approximates u within ε.
Similarly, if $1 \notin \sigma(u)$, then there is a continuous function $f \colon \sigma(u) \to i[-\pi, \pi]$ that is a right inverse to the function $it \mapsto -\exp(-it)$ and that satisfies $f(z^*) = f(z)^*$.
In the general case, suppose that $\lambda \notin \sigma(u)$ for some $\lambda \in S^1$. Let $\sigma_1 = \{w \in \sigma(u) \mid \mathrm{Re}(w) > \mathrm{Re}(\lambda)\}$ and let $\sigma_2 = \{w \in \sigma(u) \mid \mathrm{Re}(w) < \mathrm{Re}(\lambda)\}$. Then $\sigma = \sigma_1 \cup \sigma_2$. Let $u_i = u_i E_u(\sigma_i)$, where $E_u(\sigma_i)$ denotes the spectral projection of u associated with the clopen subset σ_i of σ. Then $1 \notin \sigma(u_2)$ and $-1 \notin \sigma(u_1)$. Using the results from the first two paragraphs, let v_i be a unitary that approximates u_i in $E_u(\sigma_i)AE_u(\sigma_i)$ within ε. Then since $u = u_1 + u_2$ we have that $v = v_1 + v_2$ is a unitary that approximates u within ε. □

Lemma 4.9. *Let A be a real unital simple purely infinite C*-algebra let $u \in \mathcal{U}(A)$ and let $\{\lambda_1, \dots, \lambda_n\}$ be a subset of $\sigma(u)$ that is closed under conjugation. For any $\varepsilon > 0$ there exist $v \in \mathcal{U}(A)$ and orthogonal projections $p_1, \dots, p_n \in A_{\mathbb{C}}$ such that $\|u - v\| < \varepsilon$ and $v = v_0 + \lambda_1 p_1 + \dots + \lambda_n p_n$ with $v_0 \in \mathcal{U}((p_1 + \dots + p_n)^{\perp} A (p_1 + \dots + p_n)^{\perp})$.*
Furthermore, the elements $\sum_{i=1}^n p_i$ and $\sum_{i=1}^n \lambda_i p_i$ are both in A.

Proof. Use the constructions of Lemma 3.11 above as in the proof of Lemma 6 of [34]. □

Lemma 4.10. *Let A be a real unital C*-algebra and let $u \in \mathcal{U}(A)$. For any $\varepsilon > 0$ there exists an $h \in M_2(A)_{sk}$ such that $\|u \oplus u^* - \exp(h)\| < \varepsilon$.*

Proof. As in the proof of Corollary 5 of [34], there exists a continuous path $v(t)$ of unitaries in $M_2(A)$ with $v(0) = 1$ and $v(\pi/2) = u \oplus u^*$ such that $-1 \notin \sigma(v(t))$ for $0 \le t < \pi/2$. Thus we can find a t close enough to $\pi/2$ such that $\|u \oplus u^* - v(t)\| < \varepsilon$ and $v(t) = \exp(h)$ for a skew-adjoint h. □

Lemma 4.11. *Let A be a real unital c-simple purely infinite C*-algebra such that $[1] \in 2K_0(A)$. Let e_1, e_2, e_3, e_4 be nonzero orthogonal projections in A that sum to 1. Let a be a partial isometry such that $a^*a = e_2$ and $aa^* = e_3$. Let $u \in \mathcal{U}(e_1 A e_1)$ and $v \in \mathcal{U}(e_2 A e_2)$ be unitaries with $\sigma(u) = S^1$. Then for all $\varepsilon > 0$ there is a unitary $z \in \mathcal{U}(A)$ and a unitary $w \in \mathcal{U}(e_4 A e_4)$ with finite spectrum such that*

$$\|z^*(u + 1 - e_1)z - (u + v + av^*a^* + w)\| < \varepsilon \,.$$

Proof. This proof closely follows that of Lemma 7 of [34]. By Lemma 4.10 there is a unitary in $(e_2 + e_3)A(e_2 + e_3)$ that is arbitrarily close to $v + av^*a^*$ and that has the form $\exp h$ for $h \in A_{sk}$. This in turn can be approximated by a unitary that has finite spectrum by Proposition 4.7. The general form of such a unitary is

$$\sum_{k=1}^{n} (\lambda_k q_{k1} + \lambda_k^* q_{k2}) + 1 q_{01} + (-1) q_{02}$$

where $\lambda_k^* \neq \lambda_k$, the nonzero projections $q_{ki} \in A_{\mathbb{C}}$ satisfy $\Phi(q_{k1}) = q_{k2}$ for $1 \leq k \leq n$, and the (possibly zero) projections q_{0i} are in A. Furthermore, the q_{ki} are orthogonal and sum to $e_2 + e_3$. Without loss of generality, we assume that $v + av^*a^*$ has this form. With an obvious choice of coefficients λ_{ki} we can write this as

$$v + av^*a^* = \sum_{k=0}^{n} \sum_{i=1}^{2} \lambda_{ki} q_{ki} = \sum \lambda_{ki} q_{ki} \ .$$

(Henceforth in this proof will use an undecorated $\sum$ to represent a double sum indexed as $\sum_{k=0}^{n} \sum_{i=1}^{2}$.)
Now we replace u by a nearby element of the form given by Lemma 4.9. Specifically, there are orthogonal projections $p_{ki} \in e_1 A_{\mathbb{C}} e_1$ and, setting $p = e_1 - \sum p_{ki} \in A$, there is a unitary $u_0 \in pAp$ such that

$$u = u_0 + \sum \lambda_{ki} p_{ki}$$

(where the projection $p_{0i} = 0$ if and only if $q_{0i} = 0$).
For each $k \in \{1, \ldots, n\}$ let $c_{k1} \in A_{\mathbb{C}}$ be a partial isometry such that $c_{k1}^* c_{k1} = p_{k1}$ and $c_{k1} c_{k1}^* < p_{k1}$. Then $c_{k2} = \Phi(c_{k1})$ satisfies $c_{k2}^* c_{k2} = p_{k2}$ and $c_{k2} c_{k2}^* < p_{k2}$ and $c_k = c_{k1} + c_{k2} \in A$ satisfies $c_k^* c_k = p_{k1} + p_{k2}$ and $c_k c_k^* < p_{k1} + p_{k2}$. For $k = 0$ we obtain partial isometries $c_{0k} \in A$ such that $c_{0i}^* c_{0i} = p_{0i}$ and $c_{ki} c_{ki}^* < p_{ki}$. Then $c = p + \sum c_{ki} \in A$ satisfies

$$c^*c = e_1, \qquad cc^* = e_1 - \sum (p_{ki} - c_{ki} c_{ki}^*), \qquad \text{and}$$

$$cuc^* = u_0 + \sum \lambda_{ki} c_{ki} c_{ki}^* \ .$$

Similarly we can find a collection of partial isometries d_{ki} with domain projection q_{ki} and range projection a subprojection of $p_{ki} - c_{ki}^* c_{ki}$ that also satisfy $\Phi(d_{k1}) = d_{k2}$ for $k \neq 0$ and $\Phi(d_{ki}) = d_{ki}$ for $k = 0$. Then the partial isometry $d = \sum d_{ki} \in A$ satisfies

$$d^*d = e_2 + e_3, \qquad dd^* \leq \sum (p_{ki} - c_{ki} c_{ki}^*), \qquad \text{and}$$

$$d \left(\sum \lambda_{ki} q_{ki} \right) d^* = \sum \lambda_{ki} d_{ki} d_{ki}^* \ .$$

Now, choose a partial isometry b such that

$$b^*b < e_4, \qquad bb^* = \sum (p_{ki} - c_{ki} c_{ki}^* - d_{ki} d_{ki}^*)$$

and define

$$w_0 = \sum \lambda_{ki} b^* (p_{ki} - c_{ki} c_{ki}^* - d_{ki} d_{ki}^*) b \ .$$

Then $z_0 = b + c + d$ is a partial isometry with $z_0^* z_0 = e_1 + e_2 + e_3 + b^*b$ and $z_0 z_0^* = e_1$. So in $K_0(A)$ we have $[e_1] = [e_1 + e_2 + e_3 + b^*b]$, which implies $[1 - e_1] = [e_4 - b^*b]$. By Proposition 11 of [10], there is a partial isometry $z_1 \in A$ such that $z_1 z_1^* = 1 - e_1$ and $z_1^* z_1 = e_4 - b^*b$. Then $w = w_0 + e_4 - b^*b$ is a unitary with finite spectrum in $e_4 A e_4$ and $z = z_0 + z_1$ is a unitary in A that satisfies $z^*(u + 1 - e_1)z = u + \sum \lambda_{ki} q_{ki} + w$. $\square$

Theorem 4.12. *Let A be a real unital c-simple purely infinite C*-algebra such that $[1] \in 2K_0(A)$. For every $u \in \mathcal{U}_0(A)$ and every $\varepsilon > 0$ there is a unitary v with finite spectum such that $\|u - v\| < \varepsilon$.*

Proof. With the lemmas that we have developed, the proof is now the same as that of the unital case of Theorem 1 and Corollary 2 of [34], except that wherever there is an element of the form $\exp(ih)$ where h is self-adjoint, we use $\exp(k)$ where k is skew-adjoint. □

As in the complex case, we have the following corollary concerning exponential length.

Corollary 4.13. *Let A be a real unital c-simple purely infinite C*-algebra such that $[1] \in 2K_0(A)$. Then $\mathrm{cel}(A) \leq 4$.*

Proof. By Theorem 4.12, every unitary $u \in \mathcal{U}_0(A)$ can be approximated within ε by a unitary v with finite spectrum. For ε sufficiently small, $\|v^*u - 1\| < \varepsilon$ implies there exists a skew-adjoint k_2 such that $v^*u = \exp(k_2)$ with $\|k_2\| \leq 4 - \pi$. As v has finite spectrum, there exists a skew-adjoint k_1 such that $v = \exp(k_1)$ and $\|k_1\| \leq \pi$. Then $u = \exp(k_1)\exp(k_2)$ and $\|k_1\| + \|k_2\| \leq 4$. □

5. Homomorphisms from $\mathcal{O}_n^{\mathbb{R}}$

The following theorem gives the real version of the Rokhlin property of the Bernoulli shift, established in [15] and summarized in [39]. Let $M_{2^\infty} = \lim_{k\to\infty} M_{2^k}$ be the real CAR algebra and let $\mathbb{H}$ be the real C*-algebra of quaternions.

