Divisorial Cohomology Vanishing on Toric Varieties

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Abstract. This work discusses combinatorial and arithmetic aspects of cohomology vanishing for divisorial sheaves on toric varieties. We obtain a refined variant of the Kawamata-Viehweg theorem which is slightly stronger. Moreover, we prove a new vanishing theorem related to divisors whose inverse is nef and has small Iitaka dimension. Finally, we give a new criterion for divisorial sheaves for being maximal Cohen-Macaulay.

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Contents

1 Introduction

This work is motivated by numerical experiments [Per04] related to the conjecture of King [Kin97] concerning the derived category smooth complete toric varieties. These experiments led to the calculations of [HP06], where a counterexample to King's conjecture was given. Our goal is to develop a more systematic approach to the combinatorial and arithmetic aspects of cohomology vanishing for divisorial sheaves on toric varieties and to better understand from these points of view some phenomena related to this problem.

Based on work of Bondal (see [Rud90], [Bon90]), it was conjectured [Kin97] that on every smooth complete toric variety X there exists a full strongly exceptional collection of line bundles. That is, a collection of line bundles $\mathcal{L}_1, \dots, \mathcal{L}_n$ on X which generates $D^b(X)$ and has the property that $\mathrm{Ext}^k(\mathcal{L}_i, \mathcal{L}_j) = 0$ for all $k > 0$ and all i, j. Such a collection induces an equivalence of categories $\mathrm{RHom}(\bigoplus_i \mathcal{L}_i, \,.\,) : D^b(X) \longrightarrow D^b\big(\mathrm{End}(\bigoplus_i \mathcal{L}_i) - \mathrm{mod}\big)$. This possible generalization of Beilinson's theorem (pending the existence of a full strongly exceptional collection) has attracted much interest, notably also in the context of the homological mirror conjecture [Kon95]. For line bundles, the problem of Ext-vanishing can be reformulated to a problem of cohomology vanishing for line bundles by the isomorphisms

$$\mathrm{Ext}^k(\mathcal{L}_i, \mathcal{L}_j) \cong H^k(X, \mathcal{L}_i\check{} \otimes \mathcal{L}_j) = 0 \text{ for all } k \geq 0 \text{ and all } i, j.$$

So we are facing a quite peculiar cohomology vanishing problem: let n denote the rank of the Grothendieck group of X, then we look for a certain constellation of $n(n-1)$ – not necessarily distinct – line bundles, all of which have vanishing higher cohomology groups. The strongest general vanishing theorems so far are of the Kawamata-Viehweg type (see [Mus02] and [Fuj07], and also [Mat02] for Bott type formulas for cohomologies of line bundles), but it can be seen from very easy examples, such as Hirzebruch surfaces, that these alone in general do not suffice to prove or disprove the existence of strongly exceptional collections by means of cohomology vanishing. In [HP06], on a certain toric surface X, all line bundles $\mathcal{L}$ with the property that $H^i(X, \mathcal{L}) = H^i(X, \mathcal{L}\check{}) = 0$ for all $i > 0$ were completely classified by making use of an explicit toric representation of the cohomology vanishing problem for line bundles. This approach exhibits quite complicated combinatorial as well as number theoretic conditions for cohomology vanishing which we are going to describe in general.

We will consider and partially answer the following more general problem. Let D be a Weil divisor on any toric variety X and $V \subset X$ a torus invariant closed subscheme. Then what are necessary and sufficient conditions for the (global) local cohomology modules $H^i_V\big(X, \mathcal{O}_X(D)\big)$ to vanish? Given this spectrum of cohomology vanishing problems, we have at one extreme the cohomology vanishing problem for line bundles, and at the other extreme the classification problem for maximal Cohen Macaulay (MCM) modules over semigroup rings: on an affine toric variety X, the sheaf $\mathcal{O}_X(D)$ is MCM if and only if the local cohomologies $H^i_x\big(X, \mathcal{O}_X(D)\big)$ vanish for $i \neq \dim X$, where $x \in X$ is the torus fixed point. These local cohomologies have been studied by Stanley [Sta82], [Sta96] and Bruns and Gubeladze [BG03] showed that only finitely many sheaves in this class are MCM. MCM sheaves over affine toric varieties have only been classified for some special cases (see for instance [BGS87] and [Yos90]). Our contribution will be to give a more explicit combinatorial characterization of MCM modules of rank one over normal semigroup rings and their ties to the birational geometry of toric varieties.

One important aspect of our results is that, though we will also make use of $\mathbb{Q}$-divisors, our vanishing results will completely be formulated in the integral setting. We will illustrate the effect of this by the following example. Consider the weighted projective surface $\mathbb{P}(2,3,5)$. Then the divisor class group $A_1\big(\mathbb{P}(2,3,5)\big)$ is isomorphic to $\mathbb{Z}$ and, after fixing the generator $D = 1$ of $A_1\big(\mathbb{P}(2,3,5)\big)$ to be $\mathbb{Q}$-effective, the torus invariant irreducible divisors can be identified with the integers 2, 3, and 5, and the canonical divisor has class -10. By the toric Kawamata-Viehweg theorem we obtain that $H^2\big((\mathbb{P}(2,3,5), \mathcal{O}(kD)\big) = 0$ for $k > -10$. However, as we will explain in more detail below, the set of all divisors kD with nontrivial second cohomology is given by all k with $-k = 2r + 3s + 5t$ with r, s, t positive integers. So, Kawamata-Viehweg misses the divisor $-11D$. The reason is that the toric Kawamata-Viehweg vanishing theorem tells us that the cohomology of some divisor D' vanishes if the rational equivalence class over $\mathbb{Q}$ of $D' - K_{\mathbb{P}(2,3,5)}$ is contained in the interior of the nef cone in $A_1\big(\mathbb{P}(2,3,5)\big)_{\mathbb{Q}}$. Over the integers, the domain of cohomology vanishing thus in general is larger than over $\mathbb{Q}$. Below we will see that this is a general feature of cohomology vanishing, even for smooth toric varieties, as can be seen, for instance, by considering the strict transform of the divisor $-11D$ along some toric blow-up $X \longrightarrow \mathbb{P}(2,3,5)$ such that X is smooth.

The main results. The first main result will be an integral version of the Kawamata-Viehweg vanishing theorem. Consider the nef cone $\operatorname{nef}(X) \subset A_{d-1}(X)_{\mathbb{Q}}$, then the toric Kawamata-Viehweg vanishing theorem (see Theorem 3.29) can be interpreted such that if $D - K_X$ is contained in the interior of $\operatorname{nef}(X)$, then $H^i\big(X, \mathcal{O}_X(D)\big) = 0$ for all $i > 0$. For our version we will define a set $\mathfrak{A}_{\mathrm{nef}} \subset A_{d-1}(X)$, which we call the *arithmetic core* of $\operatorname{nef}(X)$ (see definition 4.11). The set $\mathfrak{A}_{\mathrm{nef}}$ has the property that it contains all integral Weil divisors which map to the interior of the cone $K_X + \operatorname{nef}(X)$ in $A_{d-1}(X)_{\mathbb{Q}}$. But in general it is strictly larger, as in the example above. We can lift the cohomology vanishing theorem for divisors in $\mathrm{nef}(X)$ to $\mathfrak{A}_{\mathrm{nef}}$:

Theorem (4.14): *Let X be a complete toric variety and $D \in \mathfrak{A}_{\mathrm{nef}}$. Then $H^i\big(X, \mathcal{O}_X(D)\big) = 0$ for all $i > 0$.*

One can consider Theorem 4.14 as an "augmentation" of the standard vanishing theorem for nef divisors to the subset $\mathfrak{A}_{\mathrm{nef}}$ of $A_{d-1}(X)$. In general, Theorem 4.14 is slightly stronger than the toric Kawamata-Viehweg vanishing theorem and yields refined arithmetic conditions.
However, the main goal of this paper is to find vanishing results which cannot directly be derived from known vanishing theorems. Let D be a nef Cartier divisor whose Iitaka dimension is positive but smaller than d. This class of divisors is contained in nonzero faces of the nef cone of X which are contained in the intersection of the nef cone with the boundary of the effective cone of X (see Section 4.3). Let F be such a face. Similarly as with $\mathfrak{A}_{\mathrm{nef}}$, we can define for the inverse cone $-F$ an arithmetic core $\mathfrak{A}_{-F}$ (see 4.11) and associate to it a vanishing theorem, which may be considered as the principal result of this article:

Theorem (4.17): *Let X be a complete d-dimensional toric variety. Then $H^i\big(X, \mathcal{O}(D)\big) = 0$ for every i and all D which are contained in some $\mathfrak{A}_{-F}$, where F is a face of $\mathrm{nef}(X)$ which contains nef divisors of Iitaka dimension $0 < \kappa(D) < d$. If $\mathfrak{A}_{-F}$ is nonempty, then it contains infinitely many divisor classes.*

This theorem cannot be an augmentation of a vanishing theorem for $-F$, as it is not true in general that $H^i\big(X, \mathcal{O}_X(-D)\big) = 0$ for all i for D nef of Iitaka dimension smaller than d. In particular, the set of $\mathbb{Q}$-equivalence classes of elements in $\mathfrak{A}_{-F}$ does not intersect $-F$.
For the case of a toric surface X we show that above vanishing theorems combine to a nearly complete vanishing theorem for X. Recall that in the fan associated to a complete toric surface X every pair of opposite rays by projection gives rise to a morphism from X to $\mathbb{P}^1$ (e.g. such a pair does always exist if X is smooth and $X \neq \mathbb{P}^2$). Correspondingly, we obtain a family of nef divisors of Iitaka dimension 1 on X given by the pullbacks of the sheaves $\mathcal{O}_{\mathbb{P}_1}(i)$ for $i > 0$. We get:

Theorem (4.21): *Let X be a complete toric surface. Then there are only finitely many divisors D with $H^i\big(X, \mathcal{O}_X(D)\big) = 0$ for all $i > 0$ which are not contained in $\mathfrak{A}_{\mathrm{nef}} \cup \bigcup_F \mathfrak{A}_{-F}$, where the union ranges over all faces of $\mathrm{nef}(X)$ which correspond to pairs of opposite rays in the fan associated to X.*

Some more precise numerical characterizations on the sets $\mathfrak{A}_{-F}$ will be given in subsection 4.3. The final result is a birational characterization of MCM-sheaves of rank one. This is a test case to see whether point of view of birational geometry might be useful for classifying more general MCM-sheaves. The idea for this comes from the investigation of MCM-sheaves over surface singularities in terms of resolutions in the context of the McKay correspondence (see [GSV83], [AV85], [EK85]). For an affine toric variety X, in general one cannot expect to find a similar nice correspondence. However, there is a set of preferred partial resolutions of singularities $\pi : \tilde{X} \longrightarrow X$ which is parameterized by the secondary fan of X. Our result is a toric analog of a technical criterion of loc. cit.

Theorem (4.36): *Let X be a d-dimensional affine toric variety whose associated cone has simplicial facets and let $D \in A_{d-1}(X)$. If $R^i\pi_*\mathcal{O}_{\tilde{X}}(\pi^* D) = 0$ for every regular triangulation $\pi : \tilde{X} \longrightarrow X$, then $\mathcal{O}_X(D)$ is MCM. For $d = 3$ the converse is also true.*

Note that the facets of a 3-dimensional cone are always simplicial.
To prove our results we will require a lot of bookkeeping, combining various geometric, combinatorial and arithmetic aspects of toric varieties. This has the unfortunate effect that the exposition will be rather technical and incorporate many notions (though not much theory) coming from combinatorics. As this might be cumbersome to follow for the more geometrically inclined reader, we will give an overview of the key structures and explain how they fit together. From now X denotes an arbitrary d-dimensional toric variety, Δ the fan associated to X, $M \cong \mathbb{Z}^d$ the character group of the torus which acts on X. We denote N the dual module of M, $l_1, \ldots, l_n \in N$ the set of primitive vectors of the 1-dimensional cones in Δ and $D_1, \ldots, D_n$ the corresponding torus invariant prime divisors on X. By abuse of notion, we will often identify the sets $[n] := \{1, \ldots, n\}$ and $\{l_1, \ldots, l_n\}$.

The circuit geometry of a toric variety. In order to compute the cohomology $H^i_V\big(X, \mathcal{O}_X(D)\big)$ of a torus-invariant Weil divisor $D = \sum_{i=1}^n c_i D_i$ with respect to some torus-invariant support $V \subseteq X$, one uses the induced eigenspace decomposition

$$H^i_V\big(X, \mathcal{O}_X(D)\big) \cong \bigoplus_{m \in M} H^i_V\big(X, \mathcal{O}_X(D)\big)_m.$$

By a well-known formula, we can compute every eigenspace by computing the relative cohomology of a certain simplicial complex:

$$H^i_V\big(X, \mathcal{O}_X(D)\big)_m \cong H^{i-1}(\hat{\Delta}_m, \hat{\Delta}_{V,m}; k).$$

Here $\hat{\Delta}$ denotes the simplicial model of Δ, i.e. the abstract simplicial complex on the set $[n]$ such that any subset $I \subset [n]$ is in $\hat{\Delta}$ iff there exists a cone σ in Δ such that elements in I are faces of σ. Similarly, $\hat{\Delta}_V$ is a subcomplex of $\hat{\Delta}$, generated by only those cones in Δ whose associated orbits in X are not contained in V (see also Section 2). For any character $m \in M$, $\hat{\Delta}_m$ and $\hat{\Delta}_{V,m}$ are the full subcomplexes which are supported on those l_i with $l_i(m) < -c_i$ (see Theorem 2.1).

By this, for an invariant divisor $D = \sum_{i=1}^n c_i D_i$, the eigenspaces $H^i_V\big(X, \mathcal{O}_X(D)\big)_m$ depend on the simplicial complexes $\hat{\Delta}$, $\hat{\Delta}_V$ as well as on the position of the characters m with respect to the hyperplanes $H_i^{\underline{c}} = \{m \in M_\mathbb{Q} \mid l_i(m) = -c_i\}$, where $M_\mathbb{Q} = M \otimes_\mathbb{Z} \mathbb{Q}$. The chamber decomposition of $M_\mathbb{Q}$ induced by the $H_i^{\underline{c}}$ (or their intersection poset) can be interpreted as the combinatorial type of D. Our strategy will be to consider the variations of combinatorial types depending on $\underline{c} = (c_1, \ldots, c_n) \in \mathbb{Q}^n$. The solution to this discriminantal problem is given by the *discriminantal arrangement* associated to the vectors $l_1, \ldots, l_n$, which has first been considered by Crapo [Cra84] and Manin and Schechtman [MS89]. The discriminantal arrangement is constructed as follows. Consider the standard short exact sequence associated to X:

$$0 \longrightarrow M_\mathbb{Q} \xrightarrow{L} \mathbb{Q}^n \xrightarrow{D} A_\mathbb{Q} \longrightarrow 0, \tag{1}$$

where L is given by $L(m) = \big(l_1(m), \ldots, l_n(m)\big)$, and $A_\mathbb{Q} := A_{d-1}(X) \otimes_\mathbb{Z} \mathbb{Q}$ is the rational divisor class group of X. The matrix D is called the *Gale* transform of L, and its i-th column D_i is the Gale transform of l_i. The most important property of the Gale transform is that the linear dependencies among the l_i and among the D_i are inverted. That is, for any subset among the l_i which forms a basis, the complementary subset of the D_i forms a basis of $A_\mathbb{Q}$, and vice versa. Moreover, for every *circuit*, i.e. a minimal linearly dependent subset, $\mathcal{C} \subset [n]$ the complementary set $\{D_i \mid l_i \notin \mathcal{C}\}$ spans a hyperplane $H_\mathcal{C}$ in $A_\mathbb{Q}$. Then the discriminantal arrangement is given by the hyperplane arrangement

$$\{H_\mathcal{C} \mid \mathcal{C} \subset [n] \text{ circuit}\}.$$

The stratification of $A_\mathbb{Q}$ by this arrangement then is in bijection with the combinatorial types of the arrangements given by the $H_i^{\underline{c}}$ under variation of $\underline{c}$. As we will see, virtually all properties of X concerning its birational geometry and cohomology vanishing of divisorial sheaves on X depend on the discriminantal arrangement. In particular, (see Proposition 3.19), the discriminantal arrangement coincides with the hyperplane arrangement generated by the facets of the secondary fan. Ubiquitous standard constructions such as the effective cone, nef cone, and the Picard group can easily be identified as its substructures.
Another interesting aspect is that the discriminantal arrangement by itself (or the associated matroid, respectively) represents a combinatorial invariant of the variety X, which one can refer to as its *circuit geometry*. This circuit geometry refines the combinatorial information coming with the toric variety, that is, the fan Δ and the matroid structure underlying the l_i (i.e. their linear dependencies). It depends only on the l_i, and even for two combinatorially equivalent fans Δ, Δ' such that corresponding sets of primitive vectors $l_1, \ldots, l_n$ and $l'_1, \ldots, l'_n$ have the same underlying linear dependencies, their associated circuit geometries are different in general. This already is the case for surfaces, see, for instance, Crapo's example of a plane tetrahedral line configuration ([Cra84], §4). Falk ([Fal94], Example 3.2) gives a 3-dimensional example.

