The Hilbert-Chow Morphism and the Incidence Divisor

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Abstract. For a smooth projective variety P of dimension n, we construct a Cartier divisor supported on the incidence locus in the product of Chow varieties $\mathscr{C}_a(P) \times \mathscr{C}_{n-a-1}(P)$. There is a natural definition of the corresponding line bundle on a product of Hilbert schemes, and we show this bundle descends to the Chow varieties. This answers a question posed by Barry Mazur.

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Section 1. Introduction

Let $(P, \mathcal{O}_P(1))$ be a smooth projective variety of dimension n over an algebraically closed field k. The *Chow variety* $\mathscr{C}_{d,d'}(P)$ parameterizes algebraic cycles on P. In particular, there is a bijection between $\mathscr{C}_{d,d'}(P)(k)$ and the set of effective algebraic cycles on P of dimension d and degree d'. (We suppress the degree since it plays no role.) Let a and b be nonnegative integers such that $a + b + 1 = n$. The product $\mathscr{C}_a(P) \times \mathscr{C}_b(P)$ parameterizes pairs (A, B) of an a-dimensional cycle A and a b-dimensional cycle B. Since $a + b$ is less than n, one expects a generic A and B to be disjoint. But one expects that the *incidence locus* $\mathscr{I}$ parameterizing incident pairs is a codimension 1 closed subvariety in $\mathscr{C}_a(P) \times \mathscr{C}_b(P)$.

For k the field of complex numbers, Mazur [14] constructed a Weil divisor supported on the incidence locus as follows. Consider the diagram of schemes:

$$\begin{tikzcd} P \times \mathscr{C}_a(P) \times \mathscr{C}_b(P) \arrow[r, "\Delta"] \arrow[d, "pr_{23}"] & P \times P \times \mathscr{C}_a(P) \times \mathscr{C}_b(P) \\ \mathscr{C}_a(P) \times \mathscr{C}_b(P) & \end{tikzcd}$$

Let U_a, U_b denote the universal cycles on $P \times \mathscr{C}_a(P)$, $P \times \mathscr{C}_b(P)$ respectively (these exist in characteristic zero). Since Δ is a local complete intersection morphism, there is a refined Gysin homomorphism $\Delta^!$, as constructed in [7, 6.2]. Using standard operations in intersection theory, one has $pr_{23*}\Delta^!(U_a \boxtimes U_b)$, a cycle of codimension 1 on $\mathscr{C}_a(P) \times \mathscr{C}_b(P)$. The main question of [14] is whether $pr_{23*}\Delta^!(U_a \boxtimes U_b)$ is a Cartier divisor. The main result of this paper is a positive answer to Mazur's question.

Theorem 1.1. *Let $U \subset \mathscr{C}_a(P) \times \mathscr{C}_b(P)$ denote the locus of disjoint cycles, i.e., the complement of the incidence locus $\mathscr{I}$. Let $U' \subset \overline{U}$ denote the union of products $C_a \times C_b$ of irreducible components $C_a \subset \mathscr{C}_a(P), C_b \subset \mathscr{C}_b(P)$ over which the universal cycles intersect properly.*

- *There is a Cartier divisor D on $\overline{U}$ which is supported on $\overline{U} - U$.*
- *The restriction of D to U' is an effective Cartier divisor.*
- *Let T be the spectrum of a discrete valuation ring $R \supset k$, with generic point η. Let $g : T \to \overline{U} \subset \mathscr{C}_a(P) \times \mathscr{C}_b(P)$ be a morphism corresponding to cycles Z, W on $P \times T$, and such that $g(\eta) \in U$. Let s_D denote the canonical section of the line bundle $\mathcal{O}_{\overline{U}}(D)$. Then we have*

$$\operatorname{ord} g^*(s_D) = \deg(Z \cdot W) \in \mathbb{Z},$$

where $Z \cdot W \in A_0(P \times T)$ is the class constructed in [7, 20.2].

Remark 1.2. Our methods will suggest there is a line bundle on the whole of the product $\mathscr{C}_a(P) \times \mathscr{C}_b(P)$, but it does not seem reasonable to expect a Cartier divisor beyond the locus $\overline{U}$. On the locus U', the operation $\Delta^!$ is defined on the cycle level, and all of the coefficients appearing in $pr_{23*}\Delta^!(U_a \boxtimes U_b)$ are positive. On the locus $\overline{U} - U'$, negative coefficients may appear.

Techniques. Our approach to Mazur's question, initiated in [20], is to define the incidence line bundle $\mathcal{L}$ on a product of Hilbert schemes mapping to the corresponding Chow varieties, and then show $\mathcal{L}$ is the pullback of a line bundle $\mathcal{M}$ on the Chow varieties. Our $\mathcal{L}$ will be equipped with a canonical nonvanishing rational section on the locus of disjoint subschemes, and we will show this section is induced by a trivialization of $\mathcal{M}$ on U. Briefly, $\mathcal{L}$ is the determinant of a perfect complex formed from the universal flat families. Then we form a proper hypercovering of $\overline{U}$ along the Hilbert-Chow morphism, and a descent datum for $\mathcal{L}$ on this hypercovering. This amounts to an identification ϕ between two pullbacks of $\mathcal{L}$, satisfying a cocycle condition.

At first we define the descent datum ϕ over a normal base provided the incidence has the expected dimension (5.18); this boils down to the Serre Tor-formula for intersection multiplicities and basic properties of the determinant functor (additivity on short exact sequences). To extend the descent datum over families with more complicated incidence structure, we establish some moving lemmas to produce local trivializations (5.13, 5.14), then apply Grothendieck-Riemann-Roch to show these local sections glue (5.17, 5.32). A useful tool is the following result, which characterizes functions on a seminormal scheme (4.3; see also Definition 4.2): a Noetherian ring A is seminormal if and only if every pointwise function on Spec A which varies algebraically along (complete) DVRs is induced by an element of A.

As for the effectiveness of $(\mathcal{L}, \phi)$, i.e., that $\mathcal{L}$ is induced by a line bundle on the Chow varieties, an outgrowth of 4.3 is a criterion for effective descent (4.6) applicable to our Hilbert-Chow hypercovering: the bundle $\mathcal{L}$ descends to $\mathcal{M} \in \mathrm{Pic}(\overline{U})$ if it can be trivialized locally on $\overline{U}$, compatibly with the descent datum ϕ. The compatible local trivializations are built into the definition of the descent datum.

Motivation. In the classical construction, the Chow variety $\mathscr{C}_{d,d'}(\mathbb{P}^n)$ is realized as a closed subvariety of the scheme of Cartier divisors of the Grassmannian $\mathscr{G}$ of $(n-d-1)$-planes: to a d-dimensional cycle Z on $\mathbb{P}^n$ we associate the codimension one set of $(n-d-1)$-planes in $\mathbb{P}^n$ which intersect Z. Thus the natural ample line bundle on $\mathscr{G} \times \mathrm{CDiv}(\mathscr{G})$ simultaneously shows the projectivity of the Chow variety, and endows the incidence locus (in $\mathscr{C}_{d,d'}(\mathbb{P}^n) \times \mathscr{G}$, a special case of the $\mathscr{I}$ considered above) with the structure of an effective Cartier divisor. This generalizes to the case $\mathscr{C}_d(\mathbb{P}^n) \times \mathscr{C}_{n-d-1}(\mathbb{P}^n)$ using the ruled join; see [17].

This direct geometric construction does not extend to general smooth projective P. However, the Hilbert scheme (the moduli space for closed subschemes of P) and the Hilbert-Chow morphism $\mathscr{H} \to \mathscr{C}$ suggest another approach. The pullback of the line bundle associated to the incidence divisor via $\mathscr{H}(\mathbb{P}^n) \times \mathscr{G} \to \mathscr{C}(\mathbb{P}^n) \times \mathscr{G}$ is the determinant of a perfect complex formed from the universal flat families (see the end of Section 3), and the determinant construction can be defined for any smooth projective P. Thus one is naturally led to wonder, for a general pair of Hilbert schemes parameterizing subschemes of dimension a, b as above, whether the determinant line bundle descends to the corresponding product of Chow varieties. The direct geometric construction for $P = \mathbb{P}^n$ and the determinant formula are in fact compatible; see the end of Section 3.

Further motivation comes from the case of zero-cycles and divisors, where the Hilbert-Chow morphism admits a reasonably explicit description. The equality of the families of zero-cycles associated to two families of zero-dimensional subschemes has a natural expression in terms of determinants; and similarly two families of codimension one subschemes determine the same family of cycles if the determinants of their structure sheaves agree. For a detailed study of the determinant bundle in the case of zero-cycles and divisors, in particular the descent to the Chow varieties, see [20].

Contents. In Section 2 we recall background material on determinant functors and K-theory. In Section 3 we discuss the relevant properties of the Chow variety and the Hilbert-Chow morphism, and define the incidence line bundle and the Hilbert-Chow hypercovering along which the incidence bundle descends. In Section 4 we explain the role of seminormality both in defining the descent datum and demonstrating its effectiveness. In Section 5 we construct the descent datum and show it is effective.

Other work. In [24] Wang proved that a certain multiple of the incidence divisor, namely $(n-1)!({pr_{23}}_*\Delta^!(U_a \boxtimes U_b))$, is Cartier by using the Archimedean height pairing on algebraic cycles (over $\mathbb{C}$). (See the references in [24] for history of the height pairing.) Given disjoint cycles A, B on P as above, one has the pairing $\langle A, B\rangle := \int_A [G_B]$ defined by integrating a normalized Green's current for B over A. Wang views $\langle A, B\rangle$ as a function on the open set U in $\mathscr{C}_a(P) \times \mathscr{C}_b(P)$ consisting of disjoint cycles, and by studying the behavior of the function as the cycles collide, obtains [24, Thm. 1.1.2] a metrized line bundle L on $\overline{U}$ and a rational section s that is regular and nowhere zero on U, such that $\log ||s(A, B)||^2 = (\dim(P) - 1)!\langle A, B\rangle$. Using relative fundamental classes in Deligne cohomology (and again over $\mathbb{C}$), Barlet and Kaddar [4] constructed an incidence Cartier divisor in the analytic setting under the assumption that the incidence is generically finite over the parameter space. It would be interesting to "go back" from the Chow varieties to the height pairing.

Conventions. We use the definition of the Chow variety from [13]. In characteristic 0, there is a functor of effective algebraic cycles (of dimension d and degree d') defined on the category of seminormal k-schemes; and this functor is represented by a seminormal, projective k-scheme $\mathscr{C}_{d,d'}(P)$ [13, I.3.21]. In characteristic $p > 0$, there are several plausible notions of a family of effective algebraic cycles, stemming from the ambiguity of the field of definition of a cycle [13, I.4.11]. In this case we work with the seminormal, projective k-scheme $\mathscr{C}_{d,d'}(P)$ constructed in [13, I.4.13]. This coarsely represents at least two reasonable functors of effective algebraic cycles. Since we rely on the method of "seminormal descent," our methods do not apply to other definitions of Chow varieties (e.g., those of Barlet [3], Angéniol [1], and Rydh [21]) when those constructions yield Chow varieties/schemes which are not seminormal.
All schemes considered in this paper are locally Noetherian. A variety over a field k is an integral separated scheme of finite type over k.

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Section 2. Determinant functors and K-groups

A determinant functor assigns an invertible sheaf to each perfect complex. We discuss this notion following [12]. Then we quickly review some background material on K-groups.

Notation 2.1. Let X be a scheme. Let $D(X)$ denote the derived category of the abelian category $\mathrm{Mod}(X)$ of $\mathcal{O}_X$-modules. Denote by $D^+(X)$, respectively $D^-(X)$, the full subcategory of $D(X)$ whose objects are complexes of $\mathcal{O}_X$-modules which are bounded below, respectively bounded above. Denote by $D^b(X)$ the full subcategory whose objects are complexes which are both bounded below and bounded above.

We denote by $D_{(q)coh}(X)$ the full triangulated subcategory of $D(X)$ consisting of pseudo-(quasi)coherent complexes, and by $D^*_{(q)coh}(X)$ the corresponding bounded category for $* = +, -, b$ [10, 1.4 p.38, 2.1 p.85]. We denote by $\mathrm{Parf}(X) \subset D^b(X)$ the full triangulated subcategory consisting of perfect complexes [9, Exp. I Def. 4.7 p.44].
Let Parf-is(X) denote the category whose objects are perfect complexes on X, with morphisms isomorphisms in $D(X)$. Let $\mathrm{Pic}(X)$ denote the category whose objects are invertible sheaves on X, and whose morphisms are isomorphisms. This is a Picard category in the sense of [2, Exp. XVIII Def. 1.4.2].

Determinants. The main result of [12, Thm. 2] is that there exists up to canonical isomorphism a unique *determinant functor* $\det_X$: Parf-is$(X) \to \mathrm{Pic}(X)$ extending the usual determinant (top exterior power) of a locally free sheaf. Indeed the idea is to locally replace a perfect complex by a bounded complex of locally free sheaves, and take the signed tensor product of the usual determinants of the locally free terms. Then one shows this patches to give a global invertible sheaf. For every true triangle of complexes $0 \to \mathcal{F}_1 \xrightarrow{\alpha} \mathcal{F}_2 \xrightarrow{\beta} \mathcal{F}_3 \to 0$ in Parf-is(X), we require an isomorphism $i_X(\alpha, \beta) : \det(\mathcal{F}_1) \otimes \det(\mathcal{F}_3) \xrightarrow{\sim} \det(\mathcal{F}_2)$, and the isomorphisms i (extending the obvious i for short exact sequences of locally free sheaves) are required to be compatible with isomorphisms of triangles, and more generally triangles of triangles. Associated to morphisms of schemes we have base change isomorphisms interchanging the determinant with pullback, and these are required to be compatible with composition of morphisms of schemes.

Remark 2.2. When X is reduced, i extends to the class of distinguished triangles, is functorial over isomorphisms (in $D(X)$) of distinguished triangles, and is compatible with distinguished triangles of distinguished triangles [12, Prop. 7].

