# The Additivity Theorem in Algebraic $K$-Theory

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Abstract. The additivity theorem in algebraic $K$-theory, due to Quillen and Waldhausen, is a basic tool. In this paper we present a new proof, which proceeds by constructing an explicit homotopy combinatorially.

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## Introduction

In this paper[1], we present a new proof of the additivity theorem of Quillen [7, §3, Theorem 2] and Waldhausen [8, 1.3.2(4)]. See also [6] and [5]. Previous proofs used Theorem A or Theorem B of Quillen [7], but this one proceeds by constructing an explicit combinatorial homotopy, which is made possible by suitably subdividing one of the spaces involved.

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## 1. The additivity theorem

Let $Ord$ denote the category of finite nonempty ordered sets. We regard a simplicial object in a category $\mathcal{C}$ as a functor $Ord^{\mathrm{op}} \to \mathcal{C}$. For $A \in Ord$ let $\Delta^A$ denote the simplicial set it represents. For each $n \in \mathbb{N}$ let $[n]$ denote the ordered set $\{0 < 1 < \cdots < n\} \in Ord$, and let $\Delta^n$ denote the simplicial set it represents. Let $\Delta^A_{top}$ denote the corresponding topological simplex, consisting of the functions $p : A \to [0,1]$ that sum to 1; for $A = [n]$ we may also write $p = (p_0, \ldots, p_n)$.

If $X$ is a simplicial set, we let $[A, x, p]$ denote the point of the geometric realization $|X|$ corresponding to $A \in Ord$, $x \in X(A)$, and $p \in \Delta^A_{top}$.

For objects $A$ and $B$ in $Ord$, let $A * B \in Ord$ denote their concatenation; it is the disjoint union, with the ordering extended so the elements of $A$ are smaller

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than the elements of $B$. We make that precise by setting $A * B := (\{0\} \times A) \cup (\{1\} \times B)$, so $(0, a)$ and $(1, b)$ denote typical elements, and the ordering is lexicographic. We do the analogous thing with multiple concatenation, e.g., $A * B * C := (\{0\} \times A) \cup (\{1\} \times B) \cup (\{2\} \times C)$. Given functions $p : A \to \mathbb{R}$ and $q : B \to \mathbb{R}$, we let $p * q : A * B \to \mathbb{R}$ be the function defined by $(0, a) \mapsto p(a)$ and $(1, b) \mapsto q(b)$. An embedding $\Delta^A_{top} \times \Delta^B_{top} \to \Delta^{A*B}_{top}$ is defined by $(p, q) \mapsto (p/2) * (q/2)$.

The reason for using $Ord$ in this paper, instead of its full subcategory whose objects are the ordered sets $[n]$, is that it is closed under the concatenation operation $(A, B) \mapsto A * B$ and under various other constructions used later in the paper. Since the two categories are equivalent, nothing essential is changed. Since $Ord$ is not a small category, to make the definition of geometric realization of a simplicial set work, one should either replace $Ord$ by a small subcategory containing each $[n]$ and closed under the constructions used in this paper, or one should interpret the point $[A, x, p]$ introduced above as the point $[[n], \theta^* x, p\theta]$ where $\theta : [n] \to A$ is the unique isomorphism of its form.

For a simplicial object $X$, its *two-fold edge-wise subdivision* $sub_2 X$ (see [3, §4], [2], and [1]) is the simplicial object defined by $A \mapsto X(A * A)$. For a simplicial set $X$, there is a natural homeomorphism $\Psi : |sub_2 X| \xrightarrow{\cong} |X|$ (defined in [3, §4]). It can be defined on each simplex as the affine map that sends each vertex of $|sub_2 X|$ to the midpoint of the corresponding (possibly degenerate) edge of $|X|$. More precisely, it sends a point $[A, x, p] \in |sub_2 X|$ to $[A * A, x, (p/2) * (p/2)] \in |X|$.

The edges of $|sub_2 X|$ that map onto the two parts of each edge of $|X|$ are oriented in the same direction. There is another *edge-wise subdivision* where the edges are oriented in opposite directions, defined by $A \mapsto X(A * A^{\mathrm{op}})$. Subdivision into more parts can be accomplished by adding additional factors of $A$ or $A^{\mathrm{op}}$. Our use of $sub_2 X$ in this paper, rather than one of the other available subdivisions, was based on rough sketches in low dimension of the homotopy $\Theta$ produced in Lemma 7 below.