Circuits and the diophantine Frobenius problem. Circuits are also the building blocks for our arithmetic conditions on cohomology vanishing, which can easily be illustrated for the case of weighted projective spaces. Assume, for simplicity, that the first $d+1$ primitive vectors $l_1, \dots, l_{d+1}$ generate N and form a circuit. Then we have a relation

$$\sum_{i=1}^{d+1} \alpha_i l_i = 0 \tag{2}$$

where the α_i are nonzero integers whose largest common divisor is one. This relation is unique up to sign and we assume for simplicity that $\alpha_i > 0$ for at least one i. In the special case that all the α_i are positive, $l_1, \dots, l_{d+1}$ generate the fan of a weighted projective space $\mathbb{P}(\alpha_1, \dots, \alpha_{d+1})$. Denote D the unique $\mathbb{Q}$-effective generator of $A_{d-1}\big(\mathbb{P}(\alpha_1, \dots, \alpha_{d+1})\big)$. Then there is a standard construction for counting global sections

$$\begin{aligned} &\dim H^0\big(\mathbb{P}(\alpha_1, \dots, \alpha_{d+1}), \mathcal{O}_{\mathbb{P}(\alpha_1, \dots, \alpha_{d+1})}(nD)\big) \\ =&\Big|\{(k_1, \dots, k_{d+1}) \in \mathbb{N}^{d+1} \mid \sum_{i=1}^{d+1} k_i \alpha_i = n\}\Big| =: \mathrm{VP}_{\alpha_1, \dots, \alpha_{d+1}}(n), \end{aligned}$$

for any $n \in \mathbb{Z}$. Here, $\mathrm{VP}_{\alpha_1, \dots, \alpha_{d+1}}$ is the so-called vector partition function (or denumerant function) with respect to the α_i. The problem of determining the zero set of $\mathrm{VP}_{\alpha_1, \dots, \alpha_{d+1}}$ (or the maximum of this set) is quite famously known as the diophantine Frobenius problem. This problem is hard in general (though not necessarily so in specific cases) and there does not exist a general closed expression to determine the zero set (for a survey of the diophantine Frobenius problem we refer to the book [Ram05]). Analogously, one can write down similar functions for any circuit among the l_i (see subsection 4.1).

The basic idea now is to transport the discriminantal arrangement from $A_{\mathbb{Q}}$ to some diophantine analog in $A_{d-1}(X)$. For any circuit $\mathcal{C} \subset [n]$ there is a short exact sequence

$$0 \longrightarrow H_{\mathcal{C}} \longrightarrow A_{\mathbb{Q}} \longrightarrow A_{\mathcal{C}, \mathbb{Q}} \longrightarrow 0.$$

By lifting the surjection $A_{\mathbb{Q}} \to A_{\mathcal{C}, \mathbb{Q}}$ to its integral counterpart $A_{d-1}(X) \to A_{\mathcal{C}}$, we lift the zero set of the corresponding vector partition function on $A_{\mathcal{C}}$ to $A_{d-1}(X)$. By doing this for every circuit $\mathcal{C}$, we construct in $A_{d-1}(X)$ what we call the *Frobenius discriminantal arrangement*. One can consider the Frobenius discriminantal arrangement as an arithmetic thickening of the discriminantal arrangement. This thickening in general is just enough to enlarge the relevant strata in the discriminantal arrangement such that it encompasses the Kawamata-Viehweg-like theorems. To derive other vanishing results, our analysis will mostly be concerned with analyzing the birational geometry of X and its implications on the combinatorics of the discriminantal arrangement, and the transport of this analysis to the Frobenius arrangement.

Overview. Section 2 introduce some general notation and results related to toric varieties. In section 3 we survey discriminantal arrangements, secondary fans, and rational aspects of cohomology vanishing. Several technical facts will be collected which are important for the subsequent sections. Section 4 contains all the essential results of this work. In 4.3 we will prove our main arithmetic vanishing results. These will be applied in 4.4 to give a quite complete characterization of cohomology vanishing for toric surfaces. Section 4.5 is devoted to maximal Cohen-Macaulay modules.

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2 Toric Preliminaries

In this section we first introduce notions from toric geometry which will be used throughout the rest of the paper. As general reference for toric varieties we use [Oda88], [Ful93]. We will always work over an algebraically closed field k.
Let Δ be a fan in the rational vector space $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$ over a lattice $N \cong \mathbb{Z}^d$. Let M be the lattice dual to N, then the elements of N represent linear forms on M and we write $n(m)$ for the canonical pairing $N \times M \to \mathbb{Z}$, where $n \in N$ and $m \in M$. This pairing extends naturally over $\mathbb{Q}$, $M_{\mathbb{Q}} \times N_{\mathbb{Q}} \to \mathbb{Q}$. Elements of M are denoted by m, m', etc. if written additively, and by $\chi(m)$, $\chi(m')$, etc. if written multiplicatively, i.e. $\chi(m + m') = \chi(m)\chi(m')$. The lattice M is identified with the group of characters of the algebraic torus $T = \operatorname{Hom}(M, k^*) \cong (k^*)^d$ which acts on the toric variety $X = X_\Delta$ associated to Δ. Moreover, we will use the following notation:

- cones in Δ are denoted by small greek letters $\rho, \sigma, \tau, \ldots$, their natural partial order by $\prec$, i.e. $\rho \prec \tau$ iff $\rho \subseteq \tau$;
- $|\Delta| := \bigcup_{\sigma \in \Delta} \sigma$ denotes the support of Δ;
- for $0 \leq i \leq d$ we denote $\Delta(i) \subset \Delta$ the set of i-dimensional cones; for $\sigma \in \Delta$, we denote $\sigma(i)$ the set of i-dimensional faces of σ;
- U_σ denotes the associated affine toric variety for any $\sigma \in \Delta$;
- $\check{\sigma} := \{m \in M_{\mathbb{Q}} \mid n(m) \geq 0 \text{ for all } n \in \sigma\}$ is the cone *dual* to σ;
- $\sigma^\perp = \{m \in M_{\mathbb{Q}} \mid n(m) = 0 \text{ for all } n \in \sigma\}$;
- $\sigma_M := \check{\sigma} \cap M$ is the submonoid of M associated to σ.

We will mostly be interested in the structure of Δ as a combinatorial cellular complex. For this, we make a few convenient identifications. We always denote n the cardinality of $\Delta(1)$. i.e. the number of 1-dimensional cones (*rays*) and

$[n] := \{1, \ldots, n\}$. The primitive vectors along rays are denoted $l_1, \ldots, l_n$, and, by abuse of notion, we will usually identify the sets $\Delta(1)$, the set of primitive vectors, and $[n]$. Also, we will often identify $\sigma \in \Delta$ with the set $\sigma(1) \subset [n]$. With these identifications, and using the natural order of $[n]$, we obtain a combinatorial cellular complex with support $[n]$; we may consider this complex as a combinatorial model for Δ. In the case where Δ is simplicial, this complex is just a combinatorial simplicial complex in the usual sense. If Δ is not simplicial, we consider the *simplicial cover* $\hat{\Delta}$ of Δ, modelled on $[n]$: some subset $I \subset [n]$ is in $\hat{\Delta}$ iff there exists some $\sigma \in \Delta$ such that $I \subset \sigma(1)$. The identity on $[n]$ then induces a surjective morphism $\hat{\Delta} \longrightarrow \Delta$ of combinatorial cellular complexes. This morphism has a natural representation in terms of fans. We can identify $\hat{\Delta}$ with the fan in $\mathbb{Q}^n$ which is defined as follows. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{Q}^n$, then for any set $I \subset [n]$, the vectors $\{e_i\}_{i \in I}$ span a cone over $\mathbb{Q}_{\geq 0}$ iff there exists $\sigma \in \Delta$ with $I \subset \sigma(1)$. The associated toric variety $\hat{X}$ is open in $\mathbb{A}^n_k$, and the vector space homomorphism defined by mapping $e_i \mapsto l_i$ for $i \in [n]$ induces a map of fans $\hat{\Delta} \to \Delta$. The induced morphism $\hat{X} \to X$ is the quotient presentation due to Cox [Cox95]. We will not make explicit use of this construction, but it may be useful to have it in mind.

An important fact used throughout this work is the following exact sequence which exists for any toric variety X with associated fan Δ:

$$M \xrightarrow{L} \mathbb{Z}^n \longrightarrow A_{d-1}(X) \longrightarrow 0. \tag{3}$$

Here $L(m) = (l_1(m), \ldots, l_n(m))$, i.e. as a matrix, the primitive vectors l_i represent the row vectors of L. Note that L is injective iff Δ is not contained in a proper subspace of $N_{\mathbb{Q}}$. The sequence follows from the fact that every Weil divisor D on X is rationally equivalent to a T-invariant Weil divisor, i.e. $D \sim \sum_{i=1}^n c_i D_i$, where $\underline{c} = (c_1, \ldots, c_n) \in \mathbb{Z}^n$ and $D_1, \ldots, D_n$, the T-invariant irreducible divisors of X. Moreover, any two T-invariant divisors D, D' are rationally equivalent if and only if there exists $m \in M$ such that $D - D' = \sum_{i=1}^n l_i(m) D_i$. To every Weil divisor D, one associates its divisorial sheaf $\mathcal{O}_X(D) = \mathcal{O}(D)$ (we will omit the subscript X whenever there is no ambiguity), which is a reflexive sheaf of rank one and locally free if and only if D is Cartier. Rational equivalence classes of Weil divisors are in bijection with isomorphism classes of divisorial sheaves. If D is T-invariant, the sheaf $\mathcal{O}(D)$ acquires a T-equivariant structure and the equivariant isomorphism classes of sheaves $\mathcal{O}(D)$ are one-to-one with $\mathbb{Z}^n$.

Consider a closed T-invariant subscheme $V \subseteq X$. Then for any T-invariant Weil divisor D there are induced linear representations of T on the local cohomology groups $H^i_V\big(X, \mathcal{O}(D)\big)$. In particular, each such module has a natural eigenspace decomposition

$$H^i_V\big(X, \mathcal{O}(D)\big) \cong \bigoplus_{m \in M} H^i_V\big(X, \mathcal{O}(D)\big)_m.$$

The eigenspaces $H^i_V\big(X, \mathcal{O}(D)\big)_m$ can be characterized by the relative cohomologies of certain simplicial complexes. For any $I \subset [n]$ we denote $\hat{\Delta}_I$ the maximal subcomplex of $\hat{\Delta}$ which is supported on I. We denote $\hat{\Delta}_V$ the simplicial cover of the fan associated to the complement of the reduced subscheme underlying V in X. Correspondingly, for $I \subset [n]$ we denote $\hat{\Delta}_{V,I}$ the maximal subcomplex of $\hat{\Delta}_V$ which is supported on I. If $\underline{c} \in \mathbb{Z}^n$ is fixed, and $D = \sum_{i \in [n]} c_i D_i$, then every $m \in M$ determines a subset $I(m)$ of $[n]$ which is given by

$$I(m) = \{i \in [n] \mid l_i(m) < -c_i\}.$$

Then we will write $\hat{\Delta}_m$ and $\hat{\Delta}_{V,m}$ instead of $\hat{\Delta}_{I(m)}$ and $\hat{\Delta}_{V,I(m)}$, respectively. In the case where Δ is generated by just one cone σ, we will also write $\hat{\sigma}_m$, etc. With respect to these notions we get:

Theorem 2.1: *Let $D \in \mathbb{Z}^{\Delta(1)}$ be a T-invariant Weil divisor on X. Then for every T-invariant closed subscheme V of X, every $i \geq 0$ and every $m \in M$ there exists an isomorphism of k-vector spaces*

$$H^i_V\big(X, \mathcal{O}(D)\big)_m \cong H^{i-1}(\hat{\Delta}_m, \hat{\Delta}_{V,m}; k).$$

Note that here $H^{i-1}(\hat{\Delta}_m, \hat{\Delta}_{V,m})$ denotes the *reduced* relative cohomology group of the pair $(\hat{\Delta}_m, \hat{\Delta}_{V,m})$.

Proof. For $V = X$ it follows from [EMS00], §2 that $H^i\big(X, \mathcal{O}(D)\big)_m \cong H^{i-1}(\hat{\Delta}_m; k)$ and $H^i\big(X \setminus V, \mathcal{O}(D)\big)_m \cong H^{i-1}(\hat{\Delta}_{V,m}; k)$. Then the assertion follows from comparing the long exact relative cohomology sequence of the pair $(\hat{\Delta}_m, \hat{\Delta}_{V,m})$ with the long exact local cohomology sequence with respect to X and V in degree m. □

We mention a special case of this theorem, which follows from the long exact cohomology sequence.

Corollary 2.2: *Let $X = U_\sigma$ and V a T-invariant closed subvariety of X and denote $\hat{\sigma}$ the simplicial model for the fan generated by σ. Then for every $m \in M$ and every $i \in \mathbb{Z}$:*

$$H^i_V\big(X, \mathcal{O}(D)\big)_m = \begin{cases} 0 & \text{if } \hat{\sigma}_m = \emptyset, \\ H^{i-2}(\hat{\sigma}_{V,m}; k) & \text{else.} \end{cases}$$

3 Discriminants and combinatorial aspects cohomology vanishing

A toric variety X is specified by the set of primitive vectors $l_1, \ldots, l_n \in N$ and the fan Δ supported on these vectors. We can separate three properties which govern the geometry of X and are relevant for cohomology vanishing problems:

(i) the linear algebra given by the vectors $l_1, \ldots, l_n$ and their linear dependencies as $\mathbb{Q}$-vectors;

(ii) arithmetic properties, which are also determined by the l_i, but considered as integral vectors;

(iii) its combinatorics, which is given by the fan Δ.

In this section we will have a closer look into the linear algebraic and combinatorial aspects. In subsection 3.1 we will introduce the notion of oriented and non-oriented circuits associated to the vectors l_i. In subsection 3.2 we consider circuits of the matrix L and the induced stratification of $A_{d-1}(X)_{\mathbb{Q}}$. In subsection 3.3 we will collect some well-known material on secondary fans from [GKZ94], [OP91], and [BFS90] and explain their relation to discriminantal arrangements. Subsection 3.4 then applies this to certain statements about the birational geometry of toric varieties and cohomology vanishing.
For this section and the following sections we will introduce the following conventions.

Convention 3.1: We will denote L the matrix whose rows are given by the l_i. For any subset I of $[n]$ we will denote L_I the submatrix of L consisting of the rows which are given by the l_i with $i \in I$. In general, we will not distinguish between $\{l_i\}_{i \in I}$ and L_I. Similarly, we will usually identify subsets $I \subset [n]$ with the corresponding subsets of $\{l_1, \ldots, l_n\}$. If Δ is a fan in $N_{\mathbb{R}}$ such that $\Delta(1)$ is generated by some subset of the l_i, then we say that Δ is *supported on* L (resp. on $l_1, \ldots, l_n$).
Let $\mathcal{C}$ be a subset of $[n]$ which is minimal with the property that the l_i with $i \in \mathcal{C}$ are linearly dependent. Then the set $\{l_i\}_{i \in \mathcal{C}}$ is called a *circuit*. By abuse of notion we will also call $\mathcal{C}$ itself a circuit.

3.1 Circuits

Let $\mathcal{C} \subseteq [n]$ be a circuit. Then we have a relation

$$\sum_{i \in [n]} \alpha_i l_i = 0,$$

which is unique up to a common multiple of the α_i, and the α_i are nonzero. Without loss of generality, we will assume that the α_i are integral and $\gcd\{|\alpha_i|\}_{i \in [n]} = 1$. To simplify the discussion, we will further assume that $L_{\mathcal{C}}$ generates a submodule $N_{\mathcal{C}}$ of finite index in N. For a fixed choice of the α_i, we have a partition $\mathcal{C} = \mathfrak{C}^+ \coprod \mathfrak{C}^-$, where $\mathfrak{C}^{\pm} = \{i \in [n] \mid \pm\alpha_i > 0\}$. This decomposition depends only on the signs of the α_i; flipping the signs exchanges $\mathfrak{C}^+$ and $\mathfrak{C}^-$. We want to keep track of these two possibilities and call the choice of $\mathfrak{C}^+ \coprod \mathfrak{C}^- =: \mathfrak{C}$ the *oriented circuit* with underlying circuit $\mathcal{C}$ (or simply an *orientation* of $\mathfrak{C}$), and $-\mathfrak{C} := -\mathfrak{C}^+ \coprod -\mathfrak{C}^-$ its inverse, where $-\mathfrak{C}^{\pm} := \mathfrak{C}^{\mp}$.

Definition 3.2: We denote $\mathcal{C}(L)$ the set of circuits of L and $\mathfrak{C}(L)$ the set of oriented circuits of L, i.e. the set of all orientations $\mathfrak{C}, -\mathfrak{C}$ for $\mathcal{C} \in \mathcal{C}(L)$.

For a given circuit $\mathcal{C}$, the primitive vectors $L_{\mathcal{C}}$ can support at most two simplicial fans, each corresponding to an orientation of $\mathcal{C}$. For fixed orientation $\mathfrak{C}$, we denote $\Delta = \Delta_{\mathfrak{C}}$ the fan whose maximal cones are generated by $\mathcal{C} \setminus \{i\}$, where i runs over the elements of $\mathfrak{C}^+$. The only exception for this procedure is the case where $\mathfrak{C}^+$ is empty, which we leave undefined. The associated toric variety $X_{\Delta_{\mathfrak{C}}}$ is simplicial and quasi-projective.

Definition 3.3: We call a toric variety $X = X_{\Delta_{\mathfrak{C}}}$ associated to an oriented circuit $\mathfrak{C}$ a *toric* 1*-circuit variety.*

Now let us assume that the sublattice $N_{\mathcal{C}}$ of N which is generated by $L_{\mathcal{C}}$ is saturated. Then we have a short exact sequence

$$0 \longrightarrow M \xrightarrow{L_{\mathcal{C}}} \mathbb{Z}^n \xrightarrow{G_{\mathcal{C}}} A \longrightarrow 0, \tag{4}$$

such that $A \cong \mathbb{Z}$ and thus torsion free. Here, $L_{\mathcal{C}}$ is considered as a tuple of linear forms on M, $A \cong \mathbb{Z}$ and $G_{\mathcal{C}} = (\alpha_1, \dots, \alpha_n)$ is a $(1 \times n)$-matrix, i.e. we can consider the α_i as the *Gale transform* of the l_i. Conversely, if the α_i are given, then the l_i are determined up to a $\mathbb{Z}$-linear automorphism of N. We will make more extensive use of the Gale transform later on. For generalities we refer to [OP91] and [GKZ94].
If $N_{\mathcal{C}} \subsetneq N$, we can formally consider the inclusion of $N_{\mathcal{C}}$ as the image of N via an injective endomorphism ξ of N. The inverse images of the l_i with respect to ξ satisfy the same relation as the l_i. Therefore, a general toric circuit variety is completely specified by ξ and the integers α_i. More precisely, a toric 1-circuit variety is specified by the Gale duals l_i of the α_i and an injective endomorphism ξ of N with the property that $\xi(l_i)$ is primitive in N for every $i \in [n]$.

