On the Existence of Stationary Solutions for Some Non-Fredholm Integro-Differential Equations

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Abstract. We show the existence of stationary solutions for some reaction-diffusion type equations in the appropriate H^2 spaces using the fixed point technique when the elliptic problem contains second order differential operators with and without Fredholm property.

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1 Introduction

Let us recall that a linear operator L acting from a Banach space E into another Banach space F satisfies the Fredholm property if its image is closed, the dimension of its kernel and the codimension of its image are finite. As a consequence, the equation $Lu = f$ is solvable if and only if $\phi_i(f) = 0$ for a finite number of functionals ϕ_i from the dual space F^*. These properties of Fredholm operators are widely used in many methods of linear and nonlinear analysis.

Elliptic problems in bounded domains with a sufficiently smooth boundary satisfy the Fredholm property if the ellipticity condition, proper ellipticity and Lopatinskii conditions are satisfied (see e.g. [1], [9], [10]). This is the main result of the theory of linear elliptic problems. In the case of unbounded domains, these conditions may not be sufficient and the Fredholm property may not be satisfied. For example, Laplace operator, $Lu = \Delta u$, in $\mathbb{R}^d$ does not satisfy the Fredholm property when considered in Hölder spaces, $L : C^{2+\alpha}(\mathbb{R}^d) \to C^{\alpha}(\mathbb{R}^d)$, or in Sobolev spaces, $L : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.

Linear elliptic problems in unbounded domains satisfy the Fredholm property if and only if, in addition to the conditions cited above, limiting operators are invertible (see [11]). In some simple cases, limiting operators can be explicitly constructed. For example, if

$$Lu = a(x)u'' + b(x)u' + c(x)u, \quad x \in \mathbb{R},$$

where the coefficients of the operator have limits at infinity,

$$a_\pm = \lim_{x\to\pm\infty} a(x), \quad b_\pm = \lim_{x\to\pm\infty} b(x), \quad c_\pm = \lim_{x\to\pm\infty} c(x),$$

the limiting operators are:

$$L_\pm u = a_\pm u'' + b_\pm u' + c_\pm u.$$

Since the coefficients are constant, the essential spectrum of the operator, that is the set of complex numbers λ for which the operator $L-\lambda$ does not satisfy the Fredholm property, can be explicitly found by means of the Fourier transform:

$$\lambda_\pm(\xi) = -a_\pm \xi^2 + b_\pm i\xi + c_\pm, \quad \xi \in \mathbb{R}.$$

Invertibility of limiting operators is equivalent to the condition that the essential spectrum does not contain the origin.

In the case of general elliptic problems, the same assertions hold true. The Fredholm property is satisfied if the essential spectrum does not contain the origin or if the limiting operators are invertible. However, these conditions may not be explicitly written.

In the case of non-Fredholm operators the usual solvability conditions may not be applicable and solvability conditions are, in general, not known. There are some classes of operators for which solvability conditions are obtained. Let us illustrate them with the following example. Consider the equation

$$Lu \equiv \Delta u + au = f \tag{1.1}$$

in $\mathbb{R}^d$, where a is a positive constant. The operator L coincides with its limiting operators. The homogeneous equation has a nonzero bounded solution. Hence the Fredholm property is not satisfied. However, since the operator has constant coefficients, we can apply the Fourier transform and find the solution explicitly. Solvability conditions can be formulated as follows. If $f \in L^2(\mathbb{R}^d)$ and $xf \in L^1(\mathbb{R}^d)$, then there exist a solution of this equation in $H^2(\mathbb{R}^d)$ if and only if

$$\left(f(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\right)_{L^2(\mathbb{R}^d)} = 0, \quad p \in S^d_{\sqrt{a}} \quad a.e.$$

(see [19]). Here and further down S^d_r denotes the sphere in $\mathbb{R}^d$ of radius r centered at the origin. Thus, though the operator does not satisfy the Fredholm property, solvability conditions are formulated in a similar way. However, this similarity is only formal since the range of the operator is not closed.

In the case of the operator with a potential,

$$Lu \equiv \Delta u + a(x)u = f,$$

Fourier transform is not directly applicable. Nevertheless, solvability conditions in $\mathbb{R}^3$ can be obtained by a rather sophisticated application of the theory of self-adjoint operators (see [13]). As before, solvability conditions are formulated in terms of orthogonality to solutions of the homogeneous adjoint equation. There are several other examples of linear elliptic operators without Fredholm property for which solvability conditions can be obtained (see [11]-[19]).
Solvability conditions play an important role in the analysis of nonlinear elliptic problems. In the case of non-Fredholm operators, in spite of some progress in understanding of linear problems, there exist only few examples where nonlinear non-Fredholm operators are analyzed (see [4]-[6]). In the present article we consider another class of nonlinear equations, for which the Fredholm property may not be satisfied:

$$\frac{\partial u}{\partial t} = \Delta u + au + \int_\Omega G(x-y)F(u(y),y)dy = 0, \quad a \geq 0. \tag{1.2}$$

Here Ω is a domain in $\mathbb{R}^d$, $\quad d = 1, 2, 3$, the more physically interesting dimensions. In population dynamics the integro-differential equations describe models with intra-specific competition and nonlocal consumption of resources (see e.g. [2], [3], [7]). The linear part of the corresponding operator is the same as in equation (1.1) above. We will use the explicit form of solvability conditions and will study the existence of stationary solutions of the nonlinear equation.

2 Formulation of the results

The nonlinear part of equation (1.2) will satisfy the following regularity conditions.

Assumption 1. *Function $F(u,x) : \mathbb{R} \times \Omega \to \mathbb{R}$ is such that*

$$|F(u,x)| \leq k|u| + h(x) \quad for \quad u \in \mathbb{R}, \ x \in \Omega \tag{2.1}$$

with a constant $k > 0$ and $h(x) : \Omega \to \mathbb{R}^+$, $\quad h(x) \in L^2(\Omega)$. Moreover, it is a Lipschitz continuous function, such that

$$|F(u_1,x) - F(u_2,x)| \leq l|u_1 - u_2| \quad for \quad any \quad u_{1,2} \in \mathbb{R}, \quad x \in \Omega \tag{2.2}$$

with a constant $l > 0$.

Clearly, the stationary solutions of (1.2), if they exist, will satisfy the nonlocal elliptic equation

$$\Delta u + \int_\Omega G(x-y)F(u(y),y)dy + au = 0, \ a \geq 0.$$

Let us introduce the auxiliary problem

$$-\Delta u - au = \int_{\Omega} G(x-y)F(v(y),y)dy. \tag{2.3}$$

We denote $(f_1(x), f_2(x))_{L^2(\Omega)} := \int_{\Omega} f_1(x)\bar{f}_2(x)dx$, with a slight abuse of notations when these functions are not square integrable, like for instance those used in the one dimensional Lemma A1 of the Appendix. In the first part of the article we study the case of $\Omega = \mathbb{R}^d$, such that the appropriate Sobolev space is equipped with the norm

$$\|u\|^2_{H^2(\mathbb{R}^d)} := \|u\|^2_{L^2(\mathbb{R}^d)} + \|\Delta u\|^2_{L^2(\mathbb{R}^d)}.$$

The main issue for the problem above is that the operator $-\Delta - a : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d),\ a \geq 0$ does not satisfy the Fredholm property, which is the obstacle to solve equation (2.3). The similar situations but in linear problems, both self- adjoint and non self-adjoint involving non Fredholm second or fourth order differential operators or even systems of equations with non Fredholm operators have been studied extensively in recent years (see [13]-[18]). However, we manage to show that equation (2.3) in this case defines a map $T_a : H^2(\mathbb{R}^d) \to H^2(\mathbb{R}^d),\ a \geq 0$, which is a strict contraction under certain technical conditions.