Let $\mathcal{C}$ be a category. Let $Ar\,\mathcal{C}$ denote the category of arrows in $\mathcal{C}$. If $f$ is an arrow of $\mathcal{C}$, let $[f]$ denote the corresponding object of $Ar\,\mathcal{C}$.

As defined in [8, 1.1 and 1.2] a *category with cofibrations and weak equivalences* consists of a category $\mathcal{N}$ equipped with a subcategory $co\mathcal{N}$ of *cofibrations* and a subcategory $w\mathcal{N}$ of *weak equivalences* satisfying five axioms, not repeated here. Its $K$-theory space is denoted by $K\mathcal{N}$ or $Kw\mathcal{N}$, and is defined as the loop space $\Omega|wS.\mathcal{N}|$, where $wS.\mathcal{N}$ is defined in [8, (1.3)] as follows. Given $A \in Ord$, we regard it as a category in the usual way, and we let $Exact(Ar\,A, \mathcal{N})$ denote the category of functors $N : Ar\,A \to \mathcal{N}$ that are *exact* in the sense that (1) $N[a \to a] = *$ for all $a \in A$, and (2) the sequence

$$N[a \to b] \rightarrowtail N[a \to c] \twoheadrightarrow N[b \to c]$$

is a *cofibration sequence*, for all $a \le b \le c$ in $A$. (In the presence of condition (1), condition (2) is equivalent to

$$\begin{array}{ccc} N[a \rightarrowtail b] & \longrightarrow & N[b \rightarrowtail b] \\ \downarrow & & \downarrow \\ N[a \to c] & \longrightarrow & N[b \to c] \end{array}$$

being a pushout square.) Then $S.\mathcal{N}$ is the simplicial category that is defined on objects by sending $A \in \mathit{Ord}$ to $\mathit{Exact}(\mathit{Ar}\, A, \mathcal{N})$, and is defined on arrows in the natural way. Since $\mathcal{N}$ is equipped with a category of weak equivalences $w\mathcal{N}$, so is the exact category $\mathit{Exact}(\mathit{Ar}\, A, \mathcal{N})$, as Waldhausen proves, yielding a simplicial category denoted $wS.\mathcal{N}$.

Now suppose $F$ and $G$ are exact functors $\mathcal{M} \to \mathcal{N}$ between categories with cofibrations and weak equivalences. Choose a coproduct operation on $\mathcal{N}$ satisfying the identities $N \vee * = N$ and $* \vee N = N$. We define a map $\Phi = \Phi_{F,G}$ : $\mathit{sub}_2 S.\mathcal{M} \to S.\mathcal{N}$ by $(\Phi M)[a \to b] := FM[(0,a) \to (0,b)] \vee GM[(1,a) \to (1,b)]$; here we have $A \in \mathit{Ord}$, an exact functor $M : \mathit{Ar}(A * A) \to \mathcal{M}$ regarded as an element of $(\mathit{sub}_2 S.\mathcal{M})(A)$, and an arrow $a \to b$ in $A$. One extends the definition of $\Phi M$ from objects to arrows by naturality and checks that it is exact (using the identity $(\Phi M)[a \to a] = * \vee * = *$ and exactness of the coproduct of two cofibration sequences), so $\Phi$ is well defined. The idea is that each edge of $S.\mathcal{M}$ gets subdivided into two parts, and we apply $F$ to the first part and $G$ to the second. (The same thing works for two homomorphisms between abelian groups, with $S.$ replaced by the nerve of the group.) Let $\mathit{sub}_2 wS.\mathcal{M}$ denote the simplicial category obtained by applying edge-wise subdivision in the simplicial direction. The functor $\Phi$ preserves weak equivalences, because $F$, $G$, and sum do, yielding a map $\Phi : \mathit{sub}_2 wS.\mathcal{M} \to wS.\mathcal{N}$ of simplicial categories.

The following definition comes from the text above [8, Proposition 1.3.2].

Definition 1. A sequence $F \rightarrowtail G \twoheadrightarrow H : \mathcal{M} \to \mathcal{N}$ of exact functors between categories with cofibrations and weak equivalences is a *cofibration sequence* if: (1) for all $M \in \mathcal{M}$ the sequence $F(M) \rightarrowtail G(M) \twoheadrightarrow H(M)$ is a cofibration sequence of $\mathcal{M}$; and (2) for any cofibration $M' \rightarrowtail M$ in $\mathcal{M}$ the map $G(M') \cup_{F(M')} F(M) \rightarrowtail G(M)$ is a cofibration in $\mathcal{N}$.