Definition 3.4: Let $\underline{\alpha} = (\alpha_i \mid i \in \mathcal{C}) \in \mathbb{Z}^{\mathcal{C}}$ with $\alpha_i \neq 0$ for every i and $\gcd\{|\alpha_i|\}_{i \in [n]} = 1$, $\mathfrak{C}$ the associated oriented circuit with $\mathfrak{C}^+ = \{i \mid \alpha_i > 0\}$, and $\xi : N \longrightarrow N$ an injective endomorphism of N which maps the Gale duals of the α_i to primitive elements p_i in N. Then we denote $\mathbb{P}(\underline{\alpha}, \xi)$ the toric 1-circuit variety associated to the fan $\Delta_{\mathfrak{C}}$ spanned by the primitive vectors p_i.

The endomorphism ξ translates into an isomorphism

$$\mathbb{P}(\underline{\alpha}, \xi) \cong \mathbb{P}(\underline{\alpha}, \mathrm{id}_N)/H,$$

where $H \cong \operatorname{spec} k[N/N_{\mathcal{C}}]$. Note that in positive characteristic, H in general is a group scheme rather than a proper algebraic group. Moreover, in sequence (4) we can identify A with the divisor class group $A_{d-1}\big(\mathbb{P}(\underline{\alpha}, \mathrm{id}_N)\big)$. Similarly, we get $A_{d-1}\big(\mathbb{P}(\underline{\alpha}, \xi)\big) \cong A \oplus N/N_{\mathcal{C}}$ and the natural surjection from $A_{d-1}\big(\mathbb{P}(\underline{\alpha}, \xi)\big)$ onto $A_{d-1}\big(\mathbb{P}(\underline{\alpha}, \mathrm{id}_N)\big)$ just projects away the torsion part.

Remarks 3.5: (i) In the case $\alpha_i > 0$ for all i and $\xi = \mathrm{id}_N$, we just recover the usual weighted projective spaces. In many respects, the spaces $\mathbb{P}(\underline{\alpha}, \xi)$ can be treated the same way as has been done in the standard references for weighted

projective spaces, see [Del75], [Dol82], [BR86]. In our setting there is the slight simplification that we naturally can assume that $\gcd\{|\alpha_j|\}_{j\neq i} = 1$ for every $i \in [n]$, which eliminates the need to discuss reduced weights.
(ii) In the case that $L_{\mathcal{C}}$ spans a subspace N' of $N_{\mathbb{Q}}$ of positive codimension r, then for some orientation $\mathfrak{C}$ of $\mathcal{C}$ the variety $X(\Delta_{\mathfrak{C}})$ is isomorphic to $\mathbb{P}(\underline{\alpha}, \xi) \times (k^*)^r$, where $\mathbb{P}(\underline{\alpha}, \xi)$ is defined as before with respect to N'. Note that if $\mathfrak{C}^+ = \mathcal{C}$, then the fan $\Delta_{-\mathfrak{C}}$ is empty. By convention, in that case one can define $X(\Delta_{-\mathfrak{C}}) := (k^*)^r$ as the associated toric variety.
(iii) The spaces $\mathbb{P}(\underline{\alpha}, \xi)$ are building blocks for the birational geometry of general toric varieties. In fact, to every extremal curve $V(\tau)$ in some simplicial toric variety X, there is associated some variety $\mathbb{P}(\underline{\alpha}, \xi)$ whose fan $\Delta_{\mathfrak{C}}$ is a subfan of Δ and which embeds as an open invariant subvariety of X. If $|\mathfrak{C}^+| \notin \{n, n-1\}$, the primitive vectors l_i span a convex polyhedral cone, giving rise to an affine toric variety Y and a canonical morphism $\pi : \mathbb{P}(\underline{\alpha}, \xi) \longrightarrow Y$ which is a partial resolution of singularities. Sign change $\underline{\alpha} \to -\underline{\alpha}$ then encodes the transition from $\mathfrak{C}$ to $-\mathfrak{C}$ and and a birational map from $\mathbb{P}(\underline{\alpha}, \xi)$ to $\mathbb{P}(-\underline{\alpha}, \xi)$, which provides a local model for well-known combinatorial operation which called *bistellar operation* [Rei99] or *modification of a triangulation* [GKZ94]:

$$\begin{tikzcd} \mathbb{P}(\underline{\alpha}, \xi) \arrow[rr, dashed] \arrow[dr, "\pi"] & & \mathbb{P}(-\underline{\alpha}, \xi) \arrow[dl, "\pi'"'] \\ & Y & \end{tikzcd}$$

(for $|\mathfrak{C}^+| = d - 1$, one can identify $\mathbb{P}(-\underline{\alpha}, \xi)$ with Y and one just obtains a blow-down).

3.2 Circuits and discriminantal arrangements

Recall that for any torus invariant divisor $D = \sum_{i \in [n]} c_i D_i$, the isotypical components $H^i_V\big(X, \mathcal{O}(D)\big)_m$ for some cohomology group depend on simplicial complexes $\hat{\Delta}_I$, where $I = I(m) = \{i \in [n] \mid l_i(m) < -c_i\}$. So, the set of all possible subcomplexes $\hat{\Delta}_I$ depends on the chamber decomposition of $M_{\mathbb{Q}}$ which is induced by the hyperplane arrangement which is given by hyperplanes $H_1^{\underline{c}}, \ldots, H_n^{\underline{c}}$, where

$$H_i^{\underline{c}} := \{m \in M_Q \mid l_i(m) = -c_i\}.$$

The set of all relevant $I \subset [n]$ is determined by the map

$$\mathfrak{s}^{\underline{c}} : M_{\mathbb{Q}} \longrightarrow 2^{[n]}, \quad m \mapsto \{i \in [n] \mid l_i(m) < -c_i\}.$$

Definition 3.6: For $m \in M_{\mathbb{Q}}$, we call $\mathfrak{s}^{\underline{c}}$ the *signature* of m. We call the image of $M_{\mathbb{Q}}$ in $2^{[n]}$ the *combinatorial type* of $\underline{c}$.

Remark 3.7: The combinatorial type encodes what in combinatorics is known as *oriented matroid* (see [BLS+93]). We will not make use of this kind of structure, but we will find it sometimes convenient to borrow some notions.

So, given $l_1, \ldots, l_n$, we would like to classify all possible combinatorial types, depending on $\underline{c} \in \mathbb{Q}^n$. The natural parameter space for all hyperplane arrangements up to translation by some element $m \in M_\mathbb{Q}$ is given by the set $A_\mathbb{Q} \cong \mathbb{Q}^n / M_\mathbb{Q}$, which is given by the following short exact sequence:

$$0 \longrightarrow M_\mathbb{Q} \xrightarrow{L} \mathbb{Q}^n \xrightarrow{D} A_{d-1}(X)_\mathbb{Q} = A_\mathbb{Q} \longrightarrow 0.$$

Then the $D_1, \ldots, D_n$ are the images of the standard basis vectors of $\mathbb{Q}^n$. This procedure of constructing the D_i from the l_i is often called *Gale transformation*, and the D_i are the *Gale duals* of the l_i.
Now, a hyperplane arrangement $H_i^{\underline{c}}$ for some $\underline{c} \in \mathbb{Q}^n$, is considered in general position if the hyperplanes $H_i^{\underline{c}}$ intersect in the smallest possible dimension. When varying $\underline{c}$ and passing from one arrangement in general position to another one which has a different combinatorial type, this necessarily implies that has to take place some specialization for some $\underline{c} \in \mathbb{Q}^n$, i.e. where the corresponding hyperplanes $H_i^{\underline{c}}$ do not intersect in the smallest possible dimension. So we see that the combinatorial types of hyperplane arrangements with fixed L and varying induce a stratification of $A_\mathbb{Q}$, where the maximal strata correspond to hyperplane arrangements in general position. To determine this stratification is the *discriminant problem* for hyperplane arrangements. To be more precise, let $I \subset [n]$ and denote

$$H_I := \{\underline{c} + M_\mathbb{Q} \in A_\mathbb{Q} \mid \bigcap_{i \in I} H_i^{\underline{c}} \neq 0\},$$

i.e. H_I represents the set of all hyperplane arrangements (up to translation) such that the hyperplanes $\{H_i\}_{i \in I}$ have nonempty intersection. The sets H_I can be described straightforwardly by the following commutative exact diagram:

$$\begin{tikzcd}
 & & & H_I \arrow[d, hook] & \\
0 \arrow[r] & M_\mathbb{Q} \arrow[r, "L"] \arrow[d, equal] & \mathbb{Q}^n \arrow[r, "D"] \arrow[d, two heads] & A_\mathbb{Q} \arrow[r] \arrow[d, two heads] & 0 \\
 & M_\mathbb{Q} \arrow[r, "L_I"] & \mathbb{Q}^I \arrow[r, "D_I"] & A_{I,\mathbb{Q}} \arrow[r] & 0.
\end{tikzcd} \tag{5}$$

In particular, H_I is a subvector space of $A_\mathbb{Q}$. Moreover, we immediately read off diagram (5):

Lemma 3.8: *(i) H_I is generated by the D_i with $i \in [n] \setminus I$.*

(ii) $\dim H_I = n - |I| - \dim(\ker L_I)$.

(iii) If $J \subseteq I$ then $H_I \subseteq H_J$.

(iv) Let $I, J \subset [n]$, then $H_{I \cup J} \subset H_I \cap H_J$.

Note that in (iv) the reverse inclusion in general is not true. It follows that the hyperplanes among the H_I are determined by the formula:

$$|I| = \operatorname{rk} L_I + 1.$$

By Lemma 3.8 (iii), we can always consider circuits fulfilling this condition. It turns out that the hyperplane $H_\mathcal{C}$ suffice to completely describe the discriminants of L:

Lemma 3.9: *Let $I \subset [n]$, then*

$$H_I = \bigcap_{\mathcal{C} \subset I \, circuit} H_\mathcal{C},$$

where, by convention, the right hand side equals $A_\mathbb{Q}$, if the l_i with $i \in I$ are linearly independent.

Hence, the stratification of $A_\mathbb{Q}$ which we were looking for is completely determined by the hyperplanes $H_\mathcal{C}$.

Definition 3.10: We denote the set $\{H_\mathcal{C} \mid \mathcal{C} \subset [n] \text{ a circuit}\}$ the *discriminantal arrangement* of L.

Remark 3.11: The discriminantal arrangement carries a natural matroid structure. This structure can be considered as another combinatorial invariant of L (or the toric variety X, respectively), its *circuit geometry.* Discriminantal arrangements seem to have been appeared first in [Cra84], where the notion of 'circuit geometry' was coined. The notion of discriminantal arrangements stems from [MS89]. Otherwise, this subject seems to have been studied explicitly only in very few places, see for instance [Fal94], [BB97], [Ath99], [Rei99], [Coh86], [CV03], though it is at least implicit in the whole body of literature on secondary fans. Above references are mostly concerned with genericity properties of discriminantal arrangements. Unfortunately, in toric geometry, the most interesting cases (such as smooth projective toric varieties, for example) virtually never give rise to discriminantal arrangements in general position. Instead, we will focus on certain properties of nongeneric circuit geometries, though we will not undertake a thorough combinatorial study of these.

Virtually all problems related to cohomology vanishing on a toric variety X must depend on the associated discriminantal arrangement and therefore on the circuits of L. In subsection 3.3 we will see that the discriminantal arrangement is tightly tied to the geometry of X.
As we have seen in section 3.1, to every circuit $\mathcal{C} \subset [n]$ we can associate two oriented circuits. These correspond to the signature of the bounded chamber of the subarrangement in $M_\mathbb{Q}$ given by the $H_i^\mathcal{C}$ with $i \in \mathcal{C}$ (or better to the bounded chamber in $M_\mathbb{Q}/\ker L_I$, as we do no longer require that the l_i with $i \in \mathcal{C}$ span $M_\mathbb{Q}$). Lifting this to $A_\mathbb{Q}$, this corresponds to the half spaces in $A_\mathbb{Q}$ which are bounded by $H_\mathcal{C}$.

Definition 3.12: Let $\mathcal{C} \subset [n]$ be a circuit, then we denote $H_{\mathfrak{C}}$ the half space in $A_{\mathbb{Q}}$ bounded by $H_{\mathcal{C}}$ corresponding to the orientation $\mathfrak{C}$.

The following is straightforward to check:

Lemma 3.13: *Let $\mathcal{C}$ be a circuit of L and $\mathfrak{C}$ an orientation of $\mathcal{C}$. Then the hyperplane $H_{\mathcal{C}}$ is* separating*, i.e. for every $i \in [n]$ one of the following holds:*

(i) $i \in [n] \setminus \mathcal{C}$ iff $D_i \in H_{\mathcal{C}}$;

(ii) if $i \in \mathfrak{C}^+$, then $D_i \in H_{\mathfrak{C}} \setminus H_{\mathcal{C}}$;

(iii) if $i \in \mathfrak{C}^-$, then $D_i \in H_{-\mathfrak{C}} \setminus H_{\mathcal{C}}$.

Now we are going to borrow some terminology from combinatorics. Consider any subvector space U of $A_{\mathbb{Q}}$ which is the intersection of some of the $H_{\mathcal{C}}$. Then the set $\mathcal{F}_U$ of all $\mathcal{C} \in \mathcal{C}(L)$ such that $H_{\mathcal{C}}$ contains U is called a *flat*. The subvector space is uniquely determined by the flat and vice versa. We can do the same for the actual strata rather than for subvector spaces. For this, we just need to consider instead the oriented circuits and their associated half spaces in $A_{\mathbb{Q}}$: any stratum S of the discriminantal arrangement uniquely determines a finite set $\mathfrak{F}_S$ of oriented circuits $\mathfrak{C}$ such that $S \subset H_{\mathfrak{C}}$. From the set $\mathfrak{F}_S$ we can reconstruct the closure of S:

$$\overline{S} = \bigcap_{\mathfrak{C} \in \mathfrak{F}_S} H_{\mathfrak{C}},$$

We give a formal definition:

Definition 3.14: For any subvector space $U \subset A_{\mathbb{Q}}$ which is a union of strata of the discriminantal arrangement, we denote $\mathcal{F}_U := \{\mathcal{C} \in \mathcal{C}(L) \mid U \subset H_{\mathcal{C}}\}$ the associated *flat*. For any single stratum $S \subset A_{\mathbb{Q}}$ of the discriminantal arrangement, we denote $\mathfrak{F}_S := \{\mathfrak{C} \in \mathfrak{C}(L) \mid U \subset H_{\mathfrak{C}}\}$ the associated *oriented flat*.

The notion of flats gives us some flexibility in handling strata. Note that flats reverse inclusions, i.e. $S \subset T$ iff $\mathfrak{F}_T \subset \mathfrak{F}_S$. Moreover, if a stratum S is contained in some $H_{\mathcal{C}}$, then its oriented flat contains both $H_{\mathfrak{C}}$ and $H_{-\mathfrak{C}}$, and vice versa. So from the oriented flat we can reconstruct the subvector space of $A_{\mathbb{Q}}$ generated by S.

Definition 3.15: Let $\mathcal{S} := \{S_1, \dots, S_k\}$ be a collection of strata of the discriminantal arrangement. We call

$$\mathfrak{F}_{\mathcal{S}} := \bigcap_{i=1}^{k} \mathfrak{F}_{S_i}$$

the *discriminantal hull* of $\mathcal{S}$.

The discriminantal hull defines a closed cone in $A_{\mathbb{Q}}$ which is given by the intersection $\bigcap_{\mathfrak{C} \in \mathfrak{F}_{\mathcal{S}}} H_{\mathfrak{C}}$. This cone contains the union of the closures $\overline{S}_i$, but is bigger in general.

Lemma 3.16: *(i) Let $\mathcal{S} = \{S_1, \ldots, S_k\}$ be a collection of discriminantal strata whose union is a closed cone in $A_{\mathbb{Q}}$. then $\mathfrak{F}_{\mathcal{S}} = \bigcap_{i=1}^{k} \mathfrak{F}_{S_i}$.*

(ii) Let $\mathcal{S} = \{S_1, \ldots, S_k\}$ be a collection of discriminantal strata and U the subvector space of $A_{\mathbb{Q}}$ generated by the S_i. Then the forgetful map $\mathfrak{F}_{\mathcal{S}} \to \mathcal{F}_U$ is surjective iff $\mathfrak{F}_{\mathcal{S}} = \mathfrak{F}_{S_i}$ for some i.

