Multigraded Factorial Rings and Fano Varieties with Torus Action

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Abstract. In a first result, we describe all finitely generated factorial algebras over an algebraically closed field of characteristic zero that come with an effective multigrading of complexity one by means of generators and relations. This enables us to construct systematically varieties with free divisor class group and a complexity one torus action via their Cox rings. For the Fano varieties of this type that have a free divisor class group of rank one, we provide explicit bounds for the number of possible deformation types depending on the dimension and the index of the Picard group in the divisor class group. As a consequence, one can produce classification lists for fixed dimension and Picard index. We carry this out expemplarily in the following cases. There are 15 non-toric surfaces with Picard index at most six. Moreover, there are 116 non-toric threefolds with Picard index at most two; nine of them are locally factorial, i.e. of Picard index one, and among these one is smooth, six have canonical singularities and two have non-canonical singularities. Finally, there are 67 non-toric locally factorial fourfolds and two one-dimensional families of non-toric locally factorial fourfolds. In all cases, we list the Cox rings explicitly.

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Introduction

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. A first aim of this paper is to determine all finitely generated factorial $\mathbb{K}$-algebras R with an effective complexity one multigrading $R = \oplus_{u \in M} R_u$ satisfying $R_0 = \mathbb{K}$; here effective complexity one multigrading means that with $d := \dim R$ we have $M \cong \mathbb{Z}^{d-1}$ and the $u \in M$ with $R_u \neq 0$ generate M as a $\mathbb{Z}$-module. Our result extends work by Mori [23] and Ishida [17], who settled the cases $d = 2$ and $d = 3$.

An obvious class of multigraded factorial algebras as above is given by polynomial rings. A much larger class is obtained as follows. Take a sequence $A = (a_0, \ldots, a_r)$ of vectors $a_i \in \mathbb{K}^2$ such that (a_i, a_k) is linearly independent whenever $k \neq i$, a sequence $\mathfrak{n} = (n_0, \ldots, n_r)$ of positive integers and a family $L = (l_{ij})$ of positive integers, where $0 \leq i \leq r$ and $1 \leq j \leq n_i$. For every $0 \leq i \leq r$, we define a monomial

$$f_i \ := \ T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}} \ \in \ \mathbb{K}[T_{ij};\ 0 \leq i \leq r,\ 1 \leq j \leq n_i],$$

for any two indices $0 \leq i, j \leq r$, we set $\alpha_{ij} := \det(a_i, a_j)$, and for any three indices $0 \leq i < j < k \leq r$, we define a trinomial

$$g_{i,j,k} \ := \ \alpha_{jk} f_i \ + \ \alpha_{ki} f_j \ + \ \alpha_{ij} f_k \ \in \ \mathbb{K}[T_{ij};\ 0 \leq i \leq r,\ 1 \leq j \leq n_i].$$

Note that the coefficients of $g_{i,j,k}$ are all nonzero. The triple $(A, \mathfrak{n}, L)$ then defines a $\mathbb{K}$-algebra

$$R(A, \mathfrak{n}, L) \ := \ \mathbb{K}[T_{ij};\ 0 \leq i \leq r,\ 1 \leq j \leq n_i] \ / \ \langle g_{i,i+1,i+2};\ 0 \leq i \leq r-2 \rangle.$$

It turns out that $R(A, \mathfrak{n}, L)$ is a normal complete intersection, see Proposition 1.2. In particular, it is of dimension

$$\dim R(A, \mathfrak{n}, L) \quad = \quad n_0 + \ldots + n_r \ - \ r \ + \ 1.$$

If the triple $(A, \mathfrak{n}, L)$ is *admissible*, i.e., the numbers $\gcd(l_{i1}, \ldots, l_{in_i})$, where $0 \leq i \leq r$, are pairwise coprime, then $R(A, \mathfrak{n}, L)$ admits a canonical effective complexity one grading by a lattice K, see Construction 1.7. Our first result is the following.

Theorem 1.9. *Up to isomorphy, the finitely generated factorial $\mathbb{K}$-algebras with an effective complexity one grading $R = \oplus_M R_u$ and $R_0 = \mathbb{K}$ are*

(i) *the polynomial algebras $\mathbb{K}[T_1, \ldots, T_d]$ with a grading $\deg(T_i) = u_i \in \mathbb{Z}^{d-1}$ such that $u_1, \ldots, u_d$ generate $\mathbb{Z}^{d-1}$ as a lattice and the convex cone on $\mathbb{Q}^{d-1}$ generated by $u_1, \ldots, u_d$ is pointed,*

(ii) *the $(K \times \mathbb{Z}^m)$-graded algebras $R(A, \mathfrak{n}, L)[S_1, \ldots, S_m]$, where $R(A, \mathfrak{n}, L)$ is the K-graded algebra defined by an admissible triple $(A, \mathfrak{n}, L)$ and $\deg S_j \in \mathbb{Z}^m$ is the j-th canonical base vector.*

The further paper is devoted to normal (possibly singular) d-dimensional Fano varieties X with an effective action of an algebraic torus T. In the case $\dim T = d$, we have the meanwhile extensively studied class of toric Fano varieties, see [3], [27] and [4] for the initiating work. Our aim is to show that the above Theorem provides an approach to classification results for the case $\dim T = d-1$, that means Fano varieties with a complexity one torus action. Here, we treat the case of divisor class group $\mathrm{Cl}(X) \cong \mathbb{Z}$; note that in the toric setting this gives precisely the weighted projective spaces. The idea is to consider the Cox ring

$$\mathcal{R}(X) \quad = \quad \bigoplus_{D \in \mathrm{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)).$$

The ring $\mathcal{R}(X)$ is factorial, finitely generated as a $\mathbb{K}$-algebra and the T-action on X gives rise to an effective complexity one multigrading of $\mathcal{R}(X)$ refining the $\mathrm{Cl}(X)$-grading, see [5] and [15]. Consequently, $\mathcal{R}(X)$ is one of the rings listed in the first Theorem. Moreover, X can be easily reconstructed from $\mathcal{R}(X)$; it is the homogeneous spectrum with respect to the $\mathrm{Cl}(X)$-grading of $\mathcal{R}(X)$. Thus, in order to construct Fano varieties, we firstly have to figure out the Cox rings among the rings occuring in the first Theorem and then find those, which belong to a Fano variety; this is done in Propositions 1.11 and 2.5.

In order to produce classification results via this approach, we need explicit bounds on the number of deformation types of Fano varieties with prescribed discrete invariants. Besides the dimension, in our setting, a suitable invariant is the *Picard index* $[\mathrm{Cl}(X) : \mathrm{Pic}(X)]$. Denoting by $\xi(\mu)$ the number of primes less or equal to μ, we obtain the following bound, see Corollary 2.2: for any pair $(d, \mu) \in \mathbb{Z}^2_{>0}$, the number $\delta(d, \mu)$ of different deformation types of d-dimensional Fano varieties with a complexity one torus action such that $\mathrm{Cl}(X) \cong \mathbb{Z}$ and $\mu = [\mathrm{Cl}(X) : \mathrm{Pic}(X)]$ hold is bounded by

$$\delta(d, \mu) \quad \leq \quad (6d\mu)^{2\xi(3d\mu)+d-2} \mu^{\xi(\mu)^2 + 2\xi((d+2)\mu)+2d+2}.$$

In particular, we conclude that for fixed $\mu \in \mathbb{Z}_{>0}$, the number $\delta(d)$ of different deformation types of d-dimensional Fano varieties with a complexity one torus action $\mathrm{Cl}(X) \cong \mathbb{Z}$ and Picard index μ is asymptotically bounded by d^{Ad} with a constant A depending only on μ, see Corollary 2.4.

In fact, in Theorem 2.1 we even obtain explicit bounds for the discrete input data of the rings $R(A, \mathfrak{n}, L)[S_1, \ldots, S_m]$. This allows us to construct all Fano varieties X with prescribed dimension and Picard index that come with an effective complexity one torus action and have divisor class group $\mathbb{Z}$. Note that, by the approach, we get the Cox rings of the resulting Fano varieties X for free. In Section 3, we give some explicit classifications. We list all non-toric surfaces X with Picard index at most six and the non-toric threefolds X with Picard index up at most two. They all have a Cox ring defined by a single relation; in fact, for surfaces the first Cox ring with more than one relation occurs for Picard index 29, and for the threefolds this happens with Picard index 3, see Proposition 3.5 as well as Examples 3.4 and 3.7. Moreover, we determine all locally factorial fourfolds X, i.e. those of Picard index one: 67 of them occur sporadic and there are two one-dimensional families. Here comes the result on the locally factorial threefolds; in the table, we denote by w_i the $\mathrm{Cl}(X)$-degree of the variable T_i.

Theorem 3.2. *The following table lists the Cox rings $\mathcal{R}(X)$ of the three-dimensional locally factorial non-toric Fano varieties X with an effective two torus action and* $\mathrm{Cl}(X) = \mathbb{Z}$.

No.	$\mathcal{R}(X)$	$(w_1, \ldots, w_5)$	$(-K_X)^3$
1	$\mathbb{K}[T_1, \ldots, T_5] \ / \ \langle T_1T_2^5 + T_3^3 + T_4^2 \rangle$	$(1, 1, 2, 3, 1)$	8
2	$\mathbb{K}[T_1, \ldots, T_5] \ / \ \langle T_1T_2T_3^4 + T_4^3 + T_5^2 \rangle$	$(1, 1, 1, 2, 3)$	8
3	$\mathbb{K}[T_1, \ldots, T_5] \ / \ \langle T_1T_2^2T_3^3 + T_4^3 + T_5^2 \rangle$	$(1, 1, 1, 2, 3)$	8
4	$\mathbb{K}[T_1, \ldots, T_5] \ / \ \langle T_1T_2 + T_3T_4 + T_5^2 \rangle$	$(1, 1, 1, 1, 1)$	54
5	$\mathbb{K}[T_1, \ldots, T_5] \ / \ \langle T_1T_2^2 + T_3T_4^2 + T_5^3 \rangle$	$(1, 1, 1, 1, 1)$	24
6	$\mathbb{K}[T_1, \ldots, T_5] \ / \ \langle T_1T_2^3 + T_3T_4^3 + T_5^4 \rangle$	$(1, 1, 1, 1, 1)$	4
7	$\mathbb{K}[T_1, \ldots, T_5] \ / \ \langle T_1T_2^3 + T_3T_4^3 + T_5^2 \rangle$	$(1, 1, 1, 1, 2)$	16
8	$\mathbb{K}[T_1, \ldots, T_5] \ / \ \langle T_1T_2^5 + T_3T_4^5 + T_5^2 \rangle$	$(1, 1, 1, 1, 3)$	2
9	$\mathbb{K}[T_1, \ldots, T_5] \ / \ \langle T_1T_2^5 + T_3^3T_4^3 + T_5^2 \rangle$	$(1, 1, 1, 1, 3)$	2

Note that each of these varieties X is a hypersurface in the respective weighted projective space $\mathbb{P}(w_1, \ldots, w_5)$. Except number 4, none of them is quasismooth in the sense that $\operatorname{Spec} \mathcal{R}(X)$ is singular at most in the origin; quasismooth hypersurfaces of weighted projective spaces were studied in [21] and [7]. In Section 4, we take a closer look at the singularities of the threefolds listed above. It turns out that number 1,3,5,7 and 9 are singular with only canonical singularities and all of them admit a crepant resolution. Number 6 and 8 are singular with non-canonical singularities but admit a smooth relative minimal model. Number two is singular with only canonical singularities, one of them of type $\mathbf{cA_1}$, and it admits only a singular relative minimal model. Moreover, in all cases, we determine the Cox rings of the resolutions.

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1 UFDs with complexity one multigrading

As mentioned before, we work over an algebraically closed field $\mathbb{K}$ of characteristic zero. In Theorem 1.9, we describe all factorial finitely generated $\mathbb{K}$-algebras R with an effective complexity one grading and $R_0 = \mathbb{K}$. Moreover, we characterize the possible Cox rings among these algebras, see Proposition 1.11. First we recall the construction sketched in the introduction.

Construction 1.1. Consider a sequence $A = (a_0, \dots, a_r)$ of vectors $a_i = (b_i, c_i)$ in $\mathbb{K}^2$ such that any pair (a_i, a_k) with $k \neq i$ is linearly independent, a sequence $\mathfrak{n} = (n_0, \dots, n_r)$ of positive integers and a family $L = (l_{ij})$ of positive integers, where $0 \le i \le r$ and $1 \le j \le n_i$. For every $0 \le i \le r$, define a monomial

$$f_i \ := \ T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}} \ \in \ \mathbb{K}[T_{ij};\ 0 \le i \le r,\ 1 \le j \le n_i],$$

for any two indices $0 \le i, j \le r$, set $\alpha_{ij} := \det(a_i, a_j) = b_i c_j - b_j c_i$ and for any three indices $0 \le i < j < k \le r$ define a trinomial

$$g_{i,j,k} \ := \ \alpha_{jk} f_i \ + \ \alpha_{ki} f_j \ + \ \alpha_{ij} f_k \ \in \ \mathbb{K}[T_{ij};\ 0 \le i \le r,\ 1 \le j \le n_i].$$

Note that the coefficients of this trinomial are all nonzero. The triple $(A, \mathfrak{n}, L)$ then defines a ring

$$R(A, \mathfrak{n}, L) \ := \ \mathbb{K}[T_{ij};\ 0 \le i \le r,\ 1 \le j \le n_i] \ / \ \langle g_{i,i+1,i+2};\ 0 \le i \le r-2 \rangle.$$

Proposition 1.2. *For every triple $(A, \mathfrak{n}, L)$ as in 1.1, the ring $R(A, \mathfrak{n}, L)$ is a normal complete intersection of dimension*

$$\dim R(A, \mathfrak{n}, L) \ = \ n - r + 1, \qquad\qquad n \ := \ n_0 + \ldots + n_r.$$

Lemma 1.3. *In the setting of 1.1, one has for any $0 \le i < j < k < l \le r$ the identities*

$$g_{i,k,l} \ = \ \alpha_{kl} \cdot g_{i,j,k} + \alpha_{ik} \cdot g_{j,k,l}, \qquad\qquad g_{i,j,l} \ = \ \alpha_{jl} \cdot g_{i,j,k} + \alpha_{ij} \cdot g_{j,k,l}.$$

In particular, every trinomial $g_{i,j,k}$, where $0 \le i < j < k \le r$ is contained in the ideal $\langle g_{i,i+1,i+2};\ 0 \le i \le r-2 \rangle$.