Proposition 5.1. *Let σ be the one-sided Bernoulli shift on M_{2^∞}. For each $\varepsilon > 0$ and for each $r \in \mathbb{N}$ there exist $k \in \mathbb{N}$ and projections $e_0, e_1, \ldots, e_{2^r} = e_0 \in M_{2^k}$ such that $\sum_{j=1}^{2^r} e_j = 1$ and $\|\sigma(e_j) - e_{j+1}\| < \varepsilon$ for all $j = 0, 1, 2, \ldots, 2^r - 1$.*

Proof. Let $A_k = M_{2^k}$ and let $A = M_{2^\infty}$. Using the notation of Proposition 5.1.3 of [39], let S denote the unilateral shift on $\ell^2(\mathbb{N}, \mathbb{C})$, let $\omega_k = \exp(2\pi i/2^k)$ for each $k \geq 0$ and, given $\delta > 0$, let

$$f_0 = \frac{1}{\sqrt{n_0}}(1, 1, \ldots, 1, 0, 0, \ldots) \in \ell^2(\mathbb{N}, \mathbb{R})$$

be a unit vector with $\|Sf_0 - f_0\| < \delta$ and let

$$f_1 = \frac{1}{\sqrt{n_1}}(0, 0, \ldots 0, 1, -1, 1, -1, \ldots, -1, 0, 0, \ldots)$$

be a unit vector in $\ell^2(\mathbb{N}, \mathbb{R})$, orthogonal to f_0, with $\|Sf_1 + f_1\| < \delta$. Then, for $r \in \mathbb{N}$, let $f_2, \ldots, f_r \in \ell^2(\mathbb{N}, \mathbb{C})$ be defined by

$$f_j = \frac{1}{\sqrt{n_j}}(0, 0, \ldots 0, 1, \omega_j, \omega_j^2, \ldots, \omega_j^{n_j - 1}, 0, 0, \ldots)$$

where there are sufficiently many initial zeros to make f_j orthogonal to its predecessors and where n_j is chosen so that

$$\langle f_j, \overline{f_j} \rangle = 1 + \omega_j^2 + \ldots \omega_j^{2(n_j - 1)} = 0$$

and $\|Sf_j - \omega_j f_j\| < \delta$. If $f_j = g_j + ih_j$ with $g_j, h_j \in \ell^2(\mathbb{N}, \mathbb{R})$ then, from the orthogonality of f_j and $\overline{f_j}$, $\|g_j\| = \|h_j\| = 1/\sqrt{2}$.

Let $a : \ell^2(\mathbb{N}, \mathbb{C}) \to A_{\mathbb{C}}$ be the map described in [15] and [39] satisfying the canonical anticommutation relations and observe that a maps $\ell^2(\mathbb{N}, \mathbb{R})$ into A. Let $v_1 = w_1 = a(f_1)(a(f_0) + a(f_0)^*)$ and, for each $2 \le j \le r$ let $v_{2j-2} = a(f_j)(a(f_0) + a(f_0)^*)$, $v_{2j-1} = a(\overline{f_j})(a(f_0) + a(f_0)^*)$, $w_{2j-2} = a(\sqrt{2}g_j)(a(f_0) + a(f_0)^*) = (v_{2j-2} + v_{2j-1})/\sqrt{2}$ and $w_{2j-1} = a(\sqrt{2}h_j)(a(f_0) + a(f_0)^*) = -i(v_{2j-2} - v_{2j-1})/\sqrt{2}$. Note that $\{w_1, w_2, \ldots, w_{2r-1}\} \subset A_k$ for all sufficiently large k.

It is noted in the proof of Proposition 4.1 of [15] that the elements v_i for $1 \le i \le 2r-1$ satisfy the relations $v_iv_j + v_jv_i = 0$ and $v_iv_j^* + v_j^*v_i = \delta_{ij}1$. It follows from this that the elements w_i for $1 \le i \le 2r-1$ satisfy the same relations. Therefore, using the matrix units described in the proof of Proposition 4.1 of [15], the real C*-algebra B generated by $w_1, \ldots, w_{2r-1}$ is isomorphic to $M_{2^{2r-1}}$. Slightly varying the proof of Proposition 4.1 of [15], let β be the automorphism of the complexification of B determined by $\beta(v_1) = -v_1$, $\beta(v_{2j}) = \omega_j v_{2j}$ and $\beta(v_{2j+1}) = \overline{\omega_j} v_{2j+1}$ for each $1 \le j \le r-1$. Note that $\beta(w_{2j}) = \frac{1}{2}(\omega_j + \overline{\omega_j})w_{2j} + \frac{i}{2}(\omega_j - \overline{\omega_j})w_{2j+1}$ and $\beta(w_{2j+1}) = -\frac{i}{2}(\omega_j - \overline{\omega_j})w_{2j} + \frac{1}{2}(\omega_j + \overline{\omega_j})w_{2j+1}$, so that β leaves the real algebra B invariant. Identifying B with $M_{2^{2r-1}}$, there is an orthogonal matrix W implementing β. By standard linear algebra, described for example in Section 81 of [22], W is orthogonally conjugate to an orthogonal matrix consisting of diagonal elements ± 1 and diagonal 2×2 rotation matrices, determined by the eigenvalues of W.

As in [39], on the complexification of B, identified with $M_{2^{2r-1}}(\mathbb{C})$, β is implemented by a diagonal unitary with entries $1, \omega_r, \omega_r^2, \ldots, \omega_r^{2^r-1}$, each repeated 2^{r-1} times. (The unitary arises as the tensor product of one diagonal unitary with entries $1, \omega_r, \omega_r^2, \ldots, \omega_r^{2^r-1}$ and another with entries $1, \overline{\omega_r}, \overline{\omega_r}^2, \ldots, \overline{\omega_r}^{2^{r-1}-1}$.) On $B \cong M_{2^{2r-1}}$ the orthogonal matrix W implementing β is therefore conjugate to an orthogonal matrix with 2×2 diagonal blocks $\mathrm{diag}(1, -1), R, R^2, \ldots, R^{2^{r-1}-1}$, each repeated 2^{r-1} times, where

$$R = \begin{pmatrix} \cos(\pi/2^{r-1}) & -\sin(\pi/2^{r-1}) \\ \sin(\pi/2^{r-1}) & \cos(\pi/2^{r-1}) \end{pmatrix}.$$

The cyclic shift on M_{2^r} is implemented by the unitary

$$V = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix},$$

which is orthogonally conjugate to $\mathrm{diag}\left(\mathrm{diag}(1, -1), R, R^2, \ldots, R^{2^{r-1}-1}\right)$. It follows that the orthogonal element W implementing β on B is orthogonally conjugate to a direct sum of 2^{r-1} copies of V and thus that β is conjugate to a direct sum of 2^{r-1} cyclic shifts. It follows that there are 2^r orthogonal projections $e_0, e_1, \dots, e_{2^r} = e_0$ in B (each of rank 2^{r-1}) that are cyclically permuted by β. As in the proof of Proposition 4.1 of [15], a suitable choice of δ at the start of the proof ensures that $\|\sigma(e_j) - \beta(e_j)\| < \varepsilon$ for each j and therefore the projections $e_0, e_1, \dots, e_{2^r} = e_0$ have the required properties. □

Theorem 5.2. *Let D be a real unital C*-algebra satisfying*

(i) *the canonical homomorphism $\mathcal{U}(D)/\mathcal{U}_0(D) \to K_1(D)$ is an isomorphism, and*
(ii) $\operatorname{cel}(D) < \infty$.

Let n be an even integer, let ϕ, ψ be unital homomorphisms from $\mathcal{O}_n^{\mathbb{R}}$ to D, let λ be the endomorphism of D defined by $\lambda(a) = \sum_{j=1}^{n} \phi(s_j) a \phi(s_j)^*$ and let $u \in \mathcal{U}(D)$ be defined by $u = \sum_{j=1}^{n} \psi(s_j)\phi(s_j)^*$, where $s_1, \dots, s_n$ are the canonical generators of $\mathcal{O}_n^{\mathbb{R}}$. Then the following are equivalent:

(1) $u \in \overline{\{v\lambda(v)^* \mid v \in \mathcal{U}(D)\}}$,
(2) $[u] \in (n-1)K_1(D)$
(3) $[\phi] = [\psi] \in KK_0(\mathcal{O}_n^{\mathbb{R}}, D)$,
(4) *ϕ and ψ are approximately unitarily equivalent.*

In particular, these statements are equivalent if D is a real unital purely infinite c-simple C-algebra.*

Proof of Theorem 5.2. The proof of the equivalence of the four statements, assuming (i) and (ii), is similar to that of the complex case in Sections 3 and 4 of [38], modified only by the use of unitaries of the form $\exp(h)$ with $h \in A_{sk}$ in the proof of the real version of Lemma 4.6 of [38]. We note that in the proof of the real version of Lemma 3.7 of [38], the required result from [19] holds, as was observed already in the proof of Proposition 3.13 above.
Suppose D is a real unital purely infinite c-simple C*-algebra. Then condition (i) holds for D by Proposition 3.13. Since $K_0(\mathcal{O}_n^{\mathbb{R}}) = \mathbb{Z}_{n-1}$ and n is even, we have $[1_{\mathcal{O}_n^{\mathbb{R}}}] \in 2K_0(\mathcal{O}_n^{\mathbb{R}})$. Using the unital homomorphism ϕ (or ψ) we obtain $[1_D] \in 2K_0(D)$. Then condition (ii) holds by Corollary 4.13. □

Corollary 5.3.

(1) *Let A be a real unital purely infinite c-simple C*-algebra. Any two unital homomorphisms $\phi, \psi\colon \mathcal{O}_2^{\mathbb{R}} \to A$ are approximately unitarily equivalent.*
(2) *Any inductive limit of the form $\mathcal{O}_2^{\mathbb{R}} \to \mathcal{O}_2^{\mathbb{R}} \to \mathcal{O}_2^{\mathbb{R}} \to \dots$, with unital connecting homomorphisms, is isomorphic to $\mathcal{O}_2^{\mathbb{R}}$.*
(3) $\mathcal{O}_2^{\mathbb{R}} \otimes \mathcal{O}_2^{\mathbb{R}} \cong \mathcal{O}_2^{\mathbb{R}}$.
(4) $\bigotimes_{n=1}^{\infty} \mathcal{O}_2^{\mathbb{R}} \cong \mathcal{O}_2^{\mathbb{R}}$.
(5) $\mathcal{O}_2^{\mathbb{R}} \otimes M_{2^\infty} \cong \mathcal{O}_2^{\mathbb{R}}$.
(6) $\mathcal{O}_2^{\mathbb{R}} \otimes \mathbb{H} \cong \mathcal{O}_2^{\mathbb{R}}$.

Proof. We know that $K^{CRT}(\mathcal{O}_2^{\mathbb{R}}) = 0$ from Section 5 of [8] so the universal coefficient theorem (Theorem 4.1 of [9]) implies that $KK_0(\mathcal{O}_2^{\mathbb{R}}, D) = 0$. Then part (1) follows immediately from Theorem 5.2.

Parts (2) and (3) can be proven in the same way as in the complex case. See Corollary 5.1.5 and Theorem 5.2.1 in [39]. Then part (4) follows from parts (2) and (3).

There is an isomorphism $\mathcal{O}_2^{\mathbb{R}} \cong M_2(\mathcal{O}_2^{\mathbb{R}})$, established as in the complex case: if s_1 and s_2 are generators of $\mathcal{O}_2^{\mathbb{R}}$ satisfying the canonical relations $s_i^* s_j = \delta_{ij} 1_{\mathcal{O}_2^{\mathbb{R}}}$ and $\sum_{i=1}^2 s_i s_i^* = 1$, then

$$S_1 = \left(\begin{smallmatrix} s_1 & s_2 \\ 0 & 0 \end{smallmatrix}\right) \qquad \text{and} \qquad S_2 = \left(\begin{smallmatrix} 0 & 0 \\ s_1 & s_2 \end{smallmatrix}\right)$$

satisfy the same relations and generate $M_2(\mathcal{O}_2^{\mathbb{R}})$. Using that isomorphism, part (5) follows from part (2).

Finally, part (6) follows from (5) and the formula $M_{2^\infty} \otimes \mathbb{H} \cong M_{2^\infty}$, which follows from Theorem 10.1 of [21] or from Theorem 4.8 of [42]). □

6. Tensor Product Theorems

In this section, we reproduce for real C*-algebras some standard results regarding tensor products with $\mathcal{O}_2^{\mathbb{R}}$ and $\mathcal{O}_\infty^{\mathbb{R}}$.

Definition 6.1.