Proof. For (i) just note that because $\bigcup_{i=1}^{k} S_k$ is a closed cone, it must be an intersection of some $H_{\mathfrak{C}}$. For (ii): the set $\bigcap_{\mathfrak{C} \in \mathfrak{F}_{\mathcal{S}}} H_{\mathfrak{C}}$ is a cone which contains the convex hull of all the $\overline{S}_i$. If some $\mathcal{C}$ is not in the image of the forgetful map, then the hyperplane $H_{\mathcal{C}}$ must intersect the relative interior of this cone. So the assertion follows. $\square$

3.3 Secondary Fans

For any $\underline{c} \in \mathbb{Q}^n$ the arrangement $H_i^{\underline{c}}$ induces a chamber decomposition of $M_{\mathbb{Q}}$, where the closures of the chambers are given by

$$P_{\underline{c}}^I := \{m \in M_{\mathbb{Q}} \mid l_i(m) \leq -c_i \text{ for } i \in I \text{ and } l_i(m) \geq -c_i \text{ for } i \notin I\}$$

for every $I \subset [n]$ which belongs to the combinatorial type of $\underline{c}$. In particular, $\underline{c}$ represents an element $D \in A_{\mathbb{Q}}$ with

$$D \in \bigcap_{I \in \mathfrak{s}^{\underline{c}}(M_{\mathbb{Q}})} C_I,$$

where C_I is the cone in $A_{\mathbb{Q}}$ which is generated by the $-D_i$ for $i \in I$ and the D_i with $i \notin I$ for some $I \subset [n]$. For an invariant divisor $D = \sum_{i \in [n]} c_i D_i$ we will also write P_D^I instead of $P_{\underline{c}}^I$. If $I = \emptyset$, we will occasionally omit the index I. The faces of the C_I can be read off directly from the signature:

Proposition 3.17: *Let $I \subset [n]$, then C_I is an nonredundant intersection of the $H_{\mathfrak{C}}$ with $\mathfrak{C}^- \subset I$ and $\mathfrak{C}^+ \cap I = \emptyset$.*

Proof. First of all, it is clear that C_I coincides with the intersection of half spaces

$$C_I = \bigcap_{\substack{\mathfrak{C}^+ \subset I \\ \mathfrak{C}^- \cap I = \emptyset}} H_{\mathfrak{C}}.$$

For any $H_{\mathfrak{C}}$ in the intersection let $H_{\mathcal{C}}$ its boundary. Then $H_{\mathcal{C}}$ contains a cone of codimension 1 in $A_{\mathbb{Q}}$ which is spanned by D_i with $i \in [n] \setminus (\mathcal{C} \cup I)$ and by $-D_i$ with $i \in I \setminus \mathcal{C}$ which thus forms a proper facet of C_I. $\square$

Recall that the secondary fan of L is a fan in $A_{\mathbb{Q}}$ whose maximal cones are in one-to-one correspondence with the regular simplicial fans which are supported on the l_i. That is, if $\underline{c}$ is chosen sufficiently general, then the polyhedron $P^{\emptyset}_{\underline{c}}$ is simplicial and its inner normal fan is a simplicial fan which is supported on the l_i. Wall crossing in the secondary fan then corresponds locally to a transition $\Delta_{\mathfrak{C}} \longrightarrow \Delta_{-\mathfrak{C}}$ as in section 3.1. Clearly, the secondary fan is a substructure of the discriminantal arrangement in the sense that its cones are unions of strata of the discriminantal arrangements. However, the secondary fan in general is much coarser than the discriminantal arrangement, as it only keeps track of the particular chamber $P^{\emptyset}_{\underline{c}}$. In particular, the secondary fan is only supported on $C_{\emptyset}$ which in general does not coincide with $A_{\mathbb{Q}}$. Of course, there is no reason to consider only one particular type of chamber — we can consider secondary fans for every $I \subset [n]$ and every type of chamber $P^I_{\underline{c}}$. For this, observe first that, if $\mathcal{B}$ is a subset of $[n]$ such that the l_i with $i \in \mathcal{B}$ form a basis of $M_{\mathbb{Q}}$, then the complementary Gale duals $\{D_i\}_{i \notin \mathcal{B}}$ form a basis of $A_{\mathbb{Q}}$. Then we set:

Definition 3.18: Let $I \subset [n]$ and $\mathcal{B} \subset [n]$ such that the l_i with $i \in \mathcal{B}$ form a basis of $M_{\mathbb{Q}}$, then we denote $K^I_{\mathcal{B}}$ the cone in $A_{\mathbb{Q}}$ which is generated by $-D_i$ for $i \in I \setminus \mathcal{B}$ and by D_i for $i \in [n] \setminus (I \cup \mathcal{B})$. The secondary fan $\mathrm{SF}(L, I)$ of L with respect to I is the fan whose cones are the intersections of the $K^I_{\mathcal{B}}$, where $\mathcal{B}$ runs over all bases of L.

Note that $\mathrm{SF}(L, \emptyset)$ is just the secondary fan as usually defined. Clearly, the chamber structure of the discriminantal arrangement still refines the chamber structure induced by all secondary arrangements. But now we have sufficient data to even get equality:

Proposition 3.19: *The following induce identical chamber decompositions of* $A_{\mathbb{Q}}$*:*

(i) the discriminantal arrangement,

(ii) the intersection of all secondary fans $\mathrm{SF}(L, I)$*,*

(iii) the intersection of the C_I *for all* $I \subset [n]$*.*

Proof. Clearly, the facets of every orthant C_I span a hyperplane which is part of the discriminantal arrangement, so the chamber decomposition induced by the secondary fan is a refinement of the intersection of the C_I's. The C_I induce a refinement of the secondary fans as follows. Without loss of generality, it suffices to show that every $K^{\emptyset}_{\mathcal{B}}$ is the intersection of some C_I. We have

$$K^{\emptyset}_{\mathcal{B}} \subseteq \bigcap_{I \subset \mathcal{B}} C_I.$$

On the other hand, for every facet of $K^{\emptyset}_{\mathcal{B}}$, we choose I such that C_I shares this face and $K^{\emptyset}_{\mathcal{B}}$ is contained in C_I. This can always be achieved by choosing I so

that every generator of C_I is in the same half space as $K^{\emptyset}_{\mathcal{B}}$. The intersection of these C_I then is contained in $K^{\emptyset}_{\mathcal{B}}$.
Now it remains to show that the intersection of the secondary fans refines the discriminantal arrangement. This actually follows from the fact, that for every hyperplane $H_{\mathcal{C}}$, one can choose a minimal generating set which we can complete to a basis of $A_{\mathbb{Q}}$ from the D_i, where $i \notin \mathcal{C}$. By varying the signs of this generating set, we always get a simplicial cone whose generators are contained in some secondary fan, and this way $H_{\mathcal{C}}$ is covered by a set of facets of secondary cones. □

The maximal cones in the secondary fan $\mathrm{SF}(L, \emptyset)$ correspond to regular simplicial fans supported on $l_1, \dots, l_n$. More precisely, if Δ denotes such a fan, then the corresponding cone is given by $\bigcap_{\mathcal{B}} K^{\emptyset}_{\mathcal{B}}$, where $\mathcal{B}$ runs over all bases among the l_i which span a maximal cone in Δ. With respect to a simplicial model $\hat{\Delta}$ for Δ, we define:

Definition 3.20: Let Δ be a fan supported on L, then we set:

$$\mathrm{nef}(\Delta) := \bigcap_{\substack{\mathcal{B} \in \hat{\Delta} \\ \mathcal{B} \text{ basis in } L}} K^{\emptyset}_{\mathcal{B}}$$

and denote $\mathfrak{F}_{\mathrm{nef}} = \mathfrak{F}_{\mathrm{nef}(\Delta)}$ the discriminantal hull of $\mathrm{nef}(\Delta)$.

Note that by our conventions we identify $\mathcal{B} \in \hat{\Delta}$ with the set of corresponding primitive vectors, or the corresponding rows of L, respectively. Of course, $\mathrm{nef}(\Delta)$ is just the nef cone of the toric variety associated to Δ.

Proposition 3.21: *We have:*

$$\mathrm{nef}(\Delta) = \bigcap_{\hat{\Delta} \cap (\Delta_{\mathfrak{C}})_{\max} \neq \emptyset} H_{\mathfrak{C}}.$$

Proof. For some basis $\mathcal{B} \subset [n]$, the cone $K^{\emptyset}_{\mathcal{B}}$ is simplicial, and for every $i \in [n] \setminus \mathcal{B}$, the facet of $K^{\emptyset}_{\mathcal{B}}$ which is spanned by the D_j with $j \notin \mathcal{B} \cup \{i\}$, spans a hyperplane $H_{\mathcal{C}}$ in P. This hyperplane corresponds to the unique circuit $\mathcal{C} \subset \mathcal{B} \cup \{i\}$. As we have seen before, a maximal cone in $\Delta_{\mathfrak{C}}$ is of the form $\mathcal{C} \setminus \{j\}$ for some $j \in \mathfrak{C}^+$. So we have immediately:

$$K_{\mathcal{B}} = \bigcap_{\substack{\exists F \in (\Delta_{\mathfrak{C}})_{\max} \\ \text{with } F \subset \mathcal{B}}} H_{\mathfrak{C}}$$

and the assertion follows. □

Remark 3.22: If $\Delta = \hat{\Delta}$ is a regular simplicial fan, then $\mathrm{nef}(\Delta)$ is a maximal cone in the secondary fan. Let $\mathfrak{C}$ be an oriented circuit such that Δ is supported on $\Delta_{\mathfrak{C}}$ in the sense of [GKZ94], §7, Def. 2.9, and denote Δ' the fan resulting in the bistellar operation by changing $\Delta_{\mathfrak{C}}$ to $\Delta_{-\mathfrak{C}}$. Then, by [GKZ94], §7, Thm. 2.10, the hyperplane $H_{\mathcal{C}}$ is a proper wall of $\mathrm{nef}(\Delta)$ iff Δ' is regular, too.

3.4 MCM sheaves, $\mathbb{Q}$-Cartier divisors and the toric Kawamata-Viehweg vanishing theorem

Recall that a $\mathbb{Q}$-divisor on X is $\mathbb{Q}$-Cartier if an integral multiple is Cartier in the usual sense. A torus invariant Weil divisor $D = \sum_{i \in [n]} c_i D_i$ is $\mathbb{Q}$-Cartier iff for every $\sigma \in \Delta$ there exists some $m_\sigma \in M_\mathbb{Q}$ such that $c_i = l_i(m)$ for all $i \in \sigma(1)$. A result of Bruns and Gubeladze [BG03] states that every toric $\mathbb{Q}$-Cartier divisor is maximal Cohen-Macaulay. The MCM property is useful, as it allows to replace the Ext-groups by cohomologies in Serre duality:

Proposition 3.23: *Let X be a normal variety with dualizing sheaf ω_X and $\mathcal{F}$ a coherent sheaf on X such that for every $x \in X$, the stalk $\mathcal{F}_x$ is MCM over $\mathcal{O}_{X,x}$. Then for every $i \in \mathbb{Z}$ there exists an isomorphism*

$$\operatorname{Ext}^i_X\big(\mathcal{F}, \omega_X\big) \cong H^i\big(X, \mathcal{H}om(\mathcal{F}, \omega_X)\big).$$

Proof. For any two $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$ there exists the following spectral sequence

$$E_2^{pq} = H^p\big(X, \mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})\big) \Rightarrow \operatorname{Ext}^{p+q}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}).$$

We apply this spectral sequence to the case $\mathcal{G} = \omega_X$. For every closed point $x \in X$ we have the following identity of stalks:

$$\mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{F}, \omega_X)_x \cong \operatorname{Ext}^q_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \omega_{X,x}).$$

As $\mathcal{F}$ is maximal Cohen-Macaulay, the latter vanishes for all $q > 0$, and thus the sheaf $\mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{F}, \omega_X)$ is the zero sheaf for all $q > 0$. So the above spectral sequence degenerates and we obtain an isomorphism

$$H^p(X, \mathcal{H}om(\mathcal{F}, \omega_X)) \cong \operatorname{Ext}^p_X(\mathcal{F}, \omega_X)$$

for every $p \in \mathbb{Z}$. ☐

In the case where X a toric variety, we have $\omega_X \cong \mathcal{O}(K_X)$, where $K_X = -\sum_{i \in [n]} D_i$. Then, if $\mathcal{F} = \mathcal{O}(D)$ for some $D \in A$, we can identify $\mathcal{H}om(\mathcal{O}(D), \omega_X)$ with $\mathcal{O}(K_X - D)$:

Corollary 3.24: *Let X be a toric variety and D a Weil divisor such that $\mathcal{O}(D)$ is an MCM sheaf. Then there is an isomorphism:*

$$\operatorname{Ext}^i_X\big(\mathcal{O}(D), \omega_X\big) \cong H^i\big(X, \mathcal{O}_X(K_X - D)\big).$$

And by Grothendieck-Serre duality:

Corollary 3.25: *If X is a complete toric variety and D a Weil divisor such that $\mathcal{O}(D)$ is an MCM sheaf, then*

$$H^i\big(X, \mathcal{O}(D)\big) \cong H^{d-i}\big(X, \mathcal{O}(K_X - D)\big)^{\check{}}.$$

For any Cartier divisor D on some normal variety X denote $N(X, D) := \{k \in \mathbb{N} \mid H^0\big(X, \mathcal{O}(kD)\big) \neq 0\}$. Then the Iitaka dimension of D is defined as

$$\kappa(D) := \max_{k \in N(X,D)} \{\dim \phi_k(X)\},$$

where $\phi_k : X \dashrightarrow \mathbb{P}|kD|$ is the family of morphisms given by the linear series $|kD|$.
In the case where X is a toric variety and $D = \sum_{i \in [n]} c_i D_i$ invariant, the Iitaka dimension of D is just the dimension of P_{kD} for $k >> 0$. For a $\mathbb{Q}$-Cartier divisor D, we define its Iitaka dimension by $\kappa(D) := \kappa(rD)$ for $r > 0$ such that rD is Cartier.
If D is a nef divisor, then the morphism $\phi : X \longrightarrow \mathbb{P}|D|$ is torus equivariant, its image is a projective toric variety of dimension $\kappa(D)$ whose associated fan is the inner normal fan of P_D. If $\kappa(D) < d$, then necessarily D is contained in some hyperplane $H_{\mathcal{C}}$ such that $\mathfrak{C}^+ = \mathcal{C}$ for some orientation $\mathfrak{C}$ of $\mathcal{C}$. The toric variety $X_{\Delta_{\mathfrak{C}}}$ is isomorphic to a finite cover of a weighted projective space. This kind of circuit will play an important role later on, so that we will give it a distinguished name:

Definition 3.26: We call a circuit $\mathcal{C}$ such that $\mathcal{C} = \mathfrak{C}^+$ for one of its orientations, *fibrational*. For $D \in A_{d-1}(X)_{\mathbb{Q}}$ we denote $\mathrm{fib}(D) \subset \mathcal{C}(L)$ the set of fibrational circuits such that $D \in H_{\mathcal{C}}$.

By Proposition 3.17, such a divisor D is contained in the intersection of $\mathrm{nef}(X)$ with the effective cone of X, which we identify with $C_{\emptyset}$. More precisely, it follows from linear algebra that D is contained in all $H_{\mathcal{C}}$ where $\mathcal{C}$ is fibrational and $l_i(P_D) = 0$ for all $i \in \mathcal{C}$.
The fibrational circuits of a nef divisor D tell us immediately about its Iitaka dimension:

Proposition 3.27: *Let D be a nef $\mathbb{Q}$-Cartier divisor. Then $\kappa(D) = d - \mathrm{rk}\, L_T$, where $T := \bigcup_{\mathcal{C} \in \mathrm{fib}(D)} \mathcal{C}$.*

Proof. We just remark that $\mathrm{rk}\, L_T$ is the dimension of the subvector space of $M_{\mathbb{Q}}$ which is generated by the l_i which are contained in a fibrational circuit. □

Proposition 3.28: *Let X be a complete toric variety and D a nef divisor, then $H^i\big(X, \mathcal{O}(-D)\big) = 0$ for $i \neq \kappa(D)$.*

Proof. Consider the hyperplane arrangement given by the $H_i^{\underline{c}}$ in $M_{\mathbb{Q}}$. Let $m \in M_{\mathbb{Q}}$ and $I = \mathfrak{s}^{\underline{c}}(m)$. Then the simplicial complex $\hat{\Delta}_I$ can be characterized as follows. Consider $Q \subset P_D$ the union of the set faces of P_D which are contained in any $H_i^{\underline{c}}$ with $i \in I$. This is precisely the portion of P_D, which the the point m "sees", and therefore contractible, where the convex hull of Q and m provides the homotopy between Q and m. Therefore, every $\hat{\Delta}_I$ is contractible with an exception for $I = \emptyset$, because $\hat{\Delta}_{\emptyset} = \emptyset$, which is not acyclic with respect to reduced cohomology. Now we pass to the inverse, i.e. we consider the signature of $-m$ with respect to $H_i^{-\underline{c}}$. Then for any such $-m$ which does not sit in the relative interior of the polytope $P_{-\underline{c}}^{[n]}$, there exists $m' \in M_{\mathbb{Q}}$ with signature $\mathfrak{s}^{\underline{c}}(m') =: J$ such that $\hat{\Delta}_J$ is contractible and $\mathfrak{s}^{-\underline{c}}(m) = [n] \setminus J$. As $\hat{\Delta}$ is homotopic to a $d-1$-sphere, we can apply Alexander duality and thus the simplicial complex $\hat{\Delta}_{[n]\setminus J}$ is acyclic. Thus there remain only the elements in the relative interior of $P_{-\underline{c}}^{[n]}$. Let m be such an element with signature I, then $\hat{\Delta}_I$ is isomorphic to a $d - \kappa(D) - 1$-sphere, and the assertion follows. $\square$

This proposition implies the toric Kodaira and Kawamata-Viehweg vanishing theorems (see also [Mus02]):

Theorem 3.29 (Kodaira & Kawamata-Viehweg): *Let X be a complete toric variety and D, E $\mathbb{Q}$-divisors with D nef and $E = \sum_{i \in [n]} e_i D_i$ with $-1 < e_i < 0$ for all $i \in [n]$. Then:*

(i) if D is integral, then $H^i\big(X, \mathcal{O}(D + K_X)\big) = 0$ for all $i \neq 0, d - \kappa(D)$;

(ii) if $D + E$ is a Weil divisor, then $H^i\big(X, \mathcal{O}(D + E)\big) = 0$ for all $i > 0$.