Proof. The identities are easily obtained by direct computation; note that for this one may assume $a_j = (1, 0)$ and $a_k = (0, 1)$. The supplement then follows by repeated application of the identities. ☐

Lemma 1.4. *In the notation of 1.1 and 1.2, set $X := V(\mathbb{K}^n, g_0, \dots, g_{r-2})$, and let $z \in X$. If we have $f_i(z) = f_j(z) = 0$ for two $0 \le i < j \le r$, then $f_k(z) = 0$ holds for all $0 \le k \le r$.*

Proof. If $i < k < j$ holds, then, according to Lemma 1.3, we have $g_{i,k,j}(z) = 0$, which implies $f_k(z) = 0$. The cases $k < i$ and $j < k$ are obtained similarly. ☐

Proof of Proposition 1.2. Set $X := V(\mathbb{K}^n; g_0, \dots, g_{r-2})$, where $g_i := g_{i,i+1,i+2}$. Then we have to show that X is a connected complete intersection with at most normal singularities. In order to see that X is connected, set $\ell := \prod n_i \prod l_{ij}$ and $\zeta_{ij} := \ell n_i^{-1} l_{ij}^{-1}$. Then $X \subseteq \mathbb{K}^n$ is invariant under the $\mathbb{K}^*$-action given by

$$t \cdot z \quad := \quad (t^{\zeta_{ij}} z_{ij})$$

and the point $0 \in \mathbb{K}^n$ lies in the closure of any orbit $\mathbb{K}^* \cdot x \subseteq X$, which implies connectedness. To proceed, consider the Jacobian J_g of $g := (g_0, \dots, g_{r-2})$. According to Serre's criterion, we have to show that the set of points of $z \in X$ with $J_g(z)$ not of full rank is of codimension at least two in X. Note that the Jacobian J_g is of the shape

$$J_g \quad = \quad \begin{pmatrix} \delta_{0\,0} & \delta_{0\,1} & \delta_{0\,2} & 0 & & & & & & 0 \\ 0 & \delta_{1\,1} & \delta_{1\,2} & \delta_{1\,3} & 0 & & & & & \\ & & & & & \vdots & & & & \\ & & & & & & 0 & \delta_{r-3\,r-3} & \delta_{r-3\,r-2} & \delta_{r-3\,r-1} & 0 \\ 0 & & & & & & & 0 & \delta_{r-2\,r-2} & \delta_{r-2\,r-1} & \delta_{r-2\,r} \end{pmatrix}$$

where δ_{ti} is a nonzero multiple of the gradient $\delta_i := \operatorname{grad} f_i$. Consider $z \in X$ with $J_g(z)$ not of full rank. Then $\delta_i(z) = 0 = \delta_k(z)$ holds with some $0 \le i < k \le r$. This implies $z_{ij} = 0 = z_{kl}$ for some $1 \le j \le n_i$ and $1 \le l \le n_k$. Thus, we have $f_i(z) = 0 = f_k(z)$. Lemma 1.4 gives $f_s(z) = 0$, for all $0 \le s \le r$. Thus, some coordinate z_{st} must vanish for every $0 \le s \le r$. This shows that z belongs to a closed subset of X having codimension at least two in X. $\square$

Lemma 1.5. *Notation as in 1.1. Then the variable T_{ij} defines a prime ideal in $R(A, \mathfrak{n}, L)$ if and only if the numbers $\gcd(l_{k1}, \dots, l_{kn_k})$, where $k \neq i$, are pairwise coprime.*

Proof. We treat exemplarily T_{01}. Using Lemma 1.3, we see that the ideal of relations of $R(A, \mathfrak{n}, L)$ can be presented as follows

$$\langle g_{s,s+1,s+2};\ 0 \le s \le r-2 \rangle \quad = \quad \langle g_{0,s,s+1};\ 1 \le s \le r-1 \rangle.$$

Thus, the ideal $\langle T_{01} \rangle \subseteq R(A, \mathfrak{n}, L)$ is prime if and only if the following binomial ideal is prime

$$\mathfrak{a} \ := \ \langle \alpha_{s+1\,0} f_s + \alpha_{0s} f_{s+1};\ 1 \le s \le r-1 \rangle \ \subseteq \ \mathbb{K}[T_{ij};\ (i,j) \neq (0,1)].$$

Set $l_i := (l_{i1}, \dots, l_{in_i})$. Then the ideal $\mathfrak{a}$ is prime if and only if the following family can be complemented to a lattice basis

$$(l_1, -l_2, 0, \dots, 0),\ \dots,\ (0, \dots, 0, l_{r-1}, -l_r).$$

This in turn is equivalent to the statement that the numbers $\gcd(l_{k1}, \dots, l_{kn_k})$, where $1 \le k \le r$, are pairwise coprime. $\square$

Definition 1.6. We say that a triple $(A, \mathfrak{n}, L)$ as in 1.1 is *admissible* if the numbers $\gcd(l_{i1}, \ldots, l_{in_i})$, where $0 \le i \le r$, are pairwise coprime.

Construction 1.7. Let $(A, \mathfrak{n}, L)$ be an admissible triple and consider the following free abelian groups

$$E \quad := \quad \bigoplus_{i=0}^{r} \bigoplus_{j=1}^{n_i} \mathbb{Z} \cdot e_{ij}, \qquad\qquad K \quad := \quad \bigoplus_{j=1}^{n_0} \mathbb{Z} \cdot u_{0j} \;\oplus\; \bigoplus_{i=1}^{r} \bigoplus_{j=1}^{n_i - 1} \mathbb{Z} \cdot u_{ij}$$

and define vectors $u_{in_i} := u_{01} + \ldots + u_{0r} - u_{i1} - \ldots - u_{in_i-1} \in K$. Then there is an epimorphism $\lambda \colon E \to K$ fitting into a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r, "e_{ij} \mapsto l_{ij} e_{ij}", "\alpha"'] \arrow[d, "\eta", "e_{ij} \mapsto u_{ij}"'] & E \arrow[r, "e_{ij} \mapsto \bar{e}_{ij}"] \arrow[d, "\lambda"] & \bigoplus_{i,j} \mathbb{Z}/l_{ij}\mathbb{Z} \arrow[r] \arrow[d, leftrightarrow, "\cong"] & 0 \\
0 \arrow[r] & K \arrow[r, "\beta"'] & K \arrow[r] & \bigoplus_{i,j} \mathbb{Z}/l_{ij}\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Define a K-grading of $\mathbb{K}[T_{ij};\ 0 \le i \le r,\ 1 \le j \le n_i]$ by setting $\deg T_{ij} := \lambda(e_{ij})$. Then every $f_i = T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}}$ is K-homogeneous of degree

$$\deg f_i \;=\; l_{i1}\lambda(e_{i1}) + \ldots + l_{in_i}\lambda(e_{in_i}) \;=\; l_{01}\lambda(e_{01}) + \ldots + l_{0n_0}\lambda(e_{0n_0}) \;\in\; K.$$

Thus, the polynomials $g_{i,j,k}$ of 1.1 are all K-homogeneous of the same degree and we obtain an effective K-grading of complexity one of $R(A, \mathfrak{n}, L)$.

Proof. Only for the existence of the commutative diagram there is something to show. Write for short $l_i := (l_{i1}, \ldots, l_{in_i})$. By the admissibility condition, the vectors $v_i := (0, \ldots, 0, l_i, -l_{i+1}, 0, \ldots, 0)$, where $0 \le i \le r-1$, can be completed to a lattice basis for E. Consequently, we find an epimorphism $\lambda \colon E \to K$ having precisely $\mathrm{lin}(v_0, \ldots, v_{r-1})$ as its kernel. By construction, $\ker(\lambda)$ equals $\alpha(\ker(\eta))$. Using this, we obtain the induced morphism $\beta \colon K \to K$ and the desired properties. $\square$

Lemma 1.8. *Notation as in 1.7. Then $R(A, \mathfrak{n}, L)_0 = \mathbb{K}$ and $R(A, \mathfrak{n}, L)^* = \mathbb{K}^*$ hold. Moreover, the T_{ij} define pairwise nonassociated prime elements in $R(A, \mathfrak{n}, L)$.*

Proof. The fact that all elements of degree zero are constant is due to the fact that all degrees $\deg T_{ij} = u_{ij} \in K$ are non-zero and generate a pointed convex cone in $K_{\mathbb{Q}}$. As a consequence, we obtain that all units in $R(A, \mathfrak{n}, L)$ are constant. The T_{ij} are prime by the admissibility condition and Lemma 1.5, and they are pairwise nonassociated because they have pairwise different degrees and all units are constant. $\square$

Theorem 1.9. *Up to isomorphy, the finitely generated factorial* $\mathbb{K}$*-algebras with an effective complexity one grading* $R = \oplus_M R_u$ *and* $R_0 = \mathbb{K}$ *are*

(i) *the polynomial algebras* $\mathbb{K}[T_1, \ldots, T_d]$ *with a grading* $\deg(T_i) = u_i \in \mathbb{Z}^{d-1}$ *such that* $u_1, \ldots, u_d$ *generate* $\mathbb{Z}^{d-1}$ *as a lattice and the convex cone on* $\mathbb{Q}^{d-1}$ *generated by* $u_1, \ldots, u_d$ *is pointed,*

(ii) *the* $(K \times \mathbb{Z}^m)$*-graded algebras* $R(A, \mathfrak{n}, L)[S_1, \ldots, S_m]$*, where* $R(A, \mathfrak{n}, L)$ *is the* K*-graded algebra defined by an admissible triple* $(A, \mathfrak{n}, L)$ *as in 1.1 and 1.7 and* $\deg S_j \in \mathbb{Z}^m$ *is the* j*-th canonical base vector.*

Proof. We first show that for any admissible triple $(A, \mathfrak{n}, L)$ the ring $R(A, \mathfrak{n}, L)$ is a unique factorization domain. If $l_{ij} = 1$ holds for any two i, j, then, by [15, Prop. 2.4], the ring $R(A, \mathfrak{n}, L)$ is the Cox ring of a space $\mathbb{P}_1(A, \mathfrak{n})$ and hence is a unique factorization domain.

Now, let $(A, \mathfrak{n}, L)$ be arbitrary admissible data and let $\lambda \colon E \to K$ be an epimorphism as in 1.7. Set $n := n_0 + \ldots + n_r$ and consider the diagonalizable groups

$$\mathbb{T}^n \ := \ \operatorname{Spec} \mathbb{K}[E], \qquad H \ := \ \operatorname{Spec} \mathbb{K}[K], \qquad H_0 \ := \ \operatorname{Spec} \mathbb{K}[\oplus_{i,j} \mathbb{Z}/l_{ij}\mathbb{Z}].$$

Then $\mathbb{T}^n = (\mathbb{K}^*)^n$ is the standard n-torus and H_0 is the direct product of the cyclic subgroups $H_{ij} := \operatorname{Spec} \mathbb{K}[\mathbb{Z}/l_{ij}\mathbb{Z}]$. Moreover, the diagram in 1.7 gives rise to a commutative diagram with exact rows

$$\begin{tikzcd}
0 & \mathbb{T}^n \arrow[l] & \mathbb{T}^n \arrow[l, "(t_{ij}^{l_{ij}}) \mapsfrom (t_{ij})"'] & H_0 \arrow[l] & 0 \arrow[l] \\
0 & H \arrow[l] \arrow[u, "\imath"] & H \arrow[l] \arrow[u, "\jmath"] & H_0 \arrow[l] \arrow[u, leftrightarrow, "\cong"'] & 0 \arrow[l]
\end{tikzcd}$$

where $t_{ij} = \chi^{e_{ij}}$ are the coordinates of $\mathbb{T}^n$ corresponding to the characters $e_{ij} \in E$ and the maps $\imath$, $\jmath$ are the closed embeddings corresponding to the epimorphisms η, λ respectively.

Setting $\deg T_{ij} := e_{ij}$ defines an action of $\mathbb{T}^n$ on $\mathbb{K}^n = \operatorname{Spec} \mathbb{K}[T_{ij}]$; in terms of the coordinates z_{ij} corresponding to T_{ij} this action is given by $t \cdot z = (t_{ij} z_{ij})$. The torus H acts effectively on $\mathbb{K}^n$ via the embedding $\jmath \colon H \to \mathbb{T}^n$. The generic isotropy group of H along $V(\mathbb{K}^n, T_{ij})$ is the subgroup $H_{ij} \subseteq H$ corresponding to $K \to K/\lambda(E_{ij})$, where $E_{ij} \subseteq E$ denotes the sublattice generated by all e_{kl} with $(k, l) \neq (i, j)$; recall that we have $K/\lambda(E_{ij}) \cong \mathbb{Z}/l_{ij}\mathbb{Z}$.

Now, set $l'_{ij} := 1$ for any two i, j and consider the spectra $X := \operatorname{Spec} R(A, \mathfrak{n}, L)$ and $X' := \operatorname{Spec} R(A, \mathfrak{n}, L')$. Then the canonical surjections $\mathbb{K}[T_{ij}] \to R(A, \mathfrak{n}, L)$ and $\mathbb{K}[T_{ij}] \to R(A, \mathfrak{n}, L')$ define embeddings $X \to \mathbb{K}^n$ and $X' \to \mathbb{K}^n$. These embeddings fit into the following commutative diagram

$$\begin{tikzcd} \mathbb{K}^n & \mathbb{K}^n \arrow[l, "(z_{ij}^{l_{ij}}) \leftarrow\!\shortmid (z_{ij})"', "\pi"] \\ X' \arrow[u] & X \arrow[l] \arrow[u] \end{tikzcd}$$

The action of H leaves X invariant and the induced H-action on X is the one given by the K-grading of $R(A, \mathfrak{n}, L)$. Moreover, $\pi\colon \mathbb{K}^n \to \mathbb{K}^n$ is the quotient map for the induced action of $H_0 \subseteq H$ on $\mathbb{K}^n$, we have $X = \pi^{-1}(X')$, and hence the restriction $\pi\colon X \to X'$ is a quotient map for the induced action of H_0 on X.

Removing all subsets $V(X; T_{ij}, T_{kl})$, where $(i,j) \neq (k,l)$ from X, we obtain an open subset $U \subseteq X$. By Lemma 1.8, the complement $X \setminus U$ is of codimension at least two and each $V(U, T_{ij})$ is irreducible. By construction, the only isotropy groups of the H-action on U are the groups H_{ij} of the points of $V(U, T_{ij})$. The image $U' := \pi(U)$ is open in X', the complement $X' \setminus U'$ is as well of codimension at least two and H/H_0 acts freely on U'. According to [22, Cor. 5.3], we have two exact sequences fitting into the following diagram

$$\begin{tikzcd} & & 1 \arrow[d] & \\ & & \mathrm{Pic}(U') \arrow[d, "\pi^*"] & \\ 1 \arrow[r] & \mathbb{X}(H_0) \arrow[r, "\alpha"] & \mathrm{Pic}_{H_0}(U) \arrow[r, "\beta"] \arrow[d, "\delta"] & \mathrm{Pic}(U) \\ & & \prod_{i,j} \mathbb{X}(H_{ij}) & \end{tikzcd}$$

Since X' is factorial, the Picard group $\mathrm{Pic}(U')$ is trivial and we obtain that δ is injective. Since H_0 is the direct product of the isotropy groups H_{ij} of the Luna strata $V(U, T_{ij})$, we see that $\delta \circ \alpha$ is an isomorphism. It follows that δ is surjective and hence an isomorphism. This in turn shows that α is an isomorphism. Now, every bundle on U is H-linearizable. Since H_0 acts as a subgroup of H, we obtain that every bundle is H_0-linearizable. It follows that β is surjective and hence $\mathrm{Pic}(U)$ is trivial. We conclude $\mathrm{Cl}(X) = \mathrm{Pic}(U) = 0$, which means that $R(A, \mathfrak{n}, L)$ admits unique factorization.

The second thing we have to show is that any finitely generated factorial $\mathbb{K}$-algebra R with an effective complexity one multigrading satisfying $R_0 = \mathbb{K}$ is as claimed. Consider the action of the torus G on $X = \operatorname{Spec} R$ defined by the multigrading, and let $X_0 \subseteq X$ be the set of points having finite isotropy G_x.

Then [15, Prop 3.3] provides a graded splitting

$$R \quad\cong\quad R'[S_1, \ldots, S_m],$$

where the variables S_j are identified with the homogeneous functions defining the prime divisors E_j inside the boundary $X \setminus X_0$ and R' is the ring of functions of X_0, which are invariant under the subtorus $G_0 \subseteq G$ generated by the generic isotropy groups G_j of E_j.

Since $R'_0 = R_0 = \mathbb{K}$ holds, the orbit space X_0/G has only constant functions and thus is a space $\mathbb{P}_1(A, \mathfrak{n})$ as constructed in [15, Section 2]. This allows us to proceed exactly as in the proof of Theorem [15, Thm 1.3] and gives $R' = R(A, \mathfrak{n}, L)$. The admissibility condition follows from Lemma 1.5 and the fact that each T_{ij} defines a prime element in R'. □

Remark 1.10. Let $(A, \mathfrak{n}, L)$ be an admissible triple with $\mathfrak{n} = (1, \ldots, 1)$. Then $K = \mathbb{Z}$ holds, the admissibility condition just means that the numbers l_{ij} are pairwise coprime and we have

$$\dim R(A, \mathfrak{n}, L) \;=\; n_0 + \ldots + n_r - r + 1 \;=\; 2.$$

Consequently, for two-dimensional rings, Theorem 1.9 specializes to Mori's description of almost geometrically graded two-dimensional unique factorization domains provided in [23].