(1) A real (resp. complex) C*-algebra A is *amenable* if for all $\varepsilon > 0$ and all finite subsets $F \subset A$, there is a finite dimensional real (resp. complex) C*-algebra B and contractive completely positive linear maps $\phi\colon A \to B$ and $\psi\colon B \to A$ such that

$$\|\psi \circ \phi(a) - a\| < \varepsilon \quad \text{for all } a \in F.$$

(2) A real (resp. complex) C*-algebra A is *nuclear* if for all real (resp. complex) C*-algebras B the algebraic tensor product $A \otimes_{\mathbb{R}} B$ (resp. $A \otimes_{\mathbb{C}} B$) has a unique C*-norm.

(3) A real (resp. complex) C*-algebra A is *exact* if the tensor product functor $B \mapsto A \otimes_{\min} B$ is exact. Here the tensor product is over $\mathbb{R}$ (resp. $\mathbb{C}$) and B can be any real (resp. complex) C*-algebra.

Lemma 6.2. *Let A be a real C*-algebra. Then*

(1) *A is amenable if and only if $A_{\mathbb{C}}$ is amenable.*

(2) *A is nuclear if and only if $A_{\mathbb{C}}$ is nuclear.*

(3) *A is exact if and only if $A_{\mathbb{C}}$ is exact.*

Consequently, A is amenable if and only if it is nuclear; and in this case it is also exact.

Proof. Part (1) can be found in Proposition 3 of [25] and the preceding text.

We claim that there is a one-to-one correspondence between C*-norms on the algebraic tensor product $A \otimes_{\mathbb{R}} B$ and those on $A_{\mathbb{C}} \otimes_{\mathbb{C}} B_{\mathbb{C}}$. Let γ be a C*-norm on $A \otimes_{\mathbb{R}} B$, and let $A \otimes_\gamma B$ be the real C*-algebra obtained by completion. Then the complexification $(A \otimes_\gamma B)_{\mathbb{C}}$ has a unique C*-norm extending that on $A \otimes_{\mathbb{R}} B$. Thus every C*-norm on the algebraic tensor product $A \otimes_{\mathbb{R}} B$ extends uniquely to a C*-norm on $A_{\mathbb{C}} \otimes_{\mathbb{C}} B_{\mathbb{C}}$. Part (2) follows immediately from this claim.

It also follows that the restriction of the minimal C*-norm on $A_{\mathbb{C}} \otimes_{\mathbb{C}} B_{\mathbb{C}}$ gives the minimal C*-norm on $A \otimes_{\mathbb{R}} B$. This fact, plus the fact that the complexification functor $A \mapsto A_{\mathbb{C}}$ is exact, implies (3).
The final statement then follows from the corresponding statement for complex C*-algebras. See Theorem 6.1.3 of [39] and Theorem 6.5.2 of [32]. □

Proposition 6.3. *Let A be a real separable C*-algebra A. Then A is exact if and only if there is an injective homomorphism $\iota \colon A \to \mathcal{O}_2^{\mathbb{R}}$. If A is unital then ι can be chosen to be unital.*

Proof. Suppose that A is exact. Then $A_{\mathbb{C}}$ is separable and exact. Thus, by Theorem 6.3.11 of [39], there is an injective homomorphism $\iota_{\mathbb{C}} \colon A_{\mathbb{C}} \to \mathcal{O}_2$ (which is unital if $A_{\mathbb{C}}$ is unital). Then we can take ι to be the composition

$$A \hookrightarrow A_{\mathbb{C}} \xrightarrow{\iota_{\mathbb{C}}} \mathcal{O}_2 \hookrightarrow M_2(\mathcal{O}_2^{\mathbb{R}}) \cong \mathcal{O}_2^{\mathbb{R}} \; .$$

Conversely, if there is an injective homomorphism $\iota \colon A \to \mathcal{O}_2^{\mathbb{R}}$ then the complexification yields an injective homomorphism from $A_{\mathbb{C}}$ to $\mathcal{O}_2$. By Theorem 6.3.11 of [39] this implies that $A_{\mathbb{C}}$ is exact, hence A is exact. □

Lemma 6.4. *Let A be a real purely infinite c-simple nuclear unital C*-algebra. Then all unital endomorphisms on $A \otimes \mathcal{O}_2^{\mathbb{R}}$ are approximately unitarily equivalent.*

Proof. In the complex case, this result is found as Theorem 6.3.8 of [39]. We will use that result to prove the real version.
By Corollary 5.3, Part (5) it suffices to show that any unital homomorphism

$$\gamma \colon A \otimes \mathcal{O}_2^{\mathbb{R}} \otimes M_{2^\infty} \to A \otimes \mathcal{O}_2^{\mathbb{R}} \otimes M_{2^\infty}$$

is approximately unitarily equivalent to the identity. We write $A' = A \otimes \mathcal{O}_2^{\mathbb{R}}$ and let

$$\begin{aligned} \alpha_{\ell,k} \colon A' \otimes M_{2^k} &\hookrightarrow A' \otimes M_{2^\ell} \quad \text{for } k < \ell \\ \alpha_k \colon A' \otimes M_{2^k} &\hookrightarrow A' \otimes M_{2^\infty} \end{aligned}$$

be the canonical injections. Then we use the commutative diagram

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "c"] & \cdots \arrow[r] & A' \otimes M_{2^k} \arrow[r, "\alpha_{k+1,k}"] \arrow[d, "c"] & A' \otimes M_{2^{k+1}} \arrow[r] \arrow[d, "c"] & \cdots \arrow[r] & A' \otimes M_{2^\infty} \arrow[d, "c"] \\ A'_{\mathbb{C}} \arrow[r] & \cdots \arrow[r] & (A' \otimes M_{2^k})_{\mathbb{C}} \arrow[r, "\alpha_{k+1,k}"] & (A' \otimes M_{2^{k+1}})_{\mathbb{C}} \arrow[r] & \cdots \arrow[r] & (A' \otimes M_{2^\infty})_{\mathbb{C}} \end{tikzcd}$$

By Theorem 6.3.8 of [39], there is a sequence of unitaries $u_n \in (A' \otimes M_{2^\infty})_{\mathbb{C}}$ such that

$$\|u_n a u_n^* - \gamma(a)\| \to 0 \quad \text{for all } a \in A' \otimes M_{2^\infty} \; .$$

For each n find an integer $k(n)$ and a unitary $v_n \in (A' \otimes M_{2^{k(n)}})_{\mathbb{C}}$ such that $\|\alpha_{k(n)}(v_n) - u_n\| < 1/n$. Let $w_n = r(v_n) \in A' \otimes M_{2^{k(n)+1}}$, where r is induced by the realification map $M_{2^{k(n)}} \otimes \mathbb{C} \to M_{2^{k(n)+1}}$. We may assume that the sequence $\{k(n)\}_{n=1}^{\infty}$ is increasing.
Let $a \in A' \otimes M_{2^\infty}$ be given such that $\|a\| = 1$ and let $\varepsilon > 0$. Then find an integer N large enough so that, for all $n \geq N$,

- $\|\alpha_{k(n)}(v_n) - u_n\| < \varepsilon$,
- $\|u_n a u_n^* - \gamma(a)\| < \varepsilon$,
- there exist $a_n, b_n \in A' \otimes M_{2^{k(n)}}$ such that

$$\|a - \alpha_{k(n)}(a_n)\| < \varepsilon \qquad \text{and} \qquad \|\gamma(a) - \alpha_{k(n)}(b_n)\| < \varepsilon \, .$$

Then a calculation shows that, for all $n \geq N$,

$$\|v_n a_n v_n^* - b_n\| = \|\alpha_{k(n)}(v_n)\alpha_{k(n)}(a_n)\alpha_{k(n)}(v_n)^* - \alpha_{k(n)}(b_n)\| < 5\varepsilon.$$

Now for any element $x \in A' \otimes M_{2^{k(n)}}$ we have

$$\alpha_{k(n)+1} rc(x) = \alpha_{k(n)+1}\alpha_{k(n)+1,k(n)}(x) = \alpha_{k(n)}(x) \, .$$

It follows that

$$\begin{aligned}
&\|\alpha_{k(n)+1}(w_n) a \alpha_{k(n)+1}(w_n)^* - \gamma(a)\| \\
<&\|\alpha_{k(n)+1}(w_n)\alpha_{k(n)}(a_n)\alpha_{k(n)+1}(w_n)^* - \alpha_{k(n)}(b_n)\| + 2\varepsilon \\
=&\|\alpha_{k(n)+1} r(v_n)\alpha_{k(n)+1} rc(a_n)\alpha_{k(n)+1} r(v_n)^* - \alpha_{k(n)+1} rc(b_n)\| + 2\varepsilon \\
=&\|v_n c(a_n) v_n^* - c(b_n)\| + 2\varepsilon \\
=&\|v_n a_n v_n^* - b_n\| + 2\varepsilon < 7\varepsilon.
\end{aligned}$$

□

Theorem 6.5. *Let A be a real C*-algebra. Then A is c-simple, separable, unital, and nuclear if and only if $A \otimes \mathcal{O}_2^{\mathbb{R}} \cong \mathcal{O}_2^{\mathbb{R}}$*

Proof. Suppose that A is c-simple, separable, unital, and nuclear. There is a unital homomorphim $\gamma\colon \mathcal{O}_2^{\mathbb{R}} \to A \otimes \mathcal{O}_2^{\mathbb{R}}$ given by $x \mapsto 1 \otimes x$ and there is a unital homomorphism $\kappa\colon A \otimes \mathcal{O}_2^{\mathbb{R}} \to \mathcal{O}_2^{\mathbb{R}}$ by Lemma 6.2 and Proposition 6.3. Then by Theorem 5.2 we have $\kappa \circ \gamma \approx_u 1_{\mathcal{O}_2^{\mathbb{R}}}$ and by Lemma 6.4 we have $\gamma \circ \kappa \approx_u 1_{A \otimes \mathcal{O}_2^{\mathbb{R}}}$. Therefore, by (the real analog of) Corollary 2.3.4 of [39], $A \otimes \mathcal{O}_2^{\mathbb{R}} \cong \mathcal{O}_2^{\mathbb{R}}$.
Conversely, if the isomorphism $A \otimes \mathcal{O}_2^{\mathbb{R}} \cong \mathcal{O}_2^{\mathbb{R}}$ holds for a real C*-algebra A, then we have $A_{\mathbb{C}} \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ which implies by Theorem 7.1.2 of [39] that $A_{\mathbb{C}}$ is simple, separable, unital, and nuclear. Therefore A is c-simple, separable, unital, and nuclear. □

We note that the hypothesis above requiring that A be c-simple cannot be relaxed, as the result does not hold for $A = \mathcal{O}_2$ (considered as a real C*-algebra).
Theorem 6.6.

1. *Any two unital homomorphisms ϕ, ψ from $\mathcal{O}_\infty^{\mathbb{R}}$ into a real, unital, purely infinite, nuclear, c-simple C*-algebra A are approximately unitarily equivalent.*
2. *Let A be a real c-simple, separable, and nuclear C*-algebra. Then A is isomorphic to $A \otimes \mathcal{O}_\infty^{\mathbb{R}}$ if and only if A is purely infinite.*
3. $\mathcal{O}_\infty^{\mathbb{R}} \cong \bigotimes_{n=1}^{\infty} \mathcal{O}_\infty^{\mathbb{R}}$.