Proof. Because a toric $\mathbb{Q}$-Cartier divisor is MCM, we can apply Serre duality (Corollary 3.25) and obtain $H^i\big(X, \mathcal{O}(D + K_X)\big) \cong H^{d-i}\big(X, \mathcal{O}(-D)\big)$ and (i) follows from Proposition 3.28.
For (ii): $D + E$ is contained the interior of every half space $K_X + H_{\mathfrak{C}}$ for $\mathfrak{C} \in \mathfrak{F}_{\mathrm{nef}}$, and the result follows. $\square$

4 Arithmetic aspects of cohomology vanishing

In this section we want to generalize classical vanishing results for integral divisors which cannot directly be derived from the setting of $\mathbb{Q}$-divisors as in section 3.4. From now on we assume that the l_i are integral. Recall that for any integral divisor $D = \sum_{i \in [n]} c_i D_i$ and any torus invariant closed subvariety V of X, vanishing of $H^i_V\big(X, \mathcal{O}(D)\big)$ depends on two things:

(i) whether the set of signatures $\mathfrak{s}^{\underline{c}}(M_{\mathbb{Q}})$ consists of $I \subset [n]$ such that the relative cohomology groups $H^{i-1}(\hat{\Delta}_I, \hat{\Delta}_{V,I}; k)$ vanish, and,

(ii) if $H^{i-1}(\hat{\Delta}_I, \hat{\Delta}_{V,I}; k)$ for one such I, whether the corresponding polytope $P^I_{\underline{c}}$ contain lattice points m with $\mathfrak{s}^{\underline{c}}(m) = I$.

In the Gale dual picture, the signature $\mathfrak{s}^{\underline{c}}(M_{\mathbb{Q}})$ coincides with the set of $I \subset [n]$ such that the class of D in $A_{d-1}(X)_{\mathbb{Q}}$ is contained in C_I. For fixed I, the classes of divisors D in $A_{d-1}(X)$ such that the equation $l_i(m) < -c_i$ for $i \in I$ and $l_i(m) \geq -c_i$ for $i \notin I$ is satisfied, is counted by the *generalized partition function*. That is, by the function

$$D \mapsto \Big|\big\{(k_1, \dots, k_n) \in \mathbb{N}^n \mid \sum_{i \in [n] \setminus I} k_i D_i - \sum_{i \in I} k_i D_i = D \text{ where } k_i > 0 \text{ for } i \in I\big\}\Big|.$$

So, in the most general picture, we are looking for D lying in the common zero set of the vector partition function for all relevant signatures I of D. In general, this is a difficult problem to determine these zero sets, and it is hardly necessary for practical purposes.
Vector partition functions play an important role in the combinatorial theory of rational polytopes and have been considered, e.g. in [Stu95], [BV97] (see also references therein). In [BV97] closed expressions in terms of residue formulas have been obtained. Moreover it was shown that the vector partition function is a piecewise quasipolynomial function, where the domains of quasipolynomiality are chambers (or possibly unions of chambers) of the secondary fan. In particular, for if $P^{\emptyset}_{\underline{c}}$ is a rational bounded polytope, then the values of the vector partition function for $P^{\emptyset}_{k\cdot\underline{c}}$ for $k \geq 0$, is just the Ehrhart quasipolynomial. A special case which we will consider in subsection 4.1 is where the vectors $l_1, \ldots, l_n$ form circuit. In this form, the computation of generalized partition functions is essentially equivalent to the classically known diophantine Frobenius problem (also known as money change problem or denumerant problem). We refer to the book [Ram05] for a general overview.

4.1 Arithmetic cohomology vanishing for circuits

In this subsection we assume that $n = d + 1$ and $\mathcal{C} = [n]$ forms a circuit. In light of Theorem 2.1, for cohomology vanishing on a toric 1-circuit variety, we have to consider the reduced cohomology of simplicial complexes associated to its fan:

LEMMA 4.1: *Let $I \subset [n]$, such that $I \neq \mathfrak{C}^+$, then $H^i((\hat{\Delta}_{\mathfrak{C}})_I; k) = 0$ for all i. Moreover,*

$$(\hat{\Delta}_{\mathfrak{C}})_{\mathfrak{C}^+} \cong S^{|\mathfrak{C}^+|-2} \quad \textit{and} \quad (\hat{\Delta}_{\mathfrak{C}})_{\mathfrak{C}^-} \cong B^{|\mathfrak{C}^-|-1},$$

where B^k is the k-ball, with $B^{-1} := \emptyset$.

Proof. It is easy to see that $(\hat{\Delta}_{\mathfrak{C}})_{\mathfrak{C}^+}$ corresponds to the boundary of the $(|\mathfrak{C}^+| - 1)$-simplex, so it is homeomorphic to $S^{|\mathfrak{C}^+|-2}$. Similarly, $\{l_i\}_{i\in\mathfrak{C}^+}$ span a simplicial cone in $\Delta_{\mathfrak{C}}$ and thus $(\hat{\Delta}_{\mathfrak{C}})_{\mathfrak{C}^-} \cong B^{|\mathfrak{C}^-|-1}$. Now assume there exists $i \in \mathfrak{C}^+ \setminus I$, then I is a face of the cone σ_i and $(\hat{\Delta}_{\mathfrak{C}})_I$ is contractible. On the other hand, if $\mathfrak{C}^+$ is a proper subset of I, the set $I \cap \mathfrak{C}^-$ spans a cone τ in $\Delta_{\mathfrak{C}}$. The simplicial complex $\hat{\Delta}_I$ then is homeomorphic to a simplicial decomposition of the $(|\mathfrak{C}^+| - 1)$-ball with center τ and boundary $(\hat{\Delta}_{\mathfrak{C}})_{\mathfrak{C}^+}$. ☐

In this special situation, the chamber decomposition of $M_{\mathbb{Q}}$ by hyperplanes $H_i^{\underline{c}}$ as in subsection 3.2 contains at most one bounded chamber. In fact, if D is a rational divisor, all maximal chambers are unbounded. If $D \not\sim 0$, we have precisely one bounded chamber for whose signatures there are precisely two possibilities. Namely, we either have for every m in this chamber that $l_i(m) < -c_i$ for every $i \in \mathfrak{C}^-$ and $l_i(m) \geq -c_i$ for every $i \in \mathfrak{C}^+$, or vice versa. The set of rational divisor classes in $A_{d-1}\big(\mathbb{P}(\underline{\alpha}, \xi)\big)_{\mathbb{Q}} \cong \mathbb{Q}$ corresponding to torus invariant divisors whose associated bounded chamber has signature either $\mathfrak{C}^+$ or $\mathfrak{C}^-$ corresponds precisely to the two open intervals $(-\infty, 0)$ and $(0, \infty)$, respectively, in $A_{d-1}\big(\mathbb{P}(\underline{\alpha}, \xi)\big)_{\mathbb{Q}}$.

To count lattice points in the bounded chamber we can use a special case of the generalized partition function, i.e. the number of lattice points m such that $l_i(m) \geq -c_i$ for $i \in \mathfrak{C}^+$ and $l_i(m) < -c_i$ for $i \in \mathfrak{C}^-$ coincides with the cardinality of the following set:

$$\{(k_1, \dots, k_{d+1}) \in \mathbb{N}^{d+1} \mid k_i > 0 \text{ for } i \in \mathfrak{C}^- \text{ and } \sum_{i \in \mathfrak{C}^+} k_i D_i - \sum_{i \in \mathfrak{C}^-} k_i D_i = D\}.$$

For the integral case, this leads to *arithmetic thickenings* of the intervals $(-\infty, 0)$ and $(0, \infty)$ as follows:

Definition 4.2: We denote $F_{\mathfrak{C}} \subset A_{d-1}\big(\mathbb{P}(\underline{\alpha}, \xi)\big)$ the complement of the semigroup of the form $\sum_{i \in \mathfrak{C}^-} c_i D_i - \sum_{i \in \mathfrak{C}^+} c_i D_i$, where $c_i \in \mathbb{N}$ for all i with $c_i > 0$ for $i \in \mathfrak{C}^+$.

The set $F_{\mathfrak{C}}$ is the complement of the set of classes whose associated chamber has signature $\mathfrak{C}^-$ and contains a lattice point. With this we can give a complete characterization of global cohomology vanishing:

Proposition 4.3: *Let $\mathbb{P}(\underline{\alpha}, \xi)$ be as before with associated fan $\Delta_{\mathfrak{C}}$ and $D \in A_{d-1}\big(\mathbb{P}(\underline{\alpha}, \xi)\big)$, then:*

(i) $H^i\big(\mathbb{P}(\underline{\alpha}, \xi), \mathcal{O}(D)\big) = 0$ *for* $i \neq 0, |\mathfrak{C}^+| - 1$;

(ii) $H^{|\mathfrak{C}^+|-1}\big(\mathbb{P}(\underline{\alpha}, \xi), \mathcal{O}(D)\big) = 0$ *iff* $D \in F_{\mathfrak{C}}$;

(iii) if $\mathfrak{C}^+ \neq \mathcal{C}$, *then* $H^0\big(\mathbb{P}(\underline{\alpha}, \xi), \mathcal{O}(D)\big) \neq 0$;

(iv) if $\mathfrak{C}^+ = \mathcal{C}$, *then* $H^0\big(\mathbb{P}(\underline{\alpha}, \xi), \mathcal{O}(D)\big) = 0$ *iff* $D \in F_{-\mathfrak{C}}$.

Proof. The proof is immediate. Just observe that the simplicial complex $(\hat{\Delta}_{\mathfrak{C}})_m$, for m an element in the bounded chamber, coincides either with $(\hat{\Delta}_{\mathfrak{C}})_{\mathfrak{C}^+}$ or $(\hat{\Delta}_{\mathfrak{C}})_{\mathfrak{C}^-}$. $\square$

Another case of interest is where $\mathfrak{C}^+ \neq \mathcal{C}$ and $V = V(\tau)$, where τ is the cone spanned by the l_i with $i \in \mathfrak{C}^-$, i.e. V is the unique maximal complete torus invariant subvariety of $\mathbb{P}(\underline{\alpha}, \xi)$.

Proposition 4.4: *Consider $\mathbb{P}(\underline{\alpha}, \xi)$ such that $\alpha_i < 0$ for at least one i, $D \in A_{d-1}\big(\mathbb{P}(\underline{\alpha}, \xi)\big)$ and V the maximal complete torus invariant subvariety of $\mathbb{P}(\underline{\alpha}, \xi)$, then:*

(i) $H^d_V\big(\mathbb{P}(\underline{\alpha}, \xi), \mathcal{O}(D)\big) \neq 0$;

(ii) $H^{|\mathfrak{C}^-|}_V\big(\mathbb{P}(\underline{\alpha}, \xi), \mathcal{O}(D)\big) = 0$ *iff* $D \in F_{\mathfrak{C}}$;

(iii) $H^i_V\big(\mathbb{P}(\underline{\alpha}, \xi), \mathcal{O}(D)\big) = 0$ *for all* $i \neq d$, $|\mathfrak{C}^-|$.

Proof. Consider first $I = \mathcal{C}$, then $(\hat{\Delta}_{\mathfrak{C}})_I = \hat{\Delta}_{\mathfrak{C}} \cong B^{d-1}$ and $(\hat{\Delta}_{\mathfrak{C}})_{V,I} = (\hat{\Delta}_{\mathfrak{C}})_V \cong S^{d-2}$. It follows that $H^i\big(\hat{\Delta}_{\mathfrak{C}}; k\big) = 0$ for all i and $H^{i-1}\big(\hat{\Delta}_{\mathfrak{C}}, (\hat{\Delta}_{\mathfrak{C}})_V; k\big) \cong H^{i-2}\big((\hat{\Delta}_{\mathfrak{C}})_V; k\big)$. As by assumption, $\mathfrak{C}^+ \neq \mathcal{C}$, so the associated hyperplane arrangement contains an unbounded chamber such that $l_i(m) \geq -c_i$ for all $i \in \mathcal{C}$ and all m in this chamber. Hence (i) follows. As in the proof of lemma 4.1, it follows that $\hat{\Delta}_I$ is contractible whenever $\mathfrak{C}^+ \cap I \neq \emptyset$ and $\mathfrak{C}^- \cap I \neq \emptyset$. So in that case $H^i(\hat{\Delta}_I) = 0$ for all i and $H^{i-1}(\hat{\Delta}_I, \hat{\Delta}_{V,I}; k) = H^{i-2}(\hat{\Delta}_{V,I}; k)$ for all i.
Now let $I = \mathfrak{C}^+$; then $(\hat{\Delta}_{\mathfrak{C}})_I = (\hat{\Delta}\mathfrak{C})_{V,I} \cong S^{\mathfrak{C}^+ -2}$, so $H^i\big((\hat{\Delta}_{\mathfrak{C}})_I, (\hat{\Delta}\mathfrak{C})_{V,I}; k\big) = 0$ for all i. For $I = \mathfrak{C}^-$, then $(\hat{\Delta}_{\mathfrak{C}})_I \cong B^{|\mathfrak{C}^-|-1}$ and $(\hat{\Delta}_{\mathfrak{C}})_{V,I} \cong S^{|\mathfrak{C}^-|-2}$, the former by Lemma 4.1, the latter by Lemma 4.1 and the fact that $(\hat{\Delta}_{\mathfrak{C}})_{V,I}$ has empty intersection with $\mathrm{star}(\tau)$. This implies (ii) and consequently (iii). □

4.2 Arithmetic Kawamata-Viehweg vanishing

A first — trivial — approximation is given by the observation that the divisors D where the vector partition function takes a nontrivial value map to the cone C_I, shifted by the *offset* $e_I := -\sum_{i \in I} e_i$.

Definition 4.5: We denote $\mathbb{O}'(L, I)$ the saturation of the cone generated the $-D_i$ for $i \in I$ and the D_i for $i \notin I$ and $\mathbb{O}(L, I) := e_I + \mathbb{O}'(L, I)$. Moreover, we denote $\Omega(L, I)$ the zero set in $\mathbb{O}(L, I)$ of the vector partition function as defined above.

In the next step we want to approximate the sets $\Omega(L, I)$ by reducing to the classical diophantine Frobenius problem. For this, fix some $I \subset [n]$ and consider some polytope $P^I_{\underline{c}}$. It follows from Proposition 3.17 that D is contained in the intersection of half spaces $H_{\mathfrak{C}}$ for $\mathfrak{C} \in \mathfrak{C}(L)$ such that $\mathfrak{C}^- = \mathcal{C} \cap I$. In the polytope picture, we can interpret this as follows. For every $\mathfrak{C}$ and its underlying circuit $\mathcal{C}$, we set

$$P^{\mathfrak{C}}_{\underline{c}} := \{m \in M_{\mathbb{Q}} \mid l_i(m) \leq -c_i \text{ for } i \in \mathfrak{C}^- \text{ and } l_i(m) \geq -c_i \text{ for } i \in \mathfrak{C}^+\}.$$

Consequently, we get

$$P^I_{\underline{c}} = \bigcap_{\mathfrak{C}} P^{\mathfrak{C}}_{\underline{c}},$$

where the intersection runs over all $\mathfrak{C} \in \mathfrak{C}(L)$ with $\mathfrak{C}^- = \mathcal{C} \cap I$. It follows that if there exists a compatible oriented circuit $\mathfrak{C}$ such that $P^{\mathfrak{C}}_{\underline{c}}$ does not contain a lattice point, then $P^I_{\underline{c}}$ also does not contain a lattice point. We want to capture this by considering an arithmetic analogue of the discriminantal arrangement in $A_{d-1}(X)$ rather than in $A_{d-1}(X)_{\mathbb{Q}}$. Using the integral pendant to diagram (5):

DEFINITION 4.6: Consider the morphism $\eta_I : A_{d-1}(X) \twoheadrightarrow A_I$. Then we denote Z_I its kernel. For $I = \mathcal{C}$ and $\mathfrak{C}$ some orientation of $\mathcal{C}$ we denote by $F_{\mathfrak{C}}$ the preimage in $A_{d-1}(X)$ of the complement of the semigroup consisting of elements $\sum_{i \in \mathfrak{C}^-} c_i D_i - \sum_{i \in \mathfrak{C}^+} c_i D_i$, where $c_i \geq 0$ for $i \in \mathfrak{C}^-$ and $c_i > 0$ for $i \in \mathfrak{C}^+$. We set $F_{\mathcal{C}} := F_{\mathfrak{C}} \cap F_{-\mathfrak{C}}$.

So, there are two candidates for a discriminantal arrangement in $A_{d-1}(X)$, the $Z_{\mathcal{C}}$ on the one hand, and the $F_{\mathcal{C}}$ on the other.

DEFINITION 4.7: We denote:

- $\{Z_{\mathcal{C}}\}_{\mathcal{C} \in \mathcal{C}(L)}$ the *integral* discriminantal arrangement, and
- $\{F_{\mathcal{C}}\}_{\mathcal{C} \in \mathcal{C}(L)}$ the *Frobenius* discriminantal arrangement.

The integral discriminantal arrangement has similar properties as the H_I, as they give a solution to the integral discriminant problem (compare Lemma 3.9):

LEMMA 4.8: *Let $I \subset [n]$, then*

$$Z_I = \bigcap_{\mathcal{C} \in \mathcal{C}(L_I)} Z_{\mathcal{C}}.$$

We can now locate both the rational as well as the integral Picard group in $A_{d-1}(X)_{\mathbb{Q}}$ and $A_{d-1}(X)$, respectively:

THEOREM 4.9 (see [Eik92], Theorem 3.2): *Let X be any toric variety, then:*

(i) $\operatorname{Pic}(X)_{\mathbb{Q}} = \bigcap_{\sigma \in \Delta_{\max}} H_{\sigma(1)} = \bigcap_{\substack{\mathcal{C} \in \mathcal{C}(L_{\sigma(1)}) \\ \sigma \in \Delta_{\max}}} H_{\mathcal{C}}$.