Proposition 1.11. *Let $(A, \mathfrak{n}, L)$ be an admissible triple, consider the associated $(K \times \mathbb{Z}^m)$-graded ring $R(A, \mathfrak{n}, L)[S_1, \ldots, S_m]$ as in Theorem 1.9 and let $\mu\colon K \times \mathbb{Z}^m \to K'$ be a surjection onto an abelian group K'. Then the following statements are equivalent.*

(i) *The K'-graded ring $R(A, \mathfrak{n}, L)[S_1, \ldots, S_m]$ is the Cox ring of a projective variety X' with $\mathrm{Cl}(X') \cong K'$.*

(ii) *For every pair i, j with $0 \le i \le r$ and $1 \le j \le n_i$, the group K' is generated by the elements $\mu(\lambda(e_{kl}))$ and $\mu(e_s)$, where $(i,j) \neq (k,l)$ and $1 \le s \le m$, for every $1 \le t \le m$, the group K' is generated by the elements $\mu(\lambda(e_{ij}))$ and $\mu(e_s)$, where $0 \le i \le r$, $1 \le j \le n_i$ and $s \neq t$, and, finally the following cone is of full dimension in $K'_{\mathbb{Q}}$:*

$$\bigcap_{(k,l)} \mathrm{cone}(\mu(\lambda(e_{ij})), \mu(e_s);\ (i,j) \neq (k,l)) \;\cap\; \bigcap_{t} \mathrm{cone}(\mu(\lambda(e_{ij})), \mu(e_s);\ s \neq t).$$

Proof. Suppose that (i) holds, let $p\colon \widehat{X}' \to X'$ denote the universal torsor and let $X'' \subseteq X'$ be the set of smooth points. According to [14, Prop. 2.2], the group $H' = \mathrm{Spec}\,\mathbb{K}[K']$ acts freely on $p^{-1}(X'')$, which is a big open subset of the total coordinate space $\mathrm{Spec}\, R(A, \mathfrak{n}, L)[S_1, \ldots, S_m]$. This implies the first condition of (ii). Moreover, by [14, Prop. 4.1], the displayed cone is the moving cone of X' and hence of full dimension. Conversely, if (ii) holds, then the K'-graded ring $R(A, \mathfrak{n}, L)[S_1, \ldots, S_m]$ can be made into a bunched ring and hence is the Cox ring of a projective variety, use [14, Thm. 3.6]. □

2 Bounds for Fano varieties

We consider d-dimensional Fano varieties X that come with a complexity one torus action and have divisor class group $\mathrm{Cl}(X) \cong \mathbb{Z}$. Then the Cox ring $\mathcal{R}(X)$ of X is factorial [5, Prop. 8.4] and has an effective complexity one grading, which refines the $\mathrm{Cl}(X)$-grading, see [15, Prop. 2.6]. Thus, according to Theorem 1.9, it is of the form

$$\begin{aligned}\mathcal{R}(X) &\cong \mathbb{K}[T_{ij};\ 0 \le i \le r,\ 1 \le j \le n_i][S_1, \ldots, S_m] \ / \ \langle g_{i,i+1,i+2};\ 0 \le i \le r-2 \rangle, \\ g_{i,j,k} &:= \alpha_{jk} T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}} \ + \ \alpha_{ki} T_{j1}^{l_{j1}} \cdots T_{jn_j}^{l_{jn_j}} \ + \ \alpha_{ij} T_{k1}^{l_{k1}} \cdots T_{kn_k}^{l_{kn_k}}.\end{aligned}$$

Here, we may (and will) assume $n_0 \ge \ldots \ge n_r \ge 1$. With $n := n_0 + \ldots + n_r$, we have $n + m = d + r$. For the degrees of the variables in $\mathrm{Cl}(X) \cong \mathbb{Z}$, we write $w_{ij} := \deg T_{ij}$ for $0 \le i \le r$, $1 \le j \le n_i$ and $u_k = \deg S_k$ for $1 \le k \le m$. Moreover, for $\mu \in \mathbb{Z}_{>0}$, we denote by $\xi(\mu)$ the number of primes in $\{2, \ldots, \mu\}$. The following result provides bounds for the discrete data of the Cox ring.

THEOREM 2.1. *In the above situation, fix the dimension* $d = \dim(X)$ *and the Picard index* $\mu = [\mathrm{Cl}(X) : \mathrm{Pic}(X)]$. *Then we have*

$$u_k \ \le \ \mu \quad \text{for } 1 \le k \le m.$$

Moreover, for the degree γ *of the relations, the weights* w_{ij} *and the exponents* l_{ij}, *where* $0 \le i \le r$ *and* $1 \le j \le n_i$ *one obtains the following.*

(i) *Suppose that* $r = 0, 1$ *holds. Then* $n + m \le d + 1$ *holds and one has the bounds*

$$w_{ij} \ \le \ \mu \quad \text{for } 0 \le i \le r \text{ and } 1 \le j \le n_i,$$

and the Picard index is given by

$$\mu \ = \ \mathrm{lcm}(w_{ij}, u_k;\ 0 \le i \le r, 1 \le j \le n_i, 1 \le k \le m).$$

(ii) *Suppose that* $r \ge 2$ *and* $n_0 = 1$ *hold. Then* $r \le \xi(\mu) - 1$ *and* $n = r + 1$ *and* $m = d - 1$ *hold and one has*

$$w_{i1} \ \le \ \mu^r \quad \text{for } 0 \le i \le r, \qquad l_{01} \cdots l_{r1} \ | \ \mu, \qquad l_{01} \cdots l_{r1} \ | \ \gamma \ \le \ \mu^{r+1},$$

and the Picard index is given by

$$\mu \ = \ \mathrm{lcm}(\gcd(w_{j1};\ j \ne i), u_k;\ 0 \le i \le r, 1 \le k \le m).$$

(iii) *Suppose that* $r \ge 2$ *and* $n_0 > n_1 = 1$ *hold. Then we may assume* $l_{11} > \ldots > l_{r1} \ge 2$, *we have* $r \le \xi(3d\mu) - 1$ *and* $n_0 + m = d$ *and the bounds*

$$w_{01}, \ldots, w_{0n_0} \ \le \ \mu, \qquad l_{01}, \ldots, l_{0n_0} \ < \ 6d\mu,$$

$$w_{11}, l_{21} \ < \ 2d\mu, \qquad w_{21}, l_{11} \ < \ 3d\mu,$$

$$w_{i1} \; < \; 6d\mu, \quad l_{i1} \; < \; 2d\mu \quad \text{for } 2 \le i \le r,$$

$$l_{11} \cdots l_{r1} \; | \; \gamma \; < \; 6d\mu,$$

and the Picard index is given by

$$\mu \; = \; \operatorname{lcm}(w_{0j}, \gcd(w_{11}, \ldots, w_{r1}), u_k; \; 1 \le j \le n_0, 1 \le k \le m).$$

(iv) *Suppose that $n_1 > n_2 = 1$ holds. Then we may assume $l_{21} > \ldots > l_{r1} \ge 2$, we have $r \le \xi(2(d+1)\mu) - 1$ and $n_0 + n_1 + m = d+1$ and the bounds*

$$w_{ij} \; \le \; \mu \quad \text{for } i = 0, 1 \text{ and } 1 \le j \le n_i, \qquad w_{21} \; < \; (d+1)\mu,$$

$$w_{ij}, l_{ij} \; < \; 2(d+1)\mu \quad \text{for } 0 \le i \le r \text{ and } 1 \le j \le n_i,$$

$$l_{21} \cdots l_{r1} \; | \; \gamma \; < \; 2(d+1)\mu,$$

and the Picard index is given by

$$\mu \; = \; \operatorname{lcm}(w_{ij}, u_k; \; 0 \le i \le 1, 1 \le j \le n_i, 1 \le k \le m).$$

(v) *Suppose that $n_2 > 1$ holds and let s be the maximal number with $n_s > 1$. Then one may assume $l_{s+1,1} > \ldots > l_{r1} \ge 2$, we have $r \le \xi((d+2)\mu) - 1$ and $n_0 + \ldots + n_s + m = d + s$ and the bounds*

$$w_{ij} \; \le \; \mu, \quad \text{for } 0 \le i \le s,$$

$$w_{ij}, l_{ij} \; < \; (d+2)\mu \quad \text{for } 0 \le i \le r \text{ and } 1 \le j \le n_i,$$

$$l_{s+1,1} \cdots l_{r1} \; | \; \gamma \; < \; (d+2)\mu,$$

and the Picard index is given by

$$\mu \; = \; \operatorname{lcm}(w_{ij}, u_k; \; 0 \le i \le s, 1 \le j \le n_i, 1 \le k \le m).$$

Putting all the bounds of the theorem together, we obtain the following (raw) bound for the number of deformation types.

Corollary 2.2. *For any pair $(d, \mu) \in \mathbb{Z}^2_{>0}$, the number $\delta(d, \mu)$ of different deformation types of d-dimensional Fano varieties with a complexity one torus action such that $\mathrm{Cl}(X) \cong \mathbb{Z}$ and $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = \mu$ hold is bounded by*

$$\delta(d, \mu) \quad \le \quad (6d\mu)^{2\xi(3d\mu)+d-2} \mu^{\xi(\mu)^2 + 2\xi((d+2)\mu)+2d+2}.$$

Proof. By Theorem 2.1 the discrete data r, $\mathfrak{n}$, L and m occuring in $\mathcal{R}(X)$ are bounded as in the assertion. The continuous data in $\mathcal{R}(X)$ are the coefficients α_{ij}; they stem from the family $A = (a_0, \ldots, a_r)$ of points $a_i \in \mathbb{K}^2$. Varying the a_i provides flat families of Cox rings and hence, by passing to the homogeneous spectra, flat families of the resulting Fano varieties X. $\square$

Corollary 2.3. *Fix $d \in \mathbb{Z}_{>0}$. Then the number $\delta(\mu)$ of different deformation types of d-dimensional Fano varieties with a complexity one torus action, $\mathrm{Cl}(X) \cong \mathbb{Z}$ and Picard index $\mu := [\mathrm{Cl}(X) : \mathrm{Pic}(X)]$ is asymptotically bounded by $\mu^{A\mu^2/\log^2 \mu}$ with a constant A depending only on d.*

Corollary 2.4. *Fix $\mu \in \mathbb{Z}_{>0}$. Then the number $\delta(d)$ of different deformation types of d-dimensional Fano varieties with a complexity one torus action, $\mathrm{Cl}(X) \cong \mathbb{Z}$ and Picard index $\mu := [\mathrm{Cl}(X) : \mathrm{Pic}(X)]$ is asymptotically bounded by d^{Ad} with a constant A depending only on μ.*

We first recall the necessary facts on Cox rings, for details, we refer to [14]. Let X be a complete d-dimensional variety with divisor class group $\mathrm{Cl}(X) \cong \mathbb{Z}$. Then the Cox ring $\mathcal{R}(X)$ is finitely generated and the total coordinate space $\overline{X} := \mathrm{Spec}\,\mathcal{R}(X)$ is a factorial affine variety coming with an action of $\mathbb{K}^*$ defined by the $\mathrm{Cl}(X)$-grading of $\mathcal{R}(X)$. Choose a system $f_1, \ldots, f_\nu$ of homogeneous pairwise nonassociated prime generators for $\mathcal{R}(X)$. This provides an $\mathbb{K}^*$-equivariant embedding

$$\overline{X} \ \to \ \mathbb{K}^\nu, \qquad \overline{x} \ \mapsto \ (f_1(\overline{x}), \ldots, f_\nu(\overline{x})).$$

where $\mathbb{K}^*$ acts diagonally with the weights $w_i = \deg(f_i) \in \mathrm{Cl}(X) \cong \mathbb{Z}$ on $\mathbb{K}^\nu$. Moreover, X is the geometric $\mathbb{K}^*$-quotient of $\widehat{X} := \overline{X} \setminus \{0\}$, and the quotient map $p \colon \widehat{X} \to X$ is a universal torsor. By the local divisor class group $\mathrm{Cl}(X, x)$ of a point $x \in X$, we mean the group of Weil divisors $\mathrm{WDiv}(X)$ modulo those that are principal near x.

Proposition 2.5. *For any $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_\nu) \in \widehat{X}$ the local divisor class group $\mathrm{Cl}(X, x)$ of $x := p(\overline{x})$ is finite of order $\gcd(w_i;\ \overline{x}_i \neq 0)$. The index of the Picard group $\mathrm{Pic}(X)$ in $\mathrm{Cl}(X)$ is given by*

$$[\mathrm{Cl}(X) : \mathrm{Pic}(X)] \quad = \quad \mathrm{lcm}_{x \in X}(|\,\mathrm{Cl}(X, x)|).$$

Suppose that the ideal of $\overline{X} \subseteq \mathbb{K}^\nu$ is generated by $\mathrm{Cl}(X)$-homogeneous polynomials $g_1, \ldots, g_{\nu-d-1}$ of degree $\gamma_j := \deg(g_j)$. Then one obtains

$$-\mathcal{K}_X \ = \ \sum_{i=1}^{\nu} w_i - \sum_{j=1}^{\nu-d-1} \gamma_j, \quad (-\mathcal{K}_X)^d \ = \ \left(\sum_{i=1}^{\nu} w_i - \sum_{j=1}^{\nu-d-1} \gamma_j \right)^d \frac{\gamma_1 \cdots \gamma_{\nu-d-1}}{w_1 \cdots w_\nu}$$

for the anticanonical class $-\mathcal{K}_X \in \mathrm{Cl}(X) \cong \mathbb{Z}$. In particular, X is a Fano variety if and only if the following inequality holds

$$\sum_{j=1}^{\nu-d-1} \gamma_j \ < \ \sum_{i=1}^{\nu} w_i.$$

Proof. Using [14, Prop. 2.2, Thm. 4.19], we observe that X arises from the bunched ring $(R, \mathfrak{F}, \Phi)$, where $R = \mathcal{R}(X)$, $\mathfrak{F} = (f_1, \ldots, f_\nu)$ and $\Phi = \{\mathbb{Q}_{\geq 0}\}$.

The descriptions of local class groups, the Picard index and the anticanonical class are then special cases of [14, Prop. 4.7, Cor. 4.9 and Cor. 4.16]. The anticanonical self-intersection number is easily computed in the ambient weighted projective space $\mathbb{P}(w_1, \ldots, w_\nu)$, use [14, Constr. 3.13, Cor. 4.13]. $\square$

Remark 2.6. If the ideal of $\overline{X} \subseteq \mathbb{K}^\nu$ is generated by $\mathrm{Cl}(X)$-homogeneous polynomials $g_1, \ldots, g_{\nu-d-1}$, then [14, Constr. 3.13, Cor. 4.13] show that X is a well formed complete intersection in the weighted projective space $\mathbb{P}(w_1, \ldots, w_\nu)$ in the sense of [16, Def. 6.9].

We turn back to the case that X comes with a complexity one torus action as at the beginning of this section. We consider the case $n_0 = \ldots = n_r = 1$, that means that each relation $g_{i,j,k}$ of the Cox ring $\mathcal{R}(X)$ depends only on three variables. Then we may write T_i instead of T_{i1} and w_i instead of w_{i1}, etc.. In this setting, we obtain the following bounds for the numbers of possible varieties X (Fano or not).

Proposition 2.7. *For any pair $(d, \mu) \in \mathbb{Z}^2_{>0}$ there is, up to deformation, only a finite number of complete d-dimensional varieties with divisor class group $\mathbb{Z}$, Picard index $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = \mu$ and Cox ring*

$$\mathbb{K}[T_0, \ldots, T_r, S_1, \ldots, S_m] \ / \ \langle \alpha_{i+1,i+2} T_i^{l_i} + \alpha_{i+2,i} T_{i+1}^{l_{i+1}} + \alpha_{i,i+1} T_{i+2}^{l_{i+2}};\ 0 \le i \le r-2 \rangle.$$

In this situation we have $r \le \xi(\mu) - 1$. Moreover, for the weights $w_i := \deg T_i$, where $0 \le i \le r$ and $u_k := \deg S_k$, where $1 \le k \le m$, the exponents l_i and the degree $\gamma := l_0 w_0$ of the relation one has

$$l_0 \cdots l_r \ \mid \ \gamma, \qquad l_0 \cdots l_r \ \mid \ \mu, \qquad w_i \ \le \ \mu^{\xi(\mu)-1}, \qquad u_k \ \le \ \mu.$$

Proof. Consider the total coordinate space $\overline{X} \subseteq \mathbb{K}^{r+1+n}$ and the universal torsor $p\colon \widehat{X} \to X$ as discussed before. For each $0 \le i \le r$ fix a point $\overline{x}(i) = (\overline{x}_0, \ldots, \overline{x}_r, 0, \ldots, 0)$ in $\widehat{X}$ such that $\overline{x}_i = 0$ and $\overline{x}_j \ne 0$ for $j \ne i$ hold. Then, denoting $x(i) := p(\overline{x}(i))$, we obtain

$$\gcd(w_j; j \ne i) \ = \ |\operatorname{Cl}(X, x(i))| \ \mid \ \mu.$$

Consider i, j with $j \ne i$. Since all relations are homogeneous of the same degree, we have $l_i w_i = l_j w_j$. Moreover, by the admissibility condition, l_i and l_j are coprime. We conclude $l_i | w_j$ for all $j \ne i$ and hence $l_i | \gcd(w_j;\ j \ne i)$. This implies

$$l_0 \cdots l_r \ \mid \ l_0 w_0 \ = \ \gamma, \qquad\qquad l_0 \cdots l_r \ \mid \ \mu.$$

We turn to the bounds for the w_i, and first verify $w_0 \le \mu^r$. Using the relation $l_i w_i = l_0 w_0$, we obtain for every l_i a presentation

$$l_i \ = \ l_0 \cdot \frac{w_0 \cdots w_{i-1}}{w_1 \cdots w_i} \ = \ \eta_i \cdot \frac{\gcd(w_0, \ldots, w_{i-1})}{\gcd(w_0, \ldots, w_i)}$$

with suitable integers $1 \le \eta_i \le \mu$. In particular, the very last fraction is bounded by μ. This gives the desired estimate:

$$w_0 = \frac{w_0}{\gcd(w_0, w_1)} \cdot \frac{\gcd(w_0, w_1)}{\gcd(w_0, w_1, w_2)} \cdots \frac{\gcd(w_0, \ldots, w_{r-2})}{\gcd(w_0, \ldots, w_{r-1})} \cdot \gcd(w_0, \ldots, w_{r-1}) \le \mu^r.$$

Similarly, we obtain $w_i \le \mu^r$ for $1 \le i \le r$. Then we only have to show that $r+1$ is bounded by $\xi(\mu)$, but this follows immediately from the fact that $l_0, \ldots, l_r$ are pairwise coprime.