Associated Cartier divisors. If $\mathcal{F} \in \mathrm{Parf}(X)$ is acyclic at every $x \in X$ of depth 0, then [12, Ch.II] constructs a Cartier divisor $\mathrm{Div}(\mathcal{F})$ on X and a canonical isomorphism $\mathcal{O}_X(\mathrm{Div}(\mathcal{F})) \cong \det_X(\mathcal{F})$ extending the trivialization $\mathcal{O}_{X,x} \cong \det_x \mathcal{F}$ at $x \in X$ of depth 0. The formation of this divisor and isomorphism is additive on short exact sequences [12, Thm.3(ii)], and is compatible with base change $f : X' \to X$ such that $\mathrm{L}f^*(\mathcal{F})$ is acyclic at every $x' \in X'$ of depth 0 [12, Thm.3(v)]. Furthermore, in case X is normal and $x \in X$ is a point of depth 1, the coefficient of $\overline{\{x\}}$ in the Weil divisor associated to $\mathrm{Div}(\mathcal{F})$ is the alternating sum of the lengths (at x) of the cohomology sheaves, i.e., $\sum_i (-1)^i \ell_x(\mathcal{H}^i(\mathcal{F}))$. This construction is also studied in [6, Sect. 3] and [15, Ch. 5 Sect. 3].
We mention two further properties implicit in [12, Thm.3]. If the complex $\mathcal{F}$ is acyclic, then $\mathrm{Div}(\mathcal{F}) = 0$ and the canonical isomorphism $\mathcal{O}_X(\mathrm{Div}(\mathcal{F})) = \mathcal{O}_X \cong \det_X(\mathcal{F})$ is the trivialization of the determinant of an acyclic complex [12, Lemma 2]. Finally, the construction is determined by the quasi-isomorphism class of the perfect complex $\mathcal{F}$: since all filtration levels and subquotients

appearing in the canonical filtration ("good truncation") of $\mathcal{F}$ are generically acyclic so long as $\mathcal{F}$ is, the additivity implies $\mathrm{Div}(\mathcal{F}) = \sum_i (-1)^i \mathrm{Div}(\mathcal{H}^i(\mathcal{F}))$.

K-GROUPS. Let X be a variety. Then $K_0(X)$ is the Grothendieck group of X, generated by coherent sheaves on X with relations for short exact sequences of sheaves; $K^0(X)$ is the Grothendieck group of vector bundles. When X is regular, we have an isomorphism $K_0(X) \cong K^0(X)$. We have also the Chern character $\mathrm{ch} : K_0(X) \to A_*(X)$, where $A_*(X)$ is the Chow group of cycles on X, graded by dimension. We note $\mathcal{F} \in \mathrm{Parf}(X)$ determines a class in $K_0(X)$ since for any abelian category $\mathscr{A}$ (e.g., $\mathrm{Coh}(X)$), $K_0(\mathscr{A}) \cong K_0(D^b(\mathscr{A}))$; the latter group is generated by objects of the triangulated category $D^b(\mathscr{A})$ with relations for distinguished triangles.

The group $K_0(X)$ has a topological filtration: the subgroup $F_k(K_0(X))$ is generated by those $\mathcal{F} \in \mathrm{Coh}(X)$ such that $\dim(\mathrm{Supp}(\mathcal{F})) \leq k$. For a proper morphism of schemes $f : X \to T$ we obtain a homomorphism $f_* : K_0(X) \to K_0(T)$ sending (the class of) a coherent sheaf to the alternating sum of (the classes of) its higher direct image sheaves. This preserves the topological filtration. If T is a point and $\mathcal{F} \in \mathrm{Coh}(X)$, then $\chi(\mathcal{F}) = \mathrm{ch}(f_*(\mathcal{F}))$.

SECTION 3. THE HILBERT-CHOW MORPHISM AND THE INCIDENCE DIVISOR

In this section we define the Chow variety, the Hilbert-Chow morphism, and construct our proper hypercovering. Then we define the incidence line bundle on the product of Hilbert schemes.

We recall an application of the characterization of seminormal schemes [19, 5.1], where it is shown that properties (1)-(5) below characterize the Chow variety. For properties (6) and (7) we refer to [13].

DEFINITION-THEOREM 3.1 (Existence of the Chow variety). *Let P be a smooth projective variety over a field k. The Chow variety $\mathscr{C}_{d,d'}$ of P is a k-scheme with the following properties:*

(1) It is projective over k.

(2) It is seminormal.

(3) For every point $w \in \mathscr{C}_{d,d'}$ there exist purely inseparable field extensions $\kappa(w) \subset L_i$ and cycles Z_i on P_{L_i} such that:

(a) Z_i and Z_j are essentially equivalent [13, I.3.8]*: they agree as cycles over the perfection $\kappa(w)^{perf}$ of $\kappa(w)$;*

(b) the intersection of the fields L_i is $\kappa(w)$, which is the Chow field (field of definition of the Chow form in any projective embedding of P) of any of the Z_i [13, I.3.24.1]*; and*

(c) for any cycle Z on P_M defined over a subfield $k \subset M \subset \kappa(w)^{perf}$ which agrees with the Z_i over $\kappa(w)^{perf}$ (equivalently, agrees with one Z_i), we have $\kappa(w) \subset M$ (the Chow field is the intersection of all fields of definition of the cycle).

(4) Points w of $\mathscr{C}_{d,d'}$ are in bijective correspondence with systems $(k \subset \kappa(w), \{\kappa(w) \subset L_i, Z_i\}_{i \in I})$ up to an obvious equivalence relation.

(5) *For any DVR $R \supset k$ and any cycle Z on P_R of relative dimension d and degree d' in the generic fiber, we obtain a morphism $g : \operatorname{Spec} R \to \mathscr{C}_{d,d'}$ such that the generic fiber Z_η and the special fiber Z_s agree with the systems of cycles of the previous property at $g(\eta)$ and $g(s)$.*

(6) *For any numerical polynomial q of degree d and with leading coefficient $d'/(d!)$, we obtain a morphism (the Hilbert-Chow morphism)*

$$FC : (\mathscr{H}^q)^{sn}_{red} \to \mathscr{C}_{d,d'}$$

by taking the fundamental cycle of the components of maximal relative dimension $(= d)$[13, I.6.3.1]. *A finite number of $(\mathscr{H}^q)^{sn}_{red}$'s surject onto $\mathscr{C}_{d,d'}$.*

(7) *Let $\eta \in \mathscr{C}_{d,d'}$ be a generic point. Then either $\dim\overline{\{\eta\}} = 0$ or there exists a cycle Z_η on P_η defined over $\kappa(\eta)$. In particular, if k is perfect then there exists a Z_η for every generic point η of $\mathscr{C}_{d,d'}$* [13, I.4.14].

Construction 3.2. Let P be a smooth projective variety, and $r \in \mathbb{Z}_{\geq 0}$. Let $\mathscr{H}'_r$ denote the Hilbert scheme of r-dimensional subschemes of P. Let $\mathscr{H}_r$ denote the seminormalization of the (closed) subscheme of $\mathscr{H}'_r$ consisting of subschemes Z such that Z has pure r-dimensional support (this is different from the notion of a pure sheaf: Z may have embedded components of smaller dimension so long as they are set-theoretically contained in the top-dimensional components). We have the product of the Hilbert-Chow morphisms (3.1) $\pi : Y_0 = \mathscr{H}_a \times \mathscr{H}_b \to \mathscr{C}_a \times \mathscr{C}_b =: C$. Because seminormalization is a functor, we may form a proper hypercovering $\pi_\bullet : Y_\bullet \to C$ augmented towards C whose i-th term Y_i is the seminormalization of $Y_0 \times_C \ldots \times_C Y_0$ ($i+1$ factors), with the (seminormalizations of the) canonical morphisms.

Remarks 3.3. (3.3.1) We explain property (7) in more detail. For any positive-dimensional component of $\mathscr{C}_{d,d'}$, its generic point corresponds to a cycle all of whose coefficients are 1, i.e., a subscheme [13, 1.4.14]. Hence we can find a (component of some) $(\mathscr{H}^q)^{sn}_{red}$ admitting a birational morphism to that component.

(3.3.2) Over a field of characteristic zero, the seminormality of the Chow variety and [19, 4.1] imply $\mathcal{O}_{\mathscr{C}} = \pi_{\bullet *}(\mathcal{O}_{X_\bullet})$ for a proper hypercovering $X_\bullet$ augmented towards $\mathscr{C}$. In characteristic $p > 0$, we have the characterization of the residue fields on the Chow variety as the intersection of all fields of definition [13, I.4.5]. So by [19, 4.1 Corrigendum], we have $\mathcal{O}_{\mathscr{C}} = \pi_{\bullet *}(\mathcal{O}_{X_\bullet})$ for a proper hypercovering such that $X_0 = \mathscr{H}^{sn}_{red}$.

In more detail and in the language of [19, 4.1], we explain how to construct (locally) a pointwise function on $\mathscr{C}$ from a pointwise function on $\mathscr{H}^{sn}_{red}$ which belongs to $\pi_{\bullet *}(\mathcal{O}_{X_\bullet})$. So suppose $z \in \mathscr{C}(P_k)$ corresponds via a morphism $\operatorname{Spec} \kappa(z) \to \mathscr{C}(P_k)$ to the cycle Z on $P_{\kappa(z)}$. Consider an algebraically closed field $\overline{K}$ containing $\kappa(z)$ and the cycle associated to the base change $Z_{\overline{K}}$. Then by [13, I.4.5], the residue field $\kappa(z)$ is characterized as the intersection in $\overline{K}$ of all fields of definition of Z, i.e., the intersection of all E_i such that $k \subset E_i \subset \overline{K}$ and there exists a subscheme $Y_i \subset P_{E_i}$ whose associated cycle agrees with $Z_{\overline{K}}$ upon base change. Consider fields E_0, E_1 satisfying these conditions. Then we have morphisms $\operatorname{Spec} E_i \to \mathscr{H}$ with the property that the compositions $\operatorname{Spec} \overline{K} \to \operatorname{Spec} E_i \to \mathscr{H} \to \mathscr{C}$ are the same. Thus we have a commutative diagram:

$$\begin{tikzcd}
\operatorname{Spec} \overline{K} \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & (\mathscr{H} \times_{\mathscr{C}} \mathscr{H})^{sn}_{red} \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] \\
\operatorname{Spec} E_0, \operatorname{Spec} E_1 \arrow[r] & \mathscr{H}
\end{tikzcd}$$

(The morphism from $\operatorname{Spec} \overline{K}$ factors through the seminormalization.) Considering $a \in \pi_{\bullet *}(\mathcal{O}_{X_\bullet})$ as a pointwise function, we obtain elements $a_i \in E_i$. The preceding diagram shows $a_0 = a_1$ in $\overline{K}$, therefore $a_0 \in E_0 \cap E_1$. By the same argument we find $a_0 \in E_0 \cap E_i$ for all i, therefore $a_0 \in \kappa(z)$. Thus we made an element in the residue field $\kappa(z)$. It varies algebraically along DVRs by [19, 4.1].

(3.3.3) If $X = Y \cup Z$ is a reducible scheme with irreducible components Y, Z, then the field of definition of Y is contained in the field of definition of X. Also, a scheme and its seminormalization have the same residue fields. Hence to cut out the residue fields on $\mathscr{C}$, it is enough to consider the subscheme $\mathscr{H}_r \hookrightarrow \mathscr{H}'_r$ defined in 3.2. So we have $\mathcal{O}_{\mathscr{C}} = \pi_{\bullet *}(\mathcal{O}_{X_\bullet})$ for a proper hypercovering such that $X_0 = \mathscr{H}_r$.

We record the (presumably known) fact that the Hilbert-Chow morphism is compatible with products.

LEMMA 3.4. *If P, P' are smooth projective varieties over a field k, then the following diagram commutes. (We suppose p has leading coefficient $d'/(d!)$ and $\deg(p) = d$; and q has leading coefficient $e'/(e!)$ and $\deg(q) = e$.)*

$$\begin{tikzcd}
(\mathscr{H}^p(P))^{sn}_{red} \times (\mathscr{H}^q(P'))^{sn}_{red} \arrow[r] \arrow[d, "FC \times FC"] & (\mathscr{H}^{pq}(P \times P'))^{sn}_{red} \arrow[d, "FC"] \\
\mathscr{C}_{d,d'}(P) \times \mathscr{C}_{e,e'}(P') \arrow[r] & \mathscr{C}_{d+e,d'e'}(P \times P')
\end{tikzcd}$$

Proof. We describe the map in the top row: if $Z \hookrightarrow P \times T, Z' \hookrightarrow P' \times T$ constitute a T-point of $(\mathscr{H}^p(P))^{sn}_{red} \times (\mathscr{H}^q(P'))^{sn}_{red}$, then the scheme theoretic intersection $pr_{13}^* Z \cap pr_{23}^* Z'$ in $P \times P' \times T$ is a T-point of $(\mathscr{H}^{pq}(P \times P'))^{sn}_{red}$. A top-dimensional component in the product scheme is induced by a pair of top-dimensional components; and length multiplies, so the coefficients in the product cycle are the products of the coefficients of the factors. $\square$

The main goal of this paper is to construct a Cartier divisor supported on the incidence locus. Now we define an invertible sheaf (the "incidence bundle") on a product of Hilbert schemes, and show the incidence bundle is pulled back from the product of Chow varieties in the case $P = \mathbb{P}^n$.