(ii) $\operatorname{Pic}(X) = \bigcap_{\sigma \in \Delta_{\max}} Z_{\sigma(1)} = \bigcap_{\substack{\mathcal{C} \in \mathcal{C}(L_{\sigma(1)}) \\ \sigma \in \Delta_{\max}}} Z_{\mathcal{C}}$.

Proof. (i) As remarked in subsection 3.4, a $\mathbb{Q}$-Cartier divisor is specified by a collection $\{m_\sigma\}_{\sigma \in \Delta} \subset M_{\mathbb{Q}}$. In particular, all for every $\sigma \in \Delta$, the hyperplanes $H_i^{\underline{c}}$ with $i \in \sigma(1)$ have nonempty intersection. So the first equality follows. The second equality follows from Lemma 3.9.
(ii) A Cartier divisor is specified by a collection $\{m_\sigma\}_{\sigma \in \Delta} \subset M$ such that the hyperplanes $H_i^{\underline{\mathcal{C}}}$ with $i \in \sigma(1)$ intersect in integral points. So the first equality follows. The second equality follows from Lemma 4.8. □

The Frobenius discriminantal arrangement is not as straightforward. First, we note the following properties:

LEMMA 4.10: *Let $\mathcal{C} \in \mathcal{C}(L)$, then:*

(i) *$F_{\mathcal{C}}$ is nonempty;*

(ii) *the saturation of $Z_{\mathcal{C}}$ in $A_{d-1}(X)$ is contained in $F_{\mathcal{C}}$ iff $\mathcal{C}$ is not fibrational.*

Proof. The first assertion follows because $F_{\mathcal{C}}$ contains all elements which map to the open interval $(K_{\mathfrak{C}}, K_{-\mathfrak{C}})$ in $A_{\mathcal{C},\mathbb{Q}}$, where $K_{\mathfrak{C}} = -\sum_{i \in \mathfrak{C}^+} D_i$. For the second assertion, note that the set $\{m \in M \mid l_i(m) = 0 \text{ for all } i \in \mathcal{C}\}$ is in $F_{\mathcal{C}}$ iff $\mathfrak{C}^+ \neq \mathcal{C}$ for either orientation $\mathfrak{C}$ of $\mathcal{C}$. □

Lemma 4.10 shows that the $F_{\mathcal{C}}$ are thickenings of the $Z_{\mathcal{C}}$ with one notable exception, where $\mathcal{C}$ is fibrational. In this case, $F_{\mathcal{C}}$ can be considered as parallel to, but slightly shifted away from $Z_{\mathcal{C}}$. In the sequel we will not make any explicit use of the $Z_{\mathcal{C}}$ anymore, but these facts should be kept in mind.
Regarding the Frobenius discriminantal arrangement, we want also to consider integral versions of the discriminantal strata:

Definition 4.11: Let $\mathfrak{C} \in \mathfrak{C}(L)$ and let $\mathfrak{F}_{\mathcal{S}}$ be a discriminantal hull of $\mathcal{S} = \{S_1, \ldots, S_k\}$, then we denote

$$\mathfrak{A}_{\mathcal{S}} := \bigcap_{\mathfrak{C} \in \mathfrak{F}_{\mathcal{S}}} F_{\mathfrak{C}}.$$

the *arithmetic core* of $\mathfrak{F}_{\mathcal{S}}$. In the special case $\mathfrak{F}_{\mathcal{S}} = \mathfrak{F}_{\mathrm{nef}}$ we write $\mathfrak{A}_{\mathrm{nef}}$.

Remark 4.12: The notion *core* refers to the fact that we consider *all* $F_{\mathfrak{C}}$, instead of a non-redundant subset describing the set S as a convex cone.

We will use arithmetic cores to derive arithmetic versions of known vanishing theorems formulated in the setting of $\mathbb{Q}$-divisors and to get refined conditions on cohomology vanishing. This principle is reflected in the following theorem:

Theorem 4.13: *Let V be a T-invariant closed subscheme of X and S a discriminantal stratum in $A_{d-1}(X)_{\mathbb{Q}}$. If $H^i_V\big(X, \mathcal{O}(D)\big) = 0$ for some i and for all integral divisors $D \in S$, then also $H^i_V\big(X, \mathcal{O}(D)\big) = 0$ for all $D \in \mathfrak{A}_S$.*

Proof. Without loss of generality we can assume that $\dim S > 0$. Consider some nonempty $P^I_{\underline{c}}$ for some $I \subset [n]$. Then for any such I, we can choose some multiple of kD such that $P^I_{k\underline{c}}$ contains a lattice point. But if $H^i_V\big(X, \mathcal{O}(D)\big) = 0$, then also $H^i_V\big(X, \mathcal{O}(kD)\big) = 0$, hence $H^{i-1}(\hat{\Delta}_I, \hat{\Delta}_{V,I}; k) = 0$. Now, any divisor $D' \in \mathfrak{A}_S$ which does not map to S, is contained in $F_{\mathfrak{C}}$ for all $\mathfrak{C} \in \mathfrak{F}_S$ and therefore for any I which is in the signature for D' but not for D, the equations $l_i(m) < -c'_i$ for $i \in I$ and $l_i(m) \geq -c'_i$ for $i \notin I$ cannot have any integral solution. □

We apply Theorem 4.13 to $\mathfrak{A}_{\mathrm{nef}}$:

Theorem 4.14 (Arithmetic version of Kawamata-Viehweg vanishing): *Let X be a complete toric variety. Then $H^i\big(X, \mathcal{O}(D)\big) = 0$ for all $i > 0$ and all $D \in \mathfrak{A}_{\mathrm{nef}}$.*

Proof. We know that the assertion is true if D is nef. Therefore we can apply Theorem 4.13 to the maximal strata $S_1, \ldots, S_k$ of $\mathrm{nef}(X)$. Therefore the assertion is true for $D \in \bigcap_{i=1^k} \mathfrak{A}_{S_i}$. To prove the theorem, we have to get rid of the $F_{\mathfrak{C}}$, where $H_{\mathcal{C}}$ intersects the relative interior of a face of $\mathrm{nef}(X)$. Let $\mathcal{C}$ be such a circuit and R the face. Without loss of generality, $\dim R > 0$. Then we can choose elements D' in R at an arbitrary distance from $H_{\mathfrak{C}}$, i.e. such that the polytope $P_{\underline{c}}^{\mathfrak{C}}$ becomes arbitrarily big and finally contains a lattice point. Now, if we move outside $\mathrm{nef}(X)$, but stay inside $\mathfrak{A}_{\mathrm{nef}}$, the lattice points of $P_{\underline{c}}^{\mathfrak{C}}$ cannot acquire any cohomology and the assertion follows. ☐

One can imagine an analog of the set $\mathfrak{A}_S$ in $A_{d-1}(X)_{\mathbb{Q}}$ as the intersection of shifted half spaces

$$\bigcap_{\mathfrak{C} \in \mathfrak{F}_S} \big(- \sum_{i \in \mathfrak{C}^+} D_i + H_{\mathfrak{C}} \big).$$

The main difference here is that one would picture the proper facets of this convex polyhedral set as "smooth", whereas the proper "walls" of $\mathfrak{A}_S$ have "ripples", which arise both from the fact that the groups $A_{\mathcal{C}}$ may have torsion, and that we use Frobenius conditions to determine the augmentations of our half spaces.

In general, the set $\mathfrak{F}_S$ is highly redundant when it comes to determine $\overline{S}$, which implies that above intersection does not yield a cone but rather a polyhedron, whose recession cone corresponds to $\overline{S}$. In the integral situation we do not quite have a recession cone, but a similar property holds:

PROPOSITION 4.15: *Let $V \subset X$ be a closed invariant subscheme and $\mathcal{S} = \{S_1, \ldots, S_k\}$ a collection of discriminantal stata different from zero. Then for any nonzero face of its discriminantal hull $\overline{S}$ there exists the class of an integral divisor $D' \in \overline{S}$ such that the intersection of the half line $D + rD'$ for $0 \leq r \in \mathbb{Q}$ with $\mathfrak{A}_{\mathcal{S}}$ contains infinitely many classes of integral divisors.*

Proof. Let $R \subset \overline{S}$ be any face of S, then the vector space spanned by R is given by an intersection $\bigcap_{\mathcal{C} \text{ with } \mathfrak{C} \in K} H_{\mathcal{C}}$ for a certain subset $K \subset \mathfrak{F}_{\mathcal{S}}$. We assume that K is maximal with this property. The intersection $\bigcap_{\mathfrak{C} \in K} F_{\mathfrak{C}}$ is invariant with respect to translations along certain (though not necessarily all) $D' \in R$. This implies that the line (or any half line, respectively), generated by D' intersects $\bigcap_{\mathfrak{C} \in K} F_{\mathfrak{C}}$ in infinitely many points. As K is maximal, there is no other $\mathfrak{C} \in \mathfrak{F}_{\mathcal{C}}$ parallel to R and the assertion follows. ☐

The property of Proposition 4.15 is necessary for elements in $\mathfrak{A}_{\mathcal{S}}$, but not sufficient. This leads to the following definition:

DEFINITION 4.16: Let $\mathcal{S} = \{S_1, \ldots, S_k\}$ be a collection of discriminantal strata and $D \in A_{d-1}(X)$ such that the property of Proposition 4.15 holds. If D is not contained in $\mathfrak{A}_{\mathcal{S}}$, then we call D $\mathfrak{A}_{\mathcal{S}}$-residual. We call D 0-*residual* if it is in the complement of $\mathfrak{A}_0 = \bigcap_{\mathfrak{C} \in \mathfrak{C}(L)} F_{\mathfrak{C}}$.

In the next subsections we will consider several special cases of interest for cohomology vanishing, which are not directly related to Kawamata-Viehweg vanishing theorems. In subsection 4.3 we will consider global cohomology vanishing for divisors in the inverse nef cone. In subsection 4.4 we will present a more explicit determination of this type of cohomology vanishing for toric surfaces. Finally, in subsection 4.5, we will give a geometric criterion for determing maximally Cohen-Macaulay modules.

4.3 Nonstandard Cohomology Vanishing

In this subsection we want to give a qualitative description of cohomology vanishing which is related to divisors which are *inverse* to nef divisors of Iitaka dimension $0 < \kappa(D) < d$. We show the following theorem:

Theorem 4.17: *Let X be a complete d-dimensional toric variety. Then $H^i\big(X, \mathcal{O}(D)\big) = 0$ for every i and all D which are contained in some $\mathfrak{A}_{-F}$, where F is a face of* $\operatorname{nef}(X)$ *which contains nef divisors of Iitaka dimension $0 < \kappa(D) < d$. If $\mathfrak{A}_{-F}$ is nonempty, then it contains infinitely many divisor classes.*

Proof. Recall that such a divisor, as a $\mathbb{Q}$-divisor, is contained in the intersection $\bigcap_{\mathcal{C} \in \operatorname{fib}(D)} H_{\mathcal{C}}$ and therefore it is in the intersection of the nef cone with the boundary of the effective cone of X by Proposition 3.17. Denote this intersection by F. Then we claim that $H^i\big(X, \mathcal{O}(D')\big) = 0$ for all $D' \in \mathfrak{A}_{-F}$. By Corollary 3.28 we know that $H^i\big(X, \mathcal{O}(E)\big) = 0$ for $0 \leq i < d$ for any divisor E in the interior of the inverse nef cone. This implies that $H^i\big(X, \mathcal{O}(E)\big) = 0$ for any $E \in \mathfrak{A}_{-nef}$ and hence $H^i\big(X, \mathcal{O}(D')\big) = 0$ for any $D' \in \mathfrak{A}_{-F}$, because $\mathfrak{A}_{-F} \subset \mathfrak{A}_{-\mathrm{nef}}$. The latter assertion follows from the fact that the assumption on the Iitaka dimension implies that the face F has positive dimension. $\square$

Note that criterion is not very strong, as it is not clear in general whether the set $\mathfrak{A}_{-F}$ is nonempty. However, this is the case in a few interesting cases, in particular for toric surfaces, as we will see in the next subsection. The following remark shows that our condition indeed is rather weak in general:

Remark 4.18: The inverse of any big and nef divisor D with the property that P_D does not contain any lattice point in its interior has the property that $H^i\big(X, \mathcal{O}(D)\big) = 0$ for all i. This follows directly from the standard fact in toric geometry that the Euler characteristics $\chi(-D)$ counts the inner lattice points of the lattice polytope P_D.

4.4 The case of complete toric surfaces.

Let X be a complete toric surface. We assume that the l_i are cyclically ordered. We consider the integers $[n]$ as system of representatives for $\mathbb{Z}/n\mathbb{Z}$, i.e. for some $i \in [n]$ and $k \in \mathbb{Z}$, the sum $i + k$ denotes the unique element in $[n]$ modulo n.

Proposition 4.19: *Let X be a complete toric surface. Then* $\mathrm{nef}(X) = \overline{S}$, *where S is a single stratum of maximal dimension of the discriminantal arrangement.*

Proof. X is simplicial and projective and therefore $\mathrm{nef}(X)$ is a cone of maximal dimension in $A_1(X)_{\mathbb{Q}}$. We show that no hyperplane $H_{\mathcal{C}}$ intersects the interior of $\mathrm{nef}(X)$. By Proposition 3.17 we can at once exclude fibrational circuits. This leaves us with non-fibrational circuits $\mathcal{C}$ with cardinality three, having orientation $\mathfrak{C}$ with $|\mathfrak{C}^+| = 2$. Assume that D is contained in the interior of $H_{-\mathfrak{C}}$. Then there exists $m \in M_{\mathbb{Q}}$ such that $\mathfrak{C}^+ \subset \mathfrak{s}^D(m)$, which implies that the hyperplane $H_i^{\underline{c}}$ for $\{i\} = \mathfrak{C}^-$ does not intersect P_D, and thus D cannot be nef. It follows that $\mathrm{nef}(X) \subset H_{\mathfrak{C}}$. □

Now assume there exist $p, q \in [n]$ such that $l_q = -l_p$, i.e. l_p and l_q represent a one-dimensional fibrational circuit of L. Then for any nef divisors D which is contained in $H_{p,q}$, the associated polytope P_D is one-dimensional. The only possible variation for P_D is its length in terms of lattice units. So we can conclude that $\mathrm{nef}(X) \cap H_{p,q}$ is a one-dimensional face of $\mathrm{nef}(X)$.

Definition 4.20: Let X be a complete toric surface and $\mathcal{C} = \{p, q\}$ such that $l_p = -l_q$. Then we denote $S_{p,q}$ the relative interior of $-\mathrm{nef}(X) \cap H_{\mathcal{C}}$. Moreover, we denote $\mathfrak{A}_{p,q}$ the arithmetic core of $S_{p,q}$.

Our aim in this subsection is to prove the following:

Theorem 4.21: *Let X be a complete toric surface. Then there are only finitely many divisors D with $H^i\big(X, \mathcal{O}(D)\big) = 0$ for all $i > 0$ which are not contained in $\mathfrak{A}_{\mathrm{nef}} \cup \bigcup \mathfrak{A}_{p,q}$, where the union ranges over all pairs $\{p, q\}$ such that $l_p = -l_q$.*

We will prove this theorem in several steps. First we show that the interiors of the C_I such that $H^0(\hat{\Delta}_I; k) \neq 0$ cover all of $A_1(X)_{\mathbb{Q}}$ except $\mathrm{nef}(X)$ and $-\mathrm{nef}(X)$.

Proposition 4.22: *Let $D = \sum_{i \in [n]} c_i D_i$ be a Weil divisor which is not contained in* $\mathrm{nef}(X)$ *or* $-\mathrm{nef}(X)$, *then the corresponding arrangement $H_i^{\underline{c}}$ in $M_{\mathbb{Q}}$ has a two-dimensional chamber $P_{\underline{c}}^I$ such that complex $\hat{\Delta}_I$ has at least two components.*

Proof. Recall that $\mathrm{nef}(X) = \bigcap H_{\mathfrak{C}}$, where the intersection runs over all oriented circuits which are associated to extremal curves of X. As the statement is well-known for the case where X is either a 1-circuit toric variety or a Hirzebruch surface, we can assume without loss of generality, that the extremal curves belong to blow-downs, i.e. the associated oriented circuits are of the form $\mathfrak{C}^+ = \{i-1, i+1\}$, $\mathfrak{C}^- = \{i\}$ for any $i \in [n]$. Now assume that D is in the interior of $H_{\mathfrak{C}}$ for such an oriented circuit $\mathfrak{C}$. Then there exists a bounded chamber $P_{\underline{c}}^I$ in $M_{\mathbb{Q}}$ such that $\mathfrak{C}^- = \mathcal{C} \cap \mathfrak{s}^{\underline{c}}(m)$. In order for $\hat{\Delta}_{\mathfrak{s}^{\underline{c}}(m)}$ to be acyclic, it is necessary that $\mathfrak{s}^{\underline{c}}(m) \cap ([n] \setminus \mathcal{C}) = \emptyset$. Let $\{j, k, l\} =: \mathcal{D} \subset [n]$

represent any other circuit such that $\mathfrak{D}^+ = \{j, l\}$ for some orientation $\mathfrak{D}$ of $\mathcal{D}$. The hyperplane arrangement given by the three hyperplanes $H_j^{\underline{c}}, H_k^{\underline{c}}, H_l^{\underline{c}}$ has six unbounded regions, whose signatures contain any subset of $\{j, k, l\}$ except $\{j, l\}$ and $\{k\}$. In the cases $j = i-2, k = i-1, l = 1$ or $j = i, k = i+1, l = i+2$, $P_{\underline{c}}^I$ must be contained in the region with signature $\{i\}$. In every other case $P_{\underline{c}}^I$ must be contained in the region with signature $\emptyset$. In the case, say, $\{j, k, l\} = \{i-2, i-1, i\}$, the hyperplane $H_{i-2}^{\underline{c}}$ should not cross the bounded chamber related to the subarrangement given by the hyperplanes $H_{i-1}^{\underline{c}}$, $H_i^{\underline{c}}$, $H_{i+1}^{\underline{c}}$, as else we obtain a chamber whose signature contains $\{i-1, i+1\}$, but not $\{i-2, i\}$. Then the associated subcomplex of $\hat{\Delta}$ can never be acyclic. This implies that, if D is in the interior of $H_{\mathfrak{C}}$, then $D \in H_{\mathfrak{D}}$, where either $\mathcal{D} = \{i-2, i-1, i\}$ or $\mathcal{D} = \{i, i+1, i+2\}$. By iterating for every extremal (i.e. every invariant) curve, we conclude that $D \in \bigcap_{i \in [n]} H_{\mathfrak{C}} = \operatorname{nef}(X)$. Analogously, we conclude for $D \in H_{-\mathfrak{C}}$ that $D \in -\operatorname{nef}(X)$, and the statement follows. □

Let $\{p, q\} \subset [n]$ such that $l_p = -l_q$. Then these two primitive vectors span a 1-dimensional subvector space of $N_{\mathbb{Q}}$, which naturally separates the set $[n] \setminus \{p, q\}$ into two subsets.