Proof. As in Section 7.2 of [39]. □

Corollary 6.7. *Let A and B be real, c-simple, separable, nuclear C*-algebras. If A or B is purely infinite, then $A \otimes B$ is purely infinite.*

Proof. From part (2) of Theorem 6.6. $\square$

7. Homomorphisms from $\mathcal{O}_\infty^{\mathbb{R}}$

The goal of this section is to prove the following theorem, analogous to Proposition 2.2.7 of [35].

Theorem 7.1. *Let D be a real unital purely infinite simple C*-algebra, and let $\phi, \psi \colon \mathcal{O}_\infty^{\mathbb{R}} \to D$ be unital homomorphisms. Then ϕ is asymptotically unitarily equivalent to ψ.*

The proof of Theorem 7.1 will be the same as that in [35]. However, there are a couple of background topics that need to be addressed in the context of real C*-algebras.

We begin with a discussion of approximately divisible real C*-algebras, following [6]. It is sufficient to consider only separable unital C*-algebras. Also, we skirt the general topic of completely noncommutative C*-algebras by taking into account Definition 2.6 of [6] and the subsequent comment.

Definition 7.2. A separable unital real C*-algebra A is *approximately divisible* if for all $x_1, x_2, \ldots, x_n \in A$ and $\varepsilon > 0$, there is a unital subalgebra B isomorphic to M_2, M_3, or $M_2 \oplus M_3$ such that $\|x_i y - y x_i\| < \varepsilon$ for all $i = 1, 2, \ldots, n$ and all y in the unit ball of B.

The following theorem is the real version of Corollary 2.1.6 of [35].

Lemma 7.3. *The tensor product $\mathcal{O}_\infty^{\mathbb{R}} \otimes D$ is approximately divisible for any real separable unital C*-algebra D. In particular, every c-simple, separable, nuclear, purely infinite, unital real C*-algebra is approximately divisible.*

Proof. Let $A = \mathcal{O}_\infty^{\mathbb{R}} \otimes D$. Using the isomorphism $\mathcal{O}_\infty^{\mathbb{R}} \cong \bigotimes_{n=1}^\infty \mathcal{O}_\infty^{\mathbb{R}}$ of Theorem 6.6 we obtain a sequence of mutually commuting unital homomophisms $\phi_n \colon \mathcal{O}_\infty^{\mathbb{R}} \to A$ such that $\|\phi_n(a)b - b\phi_n(a)\| \to 0$ for all $a \in \mathcal{O}_\infty^{\mathbb{R}}$ and all $b \in A$. Choose a unital map $\gamma \colon M_2 \oplus M_3 \to \mathcal{O}_\infty^{\mathbb{R}}$ and let $\psi_n = \phi_n \circ \gamma$. Then for large enough n, the subalgebra $B = \psi_n(M_2 \oplus M_3)$ works.

The second statement follows from part (2) of Theorem 6.6. $\square$

Lemma 7.4. *Let p and q be full projections in $M_\infty(A)$ where A is a real, separable, unital, approximately divisible C*-algebra. Then $p \sim q$ if and only if $[p] = [q]$ in $K_0(A)$.*

Proof. The proof is the same as the proof of (the first part of) Proposition 3.10 in [6] in complex case. That proof relies on a progression of results from Section 2 of [6] which can all be proven in the real case in the same way with one minor caveat. The proof of Proposition 2.1 of [6] (which in that paper was left to the reader) relies on the fact that a complex C*-algebra is spanned by its unitaries. While this fact is not true in general for real C*-algebras, it can

easily be shown to be true for finite dimensional real C*-algebras, which is the relevant case.
The proof of Proposition 3.10 in [6] also relies on Theorem 3.1.4 of [3], which is a ring-theoretic result stated in enough generality to apply to real C*-algebras. □

We remark that a more direct proof of Lemma 7.4 can be achieved in the special case (which is sufficient for our purposes) that $A = \mathcal{O}_\infty^{\mathbb{R}} \otimes D$ where D is separable and unital. In that case, we write $A = \bigotimes_{i=1}^\infty \mathcal{O}_\infty^{\mathbb{R}} \otimes D$ and let $A_n = \bigotimes_{i=1}^n \mathcal{O}_\infty^{\mathbb{R}} \otimes D$ be the unital subalgebra of A consisting of the first n factors in the tensor product. Then for each n and each k, it is easy to find a unital subalgebra $B_n \subset A_n' \cap A$ that is isomorphic to $M_{2^k} \oplus M_{3^k}$. Thus we achieve the result of Corollary 2.10 of [6] without having to recheck all the earlier material of Section 2 of [6] in the real case.

Lemma 7.5. *Let D be a unital real C*-algebra and let p, q be any two full projections in $\mathcal{K}^{\mathbb{R}} \otimes \mathcal{O}_\infty^{\mathbb{R}} \otimes D$. Then p is Murray-von Neumann equivalent to a subprojection of q. Furthermore, p is homotopic to q if and only if they represent the same class in $K_0(\mathcal{K}^{\mathbb{R}} \otimes \mathcal{O}_\infty^{\mathbb{R}} \otimes D) \cong K_0(D)$.*

Proof. With our Lemmas 7.3 and 7.4, as well as Theorem 3.6 of [11], the proof is the same as that of Lemma 2.1.8 of [35]. □

Proof of Theorem 7.1. With these preliminary definitions and results, the proof is the same as the proof of Proposition 2.2.7 of [35] including all of the lemmas and intermediate results in Sections 2.1 and 2.2 of [35]. We note that in [35], the proofs of Propositions 2.1.9 and 2.1.10 (having to do with exact stability of the relations defining $\mathcal{O}_m^{\mathbb{R}}$ and $E_m(\delta)$) are referred back to the proofs of parts (1) and (2) of Lemma 1.3 of [30]. The proof given there for part (2) produces isometries w_j that live in the real algebra $E_n(\delta)$. Therefore the homomorphisms $\phi_\delta^{(m)}$ constructed in the complex case restrict to homomorphisms between the real algebras. The same will be true for the analogous proof of part (1).
We also note that the proofs for the real versions of Lemmas 2.2.1 and 2.2.3 of [35] rely on our Theorem 5.2 which is only established for n even. Hence for real C*-algebras, we need to take m to be even in Lemma 2.2.1 and n to be even in Lemma 2.2.3. This is however, sufficient for all subsequent arguments. □

8. Asymptotic Morphisms

We appropriate the following definition of an asymptotic morphism from Section 25.1 of [4]. The other definitions in this section and the next are adapted from [35].

Definition 8.1. Let A and B be real C*-algebras. An *asymptotic morphism* ϕ from A to B is a family $\{\phi_t\}_{t\in[0,\infty)}$ of maps $\phi_t \colon A \to B$ such that

(1) the map $t \mapsto \phi_t(a)$ is continuous for each $a \in A$, and

(2) for all $a, b \in A$ and all $\lambda \in \mathbb{R}$, the following functions vanish in norm as $t \to \infty$:
   (a) $\phi_t(a+b) - \phi_t(a) - \phi_t(b)$,
   (b) $\phi_t(\lambda a) - \lambda\phi_t(a)$,
   (c) $\phi_t(ab) - \phi_t(a)\phi_t(b)$,
   (d) $\phi_t(a^*) - \phi_t(a)^*$.

We say that two asymptotic morphisms ϕ_t and ψ_t from A to B are *equivalent* if $\|\phi_t(a) - \psi_t(a)\|$ vanishes as $t \to \infty$ for all $a \in A$. We say that ϕ_t and ψ_t are *homotopic* if there is an asymptotic morphism Φ_t from A to $C([0,1], B)$ such that $\Phi_t(a)(0) = \phi_t(a)$ and $\Phi_t(a)(1) = \psi_t(a)$ for all $a \in A$. Equivalent asymptotic morphisms are homotopy equivalent (see Remark 25.1.2 of [4]).
We leave the easy proof of the next lemma to the reader.

Lemma 8.2. *If A and B are real C^*-algebras and ϕ is an asymptotic morphism from A to B, then there is an asymptotic morphism $\phi_{\mathbb{C}} \colon A_{\mathbb{C}} \to B_{\mathbb{C}}$ defined by $(\phi_{\mathbb{C}})_t(a + ib) = \phi_t(a) + i\phi_t(b)$.*

It can be proven, then, from the same result in the complex case, that for any asymptotic morphism ϕ we have $\limsup_{t\to\infty} \|\phi_t(a)\| \leq \|a\|$ for all $a \in A$ (see Proposition 25.1.3 of [4]). Thus, an asymptotic morphism $\{\phi_t\}$ gives rise to a unique homomorphism

$$\phi \colon A \to C_b([0,\infty), B)/C_0([0,\infty), B)$$

defined in the natural way; and every such homomorphism represents an asymptotic morphism, unique up to equivalence.

Lemma 8.3. *Let A be separable and nuclear. Every asymptotic morphism from A to B is equivalent to one that is completely positive and contractive. Furthermore, if ϕ and ψ are homopic completely positive and contractive asymptotic morphisms from A to B, then in fact there is a homotopy from ϕ to ψ consisting of completely positive and contractive asymptotic morphisms.*

Proof. Let ϕ be an asymptotic morphism from A to B. Then by Proposition 1.1.5 of [35], the complexification $\phi_{\mathbb{C}}$ is equivalent to an asymptotic morphism ψ that is completely positive and contractive. The map $\alpha \colon B_{\mathbb{C}} \to B$ defined by $\alpha(a + ib) = a$ is completely positive and contractive. Then the restriction of $\alpha \circ \psi$ to A is a completely positive, contractive asymptotic morphism from A to B and is equivalent to ϕ.
The same construction can be applied to a homotopy to prove the second statement. □

Definition 8.4. Let ϕ and ψ be asymptotic morphisms from A to $\mathcal{K}^{\mathbb{R}} \otimes D$. We define an asymptotic morphism $\phi \oplus \psi$, also from A to $\mathcal{K}^{\mathbb{R}} \otimes D$, as follows. Choose an isomorphism $\delta \colon M_2(\mathcal{K}^{\mathbb{R}}) \to \mathcal{K}^{\mathbb{R}}$ and define

$$(\phi \oplus \psi)_t(a) = (\delta \otimes 1_D) \begin{pmatrix} \phi_t(a) & 0 \\ 0 & \psi_t(a) \end{pmatrix} .$$

Lemma 8.5. *The asymptotic morphism $\phi \oplus \psi$ is well defined up to unitary equivalence, as well as up to homotopy.*

Proof. As in the complex case every automorphism of $\mathcal{K}^{\mathbb{R}}$ is implemented by a unitary in $\mathcal{U}(B(\mathcal{H}^{\mathbb{R}}))$ (the proof in, for example, Lemma V.6.1 of [20] works in the real case). Furthermore, by [36], $\mathcal{U}(B(\mathcal{H}^{\mathbb{R}}))$ is path connected. (In fact, by Theorem 3 of [29], it is contractible.) $\square$

Definition 8.6. Let $\phi \colon A \to B$ be an asymptotic morphism of real C*-algebras and let $p \in A$ be a projection. A *tail projection* for $\phi(p)$ is a continuous path p_t of projections for $t \in [0, \infty)$ such that $\lim_{t\to\infty} \|\phi_t(p) - p_t\| = 0$.
We say that ϕ is *full* if there is a full projection $p \in A$ such that $\phi(p)$ has a full tail projection.

Definition 8.7. Let A and B be real C*-algebras. Two asymptotic morphisms ϕ and ψ from A and B are *asymptotically unitarily equivalent* if there is a continuous family of unitary elements $u_t \in \widetilde{B}$ such that $\lim_{t\to\infty} \|u_t\phi_t(a)u_t^* - \psi_t(a)\| = 0$ for all $a \in A$.