Theorem 1. *Let $\Omega = \mathbb{R}^d,\ G(x) : \mathbb{R}^d \to \mathbb{R},\ G(x) \in L^1(\mathbb{R}^d)$ and Assumption 1 holds.*
I) When $a > 0$ we assume that $xG(x) \in L^1(\mathbb{R}^d)$, orthogonality relations (6.4) hold if $d = 1$ and (6.9) when $d = 2, 3$ and $\sqrt{2}(2\pi)^{\frac{d}{2}} N_{a,\ d}\, l < 1$. Then the map $T_a v = u$ on $H^2(\mathbb{R}^d)$ defined by equation (2.3) has a unique fixed point v_a, which is the only stationary solution of problem (1.2) in $H^2(\mathbb{R}^d)$.
II) When $a = 0$ we assume that $x^2G(x) \in L^1(\mathbb{R}^d)$, orthogonality relations (6.10) hold, $d = 1, 2, 3$ and $\sqrt{2}(2\pi)^{\frac{d}{2}} N_{0,\ d}\, l < 1$. Then the map $T_0 v = u$ on $H^2(\mathbb{R}^d)$ defined by equation (2.3) admits a unique fixed point v_0, which is the only stationary solution of problem (1.2) with $a = 0$ in $H^2(\mathbb{R}^d)$.
In both cases I) and II) the fixed point $v_a,\ a \geq 0$ is nontrivial provided the intersection of supports of the Fourier transforms of functions $supp\widehat{F(0,x)} \cap supp\widehat{G}$ is a set of nonzero Lebesgue measure in $\mathbb{R}^d$.

In the second part of the work we study the analogous problem on the finite interval with periodic boundary conditions, i.e. $\Omega = I := [0,\ 2\pi]$ and the appropriate functional space is

$$H^2(I) = \{u(x) : I \to \mathbb{R} \mid u(x), u''(x) \in L^2(I),\ u(0) = u(2\pi),\ u'(0) = u'(2\pi)\}.$$

Let us introduce the following auxiliary constrained subspaces

$$H^2_0(I) := \{u \in H^2(I) \mid \left(u(x), \frac{e^{\pm i n_0 x}}{\sqrt{2\pi}}\right)_{L^2(I)} = 0\},\ n_0 \in \mathbb{N} \tag{2.4}$$

and

$$H^2_{0,\ 0}(I) = \{u \in H^2(I) \mid (u(x), 1)_{L^2(I)} = 0\}, \tag{2.5}$$

which are Hilbert spaces as well (see e.g. Chapter 2.1 of [8]). We prove that equation (2.3) in this situation defines a map $\tau_a,\ a \geq 0$ on the above mentioned spaces which will be a strict contraction under our assumptions.

Theorem 2. *Let $\Omega = I,\ G(x) : I \to \mathbb{R},\ G(x) \in L^1(I),\ G(0) = G(2\pi),\ F(u,0) = F(u,2\pi)$ for $u \in \mathbb{R}$ and Assumption 1 holds.*
I) When $a > 0$ and $a \neq n^2,\ n \in \mathbb{Z}$ we assume that $2\sqrt{\pi}\mathcal{N}_a l < 1$. Then the map $\tau_a v = u$ on $H^2(I)$ defined by equation (2.3) has a unique fixed point v_a, the only stationary solution of problem (1.2) in $H^2(I)$.
II) When $a = n_0^2,\ n_0 \in \mathbb{N}$ assume that orthogonality relations (6.17) hold and $2\sqrt{\pi}\mathcal{N}_{n_0^2} l < 1$. Then the map $\tau_{n_0^2} v = u$ on $H^2_0(I)$ defined by equation (2.3) has a unique fixed point $v_{n_0^2}$, the only stationary solution of problem (1.2) in $H^2_0(I)$.
III) When $a = 0$ assume that orthogonality relation (6.18) holds and $2\sqrt{\pi}\mathcal{N}_0 l < 1$. Then the map $\tau_0 v = u$ on $H^2_{0,\ 0}(I)$ defined by equation (2.3) has a unique fixed point v_0, the only stationary solution of problem (1.2) in $H^2_{0,\ 0}(I)$.
In all cases I), II) and III) the fixed point $v_a,\ a \geq 0$ is nontrivial provided the Fourier coefficients $G_n F(0,x)_n \neq 0$ for some $n \in \mathbb{Z}$.

Remark. *We use the constrained subspaces $H^2_0(I)$ and $H^2_{0,\ 0}(I)$ in cases II) and III) respectively, such that the operators $-\dfrac{d^2}{dx^2} - n_0^2 : H^2_0(I) \to L^2(I)$ and $-\dfrac{d^2}{dx^2} : H^2_{0,\ 0}(I) \to L^2(I)$, which possess the Fredholm property, have empty kernels.*

We conclude the article with the studies of our problem on the product of spaces, where one is the finite interval with periodic boundary conditions as before and another is the whole space of dimension not exceeding two, such that in our notations $\Omega = I \times \mathbb{R}^d = [0, 2\pi] \times \mathbb{R}^d,\ d = 1, 2$ and $x = (x_1, x_\perp)$ with $x_1 \in I$ and $x_\perp \in \mathbb{R}^d$. The appropriate Sobolev space for the problem is $H^2(\Omega)$ defined as

$$\{u(x) : \Omega \to \mathbb{R} \mid u(x), \Delta u(x) \in L^2(\Omega),$$
$$u(0, x_\perp) = u(2\pi, x_\perp),\ u_{x_1}(0, x_\perp) = u_{x_1}(2\pi, x_\perp)\},$$

where $x_\perp \in \mathbb{R}^d$ *a.e.* and u_{x_1} stands for the derivative of $u(x)$ with respect to the first variable x_1. As in the whole space case covered in Theorem 1, the operator $-\Delta - a : H^2(\Omega) \to L^2(\Omega),\quad a \geq 0$ does not possess the Fredholm property. Let us show that problem (2.3) in this context defines a map $t_a : H^2(\Omega) \to H^2(\Omega),\ a \geq 0$, a strict contraction under appropriate technical conditions.