Construction 3.5. Let B be a base scheme, and P a smooth projective B-scheme. Let $\mathscr{H}^1, \mathscr{H}^2$ denote the Hilbert schemes corresponding to numerical polynomials q_1, q_2, and set $\mathscr{H}^{1,2} := \mathscr{H}^1 \times_B \mathscr{H}^2$.
Over each $\mathscr{H}^i$ we have a universal flat family, a closed subscheme of $P \times_B \mathscr{H}^i$. Denote by $\mathscr{U}_i$ its pullback to $P \times_B \mathscr{H}^{1,2}$. Standard facts about the behavior of perfect complexes under certain operations (stability under tensor product; pullback; and pushfoward via a proper morphism of finite Tor-dimension; and the perfectness of a coherent sheaf on the source of a smooth morphism which is flat over the target) imply $\mathrm{R}pr_{23*}(\mathcal{O}_{\mathscr{U}_1} \otimes^{\mathrm{L}} \mathcal{O}_{\mathscr{U}_2})$ is a perfect complex on $\mathscr{H}^{1,2}$. For details on the necessary facts about perfect complexes, see section 2 of [20]. The incidence bundle $\mathcal{L}$ is defined to be its determinant:

$$\mathcal{L} := \det{}_{\mathscr{H}^{1,2}} \mathrm{R}pr_{23*}(\mathcal{O}_{\mathscr{U}_1} \otimes^{\mathrm{L}} \mathcal{O}_{\mathscr{U}_2}).$$

In fact we will be interested in this construction only on the locus Y_0 defined earlier in this section. Furthermore, since the complex $\mathrm{R}pr_{23*}(\mathcal{O}_{\mathscr{U}_1} \otimes^{\mathrm{L}} \mathcal{O}_{\mathscr{U}_2})$ is acyclic over the locus U_0 of disjoint pairs of subschemes, this construction determines a Cartier divisor on the closure of the locus of disjoint pairs of subschemes (the Hilbert scheme analogue of the locus $\overline{U}$ defined in 1.1). See [20, 2.5].
As motivation for pursuing the determinant formula (mentioned in the Introduction), we make contact with the classical construction of the Chow variety of $P = \mathbb{P}^n$. As explained in the Introduction, the construction of the Chow variety endows the incidence locus $\mathscr{I} \hookrightarrow \mathscr{C}_d(\mathbb{P}^n) \times \mathscr{G}$ with the structure of a Cartier divisor. Let $FC_{\mathbb{P}^n} : \mathscr{H}(\mathbb{P}^n) \to \mathscr{C}(\mathbb{P}^n)$ denote the Hilbert-Chow morphism (and its product with $\mathscr{G}$). In the special case of Construction 3.5 with $P = \mathbb{P}^n$ and $\mathscr{H}^2 = \mathscr{G}$, it follows from [5, Thms. 1.2, 1.4] that there is a canonical isomorphism $\mathcal{L} \cong {FC_{\mathbb{P}^n}}^* \mathcal{O}(\mathscr{I})$ of invertible sheaves on $\mathscr{H} \times \mathscr{G}$. This isomorphism is canonical in the following sense. Over the locus U_0 of disjoint subschemes, there is a canonical trivialization $\mathcal{L}|_{U_0} \cong \mathcal{O}_{U_0}$. This rational section is the pullback via $FC_{\mathbb{P}^n}$ of the canonical trivialization of $\mathcal{O}(\mathscr{I})$ on the complement of $\mathscr{I}$.

Section 4. Seminormal schemes and descent criteria

In this section we explain the role of seminormality both in defining the descent datum and demonstrating its effectiveness.

Definition 4.1 ([8]). A ring A is a *Mori ring* if it is reduced and its integral closure A^ν (in its total quotient ring Q) is finite over it; if A is a Mori ring, A^{sn} denotes its seminormalization, the largest subring $A \subset A^{sn} \subset A^\nu$ such that $\operatorname{Spec} A^{sn} \to \operatorname{Spec} A$ is bijective and all maps on residue fields are isomorphisms. The seminormalization is described elementwise in [23, 1.1]. We say A is seminormal if $A = A^{sn}$ (so we only define seminormality for Mori rings). A locally Noetherian scheme X is Mori if and only if it has an affine cover by Noetherian Mori rings [8, Def. 3.1].

DEFINITION 4.2. Let A be a ring, and let $S = \{f_y \in \kappa(y) | y \in \operatorname{Spec} A\}$ be a collection of elements, one in each residue field. Then we say S *is a pointwise function on* $\operatorname{Spec} A$. We say the pointwise function S *varies algebraically along (complete) DVRs* if it has the following property: for every specialization $\mathfrak{p}_1 \subset \mathfrak{p}_2$ in A and every (complete) discrete valuation ring R covering that specialization via a ring homomorphism $g : A \to R$, there exists a (necessarily) unique $f_R \in R$ such that $\overline{g_{\mathfrak{p}_1}}(f_{\mathfrak{p}_1}) = f_R$ (in K) and $\overline{g_{\mathfrak{p}_2}}(f_{\mathfrak{p}_2}) = \overline{f_R}$ (in k_0).

The main result of [19, 2.2, 2.6] is the following.

THEOREM 4.3. *Let A be a seminormal (in particular, Mori) ring which is Noetherian. Let $\{f_y \in \kappa(y) | y \in \operatorname{Spec} A\}$ be a pointwise function on $\operatorname{Spec} A$ which varies algebraically along (complete) DVRs. Then there exists a unique $f \in A$ whose image in $\kappa(y)$ is f_y for all $y \in \operatorname{Spec} A$.*

This simplifies greatly the problem of defining a descent datum for a line bundle on a seminormal scheme, as seen in the following corollary.

COROLLARY 4.4. *Let X be a seminormal locally Noetherian (in particular, Mori) scheme, and let $L, M \in \operatorname{Pic}(X)$. Then an isomorphism $L \cong M$ is equivalent to an "identification of fibers varying algebraically along DVRs," that is:*

> *for any field or (complete) DVR R, any $\operatorname{Spec} R \xrightarrow{f} X$, an identification $\beta_f : f^*L \cong f^*M$ compatible with restriction to the closed and generic points: if $s \xrightarrow{i} \operatorname{Spec} R, \eta \xrightarrow{j} \operatorname{Spec} R$ denote the inclusions, then $\beta_{fi} = i^*\beta_f$ and $\beta_{fj} = j^*\beta_f$.*

Proof. Fix an open cover $X = \cup_i \operatorname{Spec} S_i$ with S_i a seminormal (Noetherian and Mori) ring which trivializes both L and M, and fix trivializations $\varphi_i : L_i := L|_{\operatorname{Spec} S_i} \cong \mathcal{O}_{\operatorname{Spec} S_i}$, $\psi_i : M_i := M|_{\operatorname{Spec} S_i} \cong \mathcal{O}_{\operatorname{Spec} S_i}$. Then defining $L \cong M$ is equivalent to identifying $\Gamma(\operatorname{Spec} S_i, L_i) \cong \Gamma(\operatorname{Spec} S_i, M_i)$ as S_i-modules (for all i), compatibly with restrictions. Then considering the diagram:

$$\begin{tikzcd} \Gamma(L_i) \arrow[r, "\cong"] \arrow[d, "\varphi_i"] & \Gamma(M_i) \arrow[d, "\psi_i"] \\ S_i \arrow[r] & S_i \end{tikzcd}$$

and its pullbacks to spectra of fields and DVRs, we see that relative to the fixed φ_i, ψ_i, a family β_f as in the statement is equivalent to an invertible pointwise function on each S_i varying algebraically along DVRs. By Theorem 4.3 this is equivalent to a family of elements $f_i \in S_i{}^\times = \operatorname{Isom}_{S_i}(S_i, S_i)$. The f_i thus obtained agree on overlaps by the uniqueness statement in Theorem 4.3. Then using the above diagram again we see that relative to the fixed trivializations, the family f_i is equivalent to a family of isomorphisms $\Gamma(L_i) \cong \Gamma(M_i)$ compatible with restrictions. □

We will need the following general fact later.

Lemma 4.5. *Let X be a seminormal k-scheme and $Y \hookrightarrow X$ a closed subscheme. Suppose every $y \in Y$ admits a generization to a point in $X - Y$, i.e., that Y does not contain any generic points of X. Let S be a pointwise function on X which varies algebraically along DVRs covering specializations within $X - Y$, and along those from $X - Y$ to Y. Then S varies algebraically along DVRs.*

Proof. This follows readily from the techniques used in [19]. We may assume X is affine. Let $\nu : X^\nu \to X$ denote the normalization. Then our pointwise function S determines a pointwise function on X^ν which is constant along the fibers of ν. The normalization is birational, so identifies generic points of X^ν with those of X. Hence as a pointwise function on X^ν, S varies algebraically along specializations of the form $\eta \rightsquigarrow x$, with η generic. This is enough to conclude S is induced by an element of $\Gamma(X^\nu, \mathcal{O}_{X^\nu})$ (see [19, 2.4]). But then because S is constant along the fibers of ν, this element comes from $\Gamma(X, \mathcal{O}_X)$ and *a fortiori* varies algebraically along all DVRs. $\square$

As for the effectiveness of a descent datum, we recall the following result from [20]. Let $\pi_\bullet : X_\bullet \to X$ be a proper hypercovering augmented towards a scheme X. We denote by $(\mathcal{L}, \phi)$ an element of $\operatorname{Pic}(X_\bullet)$, i.e., $\mathcal{L}$ is an element of $\operatorname{Pic}(X_0)$ and $\phi : {p_0}^*\mathcal{L} \xrightarrow{\sim} {p_1}^*\mathcal{L}$ is an isomorphism on X_1 satisfying the cocycle condition on X_2. As in [20, 3.3], we say $(\mathcal{L}, \phi) \in \operatorname{Pic}(X_\bullet)$ is *Zariski locally effective* if for every $x \in X$, there exists an open $U \subset X$ containing x and a trivialization $T_x : \mathcal{L}|_{{\pi_0}^{-1}(U)} \xrightarrow{\sim} \mathcal{O}_{{\pi_0}^{-1}(U)}$ compatible with ϕ in the sense that the diagram

$$\begin{tikzcd}
{p_0}^*(\mathcal{L}|_{{\pi_0}^{-1}(U)}) \arrow[r, "{p_0}^*T"] \arrow[d, "\phi"] & \mathcal{O}_{(p_0)^{-1}({\pi_0}^{-1}(U))} \arrow[d, "="] \\
{p_1}^*(\mathcal{L}|_{{\pi_0}^{-1}(U)}) \arrow[r, "{p_1}^*T"] & \mathcal{O}_{(p_1)^{-1}({\pi_0}^{-1}(U))}
\end{tikzcd}$$

commutes.

Proposition 4.6. [20, 3.4] *Let X be a scheme, and let $\pi_\bullet : X_\bullet \to X$ be a proper hypercovering augmented towards X which satisfies $\mathcal{O}_X = \pi_{\bullet *}(\mathcal{O}_{X_\bullet})$. Then:*

- ${\pi_\bullet}^* : \operatorname{Pic}(X) \to \operatorname{Pic}(X_\bullet)$ *is injective; and*
- *the image of ${\pi_\bullet}^*$ consists of those $(\mathcal{L}, \phi)$ that are Zariski locally effective.*

Remark 4.7. The Proposition applies when X is seminormal and $X_\bullet$ satisfies any of the conditions in [19, 4.1 Corrigendum], for example the proper hypercovering $\pi_\bullet : Y_\bullet \to C$ defined in 3.2.

Section 5. Definition of the descent datum

In this section we prove the main result, in the following form. Having established this result, we will consider the refinements and further properties stated in 1.1.

Theorem 5.1 (5.36, 5.37). *With the notation as in 3.2, let $\mathscr{U}_r \hookrightarrow P \times \mathscr{H}_r$ denote the (pullback of the) universal flat family. Using 3.5 we may form the determinant line bundle $\mathcal{L}$ on Y_0. Now base change everything to $\overline{U} \subset C$, the closure of the locus of disjoint cycles. Then the following hold.*

- *The sheaf $\mathcal{L}$ lifts to an invertible sheaf on $Y_\bullet$, i.e., there is an isomorphism $\phi : {p_0}^*\mathcal{L} \cong {p_1}^*\mathcal{L}$ on Y_1 satisfying the cocycle condition on Y_2.*
- *The descent datum ϕ is effective: there is a unique $\mathcal{M} \in \operatorname{Pic}(\overline{U})$ such that $(\pi^*\mathcal{M}, can) \cong (\mathcal{L}, \phi)$.*

Subsection 5.1. Notation and preliminary reductions.

Definition 5.2. Let P be a smooth projective k-variety of dimension n. A *Hilbert datum for P over T* consists of the following:

(1) a seminormal k-scheme T;
(2) $Z \hookrightarrow P_T := P \times_k T$ a T-flat closed subscheme of relative dimension a, such that the support of Z has pure dimension a in every fiber; and
(3) $W \hookrightarrow P_T$ a T-flat closed subscheme of relative dimension b, such that the support of W has pure dimension b in every fiber;

such that $a + b + 1 \leq n$; and every point $t \in T$ admits a generization to the locus of disjoint subschemes. Thus a Hilbert datum (Z, W) is simply a morphism $T \to \mathscr{H}_a \times \mathscr{H}_b$ such that the image of every generic point of T lies in a component of $\mathscr{H}_a \times \mathscr{H}_b$ with at least one pair of disjoint subschemes.

Typically we will make some construction from (Z, W) and then show the construction only depends on $[Z], [W]$, the cycles underlying Z and W. Therefore we make the following definition. A *Hilbert-Chow datum for P over T* is a pair of Hilbert data $(Z, W), (Z', W')$ for P over T such that $[Z] = [Z']$ and $[W] = [W']$. Since the supports of Z, W are assumed pure-dimensional, we have also $\operatorname{Supp}(Z) = \operatorname{Supp}(Z')$ and $\operatorname{Supp}(W) = \operatorname{Supp}(W')$. Thus a Hilbert-Chow datum (Z, Z', W, W') for P over T is nothing more than a morphism $T \to (\mathscr{H}_a \times \mathscr{H}_b \times_{\mathscr{C}_a \times \mathscr{C}_b} \mathscr{H}_a \times \mathscr{H}_b)^{sn}$ such that (after projecting to either $\mathscr{H}_a \times \mathscr{H}_b$ factor) every generic point of T lands in a pair of irreducible components with at least one pair of disjoint subschemes.

Because we work on the subscheme $\mathscr{H}_r$ of the Hilbert scheme, disjointness of subschemes on $\mathscr{H}_a \times \mathscr{H}_b$ corresponds exactly to disjointness of their associated cycles on $\mathscr{C}_a \times \mathscr{C}_b$. In general, two subschemes could have disjoint associated cycles but lower-dimensional components which coincide. So in the notation above we have $Z \cap W = \emptyset$ if and only if $Z' \cap W = \emptyset$.

Note that given a morphism $S \to T$ of seminormal k-schemes and a Hilbert-Chow datum for P over T, by pullback we obtain a Hilbert-Chow datum for P over S.

Notation 5.3. The structure morphism $P_T \to T$ will be called π.