With these definitions, all the results of Sections 1.2 and 1.3 of [35] hold for real C*-algebras.

Definition 8.8. Let A and D be real C*-algebras. An asymptotic morphism $\phi \colon A \to D$ has a *standard factorization* through $\mathcal{O}_\infty^{\mathbb{R}} \otimes A$ if there is an asymptotic morphism $\psi \colon \mathcal{O}_\infty^{\mathbb{R}} \otimes A \to D$ such that the asymptotic morphisms $\phi(a)$ and $\psi(1 \otimes a)$ (both from A to D) are asymptotically unitarily equivalent.
Similarly, ϕ is *asymptotically trivially factorizable* if there is an asymptotic morphism $\psi \colon \mathcal{O}_2^{\mathbb{R}} \otimes A \to D$ such that $\phi(a)$ and $\psi(1 \otimes a)$ are asymptotically unitarily equivalent.

Theorem 8.9 (Theorem 2.3.7 of [35]). *Let A be a separable, nuclear, unital, and c-simple. Let D_0 be a unital C*-algebra, and let $D = \mathcal{O}_\infty^{\mathbb{R}} \otimes D_0$. Then two full asymptotic morphisms from A to $\mathcal{K}^{\mathbb{R}} \otimes D$ are asymptotically unitarily equivalent if and only they are homotopic.*

Proof. The proof of Theorem 2.3.7 in [35] as well as the proofs of all of the preceeding lemmas in Section 2.3 of [35] can be proven in the real case with the same proofs, with some extra attention paid to the issue of connectedness of unitary groups.
In a few places Phillips uses the fact that the unitary group of $\mathcal{O}_2$ is connected. It is also true that $\mathcal{O}_2^{\mathbb{R}}$ is connected since $K_1(\mathcal{O}_2^{\mathbb{R}}) \cong 0$. However, on page 85 of [35], Phillips also uses the fact that the unitary group of a corner algebra of $\mathcal{O}_\infty$ is connected. The corresponding statement in the real case is not true since $K_1(\mathcal{O}_\infty^{\mathbb{R}}) \cong \mathbb{Z}_2$. We will show how to adjust the proof so that it works in the real case.
At this point in the proof we are (using Phillips' notation) trying to find a path of partial isometries from $w_n + f_{n+2}$ to $v_{n+1} + w_{n+1}$ (these are partial isometries from $f_{n+1} + f_{n+2}$ to $f_{n+2} + e$). If the unitaries $(w_n + f_{n+2})^*(v_{n+1} + w_{n+1})$ and $f_{n+1} + f_{n+2}$ are not in the same connected component of $(f_{n+1} +$

$f_{n+2})\mathcal{O}_\infty^{\mathbb{R}}(f_{n+1} + f_{n+2})$, then this can be changed by by multiplying w_{n+1} on the right by a suitable unitary in $f_{n+2}\mathcal{O}_\infty^{\mathbb{R}} f_{n+2}$. Thus by re-choosing the w_n's inductively, we can be sure that there is an appropriate path of partial isometries at each step. □

9. Groups of Asymptotic Morphisms

Definition 9.1. Let A be a real, separable, nuclear, unital, c-simple C*-algebra and let D be unital. We define $E_A(D)$ to be the the set of homotopy classes of full asymptotic morphisms from A to $\mathcal{K}^{\mathbb{R}} \otimes \mathcal{O}_\infty^{\mathbb{R}} \otimes D$. That is,

$$E_A(D) = [[A, \mathcal{K}^{\mathbb{R}} \otimes \mathcal{O}_\infty^{\mathbb{R}} \otimes D]]_+ \ .$$

More generally, for D unital or not, we define

$$\widetilde{E}_A(D) = \ker\left(E_A(D^+) \to E_A(\mathbb{R})\right) \ .$$

Proposition 9.2. *Let A be real, separable, nuclear, unital, and c-simple. Then $\widetilde{E}_A(-)$ is a functor from the category of separable real C*-algebras with homotopy classes of asymptotic morphisms to abelian groups, that is homotopy invariant, stable, half exact, and split exact.*

Proof. In the complex case, these results are proven in Section 3.1 of [35]. In the real case, they are proven the same way. Note that split exactness follows from homotopy invariance and half exactness by Corollary 3.5 of [12]. □

Lemma 9.3. *Let A and B be C*-algebras (real or complex). Let $\phi\colon A \to B$ be an asymptotic morphism. If p, q are projections in A with $p \leq q$, then there are tail projections p_t (for $\phi(p)$) and q_t (for $\phi(q)$) in B with $p_t \leq q_t$ for all t.*

Proof. Let $\widetilde{p_t}$ and q_t be arbitrary tail projections corresponding to $\phi(p)$ and $\phi(q)$, respectively (these exist as in Remark 1.2.2 of [35]). One can easily show that

$$\lim_{t\to\infty} \|\widetilde{p_t} - q_t \widetilde{p}_t q_t\| = 0 \ .$$

For each t, the element $q_t \widetilde{p}_t q_t$ is a self adjoint and asymptotically idempotent element of $q_t B q_t$. Therefore, there is a continuous path of projections $p_t \in q_t B q_t$ such that

$$\lim_{t\to\infty} \|q_t \widetilde{p}_t q_t - p_t\| = 0 \ .$$

The tail projections p_t and q_t have the desired properties. □

We note that if A and D are complex C*-algebras there are two groups one might consider: we let $\widetilde{E}^{\mathbb{C}}_A(D)$ denote the functor of [35] that is based on complex asymptotic morphisms. On the other hand, according to the notation established in Definition 9.1, the asymptotic morphisms comprising $\widetilde{E}_A(D)$ are only required to be asymptotically linear over $\mathbb{R}$ (thus the complex structures of A and D are forgotten). The following theorem relates the two groups.

Proposition 9.4. *If A is a real C*-algebra satisfying the hypotheses of Definition 9.1 and D is a complex unital C*-algebra, then there is a isomorphism*

$$\widetilde{E}_A(D) \cong \widetilde{E}^{\mathbb{C}}_{A_{\mathbb{C}}}(D)$$

which is natural with respect to complex homomorphisms.

Proof. We show that for a real unital C*-algebra A and a complex C*-algebra B, there is a bijection

$$[[A,B]]_+ \cong [[A_{\mathbb{C}},B]]^{\mathbb{C}}_+$$

of equivalence classes of full asymptotic morphisms.
Given a complex asymptotic morphism ϕ from $A_{\mathbb{C}}$ to B, then we let $\Gamma(\phi)$ be the restriction of ϕ to A. If ϕ is full, then we claim that $\Gamma(\phi)$ is full. Since ϕ is full, there is a full projection $p \in A_{\mathbb{C}}$ and a full tail projection $r_t \in B$ such that $\|\phi_t(p) - r_t\| \to 0$. Applying Lemma 9.3 to $p \leq 1$ we obtain tail projections p_t and q_t for p and 1, respectively, such that $p_t \leq q_t$ for all t. Since the tail projections p_t and r_t are asymptotically equal, it must be that p_t are full projections. It follows that q_t are also full projections; and since they are tail projections for the full projection 1_A in A, it follows that $\Gamma(\phi)$ is full.
Given a real asymptotic morphism ψ from A to B, then

$$\Delta(\psi)_t(a+ib) = \psi_t(a) + i\psi_t(b)$$

defines a complex asymptotic morphism from $A_{\mathbb{C}}$ to B. Suppose that ψ is full. Let p be a full projection in A and let $q_t \in B$ be a full tail projection for $\psi(p)$. Then clearly p is full in $A_{\mathbb{C}}$ and q_t is a full tail projection for $\Delta(\psi(p))$. Hence $\Delta(\psi)$ is full.
It is immediate that Δ is a two-sided inverse for Γ. Furthermore, in the case that B is stable, it is easy to see that Γ preserves the semigroup operation of Definition 8.4. Therefore, under the hypotheses of the theorem, there is an group isomorphism $\widetilde{E}_A(D) \cong \widetilde{E}^{\mathbb{C}}_{A_{\mathbb{C}}}(D)$. □

Proposition 9.5. *Let A be a separable, nuclear, c-simple unital, real C*-algebra. Let B be a separable real C*-algebra. Then there is a natural isomophism $KK(A,B) \cong \widetilde{E}_A(B)$.*

The proof in the complex case takes place in Section 3.2 of [35]. Rather than reconstructing all of the arguments in the real case, we give a proof that uses results from [12] to reduce the real case to the complex case.

Proof of Proposition 9.5. Fix A satisfying the hypotheses above. Let e be a rank one projection in $\mathcal{K}^{\mathbb{R}}$ and let $\iota_A \colon A \to \mathcal{K}^{\mathbb{R}} \otimes \mathcal{O}^{\mathbb{R}}_\infty \otimes A$ be the homomorphism defined by $\iota_A(a) = e \otimes 1 \otimes a$. Let $[[\iota_A]]$ be the induced element of $\widetilde{E}_A(A)$. Let $[1_A] \in KK(A,A)$ be the class of the identity. By Corollary 3.3 of [12], there is a unique natural transformation α from $KK(A,-)$ to $\widetilde{E}_A(-)$ such that $\alpha([1_A]) = [[\iota_A]]$. We will show that

$$\alpha \colon KK(A,B) \to \widetilde{E}_A(B)$$

is an isomorphism for all separable real C*-algebras B. By Theorem 3.9 of [12] it suffices to show that α is an isomorphism when B is complex.
In the complex case we have the element $[1_{A_\mathbb{C}}] \in KK^\mathbb{C}(A_\mathbb{C}, A_\mathbb{C})$ and the homomorphism

$$(\iota_A)_\mathbb{C} \colon A_\mathbb{C} \to \mathcal{K}^\mathbb{R} \otimes \mathcal{O}_\infty^\mathbb{R} \otimes A_\mathbb{C} \cong \mathcal{K} \otimes \mathcal{O}_\infty \otimes A_\mathbb{C} \ .$$

By Theorem 3.7 of [24] there is a unique natural transformation $\alpha^\mathbb{C}$ from $KK^\mathbb{C}(A_\mathbb{C}, -)$ to $\widetilde{E}^\mathbb{C}_{A_\mathbb{C}}(-)$ such that $\alpha^\mathbb{C}([1_{A_\mathbb{C}}]) = [[\iota^\mathbb{C}_A]]$. A special case of Theorem 3.2.6 of [35] shows that $\alpha^\mathbb{C}$ is an isomorphism for all separable complex C*-algebras B.
Consider the following diagram for a complex C*-algebra B,

$$\begin{tikzcd}
KK^\mathbb{C}(A_\mathbb{C}, B) \arrow[r, "\alpha^\mathbb{C}"] \arrow[d, "\nu"] & \widetilde{E}^\mathbb{C}_{A_\mathbb{C}}(B) \arrow[d, "\mu"] \\
KK(A, B) \arrow[r, "\alpha"] & \widetilde{E}_A(B)
\end{tikzcd}$$

where μ is the isomorphism of Proposition 9.4 above and ν is the isomorphism of Lemma 4.3 of [9]. To complete the proof, we only need to show that the diagram commutes. Since the homomorphism $\alpha^\mathbb{C}$ is characterized by the value of $\alpha^\mathbb{C}([1_{A_\mathbb{C}}])$ it suffices to consider the case $B = A_\mathbb{C}$ as in the diagram

$$\begin{tikzcd}
KK^\mathbb{C}(A_\mathbb{C}, A_\mathbb{C}) \arrow[r, "\alpha^\mathbb{C}"] \arrow[d, "\nu"] & \widetilde{E}^\mathbb{C}_{A_\mathbb{C}}(A_\mathbb{C}) \arrow[d, "\mu"] \\
KK(A, A_\mathbb{C}) \arrow[r, "\alpha"] & \widetilde{E}_A(A_\mathbb{C})
\end{tikzcd}$$

and to show that $\alpha^\mathbb{C}([1_{A_\mathbb{C}}]) = (\mu^{-1} \circ \alpha \circ \nu)([1_{A_\mathbb{C}}])$ or, equivalently, $(\mu \circ \alpha^\mathbb{C})([1_{A_\mathbb{C}}]) = (\alpha \circ \nu)([1_{A_\mathbb{C}}])$.
From the construction of ν in the proof of Lemma 4.3 of [9] it is apparent that $\nu([1_{A_\mathbb{C}}]) = [c_A] = (c_A)_*([1_A])$ where $c_A \colon A \to A_\mathbb{C}$ is the real C*-algebra homomorphism induced by the unital inclusion $c \colon \mathbb{R} \hookrightarrow \mathbb{C}$. Thus

$$(\alpha \circ \nu)([1_{A_\mathbb{C}}]) = \alpha((c_A)_*([1_A])) = (c_A)_*(\alpha([1_A])) = (c_A)_*([[\iota_A]]) = [[c_A]] \ .$$