Finally, to estimate the u_k, consider the points $\overline{x}(k) \in \widehat{X}$ having the $(r+k)$-th coordinate one and all others zero. Set $x(k) := p(\overline{x}(k))$. Then $\mathrm{Cl}(X, x(k))$ is of order u_k, which implies $u_k \le \mu$. □

Lemma 2.8. *Consider the ring $\mathbb{K}[T_{ij};\ 0 \le i \le 2,\ 1 \le j \le n_i][S_1, \ldots, S_k]/\langle g \rangle$ where $n_0 \ge n_1 \ge n_2 \ge 1$ holds. Suppose that g is homogeneous with respect to a $\mathbb{Z}$-grading of $\mathbb{K}[T_{ij}, S_k]$ given by $\deg T_{ij} = w_{ij} \in \mathbb{Z}_{>0}$ and $\deg S_k = u_k \in \mathbb{Z}_{>0}$, and assume*

$$\deg g \ < \ \sum_{i=0}^{2} \sum_{j=1}^{n_i} w_{ij} \ + \ \sum_{i=1}^{m} u_i.$$

Let $\mu \in \mathbb{Z}_{>1}$, assume $w_{ij} \le \mu$ whenever $n_i > 1$, $1 \le j \le n_i$ and $u_k \le \mu$ for $1 \le k \le m$ and set $d := n_0 + n_1 + n_2 + m - 2$. Depending on the shape of g, one obtains the following bounds.

(i) *Suppose that $g = \eta_0 T_{01}^{l_{01}} \cdots T_{0n_0}^{l_{0n_0}} + \eta_1 T_{11}^{l_{11}} + \eta_2 T_{21}^{l_{21}}$ with $n_0 > 1$ and coefficients $\eta_i \in \mathbb{K}^*$ holds, we have $l_{11} \ge l_{21} \ge 2$ and l_{11}, l_{21} are coprime. Then, one has*

$$w_{11}, l_{21} \ < \ 2d\mu, \qquad w_{21}, l_{11} \ < \ 3d\mu, \qquad \deg g \ < \ 6d\mu.$$

(ii) *Suppose that $g = \eta_0 T_{01}^{l_{01}} \cdots T_{0n_0}^{l_{0n_0}} + \eta_1 T_{11}^{l_{11}} \cdots T_{1n_1}^{l_{1n_1}} + \eta_2 T_{21}^{l_{21}}$ with $n_1 > 1$ and coefficients $\eta_i \in \mathbb{K}^*$ holds and we have $l_{21} \ge 2$. Then one has*

$$w_{21} \ < \ (d+1)\mu, \qquad \deg g \ < \ 2(d+1)\mu.$$

Proof. We prove (i). Set for short $c := (n_0 + m)\mu = d\mu$. Then, using homogeneity of g and the assumed inequality, we obtain

$$l_{11} w_{11} \ = \ l_{21} w_{21} \ = \ \deg g \ < \ \sum_{i=0}^{2} \sum_{j=1}^{n_i} w_{ij} + \sum_{i=1}^{m} u_i \ \le \ c + w_{11} + w_{21}.$$

Since l_{11} and l_{21} are coprime, we have $l_{11} > l_{21} \ge 2$. Plugging this into the above inequalities, we arrive at $2w_{11} < c + w_{21}$ and $w_{21} < c + w_{11}$. We conclude

$w_{11} < 2c$ and $w_{21} < 3c$. Moreover, $l_{11}w_{11} = l_{21}w_{21}$ and $\gcd(l_{11}, l_{21}) = 1$ imply $l_{11}|w_{21}$ and $l_{21}|w_{11}$. This shows $l_{11} < 3c$ and $l_{21} < 2c$. Finally, we obtain

$$\deg g \ < \ c + w_{11} + w_{21} \ < \ 6c.$$

We prove (ii). Here we set $c := (n_0 + n_1 + m)\mu = (d+1)\mu$. Then the assumed inequality gives

$$l_{21}w_{21} \ = \ \deg g \ < \ \sum_{i=0}^{1}\sum_{j=1}^{n_i} w_{ij} + \sum_{i=1}^{m} u_i + w_{21} \ \leq \ c + w_{21}.$$

Since we assumed $l_{21} \geq 2$, we can conclude $w_{21} < c$. This in turn gives us $\deg g < 2c$ for the degree of the relation. □

Proof of Theorem 2.1. As before, we denote by $\overline{X} \subseteq \mathbb{K}^{n+m}$ the total coordinate space and by $p\colon \widehat{X} \to X$ the universal torsor.

We first consider the case that X is a toric variety. Then the Cox ring is a polynomial ring, $\mathcal{R}(X) = \mathbb{K}[S_1, \ldots, S_m]$. For each $1 \leq k \leq m$, consider the point $\overline{x}(k) \in \widehat{X}$ having the k-th coordinate one and all others zero and set $x(k) := p(\overline{x}(k))$. Then, by Proposition 2.5, the local class group $\mathrm{Cl}(X, x(k))$ is of order u_k where $u_k := \deg S_k$. This implies $u_k \leq \mu$ for $1 \leq k \leq m$ and settles Assertion (i).

Now we treat the non-toric case, which means $r \geq 2$. Note that we have $n \geq 3$. The case $n_0 = 1$ is done in Proposition 2.7. So, we are left with $n_0 > 1$. For every i with $n_i > 1$ and every $1 \leq j \leq n_i$, there is the point $\overline{x}(i,j) \in \widehat{X}$ with ij-coordinate T_{ij} equal to one and all others equal to zero, and thus we have the point $x(i,j) := p(\overline{x}(i,j)) \in X$. Moreover, for every $1 \leq k \leq m$, we have the point $\overline{x}(k) \in \widehat{X}$ having the k-coordinate S_k equal to one and all others zero; we set $x(k) := p(\overline{x}(k))$. Proposition 2.5 provides the bounds

$$w_{ij} \ = \ \deg T_{ij} \ = \ |\,\mathrm{Cl}(X, x(i,j))| \ \leq \ \mu \qquad \text{for } n_i > 1,\ 1 \leq j \leq n_i,$$

$$u_k \ = \ \deg S_k \ = \ |\,\mathrm{Cl}(X, x(k))| \ \leq \ \mu \qquad \text{for } 1 \leq k \leq m.$$

Let $0 \leq s \leq r$ be the maximal number with $n_s > 1$. Then $g_{s-2,s-1,s}$ is the last polynomial such that each of its three monomials depends on more than one variable. For any $t \geq s$, we have the "cut ring"

$$R_t \quad := \quad \mathbb{K}[T_{ij};\ 0 \leq i \leq t,\ 1 \leq j \leq n_i][S_1, \ldots, S_m] \ / \ \langle g_{i,i+1,i+2};\ 0 \leq i \leq t-2 \rangle$$

where the relations $g_{i,i+1,i+2}$ depend on only three variables as soon as $i > s$

holds. For the degree γ of the relations we have

$$\begin{aligned}(r-1)\gamma &= (t-1)\gamma \;+\; (r-t)\gamma \\ &= (t-1)\gamma \;+\; l_{t+1,1}w_{t+1,1}+\ldots+l_{r1}w_{r1} \\ &< \sum_{i=0}^{r}\sum_{j=1}^{n_i} w_{ij} \;+\; \sum_{i=1}^{m} u_i \\ &= \sum_{i=0}^{t}\sum_{j=1}^{n_i} w_{ij} \;+\; w_{t+1,1}+\ldots+w_{r1} \;+\; \sum_{i=1}^{m} u_i.\end{aligned}$$

Since $l_{i1}w_{i1} > w_{i1}$ holds in particular for $t+1 \le i \le r$, we derive from this the inequality

$$\gamma \;<\; \frac{1}{t-1}\left(\sum_{i=0}^{t}\sum_{j=1}^{n_i} w_{ij} \;+\; \sum_{i=1}^{m} u_i\right).$$

To obtain the bounds in Assertions (iii) and (iv), we consider the cut ring R_t with $t = 2$ and apply Lemma 2.8; note that we have $d = n_0+n_1+n_2+m-2$ for the dimension $d = \dim(X)$ and that $l_{22} \ge 0$ is due to the fact that X is non-toric. The bounds $w_{ij}, l_{0j} < 6d\mu$ in Assertion (iii) follow from $l_{ij}w_{ij} = \gamma < 6d\mu$ and $l_{i1} < 2d\mu$ follows from $l_{i1} \mid w_{21}$ for $3 \le i \le r$. Moreover, $l_{i1} \mid w_{11}$ for $2 \le i \le r$ implies $l_{11}\cdots l_{r1} \mid \gamma = l_{11}w_{11}$. Similarly $w_{ij}, l_{ij} < 2(d+1)\mu$ in Assertion (iv) follow from $l_{ij}w_{ij} = \gamma < 2(d+1)d\mu$ and $l_{21}\cdots l_{r1} \mid \gamma = l_{21}w_{21}$ follows from $l_{i1} \mid w_{21}$ for $3 \le i \le r$. The bounds on r in (iii) in (iv) are as well consequences of the admissibility condition.

To obtain the bounds in Assertion (v), we consider the cut ring R_t with $t = s$. Using $n_i = 1$ for $i \ge t+1$, we can estimate the degree of the relation as follows:

$$\gamma \;\le\; \frac{(n_0+\ldots+n_t+m)\mu}{t-1} \;=\; \frac{(d+t)\mu}{t-1} \;\le\; (d+2)\mu.$$

Since we have $w_{ij}l_{ij} \le \deg g_0$ for any $0 \le i \le r$ and any $1 \le j \le n_i$, we see that all w_{ij} and l_{ij} are bounded by $(d+2)\mu$. As before, $l_{s+1,1}\cdots l_{r1} \mid \gamma$ is a consequence of $l_{i1} \mid \gamma$ for $i = s+2,\ldots,r$ and also the bound on r follows from the admissibility condition.

Finally, we have to express the Picard index μ in terms of the weights w_{ij} and u_k as claimed in the Assertions. This is a direct application of the formula of Proposition 2.5. Observe that it suffices to work with the p-images of the following points: For every $0 \le i \le r$ with $n_i > 1$ take a point $\overline{x}(i,j) \in \widehat{X}$ with ij-coordinate T_{ij} equal to one and all others equal to zero, for every $0 \le i \le r$ with $n_i = 1$ whenever $n_i = 1$ take $\overline{x}(i,j) \in \widehat{X}$ with ij-coordinate T_{ij} equal to zero, all other T_{st} equal to one and coordinates S_k equal to zero, and, for every $1 \le k \le m$, take a point $\overline{x}(k) \in \widehat{X}$ having the k-coordinate S_k equal to one and all others zero. $\square$

We conclude the section with discussing some aspects of the not necessarily Fano varieties of Proposition 2.7. Recall that we considered admissible triples $(A, \mathfrak{n}, L)$ with $n_0 = \ldots = n_r = 1$ and thus rings R of the form

$$\mathbb{K}[T_0, \ldots, T_r, S_1, \ldots, S_m] \;/\; \langle \alpha_{i+1,i+2} T_i^{l_i} + \alpha_{i+2,i} T_{i+1}^{l_{i+1}} + \alpha_{i,i+1} T_{i+2}^{l_{i+2}};\; 0 \le i \le r-2 \rangle.$$

Proposition 2.9. *Suppose that the ring R as above is the Cox ring of a non-toric variety X with $\mathrm{Cl}(X) = \mathbb{Z}$. Then we have $m \ge 1$ and $\mu := [\mathrm{Cl}(X) : \mathrm{Pic}(X)] \ge 30$. Moreover, if X is a surface, then we have $m = 1$ and $w_i = l_i^{-1} l_0 \cdots l_r$.*

Proof. The homogeneity condition $l_i w_i = l_j w_j$ together with the admissibility condition $\gcd(l_i, l_j) = 1$ for $0 \le i \ne j \le r$ gives us $l_i \mid \gcd(w_j; j \ne i)$. Moreover, by Proposition 1.11, every set of $m + r$ weights w_i has to generate the class group $\mathbb{Z}$, so they must have greatest common divisor one. Since X is non-toric, $l_i \ge 2$ holds and we obtain $m \ge 1$. To proceed, we infer $l_0 \cdots l_r \mid \mu$ and $l_0 \cdots l_r \mid \deg g_{ijk}$ from Proposition 2.5. As a consequence, the minimal value for μ and $\deg g_{ijk}$ is obviously $2 \cdot 3 \cdot 5 = 30$. what really can be received as the following example shows. Note that if X is a surface we have $m = 1$ and $\gcd(w_i; 0 \le i \le r) = 1$. Thus, $l_i w_i = l_j w_j$ gives us $\deg g_{ijk} = l_0 \cdots l_r$ and $w_i = l_i^{-1} l_0 \cdots l_r$. □

The bound $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] \ge 30$ given in the above proposition is even sharp; the surface discussed below realizes it.

Example 2.10. Consider X with $\mathcal{R}(X) = \mathbb{K}[T_0, T_1, T_2, T_3]/\langle g \rangle$ with $g = T_0^2 + T_1^3 + T_2^5$ and the grading

$$\deg T_0 \;=\; 15, \quad \deg T_1 \;=\; 10, \quad \deg T_2 \;=\; 6, \quad \deg T_3 \;=\; 1.$$

Then we have $\gcd(15, 10) = 5$, $\gcd(15, 6) = 3$ and $\gcd(10, 6) = 2$ and therefore $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 30$. Further X is Fano because of

$$\deg g \;=\; 30 \;<\; 32 \;=\; \deg T_0 + \ldots + \deg T_3.$$

Let us have a look at the geometric meaning of the condition $n_0 = \ldots = n_r = 1$. For a variety X with an action of a torus T, we denote by $X_0 \subseteq X$ the union of all orbits with at most finite isotropy. Then there is a possibly non-separated orbit space X_0/T; we call it the maximal orbit space. From [15], we infer that $n_0 = \ldots = n_r = 1$ holds if and only if X_0/T is separated. Combining this with Propositions 2.7 and 2.9 gives the following.

Corollary 2.11. *For any pair $(d, \mu) \in \mathbb{Z}^2_{>0}$ there is, up to deformation, only a finite number of d-dimensional complete varieties X with a complexity one torus action having divisor class group $\mathbb{Z}$, Picard index $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = \mu$ and maximal orbit space $\mathbb{P}_1$ and for each of these varieties the complement $X \setminus X_0$ contains divisors.*

Finally, we present a couple of examples showing that there are also non-Fano varieties with a complexity one torus action having divisor class group $\mathbb{Z}$ and maximal orbit space $\mathbb{P}_1$.

Example 2.12. Consider X with $\mathcal{R}(X) = \mathbb{K}[T_0, T_1, T_2, T_3]/\langle g \rangle$ with $g = T_0^2 + T_1^3 + T_2^7$ and the grading

$$\deg T_0 \ = \ 21, \quad \deg T_1 \ = \ 14, \quad \deg T_2 \ = \ 6, \quad \deg T_3 \ = \ 1.$$

Then we have $\gcd(21, 14) = 7$, $\gcd(21, 6) = 3$ and $\gcd(14, 6) = 2$ and therefore $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 42$. Moreover, X is not Fano, because its canonical class $\mathcal{K}_X$ is trivial

$$\mathcal{K}_X \ = \ \deg g - \deg T_0 - \ldots - \deg T_3 \ = \ 0.$$

Example 2.13. Consider X with $\mathcal{R}(X) = \mathbb{K}[T_0, T_1, T_2, T_3]/\langle g \rangle$ with $g = T_0^2 + T_1^3 + T_2^{11}$ and the grading

$$\deg T_0 \ = \ 33, \quad \deg T_1 \ = \ 22, \quad \deg T_2 \ = \ 6, \quad \deg T_3 \ = \ 1.$$

Then we have $\gcd(22, 33) = 11$, $\gcd(33, 6) = 3$ and $\gcd(22, 6) = 2$ and therefore $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 66$. The canonical class $\mathcal{K}_X$ of X is even ample:

$$\mathcal{K}_X \ = \ \deg g - \deg T_0 - \ldots - \deg T_3 \ = \ 4.$$

The following example shows that the Fano assumption is essential for the finiteness results in Theorem 2.1.