For $\mathcal{F}, \mathcal{G} \in \operatorname{Parf}(P_T)$, we set $f_T(\mathcal{F}, \mathcal{G}) := \det_T \mathrm{R}\pi_*(\mathcal{F} \otimes^{\mathrm{L}} \mathcal{G}) \in \operatorname{Pic}(T)$.

If α is a b-dimensional cycle on P_T with $\alpha = \sum a_i W_i$, we put $f_T(\mathcal{O}_Z, \alpha) := \otimes_i (f_T(\mathcal{O}_Z, \mathcal{O}_{W_i})^{\otimes a_i})$. In general we use the notation $[-]$ to denote the cycle associated to a subscheme or coherent sheaf: this means the top-dimensional components and their geometric multiplicities, even if, for example, $b < n - a - 1$. (In fact we used this in the preceding definition.)

If T is affine and equal to $\operatorname{Spec} R$, we may write f_R for f_T.

We will use the subscripts $(-)_0$ and $(-)_\eta$ to denote the base change of some object to closed and generic fibers, respectively.

By the incidence $Z \cap W$, we mean the underlying reduced algebraic subset $\operatorname{Supp}(Z) \cap \operatorname{Supp}(W)$. Stated properties of $Z \cap W$ will depend only on the underlying supports $\operatorname{Supp}(Z), \operatorname{Supp}(W)$.

Goal. For every Hilbert-Chow datum, we aim to construct an isomorphism $\phi_T^{Z,Z',W,W'} : f_T(\mathcal{O}_Z, \mathcal{O}_W) \cong f_T(\mathcal{O}_{Z'}, \mathcal{O}_{W'})$ varying functorially in T, in such a way that the resulting descent datum $\{\phi_T\}$ is Zariski locally effective. The essential case is $b = n - a - 1$.

Proposition 5.4 (reduction to fields and DVRs). *To define an isomorphism $\phi_T : f_T(\mathcal{O}_Z, \mathcal{O}_W) \cong f_T(\mathcal{O}_{Z'}, \mathcal{O}_{W'})$ for each Hilbert-Chow datum, for all smooth projective P, so that for each P, the collection $\{\phi_T\}$*

(1) is compatible with base change $S \to T$; and
(2) satisfies the cocycle condition;

it is sufficient to define an isomorphism ϕ_T for each Hilbert-Chow datum with T the spectrum of a field or complete DVR, compatible with base change among fields and complete DVRs, and satisfying the cocycle condition on fields.

Proof. This is a consequence of 4.4. □

Proposition 5.5 (reduction to the diagonal). *To define an isomorphism $\phi_T : f_T(\mathcal{O}_Z, \mathcal{O}_W) \cong f_T(\mathcal{O}_{Z'}, \mathcal{O}_{W'})$ for each Hilbert-Chow datum, for all smooth projective P, so that for each P, the collection $\{\phi_T\}$:*

(1) is compatible with base change $S \to T$;
(2) satisfies the cocycle condition; and
(3) is Zariski locally effective;

it is sufficient to define an isomorphism ϕ_T for each Hilbert-Chow datum with $W = W'$, for all P, so that for each P, the collection $\{\phi_T\}$ has the stated properties.

Proof. On $P \times P \times T$, let $\mathcal{O}_\Delta$ denote the structure sheaf of the diagonal $(\times T)$, i.e., the image of the closed immersion $P \times T \xrightarrow{\Delta \times 1_T} P \times P \times T$. Given T-flat closed subschemes $Z, W \hookrightarrow P \times T$, we let $Z \times W \hookrightarrow P \times P \times T$ denote the scheme-theoretic intersection $pr_{13}^* Z \cap pr_{23}^* W$. Then there is a canonical isomorphism of line bundles on T: $f_T(\mathcal{O}_Z, \mathcal{O}_W; P) \cong f_T(\mathcal{O}_{Z \times W}, \mathcal{O}_\Delta; P \times P)$. Then the proposition follows from the fact that the Hilbert-Chow morphism is compatible with products (3.4). □

Remark 5.6. We may even assume W is constant, i.e., there is a k-subscheme W_k such that $W = W_k \times_k T$; and we may assume W_k is integral (even smooth).

NOTATION 5.7. When W and W' are omitted from the notations, this means $W = W'$.

We start with some easy cases of our goal.

LEMMA 5.8. *Let (Z, W) be a Hilbert datum over any base T such that $Z \cap W = \emptyset$. Then there is a canonical trivialization $\varphi_T^Z : f_T(\mathcal{O}_Z, \mathcal{O}_W) \cong \mathcal{O}_T$ which is compatible with base change.*

Proof. The hypothesis implies $\mathcal{O}_Z \otimes^{\mathrm{L}} \mathcal{O}_W$, hence also $\mathrm{R}\pi_*(\mathcal{O}_Z \otimes^{\mathrm{L}} \mathcal{O}_W)$, is acyclic. The pullback via $S \to T$ is also acyclic, and the trivialization of the determinant of an acyclic complex is compatible with base change. □

Remark 5.9. The canonical isomorphism φ of the lemma has an additivity property in each variable. For example, if $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to^{+1}$ is a distinguished triangle in $\mathrm{Parf}(P_T)$ such that $\mathrm{Supp}(\mathcal{F}_i) \cap W = \emptyset$ for all i, then the isomorphism $f_T(\mathcal{F}_1, \mathcal{O}_W) \otimes f_T(\mathcal{F}_3, \mathcal{O}_W) \cong f_T(\mathcal{F}_2, \mathcal{O}_W)$ induced by the triangle corresponds, via the identifications $\varphi_T^{\mathcal{F}_i}$, to multiplication $\mathcal{O}_T \otimes \mathcal{O}_T \to \mathcal{O}_T$.

COROLLARY 5.10. *Among Hilbert-Chow data satisfying $Z \cap W = Z' \cap W = \emptyset$, there exists a collection of isomorphisms $\phi_T^{Z,Z'} : f_T(\mathcal{O}_Z, \mathcal{O}_W) \cong f_T(\mathcal{O}_{Z'}, \mathcal{O}_W)$ which is compatible with base change and satisfies the cocycle condition.*

Proof. We define $\phi_T^{Z,Z';W} := (\varphi_T^{Z'})^{-1} \circ \varphi_T^Z : f_T(\mathcal{O}_Z, \mathcal{O}_W) \cong f_T(\mathcal{O}_{Z'}, \mathcal{O}_W)$ to be the composition of the canonical trivializations. This is compatible with base change because each φ_T^Z is. We check the cocycle condition:

$$\phi_T^{Z',Z'';W} \circ \phi_T^{Z,Z';W} = ((\varphi_T^{Z''})^{-1} \circ \varphi_T^{Z'}) \circ ((\varphi_T^{Z'})^{-1} \circ \varphi_T^Z) = (\varphi_T^{Z''})^{-1} \circ \varphi_T^Z = \phi_T^{Z,Z'';W}.$$

□

From now on we keep the collection $\{\phi_T\}$ whose existence is asserted in 5.10. The idea is to gradually extend it to a collection over Hilbert-Chow data with increasingly complicated incidence structure, until we have covered the whole moduli space. Note that an isomorphism of line bundles on a reduced (e.g., seminormal) scheme is determined by its restriction to generic points (i.e., points of depth 0). Since our base T is always reduced, when we have defined an isomorphism ϕ for a more general class of Hilbert-Chow data, to check agreement with previously defined isomorphisms it suffices to check agreement on generic points.

LEMMA 5.11. *Let (Z, W) be a Hilbert datum over T, and suppose that $(Z \cap W)_\eta = \emptyset$ for all generic points $\eta \in T$. (This holds, for example, whenever Z and W intersect properly on P_T.) Then there exists a Cartier divisor $D_{Z,W}$ on T and a canonical isomorphism $\varphi_T^Z : f_T(\mathcal{O}_Z, \mathcal{O}_W) \cong \mathcal{O}_T(D_{Z,W})$ characterized by agreeing with the trivialization $\varphi_\eta^{Z_\eta}$ for every generic point $\eta \in T$. When $Z \cap W = \emptyset$, $D_{Z,W} = 0$ and φ_T^Z is the canonical trivialization. The formation of the divisor $D_{Z,W}$ and the isomorphism φ_T^Z are compatible with base change $S \to T$ preserving the generic disjointness.*

Furthermore, the formation of $D_{Z,W}$ is additive in each variable: if $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to^{+1}$ is a distinguished triangle in $\mathrm{Parf}(P_T)$ *such that* $(\mathrm{Supp}(\mathcal{F}_i) \cap W)_\eta = \emptyset$ *for all generic points $\eta \in T$, all i; then $D_{\mathcal{F}_1,W} + D_{\mathcal{F}_3,W} = D_{\mathcal{F}_2,W}$; and the triangle induces, upon application of $\varphi_T^{\mathcal{F}_i}$, the canonical isomorphism $\mathcal{O}_T(D_{\mathcal{F}_1,W}) \otimes \mathcal{O}_T(D_{\mathcal{F}_3,W}) \cong \mathcal{O}_T(D_{\mathcal{F}_2,W})$. Similarly we have an additivity property in the variable W.*

Proof. To see the generic disjointness is satisfied when Z and W intersect properly, note that Z (resp. W) has codimension $\geq b+1$ (resp. $\geq a+1$) in P_T, hence $Z \cap W$ has codimension $\geq a+b+2$ in P_T. Therefore $\dim(Z \cap W) < \dim(T)$, so the support of $\mathcal{O}_Z \otimes^{\mathrm{L}} \mathcal{O}_W$ cannot dominate any component of T. The hypothesis on the incidence means the construction of [12, Ch.II] applies. The compatibility with base change is a consequence of [12, Thm.3(v)]; and the additivity is inherited from [12, Thm.3(ii)]. □

Remark 5.12. Our essential task is to show that given a Hilbert-Chow datum (Z, Z', W), we have $D_{Z,W} = D_{Z',W}$.

Subsection 5.2. Moving lemmas.

Proposition 5.13. *Let (Z, W) be a Hilbert datum over T the spectrum of a local ring, with $W = W_k \times_k T$ for a b-dimensional k-scheme W_k, and suppose $(Z \cap W)_\eta = \emptyset$ for every generic point $\eta \in T$. Then there exist subvarieties $B_1, \ldots, B_n \subset P$ of dimension $b+1$, $M_i \in \mathrm{Pic}(B_i)$, and short exact sequences:*

$$0 \to M_i \xrightarrow{s_0^i} \mathcal{O}_{B_i} \to Q_0^i \to 0$$

$$0 \to M_i \xrightarrow{s_\infty^i} \mathcal{O}_{B_i} \to Q_\infty^i \to 0$$

such that:

1. *$(Z \cap \mathrm{Supp}(Q_*^i))_\eta = \emptyset$ for all i, all generic η; and*
2. *the b-dimensional cycle $[W] + \sum_i([Q_0^i] - [Q_\infty^i])$ is disjoint from Z.*

Proof. We let Z_0 denote the cycle over the closed fiber of T. By Chow's moving lemma [16, Thm.], we can find a cycle α rationally equivalent to $[W_k]$ and satisfying $\alpha \cap Z_0 = \emptyset$; hence also $\alpha \cap Z = \emptyset$ on P_T. This shows we can achieve the second property; the issue is to show we can move W in such a way that the first property is satisfied.

Suppose we have a closed immersion $P \hookrightarrow \mathbb{P}^{2n+1}$. Then every step in moving a cycle involves essentially two choices: a linear space $L \cong \mathbb{P}^n \hookrightarrow \mathbb{P}^{2n+1}$, disjoint from P, from which projection induces a finite morphism $\pi_L : P \to \mathbb{P}^n$; and an element $g \in PGL(n+1)$. The excess intersection $e(Z_0, \pi_L^*\pi_{L*}[W] - [W])$ is smaller than $e(Z_0, [W])$ for generic L; and $\pi_L^*(g \cdot \pi_{L*}W)$ is disjoint from Z_0 for generic g.

If $(Z \cap W)_\eta = \emptyset$ for all generic points $\eta \in T$, then for generic choices of L, g, we have $(Z \cap \pi_L^*(g \cdot \pi_{L*}[W]))_\eta = (Z \cap \pi_L^*(\pi_{L*}[W]))_\eta = \emptyset$. The Q_*s are supported in subsets of the form $\pi_L^*(g \cdot \pi_{L*}W)$ and $\pi_L^*(\pi_{L*}W)$, hence the result. □

We need a slight variation for subvarieties W as in 5.13 of dimension strictly smaller than $n-a-1$; eventually we need that such subvarieties do not contribute to $D_{Z,W}$.

Proposition 5.14. *Let (Z, W) be a Hilbert datum over T a base of dimension ≤ 1, with $W = W_k \times_k T$ for a k-scheme W_k, and suppose $(Z \cap W)_\eta = \emptyset$ for every generic point $\eta \in T$.*
Suppose further that $\dim(W_k) = b \leq n-a-2$. Then there exist subvarieties $B_1, \ldots, B_n \subset P$ of dimension $b+1$, $M_i \in \mathrm{Pic}(B_i)$, and short exact sequences:

$$0 \to M_i \xrightarrow{s_0^i} \mathcal{O}_{B_i} \to Q_0^i \to 0$$

$$0 \to M_i \xrightarrow{s_\infty^i} \mathcal{O}_{B_i} \to Q_\infty^i \to 0$$

such that:

(1) $(Z \cap B_i)_\eta = \emptyset$ for all i, all generic η; and
(2) the b-dimensional cycle $[W] + \sum_i([Q_0^i] - [Q_\infty^i])$ is disjoint from Z.

Remark 5.15. The first condition in the conclusion implies $(Z \cap \mathrm{Supp}(Q_*^i))_\eta = \emptyset$ for all i, all generic η.