On the other hand, it is apparent from the construction of μ in the proof of Proposition 9.4 above that $\mu([[\iota^\mathbb{C}_A]]) = [[c_A]]$. Thus

$$(\mu \circ \alpha^\mathbb{C})([1_{A_\mathbb{C}}]) = \mu(\alpha^\mathbb{C}([1_{A_\mathbb{C}}])) = \mu([[\iota^\mathbb{C}_A]]) = [[c_A]] \ .$$

□

The following is the real version of Theorems 4.1.1 and 4.1.3 of [35].

Theorem 9.6. *Let A be a real separable unital nuclear c-simple C*-algebra and let D be a separable unital C*-algebra. Then the following groups are naturally isomorphic, via the obvious maps.*

(1) $KK(A, D)$

(2) *The set of asymptotic unitary equivalence classes of full homomorphisms from A to $\mathcal{K}^{\mathbb{R}} \otimes \mathcal{O}_\infty^{\mathbb{R}} \otimes D$.*
(3) *The set of homotopy classes of full homomorphisms from A to $\mathcal{K}^{\mathbb{R}} \otimes \mathcal{O}_\infty^{\mathbb{R}} \otimes D$.*
(4) *The set of asymptotic unitary equivalence classes of full homomorphisms from $\mathcal{K}^{\mathbb{R}} \otimes \mathcal{O}_\infty^{\mathbb{R}} \otimes A$ to $\mathcal{K}^{\mathbb{R}} \otimes \mathcal{O}_\infty^{\mathbb{R}} \otimes D$.*
(5) *The set of homotopy classes of full homomorphisms from $\mathcal{K}^{\mathbb{R}} \otimes \mathcal{O}_\infty^{\mathbb{R}} \otimes A$ to $\mathcal{K}^{\mathbb{R}} \otimes \mathcal{O}_\infty^{\mathbb{R}} \otimes D$.*

Proof. The proof of the isomorphism of (1), (2), and (3) is the same as the proof of Theorem 4.1.1 in [35]. The proof of the isomorphism of (1), (4), and (5) relies on Lemma 9.7 below (which is the real version of Lemma 4.1.2 of [35]). Once that lemma is established, the proof of the isomorphism of (1), (4), and (5) is the same as the proof of Theorem 4.1.3 of [35]. □

Lemma 9.7. *Let A be separable, nuclear, unital, and c-simple; let D_0 be separable and unital; and let $D = \mathcal{O}_\infty^{\mathbb{R}} \otimes D_0$. Let $t \mapsto \phi_t$, for $t \in [0, \infty)$, be a continuous path of full homomorphisms from $\mathcal{K}^{\mathbb{R}} \otimes A$ to $\mathcal{K}^{\mathbb{R}} \otimes D$, and let $\psi \colon \mathcal{K}^{\mathbb{R}} \otimes A \to \mathcal{K}^{\mathbb{R}} \otimes D$ be a full homomorphism. Assume that $[\phi_0] = [\psi]$ in $KK_0(A, D)$. Then there is an asymptotic unitary equivalence from ϕ to ψ that consists of unitaries in $\mathcal{U}_0((\mathcal{K}^{\mathbb{R}} \otimes D)^+)$.*

The proof will be essentially the same as the proof of Lemma 4.1.2 of [35]. However, that proof has an error in the third paragraph. The element w_t introduced there does not seem to be a unitary as purported. Also, the order of the product in the definition of z_t seems wrong. Fortunately, there is an easy fix and most of the proof can be left as it is. For clarity and completeness we present the entire proof, but the only significant difference is the unitary w in the third paragraph and following. In places where the proof does not change (such as the entire first and second paragraphs, and most of the final paragraph), we use exactly the same language as in [35], except for the references to previous results in the present paper.

Proof of Lemma 9.7. Let $\{e_{ij}\}$ be a system of matrix units for $\mathcal{K}^{\mathbb{R}}$. Identify A with the subalgebra $e_{11} \otimes A$ of $\mathcal{K}^{\mathbb{R}} \otimes A$. Define $\psi_t^{(0)}$ and $\psi^{(0)}$ to be the restrictions of ϕ_t and ψ to A. Then $[\phi_0^{(0)}] = [\psi^{(0)}]$ in $KK_0(A, D)$. It follows from (the equivalence of (1) and (3) of) Theorem 9.6 that $\phi_0^{(0)}$ is homotopic to $\psi^{(0)}$. Therefore $\phi_0^{(0)}$ and $\psi^{(0)}$ are homotopic as asymptotic morphisms, and Theorem 8.9 provides an asymptotic unitary equivalence $t \mapsto u_t$ in $\mathcal{U}((\mathcal{K}^{\mathbb{R}} \otimes D)^+)$ from $\phi^{(0)}$ to $\psi^{(0)}$. Let $c \in \mathcal{U}((\mathcal{K}^{\mathbb{R}} \otimes D)^+)$ be a unitary with $c\psi^{(0)}(1) = \psi^{(0)}(1)c = \psi^{(0)}(1)$ and such that c is homotopic to u_0^{-1}. Then c commutes with every $\psi^{(0)}(a)$. Replacing u_t by cu_t, we obtain an asymptotic unitary equivalence, which we again call $t \mapsto u_t$, from $\phi^{(0)}$ to $\psi^{(0)}$ which is in $\mathcal{U}_0(\mathcal{K}^{\mathbb{R}} \otimes D)^+)$.

Define $\overline{e}_{ij} = e_{ij} \otimes 1$. Then in particular $u_t\phi_t(\overline{e}_{11})u_t^* \to \psi(\overline{e}_{11})$ as $t \to \infty$. Therefore there is a continuous path $t \to z_t^{(1)} \in \mathcal{U}_0((\mathcal{K}^{\mathbb{R}} \otimes D)^+)$ such that $z_t^{(1)} \to$

1 and $z_t^{(1)}u_t\phi_t(\overline{e}_{11})u_t^*(z_t^{(1)})^* = \psi(\overline{e}_{11})$ for all t. We still have $z_t^{(1)}u_t\phi_t(e_{11}\otimes a)u_t^*(z_t^{(1)})^* \to \psi(e_{11}\otimes a)$ for $a\in A$.

For convenience, set $f_{ijt}^{(1)} = z_t^{(1)}u_t\phi_t(\overline{e}_{ij})u_t^*(z_t^{(1)})^*$ and set $g_{ij} = \psi(\overline{e}_{ij})$. For each fixed t, the $f_{ijt}^{(1)}$ are matrix units for $\mathcal{K}^{\mathbb{R}}$ as are the g_{ij}. Also, we have $f_{11t}^{(1)} = g_{11}$. The projections $f_{11t}^{(1)}+f_{22t}^{(1)}$ and $g_{11}+g_{22}$ represent the same element of $K_0(D)$ so (using Lemma 7.4) there is a continuous path of partial isometries $x_t^{(1)}$ in $\mathcal{K}^{\mathbb{R}}\otimes D$ such that $x_t^{(1)}(x_t^{(1)})^* = 1-g_{11}-g_{22}$ and $(x_t^{(1)})^*x_t^{(1)} = 1-f_{11t}^{(1)}-f_{22t}^{(1)}$. Set $w_t^{(1)} = g_{11}+g_{21}f_{12t}^{(1)}+x_t^{(1)} \in \mathcal{U}((\mathcal{K}^{\mathbb{R}}\otimes D)^+)$. Then one checks that $w_t^{(1)}f_{ijt}^{(1)}(w_t^{(1)})^* = g_{ij}$ for all t and for $1\le i,j\le 2$. Choose $c^{(1)}\in\mathcal{U}((\mathcal{K}^{\mathbb{R}}\otimes D)^+)$ with

$$c^{(1)}(g_{11}+g_{22}) = (g_{11}+g_{22})c^{(1)} = g_{11}+g_{22}$$

$$\text{and}\qquad c^{(1)}w_1^{(1)} \in \mathcal{U}_0((\mathcal{K}^{\mathbb{R}}\otimes D)^+)\ .$$

Set $z^{(2)} = c^{(1)}w_t^{(1)}$ for $t\ge 1$ and extend $z_t^{(2)}$ continuously over $[0,1]$ through unitaries so that $z_0^{(2)} = 1$, retaining the property that $z_t^{(2)}g_{11} = g_{11}z_t^{(2)} = g_{11}$. This gives $z_t^{(2)} = 1$ for $t=0$, $z_t^{(2)}g_{11} = g_{11}z_t^{(2)} = g_{11}$ for all t, and

$$z_t^{(2)}z_t^{(1)}u_t\phi_t(\overline{e}_{ij})u_t^*(z_t^{(1)})^*(z_t^{(2)})^* = \psi(\overline{e}_{ij})$$

for $t\ge 1$ and $1\le i,j\le 2$.

Set $p^{(m)} = \sum_{k=1}^m g_{kk}$ for all postive integers m. For the induction step, assume that we have continuous paths unitaries $z_t^{(1)}, z_t^{(2)}, \dots, z_t^{(n)}$ defined on $[0,\infty)$ such that

- $z_t^{(n)} = 1$ for $0\le t\le n-2$,
- $z_t^{(n)}p^{(n-1)} = p^{(n-1)}z_t^{(n)} = p^{(n-1)}$ for all $t\ge 0$,
- $z_t^{(n)}\cdots z_t^{(1)}u_t\phi_t(\overline{e}_{ij})u_t^*(z_t^{(1)})^*\cdots(z_t^{(n)})^* = \psi(\overline{e}_{ij})$ for $t\ge n-1$ and $1\le i,j\le n$.