Remark 2.14. For any pair p, q of coprime positive integers, we obtain a locally factorial $\mathbb{K}^*$-surface $X(p, q)$ with $\mathrm{Cl}(X) = \mathbb{Z}$ and Cox ring

$$\mathcal{R}(X(p,q)) \ = \ \mathbb{K}[T_{01}, T_{02}, T_{11}, T_{21}] \ / \ \langle g \rangle, \qquad\qquad g \ = \ T_{01}T_{02}^{pq-1} + T_{11}^q + T_{21}^p;$$

the $\mathrm{Cl}(X)$-grading is given by $\deg T_{01} = \deg T_{02} = 1$, $\deg T_{11} = p$ and $\deg T_{21} = q$. Note that $\deg g = pq$ holds and for $p, q \geq 3$, the canonical class $\mathcal{K}_X$ satisfies

$$\mathcal{K}_X \ = \ \deg g - \deg T_{01} - \deg T_{02} - \deg T_{11} - \deg T_{21} \ = \ pq - 2 - p - q \ \geq \ 0.$$

3 Classification results

In this section, we give classification results for Fano varieties X with $\mathrm{Cl}(X) \cong \mathbb{Z}$ that come with a complexity one torus action; note that they are necessarily rational. The procedure to obtain classification lists for prescribed dimension $d = \dim X$ and Picard index $\mu = [\mathrm{Cl}(X) : \mathrm{Pic}(X)]$ is always the following. By Theorem 1.9, we know that their Cox rings are of the form $\mathcal{R}(X) \cong R(A, \mathfrak{n}, L)[S_1, \ldots, S_m]$ with admissible triples $(A, \mathfrak{n}, L)$. Note that for the family $A = (a_0, \ldots, a_r)$ of points $a_i \in \mathbb{K}^2$, we may assume

$$a_0 \ = \ (1,0), \qquad a_1 \ = \ (1,1), \qquad a_2 \ = \ (0,1).$$

The bounds on the input data of $(A, \mathfrak{n}, L)$ provided by Theorem 2.1 as well as the criteria of Propositions 1.11 and 2.5 allow us to generate all the possible Cox rings $\mathcal{R}(X)$ of the Fano varieties X in question for fixed dimension d and Picard index μ. Note that X can be reconstructed from $\mathcal{R}(X) = R(A, \mathfrak{n}, L)[S_1, \ldots, S_n]$ as the homogeneous spectrum with respect to the $\mathrm{Cl}(X)$-grading. Thus X is classified by its Cox ring $\mathcal{R}(X)$.

In the following tables, we present the Cox rings as $\mathbb{K}[T_1, \ldots, T_s]$ modulo relations and fix the $\mathbb{Z}$-gradings by giving the weight vector $(w_1, \ldots, w_s)$, where $w_i := \deg T_i$. The first classification result concerns surfaces.

Theorem 3.1. *Let X be a non-toric Fano surface with an effective $\mathbb{K}^*$-action such that $\mathrm{Cl}(X) = \mathbb{Z}$ and $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] \le 6$ hold. Then its Cox ring is precisely one of the following.*

$[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 1$

No.	$\mathcal{R}(X)$	$(w_1, \ldots, w_4)$	$(-K_X)^2$
1	$\mathbb{K}[T_1, \ldots, T_4]/\langle T_1T_2^5 + T_3^3 + T_4^2 \rangle$	$(1, 1, 2, 3)$	1

$[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 2$

No.	$\mathcal{R}(X)$	$(w_1, \ldots, w_4)$	$(-K_X)^2$
2	$\mathbb{K}[T_1, \ldots, T_4]/\langle T_1^4T_2 + T_3^3 + T_4^2 \rangle$	$(1, 2, 2, 3)$	2

$[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 3$

No.	$\mathcal{R}(X)$	$(w_1, \ldots, w_4)$	$(-K_X)^2$
3	$\mathbb{K}[T_1, \ldots, T_4]/\langle T_1^3T_2 + T_3^3 + T_4^2 \rangle$	$(1, 3, 2, 3)$	3
4	$\mathbb{K}[T_1, \ldots, T_4]/\langle T_1T_2^3 + T_3^5 + T_4^2 \rangle$	$(1, 3, 2, 5)$	1/3
5	$\mathbb{K}[T_1, \ldots, T_4]/\langle T_1^7T_2 + T_3^5 + T_4^2 \rangle$	$(1, 3, 2, 5)$	1/3

$[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 4$

No.	$\mathcal{R}(X)$	$(w_1, \ldots, w_4)$	$(-K_X)^2$
6	$\mathbb{K}[T_1, \ldots, T_4]/\langle T_1^2T_2 + T_3^3 + T_4^2 \rangle$	$(1, 4, 2, 3)$	4
7	$\mathbb{K}[T_1, \ldots, T_4]/\langle T_1^6T_2 + T_3^5 + T_4^2 \rangle$	$(1, 4, 2, 5)$	1

$[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 5$

No.	$\mathcal{R}(X)$	$(w_1, \ldots, w_4)$	$(-K_X)^2$

8	$\mathbb{K}[T_1,\ldots,T_4]/\langle T_1T_2+T_3^3+T_4^2\rangle$	$(1,5,2,3)$	5
9	$\mathbb{K}[T_1,\ldots,T_4]/\langle T_1^5T_2+T_3^5+T_4^2\rangle$	$(1,5,2,5)$	$9/5$
10	$\mathbb{K}[T_1,\ldots,T_4]/\langle T_1^9T_2+T_3^7+T_4^2\rangle$	$(1,5,2,7)$	$1/5$
11	$\mathbb{K}[T_1,\ldots,T_4]/\langle T_1^7T_2+T_3^4+T_4^3\rangle$	$(1,5,3,4)$	$1/5$

$[\mathrm{Cl}(X):\mathrm{Pic}(X)]=6$

No.	$\mathcal{R}(X)$	$(w_1,\ldots,w_4)$	$(-K_X)^2$
12	$\mathbb{K}[T_1,\ldots,T_4]/\langle T_1^4T_2+T_3^5+T_4^2\rangle$	$(1,6,2,5)$	$8/3$
13	$\mathbb{K}[T_1,\ldots,T_4]/\langle T_1^8T_2+T_3^7+T_4^2\rangle$	$(1,6,2,7)$	$2/3$
14	$\mathbb{K}[T_1,\ldots,T_4]/\langle T_1^6T_2+T_3^4+T_4^3\rangle$	$(1,6,3,4)$	$2/3$
15	$\mathbb{K}[T_1,\ldots,T_4]/\langle T_1^9T_2+T_3^3+T_4^2\rangle$	$(1,3,4,6)$	$2/3$

Proof. As mentioned, Theorems 1.9, 2.1 and Propositions 1.11, 2.5 produce a list of all Cox rings of surfaces with the prescribed data. Doing this computation, we obtain the list of the assertion. Note that none of the Cox rings listed is a polynomial ring and hence none of the resulting surfaces X is a toric variety. To show that different members of the list are not isomorphic to each other, we use the following two facts. Firstly, observe that any two minimal systems of homogeneous generators of the Cox ring have (up to reordering) the same list of degrees, and thus the list of generator degrees is invariant under isomorphism (up to reordering). Secondly, by Construction 1.7, the exponents $l_{ij}>1$ are precisely the orders of the non-trivial isotropy groups of one-codimensional orbits of the action of the torus T on X. Using both principles and going through the list, we see that different members X cannot be T-equivariantly isomorphic to each other. Since all listed X are non-toric, the effective complexity one torus action on each X corresponds to a maximal torus in the linear algebraic group $\mathrm{Aut}(X)$. Any two maximal tori in the automorphism group are conjugate, and thus we can conclude that two members are isomorphic if and only if they are T-equivariantly isomorphic. □

We remark that in [28, Section 4], log del Pezzo surfaces with an effective $\mathbb{K}^*$-action and Picard number 1 and Gorenstein index less than 4 were classified. The above list contains six such surfaces, namely no. 1-4, 6 and 8; these are exactly the ones where the maximal exponents of the monomials form a platonic triple, i.e., are of the form $(1,k,l)$, $(2,2,k)$, $(2,3,3)$, $(2,3,4)$ or $(2,3,5)$. The remaining ones, i.e., no. 5, 7, and 9-15 have non-log-terminal and thus non-rational singularities; to check this one may compute the resolutions via resolution of the ambient weighted projective space as in [14, Ex. 7.5].

With the same scheme of proof as in the surface case, one establishes the following classification results on Fano threefolds.

Theorem 3.2. *Let X be a three-dimensional locally factorial non-toric Fano variety with an effective two torus action such that $\mathrm{Cl}(X) = \mathbb{Z}$ holds. Then its Cox ring is precisely one of the following.*

No.	$\mathcal{R}(X)$	$(w_1, \ldots, w_5)$	$(-K_X)^3$
1	$\mathbb{K}[T_1, \ldots, T_5] \ / \ \langle T_1T_2^5 + T_3^3 + T_4^2 \rangle$	$(1, 1, 2, 3, 1)$	8
2	$\mathbb{K}[T_1, \ldots, T_5] \ / \ \langle T_1T_2T_3^4 + T_4^3 + T_5^2 \rangle$	$(1, 1, 1, 2, 3)$	8
3	$\mathbb{K}[T_1, \ldots, T_5] \ / \ \langle T_1T_2^2T_3^3 + T_4^3 + T_5^2 \rangle$	$(1, 1, 1, 2, 3)$	8
4	$\mathbb{K}[T_1, \ldots, T_5] \ / \ \langle T_1T_2 + T_3T_4 + T_5^2 \rangle$	$(1, 1, 1, 1, 1)$	54
5	$\mathbb{K}[T_1, \ldots, T_5] \ / \ \langle T_1T_2^2 + T_3T_4^2 + T_5^3 \rangle$	$(1, 1, 1, 1, 1)$	24
6	$\mathbb{K}[T_1, \ldots, T_5] \ / \ \langle T_1T_2^3 + T_3T_4^3 + T_5^4 \rangle$	$(1, 1, 1, 1, 1)$	4
7	$\mathbb{K}[T_1, \ldots, T_5] \ / \ \langle T_1T_2^3 + T_3T_4^3 + T_5^2 \rangle$	$(1, 1, 1, 1, 2)$	16
8	$\mathbb{K}[T_1, \ldots, T_5] \ / \ \langle T_1T_2^5 + T_3T_4^5 + T_5^2 \rangle$	$(1, 1, 1, 1, 3)$	2
9	$\mathbb{K}[T_1, \ldots, T_5] \ / \ \langle T_1T_2^5 + T_3^3T_4^3 + T_5^2 \rangle$	$(1, 1, 1, 1, 3)$	2

The singular threefolds listed in this theorem are rational degenerations of smooth Fano threefolds from [18]. The (smooth) general Fano threefolds of the corresponding families are non-rational see [12] for no. 1-3, [8] for no. 5, [20] for no. 6, [30, 29] for no. 7 and [19] for no. 8-9. Even if one allows certain mild singularities, one still has non-rationality in some cases, see [13], [9, 25], [10], [6].

Theorem 3.3. *Let X be a three-dimensional non-toric Fano variety with an effective two torus action such that $\mathrm{Cl}(X) = \mathbb{Z}$ and $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 2$ hold. Then its Cox ring is precisely one of the following.*

No.	$\mathcal{R}(X)$	$(w_1, \ldots, w_5)$	$(-K_X)^3$
1	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1^4T_2 + T_3^3 + T_4^2 \rangle$	$(1, 2, 2, 3, 1)$	27/2
2	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1^4T_2^3 + T_3^5 + T_4^2 \rangle$	$(1, 2, 2, 5, 1)$	1/2
3	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1^8T_2 + T_3^5 + T_4^2 \rangle$	$(1, 2, 2, 5, 1)$	1/2
4	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1^4T_2 + T_3^3 + T_4^2 \rangle$	$(1, 2, 2, 3, 2)$	16

5	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2^3+T_3^5+T_4^2\rangle$	$(1,2,2,5,2)$	2
6	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^8T_2+T_3^5+T_4^2\rangle$	$(1,2,2,5,2)$	2
7	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5+T_3^3+T_4^2\rangle$	$(1,1,2,3,2)$	27/2
8	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^9+T_3^5+T_4^2\rangle$	$(1,1,2,5,2)$	1/2
9	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2^7+T_3^5+T_4^2\rangle$	$(1,1,2,5,2)$	1/2
10	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^{11}+T_3^3+T_4^2\rangle$	$(1,1,4,6,1)$	1/2
11	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^5T_2^7+T_3^3+T_4^2\rangle$	$(1,1,4,6,1)$	1/2
12	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^{11}+T_3^3+T_4^2\rangle$	$(1,1,4,6,2)$	2
13	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^5T_2^7+T_3^3+T_4^2\rangle$	$(1,1,4,6,2)$	2
14	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^5+T_3^3+T_4^2\rangle$	$(1,2,4,6,1)$	2
15	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^{10}T_2+T_3^3+T_4^2\rangle$	$(1,2,4,6,1)$	2
16	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^2+T_3^3+T_4^2\rangle$	$(2,2,2,3,1)$	16
17	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^4+T_3^5+T_4^2\rangle$	$(2,2,2,5,1)$	2
18	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^3+T_3^5+T_4^2\rangle$	$(2,2,2,5,1)$	2
19	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^2+T_3T_4+T_5^3\rangle$	$(1,1,1,2,1)$	81/2
20	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^4+T_3T_4^2+T_5^5\rangle$	$(1,1,1,2,1)$	5/2
21	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^3+T_3T_4^2+T_5^5\rangle$	$(1,1,1,2,1)$	5/2
22	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^3+T_3^2T_4+T_5^4\rangle$	$(1,1,1,2,1)$	16
23	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^4+T_3^3T_4+T_5^5\rangle$	$(1,1,1,2,1)$	5/2
24	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^3+T_3^3T_4+T_5^5\rangle$	$(1,1,1,2,1)$	5/2
25	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^3+T_3^2T_4+T_5^2\rangle$	$(1,1,1,2,2)$	27
26	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5+T_3^2T_4^2+T_5^3\rangle$	$(1,1,1,2,2)$	3/2
27	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5+T_3^4T_4+T_5^3\rangle$	$(1,1,1,2,2)$	3/2
28	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^4+T_3^4T_4+T_5^3\rangle$	$(1,1,1,2,2)$	3/2
29	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5+T_3^4T_4+T_5^2\rangle$	$(1,1,1,2,3)$	8
30	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2^3+T_3^4T_4+T_5^2\rangle$	$(1,1,1,2,3)$	8
31	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^7+T_3^2T_4^3+T_5^2\rangle$	$(1,1,1,2,4)$	1
32	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2^5+T_3^2T_4^3+T_5^2\rangle$	$(1,1,1,2,4)$	1
33	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^7+T_3^6T_4+T_5^2\rangle$	$(1,1,1,2,4)$	1
34	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2^5+T_3^6T_4+T_5^2\rangle$	$(1,1,1,2,4)$	1
35	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^3+T_3T_4+T_5^4\rangle$	$(1,1,2,2,1)$	27

36	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5+T_3T_4^2+T_5^6\rangle$	$(1,1,2,2,1)$	$3/2$
37	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^3+T_3T_4+T_5^2\rangle$	$(1,1,2,2,2)$	16
38	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5+T_3T_4^2+T_5^3\rangle$	$(1,1,2,2,2)$	6
39	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^4+T_3T_4^2+T_5^3\rangle$	$(1,1,2,2,2)$	6
40	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2^3+T_3T_4^2+T_5^2\rangle$	$(1,1,2,2,2)$	$27/2$
41	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2^5+T_3T_4^3+T_5^2\rangle$	$(1,1,2,2,2)$	32
42	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5+T_3T_4^2+T_5^2\rangle$	$(1,1,2,2,3)$	4
43	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^7+T_3T_4^3+T_5^2\rangle$	$(1,1,2,2,4)$	32
44	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^9+T_3T_4^4+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$
45	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^9+T_3^2T_4^3+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$
46	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2^7+T_3T_4^4+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$
47	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2^7+T_3^2T_4^3+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$
48	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^5T_2^5+T_3T_4^4+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$
49	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^5T_2^5+T_3^2T_4^3+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$
50	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3T_4+T_5^3\rangle$	$(1,2,1,2,1)$	48
51	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2+T_3^2T_4+T_5^4\rangle$	$(1,2,1,2,1)$	27
52	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^2+T_3T_4^2+T_5^5\rangle$	$(1,2,1,2,1)$	10
53	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^2+T_3^3T_4+T_5^5\rangle$	$(1,2,1,2,1)$	10
54	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2+T_3^3T_4+T_5^5\rangle$	$(1,2,1,2,1)$	10
55	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2+T_3^4T_4+T_5^6\rangle$	$(1,2,1,2,1)$	$3/2$
56	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2+T_3^2T_4+T_5^2\rangle$	$(1,2,1,2,2)$	32
57	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^2+T_3^4T_4+T_5^3\rangle$	$(1,2,1,2,2)$	6
58	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2+T_3^4T_4+T_5^3\rangle$	$(1,2,1,2,2)$	6
59	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2+T_3^4T_4+T_5^2\rangle$	$(1,2,1,2,3)$	$27/2$
60	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^3+T_3^2T_4^3+T_5^2\rangle$	$(1,2,1,2,4)$	4
61	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^3+T_3^6T_4+T_5^2\rangle$	$(1,2,1,2,4)$	4
62	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^6T_2+T_3^6T_4+T_5^2\rangle$	$(1,2,1,2,4)$	4
63	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2^3+T_3^4T_4^3+T_5^2\rangle$	$(1,2,1,2,5)$	$1/2$
64	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^8T_2+T_3^4T_4^3+T_5^2\rangle$	$(1,2,1,2,5)$	$1/2$
65	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^8T_2+T_3^8T_4+T_5^2\rangle$	$(1,2,1,2,5)$	$1/2$
66	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2+T_3T_4+T_5^4\rangle$	$(1,2,2,2,1)$	32
67	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2+T_3T_4^2+T_5^6\rangle$	$(1,2,2,2,1)$	6