Proof. Again we are intersecting a finite number of open conditions. Without loss of generality we may assume W_k is an integral subscheme of dimension $n-a-2$. Let $pr_1(Z) \hookrightarrow P$ denote the "sweep" of the family Z (with the reduced structure); this is a subscheme of dimension $\leq a+1$.
Now $pr_1(Z)$ and W_k are not expected to meet, and we have an open dense $U \subset T$ such that $pr_1(Z_U) \cap W_k = \emptyset$. For a generic finite morphism $\pi : P \to \mathbb{P}^n$ (as in the proof of 5.13) we have, possibly after shrinking U, that $\pi(pr_1(Z_U)) \cap \pi(W_k) = \emptyset$; and that the pair $(Z, \pi^*\pi_*(W_k) - W_k)$ has smaller excess intersection than does (Z, W_k). Now we move $\pi_*(W_k)$ along a general smooth (affine) rational curve $C \hookrightarrow PGL(n+1)$. Let $\mathscr{Y} \hookrightarrow \mathbb{P}^n \times C$ note the total space of the resulting family,
Write $\mathscr{Y} = \sum m_i Y_i$, and let $Y_i^{fl} \hookrightarrow \mathbb{P}^n \times \mathbb{P}^1$ denote the flat limit of the family $Y_i \hookrightarrow \mathbb{P}^n \times C$. Then $\mathscr{Y}^{fl} := \sum m_i Y_i^{fl}$ is the unique way to complete $\mathscr{Y}$ to a family of cycles over $\mathbb{P}^1$. Let $pr_1(\mathscr{Y}^{fl}) \hookrightarrow \mathbb{P}^n$ be the sweep; this is a subscheme of dimension $n-a-1$. Choose some $t \in T$ such that $Z_t \cap W_k = \emptyset$. For a general choice of C, since $\dim(Z_t) + \dim(\mathscr{Y}^{fl}) = a + (n-a-1) < n$, we will have $\pi(pr_1(Z_t)) \cap \mathscr{Y}^{fl} = \emptyset$. Hence the disjointness holds on an open dense of T. This process can be iterated until we have a cycle $\alpha \sim W_k$ such that $pr_1(Z) \cap \alpha = \emptyset$.
The subvarieties $B_i \hookrightarrow P$ lie in subsets of the form $\pi^{-1}(\mathscr{Y}^{fl})$. (This follows from the proof that flat pullback preserves rational equivalence [7, 1.7].) Since a general $\mathscr{Y}^{fl}$ used in one step of the moving process is disjoint from a general member of the family Z, this holds after pullback by π as well. □

Subsection 5.3. Grothendieck-Riemann-Roch.

Lemma 5.16. *Let T be the spectrum of a field, and suppose $\mathcal{F}, \mathcal{G} \in \mathrm{Parf}(P_T)$ satisfy $\dim(\mathrm{Supp}(\mathcal{F})) + \dim(\mathrm{Supp}(\mathcal{G})) < n = \dim(P)$. Then $\chi(\mathcal{F} \otimes^{\mathrm{L}} \mathcal{G}) = 0$.*

Proof. Since $F_a(K_0(P))$ is generated by $[\mathcal{O}_V]$, $V \subset P$ a subvariety of dimension $\leq a$ [7, Ex. 15.1.5], we may assume $\mathcal{F}, \mathcal{G}$ are structure sheaves of subvarieties of dimensions a, b respectively, with $a + b < n$.

Now since P is smooth, any coherent sheaf has a finite length resolution by finite rank locally free sheaves, so we may apply [7, 18.3.1 (c)] to the closed immersion $i : V \to P$ with $\beta = \mathcal{O}_V$. This gives, in the Chow group $A_*(P)$:

$$i_*(\mathrm{ch}(\mathcal{O}_V) \cap \mathrm{Td}(V)) = \mathrm{ch}(i_*\mathcal{O}_V) \cap \mathrm{Td}(P).$$

As $\mathrm{ch}(\mathcal{O}_V) = 1$ and $\mathrm{Td}(V) = [V] + r_V$ with $r_V \in A_{<a}(V)$, the left hand side lies in $A_{\leq a}(P)$.

Since P is smooth, $A^p(P) \cap A_q(P) \subset A_{q-p}(P)$ by [7, 8.3 (b)]. As $\mathrm{Td}(P) = [P] + r_P$ with $r_P \in A_{<n}(P)$, by equating terms in each degree, we find $\mathrm{ch}(i_*\mathcal{O}_V) \in A^{\geq n-a}(P)$.

By Grothendieck-Riemann-Roch (for the smooth P, as in [7, 15.2.1]) and the action of ch on $\otimes$, $\chi(\mathcal{F} \otimes^{\mathrm{L}} \mathcal{G}) = \int_P \mathrm{ch}(\mathcal{F}) \cdot \mathrm{ch}(\mathcal{G}) \cdot \mathrm{Td}_P$. Here $\cdot$ means intersection product of cycle classes. The first possible nonzero term in $\mathrm{ch}(\mathcal{F}) \cdot \mathrm{ch}(\mathcal{G})$ would come from $\mathrm{ch}_{n-a}(\mathcal{F}) \cdot \mathrm{ch}_{n-b}(\mathcal{G})$, but this term is zero for degree reasons. $\square$

Proposition 5.17. *Let T be the spectrum of a field, and $Z \hookrightarrow P_T$ an a-dimensional subscheme. Let $M, N \in \mathrm{Coh}(P_T)$ be invertible sheaves on some subvariety of P_T of dimension $\leq n - a$, and suppose we have exact sequences:*

$$0 \to M \xrightarrow{s} N \to Q_s \to 0$$

$$0 \to M \xrightarrow{t} N \to Q_t \to 0$$

such that $Z \cap \mathrm{Supp}(Q_s) = Z \cap \mathrm{Supp}(Q_t) = \emptyset$.
Then the unique $a_Z \in \Gamma(T, \mathcal{O}_T^*)$ making the following diagram commute:

$$\begin{tikzcd}
f_T(\mathcal{O}_Z, Q_s) \arrow[r, "\varphi_T^Z"] \arrow[d, "via\ s"'] & \mathcal{O}_T \arrow[dd, "a_Z"] \\
(f_T(\mathcal{O}_Z, M))^{-1} \otimes f_T(\mathcal{O}_Z, N) & \\
f_T(\mathcal{O}_Z, Q_t) \arrow[r, "\varphi_T^Z"] \arrow[u, "via\ t"] & \mathcal{O}_T
\end{tikzcd}$$

depends only on $[Z]$, i.e., $a_Z = a_{Z'}$ if $[Z] = [Z']$.

Proof. To prove the claim it is equivalent to show that the difference between $f(1 \otimes s), f(1 \otimes t) : f_T(\mathcal{O}_Z, M) \cong f_T(\mathcal{O}_Z, N)$ depends only on $[Z]$.

Step 1. By taking a filtration of $\mathcal{O}_Z$ such that the graded pieces are isomorphic to (twists of) structure sheaves of subvarieties, and using the additivity of the determinant on filtrations, we are reduced to showing that if $\mathcal{F} \in \mathrm{Coh}(P_T)$ with $\mathrm{Supp}(\mathcal{F}) \subset \mathrm{Supp}(Z)$ and $\dim(\mathrm{Supp}(\mathcal{F})) \leq a - 1$, the induced isomorphisms $f(1 \otimes s), f(1 \otimes t) : f(\mathcal{F}, M) \cong f(\mathcal{F}, N)$ are equal. (Such a filtration exists by

[11, I.7.4].) The subquotients of the filtration of $\mathcal{O}_Z$ depend on the filtration chosen, but the top-dimensional components always appear with their correct multiplicities, i.e., the cycle $[Z]$ can be extracted from the filtration.

Step 2. Let Q_U denote the cokernel of the universal $\mathcal{O}_P$-homomorphism $M \to N$; note Q_U is flat over $\mathrm{Hom}_{\mathcal{O}_P}(M,N) \setminus 0$ since a morphism between invertible sheaves is either injective or zero, hence the Euler characteristic of every cokernel is $\chi(M) - \chi(N)$. We consider the line bundle $\det \mathrm{R}\pi_*(p_1^*(\mathcal{F}) \otimes^{\mathrm{L}} Q_U)$ on $\mathrm{Hom}_{\mathcal{O}_P}(M,N) \setminus 0$. Its fiber over $s \in \mathrm{Hom}_{\mathcal{O}_P}(M,N)$ is precisely $f(\mathcal{F}, Q_s)$. Since $Q_s = Q_{\lambda s}$ for $\lambda \in \Gamma(T, \mathcal{O}_T^*)$, we consider $\det \mathrm{R}\pi_*(p_1^*(\mathcal{F}) \otimes^{\mathrm{L}} Q_U)$ as a line bundle on the projective space $\mathbb{P}(\mathrm{Hom}_{\mathcal{O}_P}(M,N) \setminus 0)$.

We claim this line bundle is trivial. To prove this it suffices to show it is trivial along a line $\mathbb{P}^1 \cong L \hookrightarrow \mathbb{P}(\mathrm{Hom}_{\mathcal{O}_P}(M,N) \setminus 0)$. For this purpose Grothendieck-Riemann-Roch (i.e., ignoring torsion) is adequate. More precisely, we consider the GRR diagram:

$$\begin{tikzcd}
K_0(P \times L) \arrow[r, "\mathrm{ch}(-)\cdot \mathrm{Td}(P)\cdot \mathrm{Td}(L)"] \arrow[d, "\mathrm{R}\pi_*"] & A_*(P \times L)_{\mathbb{Q}} \arrow[d, "\pi_*"] \\
K_0(L) \arrow[r, "\mathrm{ch}(-)\cdot \mathrm{Td}(L)"] \arrow[d, "\det"] & A_*(L)_{\mathbb{Q}} \\
\mathrm{Pic}(L) \arrow[ur, "c_1"] &
\end{tikzcd}$$

We have $\dim(\mathrm{Supp}(p_1^*(\mathcal{F}))) \le a, \dim(\mathrm{Supp}(Q_U)) \le n-a$, and $\dim(P \times L) = n+1$. Hence $\mathrm{ch}(p_1^*(\mathcal{F})) \cdot \mathrm{ch}(Q_U) = 0$ in $A_*(P \times L)_{\mathbb{Q}}$ for degree reasons (as in the proof of 5.16), so $p_1^*(\mathcal{F}) \otimes^{\mathrm{L}} Q_U \in K_0(P \times L)$ maps to 0 in the top row. Hence $c_1(\det \mathrm{R}\pi_*(p_1^*(\mathcal{F}) \otimes^{\mathrm{L}} Q_U))$ is a torsion class, and therefore $\det \mathrm{R}\pi_*(p_1^*(\mathcal{F}) \otimes^{\mathrm{L}} Q_U)$ is trivial.

Step 3. For $s \in \mathrm{Hom}_{\mathcal{O}_P}(M,N) \setminus 0$, consider the induced identification $f(s) : f(\mathcal{F}, Q_s) \otimes f(\mathcal{F}, M) \cong f(\mathcal{F}, N)$. Since $f(\mathcal{F}, Q_s)$ is canonically trivial, i.e., the trivialization induced by $\mathrm{Supp}(\mathcal{F}) \cap \mathrm{Supp}(Q_s) = \emptyset$ extends over all $\mathrm{Hom}_{\mathcal{O}_P}(M,N) \setminus 0$, we consider $f(s)$ as an isomorphism $f(\mathcal{F}, M) \cong f(\mathcal{F}, N)$. Since $\chi(\mathcal{F} \otimes^{\mathrm{L}} M) = \chi(\mathcal{F} \otimes^{\mathrm{L}} N) = 0$ by 5.16, we have $f(s) = f(\lambda s)$ for $\lambda \in \Gamma(T, \mathcal{O}_T^*)$. Therefore we have a commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}(M,N) \setminus 0 \arrow[r, "s \mapsto f(s)"] \arrow[d] & \mathrm{Isom}(f(\mathcal{F}, M), f(\mathcal{F}, N)) \\
\mathbb{P}(\mathrm{Hom}(M,N) \setminus 0) \arrow[ur] &
\end{tikzcd}$$

But there are no nonconstant functions on $\mathbb{P}(\mathrm{Hom}(M,N) \setminus 0)$, hence $f(s) = f(t) : f(\mathcal{F}, M) \cong f(\mathcal{F}, N)$.

□

Subsection 5.4. Further properties of $D_{Z,W}$. Assuming the base and incidence are optimal, we can already prove the following important property of $D_{Z,W}$.

Proposition 5.18. *Let (Z, Z', W) be a Hilbert-Chow datum over a normal base T, and suppose the incidence $Z \cap W$ satisfies:*

(1) $(Z \cap W)_\eta = \emptyset$ for all generic points $\eta \in T$; and

(2) $Z \cap W$ is finite over all points of depth 1 in T.

Then the Cartier divisors $D_{Z,W}, D_{Z',W}$ are equal.

Proof. Since the smooth locus of T contains all points of depth 1, and the formation of $D_{Z,W}$ is compatible with the inclusion $T^{sm} \subset T$, we may assume T is smooth. For T regular there is a canonical isomorphism [12, Prop. 8]:

$$f_T(\mathcal{O}_Z, \mathcal{O}_W) \cong \otimes_{p,q}(\det{}_T R^q \pi_*(\mathcal{H}^p(\mathcal{O}_Z \otimes^{\mathrm{L}} \mathcal{O}_W)))^{(-1)^{p+q}}.$$

To calculate the coefficient of a depth 1 point $t \in T$ in $D_{Z,W}$, we may replace T with the spectrum of the DVR $\mathcal{O}_{T,t}$. Then the support of $\mathcal{H}^p(\mathcal{O}_Z \otimes^{\mathrm{L}} \mathcal{O}_W)$ is finite over T (indeed, over t), so in the displayed expression only the terms with $q = 0$ can contribute.

By [12, Thm.3(vi)], the multiplicity of a depth one point is determined by the sum $\sum_i (-1)^i \ell_t(\mathcal{H}^i(\mathrm{R}\pi_*(\mathcal{O}_Z \otimes^{\mathrm{L}} \mathcal{O}_W)))$. This last sum is equal to

$$\sum_i (-1)^i \ell_t(\pi_* \mathcal{H}^i(\mathcal{O}_Z \otimes^{\mathrm{L}} \mathcal{O}_W)) = (\deg \pi)(\sum_{p, t' \to t} (-1)^p \ell_{t'}(\mathcal{H}^p(\mathcal{O}_Z \otimes^{\mathrm{L}} \mathcal{O}_W))),$$

hence it suffices to show $\gamma(\mathcal{O}_Z) := \sum_{p,t' \to t} (-1)^p \ell_{t'}(\mathcal{H}^p(\mathcal{O}_Z \otimes^{\mathrm{L}} \mathcal{O}_W))$ depends only the underlying cycle $[Z]$. We remark that if Z and W are integral and $b = n - a - 1$, then the contribution of a point t' (lying over t) to $\gamma(\mathcal{O}_Z)$ is exactly Serre's Tor-formula for the intersection index of Z and W at t' [22, V.C.Thm.1(b)].