We must construct a $z_t^{(n+1)}$ with the corresponding properties. Initially, working with $t\in[n,\infty)$, set

$$f_{ijt}^{(n)} = z_t^{(n)}\dots z_t^{(1)}u_t\phi_t(\overline{e}_{ij})u_t^*(z_t^{(1)})^*\dots(z_t^{(n)})^*$$

and let $x_t^{(n)}$ be a continuous path of partial isometries such that $x_t^{(n)}(x_t^{(n)})^* = 1-\sum_{k=1}^n g_{kk} = 1-p^{(n)}$ and $(x_t^{(n)})^*x_t^{(n)} = 1-\sum_{k=1}^n f_{kkt}$. Set $w_t^{(n)} = p^{(n)} + g_{(n+1)\,1}f_{1\,(n+1)\,t}^{(n)} + x_t^{(n)}$. This continuous path of unitaries satisfies $w_t^{(n)}p^{(n)} = p^{(n)}w_t^{(n)} = p^{(n)}$ and $w_t^{(n)}f_{ijt}^{(n)}(w_t^{(n)})^* = g_{ij}$ for all $t\ge n$ and all $1\le i,j\le n+1$. As above, we can find a unitary $c^{(n)}$ such that $z_t^{(n+1)} = c^{(n)}w_t^{(n)}$ is in the connected component of the identity and $c^{(n)}p^{(n+1)} = p^{(n+1)}c^{(n)} = c^{(n)}$. Then extend $z_t^{(n+1)}$ so that it is defined for all $t\ge 0$ and $z_t^{(n+1)} = 1$ for $0\le t\le n-1$. Check that this $z^{(n+1)}$ satisfies the corresponding properties listed above.

Now define

$$z_t = \left(\lim_{n\to\infty} z_t^{(n)}\cdots z_t^{(2)}z_t^{(1)}\right)u_t\ .$$

In a neighborhood of each t, all but finitely many of the $z_t^{(k)}$ are equal to 1, so this limit of products yields a continuous path of unitaries of $\mathcal{U}_0((\mathcal{K}^{\mathbb{R}} \otimes D)^+)$. Moreover, $z_t\phi_t(\overline{e}_{ij})z_t^* = \psi(\overline{e}_{ij})$ whenever $t \geq i, j$, so that $\lim_{t\to\infty} z_t\phi_t(\overline{e}_{ij})z_t^* = \psi(\overline{e}_{ij})$ for all i and j, while

$$\lim_{t\to\infty} z_t\phi_t(e_{11} \otimes a)z_t^* = \lim_{t\to\infty} z_t^{(1)}u_t\phi_t(e_{11} \otimes a)u_t^*(z_t^{(1)})^* = \psi(e_{11} \otimes a)$$

for all $a \in A$. Since the $\overline{e}_{ij}$ and $e_{11} \otimes a$ generate $\mathcal{K}^{\mathbb{R}} \otimes A$, this shows that $t \mapsto z_t$ is an asymptotic unitary equivalence. □

10. Classification of Real Kirchberg Algebras

We now present our main classification theorems for real Kirchberg algebras, analogous to the results of Section 4.2 of [35].

Theorem 10.1. *Let A and B be unital separable nuclear purely infinite c-simple C^*-algebras.*

1. *Let η be an invertible element in $KK(A, B)$. Then there is an isomorphism $\phi\colon \mathcal{K}^{\mathbb{R}} \otimes A \to \mathcal{K}^{\mathbb{R}} \otimes B$ such that $[\phi] = \eta$.*
2. *Let η be an invertible element in $KK(A, B)$ such that $[1_A] \times \eta = [1_B]$. Then there is an isomorphism $\phi\colon A \to B$ such that $[\phi] = \eta$.*

Proof. As in the proofs of Theorem 4.2.1 and Corollary 4.2.2 of [35]. □

Theorem 10.2. *Let A and B be unital separable nuclear purely infinite c-simple C^*-algebras that satisfy the universal coefficient theorem.*

1. *The stable C^*-algebras $\mathcal{K}^{\mathbb{R}} \otimes A$ and $\mathcal{K}^{\mathbb{R}} \otimes B$ are isomorphic if and only if $K^{CRT}(A)$ and $K^{CRT}(B)$ are isomorphic CRT-modules.*
2. *The unital C^*-algebras A and B are isomorphic if and only if the invariants $(K^{CRT}(A), [1_A])$ and $(K^{CRT}(B), [1_B])$ are isomorphic.*
3. *The stable C^*-algebras $\mathcal{K}^{\mathbb{R}} \otimes A$ and $\mathcal{K}^{\mathbb{R}} \otimes B$ are isomorphic if and only if $K^{CR}(A)$ and $K^{CR}(B)$ are isomorphic CR-modules.*
4. *The unital C^*-algebras A and B are isomorphic if and only if the invariants $(K^{CR}(A), [1_A])$ and $(K^{CR}(B), [1_B])$ are isomorphic.*

Proof. Parts (1) and (2) are proven as in the proof of Theorem 4.2.4 of [35], using Proposition 2.3. Parts (3) and (4) then follow by Proposition 2.5. □

Corollary 10.3.

1. *The functor $A \mapsto K^{CRT}(A)$ is a bijection from isomorphism classes of real stable separable nuclear purely infinite c-simple C^*-algebras that satisfy the universal coefficient theorem to isomorphism classes of countable acyclic CRT-modules.*
2. *The functor $A \mapsto (K^{CRT}(A), [1_A])$ is a bijection from isomorphism classes of real unital separable nuclear purely infinite c-simple C^*-algebras that satisfy the universal coefficient theorem to isomorphism classes of countable acyclic CRT-modules M with distinguished element $m \in M_0^O$.*

Proof. Combine Theorem 10.2 above with Theorem 1 of [10]. □

Definition 10.4.

(1) Let A be a complex C*-algebra. A *real form* of A is a real C*-algebra B such that $B_{\mathbb{C}} \cong A$.
(2) Let $G_* = (G_0, G_1)$ be a pair of groups. A *real form* of G_* is an acyclic *CRT*-module such that $M^U_* \cong G_*$.
(3) Let $G_* = (G_0, G_1, g)$ be a pair of groups with a distinguished element $g \in G_0$. A *real form* of G_* is a pair (M, m) where M is an acyclic *CRT*-module and m is a distinguished element of M^O_0 such that $(M^U_0, M^U_1, c(m)) \cong (G_0, G_1, g)$.

Corollary 10.5. *Let A be a complex unital separable nuclear purely infinite simple C*-algebra satisfying the universal coefficient theorem.*

(1) *The functor $B \mapsto K^{\scriptscriptstyle CRT}(B)$ is a bijection from isomorphism classes of real forms of $\mathcal{K}^{\mathbb{R}} \otimes A$ to isomorphism classes of real forms of $K_*(A)$.*
(2) *The functor $B \mapsto (K^{\scriptscriptstyle CRT}(B), [1_B])$ is a bijection from isomorphism classes of real forms of A to isomorphism classes of real forms of $(K_*(A), [1_A])$.*

Proof. If B is a real form of $\mathcal{K}^{\mathbb{R}} \otimes A$, then B is necessarily stable separable nuclear purely infinite and c-simple. Then $KU_*(B) = K_*(B_{\mathbb{C}}) \cong K_*(A)$, so $K^{\scriptscriptstyle CRT}(B)$ is a real form of $K_*(A)$. Conversely, suppose M is a real form of $K_*(A)$. Since $K_*(A)$ is countable, the exact sequences of Section 2.3 of [14] imply that M is countable. Then by Corollary 10.3, $M \cong K^{\scriptscriptstyle CRT}(B)$ for some real stable separable nuclear purely infinite c-simple C*-algebra satisfying the universal coefficient theorem. Since $K_*(B_{\mathbb{C}}) \cong K_*(A)$, it follows from Theorem 4.2.4 of [35] that $B_{\mathbb{C}} \cong A$ hence B is a real form of A. Furthermore, Corollary 10.3 also implies that B is unique up to isomorphism.

In the unital case, suppose that B is a real form of A. As there is a isomorphism $B_{\mathbb{C}} \cong A$ and the unit of $B_{\mathbb{C}}$ is $c(1_B)$, there is an isomorphism $\phi\colon KU_*(B) \to K_*(A)$ such that $\phi_*(c([1_B])) = [1_A]$. Thus $(K^{\scriptscriptstyle CRT}(B), [1_B])$ is a real form of $(K_*(A), [1_A])$. Conversely, if (M, m) is a real form of $K_*(A)$, then let B be a real unital separable nuclear purely infinite c-simple C*-algebra such that $(K^{\scriptscriptstyle CRT}(B), [1_B]) \cong (M, m)$. Again, Theorem 4.2.4 of [35], implies that B is a real form of A. □

11. Real Forms of Cuntz Algebras

In this section, we use Corollary 10.5 to give a complete description of all real forms of the complex Cuntz algebras $\mathcal{O}_n$ for $n \in \{2, \ldots, \infty\}$. The natural real form of $\mathcal{O}_n$ is the real Cuntz algebra $\mathcal{O}^{\mathbb{R}}_n$, but we will find that there are others when n is odd. For reference, we show in Table 1 the groups making up $K^{\scriptscriptstyle CRT}(\mathcal{O}^{\mathbb{R}}_n)$. In the case of $n = \infty$ this arises from the isomorphism $K^{\scriptscriptstyle CRT}(\mathbb{R}) \cong K^{\scriptscriptstyle CRT}(\mathcal{O}^{\mathbb{R}}_\infty)$ of Proposition 2.2; while for finite n, these *CRT*-modules were computed in Section 5.1 of [8].

Table 1

$K^{CRT}(\mathcal{O}_\infty^{\mathbb{R}})$

	0	1	2	3	4	5	6	7	8
KO_*	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
KU_*	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
KT_*	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}$

$K^{CRT}(\mathcal{O}_n^{\mathbb{R}})$ for n even

	0	1	2	3	4	5	6	7	8
KO_*	$\mathbb{Z}_{n-1}$	0	0	0	$\mathbb{Z}_{n-1}$	0	0	0	$\mathbb{Z}_{n-1}$
KU_*	$\mathbb{Z}_{n-1}$	0	$\mathbb{Z}_{n-1}$	0	$\mathbb{Z}_{n-1}$	0	$\mathbb{Z}_{n-1}$	0	$\mathbb{Z}_{n-1}$
KT_*	$\mathbb{Z}_{n-1}$	0	0	$\mathbb{Z}_{n-1}$	$\mathbb{Z}_{n-1}$	0	0	$\mathbb{Z}_{n-1}$	$\mathbb{Z}_{n-1}$

$K^{CRT}(\mathcal{O}_n^{\mathbb{R}})$ for $n-1 \equiv 2 \pmod 4$

	0	1	2	3	4	5	6	7	8
KO_*	$\mathbb{Z}_{n-1}$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2$	$\mathbb{Z}_{n-1}$	0	0	0	$\mathbb{Z}_{n-1}$
KU_*	$\mathbb{Z}_{n-1}$	0	$\mathbb{Z}_{n-1}$	0	$\mathbb{Z}_{n-1}$	0	$\mathbb{Z}_{n-1}$	0	$\mathbb{Z}_{n-1}$
KT_*	$\mathbb{Z}_{n-1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{n-1}$	$\mathbb{Z}_{n-1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{n-1}$	$\mathbb{Z}_{n-1}$

$K^{CRT}(\mathcal{O}_n^{\mathbb{R}})$ for $n-1 \equiv 0 \pmod 4$

	0	1	2	3	4	5	6	7	8
KO_*	$\mathbb{Z}_{n-1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_{n-1}$	0	0	0	$\mathbb{Z}_{n-1}$
KU_*	$\mathbb{Z}_{n-1}$	0	$\mathbb{Z}_{n-1}$	0	$\mathbb{Z}_{n-1}$	0	$\mathbb{Z}_{n-1}$	0	$\mathbb{Z}_{n-1}$
KT_*	$\mathbb{Z}_{n-1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{n-1}$	$\mathbb{Z}_{n-1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{n-1}$	$\mathbb{Z}_{n-1}$

Theorem 11.1. (1) *For n even or $n = \infty$, there is up to isomorphism only one real form of $\mathcal{O}_n$: the real Cuntz algebra $\mathcal{O}_n^{\mathbb{R}}$.*

(2) *For n odd, there are up to isomorphism two real forms of $\mathcal{O}_n$: the real Cuntz algebra $\mathcal{O}_n^{\mathbb{R}}$ and an exotic real form $\mathcal{E}_n$.*

Proof. First check that for odd integers n, $n \geq 3$, the groups and operations shown in Table 2 form an acyclic *CRT*-module. Using Corollary 10.3 (that is, Theorem 1 of [10]), let $\mathcal{E}_n$ be the unique real unital separable nuclear c-simple purely infinite C*-algebra satisfying the universal coefficient theorem with united K-theory as shown in Table 2 and such that $[1_{\mathcal{E}_n}]$ corresponds to a generator of the group in the real part in degree 0.