68	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2+T_3T_4^2+T_5^2\rangle$	$(1,2,2,2,3)$	16
69	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2^3+T_3T_4^4+T_5^2\rangle$	$(1,2,2,2,5)$	2
70	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2^3+T_3^2T_4^3+T_5^2\rangle$	$(1,2,2,2,5)$	2
71	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^8T_2+T_3T_4^4+T_5^2\rangle$	$(1,2,2,2,5)$	2
72	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^8T_2+T_3^2T_4^3+T_5^2\rangle$	$(1,2,2,2,5)$	2
73	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2T_3^{10}+T_4^3+T_5^2\rangle$	$(1,1,1,4,6)$	1/2
74	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^2T_3^9+T_4^3+T_5^2\rangle$	$(1,1,1,4,6)$	1/2
75	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^3T_3^8+T_4^3+T_5^2\rangle$	$(1,1,1,4,6)$	1/2
76	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^4T_3^7+T_4^3+T_5^2\rangle$	$(1,1,1,4,6)$	1/2
77	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5T_3^6+T_4^3+T_5^2\rangle$	$(1,1,1,4,6)$	1/2
78	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^3T_3^7+T_4^3+T_5^2\rangle$	$(1,1,1,4,6)$	1/2
79	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^5T_3^5+T_4^3+T_5^2\rangle$	$(1,1,1,4,6)$	1/2
80	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2^4T_3^5+T_4^3+T_5^2\rangle$	$(1,1,1,4,6)$	1/2
81	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2T_3^2+T_4^3+T_5^2\rangle$	$(1,1,2,2,3)$	27/2
82	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^3T_3+T_4^3+T_5^2\rangle$	$(1,1,2,2,3)$	27/2
83	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^2T_3+T_4^3+T_5^2\rangle$	$(1,1,2,2,3)$	27/2
84	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2T_3^4+T_4^5+T_5^2\rangle$	$(1,1,2,2,5)$	1/2
85	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^3T_3^3+T_4^5+T_5^2\rangle$	$(1,1,2,2,5)$	1/2
86	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5T_3^2+T_4^5+T_5^2\rangle$	$(1,1,2,2,5)$	1/2
87	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^7T_3+T_4^5+T_5^2\rangle$	$(1,1,2,2,5)$	1/2
88	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^2T_3^3+T_4^5+T_5^2\rangle$	$(1,1,2,2,5)$	1/2
89	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^6T_3+T_4^5+T_5^2\rangle$	$(1,1,2,2,5)$	1/2
90	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2^3T_3^2+T_4^5+T_5^2\rangle$	$(1,1,2,2,5)$	1/2
91	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2^5T_3+T_4^5+T_5^2\rangle$	$(1,1,2,2,5)$	1/2
92	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2^4T_3+T_4^5+T_5^2\rangle$	$(1,1,2,2,5)$	1/2
93	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2T_3^5+T_4^3+T_5^2\rangle$	$(1,1,2,4,6)$	2
94	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^3T_3^4+T_4^3+T_5^2\rangle$	$(1,1,2,4,6)$	2
95	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5T_3^3+T_4^3+T_5^2\rangle$	$(1,1,2,4,6)$	2
96	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^7T_3^2+T_4^3+T_5^2\rangle$	$(1,1,2,4,6)$	2
97	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^9T_3+T_4^3+T_5^2\rangle$	$(1,1,2,4,6)$	2
98	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^4T_3^3+T_4^3+T_5^2\rangle$	$(1,1,2,4,6)$	2

99	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^8T_3+T_4^3+T_5^2\rangle$	$(1,1,2,4,6)$	2
100	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2^5T_3^2+T_4^3+T_5^2\rangle$	$(1,1,2,4,6)$	2
101	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2^7T_3+T_4^3+T_5^2\rangle$	$(1,1,2,4,6)$	2
102	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2^6T_3+T_4^3+T_5^2\rangle$	$(1,1,2,4,6)$	2
103	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^5T_2^5T_3+T_4^3+T_5^2\rangle$	$(1,1,2,4,6)$	2
104	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2T_3+T_4^3+T_5^2\rangle$	$(1,2,2,2,3)$	16
105	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2T_3^3+T_4^5+T_5^2\rangle$	$(1,2,2,2,5)$	2
106	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2T_3^2+T_4^5+T_5^2\rangle$	$(1,2,2,2,5)$	2
107	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^6T_2T_3+T_4^5+T_5^2\rangle$	$(1,2,2,2,5)$	2

The varieties no. 2,3 and 25, 26 are rational degenerations of quasismooth varieties from the list in [16]. In [11] the non-rationality of a general (quasismooth) element of the corresponding family was proved.

The varieties listed so far might suggest that we always obtain only one relation in the Cox ring. We discuss now some examples, showing that for a Picard index big enough, we need in general more than one relation, where this refers always to a presentation as in Theorem 1.9 (ii).

Example 3.4. A Fano $\mathbb{K}^*$-surface X with $\mathrm{Cl}(X)=\mathbb{Z}$ such that the Cox ring $\mathcal{R}(X)$ needs two relations. Consider the $\mathbb{Z}$-graded ring

$$R \quad = \quad \mathbb{K}[T_{01},T_{02},T_{11},T_{21},T_{31}]/\langle g_0,g_1\rangle,$$

where the degrees of $T_{01},T_{02},T_{11},T_{21},T_{31}$ are $29,1,6,10,15$, respectively, and the relations g_0,g_1 are given by

$$g_0 \ := \ T_{01}T_{02}+T_{11}^5+T_{21}^3, \qquad g_1 \ := \ \alpha_{23}T_{11}^5+\alpha_{31}T_{21}^3+\alpha_{12}T_{31}^2$$

Then R is the Cox ring of a Fano $\mathbb{K}^*$-surface. Note that the Picard index is given by $[\mathrm{Cl}(X):\mathrm{Pic}(X)]=\mathrm{lcm}(29,1)=29$.

Proposition 3.5. *Let X be a non-toric Fano surface with an effective $\mathbb{K}^*$-action such that $\mathrm{Cl}(X)\cong\mathbb{Z}$ and $[\mathrm{Cl}(X):\mathrm{Pic}(X)]<29$ hold. Then the Cox ring of X is of the form*

$$\mathcal{R}(X) \quad \cong \quad \mathbb{K}[T_1,\dots,T_4]/\langle T_1^{l_1}T_2^{l_2}+T_3^{l_3}+T_4^{l_4}\rangle.$$

Proof. The Cox ring $\mathcal{R}(X)$ is as in Theorem 1.9, and, in the notation used there, we have $n_0+\ldots+n_r+m=2+r$. This leaves us with the possibilities $n_0=m=1$ and $n_0=2$, $m=0$. In the first case, Proposition 2.9 tells us that the Picard index of X is at least 30.

So, consider the case $n_0 = 2$ and $m = 0$. Then, according to Theorem 1.9, the Cox ring $\mathcal{R}(X)$ is $\mathbb{K}[T_{01}, T_{02}, T_1 \ldots, T_r]$ divided by relations

$$g_{0,1,2} = T_{01}^{l_{01}} T_{02}^{l_{02}} + T_1^{l_1} + T_2^{l_2}, \; g_{i,i+1,i+2} = \alpha_{i+1,i+2} T_i^{l_i} + \alpha_{i+2,i} T_{i+1}^{l_{i+1}} + \alpha_{i,i+1} T_{i+2}^{l_{i+2}},$$

where $1 \le i \le r-2$. We have to show that $r = 2$ holds. Set $\mu := [\mathrm{Cl}(X) : \mathrm{Pic}(X)]$ and let $\gamma \in \mathbb{Z}$ denote the degree of the relations. Then we have $\gamma = w_i l_i$ for $1 \le i \le r$, where $w_i := \deg T_i$. With $w_{0i} := \deg T_{0i}$, Proposition 2.5 gives us

$$(r-1)\gamma \quad < \quad w_{01} + w_{02} + w_1 + \ldots + w_r.$$

We claim that w_{01} and w_{02} are coprime. Otherwise they had a common prime divisor p. This p divides $\gamma = l_i w_i$. Since $l_1, \ldots, l_r$ are pairwise coprime, p divides at least $r-1$ of the weights $w_1, \ldots, w_r$. This contradicts the Cox ring condition that any $r+1$ of the $r+2$ weights generate the class group $\mathbb{Z}$. Thus, w_{01} and w_{02} are coprime and we obtain

$$\mu \;\ge\; \mathrm{lcm}(\mathrm{w}_{01}, \mathrm{w}_{02}) \;=\; \mathrm{w}_{01} \cdot \mathrm{w}_{02} \;\ge\; \mathrm{w}_{01} + \mathrm{w}_{02} - 1.$$

Now assume that $r \ge 3$ holds. Then we can conclude

$$2\gamma \;<\; w_{01} + w_{02} + w_1 + w_2 + w_3 \;\le\; \mu + 1 + \gamma\left(\frac{1}{l_1} + \frac{1}{l_2} + \frac{1}{l_3}\right)$$

Since the numbers l_i are pairwise coprime, we obtain $l_1 \ge 5$, $l_2 \ge 3$ and $l_3 \ge 2$. Moreover, $l_i w_i = l_j w_j$ implies $l_i \mid w_j$ and hence $l_1 l_2 l_3 \mid \gamma$. Thus, we have $\gamma \ge 30$. Plugging this in the above inequality gives

$$\mu \;\ge\; \gamma\left(2 - \frac{1}{l_1} - \frac{1}{l_2} - \frac{1}{l_3}\right) - 1 \;=\; 29.$$

□

The Fano assumption is essential in this result; if we omit it, then we may even construct locally factorial surfaces with a Cox ring that needs more then one relation.

Example 3.6. A locally factorial $\mathbb{K}^*$-surface X with $\mathrm{Cl}(X) = \mathbb{Z}$ such that the Cox ring $\mathcal{R}(X)$ needs two relations. Consider the $\mathbb{Z}$-graded ring

$$R \quad = \quad \mathbb{K}[T_{01}, T_{02}, T_{11}, T_{21}, T_{31}]/\langle g_0, g_1\rangle,$$

where the degrees of $T_{01}, T_{02}, T_{11}, T_{21}, T_{31}$ are $1, 1, 6, 10, 15$, respectively, and the relations g_0, g_1 are given by

$$g_0 \;:=\; T_{01}^7 T_{02}^{23} + T_{11}^5 + T_{21}^3, \qquad\qquad g_1 \;:=\; \alpha_{23} T_{11}^5 + \alpha_{31} T_{21}^3 + \alpha_{12} T_{31}^2$$

Then R is the Cox ring of a non Fano $\mathbb{K}^*$-surface X of Picard index one, i.e, X is locally factorial.

For non-toric Fano threefolds X with an effective 2-torus action $\mathrm{Cl}(X) \cong \mathbb{Z}$, the classifications 3.2 and 3.3 show that for Picard indices one and two we only obtain hypersurfaces as Cox rings. The following example shows that this stops at Picard index three.

Example 3.7. A Fano threefold X with $\mathrm{Cl}(X) = \mathbb{Z}$ and a 2-torus action such that the Cox ring $\mathcal{R}(X)$ needs two relations. Consider

$$R = \mathbb{K}[T_{01}, T_{02}, T_{11}, T_{12}, T_{21}, T_{31}]/\langle g_0, g_1\rangle$$

where the degrees of $T_{01}, T_{02}, T_{11}, T_{12}, T_{21}, T_{31}$ are $1, 1, 3, 3, 2, 3$, respectively, and the relations are given by

$$g_0 = T_{01}^5 T_{02} + T_{11}T_{12} + T_{21}^3, \qquad g_1 = \alpha_{23}T_{11}T_{12} + \alpha_{31}T_{21}^3 + \alpha_{12}T_{31}^2.$$

Then R is the Cox ring of a Fano threefold with a 2-torus action. Note that the Picard index is given by

$$[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = \mathrm{lcm}(1, 1, 3, 3) = 3.$$

Finally, we turn to locally factorial Fano fourfolds. Here we observe more than one relation in the Cox ring even in the locally factorial case.

Theorem 3.8. *Let X be a four-dimensional locally factorial non-toric Fano variety with an effective three torus action such that $\mathrm{Cl}(X) = \mathbb{Z}$ holds. Then its Cox ring is precisely one of the following.*

No.	$\mathcal{R}(X)$	$(w_1, \ldots, w_6)$	$(-K_X)^4$
1	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2^5 + T_3^3 + T_4^2\rangle$	$(1, 1, 2, 3, 1, 1)$	81
2	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2^9 + T_3^2 + T_4^5\rangle$	$(1, 1, 2, 5, 1, 1)$	1
3	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1^3T_2^7 + T_3^2 + T_4^5\rangle$	$(1, 1, 2, 5, 1, 1)$	1
4	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2T_3^4 + T_4^3 + T_5^2\rangle$	$(1, 1, 1, 2, 3, 1)$	81
5	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2^2T_3^3 + T_4^3 + T_5^2\rangle$	$(1, 1, 1, 2, 3, 1)$	81
6	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2T_3^8 + T_4^5 + T_5^2\rangle$	$(1, 1, 1, 2, 5, 1)$	1
7	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2^2T_3^7 + T_4^5 + T_5^2\rangle$	$(1, 1, 1, 2, 5, 1)$	1
8	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2^3T_3^6 + T_4^5 + T_5^2\rangle$	$(1, 1, 1, 2, 5, 1)$	1
9	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2^4T_3^5 + T_4^5 + T_5^2\rangle$	$(1, 1, 1, 2, 5, 1)$	1
10	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1^2T_2^3T_3^5 + T_4^5 + T_5^2\rangle$	$(1, 1, 1, 2, 5, 1)$	1
11	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1^3T_2^3T_3^4 + T_4^5 + T_5^2\rangle$	$(1, 1, 1, 2, 5, 1)$	1
12	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2 + T_3T_4 + T_5^2\rangle$	$(1, 1, 1, 1, 1, 1)$	512
13	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2^2 + T_3T_4^2 + T_5^3\rangle$	$(1, 1, 1, 1, 1, 1)$	243