Without loss of generality we assume W is integral and $\dim(W) = b = n-a-1$; we will see in the proof all sums are 0 if $b < n-a-1$. Given an exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ of coherent sheaves on P_T with support of relative dimension $\leq a$ and satisfying the incidence hypothesis with respect to W, by the long exact cohomology sequence we obtain $\gamma(\mathcal{F}_1) + \gamma(\mathcal{F}_3) = \gamma(\mathcal{F}_2)$. It then follows γ is additive on filtrations.

Write $[Z] = \sum_i a_i Z_i$. Again by [11, I.7.4], the sheaf $\mathcal{O}_Z$ admits a filtration whose subquotients are invertible sheaves L_i on subvarieties contained in Z; and each top-dimensional component Z_i appears exactly a_i times. Since every L_i is some tensor power of a very ample class on P_T, we may assume there is either an injective map $\mathcal{O}_{Z_i} \to L_i$ or an injective map $L_i \to \mathcal{O}_{Z_i}$. Therefore $\gamma(\mathcal{O}_Z) = \sum_i \gamma(L_i) = \sum_i a_i \gamma(\mathcal{O}_{Z_i})$ modulo summands of the form $\gamma(\mathcal{F})$ where $\mathcal{F}$ is a sheaf on P_T whose support over the generic point of T has dimension $\leq a - 1$. So it suffices to show γ vanishes on sheaves of this type. Again we may assume $\mathcal{F}$ is isomorphic to the structure sheaf of a subvariety $Y \hookrightarrow Z$ in P_T. Note that $\dim Y \leq a$, else, being contained in Z, Y would dominate T and have all fibers of dimension $\geq a$; and then Y would contribute to the cycle $[Z]$ of Z.

There are two cases to consider: if $Y \cap W = \emptyset$, then $\gamma(\mathcal{F}) = 0$ since $\mathcal{F} \otimes^{\mathrm{L}} \mathcal{O}_W$ is acyclic. If $Y \cap W \neq \emptyset$, then the intersection $Y \cap W$ is improper, and the Tor-formula vanishes at components of improper intersection (due to Serre in the equal characteristic case [22, V.C.Thm.1(a)]). □

Remarks 5.19. (5.19.1) The proof shows the relation between the incidence line bundle and Serre's Tor-formula for intersection multiplicities; and also that our construction agrees with Mazur's at least on the normalization of the locus U'. Thus the essential tasks are to extend the divisor through the locus where the expected incidence condition (the hypothesis in 5.18) fails, and to remove the assumption of normality.

(5.19.2) The assumption (C2) of [4] is that the incidence is generically finite over, and nowhere dense in, its image in the base S. These assumptions imply the map $S \to \mathscr{C}_a(P) \times \mathscr{C}_{n-a-1}(P)$ factors through U'.

(5.19.3) Considering the GRR diagram as in the proof of 5.17 with L replaced by a general regular base T, one sees that the first Chern class of the incidence line bundle $f_T(\mathcal{O}_Z, \mathcal{O}_W)$ modulo torsion depends only on the underlying cycles $[Z], [W]$, independent of any assumption of properness of intersection. By contrast in 5.18 we have the result integrally.

(5.19.4) We may write $D_{[Z],W}$ in the case we have a Hilbert datum as in 5.18, e.g., with proper intersection over a regular base T.

COROLLARY 5.20. *Among Hilbert-Chow data:*

(1) over normal bases T; and

(2) such that Z, W (and hence Z', W) are generically disjoint and have finite incidence over points of depth 1 in T;

there exists a collection of isomorphisms $\phi_T^{Z,Z'} : f_T(\mathcal{O}_Z, \mathcal{O}_W) \cong f_T(\mathcal{O}_{Z'}, \mathcal{O}_W)$ which:

(1) is compatible with base change preserving the incidence condition (2);

(2) satisfies the cocycle condition; and

(3) agrees with the collection on disjoint families defined in 5.10.

Proof. We define $\phi_T^{Z,Z'} := \left(\varphi_T^{Z'}\right)^{-1} \circ \varphi_T^{Z} : f_T(\mathcal{O}_Z, \mathcal{O}_W) \cong \mathcal{O}_T(D_{[Z],W}) \cong f_T(\mathcal{O}_{Z'}, \mathcal{O}_W)$. □

CONSTRUCTION-NOTATION 5.21. We continue with the Cartier divisor $D_{Z,W} \hookrightarrow T$ associated to a Hilbert datum (5.11). For $Z \hookrightarrow P_T$ a T-flat family of a-dimensional subschemes of P and $s \in \mathbb{Z}_{\geq 0}$, let $\mathrm{Coh}_{\leq s; Z_T}(P)$ denote the abelian category of coherent sheaves $\mathcal{G}$ on P such that $\dim(\mathrm{Supp}(\mathcal{G})) \leq s$ and $(Z \cap \mathrm{Supp}(\mathcal{G}))_\eta = \emptyset$ for all generic points $\eta \in T$. For $\mathcal{G} \in \mathrm{Coh}_{\leq n-a-1; Z_T}(P)$, we obtain a Cartier divisor $D_{Z,\mathcal{G}} \hookrightarrow T$ and a canonical isomorphism $f_T(\mathcal{O}_Z, \mathcal{G}) \cong \mathcal{O}_T(D_{Z,\mathcal{G}})$.

We denote by $K_0^{s;Z}(P)$ the K_0-group of the abelian category $\mathrm{Coh}_{\leq s; Z_T}(P)$: we take the free abelian group on sheaves in $\mathrm{Coh}_{\leq s; Z_T}(P)$, then impose relations from short exact sequences whose terms all lie in $\mathrm{Coh}_{\leq s;Z_T}(P)$. Let $\mathrm{CDiv}(T)$ denote the group of Cartier divisors on T.

We summarize some elementary properties of this construction in the following proposition.

Proposition 5.22. *(1) The map $D_{Z,-} : \mathrm{Coh}_{\leq n-a-1;Z_T}(P) \to \mathrm{CDiv}(T)$ defined by $\mathcal{G} \mapsto D_{Z,\mathcal{G}}$ descends to a homomorphism $K_0^{n-a-1;Z}(P) \to \mathrm{CDiv}(T)$ (which we also denote by $D_{Z,-}$).*

(2) If $f : S \to T$ is a morphism and $\mathcal{G} \in \mathrm{Coh}_{\leq s;Z_S}(P)$, then $f^*(D_{Z,\mathcal{G}}) = D_{Z,\mathcal{G}}$ as divisors on S.

(3) If $\mathcal{G} \in \mathrm{Coh}_{\leq n-a-1;Z_T}(P)$ satisfies $Z \cap \mathcal{G} = \emptyset$, then $D_{Z,\mathcal{G}} = 0$.

(4) If $\mathcal{G} \in \mathrm{Coh}_{\leq 0;Z_T}(P)$ and the divisorial part of the family $Z \hookrightarrow P_T$ is trivial (i.e., $a \leq \dim(P) - 2$), then $D_{Z,\mathcal{G}} = 0$.

Proof. The first three properties follow immediately from the additivity, compatibility with base change, and compatibility with the trivialization of an acyclic complex, of the associated divisor construction discussed in Section 2. To prove the last property, by the first property we may assume $\mathcal{G}$ is the structure sheaf of a zero-dimensional subvariety $W \hookrightarrow P$. (If $k = \overline{k}$, this is just a single closed point, but we give an argument here valid for any T-flat family $W \hookrightarrow P_T$ of zero-dimensional subschemes, such that W is integral.)
By [20, 5.3] there is a canonical isomorphism:

$$\det{}_T \mathrm{R}\pi_*(\mathcal{O}_Z \otimes^{\mathrm{L}} \mathcal{G}) \cong (\det{}_T \pi_* \mathcal{G})^{\mathrm{rk}(\mathcal{O}_Z)-1} \otimes \det{}_T(\pi_*(\det{}_{P_T}(\mathcal{O}_Z)|_W)).$$

Since the divisorial part of $\mathcal{O}_Z$ was assumed to be empty, the line bundle $\det_{P_T}(\mathcal{O}_Z)$ is canonically trivial; and $\mathrm{rk}(\mathcal{O}_Z) - 1 = -1$. Therefore the right hand side is canonically trivial, so $D_{Z,\mathcal{G}} = 0$. □

Remark 5.23. In case $a = \dim(P) - 1$, we refer to [20]. The reduction to the diagonal shows we may assume $\dim(P) = 1$ or $a \leq \dim(P) - 2$; but to the extent we rely on (4) in 5.22, we must understand the case of zero-cycles and divisors.

Proposition 5.24. *Suppose T has dimension ≤ 1 and $\mathcal{G} \in \mathrm{Coh}_{\leq n-a-2;Z_T}(P)$. Then $D_{Z,\mathcal{G}} = 0$.*

Remark 5.25. The condition $D_{Z,\mathcal{G}} = 0$ is equivalent to the canonical trivialization (induced by the generic acyclicity of $\mathrm{R}\pi_*(\mathcal{O}_Z \otimes^{\mathrm{L}} \mathcal{G})$) extending over all of T.

Proof. We may also assume $\mathcal{G}$ is the structure sheaf of a subvariety $W \hookrightarrow P$ of dimension $b \leq n - a - 2$, as these sheaves generate the K_0-group.
Since $D_{Z,\mathcal{G}} = 0$ for $\dim(\mathrm{Supp}(\mathcal{G})) \leq 0$ (again by 5.22), it suffices to prove the following claim: if $D_{Z,\mathcal{G}} = 0$ for all $\mathcal{G} \in \mathrm{Coh}_{\leq b-1;Z}(P)$, and $1 \leq b \leq n - a - 2$, then $D_{Z,\mathcal{G}} = 0$ for all $\mathcal{G} \in \mathrm{Coh}_{\leq b;Z}(P)$.
We prove this claim: by 5.14 there are short exact sequences:

$$0 \to M_i \xrightarrow{s_0^i} \mathcal{O}_{B_i} \to Q_0^i \to 0$$

$$0 \to M_i \xrightarrow{s^i_\infty} \mathcal{O}_{B_i} \to Q^i_\infty \to 0$$

in $\mathrm{Coh}_{\leq n-a-1;Z_T}(P)$, such that the b-dimensional cycle $W + \sum_i([Q^i_0] - [Q^i_\infty])$ is disjoint from Z.

If $[Q] = \sum_j a_j W_j$, then we write $D_{Z,[Q]} := \sum_j a_j D_{Z,W_j}$. The short exact sequences give $\sum_i D_{Z,Q^i_0} = \sum_i D_{Z,Q^i_\infty}$ in $K_0^{n-a-1;Z}(P)$. Furthermore the difference $Q^i_* - [Q^i_*]$ lies in $F_{b-1}(K_0(P))$.

But then

$D_{Z,W} = D_{Z,W} + \sum_i(D_{Z,Q^i_0} - D_{Z,Q^i_\infty})$ from the SES

$= D_{Z,W} + \sum_i(D_{Z,[Q^i_0]} - D_{Z,[Q^i_\infty]})$ since we assumed $D_{Z,-}$ vanishes on $F_{b-1}(K_0(P))$

$= D_{Z,W+\sum_i([Q^i_0]-[Q^i_\infty])} = 0$ by the disjointness from Z. □

COROLLARY 5.26. *Suppose T is normal and $\mathcal{G} \in \mathrm{Coh}_{\leq n-a-2;Z_T}(P)$. Then $D_{Z,\mathcal{G}} = 0$.*

Proof. For $t \in T$ of depth 1, the formation of $D_{Z,\mathcal{G}}$ is compatible with the morphism $g_t : \mathrm{Spec}\,\mathcal{O}_{T,t} \to T$. By 5.24, we have $g_t^*(D_{Z,\mathcal{G}}) = 0$ for all such t. Since $D_{Z,\mathcal{G}}$ is determined its restriction to points of depth 1, the result follows. □

PROPOSITION 5.27. *Suppose T is seminormal and $\mathcal{G} \in \mathrm{Coh}_{\leq n-a-2;Z_T}(P)$. Then $D_{Z,\mathcal{G}} = 0$.*

Proof. The formation of the divisor $D_{Z,\mathcal{G}}$ is compatible with the (finite, birational) normalization morphism $\nu : T^\nu \to T$. By the previous result we know local equations for $\nu^*(D_{Z,\mathcal{G}})$ are units. We need to show these units are constant along the fibers of ν. Suppose $t \in T$ has branches $b_1, \ldots, b_r$ in T^ν. For each b_i there exists a DVR R_i and a morphism $g_i : \mathrm{Spec}\, R_i \to T$ such that: R_i has residue field $\kappa(t)$, and g_i covers a generization of t to the locus of subschemes disjoint from $\mathrm{Supp}(\mathcal{G})$. Denote by S the union of the $\mathrm{Spec}\, R_i$s glued along $\mathrm{Spec}\,\kappa(t)$. Then we have a morphism $g : S \to T$, and by 5.24 we conclude $g^*(D_{Z,\mathcal{G}}) = 0$. The corresponding trivialization $f_S(\mathcal{O}_Z, \mathcal{G}) \cong \mathcal{O}_S$ is our candidate for extending the trivialization through t.