Given a cofibration sequence $F \rightarrowtail G \twoheadrightarrow H$ as in the definition above, the additivity theorem (Theorem 8 below) states that $F \vee H$ and $G$ yield homotopic maps $wS.\mathcal{M} \to wS.\mathcal{N}$. We will prove it by showing first that $G$ and $\Phi_{H,F}$ yield homotopic maps, and then composing two such homotopies. To construct this homotopy we need a new triangulation of the cylinder $[0,1] \times |wS.\mathcal{M}|$ that agrees with that of $|wS.\mathcal{M}|$ at one end and with that of $|\mathit{sub}_2 wS.\mathcal{M}|$ at the other end. Geometrically, it's sort of clear that such a thing should exist, for another description of the triangulation on $|\mathit{sub}_2 X|$ for a simplicial set $X$, or rather of its bisimplicial variant, is that it comes by intersecting the simplices of $|\Delta^1 \times X| \cong |\Delta^1| \times |X|$ with $\{p\} \times |X|$, where $p$ denotes the midpoint of $|\Delta^1|$. The new triangulation (called $IX$ in Definition 4 below), or rather a bisimplicial variant of it, arises by intersecting the simplices of $|\Delta^2 \times X| \cong |\Delta^2| \times |X|$ with

$\ell \times |X|$, where $\ell$ is the line segment in $|\Delta^2|$ joining the first vertex with the midpoint of the opposite edge. However, we ignore that interpretation and give a direct construction, as follows.

Definition 2. Given objects $A$ and $B$ of $Ord$, define $A \ltimes B \in Ord$ to be $A \times B$ equipped with the lexicographic ordering, where $(a, b) \le (a', b')$ if and only if (1) $a < a'$, or (2) $a = a'$ and $b \le b'$. (The notation is chosen to suggest that the projection $A \ltimes B \to A$ is an order preserving map, but the projection $A \ltimes B \to B$ is, in general, not.)

Definition 3. Given maps $A \xrightarrow{\sigma} C \xleftarrow{\varphi} B$ in $Ord$, define $\varphi^{-1}(\sigma) \in Ord$ to be the ordered subset $\{(a, b) \mid \sigma a = \varphi b\} \subseteq A \ltimes B$. (The notation is chosen as a reminder that when $\sigma$ is injective, then projection to the second factor gives an isomorphism $\varphi^{-1}(\sigma) \xrightarrow{\cong} \varphi^{-1}(\sigma(A)) \subseteq B$. On the other hand, if $\sigma$ is the map $[n] \to [0]$, then $\varphi^{-1}(\sigma) = B * \ldots * B$, the concatenation of $n+1$ copies of $B$.)

Definition 4. Let $s : [2] \to [1]$ be the map in $Ord$ defined by $s(0) = 0$, $s(1) = 1$, and $s(2) = 1$. For a simplicial set $X$ we define a simplicial set $IX$ on objects by setting $IX(A) := \{(\varphi, x) \mid \varphi : A \to [1], x \in X(\varphi^{-1}(s))\}$ for $A \in Ord$; its definition on arrows arises from naturality. We point out that $\varphi^{-1}(s) = \varphi^{-1}\{0\} * \varphi^{-1}\{1\} * \varphi^{-1}\{1\}$, so $\varphi^{-1}(s) \cong A$ if $\varphi = 0$, and $\varphi^{-1}(s) \cong A * A$ if $\varphi = 1$. Consequently, the simplicial subset of $IX$ defined by the equation $\varphi = 0$ is isomorphic to $X$, and the simplicial subset of $IX$ defined by the equation $\varphi = 1$ is isomorphic to $sub_2 X$. We regard those isomorphisms as identifications.