14	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^3+T_3T_4^3+T_5^4\rangle$	$(1,1,1,1,1,1)$	64
15	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^4+T_3T_4^4+T_5^5\rangle$	$(1,1,1,1,1,1)$	5
16	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^4+T_3^2T_4^3+T_5^5\rangle$	$(1,1,1,1,1,1)$	5
17	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1^2T_2^3+T_3^2T_4^3+T_5^5\rangle$	$(1,1,1,1,1,1)$	5
18	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^3+T_3T_4^3+T_5^2\rangle$	$(1,1,1,1,2,1)$	162
19	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^5+T_3T_4^5+T_5^3\rangle$	$(1,1,1,1,2,1)$	3
20	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^5+T_3^2T_4^4+T_5^3\rangle$	$(1,1,1,1,2,1)$	3
21	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^5+T_3T_4^5+T_5^2\rangle$	$(1,1,1,1,3,1)$	32
22	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^5+T_3^3T_4^3+T_5^2\rangle$	$(1,1,1,1,3,1)$	32
23	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^7+T_3T_4^7+T_5^2\rangle$	$(1,1,1,1,4,1)$	2
24	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^7+T_3^3T_4^5+T_5^2\rangle$	$(1,1,1,1,4,1)$	2
25	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1^3T_2^5+T_3^3T_4^5+T_5^2\rangle$	$(1,1,1,1,4,1)$	2
26	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2T_3T_4^3+T_5^3+T_6^2\rangle$	$(1,1,1,1,2,3)$	81
27	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2T_3^2T_4^2+T_5^3+T_6^2\rangle$	$(1,1,1,1,2,3)$	81
28	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2T_3T_4^7+T_5^5+T_6^2\rangle$	$(1,1,1,1,2,5)$	1
29	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2T_3^2T_4^6+T_5^5+T_6^2\rangle$	$(1,1,1,1,2,5)$	1
30	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2T_3^3T_4^5+T_5^5+T_6^2\rangle$	$(1,1,1,1,2,5)$	1
31	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2T_3^4T_4^4+T_5^5+T_6^2\rangle$	$(1,1,1,1,2,5)$	1
32	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^2T_3^2T_4^5+T_5^5+T_6^2\rangle$	$(1,1,1,1,2,5)$	1
33	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^2T_3^3T_4^4+T_5^5+T_6^2\rangle$	$(1,1,1,1,2,5)$	1
34	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^3T_3^3T_4^3+T_5^5+T_6^2\rangle$	$(1,1,1,1,2,5)$	1
35	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1^2T_2^2T_3^3T_4^3+T_5^5+T_6^2\rangle$	$(1,1,1,1,2,5)$	1
36	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2T_3+T_4T_5^2+T_6^3\rangle$	$(1,1,1,1,1,1)$	243
37	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2T_3^2+T_4T_5^3+T_6^4\rangle$	$(1,1,1,1,1,1)$	64
38	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2T_3^3+T_4T_5^4+T_6^5\rangle$	$(1,1,1,1,1,1)$	5
39	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2T_3^3+T_4^2T_5^3+T_6^5\rangle$	$(1,1,1,1,1,1)$	5
40	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^2T_3^2+T_4T_5^4+T_6^5\rangle$	$(1,1,1,1,1,1)$	5
41	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^2T_3^2+T_4^2T_5^3+T_6^5\rangle$	$(1,1,1,1,1,1)$	5
42	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2T_3^2+T_4T_5^3+T_6^2\rangle$	$(1,1,1,1,1,2)$	162
43	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2T_3^4+T_4T_5^5+T_6^3\rangle$	$(1,1,1,1,1,2)$	3
44	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2T_3^4+T_4^2T_5^4+T_6^3\rangle$	$(1,1,1,1,1,2)$	3

45	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^2T_3^3+T_4T_5^5+T_6^3\rangle$	$(1,1,1,1,1,2)$	3
46	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^2T_3^3+T_4^2T_5^4+T_6^3\rangle$	$(1,1,1,1,1,2)$	3
47	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1^2T_2^2T_3^2+T_4T_5^5+T_6^3\rangle$	$(1,1,1,1,1,2)$	3
48	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^2T_3^3+T_4^3T_5^3+T_6^2\rangle$	$(1,1,1,1,1,3)$	32
49	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^2T_3^3+T_4T_5^5+T_6^2\rangle$	$(1,1,1,1,1,3)$	32
50	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2T_3^4+T_4^3T_5^3+T_6^2\rangle$	$(1,1,1,1,1,3)$	32
51	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2T_3^4+T_4T_5^5+T_6^2\rangle$	$(1,1,1,1,1,3)$	32
52	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2T_3^6+T_4T_5^7+T_6^2\rangle$	$(1,1,1,1,1,4)$	2
53	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2T_3^6+T_4^3T_5^5+T_6^2\rangle$	$(1,1,1,1,1,4)$	2
54	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^2T_3^5+T_4T_5^7+T_6^2\rangle$	$(1,1,1,1,1,4)$	2
55	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^2T_3^5+T_4^3T_5^5+T_6^2\rangle$	$(1,1,1,1,1,4)$	2
56	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^3T_3^4+T_4T_5^7+T_6^2\rangle$	$(1,1,1,1,1,4)$	2
57	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^3T_3^4+T_4^3T_5^5+T_6^2\rangle$	$(1,1,1,1,1,4)$	2
58	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1^2T_2^3T_3^3+T_4T_5^7+T_6^2\rangle$	$(1,1,1,1,1,4)$	2
59	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1^2T_2^3T_3^3+T_4^3T_5^5+T_6^2\rangle$	$(1,1,1,1,1,4)$	2
60	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2+T_3T_4+T_5T_6\rangle$	$(1,1,1,1,1,1)$	512
61	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^2+T_3T_4^2+T_5T_6^2\rangle$	$(1,1,1,1,1,1)$	243
62	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^3+T_3T_4^3+T_5T_6^3\rangle$	$(1,1,1,1,1,1)$	64
63	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^3+T_3T_4^3+T_5^2T_6^2\rangle$	$(1,1,1,1,1,1)$	64
64	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^4+T_3T_4^4+T_5T_6^4\rangle$	$(1,1,1,1,1,1)$	5
65	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^4+T_3T_4^4+T_5^2T_6^3\rangle$	$(1,1,1,1,1,1)$	5
66	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^4+T_3^2T_4^3+T_5^2T_6^3\rangle$	$(1,1,1,1,1,1)$	5
67	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1^2T_2^3+T_3^2T_4^3+T_5^2T_6^3\rangle$	$(1,1,1,1,1,1)$	5
68	$\mathbb{K}[T_1,\dots,T_7]/\left\langle \begin{smallmatrix} T_1T_2+T_3T_4+T_5T_6, \\ \alpha T_3T_4+T_5T_6+T_7^2 \end{smallmatrix} \right\rangle$	$(1,1,1,1,1,1,1)$	324
69	$\mathbb{K}[T_1,\dots,T_7]/\left\langle \begin{smallmatrix} T_1T_2^2+T_3T_4^2+T_5T_6^2, \\ \alpha T_3T_4^2+T_5T_6^2+T_7^3 \end{smallmatrix} \right\rangle$	$(1,1,1,1,1,1,1)$	9

where in the last two rows of the table the parameter α can be any element from $\mathbb{K}^* \setminus \{1\}$.

By the result of [26], the singular quintics of this list are rational degenerations of smooth non-rational Fano fourfolds.

4 Geometry of the locally factorial threefolds

In this section, we take a closer look at the (factorial) singularities of the Fano varieties X listed in Theorem 3.2. Recall that the discrepancies of a resolution $\varphi\colon \widetilde{X} \to X$ of a singularity are the coefficients of $K_{\widetilde{X}} - \varphi^* K_X$, where K_X and $K_{\widetilde{X}}$ are canonical divisors such that $K_{\widetilde{X}} - \varphi^* K_X$ is supported on the exceptional locus of φ. A resolution is called crepant, if its discrepancies vanish and a singularity is called canonical (terminal), if it admits a resolution with nonnegative (positive) discrepancies. By a relative minimal model we mean a projective morphism $\widetilde{X} \to X$ such that $\widetilde{X}$ has at most terminal singularities and its relative canonical divisor is relatively nef.

Theorem 4.1. *For the nine 3-dimensional Fano varieties listed in Theorem 3.2, we have the following statements.*

(i) *No. 4 is a smooth quadric in $\mathbb{P}^4$.*

(ii) *Nos. 1,3,5,7 and 9 are singular with only canonical singularities and all admit a crepant resolution.*

(iii) *Nos. 6 and 8 are singular with non-canonical singularities but admit a smooth relative minimal model.*

(iv) *No. 2 is singular with only canonical singularities, one of them of type $\mathbf{cA_1}$, and admits only a singular relative minimal model.*

The Cox ring of the relative minimal model $\widetilde{X}$ as well as the the Fano degree of X itself are given in the following table.

No.	$\mathcal{R}(\widetilde{X})$	$(-K_X)^3$
1	$\mathbb{K}[T_1,\ldots,T_{14}]/\langle T_1T_2T_3^2T_4^3T_5^4T_6^5 + T_7^3T_8^2T_9 + T_{10}^2T_{11}\rangle$	8
2	$\mathbb{K}[T_1,\ldots,T_9]/\langle T_1T_2T_3^2T_4^4 + T_5T_6^2T_7^3 + T_8^2\rangle$	8
3	$\mathbb{K}[T_1,\ldots,T_8]/\langle T_1T_2^2T_3^3 + T_4T_5^3 + T_6T_7^2\rangle$	8
4	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2 + T_3T_4 + T_5^2\rangle$	54
5	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^2 + T_3T_4^2 + T_5^3T_6\rangle$	24
6	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^3 + T_3T_4^3 + T_5^4T_6\rangle$	4
7	$\mathbb{K}[T_1,\ldots,T_7]/\langle T_1T_2^3 + T_3T_4^3 + T_5^2T_6\rangle$	16
8	$\mathbb{K}[T_1,\ldots,T_7]/\langle T_1T_2^5 + T_3T_4^5 + T_5^2T_6\rangle$	2
9	$\mathbb{K}[T_1,\ldots,T_{46}]/\left\langle \begin{array}{c} T_1T_2T_3T_4^2T_5^2T_6^3T_7^3T_8^4T_9^4T_{10}^5 + \\ + T_{11}\cdots T_{18}T_{19}^2\cdots T_{24}^2T_{25}^3T_{26}^3 + T_{27}\cdots T_{32}T_{33}^2 \end{array}\right\rangle$	2

For the proof, it is convenient to work in the language of polyhedral divisors introduced in [1] and [2]. As we are interested in rational varieties with a complexity one torus action, we only have to consider polyhedral divisors on the projective line $Y = \mathbb{P}^1$. This considerably simplifies the general definitions and allows us to give a short summary. In the sequel, $N \cong \mathbb{Z}^n$ denotes a lattice and $M = \mathrm{Hom}(N, \mathbb{Z})$ its dual. For the associated rational vector spaces we write $N_{\mathbb{Q}}$ and $M_{\mathbb{Q}}$. A *polyhedral divisor* on the projective line $Y := \mathbb{P}^1$ is a formal sum

$$\mathcal{D} \quad = \quad \sum_{y \in Y} \mathcal{D}_y \cdot y,$$

where the coefficients $\mathcal{D}_y \subseteq N_{\mathbb{Q}}$ are (possibly empty) convex polyhedra all sharing the same tail (i.e. recession) cone $\mathcal{D}_Y = \sigma \subseteq N_{\mathbb{Q}}$, and only finitely many $\mathcal{D}_y$ differ from σ. The *locus* of $\mathcal{D}$ is the open subset $Y(\mathcal{D}) \subseteq Y$ obtained by removing all points $y \subseteq Y$ with $\mathcal{D}_y = \emptyset$. For every $u \in \sigma^\vee \cap M$ we have the *evaluation*

$$\mathcal{D}(u) \quad := \quad \sum_{y \in Y} \min_{v \in \mathcal{D}_y} \langle u, v \rangle \cdot y,$$

which is a usual rational divisor on $Y(\mathcal{D})$. We call the polyhedral divisor $\mathcal{D}$ on Y *proper* if $\deg \mathcal{D} \subsetneq \sigma$ holds, where the *polyhedral degree* is defined by

$$\deg \mathcal{D} \quad := \quad \sum_{y \in Y} \mathcal{D}_y.$$

Every proper polyhedral divisor $\mathcal{D}$ on Y defines a normal affine variety $X(\mathcal{D})$ of dimension $\mathrm{rk}\,(N) + 1$ coming with an effective action of the torus $T = \mathrm{Spec}\,\mathbb{K}[M]$: set $X(\mathcal{D}) := \mathrm{Spec}\,A(\mathcal{D})$, where

$$A(\mathcal{D}) \;:=\; \bigoplus_{u \in \sigma^\vee \cap M} \Gamma(Y(\mathcal{D}), \mathcal{O}(\mathcal{D}(u))) \;\subseteq\; \bigoplus_{u \in M} \mathbb{K}(Y) \cdot \chi^u.$$

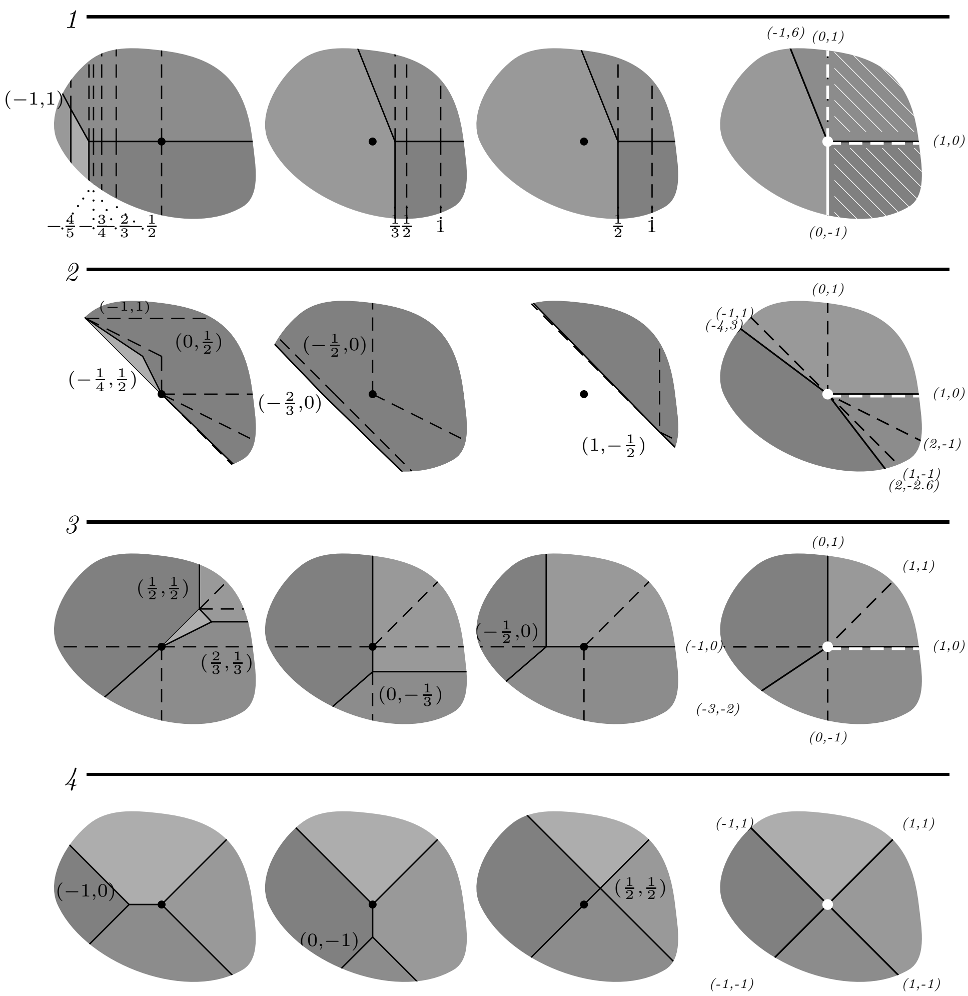
A *divisorial fan*, is a finite set Ξ of polyhedral divisors $\mathcal{D}$ on Y, all having their polyhedral coefficients $\mathcal{D}_y$ in the same $N_{\mathbb{Q}}$ and fulfilling certain compatibility conditions, see [2]. In particular, for every point $y \in Y$, the *slice*

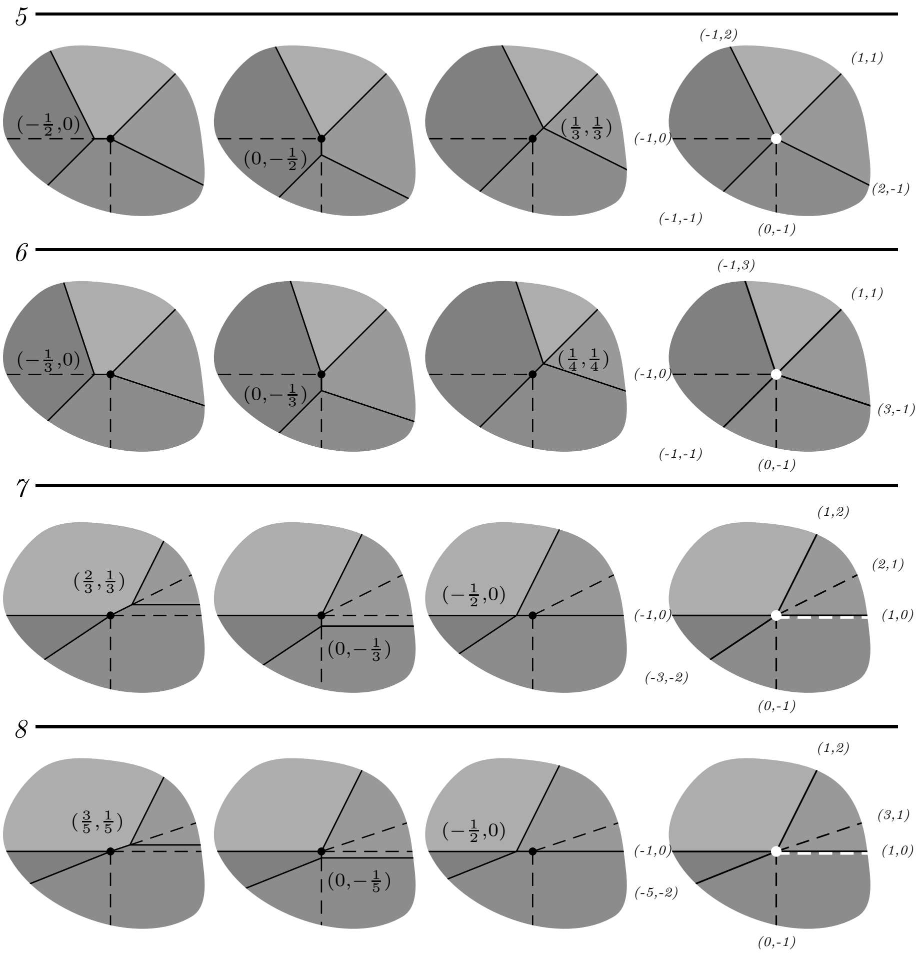
$$\Xi_y \quad := \quad \{\mathcal{D}_y;\ \mathcal{D} \in \Xi\}$$

must be a polyhedral subdivision. The *tail fan* is the set Ξ_Y of the tail cones $\mathcal{D}_Y$ of the $\mathcal{D} \in \Xi$; it is a fan in the usual sense. Given a divisorial fan Ξ, the affine varieties $X(\mathcal{D})$, where $\mathcal{D} \in \Xi$, glue equivariantly together to a normal variety $X(\Xi)$, and we obtain every rational normal variety with a complexity one torus action this way.