Theorem 3. *Let* $\Omega = I \times \mathbb{R}^d$, $d = 1, 2$, $G(x) : \Omega \to \mathbb{R}$, $G(x) \in L^1(\Omega)$, $G(0, x_\perp) = G(2\pi, x_\perp)$, $F(u, 0, x_\perp) = F(u, 2\pi, x_\perp)$ *for* $x_\perp \in \mathbb{R}^d$ *a.e. and* $u \in \mathbb{R}$ *and Assumption 1 holds.*
I) When $n_0^2 < a < (n_0+1)^2$, $n_0 \in \mathbb{Z}^+ = \mathbb{N} \cup \{0\}$ *let* $x_\perp G(x) \in L^1(\Omega)$, *condition (6.29) holds if dimension* $d = 1$ *and (6.30) if* $d = 2$ *and* $\sqrt{2}(2\pi)^{\frac{d+1}{2}} M_a l < 1$. *Then the map* $t_a v = u$ *on* $H^2(\Omega)$ *defined by equation (2.3) has a unique fixed point* v_a, *the only stationary solution of problem (1.2) in* $H^2(\Omega)$.
II) When $a = n_0^2$, $n_0 \in \mathbb{N}$ *let* $x_\perp^2 G(x) \in L^1(\Omega)$, *conditions (6.25), (6.27) hold if dimension* $d = 1$ *and conditions (6.26), (6.27) hold if* $d = 2$ *and* $\sqrt{2}(2\pi)^{\frac{d+1}{2}} M_{n_0^2} l < 1$. *Then the map* $t_{n_0^2} v = u$ *on* $H^2(\Omega)$ *defined by equation (2.3) has a unique fixed point* $v_{n_0^2}$, *the only stationary solution of problem (1.2) in* $H^2(\Omega)$.
III) When $a = 0$ *let* $x_\perp^2 G(x) \in L^1(\Omega)$, *conditions (6.23) hold and* $\sqrt{2}(2\pi)^{\frac{d+1}{2}} M_0 l < 1$. *Then the map* $t_0 v = u$ *on* $H^2(\Omega)$ *defined by equation (2.3) has a unique fixed point* v_0, *the only stationary solution of problem (1.2) in* $H^2(\Omega)$.
In all cases I), II) and III) the fixed point v_a, $a \geq 0$ *is nontrivial provided that for some* $n \in \mathbb{Z}$ *the intersection of supports of the Fourier images of functions* $supp\widehat{F(0,x)}_n \cap supp\widehat{G}_n$ *is a set of nonzero Lebesgue measure in* $\mathbb{R}^d$.

Remark. *Note that the maps discussed above act on real valued functions due to the assumptions on* $F(u, x)$ *and* $G(x)$ *involved in the nonlocal term of (2.3).*

3 The Whole Space Case

Proof of Theorem 1. We present the proof of the theorem in case I) and when $a = 0$ the argument will be similar. Let us first suppose that in the case of $\Omega = \mathbb{R}^d$ for some $v \in H^2(\mathbb{R}^d)$ there exist two solutions $u_{1,2} \in H^2(\mathbb{R}^d)$ of problem (2.3). Then their difference $w := u_1 - u_2 \in H^2(\mathbb{R}^d)$ will satisfy the homogeneous problem $-\Delta w = aw$. Since the Laplacian operator acting in the whole space does not have any nontrivial square integrable eigenfunctions, $w(x)$ vanishes a.e. in $\mathbb{R}^d$. Let $v(x) \in H^2(\mathbb{R}^d)$ be arbitrary. We apply the standard Fourier transform to both sides of (2.3) and arrive at

$$\widehat{u}(p) = (2\pi)^{\frac{d}{2}} \frac{\widehat{G}(p)\widehat{f}(p)}{p^2 - a} \tag{3.1}$$

with $\widehat{f}(p)$ denoting the Fourier image of $F(v(x), x)$. Clearly, we have the upper bounds

$$|\widehat{u}(p)| \leq (2\pi)^{\frac{d}{2}} N_{a,\ d} |\widehat{f}(p)| \quad \text{and} \quad |p^2 \widehat{u}(p)| \leq (2\pi)^{\frac{d}{2}} N_{a,\ d} |\widehat{f}(p)|$$

with $N_{a,\ d} < \infty$ by means of Lemma A1 of the Appendix in one dimension and via Lemma A2 for $d = 2, 3$ under orthogonality relations (6.4) and (6.9) respectively. This enables us to estimate the norm

$$\|u\|^2_{H^2(\mathbb{R}^d)} = \|\widehat{u}(p)\|^2_{L^2(\mathbb{R}^d)} + \|p^2\widehat{u}(p)\|^2_{L^2(\mathbb{R}^d)} \leq 2(2\pi)^d N^2_{a,\ d}\|F(v(x),x)\|^2_{L^2(\mathbb{R}^d)},$$

which is finite by means of (2.1) of Assumption 1. Therefore, for any $v(x) \in H^2(\mathbb{R}^d)$ there is a unique solution $u(x) \in H^2(\mathbb{R}^d)$ of problem (2.3) with its Fourier image given by (3.1) and the map $T_a : H^2(\mathbb{R}^d) \to H^2(\mathbb{R}^d)$ is well defined. This enables us to choose arbitrarily $v_{1,2}(x) \in H^2(\mathbb{R}^d)$ such that their images $u_{1,2} = T_a v_{1,2} \in H^2(\mathbb{R}^d)$ and estimate

$$|\widehat{u}_1(p) - \widehat{u}_2(p)| \leq (2\pi)^{\frac{d}{2}} N_{a,\ d}|\widehat{f}_1(p) - \widehat{f}_2(p)|,$$
$$|p^2\widehat{u}_1(p) - p^2\widehat{u}_2(p)| \leq (2\pi)^{\frac{d}{2}} N_{a,\ d}|\widehat{f}_1(p) - \widehat{f}_2(p)|,$$

where $\widehat{f}_{1,2}(p)$ stand for the Fourier images of $F(v_{1,2}(x),x)$. For the appropriate norms of functions this yields

$$\|u_1 - u_2\|^2_{H^2(\mathbb{R}^d)} \leq 2(2\pi)^d N^2_{a,\ d}\|F(v_1(x),x) - F(v_2(x),x)\|^2_{L^2(\mathbb{R}^d)}.$$

Note that $v_{1,2}(x) \in H^2(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d),\ d \leq 3$ by means of the Sobolev embedding. Using condition (2.2) we easily arrive at

$$\|T_a v_1 - T_a v_2\|_{H^2(\mathbb{R}^d)} \leq \sqrt{2}(2\pi)^{\frac{d}{2}} N_{a,\ d}\, l\|v_1 - v_2\|_{H^2(\mathbb{R}^d)}$$

with the constant in the right side of this estimate less than one by the assumption of the theorem. Therefore, by means of the Fixed Point Theorem, there exists a unique function $v_a \in H^2(\mathbb{R}^d)$ with the property $T_a v_a = v_a$, which is the only stationary solution of equation (1.2) in $H^2(\mathbb{R}^d)$. Suppose $v_a(x)$ vanishes a.e. in $\mathbb{R}^d$. This will contradict to the assumption that the Fourier images of $G(x)$ and $F(0,x)$ do not vanish on a set of nonzero Lebesgue measure in $\mathbb{R}^d$. ■

4 The Problem on the Finite Interval

Proof of Theorem 2. Let us demonstrate the proof of the theorem in case I) and when $a = n_0^2,\ n_0 \in \mathbb{N}$ or $a = 0$ the ideas will be similar, using the constrained subspaces (2.4) and (2.5) respectively instead of $H^2(I)$. First we suppose that for $v \in H^2(I)$ there are two solutions $u_{1,2} \in H^2(I)$ of problem (2.3) with $\Omega = I$. Then function $w := u_1 - u_2 \in H^2(I)$ will be a solution to the problem $-w'' = aw$. But $a \neq n^2,\ n \in \mathbb{Z}$ and therefore, it is not an eigenvalue of the operator $-\dfrac{d^2}{dx^2}$ on $L^2(I)$ with periodic boundary conditions. Therefore, $w(x)$ vanishes a.e. in I. Suppose $v(x) \in H^2(I)$ is arbitrary. Let us apply the Fourier transform to problem (2.3) considered on the interval I which yields

$$u_n = \sqrt{2\pi}\frac{G_n f_n}{n^2 - a}, \quad n \in \mathbb{Z} \tag{4.1}$$

with $f_n := F(v(x),x)_n$. Clearly for the transform of the second derivative we have