Definition 5. We define a map $\Psi : |IX| \to |\Delta^1| \times |X|$ as follows. The first component $|IX| \to |\Delta^1|$ arises from the simplicial map $IX \to \Delta^1$ defined by $(\varphi, x) \mapsto \varphi$, and thus it sends a point $[A, (\varphi, x), p]$ to the point $[A, \varphi, p]$. The second component $|IX| \to |X|$ is the unique map, affine on each simplex, whose behavior on vertices (each of which has either $\varphi = 0$ or $\varphi = 1$) is that it sends those with $\varphi = 0$ to the corresponding vertex of $|X|$ and those with $\varphi = 1$ to the midpoint of the corresponding (possibly degenerate) edge of $|X|$. More precisely, the map sends a point $[A, (\varphi, x), p] \in |IX|$ to $[\varphi^{-1}(s), x, \varphi \diamond p] \in |X|$, where $\varphi \diamond p \in \Delta_{top}^{\varphi^{-1}(s)}$ is defined by $(0, a) \mapsto p(a)$ for $a \in \varphi^{-1}(0)$, and by $(1, a) \mapsto p(a)/2$ and $(2, a) \mapsto p(a)/2$ for $a \in \varphi^{-1}(1)$. (Writing $p'$ for the restriction of $p$ to $\varphi^{-1}(0)$ and $p''$ for the restriction of $p$ to $\varphi^{-1}(1)$, we see that $\varphi \diamond p = p' * (p''/2) * (p''/2)$.)

Lemma 6. *For a simplicial set $X$, the map $\Psi : |IX| \to |\Delta^1| \times |X|$ is a homeomorphism.*

Proof. By commutativity with colimits, we may assume $X = \Delta^n$. The simplicial set $IX$ has only a finite number of nondegenerate simplices, so the source and target of $\Psi$ are compact Hausdorff spaces, and thus it is enough to show that $\Psi$ is a bijection.

To show surjectivity, consider a point $([[1], \beta, q], [[t], x, r])$ in $|\Delta^1| \times |X|$, with $r$ in the interior of $\Delta_{top}^t$. Let $k = q(0)$. We may assume that the partial sums $s_j :=$

$\sum_{i=0}^{j-1} r_i$, for $j = 0, \dots, t+1$, include $k$, for if not, then picking $j$ so that $s_j < k < s_{j+1}$, we may construct $r' = (r_0, \dots, r_{j-1}, k - s_j, s_{j+1} - k, r_{j+1}, \dots, r_t) \in \Delta^{t+1}_{top}$; its partial sums are those of $r$, together with $k$, and there is a surjective map $f : [t+1] \to [t]$ that collapses $r'$ to $r$. Letting $x' = f^*(x) = x \circ f$ be the corresponding degeneracy of $x$, we have $[[t], x, r] = [[t+1], x', r']$. Similarly, we may assume that each number $w$ with $k \leq w \leq k + (1-k)/2$ is a partial sum of $r$ if and only if $w + (1-k)/2$ is. Pick $b$ with $s_b = k$ and $c$ with $s_{b+c} = k + (1-k)/2$. Then, due to the symmetry of the partial sums, $r_{b+i} = r_{b+c+i}$ if $0 \leq i < c$, and $b + 2c = t + 1$. In more detail, one deduces the equality as follows: one has $r_{b+i} = s_{b+i+1} - s_{b+i}$, in which $s_{b+i+1}$ and $s_{b+i}$ are adjacent partial sums between $k$ and $k + (1-k)/2$, so by symmetry of the partial sums, $s_{b+i+1} + (1-k)/2 = s_{b+c+i+1}$ and $s_{b+i} + (1-k)/2 = s_{b+c+i}$, hence $r_{b+c+i} = s_{b+c+i+1} - s_{b+c+i} = s_{b+i+1} - s_{b+i} = r_{b+i}$. Now let $p \in \Delta^{b+c-1}_{top}$ be defined by $p = (r_0, \dots, r_{b-1}, 2r_b, \dots, 2r_{b+c-1})$, and let $\varphi : [b+c-1] \to [1]$ be defined by $\varphi(i) = 0$ for $0 \leq i < b$ and $\varphi(i) = 1$ for $b \leq i < b + c$. Then $([[1], \beta, q], [[t], x, r]) = \Psi([[b+c-1], (\varphi, x'), p])$, where $x' \in X(\varphi^{-1}(s))$ corresponds to $x \in X([t])$ via the unique isomorphism $\varphi^{-1}(s) \cong [t]$.