By Corollary 10.5, the problem of classifying real forms of $\mathcal{O}_n$ (for $n \in \{2, 3, \ldots, \infty\}$) reduces to the algebraic problem of classifying real forms of $(K_*(\mathcal{O}_n), [1_{\mathcal{O}_n}])$. Suppose that (M, m) is such a real form. For n even (respectively $n = \infty$) we will show that (M, m) is isomorphic to $(K^{CRT}(\mathcal{O}_n^{\mathbb{R}}), [1_{\mathcal{O}_n^{\mathbb{R}}}])$ (respectively $(K^{CRT}(\mathcal{O}_\infty^{\mathbb{R}}), [1_{\mathcal{O}_\infty^{\mathbb{R}}}])$). For n odd we will show that (M, m) is either isomorphic to $(K^{CRT}(\mathcal{O}_n^{\mathbb{R}}), [1_{\mathcal{O}_n^{\mathbb{R}}}])$ or to $(K^{CRT}(\mathcal{E}_n), [1_{\mathcal{E}_n}])$. Furthermore, by

TABLE 2. $K^{CRT}(\mathcal{E}_n)$, for n odd and $n \geq 3$.

	0	1	2	3	4	5	6	7	8
KO_*	$\mathbb{Z}_{2(n-1)}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}_{(n-1)/2}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{2(n-1)}$
KU_*	$\mathbb{Z}_{n-1}$	0	$\mathbb{Z}_{n-1}$	0	$\mathbb{Z}_{n-1}$	0	$\mathbb{Z}_{n-1}$	0	$\mathbb{Z}_{n-1}$
KT_*	$\mathbb{Z}_{n-1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{n-1}$	$\mathbb{Z}_{n-1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{n-1}$	$\mathbb{Z}_{n-1}$
c_*	1	0	0	0	2	0	$\frac{n-1}{2}$	0	1
r_*	2	0	1	0	1	0	0	0	2
ε_*	1	1	0	0	2	0	1	$\frac{n-1}{2}$	1
ζ_*	1	0	$\frac{n-1}{2}$	0	1	0	$\frac{n-1}{2}$	0	1
$(\psi_U)_*$	1	0	-1	0	1	0	-1	0	1
$(\psi_T)_*$	1	1	1	-1	1	1	1	-1	1
γ_*	1	0	1	0	1	0	1	0	1
τ_*	1	1	0	1	0	0	1	2	1

Proposition 2.5 it suffices to restrict our attention to the *CR*-module consisting of the real and complex parts of M.

Since (M, m) is a real form of $(K_*(\mathcal{O}_n), [1_{\mathcal{O}_n}])$ we know that $M_0^U \cong \mathbb{Z}_{n-1}$ (respectively $M_0^U \cong \mathbb{Z}$ when $n = \infty$), $M_1^U = 0$, and $m \in M_0^O$. We further suppose that $c_0(m) \in M_0^U$ is a generator (corresponding to the class of the unit in $K_0(\mathcal{O}_n)$).

We will compute the real part of M (and the behavior of the operations $\eta_O, \xi, r, c, \psi_U$) using the long exact sequence

$$\cdots \to M_n^O \xrightarrow{\eta_O} M_{n+1}^O \xrightarrow{c} M_{n+1}^U \xrightarrow{r\beta_U^{-1}} M_{n-1}^O \to \cdots$$

and the *CRT*-relations described in Section 2.

Since $M_k^U = 0$ for k odd it follows that $(\eta_O)_k$ is injective for k odd and surjective for k even. Furthermore, our hypothesis that $c_0(m)$ generates M_0^U implies that c_0 is surjective, which implies that $r_{-2} = 0$ and that $(\eta_O)_{-2}$ is injective. Thus $(\eta_O)_{-2} \colon M_{-2}^O \to M_{-1}^O$ is an isomorphism and $\eta_O^3 \colon M_{-3}^O \to M_0^O$ is injective. Then the relations $\eta_O^3 = 0$ and $2\eta_O = 0$ imply that $M_{-3}^O = 0$ and that M_{-2}^O consists only of 2-torsion.

Suppose first that $M_{-2}^O \cong M_{-1}^O = 0$. Then using the long exact sequence above and the relation $rc = 2$, the rest of the groups of M^O can be easily computed; except that in the case that n is odd we encounter an extension problem wherein M_2^O is either isomorphic to $\mathbb{Z}_4$ or to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. In that case, the same argument as in the computation of $K^{CRT}(\mathcal{O}_n^{\mathbb{R}})$ in Section 5.1 of [8] shows that $M_2^O \cong \mathbb{Z}_4$ exactly when $n - 1 \equiv 0 \pmod 4$ and $M_2^O \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ exactly when $n - 1 \equiv 2 \pmod 4$. Thus we find that the real and complex parts of M (as well as the operations η_O ξ, r, c, ψ_U) are isomorphic to the real and complex parts of $K^{CRT}(\mathcal{O}_n^{\mathbb{R}})$ (respectively $K^{CRT}(\mathcal{O}_\infty^{\mathbb{R}})$).

For the remaining case, suppose that $M^O_{-2} \neq 0$. Since this leads to the exotic *CRT*-module $K^{\scriptscriptstyle CRT}(\mathcal{E}_n)$, we will include all of the details of this computation. Since c_0 is surjective, the relation $\psi_{\scriptscriptstyle U} c = c$ implies that $(\psi_{\scriptscriptstyle U})_0 = 1$. Then the relation $\beta_{\scriptscriptstyle U}\psi_{\scriptscriptstyle U} = -\psi_{\scriptscriptstyle U}\beta_{\scriptscriptstyle U}$ implies that $\psi_{\scriptscriptstyle U} = 1$ in degrees congruent to $0 \pmod 4$ and $\psi_{\scriptscriptstyle U} = -1$ in degrees congruent to $2 \pmod 4$.

From $M^O_{-3} = 0$ it follows that c_{-2} is injective. But the only non-trivial 2-torsion subgroup of M^U_{-2} is isomorphic to $\mathbb{Z}_2$, and that occurs only when n is finite and odd. Thus $M^O_{-2} \cong M^O_{-1} \cong \mathbb{Z}_2$ and the complexification map $c_{-2} \colon \mathbb{Z}_2 \to \mathbb{Z}_{n-1}$ is multiplication by $(n-1)/2$ (in terms of chosen generators). The map r_{-4} is surjective and has kernel equal to $((n-1)/2)\mathbb{Z}_{n-1} \cong \mathbb{Z}_2$ so $M^O_{-4} \cong \mathbb{Z}_{(n-1)/2}$. The relation $c_{-4}r_{-4} = 1 + (\psi_{\scriptscriptstyle U})_{-4} = 2$ implies that the map $c_{-4} \colon \mathbb{Z}_{(n-1)/2} \to \mathbb{Z}_{n-1}$ is multiplication by 2.

Continuing to work our way down, the fact that c_{-4} is injective implies that $M^O_{-5} = 0$. The fact that the image of c_{-4} is $2\mathbb{Z}_{n-1}$ implies that $M^O_{-6} \cong \mathbb{Z}_2$ and r_{-6} is surjective. The relation $c_{-6}r_{-6} = 1 + (\psi_{\scriptscriptstyle U})_{-6} = 0$ implies that $c_{-6} = 0$ from which we see that η_{-7} is an isomorphism. Thus $M^O_{-7} \cong \mathbb{Z}_2$.

Finally, we compute $M^O_{-8} \cong M^O_0$. The exact sequence indicates that it is an extension of $\mathbb{Z}_2$ by $\mathbb{Z}_{n-1}$. We will prove that it is isomorphic to $\mathbb{Z}_{2(n-1)}$. If not, then $M^O_0 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{n-1}$ and we can arrange the direct sum decomposition so that $\eta_{-1} = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $c_0 = \left(\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right)$. Then the relation $rc = 2$ implies that $r_0 = \left(\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right)$. But then there is no isomorphism from $M^O_0/\text{image}(r_0) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ to $M^O_1 \cong \mathbb{Z}_2$ as required by the long exact sequence.

Thus in the case that $M^O_{-2} \neq 0$ it must be that n is odd and it must be that the real and complex parts of M are isomorphic to the real and complex parts of $K^{\scriptscriptstyle CRT}(\mathcal{E}_n)$ as in Table 2, completing the proof. □

We remark that the above result can instead be obtained using the analysis of acyclic *CRT*-modules in [23]. Indeed, let M be an acyclic *CRT*-module such that M^U_0 is isomorphic to $\mathbb{Z}_{k-1}$ or $\mathbb{Z}$, $M^U_1 = 0$, and $c_0 \colon M^O_0 \to M^U_0$ is surjective (hence $(\psi_{\scriptscriptstyle U})_0 = 1$). By Lemma 8.3.1, Proposition 8.3.2, and Theorem 8.3.3 of [23], there are isomorphisms

$$h_k(M) := \ker(1 - (\psi_{\scriptscriptstyle U})_k)/\text{image}(1 + (\psi_{\scriptscriptstyle U})_k) \cong \eta_{\scriptscriptstyle O} M^O_k \oplus \eta_{\scriptscriptstyle O} M^O_{k+4}$$

and, furthermore, M is determined up to isomorphism by M^U, $\psi_{\scriptscriptstyle U}$, and the resulting decompositions of $h_k(M)$ for $k = 0$ and $k = 2$. Using $(\psi_{\scriptscriptstyle U})_0 = 1$ and $(\psi_{\scriptscriptstyle U})_2 = -1$, we obtain

$$(h_0(M), h_2(M)) = \begin{cases} (\mathbb{Z}_2, 0) & \text{if } M^U_0 = \mathbb{Z} \\ (0,0) & \text{if } M^U_0 = \mathbb{Z}_{n-1} \text{ with } n \text{ even} \\ (\mathbb{Z}_2, \mathbb{Z}_2) & \text{if } M^U_0 = \mathbb{Z}_{n-1} \text{ with } n \text{ odd.} \end{cases}$$

The resulting possibilities for M are realized by the united K-theory of $\mathcal{O}^{\mathbb{R}}_\infty$ and $\mathbb{H} \otimes \mathcal{O}^{\mathbb{R}}_\infty$ in the first case; by that of $\mathcal{O}^{\mathbb{R}}_n$ in the second case; and by that of $\mathcal{O}^{\mathbb{R}}_n$, $\mathbb{H} \otimes \mathcal{O}^{\mathbb{R}}_n$, $\mathcal{E}_n$, and $\mathbb{H} \otimes \mathcal{E}_n$ in the third case. The assumption that c_0 is surjective reduces the possibilities to the united K-theory of $\mathcal{O}^{\mathbb{R}}_\infty$, $\mathcal{O}^{\mathbb{R}}_n$, or $\mathcal{E}_n$.

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