$$(-u'')_n = \sqrt{2\pi}\frac{n^2 G_n f_n}{n^2-a}, \quad n \in \mathbb{Z},$$

which enables us to estimate

$$\|u\|^2_{H^2(I)} = \sum_{n=-\infty}^{\infty} |u_n|^2 + \sum_{n=-\infty}^{\infty} |n^2 u_n|^2 \le 4\pi \mathcal{N}_a^2 \|F(v(x),x)\|^2_{L^2(I)} < \infty$$

due to (2.1) of Assumption 1 and Lemma A3 of the Appendix. Hence, for an arbitrary $v(x) \in H^2(I)$ there is a unique $u(x) \in H^2(I)$ solving equation (2.3) with its Fourier image given by (4.1) and the map $\tau_a : H^2(I) \to H^2(I)$ in case I) is well defined. Let us consider any $v_{1,2} \in H^2(I)$ with their images under the map mentioned above $u_{1,2} = \tau_a v_{1,2} \in H^2(I)$ and arrive easily at the upper bound

$$\|u_1 - u_2\|^2_{H^2(I)} = \sum_{n=-\infty}^{\infty} |u_{1n} - u_{2n}|^2 + \sum_{n=-\infty}^{\infty} |n^2(u_{1n} - u_{2n})|^2 \le$$

$$\le 4\pi \mathcal{N}_a^2 \|F(v_1(x),x) - F(v_2(x),x)\|^2_{L^2(I)}.$$

Obviously $v_{1,2}(x) \in H^2(I) \subset L^\infty(I)$ due to the Sobolev embedding. By means of (2.2) we easily obtain

$$\|\tau_a v_1 - \tau_a v_2\|_{H^2(I)} \le 2\sqrt{\pi}\mathcal{N}_a l \|v_1 - v_2\|_{H^2(I)},$$

such that the constant in the right side of this upper bound is less than one as assumed. Thus, the Fixed Point Theorem implies the existence and uniqueness of a function $v_a \in H^2(I)$ satisfying $\tau_a v_a = v_a$, which is the only stationary solution of problem (1.2) in $H^2(I)$. Suppose $v_a(x) = 0$ a.e. in I. Then we obtain the contradiction to the assumption that $G_n F(0,x)_n \neq 0$ for some $n \in \mathbb{Z}$. Note that in the case of $a \neq n^2,\ n \in \mathbb{Z}$ the argument does not require any orthogonality conditions. ∎

5 The Problem on the Product of Spaces

Proof of Theorem 3. We present the proof of the theorem for case II) since when the parameter a vanishes or is located on the open interval between squares of two nonnegative integers the ideas are similar. Suppose there exists $v(x) \in H^2(\Omega)$ which generates $u_{1,2}(x) \in H^2(\Omega)$ solving equation (2.3). Then the difference $w := u_1 - u_2 \in H^2(\Omega)$ will satisfy $-\Delta w = n_0^2 w$ in our domain Ω. By applying the partial Fourier transform to this equation we easily arrive at $-\Delta_\perp w_n(x_\perp) = (n_0^2 - n^2) w_n(x_\perp)$. Clearly $\|w\|^2_{L^2(\Omega)} = \sum_{n=-\infty}^{\infty} \|w_n\|^2_{L^2(\mathbb{R}^d)}$ such that $w_n(x_\perp) \in L^2(\mathbb{R}^d),\ n \in \mathbb{Z}$. Since the transversal Laplacian operator $-\Delta_\perp$ on $L^2(\mathbb{R}^d)$ does not have any nontrivial square integrable eigenfunctions,

$w(x)$ is vanishing a.e. in Ω. Let $v(x) \in H^2(\Omega)$ be arbitrary. We apply the Fourier transform to both sides of problem (2.3) and obtain

$$\widehat{u}_n(p) = (2\pi)^{\frac{d+1}{2}} \frac{\widehat{G}_n(p)\widehat{f}_n(p)}{p^2 + n^2 - n_0^2}, \quad n \in \mathbb{Z}, \quad p \in \mathbb{R}^d, \quad d = 1, 2, \tag{5.1}$$

where $\widehat{f}_n(p)$ stands for the Fourier image of $F(v(x), x)$. Obviously,

$$|\widehat{u}_n(p)| \leq (2\pi)^{\frac{d+1}{2}} M_{n_0^2} |\widehat{f}_n(p)| \quad \textit{and} \quad |(p^2 + n^2)\widehat{u}_n(p)| \leq (2\pi)^{\frac{d+1}{2}} M_{n_0^2} |\widehat{f}_n(p)|,$$

where $M_{n_0^2} < \infty$ by means of Lemma A5 of the Appendix under the appropriate orthogonality conditions stated in it. Thus

$$\|u\|^2_{H^2(\Omega)} = \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}^d} |\widehat{u}_n(p)|^2 dp + \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}^d} |(p^2 + n^2)\widehat{u}_n(p)|^2 dp \leq$$
$$\leq 2(2\pi)^{d+1} {M_{n_0^2}}^2 \|F(v(x), x)\|^2_{L^2(\Omega)} < \infty$$

by means of (2.1) of Assumption 1, such that for any $v(x) \in H^2(\Omega)$ there exists a unique $u(x) \in H^2(\Omega)$ solving equation (2.3) with its Fourier image given by (5.1) and the map $t_a : H^2(\Omega) \to H^2(\Omega)$ in case II) of the Theorem is well defined. Then we consider arbitrary $v_{1,2} \in H^2(\Omega)$ such that their images under the map are $u_{1,2} = t_{n_0^2} v_{1,2} \in H^2(\Omega)$ and obtain

$$\|u_1 - u_2\|^2_{H^2(\Omega)} =$$
$$= \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}^d} |\widehat{u_1}_n(p) - \widehat{u_2}_n(p)|^2 dp + \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}^d} |(p^2 + n^2)(\widehat{u_1}_n(p) - \widehat{u_2}_n(p))|^2 dp \leq$$
$$\leq 2(2\pi)^{d+1} {M_{n_0^2}}^2 \|F(v_1(x), x) - F(v_2(x), x)\|^2_{L^2(\Omega)}.$$

Clearly $v_{1,2} \in H^2(\Omega) \subset L^\infty(\Omega)$ via the Sobolev embedding theorem. Using (2.2) we easily arrive at the estimate

$$\|t_{n_0^2} v_1 - t_{n_0^2} v_2\|_{H^2(\Omega)} \leq \sqrt{2}(2\pi)^{\frac{d+1}{2}} M_{n_0^2} l \|v_1 - v_2\|_{H^2(\Omega)}$$

with the constant in the right side of it less than one by assumption. Therefore, the Fixed Point Theorem yields the existence and uniqueness of a function $v_{n_0^2} \in H^2(\Omega)$ which satisfies $t_{n_0^2} v_{n_0^2} = v_{n_0^2}$ and is the only stationary solution of problem (1.2) in $H^2(\Omega)$ in case II) of the theorem. Suppose $v_{n_0^2}(x) = 0$ a.e. in Ω. This yields the contradiction to the assumption that there exists $n \in \mathbb{Z}$ for which $supp \widehat{G}_n \cap supp \widehat{F(0, x)}_n$ is a set of nonzero Lebesgue measure in $\mathbb{R}^d$. ■

6 Appendix

Let $G(x)$ be a function, $G(x) : \mathbb{R}^d \to \mathbb{R}, \quad d \leq 3$ for which we denote its standard Fourier transform using the hat symbol as

$$\widehat{G}(p) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} G(x) e^{-ipx} dx, \; p \in \mathbb{R}^d$$

such that

$$\|\widehat{G}(p)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{(2\pi)^{\frac{d}{2}}}\|G\|_{L^1(\mathbb{R}^d)} \tag{6.1}$$

and $G(x) = \dfrac{1}{(2\pi)^{\frac{d}{2}}}\displaystyle\int_{\mathbb{R}^d} \widehat{G}(q)e^{iqx}dq,\ x \in \mathbb{R}^d$. Let us define the auxiliary quantities

$$N_{a,\ d} := max\Big\{\Big\|\frac{\widehat{G}(p)}{p^2-a}\Big\|_{L^\infty(\mathbb{R}^d)}, \quad \Big\|\frac{p^2\widehat{G}(p)}{p^2-a}\Big\|_{L^\infty(\mathbb{R}^d)}\Big\} \tag{6.2}$$

for $a > 0$ and

$$N_{0,\ d} := max\Big\{\Big\|\frac{\widehat{G}(p)}{p^2}\Big\|_{L^\infty(\mathbb{R}^d)}, \quad \Big\|\widehat{G}(p)\Big\|_{L^\infty(\mathbb{R}^d)}\Big\} \tag{6.3}$$

when $a = 0$.