To show injectivity, consider a point $[A, (\varphi, x), p] \in |IX|$ where $(\varphi, x)$ is nondegenerate and $p$ is an interior point of $\Delta^A_{top}$. Observe that $x$ is a function $\varphi^{-1}(s) \to [n]$, and that $\varphi \diamond p$ is an interior point of its simplex. The deterministic procedure described in the previous paragraph recovers $A$, $\varphi$, $x$, and $p$, up to isomorphism, from the unique nondegenerate interior representatives of the two components of $\Psi([A, (\varphi, x), p])$, showing injectivity. $\square$

Lemma 7. *Let $F \rightarrowtail G \twoheadrightarrow H : \mathcal{M} \to \mathcal{N}$ be a cofibration sequence of exact functors between categories with cofibrations and weak equivalences. There is a map $\Theta : IwS.\mathcal{M} \to wS.\mathcal{N}$ such that $\Theta$ agrees with $G$ on the simplicial subset of $IwS.\mathcal{M}$ where $\varphi = 0$ and with $\Phi_{H,F}$ on the simplicial subset of $IwS.\mathcal{M}$ where $\varphi = 1$.*

Proof. The construction will be natural in the direction of the nerve of the weak equivalences, so we don't explicitly mention the weak equivalences in the rest of the proof. For each object $[M' \xrightarrow{f} M]$ of $Ar\,\mathcal{M}$ we choose a value in $\mathcal{N}$ for

$$P[f] := colim \begin{pmatrix} F(M') \xrightarrow{F(f)} F(M) \\ \downarrow \\ G(M') \end{pmatrix}.$$

The colimit exists because the vertical map in the diagram is a cofibration, and, in the case where $f$ is a cofibration, is the same as the pushout referred to in part (2) of definition 1. We may ensure $P[f] = *$ if $M' = M = *$. Having made those choices, one defines $P$ on maps in $Ar\,\mathcal{M}$ to get a functor $P : Ar\,\mathcal{M} \to \mathcal{N}$. Recall from [8, Lemma 1.1.1] that the full subcategory $\mathcal{F}_1\mathcal{N}$ of $Ar\,\mathcal{N}$, consisting of the arrows of $\mathcal{N}$ that are cofibrations, is a category with cofibrations, where a *cofibration* $[A \rightarrowtail B] \rightarrowtail [A' \rightarrowtail B']$ is an arrow having the property that both $A \rightarrowtail A'$ and $A \cup_{A'} B \rightarrowtail B'$ are cofibrations; the latter part of the condition ensures that cofibrations are stable under pushout. It follows that $P$ sends each (horizontal) *cofibration sequence*

$$\begin{tikzcd} L' \arrow[r, tail] \arrow[d, "f"] & M' \arrow[r, two heads] \arrow[d, "g"] & N' \arrow[d, "h"] \\ L \arrow[r, tail] & M \arrow[r, two heads] & N \end{tikzcd}$$

of (vertical) maps (in which the rows are cofibration sequences of $\mathcal{M}$) to a cofibration sequence $P[f] \rightarrowtail P[g] \twoheadrightarrow P[h]$ of $\mathcal{N}$. The point is that, according to definition 1, the left vertical map in the pushout diagram

$$\begin{tikzcd} {[FL' \rightarrowtail FM']} \arrow[r] \arrow[d, tail] & {[FL \rightarrowtail FM]} \arrow[d, tail] \\ {[GL' \rightarrowtail GM']} \arrow[r] & {[P[f] \rightarrowtail P[g]]} \end{tikzcd}$$

is a cofibration in $\mathcal{F}_1\mathcal{N}$, that the upper horizontal map is an arrow in $\mathcal{F}_1\mathcal{N}$, and thus that the pushout $[P[f] \rightarrowtail P[g]]$ lies in $\mathcal{F}_1\mathcal{N}$ and is therefore a cofibration. One also sees, using the gluing lemma [8, 1.2: Weq 2], that $P$ sends each (horizontal) *weak equivalence*

$$\begin{tikzcd} L' \arrow[r, "\sim"] \arrow[d, "f"] & M' \arrow[d, "g"] \\ L \arrow[r, "\sim"] & M \end{tikzcd}$$

of (vertical) maps (in which the horizontal maps are weak equivalences of $\mathcal{M}$) to a weak equivalence $P[f] \xrightarrow{\sim} P[g]$ in $w\mathcal{N}$.
We say that $P$ is an *exact functor*, in the sense that it preserves cofibration sequences and weak equivalences, as proved above.
We point out two special cases.