Definition 4.23: Let $\{p, q\} \subset [n]$ such that $l_p = -l_q$. Then we denote $A_{p,q}^1, A_{p,q}^2 \subset [n]$ the two subsets of $[n] \setminus \{p, q\}$ separated by the line spanned by l_p, l_q.

For some fibrational circuit $\{p, q\}$, the closure $\overline{S}_{p,q}$ is a one-dimensional cone in $A_1(X)_{\mathbb{Q}}$ which has a unique primitive vector:

Definition 4.24: Consider$\{p, q\}$ as before. Then the closure $\overline{S}_{p,q}$ is a one-dimensional cone with primitive lattice vector $D_{p,q} := \sum_{i \in A_{p,q}^1} l_i(m) D_i$, where $m \in M$ the unique primitive vector on the ray in $M_{\mathbb{Q}}$ with $l_p(m) = l_q(m) = 0$ and $l_i(m) < 0$ for $i \in A_{p,q}^1$.

Proposition 4.25: *Let X be a complete toric surface. Then every $\mathfrak{A}_{p,q}$-residual divisor on X is either contained in $\mathfrak{A}_{\text{nef}}$, or in some $\mathfrak{A}_{p,q}$, or is $\mathfrak{A}_{\text{nef}}$-residual.*

Proof. For any nef divisor $D \in -S_{p,q}$, the polytope P_D is a line segment such that all $H_i^{\underline{c}}$ intersect this line segment in one of its two end points, depending on whether $i \in A_{p,q}^1$ or $i \in A_{p,q}^2$. This implies that the line spanned by $S_{p,q}$ is the intersection of all $H_{\mathcal{C}}$, where $\mathcal{C} \subset A_{p,q}^1 \cup \{p, q\}$ or $\mathcal{C} \subset A_{p,q}^2 \cup \{p, q\}$. Let D be $\mathfrak{A}_{p,q}$-residual and assume that $H^i\big(X, \mathcal{O}(D + rD_{p,q})\big) = 0$ for all i and for infinitely many r. We first show that $D \in F_{\{p,q\}}$, i.e. that $c_p + c_q = -1$ for any torus invariant representative $D = \sum_{i \in [n]} c_i D_i$. Assume that $c_p + c_q > -1$. Then there exists $m \in M$ such that $p, q \notin \mathfrak{s}^{\underline{c}}(m)$. By adding sufficiently high multiples of $D_{p,q}$ such that $D + rD_{p,q} = \sum c_i' D_i$, we can even find such an m such that $A_1 \cup A_2 \subset \mathfrak{s}^{\underline{c}'}(m)$, hence $H^1\big(X, \mathcal{O}(D + rD_{p,q})\big) \neq 0$ for large r and thus D is not $\mathfrak{A}_{p,q}$-residual. If $c_p + c_q < -1$, there is an $m \in M$ with

$\{p,q\} \subset \mathfrak{s}^{\underline{c}}(m)$, and by the same argument, we get $H^2\big(X, \mathcal{O}(D + rD_{p,q})\big) \neq 0$ for large r. Hence $c_p + c_q = -1$, i.e. $D \in F_{\{p,q\}}$. This implies that for every $m \in M$ either $p \in \mathfrak{s}^{\underline{c}}(m)$ and $q \notin \mathfrak{s}^{\underline{c}}(m)$, or $q \in \mathfrak{s}^{\underline{c}}(m)$ and $p \notin \mathfrak{s}^{\underline{c}}(m)$. Now assume that $D \notin \mathcal{F}_{\mathcal{C}}$ for some $\mathcal{C} = \{i,j,k\} \subset A_1 \cup \{p,q\}$ such that $\mathfrak{C}^+ = \{i,k\}$ for some orientation. Then there exists some $m \in M$ with $\{i,k\} \subset \mathfrak{s}^{\underline{c}}(m)$ or $\{j\} \subset \mathfrak{s}^{\underline{c}}(m)$. In the first case, as before we can simply add some multiple of $D_{p,q}$ such that $i \in \mathfrak{s}^{\underline{c}'}(m)$ and $i \in A_2$, hence $\mathfrak{s}^{\underline{c}'}(m)$ contains at least two $-$-intervals. In the second case, we have either $p \notin \mathfrak{s}^{\underline{c}}(m)$ or $q \notin \mathfrak{s}(m)$, thus at least two $-$-intervals, too. Hence $D \in \mathfrak{A}_{p,q}$ and the assertion follows. $\square$

Proposition 4.26: *Let X be a complete toric surface. Then X has only a finite number of $\mathfrak{A}_{\mathrm{nef}}$-residual divisors.*

Proof. We can assume without loss of generality that X is not $\mathbb{P}^2$ nor a Hirzebruch surface. Assume there is $D \in A_1(X)$ which is not contained in $F_{\mathfrak{C}}$ for some circuit $\mathcal{C} = \{i-1, i, i+1\}$ corresponding to an extremal curve on X. Then there exists a chamber in the corresponding arrangement whose signature contains $\{i-1, i+1\}$. To have this signature to correspond to an acyclic subcomplex of $\hat{\Delta}$, the rest of the signature must contain $[n] \setminus \mathcal{C}$. Now assume we have some integral vector $D_{\mathcal{C}} \in H_{\mathcal{C}}$, then we can add a multiple of $D_{\mathcal{C}}$ to D such that D is parallel translated to $\mathrm{nef}(X)$. In this process necessarily at least one hyperplane passes the critical chamber and thus creates cohomology. Now, D might be outside of $F_{\mathfrak{D}}$ for some $\mathfrak{D} \in \mathfrak{C}(L)$ not corresponding to an extremal curve. If the underlying circuit is not fibrational, then D being outside $F_{\mathfrak{D}}$ implies $F_{\mathfrak{C}}$ for some extremal circuit $\mathfrak{C}$. If $\mathfrak{D}$ is fibrational and $\mathcal{D} = \{p,q\}$, then we argue as in Proposition 4.25 that D has cohomology. If $\mathcal{D}$ is fibrational of cardinality three, the corresponding hypersurface $H_{\mathcal{D}}$ is not parallel to any nonzero face of $\mathrm{nef}(X)$ and there might be a finite number of divisors lying outside $F_{\mathfrak{D}}$ but in the intersection of all $F_{\mathfrak{C}}$, where $\mathcal{C}$ corresponds to an extremal curve. $\square$

Proposition 4.27: *Let X be a complete toric surface. Then X has only a finite number of 0-residual divisors.*

Proof. Let us consider some vector partition function $\mathrm{VP}(L,I) : \mathbb{O}_I \longrightarrow \mathbb{N}$, for I such that C_I does not contain a nonzero subvector space. Let $D = \sum_{i \in [n]} c_i D_i \in \Omega(L,I)$ and let P_D the polytope in $M_{\mathbb{Q}}$ such that $m \in M_{\mathbb{Q}}$ is in P_D iff $l_i(m) < -c_i$ for $i \in I$ and $l_i(m) \geq -c_i$ for $i \in [n] \setminus I$. For any $J \subset [n]$ we denote $P_{D,J}$ the polytope defined by the same inequalities, but only for $i \in J$. Clearly, $P_D \subset P_{D,J}$. Let $J \subset [n]$ be maximal with respect to the property that $P_{D,J}$ does not contain any lattice points. If $J \neq [n]$, then we can freely move the hyperplanes given by $l_i(m) = -c_i$ for $i \in [n] \setminus I$ such that $P_{D,J}$ remains constant and thus lattice point free. This is equivalent to say that there exists a nonzero $D' \in \bigcap_{\mathcal{C} \in \mathcal{C}(L_J)} H_{\mathcal{C}}$ and for every such D' the polytope $P_{D+jD'}$ does not contain any lattice point for any $j \in \mathbb{Q}_{>0}$.

Now assume that $J = [n]$. This implies that the defining inequalities of P_D are irredundant and thus there exists a unique maximal chamber in C_I which contains D (if $I = \emptyset$ this would be the nef cone by 4.19) and thus the combinatorial type of P_D is fixed. Now, clearly, the number of polygons of shape P_D with parallel faces given by integral linear inequalities and which do not contain a lattice point is finite.
By applying this to all (and in fact finitely many) cones $\mathbb{O}_I$ such that C_I does not contain a nontrivial subvector space of $A_{\mathbb{Q}}$, we see that there are only finitely many divisors D which are not contained in $\mathfrak{A}_{\text{nef}}$ or $\mathfrak{A}_{p,q}$. $\square$

Proof of theorem 4.21. By 4.22, $\text{nef}(X)$ and the $S_{p,q}$ are indeed the only relevant strata, which by 4.25 and 4.26 admit only finitely many residual elements. Hence, we are left with the 0-residuals, of which exist only finitely many by 4.27. $\square$

Example 4.28: Figure 1 shows the cohomology free divisors on the Hirzebruch surface $\mathbb{F}_3$ which is given by four rays, say $l_1 = (1,0)$, $l_2 = (0,1)$, $l_3 = (-1,3)$, $l_4 = (0,-1)$ with respect to some choice of coordinates for N. In $\text{Pic}(\mathbb{F}_3) \cong \mathbb{Z}^2$ there are two cones such that $H^1\big(X, \mathcal{O}(D)\big) \neq 0$ for every D which is contained in one of these cones. Moreover, there is one cone such that $H^2\big(X, \mathcal{O}(D)\big) \neq 0$ for every D; its tip is sitting at $K_{\mathbb{F}_3}$. The nef cone is indicated by the dashed lines.

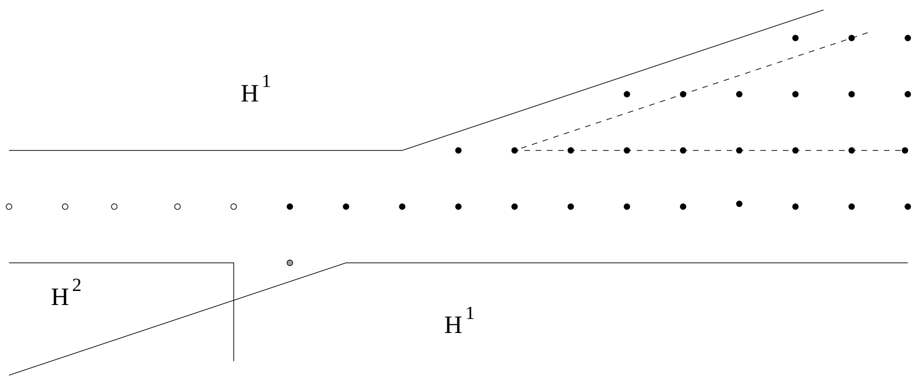

Figure 1: Cohomology free divisors on $\mathbb{F}_3$.

The picture shows the divisors contained in $\mathfrak{A}_{\text{nef}}$ as black dots. The white dots indicate the divisors in $\mathfrak{A}_{2,4}$. There is one 0-residual divisor indicated by the grey dot.
The classification of smooth complete toric surfaces implies that every such surface which is not $\mathbb{P}^2$, has a fibrational circuit of rank one. Thus the theorem implies that on every such surface there exist families of line bundles with vanishing cohomology along the inverse nef cone. For a given toric surface X,

these families can be explicitly computed by checking for every $\mathcal{C} \subset A_1 \cup \{p, q\}$ and every $\mathcal{C} \subset A_2 \cup \{p, q\}$, respectively, whether the inequalities

$$c_i + l_i(m) \begin{cases} \geq 0 & \text{for } i \in \mathfrak{C}^+ \\ < 0 & \text{for } i \in \mathfrak{C}^-, \end{cases} \qquad c_i + l_i(m) \begin{cases} \geq 0 & \text{for } i \in -\mathfrak{C}^+ \\ < 0 & \text{for } i \in -\mathfrak{C}^- \end{cases}$$

have solutions $m \in M$ for at least one of the two orientations $\mathfrak{C}$, $-\mathfrak{C}$ of $\mathcal{C}$. This requires to deal with $\binom{|A_1|+2}{3} + \binom{|A_2|+2}{3}$, i.e. of order $\sim n^3$, linear inequalities. We can reduce this number to order $\sim n^2$ as a corollary from our considerations above:

Corollary 4.29: *Let $\mathcal{C} \in A_i$ for $i = 1$ or $i = 2$. Then there exist $\{i, j\} \subset \mathcal{C}$ such that $F_{\{p,q\}} \cap F_{\mathfrak{C}} \supset F_{\{p,q\}} \cap F_{\{i,j,p\}} \cap F_{\{i,j,q\}}$.*

Proof. Assume first that there exists $m \in M$ which for the orientation $\mathfrak{C}$ of $\mathcal{C} = \{i_1, i_2, i_3\}$ with $\mathfrak{C}^+ = \{i_1, i_3\}$ which fulfills the inequalities $l_{i_k}(m) + c_{i_k} \geq 0$ for $k = 1, 3$ and $l_{i_2}(m) + c_{i_2} < 0$. This implies that $H^1\big(X, \mathcal{O}(D)\big) \neq 0$, independent of the configuration of the other hyperplanes, as long as $c_p + c_q = -1$. It is easy to see that we can choose $i, j \in \mathcal{C}$ such that $\{i, j, p\}$ and $\{i, j, q\}$ form circuits. We can choose one of those such that m is contained in the triangle, fulfilling the respective inequalities, and which is not fibrational. For the inverse orientation $-\mathfrak{C}$, we can the same way replace one of the elements of $\mathcal{C}$ by one of p, q. By adding a suitable positive multiple of $D_{p,q}$, we can rearrange the hyperplanes such that $H^1\big(X, \mathcal{O}(D + rD_{p,q})\big) \neq 0$. □

One should read the corollary the way that for any pair i, j in A_1 or in A_2, one has only to check whether a given divisor fulfills certain inequalities for triples $\{i, j, q\}$ and $\{i, j, p\}$. It seems that it is not possible to reduce further the number of equations in general. However, there is a criterion which gives a good reduction of cases for practical purposes:

Corollary 4.30: *Let X be a smooth and complete toric surface and $D = \sum_{i \in [n]} c_i D_i \in \mathfrak{A}_{p,q}$, then for every $i \in A_1 \cup A_2$, we have:*

$$c_{i-1} + c_{i+1} - a_i c_i \in [-1, a_i - 1],$$

where the a_i are the self-intersection numbers of the D_i.

Proof. The circuit $\mathcal{C} = \{i-1, i, i+1\}$ comes with the integral relation $l_{i-1} + l_{i+1} + a_i l_i = 0$. So the Frobenius problem for these circuits is trivial and we have only to consider the offset part. □

The following example shows that these equalities are necessary, but not sufficient in general:

Example 4.31: We choose some coordinates on $N \cong \mathbb{Z}^2$ and consider the complete toric surface defined by 8 rays $l_1 = (0, -1)$, $l_2 = (1, -2)$, $l_3 = (1, -1)$,

$l_4 = (1, 0)$, $l_5 = (1, 1)$, $l_6 = (1, 2)$, $l_7 = (0, 1)$, $l_8 = (-1, 0)$. Then any divisor $D = c_1 D_1 + \cdots + c_8 D_8$ with $\underline{c} = (-1, 1, 1, 0, 0, 1, 0, -k)$ for some $k \gg 0$ has nontrivial H^1, though it fulfills the conditions of corollary 4.30.

An interesting and more restricting case is the additional requirement that also $H^i\big(X, \mathcal{O}(-D)\big) = 0$ for all $i > 0$. One may compare the following with the classification of bundles of type B in [HP06].

Corollary 4.32: *Let X be a smooth and complete toric surface and $D \in \mathfrak{A}_{p,q}$ such that $H^i\big(X, \mathcal{O}(D)\big) = H^i\big(X, \mathcal{O}(-D)\big) = 0$ for all $i > 0$. Then for every $i \in A_1 \cup A_2$, we have:*

$$c_{i-1} + c_{i+1} - a_i c_i \in \begin{cases} \{\pm 1, 0\} & \text{if } a_i < -1 \\ \{-1, 0\} & \text{if } a_i = -1, \end{cases}$$

where the a_i are the self-intersection numbers of the D_i.

Proof. For $-D$, we have $c_p + c_q = 1$. Assume that there exists a circuit circuit $\mathcal{C}$ with orientation $\mathfrak{C}$ and $\mathfrak{C}^+ = \{i, j\}$ and $\mathfrak{C}^- = \{k\}$, and morover, some lattice point m such that $\mathfrak{s}^{\underline{c}}(m) \cap \mathcal{C} = \mathfrak{C}^-$. Then we get $\mathfrak{s}^{-\underline{c}}(-m) \cap \mathcal{C} = \mathfrak{C}^+$. this implies that $H^1\big(X, \mathcal{O}(-D)\big) \neq 0$. This implies the restriction $c_{i-1} + c_{i+1} - a_i c_i \in [-1, \min\{1, a_i - 1\}]$. $\square$

Note that example 4.31 also fulfills these more restrictive conditions.