Lemma A1. *Let $G(x) \in L^1(\mathbb{R})$.*
a) If $a > 0$ and $xG(x) \in L^1(\mathbb{R})$ then $N_{a,\ 1} < \infty$ if and only if

$$\Big(G(x), \frac{e^{\pm i\sqrt{a}x}}{\sqrt{2\pi}}\Big)_{L^2(\mathbb{R})} = 0. \tag{6.4}$$

b) If $a = 0$ and $x^2G(x) \in L^1(\mathbb{R})$ then $N_{0,\ 1} < \infty$ if and only if

$$(G(x), 1)_{L^2(\mathbb{R})} = 0 \quad \textit{and} \quad (G(x), x)_{L^2(\mathbb{R})} = 0. \tag{6.5}$$

Proof. In order to prove part a) of the lemma we write the function

$$\frac{\widehat{G}(p)}{p^2-a} = \frac{\widehat{G}(p)}{p^2-a}\chi_{I_\delta} + \frac{\widehat{G}(p)}{p^2-a}\chi_{{I_\delta}^c}, \tag{6.6}$$

where χ_A here and further down stands for the characteristic function of a set A, A^c for its complement, the set $I_\delta = I_\delta^+ \cup I_\delta^-$ with $I_\delta^+ = \{p \in \mathbb{R} \mid \sqrt{a}-\delta < p < \sqrt{a}+\delta\}$, $I_\delta^- = \{p \in \mathbb{R} \mid -\sqrt{a}-\delta < p < -\sqrt{a}+\delta\}$ and $0 < \delta < \sqrt{a}$. The second term in the right side of (6.6) can be easily estimated in absolute value from above using (6.1) as $\dfrac{1}{\sqrt{2\pi}\delta^2}\|G\|_{L^1(\mathbb{R})} < \infty$ and the remaining term in the right side of (6.6) can be written as

$$\frac{\widehat{G}(p)}{p^2-a}\chi_{I_\delta^+} + \frac{\widehat{G}(p)}{p^2-a}\chi_{I_\delta^-}.$$

We will use the expansions near the points of singularity given by

$$\widehat{G}(p) = \widehat{G}(\sqrt{a}) + \int_{\sqrt{a}}^p \frac{d\widehat{G}(s)}{ds}ds, \quad \widehat{G}(p) = \widehat{G}(-\sqrt{a}) + \int_{-\sqrt{a}}^p \frac{d\widehat{G}(s)}{ds}ds$$

with $\left\| \dfrac{d\widehat{G}(p)}{dp} \right\|_{L^\infty(\mathbb{R})} \leq \dfrac{1}{\sqrt{2\pi}} \|xG\|_{L^1(\mathbb{R})} < \infty$ by the assumption of the lemma. This enables us to obtain the bound

$$\left| \frac{\int_{\sqrt{a}}^{p} \frac{d\widehat{G}(s)}{ds} ds}{p^2 - a} \chi_{I_\delta^+} \right| \leq \frac{C}{2\sqrt{a} - \delta} < \infty, \quad \left| \frac{\int_{-\sqrt{a}}^{p} \frac{d\widehat{G}(s)}{ds} ds}{p^2 - a} \chi_{I_\delta^-} \right| \leq \frac{C}{2\sqrt{a} - \delta} < \infty.$$

Therefore it remains to estimate

$$\frac{\widehat{G}(\sqrt{a})}{p^2 - a} \chi_{I_\delta^+} + \frac{\widehat{G}(-\sqrt{a})}{p^2 - a} \chi_{I_\delta^-},$$

which belongs to $L^\infty(\mathbb{R})$ if and only if $\widehat{G}(\pm\sqrt{a}) = 0$, which is equivalent to the orthogonality relations (6.4). To estimate the second term in the right side of (6.2) under these orthogonality relations we consider the two situations. The first one is when $|p| \leq \sqrt{a} + \delta$ and we have the bound

$$\left| \frac{p^2 \widehat{G}(p)}{p^2 - a} \right| \leq (\sqrt{a} + \delta)^2 \left\| \frac{\widehat{G}(p)}{p^2 - a} \right\|_{L^\infty(\mathbb{R})} < \infty.$$

In the second one $|p| > \sqrt{a} + \delta$ which yields $\dfrac{p^2}{p^2 - a} \in L^\infty(\mathbb{R})$ and $\widehat{G}(p)$ is bounded via (6.1), which completes the proof of part a) of the lemma. Then we turn our attention to the situation of $a = 0$, such that

$$\frac{\widehat{G}(p)}{p^2} = \frac{\widehat{G}(p)}{p^2} \chi_{\{|p| \leq 1\}} + \frac{\widehat{G}(p)}{p^2} \chi_{\{|p| > 1\}}. \tag{6.7}$$

The second term in the right side of the identity above can be easily estimated as

$$\left| \frac{\widehat{G}(p)}{p^2} \chi_{\{|p| > 1\}} \right| \leq \|\widehat{G}(p)\|_{L^\infty(\mathbb{R})} < \infty \tag{6.8}$$

due to (6.1). We will make use of the representation

$$\widehat{G}(p) = \widehat{G}(0) + p \frac{d\widehat{G}}{dp}(0) + \int_0^p \Big(\int_0^s \frac{d^2 \widehat{G}(q)}{dq^2} dq \Big) ds.$$

Obviously $\left| \dfrac{d^2 \widehat{G}(p)}{dp^2} \right| \leq \dfrac{1}{\sqrt{2\pi}} \|x^2 G(x)\|_{L^1(\mathbb{R})} < \infty$ by the assumption of the lemma. Hence

$$\left| \frac{\int_0^p \Big(\int_0^s \frac{d^2 \widehat{G}(q)}{dq^2} dq \Big) ds}{p^2} \chi_{\{|p| \leq 1\}} \right| \leq \frac{C}{2} < \infty$$

and the only expression which remains to estimate is given by $\left[\dfrac{\widehat{G}(0)}{p^2} + \dfrac{\frac{d\widehat{G}}{dp}(0)}{p} \right] \chi_{\{|p| \leq 1\}}$, which is contained in $L^\infty(\mathbb{R})$ if and only if $\widehat{G}(0)$ and

$\frac{d\widehat{G}}{dp}(0)$ vanish. This is equivalent to the orthogonality relations (6.5). Note that $\|\widehat{G}(p)\|_{L^\infty(\mathbb{R})} < \infty$ by means of (6.1). ∎

The proposition above can be generalized to higher dimensions in the following statement.