Let $\mathrm{Exc}(\nu) \hookrightarrow T$ denote the locus over which ν is not an isomorphism, and let $I_\mathcal{G} \hookrightarrow T$ denote the locus of subschemes Z such that $Z \cap \mathrm{Supp}(\mathcal{G}) \neq \emptyset$. Set $U := T - (\mathrm{Exc}(\nu) \cap I_\mathcal{G})$. Then we have a trivialization $\varphi_U : f_U(\mathcal{O}_Z, \mathcal{G}) \cong \mathcal{O}_U$, and this extends to $\varphi_{T^\nu} : f_{T^\nu}(\mathcal{O}_Z, \mathcal{G}) \cong \mathcal{O}_{T^\nu}$. For $t \notin U$, we constructed in the previous paragraph the isomorphism $t_S : f_S(\mathcal{O}_Z, \mathcal{G}) \cong \mathcal{O}_S$, which we may restrict to t. Together these define a pointwise trivialization $f_T(\mathcal{O}_Z, \mathcal{G}) \cong \mathcal{O}_T$, i.e., a nonzero element in every fiber of the line bundle $f_T(\mathcal{O}_Z, \mathcal{G})$. We form the cartesian diagram:

$$\begin{tikzcd} S^\nu \arrow[r, "g^\nu"] \arrow[d, "\nu|_S"'] & T^\nu \arrow[d, "\nu"] \\ S \arrow[r, "g"] & T \end{tikzcd}$$

Since the formation of $D_{Z,\mathcal{G}}$ is compatible with all of the morphisms appearing in this diagram, we obtain $(\nu|_S)^*(t_S) = (g^\nu)^*(\varphi_{T^\nu})$. It then follows φ_{T^ν} is constant along the fibers of ν, hence it descends to a trivialization $\varphi_T : f_T(\mathcal{O}_Z, \mathcal{G}) \cong \mathcal{O}_T$. □

Corollary 5.28. *Suppose T is seminormal and $\mathcal{G} \in \mathrm{Coh}_{\leq n-a-1;Z_T}(P)$. Then $D_{Z,\mathcal{G}} = D_{Z,[\mathcal{G}]}$.*

Remark 5.29. For T smooth and $\mathcal{G} \in \mathrm{Coh}_{\leq n-a-2;Z_T}(P)$, one can deduce the line bundle $f_T(\mathcal{O}_Z, \mathcal{G})$ is trivial as follows. The filtration of the $K_0(X)$-group by dimension of support is compatible with multiplication, if X is a smooth quasi-projective scheme over a field [9, Exp.0 Ch.2 Sect.4 Thm.2.12 Cor.1]. From $\mathcal{O}_Z \in F_{a+\dim T}(K_0(P_T))$ and $\mathcal{G} \in F_{n-a-2+\dim T}(K_0(P_T))$ it follows that $\mathcal{O}_Z \otimes^{\mathrm{L}} \mathcal{G} \in F_{\dim T-2}(K_0(P_T))$. Therefore $\mathrm{R}\pi_*(\mathcal{O}_Z \otimes^{\mathrm{L}} \mathcal{G}) \in F_{\dim T-2}(K_0(T))$, and hence $f_T(\mathcal{O}_Z, \mathcal{G}) \cong \mathcal{O}_T$.
For a general base T, this reasoning is valid rationally, hence we can conclude $f_T(\mathcal{O}_Z, \mathcal{G})$ is a torsion line bundle.
Since the dimension filtration's compatibility with multiplication can be viewed as a consequence of the moving lemma, in some sense we have given this proof. Because we need to keep track of the trivialization and not just the abstract invertible sheaf, we work with Cartier divisors rather than line bundles.

Corollary 5.30. *Let (Z, W) be a Hilbert datum over any base T such that $(Z \cap W)_\eta = \emptyset$ for every generic point $\eta \in T$. Suppose further that $W = W_k \times_k T$ for a k-scheme W_k with $[W_k] = \sum_i a_i W_i$. Then there is a canonical isomorphism $f_T(\mathcal{O}_Z, \mathcal{O}_W) \cong \otimes_i (f_T(\mathcal{O}_Z, \mathcal{O}_{W_i}))^{\otimes a_i}$.*

Subsection 5.5. Application to the incidence line bundle.

Construction 5.31. The facts 5.13 and 5.30 produce a construction; in the description we suppress the base T and we use $f(-) := f(\mathcal{O}_Z, -)$ as before, since the first factor is constant. We use the identification of 5.30: $f(\mathcal{O}_W) \cong \otimes_i (f(\mathcal{O}_{W_i}))^{\otimes a_i} =: f([\mathcal{O}_W])$. (In our situation W is a Cartier divisor on a $(b+1)-$dimensional subvariety $B \hookrightarrow P$ with $[W] = \sum_i a_i W_i$.)
Now given $(Z, W); B_i, M_i, s^i_{0,\infty}$ as in 5.13, let $f(s_*) : f(M) \otimes f(Q_*) \cong f(\mathcal{O}_B)$ denote the isomorphism induced by the short exact sequence. We set

$$\alpha := [W] + \sum_i ([Q^i_0] - [Q^i_\infty])$$

to be the moved b-dimensional cycle. Then we have a canonical isomorphism $f([\mathcal{O}_W]) \otimes (\otimes_i f([Q^i_0])) = f(\alpha) \otimes (\otimes_i f([Q^i_\infty]))$.
Let $\beta^Z_{W,\alpha} : f([\mathcal{O}_W]) \cong f(\alpha)$ be the unique isomorphism making the following diagram commute:

$$\begin{tikzcd}
f([\mathcal{O}_W]) \otimes (\otimes_i (f([Q^i_0]) \otimes f(M_i))) \arrow[r, "="] \arrow[d, "1\otimes(\otimes_i f(s^i_0))"] & f(\alpha) \otimes (\otimes_i (f([Q^i_\infty]) \otimes f(M_i))) \arrow[d, "1\otimes(\otimes_i f(s^i_\infty))"] \\
f([\mathcal{O}_W]) \otimes (\otimes_i f(\mathcal{O}_{B_i})) \arrow[r, "\beta^Z_{W,\alpha}\otimes 1"] & f(\alpha) \otimes (\otimes_i f(\mathcal{O}_{B_i}))
\end{tikzcd}$$

In the top row, we tensored the canonical isomorphism with the identity on the $f(M_i)$ factors.

This construction extends the collection $\{\phi_T\}$ to spectra of local rings T.

PROPOSITION 5.32. *Let (Z, Z', W) be a Hilbert-Chow datum over T the spectrum of a local ring with $W = W_k \times_k T$, and suppose $(Z \cap W)_\eta = (Z' \cap W)_\eta = \emptyset$ for all generic points $\eta \in T$. Suppose subvarieties $B_1, \ldots, B_n \subset P$ and short exact sequences as in 5.13 have been chosen. Then the isomorphism:*

$$\left(\varphi^{Z';\alpha} \circ \beta^{Z'}_{W,\alpha}\right)^{-1} \circ \left(\varphi^{Z;\alpha} \circ \beta^{Z}_{W,\alpha}\right) : f_T(\mathcal{O}_Z, \mathcal{O}_W) \cong$$
$$\cong f_T(\mathcal{O}_Z, \alpha) \cong \mathcal{O}_T \cong f_T(\mathcal{O}_{Z'}, \alpha) \cong f_T(\mathcal{O}_{Z'}, \mathcal{O}_W)$$

is independent of the choice of move: given another collection of data $(\hat{B}_i, \hat{M}_i, \hat{s^i}_{0,\infty})$ producing a b-dimensional cycle $\hat{\alpha}$ (also disjoint from Z), we have:

$$\left(\varphi^{Z';\alpha} \circ \beta^{Z'}_{W,\alpha}\right)^{-1} \circ \left(\varphi^{Z;\alpha} \circ \beta^{Z}_{W,\alpha}\right) = \left(\varphi^{Z';\hat{\alpha}} \circ \beta^{Z'}_{W,\hat{\alpha}}\right)^{-1} \circ \left(\varphi^{Z;\hat{\alpha}} \circ \beta^{Z}_{W,\hat{\alpha}}\right).$$

Furthermore $\left(\varphi^{Z';\alpha} \circ \beta^{Z'}_{W,\alpha}\right)^{-1} \circ \left(\varphi^{Z;\alpha} \circ \beta^{Z}_{W,\alpha}\right)$ agrees with the canonical identifications at every generic $\eta \in T$. If in addition T is the spectrum of a regular local ring and the incidence $Z \cap W$ satisfies the hypotheses of 5.20, then $\phi^{Z,Z'}_T = \left(\varphi^{Z';\alpha} \circ \beta^{Z'}_{W,\alpha}\right)^{-1} \circ \left(\varphi^{Z;\alpha} \circ \beta^{Z}_{W,\alpha}\right)$.

Proof. We claim any choice of moving data produces the canonical isomorphism over the generic points of T. Now 5.17 shows the choice of short exact sequences affects the map $\varphi^{Z;\alpha}_\eta \circ \left(\beta^{Z}_{W,\alpha}\right)_\eta : f_\eta(\mathcal{O}_Z, \mathcal{O}_W) \cong f_\eta(\mathcal{O}_Z, \alpha) \cong \mathcal{O}_\eta$ by a constant depending only on $[Z]$, and this is exactly canceled by the map back to $f_T(\mathcal{O}_{Z'}, \mathcal{O}_W)$. In other words, since $a_Z = a_{Z'}$, the following diagram commutes.

$$\begin{tikzcd}
f_\eta(\mathcal{O}_Z, \mathcal{O}_W) \arrow[r, "\varphi^{Z;W}_\eta"] \arrow[d, "(\beta^{Z}_{W,\alpha})_\eta"'] \arrow[rrr, bend left, "\phi^{Z,Z'}_\eta"] & \mathcal{O}_{T,\eta} \arrow[r, "="] \arrow[d, "a_Z"'] & \mathcal{O}_{T,\eta} \arrow[d, "a_{Z'}"] & f_\eta(\mathcal{O}_{Z'}, \mathcal{O}_W) \arrow[l, "\varphi^{Z';W}_\eta"'] \arrow[d, "(\beta^{Z'}_{W,\alpha})_\eta"] \\
f_\eta(\mathcal{O}_Z, \alpha) \arrow[r, "\varphi^{Z;\alpha}_\eta"'] & \mathcal{O}_{T,\eta} \arrow[r, "="'] & \mathcal{O}_{T,\eta} & f_\eta(\mathcal{O}_{Z'}, \alpha) \arrow[l, "\varphi^{Z';\alpha}_\eta"]
\end{tikzcd}$$

From this our claims follow: two choices of moving data produce isomorphisms which agree at the generic points of T, hence they agree; and the isomorphism $\phi^{Z,Z'}_T$ for T regular is characterized by agreeing with the composition of the canonical trivializations over the generic points of T. $\square$

COROLLARY 5.33. *Among Hilbert-Chow data over bases T satisfying:*

(1) T is either regular, or local; and
(2) all generic points of T correspond to disjoint subschemes;

there exists a collection of isomorphisms $\{\phi_T\}$ *which:*

(1) *is compatible with base change preserving generic disjointness;*
(2) *satisfies the cocycle condition; and*
(3) *extends the collection of 5.20.*

Proof. To see our construction commutes with base change preserving generic disjointness, note that moves valid over T (i.e., producing $\beta_{W,\alpha}$) pullback via $S \to T$ to suitable moves on S. The cocycle condition (an equality of two isomorphisms of line bundles on a reduced scheme) holds at the generic points, hence it holds everywhere. □

The previous corollary provides us an isomorphism $f_s(\mathcal{O}_Z, \mathcal{O}_W) \cong f_s(\mathcal{O}_{Z'}, \mathcal{O}_W)$ for a Hilbert-Chow datum over s the spectrum of a field corresponding to a point in the incidence locus, namely the restriction of the isomorphism over the local ring, possibly followed by a field extension. In the next proposition we observe this is compatible with specializations from the locus of disjoint subschemes into the incidence locus.

Proposition 5.34. *Let* (Z, Z', W) *be a Hilbert-Chow datum over* s *the spectrum of a field* $\kappa(s)$ *corresponding to a point of incidence, i.e.,* $Z \cap W, Z' \cap W \neq \emptyset$. *Let* (Z_T, Z'_T, W) *be a Hilbert-Chow datum over* T *the spectrum of a DVR covering a generization from* s *to the locus of disjoint subschemes, and with* $T_0 = s$. *Then the isomorphism*

$$f_s(\mathcal{O}_Z, \mathcal{O}_W) \xrightarrow{can} f_T(\mathcal{O}_{Z_T}, \mathcal{O}_W) \times_T s$$
$$\xrightarrow{(\phi_T^{Z_T, Z'_T}) \times_T s} f_T(\mathcal{O}_{Z'_T}, \mathcal{O}_W) \times_T s \xleftarrow{can} f_s(\mathcal{O}_{Z'}, \mathcal{O}_W)$$

is equal to the isomorphism induced by $\phi_R := \phi_{\operatorname{Spec} R}$, *where* $(R, \mathfrak{m}, K = R/\mathfrak{m})$ *is the (seminormal) local ring of the image of* s *on* $(\mathscr{H}_a \times_{\mathscr{C}_a} \mathscr{H}_a)^{sn}$. *In other words, the previously displayed isomorphism is equal to* $(-) \otimes_K \kappa(s)$ *of the following isomorphism:*

$$f_K(\mathcal{O}_Z, \mathcal{O}_W) \xrightarrow{can} f_R(\mathcal{O}_{Z_R}, \mathcal{O}_W) \otimes_R K$$
$$\xrightarrow{(\phi_R^{Z_R, Z'_R}) \otimes_R K} f_R(\mathcal{O}_{Z'_R}, \mathcal{O}_W) \otimes_R K \xleftarrow{can} f_K(\mathcal{O}_{Z'}, \mathcal{O}_W).$$

In particular, if we generize to the locus of disjoint subschemes and then restrict, the resulting isomorphism at s *is independent of the choice of generization.*

Proof. Since the collection $\{\phi_T\}$ is compatible with base change preserving generic disjointness, we may replace R by R/I for some ideal $I \subset R$ such that R/I is a local domain whose generic point $\operatorname{Spec} L$ is the image of the generic point of T. Therefore we have commutative squares:

$$\begin{tikzcd} L \arrow[r] & \kappa(\eta) \\ R \arrow[u] \arrow[r] \arrow[d] & \Gamma(T, \mathcal{O}_T) \arrow[u] \arrow[d] \\ K \arrow[r] & \kappa(s) \end{tikzcd}$$

To show $\phi_R \times_R T = \phi_T$, it suffices to show they agree at η, i.e., that $(\phi_R \times_R T) \times_T \eta = \phi_T \times_T \eta$. But this is equivalent to $(\phi_L) \times_L \eta = \phi_\eta$, which is a consequence of the compatibility with base change on pairs of disjoint subschemes (5.10). $\square$

Corollary 5.35. *Among Hilbert-Chow data satisfying at least one of the following conditions:*

(1) the conditions of 5.33;
(2) the base T is a field;

there exists a collection of isomorphisms $\{\phi_T\}$ which:

(1) satisfies the cocycle condition;
(2) extends the collection of 5.33 (so is compatible with base change preserving generic disjointness); and
(3) is compatible with specialization from the locus of disjoint subschemes to the incidence locus.