4.5 Maximal Cohen-Macaulay Modules of Rank One

The classification of maximal Cohen-Macaulay modules can sometimes be related to resolution of singularities, the most famous example for this being the McKay correspondence in the case of certain surface singularities ([GSV83], [AV85], see also [EK85]). In the toric case, in general one cannot expect to arrive at such a nice picture, as there does not exist a canonical way to construct resolutions. However, there is a natural set of preferred partial resolutions, which is parameterized by the secondary fan.
Let X be a d-dimensional affine toric variety whose associated convex polyhedral cone σ has dimension d. Denote $x \in X$ torus fixed point. For any Weil divisor D on X, the sheaf $\mathcal{O}_X(D)$ is MCM if and only if $H^i_x\big(X, \mathcal{O}_X(D)\big)$ for all $i < d$. It was shown in [BG03] (see also [BG02]) that there exists only a finite number of such modules.
Now let $\tilde{X}$ be a toric variety given by some triangulation of σ. The natural map $\pi : \tilde{X} \longrightarrow X$ is a partial resolution of the singularities of X which is an isomorphism in codimension two and has at most quotient singularities. In particular, the map of fans is induced by the identity on N and, in turn, induces a bijection on the set of torus invariant Weil divisors. This bijection induces a natural isomorphism $\pi^{-1} : A_{d-1}(X) \longrightarrow A_{d-1}(\tilde{X})$ which can be represented by the identity morphism on the invariant divisor group $\mathbb{Z}^n$. This allows us to

identify a torus invariant divisor D on X with its strict transform $\pi^{-1}D$ on $\tilde{X}$. Moreover, there are the natural isomorphisms

$$\pi_* \mathcal{O}_{\tilde{X}}(\pi^{-1}D) \cong \mathcal{O}_X(D) \quad \text{ and } \quad \mathcal{O}_{\tilde{X}}(\pi^{-1}D) \cong \big(\pi^* \mathcal{O}_X(D)\big)^{\check{}\check{}}.$$

Our aim is to compare local cohomology and global cohomology, i.e. $H^i_x\big(X, \mathcal{O}_X(D)\big)$ and $H^i\big(\tilde{X}, \mathcal{O}_{\tilde{X}}(D)\big)$.
Probably the easiest class of cones σ which one can consider is where the primitive vectors $l_1, \dots, l_n$ form a circuit $\mathcal{C} = [n]$. Associated to this data are two small resolutions of singularities $\pi : \mathbb{P}(\underline{\alpha}, \xi) \longrightarrow X$ and $\pi' : \mathbb{P}(-\underline{\alpha}, \xi)) \longrightarrow X$ which are induced by triangulations $\Delta_{\mathfrak{C}}$ and $\Delta_{-\mathfrak{C}}$, respectively.
Now, the question whether $\mathcal{O}(D)$ is a maximal Cohen-Macaulay sheaf can be decided directly on Y or, equivalently, on the resolutions:

Theorem 4.33: *Let X be an affine toric variety whose associated cone σ is spanned by a circuit $\mathcal{C}$ and denote $\mathbb{P}(\underline{\alpha}, \xi)$ and $\mathbb{P}(-\underline{\alpha}, \xi)$ the two canonical small toric resolution of singularities. Then the sheaf $\mathcal{O}(D)$ is maximal Cohen-Macaulay if and only if $R^i\pi_*\mathcal{O}(\pi^{-1}D) = R^i\pi'_*\mathcal{O}((\pi')^{-1}D) = 0$ for all $i > 0$.*

Proof. This toric variety corresponds to the toric subvariety of Y which is the complement of its unique fixed point, which we denote y. We have to show that $H^i_y\big(Y, \mathcal{O}(D)\big) = 0$ for all $i < d$. By Corollary 2.2, we have

$$H^i_y\big(Y, \mathcal{O}(D)\big)_m = H^{i-2}(\hat{\sigma}_{y,m}; k)$$

for every $m \in M$, where $\hat{\sigma}_y$ denotes the simplicial model for the fan associated to $Y \setminus \{y\}$. Denote τ and τ' the cones corresponding to the minimal orbits of $\mathbb{P}(\underline{\alpha}, \xi)$ and $\mathbb{P}(-\underline{\alpha}, \xi)$, respectively. We observe that $(\hat{\Delta}_{\mathfrak{C}})_{V(\tau)} = (\hat{\Delta}_{\mathfrak{C}})_{V(\tau')}$ both coincide with the subfan of σ generated by its facets. It follows that the simplicial complexes relevant for computing the isotypical decomposition of $H^i_y\big(Y, \mathcal{O}(D)\big)$ coincide with the simplicial complexes relevant for computing the $H^i_V\big(\mathbb{P}(\underline{\alpha}, \xi), \mathcal{O}(\pi^{-1}D)\big)$ and $H^i_{V'}\big(\mathbb{P}(-\underline{\alpha}, \xi), \mathcal{O}((\pi')^{-1}D)\big)$, respectively, where V, V' denote the exceptional sets of the morphisms π and π', respectively. By Proposition 4.4 the corresponding cohomologies vanish for $i < d$ iff $D \in F_{\mathfrak{C}} \cap F_{-\mathfrak{C}}$. Now we observe that $\Gamma\big(Y, R^i\pi_*\mathcal{O}(\pi^{-1}D)\big) = H^i\big(\mathbb{P}(\underline{\alpha}, \xi), \mathcal{O}(\pi^{-1}D)\big)$ and $\Gamma\big(Y, R^i\pi'_*\mathcal{O}((\pi')^{-1}D)\big) = H^i\big(\mathbb{P}(-\underline{\alpha}, \xi), \mathcal{O}((\pi')^{-1}D)\big)$. By Proposition 4.3, both cohomologies vanish for $i > 0$ iff $D \in F_{\mathfrak{C}} \cap F_{-\mathfrak{C}}$. $\square$

Remark 4.34: The relation between maximal Cohen-Macaulay modules and the diophantine Frobenius problem has also been discussed in [Sta96]. See [Yos90] for a discussion of MCM-finiteness of toric 1-circuit varieties.

More generally, we have the following easy statement about general (i.e. non-regular) triangulations:

Theorem 4.35: *Let X be an affine toric variety of dimension d and $D \in A_{d-1}(X)$. If D is 0-essential, then $R^i\pi_*\mathcal{O}_{\tilde{X}}(\pi^*D) = 0$ for every triangulation $\pi : \tilde{X} \longrightarrow X$.*

Proof. If D is 0-essential, then it is contained in the intersection of all $F_{\mathcal{C}}$, where $\mathcal{C} \in \mathcal{C}(L)$, thus it represents a cohomology-free divisor. □

Note that the statement does hold for any triangulation and not only for regular triangulations. We have a refined statement for affine toric varieties whose associated cone σ has simplicial facets:

Theorem 4.36: *Let X be a d-dimensional affine toric variety whose associated cone σ has simplicial facets and let $D \in A_{d-1}(X)$. If $R^i\pi_*\mathcal{O}_{\tilde{X}}(\pi^*D) = 0$ for every regular triangulation $\pi : \tilde{X} \longrightarrow X$ then $\mathcal{O}_X(D)$ is MCM. For $d = 3$ the converse is also true.*

Proof. Recall that $H^i_x\big(X, \mathcal{O}(D)\big)_m = H^{i-2}(\hat{\sigma}_{V,m}; k)$ for some $m \in M$ and $D \in A$. We are going to show that for every subset $I \subsetneq [n]$ there exists a regular triangulation $\tilde{\Delta}$ of σ such that the simplicial complexes $\hat{\sigma}_{V,I}$ and $\tilde{\Delta}_I$ coincide. This implies that if $H^i_x\big(X, \mathcal{O}_X(D)\big)_m \neq 0$ for some $m \in M$, then also $H^{i+1}\big(\tilde{X}, \mathcal{O}_{\tilde{X}}(D)\big)_m \neq 0$, i.e. if $\mathcal{O}_X(D)$ is not MCM, then $H^i\big(\tilde{X}, \mathcal{O}_{\tilde{X}}(D)\big) \neq 0$ for some $i > 0$.
For given $I \subset [n]$ we get such a triangulation as follows. Let $i \in [n] \setminus I$ and consider the dual cone $\check{\sigma}$. Denote $\rho_i := \mathbb{Q}_{\geq 0} l_i$ and recall that $\check{\rho}_i$ is a halfspace which contains $\check{\sigma}$ and which defines a facet of $\check{\sigma}$ given by $\rho^\perp \cap \check{\sigma}$. Now we move $\check{\rho}_i$ to $m + \check{\rho}$, where $l_i(m) > 0$. So we obtain a new polytope $P := \check{\sigma} \cap (m + \check{\rho})$. As $\rho^\perp$ is not parallel to any face of $\check{\sigma}$, the hyperplane $m + \rho^\perp$ intersects every face of $\check{\sigma}$. This way the inner normal fan of P is a triangulation $\tilde{\Delta}$ of σ which has the property that every maximal cone is spanned by ρ_i and some facet of σ. This implies $\tilde{\Delta}_I = \hat{\sigma}_{V,I}$ and the first assertion follows.
For $d = 3$, a sheaf $\mathcal{O}(D)$ is MCM iff $H^2_x\big(X, \mathcal{O}(D)\big) = 0$, i.e. $H^0(\sigma_{V,m}; k) = 0$ for every $m \in M$. The latter is only possible if $\sigma_{V,m}$ represents an interval on S^1. To compare this with $H^2\big(\tilde{X}, \mathcal{O}(D)\big)$ for some regular triangulation $\tilde{X}$, we must show that $H^1(\tilde{\Delta}_m; k) = 0$ for the corresponding complex $\tilde{\Delta}_m$. To see this, we consider some cross-section $\sigma \cap H$, where $H \subset N \otimes_Z \mathbb{R}$ is some hyperplane which intersects σ nontrivially and is not parallel to any of its faces. Then this cross-section can be considered as a planar polygon and $\sigma_{V,m}$ as some connected sequence of faces of this polygon. Now with respect to the triangulation $\tilde{\Delta}$ of this polygon, we can consider two vertices $p, q \in \sigma_{V,m}$ which are connected by a line belonging to the triangulation and going through the interior of the polygon. We assume that p and q have maximal distance in $\sigma_{V,m}$ with this property. Then it is easy to see that the triangulation of σ induces a triangulation of the convex hull of the line segments connecting p and q. Then $\tilde{\Delta}_m$ is just the union of this convex hulls with respect all such pairs p, q and the remaining line segments and thus has trivial topology. Hence $H^2_x\big(X, \mathcal{O}(D)\big) = 0$ implies $H^2\big(\tilde{X}, \mathcal{O}(D)\big) = 0$ for every triangulation $\tilde{\Delta}$ of σ. □

Example 4.37: Consider the 3-dimensional cone spanned over the primitive vectors $l_1 = (1, 0, 1)$, $l_2 = (0, 1, 1)$, $l_3 = (-1, 0, 1)$, $l_4 = (-1, -1, 1)$, $l_5 =$

$(1, -1, 1)$. The corresponding toric variety X is Gorenstein and its divisor class group is torsion free. For $A_2(X) \cong \mathbb{Z}^2$ we choose the basis $D_1 + D_2 + D_5$, D_5. In this basis, the Gale duals of the l_i are $D_1 = (-1, -1)$, $D_2 = (2, 0)$, $D_3 = (-3, 1)$, $D_4 = (2, -1)$, $D_5 = (0, 1)$. Figure 2 shows the set of MCM modules in $A_2(X)$ which are indicated by circles which are supposed to sit on the lattice $A_2(X) \cong \mathbb{Z}^2$. The picture also indicates the cones C_I with vertices $-e_I$, where $I \in \{\{1,3\}, \{1,4\}, \{2,4\}, \{2,5\}, \{3,5\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{2,3,5\}, \{2,4,5\}\}$. Note that the picture has a reflection symmetry, due to the fact that X is Gorenstein. Altogether, there are 19 MCM modules of rank one, all of which are 0-essential. For $\mathcal{C} = \{l_1, l_3, l_4, l_5\}$, the group $A_2(X)_{\mathcal{C}} \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ has torsion. The two white circles indicate modules are contained in the $\mathbb{Q}$-hyperplanes $D_1 + D_4 + H_{\mathcal{C}}$ and $D_2 + D_3 + D_5 + H_{\mathcal{C}}$, respectively, but not in the sets $D_1 + D_4 + Z_{\mathcal{C}}$ and $D_2 + D_3 + D_5 + Z_{\mathcal{C}}$, respectively. Some of the $\mathbb{O}_I$ are not saturated; however, every divisor which is contained in some $(-e_I + C_I) \cap \Omega(L, I)$ is also contained in some $\mathbb{O}_{I'} \setminus \Omega(L, I')$ for some other $I' \neq I$. So for this example, the Frobenius arrangement gives a full description of MCM modules of rank one.

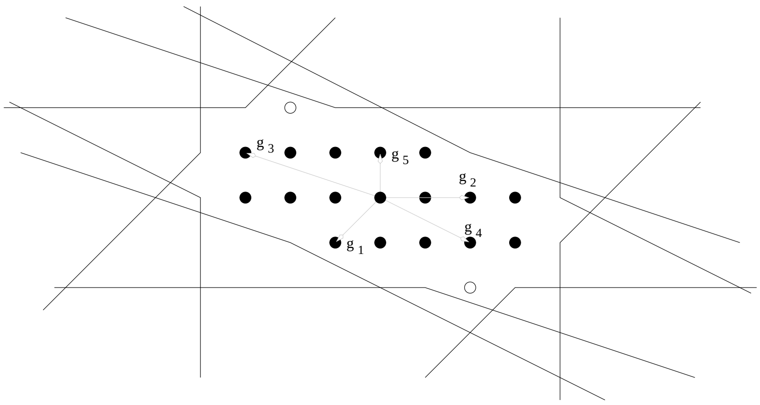

Figure 2: The 19 MCM modules of example 4.37.

Example 4.38: To give a counterexample to the reverse direction of theorem 4.36 for $d > 3$, we consider the four-dimensional cone spanned over the primitive vectors $l_1 = (0, -1, -1, 1)$, $l_2 = (-1, 0, 1, 1)$, $l_3 = (0, 1, 0, 1)$, $l_4 = (-1, 0, 0, 1)$, $l_5 = (-1, -1, 0, 1)$, $l_6 = (1, 0, 0, 1)$. The corresponding variety X has 31 MCM modules of rank one which are shown in figure 3. Here, with basis D_1 and D_6, we have $D_1 = (1, 0)$, $D_2 = (1, 0)$, $D_3 = (-1, -2)$, $D_4 = (3, 1)$, $D_5 = (-2, -2)$, $D_6 = (0, 1)$. There are six cohomology cones corresponding to $I \in \big\{\{1,2\}, \{3,5\}, \{4,6\}, \{1,2,3,5\}, \{1,2,4,6\}, \{3,4,5,6\}\big\}$.
The example features two modules which are not 0-essential, indicated by the grey dots sitting on the boundary of the cones $-e_I + C_I$, where $I \in$

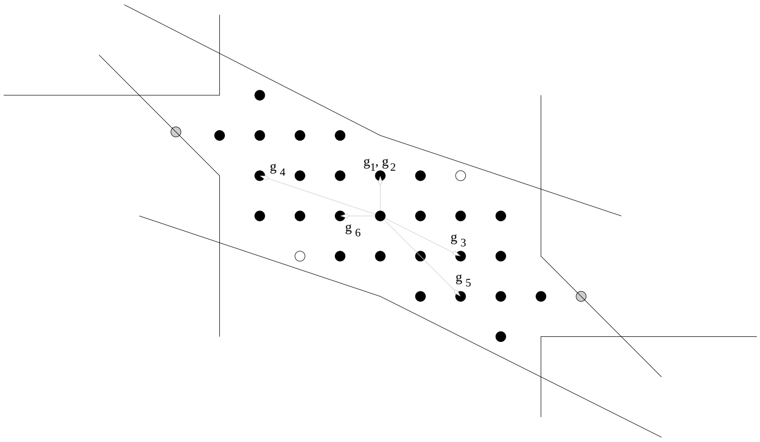

Figure 3: The 31 MCM modules of example 4.38.

$\big\{\{4,6\},\{1,2,3,5\}\big\}$. The white dots denote MCM divisors $D, -D$ such that there exists a triangulation of the cone of X such that on the associated variety $\tilde{X}$ we have $H^i\big(\tilde{X}, \mathcal{O}(\pm D)\big) \neq 0$ for some $i > 0$. Namely, we consider the triangulation which is given by the maximal cones spanned by $\{l_1, l_2, l_4, l_5\}$, $\{l_1, l_2, l_4, l_6\}$, $\{l_1, l_2, l_5, l_6\}$, $\{l_1, l_3, l_4, l_6\}$, $\{l_2, l_3, l_4, l_6\}$. Figure 4.38 indicates the two-dimensional faces of this triangulation via a three-dimensional cross-section of the cone.

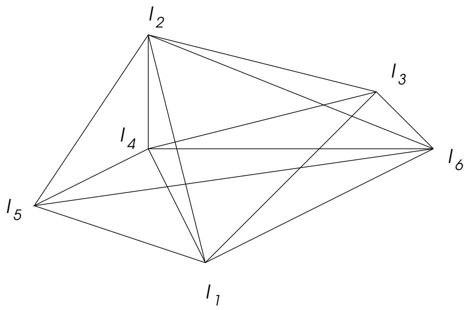

Figure 4: The triangulation for $\tilde{X}$ in example 4.38.

We find that we have six cohomology cones corresponding to $I \in \big\{\{1,2\},\{3,5\}$, $\{1,2,3\}$, $\{4,5,6\}$, $\{1,2,3,5\}$, $\{3,4,5,6\}\big\}$. In particular, we have non-vanishing H^1 for the points $-D_1 - D_2 - D_3$ and for $-D_4 - D_5 - D_6$, which correspond to D and $-D$.

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