Lemma A2. *Let $G(x) \in L^1(\mathbb{R}^d)$, $d = 2, 3$.*
a) If $a > 0$ and $xG(x) \in L^1(\mathbb{R}^d)$ then $N_{a,\ d} < \infty$ if and only if

$$\Big(G(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\Big)_{L^2(\mathbb{R}^d)} = 0 \quad for \quad p \in S^d_{\sqrt{a}} \quad a.e. \tag{6.9}$$

b) If $a = 0$ and $x^2G(x) \in L^1(\mathbb{R}^d)$ then $N_{0,\ d} < \infty$ if and only if

$$(G(x), 1)_{L^2(\mathbb{R}^d)} = 0 \quad and \quad (G(x), x_k)_{L^2(\mathbb{R}^d)} = 0, \quad 1 \le k \le d. \tag{6.10}$$

Proof. To prove part a) of the lemma we introduce the auxiliary spherical layer in the space of $d = 2, 3$ dimensions

$$A_\delta := \{p \in \mathbb{R}^d \mid \sqrt{a} - \delta < |p| < \sqrt{a} + \delta\}, \quad 0 < \delta < \sqrt{a}$$

and write

$$\frac{\widehat{G}(p)}{p^2 - a} = \frac{\widehat{G}(p)}{p^2 - a}\chi_{A_\delta} + \frac{\widehat{G}(p)}{p^2 - a}\chi_{A^c_\delta}. \tag{6.11}$$

For the second term in the right side of (6.11) we have the upper bound in the absolute value as $\frac{\|\widehat{G}(p)\|_{L^\infty(\mathbb{R}^d)}}{\sqrt{a}\delta} < \infty$ due to (6.1). Let us expand

$$\widehat{G}(p) = \int_{\sqrt{a}}^{|p|} \frac{\partial \widehat{G}(|s|, \sigma)}{\partial |s|} d|s| + \widehat{G}(\sqrt{a}, \sigma),$$

where σ stands for the angle variables on the sphere. Using the elementary inequality $\left|\frac{\partial \widehat{G}(p)}{\partial |p|}\right| \le \frac{1}{(2\pi)^{\frac{d}{2}}}\|xG(x)\|_{L^1(\mathbb{R}^d)}$ with its right side finite by the assumption of the lemma we estimate

$$\left|\frac{\int_{\sqrt{a}}^{|p|} \frac{\partial \widehat{G}(|s|,\sigma)}{\partial |s|} d|s|}{p^2 - a}\chi_{A_\sigma}\right| \le \frac{C}{\sqrt{a}} < \infty.$$

The only remaining term $\frac{\widehat{G}(\sqrt{a}, \sigma)}{p^2 - a}\chi_{A_\delta} \in L^\infty(\mathbb{R}^d)$, $\ d = 2, 3$ if and only if $\widehat{G}(\sqrt{a}, \sigma)$ vanishes a.e. on the sphere $S^d_{\sqrt{a}}$, which is equivalent to orthogonality relations (6.9). The proof of the fact that the second norm in the right side of (6.2) under conditions (6.9) is finite is analogous to the one presented in Lemma A1 in one dimension. For the proof of part b) of the lemma we apply the two and three dimensional analog of formula (6.7), such that for the second term in its right side there is a bound analogous to (6.8). Let us use the representation formula

$$\widehat{G}(p) = \widehat{G}(0) + |p|\frac{\partial \widehat{G}}{\partial |p|}(0,\sigma) + \int_0^{|p|}\left(\int_0^s \frac{\partial^2 \widehat{G}(|q|,\sigma)}{\partial |q|^2} d|q|\right) ds.$$

Apparently

$$\frac{\partial \widehat{G}}{\partial |p|}(0,\sigma) = -\frac{i}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} G(x)|x|cos\theta dx, \tag{6.12}$$

where θ is the angle between vectors p and x in $\mathbb{R}^d$ and for the second derivative

$$\left|\frac{\partial^2 \widehat{G}(p)}{\partial |p|^2}\right| \leq \frac{1}{(2\pi)^{\frac{d}{2}}}\|x^2 G(x)\|_{L^1(\mathbb{R}^d)} < \infty$$

by the assumption of the lemma. This yields

$$\left|\frac{\int_0^{|p|}\left(\int_0^s \frac{\partial^2 \widehat{G}(|q|,\sigma)}{\partial |q|^2} d|q|\right) ds}{p^2}\chi_{\{|p|\leq 1\}}\right| \leq \frac{C}{2} < \infty,$$

such that the only expression remaining to estimate is given by

$$\left[\frac{\widehat{G}(0)}{p^2} + \frac{\frac{\partial \widehat{G}}{\partial |p|}(0,\sigma)}{|p|}\right]\chi_{\{|p|\leq 1\}} \tag{6.13}$$

with the first derivative (6.12) containing the angular dependence. We consider first the case of $d = 2$ such that $p = (|p|, \theta_p)$, $x = (|x|, \theta_x) \in \mathbb{R}^2$ and the angle between them $\theta = \theta_p - \theta_x$. A straightforward computation yields that the right side of (6.12) is given by $-\frac{i}{2\pi}\sqrt{Q_1^2 + Q_2^2}cos(\theta_p - \alpha)$ with

$$Q_1 := \int_{\mathbb{R}^2} G(x)x_1 dx, \quad Q_2 := \int_{\mathbb{R}^2} G(x)x_2 dx, \quad \tan\alpha := \frac{Q_2}{Q_1} \tag{6.14}$$

and $x = (x_1, x_2) \in \mathbb{R}^2$ such that (6.13) is equal to

$$\left[\frac{\widehat{G}(0)}{p^2} - \frac{i}{2\pi}\sqrt{Q_1^2 + Q_2^2}\frac{cos(\theta_p - \alpha)}{|p|}\right]\chi_{\{|p|\leq 1\}}.$$

Note that the situation of $Q_1 = 0$ and $Q_2 \neq 0$ corresponds to the cases of α equal to $\frac{\pi}{2}$ or $-\frac{\pi}{2}$. Obviously, the expression above is contained in $L^\infty(\mathbb{R}^2)$

if and only if the quantities $\widehat{G}(0)$, Q_1 and Q_2 vanish, which is equivalent to orthogonality relations (6.10) in two dimensions. In the case of $d = 3$ the argument is quite similar. The coordinates of vectors

$$x = (x_1, x_2, x_3) = (|x| sin\theta_x cos\varphi_x, |x| sin\theta_x sin\varphi_x, |x| cos\theta_x) \in \mathbb{R}^3$$

and

$$p = (|p| sin\theta_p cos\varphi_p, |p| sin\theta_p sin\varphi_p, |p| cos\theta_p) \in \mathbb{R}^3$$

are being used to compute $cos\theta = \frac{(p,x)_{\mathbb{R}^3}}{|p||x|}$ involved in the right side of (6.12). Here $(p,x)_{\mathbb{R}^3}$ stands for the scalar product of the vectors in three dimensions. An easy calculation shows that when $d = 3$ the right side of (6.12) can be written as

$$-\frac{i}{(2\pi)^{\frac{3}{2}}}\{\sqrt{Q_1^2+Q_2^2} sin\theta_p cos(\varphi_p - \alpha) + Q_3 cos\theta_p\}$$

with α given by (6.14) and here $Q_k = \int_{\mathbb{R}^3} G(x) x_k dx,\ k = 1,2,3,$ which are the three dimensional generalizations of the correspondent expressions given by (6.14) and term (6.13) will be equal to

$$\left[\frac{\widehat{G}(0)}{p^2} - \frac{i}{(2\pi)^{\frac{3}{2}}|p|}\{\sqrt{Q_1^2+Q_2^2} sin\theta_p cos(\varphi_p - \alpha) + Q_3 cos\theta_p\}\right]\chi_{\{|p|\le 1\}}$$

and will belong to $L^\infty(\mathbb{R}^3)$ if and only if $\widehat{G}(0)$ along with $Q_k,\ k = 1,2,3$ vanish, which is equivalent to orthogonality conditions (6.10) in three dimensions. The second norm in the right side of (6.3) is finite under relations (6.1). ■