(A) if $f = 1$ is an identity map (or an isomorphism), then there is a natural isomorphism $P[f] \cong G(M')$
(B) if $f = 0$ is a map that factors through $*$, then there is a natural isomorphism $P[f] \cong F(M) \vee H(M')$

Thus, in a precise sense, $P$ includes $G$ and $F \vee H$ as special cases, allowing it to play the lead role in the construction of $\Theta$, which somehow deforms $f = 1$ to $f = 0$ continuously. (This basic idea was also used in [4, (10.3) and (10.4)] to prove a different sort of additivity theorem.)
We define $\Theta : IwS.\mathcal{M} \to wS.\mathcal{N}$ as follows. Given $A \in Ord$ and $(\varphi, M) \in (IwS.\mathcal{M})(A)$, we define $\Theta(\varphi, M) \in (wS.\mathcal{N})(A)$ as follows. Recall from definition 4 that $\varphi$ is a map $A \to [1]$, that $s$ is a certain map $s : [2] \to [1]$, and that $M \in (wS.\mathcal{M})(\varphi^{-1}(s))$. Introduce maps $d \le e : [1] \to [2]$ defined by $d(0) = e(0) = 0$, $d(1) = 1$, and $e(1) = 2$; they are the sections of $s$, and thus,

for any $a \in A$, we have $(d\varphi a, a) \in \varphi^{-1}(s)$ and $(e\varphi a, a) \in \varphi^{-1}(s)$. Our task is to define an exact functor $\Theta(\varphi, M) : Ar\,A \to \mathcal{N}$, so given an object $[a \to b]$ in $Ar\,A$, we define an object of $\mathcal{N}$ as follows, introducing the label $f$ for future reference.

$$(\Theta(\varphi, M))[a \to b] := P[M[(d\varphi a, a) \to (d\varphi b, b)] \xrightarrow{f} M[(e\varphi a, a) \to (e\varphi b, b)]]$$

We extend the definition of $\Theta(\varphi, M)$ to arrows by naturality and by pointing out that the construction preserves weak equivalences. Exactness of $\Theta(\varphi, M)$ follows from exactness of $M$ and of $P$, completing the definition of $\Theta$.
The rest of the statement follows from the following two special cases, which result from the previous ones.

(A) if $\varphi a = \varphi b = 0$ then $f = 1$ is an identity map, and thus there is a natural isomorphism

$$(\Theta(\varphi, M))[a \to b] \cong GM[(0, a) \to (0, b)]$$

(B) if $\varphi a = \varphi b = 1$, then $(d\varphi b, b) = (1, b) < (2, a) = (e\varphi a, a)$, which implies that $f = 0$ (because it factors through the object $M[(1, b) \to (1, b)] = *$), and thus that there is a natural isomorphism

$$(\Theta(\varphi, M))[a \to b] \cong HM[(1, a) \to (1, b)] \vee FM[(2, a) \to (2, b)]$$

□

Theorem 8 (Additivity, [8, 1.3.2(4)]). *Let $F \rightarrowtail G \twoheadrightarrow H$ be a cofibration sequence of exact functors $\mathcal{M} \to \mathcal{N}$ between categories with cofibrations and weak equivalences. Then $F \vee H$ and $G$ induce homotopic maps $K\mathcal{M} \to K\mathcal{N}$.*

Proof. Combining lemma 7 and lemma 6 we see that $G$ and $\Phi_{H,F}$ induce homotopic maps $|wS.\mathcal{M}| \to |wS.\mathcal{N}|$. There is a cofibration sequence $F \rightarrowtail F \vee H \twoheadrightarrow H$, so $F \vee H$ and $\Phi_{H,F}$ also induce homotopic maps. Composing the two homotopies (after reversing one of them) yields the result. □

Remark 9. Waldhausen's Additivity Theorem provides four equivalent formulations of the result, so it is sufficient to prove only the fourth of them, as we do here. Quillen's version [7, §3, Theorem 2] of the additivity theorem was stated for the $Q$-construction as a homotopy equivalence $(s, q) : Q\mathcal{E} \to Q\mathcal{M} \times Q\mathcal{M}$, where $\mathcal{M}$ is an exact category, and $\mathcal{E}$ is the exact category of short exact sequences $E = (0 \to sE \to tE \to qE \to 0)$ in $\mathcal{M}$. Here $s, q : \mathcal{E} \to \mathcal{M}$ are the exact functors that extract $sE$ and $qE$ from the exact sequence $E$. Quillen's formulation is analogous to Waldhausen's first formulation [8, 1.3.2(1)] and is implied by it.

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