Proof. The compatibility with specialization is built into the construction. The new feature to check is the cocycle condition on field points mapping to the incidence locus. But if the cocycle condition holds after generization and the covering isomorphism is compatible with base change, then the collection must also satisfy the cocycle condition at new (field) points. $\square$

We augment 5.35 to include specializations fully within the incidence locus, hence we have the first part of 5.1.

Theorem 5.36. *Among Hilbert-Chow data satisfying at least one of the following conditions:*

(1) the conditions of 5.33;
(2) the base T is a field;
(3) the base T is a DVR;

there exists a collection of isomorphisms $\{\phi_T\}$ which:

(1) satisfies the cocycle condition;
(2) extends the collection of 5.35 (so is compatible with base change preserving generic disjointness); and
(3) is compatible with arbitrary specialization.

Therefore, in the notation of 5.1, the incidence bundle $\mathcal{L} \in \mathrm{Pic}(Y_0)$ lifts to an element $(\mathcal{L}, \phi) \in \mathrm{Pic}(Y_\bullet)$.

Proof. This follows from the general lemma 4.5, the preceding result 5.35, and the hypothesis that T is seminormal. $\square$

Now we conclude the proof of 5.1.

Theorem 5.37 (Zariski local effectiveness). *The element $(\mathcal{L}, \phi = \{\phi_T\})$ of 5.36 is Zariski locally effective: for any cycle $(z, W) \in \overline{U} \subset \mathscr{C}_a \times \mathscr{C}_b$ there exists an open subscheme $V \subset \overline{U}$ containing (z, W) and an isomorphism $t : \mathcal{L}|_{\pi_0^{-1}(V)} \cong \mathcal{O}_{\pi_0^{-1}(V)}$ which is compatible with ϕ. Therefore, the incidence bundle $\mathcal{L}$ descends to $\overline{U}$.*

Proof. This structure is built into the definition of the descent datum $\phi = \{\phi_T\}$. By 5.5 we may assume W is fixed. Suppose z and W are disjoint. Then W is disjoint from all cycles in a neighborhood V of z, and also from all subschemes in $V_0 := \pi_0^{-1}(V) \subset \mathscr{H}_a$. We let $Z_{V_0} \hookrightarrow P \times V_0$ denote the corresponding family. On V_0 we use the canonical trivialization $\varphi_{V_0}^{Z_{V_0}} : f_{V_0}(\mathcal{O}_{Z_{V_0}}, \mathcal{O}_W) := \det_{V_0}(\mathrm{R}\pi_*(\mathcal{O}_{Z_{V_0}} \otimes^{\mathrm{L}} \mathcal{O}_W)) \cong \mathcal{O}_{V_0}$ induced by the acyclicity of $\mathcal{O}_{Z_{V_0}} \otimes^{\mathrm{L}} \mathcal{O}_W$. Then by our definition of ϕ on disjoint subschemes, the following diagram commutes:

$$\begin{array}{ccc} f_{V_0}(\mathcal{O}_{Z_{V_0}}, \mathcal{O}_W) & \xrightarrow{\varphi_{V_0}^{Z_{V_0}}} & \mathcal{O}_{V_0} \\ \Big\downarrow \phi_{V_0}^{Z,Z'} & & \Big\downarrow = \\ f_{V_0}(\mathcal{O}_{Z'_{V_0}}, \mathcal{O}_W) & \xrightarrow{\varphi_{V_0}^{Z'_{V_0}}} & \mathcal{O}_{V_0} \end{array}$$

For a pair (z, W) in the incidence locus, choose a collection of short exact sequences as in 5.13 moving W to a rationally equivalent α such that $z \cap \alpha = \emptyset$. Then also $z' \cap \alpha = \emptyset$ for z' in a neighborhood $V \ni z$, and α is disjoint from all subschemes parameterized by $V_0 := \pi_0^{-1}(V)$. Then we define t to be the trivialization induced by the move, then the acyclicity of $\mathcal{O}_{Z_{V_0}} \otimes^{\mathrm{L}} \mathcal{O}_{W_i}$ (for all $W_i \in \mathrm{Supp}(\alpha)$):

$$t : f_{V_0}(\mathcal{O}_{Z_{V_0}}, \mathcal{O}_W) \cong f_{V_0}(\mathcal{O}_{Z_{V_0}}, \alpha) \xrightarrow{\varphi_{V_0}^{Z_{V_0}}} \mathcal{O}_{V_0}.$$

This is compatible with ϕ by 5.32. Zariski local effectiveness implies effectiveness by 3.1 and 4.6. $\square$

Remark 5.38. In [18] we proved 1.1 for pairs of 1-dimensional cycles on a threefold P (the case $n = 3, a = b = 1$) by constructing isomorphisms $\varphi_T^{Z,W}$: $f_T(\mathcal{O}_Z, \mathcal{O}_W) \cong f_T([Z], [W])$ for all Hilbert data (Z, W) for P over spectra of fields and complete DVRs T. (We use the evident extension of the "determinant of a cycle" notation from 5.3 when both variables are cycles.) Thus we obtained for all Hilbert-Chow data $(Z, W), (Z', W')$ over spectra of fields and DVRs a system of isomorphisms $\phi_T^{Z,Z';W,W'} := \left(\varphi_T^{Z',W'}\right)^{-1} \circ \varphi_T^{Z,W} : f_T(\mathcal{O}_Z, \mathcal{O}_W) \cong f_T(\mathcal{O}_{Z'}, \mathcal{O}_{W'})$.

We explain the strategy used in [18] to construct the φ isomorphisms. Consider first the case where T is the spectrum of a field, and write $[Z] = \sum n_i Z_i$. Then $\mathcal{O}_Z, \mathcal{O}_{Z_i} \in F_1(K_0(P_T))$ and the difference $\mathcal{O}_Z - \sum n_i \mathcal{O}_{Z_i}$ lies in $F_0(K_0(P_T))$. Since the determinant is additive on filtrations, to define $\varphi_T^{Z,W}$ it is enough to trivialize in a sufficiently canonical way the determinant $f_T(\mathcal{O}_Z, \mathcal{O}_W)$ when at least one of Z, W is zero-dimensional. To achieve this we used the explicit form of the isomorphism in [20, 5.3]. For T the spectrum of a DVR, one has also to trivialize the determinant $f_T(\mathcal{O}_Z, \mathcal{O}_W)$ when at least one of Z, W is supported in the closed fiber. For this we used an exact sequence given by a uniformizer. To summarize, as far as the incidence bundle is concerned, [20, 5.3] trivializes canonically the difference between the Hilbert scheme and the Chow variety. Thus what was lacking in [18] was a generalization of [20, 5.3] to higher dimensions.

To obtain the functoriality of the resulting collection $\{\phi_T\}$ in [18], we deduced from the construction of the φ isomorphisms a natural list of properties (essentially: being additive on triangles, and agreeing with prescribed normalizations on structure sheaves of subvarieties) sufficient to characterize them, then checked the properties were stable under base change. This method worked on all of $\mathscr{C} \times \mathscr{C}$, not just the locus $\overline{U}$. By contrast, our approach here is to restrict to the locus $\overline{U}$ and compare with the canonical trivialization on the locus U of disjoint cycles. As for the effectiveness of the descent datum, both in [18] and in the present work the verification of the Zariski local effectiveness of the collection $\{\phi_T\}$ employs Grothendieck-Riemann-Roch to analyze the effect on φ of a choice of "moving collection" of short exact sequences.

As a final point of contrast, in [18] we made use of the product structure of the Hilbert-Chow proper hypercovering and showed, using [20, 3.12], that the incidence bundle could be descended to the product of a Hilbert scheme and a Chow variety. Then we showed the descended bundle inherited a Zariski locally effective descent datum of its own. In other words, we wrote $\pi = (FC \times \mathrm{Id}) \circ (\mathrm{Id} \times FC) : \mathscr{H} \times \mathscr{H} \to \mathscr{C} \times \mathscr{C}$ and descended along $(FC \times \mathrm{Id})$ and $(\mathrm{Id} \times FC)$ separately.

SUBSECTION 5.6. CONCLUSION OF THE PROOF OF 1.1. Finally we verify the properties stated in 1.1.

DESCENT OF RATIONAL SECTION. To see we have actually constructed a Cartier divisor in $\overline{U} \subset \mathscr{C}_a \times \mathscr{C}_b$ supported on the incidence locus, consider the diagram whose vertical arrows are the restriction maps:

$$\begin{tikzcd}
\mathrm{Pic}(\overline{U}) \arrow[r] \arrow[d] & \mathrm{Pic}(Y_\bullet) \arrow[d] \\
\mathrm{Pic}(U) \arrow[r] & \mathrm{Pic}((\pi^{-1}(U))_\bullet)
\end{tikzcd}$$

The arrow in the bottom row is injective by 4.6. On the locus of disjoint subschemes, the isomorphism constructed in 5.8, $\varphi : \mathcal{L}|_{U_0} \cong \mathcal{O}_{U_0}$, is an isomorphism of pairs $\mathcal{L}|_{(\pi^{-1}(U))_\bullet} = (\mathcal{L}|_{U_0}, \phi|_{U_0 \times_U U_0}) \cong (\mathcal{O}_{U_0}, id)$. Hence by the injectivity of the bottom row, the trivialization φ descends to $U \subset \overline{U}$.

Restriction to U' is effective. To check that the restriction $D|_{U'}$ is effective, we may replace U' with its normalization U'^{ν}. Then we may replace U'^{ν} with the local ring of some depth 1 point t on U'^{ν}. By the assumption that the universal cycles intersect properly, over a given component $C \subset U'$, the incidence has dimension $\dim(C) - 1$. This is preserved by the finite base change $U'^{\nu} \to U'$.

Suppose first the incidence is generically finite onto its image. Then 5.18 applies, and the coefficient of t in $D|_{U'^{\nu}}$ is a sum of intersection multiplicities of properly intersecting components (weighted with positive coefficients). If the incidence dominates t, this coefficient is positive by [22, V.C.Thm.1(b)]; in any case the coefficient is nonnegative.

If the incidence has generic positive dimension over its image, then its image must have dimension $\leq \dim(C)-2$. Hence in this case the associated coefficient is 0.

Intersection multiplicity. On the Chow varieties we have the incidence bundle $\mathcal{M}$ and its rational section over the locus of disjoint cycles, giving the Cartier divisor $D \hookrightarrow \overline{U}$. This pulls back via the Hilbert-Chow morphism $\pi : Y_0 \to \overline{U}$ to the determinant line bundle $\mathcal{L}$ and its rational section over the locus of disjoint subschemes. Our goal is to relate the order of vanishing of a local defining equation of D, to intersection numbers. So let s_D be the canonical (rational) section of the line bundle $\mathcal{O}_{\overline{U}}(D)$.

If $g : T \to \overline{U}$ is a morphism from the spectrum of a discrete valuation ring $R \supset k$ (corresponding to cycles Z, W), there exists a discrete valuation ring R' which is finite over R, and such that the composition $g' : T' := \operatorname{Spec} R' \to T \to \overline{U}$ factors through Y_0. (Note that if we start with a specialization from a generic point of $\overline{U}$, we can find a component of the Hilbert scheme so that no generic extension is necessary.) If $\operatorname{ord} g'^*(s_D) = \deg(Z_{T'} \cdot W_{T'})$, it follows that $\operatorname{ord} g^*(s_D) = \deg(Z \cdot W)$. Thus we may assume our specialization factors through the Hilbert scheme, corresponding to subschemes $\widetilde{Z}, \widetilde{W}$ such that $[\widetilde{Z}] = Z$ and $[\widetilde{W}] = W$. Now we assume disjointness over the generic point $\eta \in T$, i.e., $g(\eta) \in U$. Let $t \in T$ denote the closed point.

First we have:

$$\operatorname{ord}_T(s_D) = \operatorname{ord}_T(s_{\pi^*D}) = \sum_p (-1)^p \ell_t(\mathcal{H}^p(\mathrm{R}\pi_*(\mathcal{O}_{\widetilde{Z}} \otimes^{\mathrm{L}} \mathcal{O}_{\widetilde{W}})))$$

since each $\mathcal{H}^p(\mathrm{R}\pi_*(\mathcal{O}_{\widetilde{Z}} \otimes^{\mathrm{L}} \mathcal{O}_{\widetilde{W}}))$ is a torsion T-module, and by [12, Thm.3(v)]. Since the scheme P_T is smooth, the filtration of the K_0-groups by dimension is compatible with multiplication, thus $\mathcal{O}_{\widetilde{Z}} \otimes^{\mathrm{L}} \mathcal{O}_{\widetilde{W}}$ and $\mathrm{R}\pi_*(\mathcal{O}_{\widetilde{Z}} \otimes^{\mathrm{L}} \mathcal{O}_{\widetilde{W}})$ are classes of dimension zero. Then $\sum_p (-1)^p \ell_t(\mathcal{H}^p(\mathrm{R}\pi_*(\mathcal{O}_{\widetilde{Z}} \otimes^{\mathrm{L}} \mathcal{O}_{\widetilde{W}})))$ is equal to

the degree of the K_0-classes $\mathcal{O}_{\widetilde{Z}} \otimes^{\mathrm{L}} \mathcal{O}_{\widetilde{W}}$ and $\mathrm{R}\pi_*(\mathcal{O}_{\widetilde{Z}} \otimes^{\mathrm{L}} \mathcal{O}_{\widetilde{W}})$. Note also the refined class $Z \cdot W$ is of the expected dimension, i.e., $Z \cdot W \in A_0(P_T)$.
We have $\mathcal{O}_{\widetilde{Z}} \otimes^{\mathrm{L}} \mathcal{O}_{\widetilde{W}} = \sum_i (-1)^i [\mathrm{Tor}_i^{P_T}(\mathcal{O}_{\widetilde{Z}}, \mathcal{O}_{\widetilde{W}})] \in K_0(P_T)$. The degree of this class is computed by [7, 20.4]: it is simply the degree of the refined class $Z \cdot W \in A_0(P_T)$, since the terms of dimension < 0 necessarily vanish.

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