Let the function $G(x) : I \to \mathbb{R}, \quad G(0) = G(2\pi)$ and its Fourier transform on the finite interval is given by

$$G_n := \int_0^{2\pi} G(x)\frac{e^{-inx}}{\sqrt{2\pi}}dx, \quad n \in \mathbb{Z}$$

and $G(x) = \sum_{n=-\infty}^{\infty} G_n \frac{e^{inx}}{\sqrt{2\pi}}$. Similarly to the whole space case we define

$$\mathcal{N}_a := max\left\{\left\|\frac{G_n}{n^2-a}\right\|_{l^\infty}, \quad \left\|\frac{n^2 G_n}{n^2-a}\right\|_{l^\infty}\right\} \tag{6.15}$$

for $a > 0$. In the situation of $a = 0$

$$\mathcal{N}_0 := max\left\{\left\|\frac{G_n}{n^2}\right\|_{l^\infty}, \quad \left\|G_n\right\|_{l^\infty}\right\}. \tag{6.16}$$

We have the following elementary statement.

Lemma A3. *Let $G(x) \in L^1(I)$ and $G(0) = G(2\pi)$.*
a) If $a > 0$ and $a \neq n^2,\ n \in \mathbb{Z}$ then $\mathcal{N}_a < \infty$.
b) If $a = n_0^2,\ n_0 \in \mathbb{N}$ then $\mathcal{N}_a < \infty$ if and only if

$$\left(G(x), \frac{e^{\pm i n_0 x}}{\sqrt{2\pi}} \right)_{L^2(I)} = 0. \tag{6.17}$$

c) If $a = 0$ then $\mathcal{N}_0 < \infty$ if and only if

$$(G(x), 1)_{L^2(I)} = 0. \tag{6.18}$$

Proof. Clearly we have the bound

$$\|G_n\|_{l^\infty} \leq \frac{1}{\sqrt{2\pi}} \|G\|_{L^1(I)} < \infty. \tag{6.19}$$

Thus in case a) when $a \neq n^2,\ n \in \mathbb{Z}$ the expressions under the norms in the right side of (6.15) do not contain any singularities and the result of the lemma is obvious. When $a = n_0^2$ for some $n_0 \in \mathbb{N}$ or $a = 0$ conditions (6.17) and (6.18) respectively are necessary and sufficient for eliminating the existing singularities by making the corresponding Fourier coefficients equal to zero: $G_{\pm n_0}$ in case b) and G_0 in case c). ■

Let $G(x)$ be a function on the product of spaces studied in Theorem 3, $G(x) : \Omega = I \times \mathbb{R}^d \to \mathbb{R},\ d = 1, 2,\ G(0, x_\perp) = G(2\pi, x_\perp)$ for $x_\perp \in \mathbb{R}^d$ a.e. and its Fourier transform on the product of spaces equals to

$$\widehat{G}_n(p) := \frac{1}{(2\pi)^{\frac{d+1}{2}}} \int_{\mathbb{R}^d} dx_\perp e^{-ipx_\perp} \int_0^{2\pi} G(x_1, x_\perp) e^{-inx_1} dx_1, \quad p \in \mathbb{R}^d,\ n \in \mathbb{Z}$$

such that

$$\|\widehat{G}_n(p)\|_{L^\infty_{n,p}} := \sup_{\{p \in \mathbb{R}^d,\ n \in \mathbb{Z}\}} |\widehat{G}_n(p)| \leq \frac{1}{(2\pi)^{\frac{d+1}{2}}} \|G\|_{L^1(\Omega)} \tag{6.20}$$

and $G(x) = \frac{1}{(2\pi)^{\frac{d+1}{2}}} \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}^d} \widehat{G}_n(p) e^{ipx_\perp} e^{inx_1} dp$. It is also useful to consider the Fourier transform only in the first variable, such that

$$G_n(x_\perp) := \int_0^{2\pi} G(x_1, x_\perp) \frac{e^{-inx_1}}{\sqrt{2\pi}} dx_1, \quad n \in \mathbb{Z}.$$

Let us introduce $\xi_n^a(p) := \dfrac{\widehat{G}_n(p)}{p^2 + n^2 - a}$ and define

$$M_a := max\{\|\xi_n^a(p)\|_{L^\infty_{n,p}}, \quad \|(p^2 + n^2)\xi_n^a(p)\|_{L^\infty_{n,p}}\} \tag{6.21}$$

for $a > 0$ and

$$M_0 := max\left\{ \left\| \frac{\widehat{G}_n(p)}{p^2+n^2} \right\|_{L^\infty_{n,p}}, \quad \|\widehat{G}_n(p)\|_{L^\infty_{n,p}} \right\} \tag{6.22}$$

when $a = 0$. Here the momentum vector $p \in \mathbb{R}^d$.

Lemma A4. *Let $G(x) \in L^1(\Omega)$, $x_\perp^2 G(x) \in L^1(\Omega)$ and $G(0, x_\perp) = G(2\pi, x_\perp)$ for $x_\perp \in \mathbb{R}^d$ a.e., $d = 1, 2$.* Then $M_0 < \infty$ if and only if

$$(G(x), 1)_{L^2(\Omega)} = 0, \ (G(x), x_{\perp,\ k})_{L^2(\Omega)} = 0, \ 1 \le k \le d, \ d = 1, 2. \tag{6.23}$$

Proof. Let us expand

$$\frac{\widehat{G}_n(p)}{p^2+n^2} = \frac{\widehat{G}_0(p)}{p^2}\chi_{\{p\in\mathbb{R}^d,\ n=0\}} + \frac{\widehat{G}_n(p)}{p^2+n^2}\chi_{\{p\in\mathbb{R}^d,\ n\in\mathbb{Z},\ n\neq 0\}}.$$

The second term in the right side of this identity can be estimated above in the absolute value by means of (6.20) by $\frac{1}{(2\pi)^{\frac{d+1}{2}}}\|G\|_{L^1(\Omega)} < \infty$. Clearly we have the bound on the norm

$$\|x_\perp^2 G_n(x_\perp)\|_{L^1(\mathbb{R}^d)} \le \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} dx_1 \int_{\mathbb{R}^d} dx_\perp x_\perp^2 |G(x)| < \infty, \quad n \in \mathbb{Z} \tag{6.24}$$

by the assumption of the lemma. Thus the term $\frac{\widehat{G}_0(p)}{p^2} \in L^\infty(\mathbb{R}^d)$ if and only if the orthogonality conditions (6.23) hold, which is guaranteed for $d = 1$ by Lemma A1 and when dimension $d = 2$ by Lemma A2. Note that the last term in the right side of (6.22) is bounded via (6.20). ∎

Next we turn our attention to the situation when the parameter a is nontrivial.

Lemma A5. *Let $G(x) \in L^1(\Omega)$, $x_\perp^2 G(x) \in L^1(\Omega)$ and $G(0, x_\perp) = G(2\pi, x_\perp)$ for $x_\perp \in \mathbb{R}^d$ a.e., $d = 1, 2$ and $a = n_0^2$, $n_0 \in \mathbb{N}$. Then $M_a < \infty$ if and only if*

$$\left(G(x_1, x_\perp), \frac{e^{inx_1}}{\sqrt{2\pi}}\frac{e^{\pm i\sqrt{n_0^2-n^2}x_\perp}}{\sqrt{2\pi}}\right)_{L^2(\Omega)} = 0, \quad |n| \le n_0 - 1, \quad d = 1, \tag{6.25}$$

$$\left(G(x_1, x_\perp), \frac{e^{inx_1}}{\sqrt{2\pi}}\frac{e^{ipx_\perp}}{2\pi}\right)_{L^2(\Omega)} = 0, \quad p \in S^2_{\sqrt{n_0^2-n^2}} \quad a.e., \quad |n| \le n_0 - 1, \ d = 2, \tag{6.26}$$

$$\left(G(x_1, x_\perp), \frac{e^{\pm in_0x_1}}{\sqrt{2\pi}}\right)_{L^2(\Omega)} = 0, \ \left(G(x_1, x_\perp), \frac{e^{\pm in_0x_1}}{\sqrt{2\pi}}x_{\perp,\ k}\right)_{L^2(\Omega)} = 0, \ 1 \le k \le d. \tag{6.27}$$

Proof. We will use the representation of the function $\xi_n^a(p),\ n\in\mathbb{Z},\ p\in\mathbb{R}^d$ as the sum

$$\xi_n^a(p)\chi_{\{p\in\mathbb{R}^d,\ |n|>n_0\}}+\xi_n^a(p)\chi_{\{p\in\mathbb{R}^d,\ |n|<n_0\}}+ \\ +\xi_{n_0}^a(p)\chi_{\{p\in\mathbb{R}^d,\ n=n_0\}}+\xi_{-n_0}^a(p)\chi_{\{p\in\mathbb{R}^d,\ n=-n_0\}}. \tag{6.28}$$

Obviously $|\xi_n^a(p)\chi_{\{p\in\mathbb{R}^d,\ |n|>n_0\}}|\le\|\widehat{G}_n(p)\|_{L^\infty_{n,p}}<\infty$ by means of (6.20). We have trivial estimates on the norms for $n\in\mathbb{Z}$

$$\|G_n(x_\perp)\|_{L^1(\mathbb{R}^d)}\le\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}dx_1\int_{\mathbb{R}^d}dx_\perp|G(x_1,x_\perp)|<\infty$$

and

$$\|x_\perp G_n(x_\perp)\|_{L^1(\mathbb{R}^d)}\le\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}dx_1\int_{\mathbb{R}^d}dx_\perp|x_\perp||G(x_1,x_\perp)|<\infty.$$

Note that $G(x)\in L^1(\Omega)$ and $x_\perp^2G(x)\in L^1(\Omega)$ by the assumptions of the lemma, which yields $x_\perp G(x)\in L^1(\Omega)$. Thus when dimension $d=1$ by means of Lemma A1 $\xi_n^a(p)\chi_{\{p\in\mathbb{R}^d,\ |n|<n_0\}}\in L^\infty_{n,p}$ if and only if orthogonality relations (6.25) hold. For $d=2$ the necessary and sufficient conditions for the boundedness of the second term in (6.28) via Lemma A2 are given by (6.26). Lemmas A1 and A2 yield that the third term in (6.28) belongs to $L^\infty_{n,p}$ if and only if conditions (6.27) with the positive sign under the exponents are satisfied. Clearly $x_\perp^2G_n(x_\perp)\in L^1(\mathbb{R}^d)$ due to the assumption of the lemma and estimate (6.24). Similarly, we obtain that the necessary and sufficient conditions for the the last term in (6.28) to be contained in $L^\infty_{n,p}$ are given by (6.27) with the negative sign under the exponents. Then we represent $(p^2+n^2)\xi_n^a(p)$ as the sum

$$(p^2+n^2)\xi_n^a(p)\chi_{\{p\in\mathbb{R}^d,\ n\in\mathbb{Z},\ p^2+n^2\le n_0^2+1\}}+ \\ (p^2+n^2)\xi_n^a(p)\chi_{\{p\in\mathbb{R}^d,\ n\in\mathbb{Z},\ p^2+n^2>n_0^2+1\}}$$

in which the absolute value of the first term has the upper bound $(n_0^2+1)\|\xi_n^a(p)\|_{L^\infty_{n,p}}<\infty$ under the orthogonality conditions of the lemma and of the second one $(1+n_0^2)\|\widehat{G}_n(p)\|_{L^\infty_{n,p}}<\infty$ via (6.20). ■

Finally, we study the case when the parameter a is located on an open interval between the squares of two consecutive nonnegative integers.

Lemma A6. *Let $G(x)\in L^1(\Omega)$, $\quad x_\perp G(x)\in L^1(\Omega)$ and $G(0,x_\perp)=G(2\pi,x_\perp)$ for $x_\perp\in\mathbb{R}^d$ a.e., $d=1,2$ and $n_0^2<a<(n_0+1)^2$, $n_0\in\mathbb{Z}^+=\mathbb{N}\cup\{0\}$. Then $M_a<\infty$ if and only if*

$$\left(G(x_1,x_\perp),\frac{e^{inx_1}}{\sqrt{2\pi}}\frac{e^{\pm i\sqrt{a-n^2}x_\perp}}{\sqrt{2\pi}}\right)_{L^2(\Omega)}=0,\quad |n|\le n_0,\quad d=1, \tag{6.29}$$

$$\left(G(x_1,x_\perp),\frac{e^{inx_1}}{\sqrt{2\pi}}\frac{e^{ipx_\perp}}{2\pi}\right)_{L^2(\Omega)}=0,\quad p\in S^2_{\sqrt{a-n^2}}\ a.e.,\quad |n|\le n_0,\quad d=2. \tag{6.30}$$

Proof. Let us expand $\xi_n^a(p)$ as the sum of two terms

$$\xi_n^a(p)\chi_{\{p\in\mathbb{R}^d,\ n\in\mathbb{Z},\ |n|\ge n_0+1\}}+\xi_n^a(p)\chi_{\{p\in\mathbb{R}^d,\ n\in\mathbb{Z},\ |n|\le n_0\}},$$

such that the absolute value of the first one is bounded above by $\dfrac{\|\widehat{G}_n(p)\|_{L^\infty_{n,p}}}{(n_0+1)^2-a}<\infty$ and the second one belongs to $L^\infty_{n,p}$ if and only if orthogonality relations (6.29) are satisfied in one dimension by means of Lemma A1 and if and only if conditions (6.30) are fulfilled in two dimensions via Lemma A2. We write $(p^2+n^2)\xi_n^a(p)$ as the sum

$$(p^2+n^2)\xi_n^a(p)\chi_{\{p\in\mathbb{R}^d,\ n\in\mathbb{Z},\ p^2+n^2\ge(n_0+1)^2\}}+$$
$$+(p^2+n^2)\xi_n^a(p)\chi_{\{p\in\mathbb{R}^d,\ n\in\mathbb{Z},\ p^2+n^2<(n_0+1)^2\}}$$

in which the first and the second terms can be easily bounded above in their absolute values by the quantities finite under the conditions of the lemma, namely

$$\left(1+\frac{a}{(n_0+1)^2-a}\right)\|\widehat{G}_n(p)\|_{L^\infty_{n,p}}\quad and \quad (n_0+1)^2\|\xi_n^a(p)\|_{L^\infty_{n,p}}$$

respectively. ■

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