Smoothness of $X = X(\Xi)$ is checked locally. For a proper polyhedral divisor $\mathcal{D}$ on Y, we infer the following from [28, Theorem 3.3]. If $Y(\mathcal{D})$ is affine, then $X(\mathcal{D})$ is smooth if and only if $\operatorname{cone}(\{1\} \times \mathcal{D}_y) \subseteq \mathbb{Q} \times N_{\mathbb{Q}}$, the convex, polyhedral cone generated by $\{1\} \times \mathcal{D}_y$, is regular for every $y \in Y(\mathcal{D})$. If $Y(\mathcal{D}) = Y$ holds, then $X(\mathcal{D})$ is smooth if and only if there are $y, z \in Y$ such that $\mathcal{D} = \mathcal{D}_y y + \mathcal{D}_z z$ holds and $\operatorname{cone}(\{1\} \times \mathcal{D}_y) + \operatorname{cone}(\{-1\} \times \mathcal{D}_z)$ is a regular cone in $\mathbb{Q} \times N_{\mathbb{Q}}$. Similarly to toric geometry, singularities of $X(\mathcal{D})$ are resolved by means of subdividing $\mathcal{D}$. This means to consider divisorial fans Ξ such that for any $y \in Y$, the slice Ξ_y is a subdivision of $\mathcal{D}_y$. Such a Ξ defines a dominant morphism $X(\Xi) \to X(\mathcal{D})$ and a slight generalization of [2, Thm. 7.5.] yields that this morphism is proper.

Proposition 4.2. *The 3-dimensional Fano varieties No. 1-8 listed in Theorem 3.2 and their relative minimal models arise from divisorial fans having the following slices and tail cones.*





The above table should be interpreted as follows. The first three pictures in each row are the slices at 0, 1 and ∞ and the last one is the tail fan. The divisorial fan of the fano variety itself is given by the solid polyhedra in the pictures. Here, all polyhedra of the same gray scale belong to the same polyhedral divisor. The subdivisions for the relative minimal models are sketched with dashed lines. In general, polyhedra with the same tail cone belong all to a unique polyhedral divisor with complete locus. For the white cones inside the tail fan we have another rule: for every polyhedron $\Delta \in \Xi_y$ with the given white cone as its tail there is a polyhedral divisor $\Delta \cdot y + \emptyset \cdot z \in \Xi$, with $z \in \{0, 1, \infty\} \setminus \{y\}$. Here, different choices of z lead to isomorphic varieties, only the affine covering given by the $X(\mathcal{D})$ changes.

In order to prove Theorem 4.1, we also have to understand invariant divisors on $X = X(\Xi)$ in terms of Ξ, see [15, Prop. 4.11 and 4.12] for details. A first type of invariant prime divisors, is in bijection $D_{y,v} \leftrightarrow (y, v)$ with the vertices (y, v), where $y \in Y$ and $v \in \Xi_y$ is of dimension zero. The order of the generic isotropy group along $D_{y,v}$ equals the minimal positive integer $\mu(v)$ with $\mu(v)v \in N$. A second type of invariant prime divisors, is in $D_\varrho \leftrightarrow \varrho$ with the extremal rays $\varrho \in \Xi_Y$, where a ray $\varrho \in \Xi_Y$ is called extremal if there is a $\mathcal{D} \in \Xi$ such that $\varrho \subseteq \mathcal{D}_Y$ and $\deg \mathcal{D} \cap \varrho = \emptyset$ holds. The set of extremal rays is denoted by $\Xi_Y^\times$. The divisor of a semi-invariant function $f \cdot \chi^u \in \mathbb{K}(X)$ is then given by

$$\operatorname{div}(f \cdot \chi^u) \;=\; -\sum_{y \in Y} \sum_{v \in \Xi_y^{(0)}} \mu(v) \cdot (\langle v, u\rangle + \operatorname{ord}_y f) \cdot D_{y,v} \;-\; \sum_{\varrho \in \Xi_Y^\times} \langle n_\varrho, u\rangle \cdot D_\varrho.$$

Next we describe the canonical divisor. Choose a point $y_0 \in Y$ such that $\Xi_{y_0} = \Xi_Y$ holds. Then a canonical divisor on $X = X(\Xi)$ is given by

$$K_X \;=\; (s-2) \cdot y_0 \;-\; \sum_{\Xi_y \neq \Xi_Y} \sum_{v \in \Xi_i^{(0)}} D_{y,v} \;-\; \sum_{\varrho \in \Xi_Y^\times} E_\varrho.$$

Proposition 4.3. *Let $\mathcal{D}$ be a proper polyhedral divisor with $Y(\mathcal{D}) = \mathbb{P}_1$, let Ξ be a refinement of $\mathcal{D}$ and denote by $y_1, \ldots, y_s \in Y$ the points with $\Xi_{y_i} \neq \Xi_Y$. Then the associated morphism $\varphi\colon X(\Xi) \to X(\mathcal{D})$ satisfies the following.*

(i) *The prime divisors in the exceptional locus of φ are the divisors $D_{y_i,v}$ and D_ϱ corresponding to $v \in \Xi_{y_i}^{(0)} \setminus \mathcal{D}_{y_i}^{(0)}$ and $\varrho \in \Xi_Y^\times \setminus \mathcal{D}^\times$ respectively.*

(ii) *Then the discrepancies along the prime divisors $D_{y_i,v}$ and D_ϱ of (i) are computed as*

$$d_{y_i,v} \;=\; -\mu(v) \cdot (\langle v, u'\rangle + \alpha_y) - 1, \qquad\qquad d_\varrho \;=\; -\langle v_\varrho, u'\rangle - 1,$$

where the numbers α_i are determined by

$$\left(\begin{array}{cccc|c} -1 & -1 & \ldots & -1 & 0 \\ \hline \mu(v_1^1) & 0 & \ldots & 0 & \mu(v_1^1)v_1^1 \\ \vdots & \vdots & & \vdots & \vdots \\ \mu(v_1^{r_1}) & 0 & \ldots & 0 & \mu(v_1^{r_1})v_1^{r_1} \\ & & \ddots & & \\ 0 & 0 & \ldots & \mu(v_s^1) & \mu(v_s^1)v_s^1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & \mu(v_s^{r_s}) & \mu(v_s^{r_s})v_s^{r_s} \\ \hline 0 & 0 & \ldots & 0 & n_{\varrho_1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & n_{\varrho_r} \end{array}\right) \cdot \begin{pmatrix} \alpha_{y_1} \\ \vdots \\ \alpha_{y_s} \\ u \end{pmatrix} = \left(\begin{array}{c} 2-s \\ 1 \\ \vdots \\ 1 \\ \hline 1 \\ \vdots \\ 1 \end{array}\right)$$

Proof. The first claim is obvious by the characterization of invariant prime divisors. For the second claim note that by [24, Theorem 3.1] every Cartier divisor on $X(\mathcal{D})$ is principal. Hence, we may assume

$$\ell \cdot K_X \;=\; \operatorname{div}(f \cdot \chi^u), \qquad\qquad \operatorname{div}(f) \;=\; \sum_y \alpha_y \cdot y.$$

Then our formulæ for $\operatorname{div}(f \cdot \chi^u)$ and K_X provide a row for every vertex $v_i^j \in \Xi_{y_i}$, $i = 0, \ldots, s$, and for every extremal ray $\varrho_i \in \Xi^\times$, and $\ell^{-1}(\alpha, u)$ is the (unique) solution of the above system. □

Note, that in the above Proposition, the variety $X(\mathcal{D})$ is $\mathbb{Q}$-Gorenstein if and only if the linear system of equations has a solution.

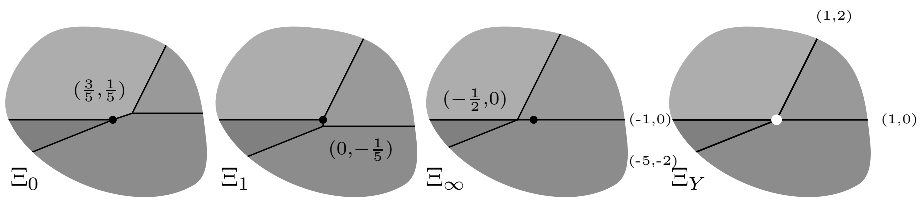
Proof of Theorem 4.1 and Proposition 4.2. We exemplarily discuss variety number eight. Recall that its Cox ring is given as

$$\mathcal{R}(X) \quad = \quad \mathbb{K}[T_1, \ldots, T_5]/(T_1T_2^5 + T_3T_4^5 + T_5^2)$$

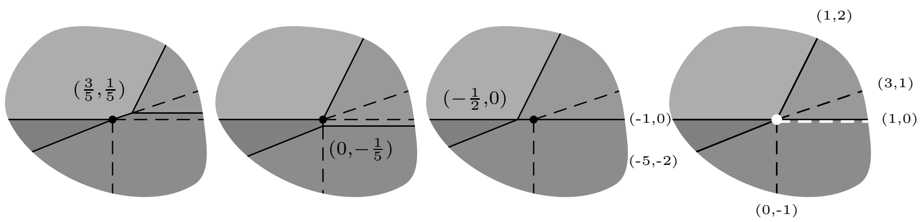
with the degrees $1, 1, 1, 1, 3$. In particular, X is a hypersurface of degree 6 in $\mathbb{P}(1, 1, 1, 1, 3)$, and the self-intersection of the anti-canonical divisor can be calculated as

$$(-K_X^3) \ = \ 6 \cdot \frac{(1+1+1+1+3-6)^3}{1 \cdot 1 \cdot 1 \cdot 1 \cdot 3} \ = \ 2.$$

The embedding $X \subseteq \mathbb{P}(1, 1, 1, 1, 3)$ is equivariant, and thus we can use the technique described in [1, Sec. 11] to calculate a divisorial fan Ξ for X. The result is the following divisorial fan; we draw its slices and indicate the polyhedral divisors with affine locus by colouring their tail cones $\mathcal{D}_Y \in \Xi_Y$ white:



One may also use [15, Cor. 4.9.] to verify that Ξ is the right divisorial fan: it computes the Cox ring in terms of Ξ, and, indeed, we obtain again $\mathcal{R}(X)$. Now we subdivide and obtain a divisorial fan having the refined slices as indicated in the following picture.



Here, the white ray $\mathbb{Q}_{\geq 0} \cdot (1, 0)$ indicates that the polyhedral divisors with that tail have affine loci. According to [15, Cor. 4.9.], the corresponding Cox ring is given by

$$\mathcal{R}(\widetilde{X}) \quad = \quad \mathbb{K}[T_1, \ldots, T_7]/\langle T_1T_2^5 + T_3T_4^5 + T_5^2T_6 \rangle.$$

We have to check that $\widetilde{X}$ is smooth. Let us do this explicitly for the affine chart defined by the polyhedral divisor $\mathcal{D}$ with tail cone $\mathcal{D}_Y = \operatorname{cone}((1,2),(3,1))$. Then $\mathcal{D}$ is given by

$$\mathcal{D} \quad = \quad \left(\left(\frac{3}{5},\frac{1}{5}\right)+\sigma\right)\cdot\{0\} \;+\; \left(\left[-\frac{1}{2},0\right]\times 0+\sigma\right)\cdot\{\infty\}.$$

Thus, $\operatorname{cone}(\{1\}\times\mathcal{D}_0)+\operatorname{cone}(\{-1\}\times\mathcal{D}_\infty)$ is generated by $(5,3,1)$, $(-2,-1,0)$ and $(-1,0,0)$; in particular, it is a regular cone. This implies smoothness of the affine chart $X(\mathcal{D})$. Furthermore, we look at the affine charts defined by the polyhedral divisors $\mathcal{D}$ with tail cone $\mathcal{D}_Y = \operatorname{cone}(1,0)$. Since they have affine locus, we have to check $\operatorname{cone}(\{1\}\times\mathcal{D}_y)$, where $y\in Y$. For $y\neq 0,1$, we have $\mathcal{D}_y=\mathcal{D}_Y$. In this case, $\operatorname{cone}(\{1\}\times\mathcal{D}_y)$ is generated by $(1,1,0)$, $(0,1,0)$ and thus is regular. For $y=0$, we obtain that $\operatorname{cone}(\{1\}\times\mathcal{D}_y)$ is generated by $(5,3,1)$, $(1,0,0)$, $(0,1,0)$ and this is regular. For $y=1$ we get the same result. Hence, the polyhedral divisors with tail cone $\mathcal{D}_y=\operatorname{cone}(1,0)$ give rise to smooth affine charts.

Now we compute the discrepancies according to Proposition 4.3. The resolution has two exceptional divisors $D_{\infty,\mathbf{0}}$ and $E_{(1,0)}$. We work in the chart defined by the divisor $\mathcal{D}\in\Xi$ with tail cone $\mathcal{D}_Y=\operatorname{cone}((1,2),(1,0))$. The resulting system of linear equations and its unique solution are given by

$$\left(\begin{array}{ccccc|c} -1 & -1 & -1 & 0 & 0 & -1\\ 5 & 0 & 0 & 3 & 1 & 1\\ 0 & 1 & 0 & 0 & 0 & 1\\ 0 & 5 & 0 & 0 & -1 & 1\\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right), \qquad \left(\begin{array}{c}\alpha_0\\ \alpha_1\\ \alpha_\infty\\ \hline u\end{array}\right) = \left(\begin{array}{c}0\\1\\0\\ \hline -1\\4\end{array}\right).$$

The formula for the discrepancies yields $d_{\infty,\mathbf{0}}=-1$ and $d_{(1,0)}=-2$. In particular, X has non-canonical singularities. By a criterion from [24, Sec. 3.4.], we know that $D_{\infty,\mathbf{0}}+2\cdot E_{(1,0)}$ is a nef divisor. It follows that $\widetilde{X}$ is a minimal model over X. ∎

References

[1] K. Altmann, J. Hausen: Polyhedral divisors and algebraic torus actions. Math. Ann. 334 (2006), no. 3, 557–607.

[2] K. Altmann, J. Hausen, H. Süß: Gluing Affine Torus Actions Via Divisorial Fans. Transformation Groups 13 (2008), no. 2, 215–242.

[3] V.V. Batyrev: Toric Fano threefolds. Izv. Akad. Nauk SSSR Ser. Mat. 45 (1981), no. 4, 704–717, 927.

[4] V.V. Batyrev: On the classification of toric Fano 4-folds. J. Math. Sci. New York 94, 1021–1050 (1999).

[5] F. Berchtold, J. Hausen: Homogeneous coordinates for algebraic varieties. J. Algebra 266 (2003), no. 2, 636–670.

[6] Cheltsov, Ivan; Park, Jihun Sextic double solids. In. Bogomolov, Fedor (ed.) et al.: Cohomological and geometric approaches to rationality problems. New Perspectives. Boston, MA: Birkhäuser. Progress in Mathematics 282, 75–132 (2010).

[7] J.J. Chen, J.A. Chen, M. Chen: On quasismooth weighted complete intersections. Preprint, arXiv:0908.1439.

[8] C.H. Clemens, P.A. Griffiths: The intermediate Jacobian of the cubic threefold. Ann. Math. (2) 95 (1972), 281–356.

[9] A. Corti: Singularities of linear systems and 3-fold birational geometry. Explicit birational geometry of 3-folds, 259 -312, London Math. Soc. Lecture Note Ser., 281, Cambridge Univ. Press, Cambridge, 2000.

[10] A. Corti, M. Mella: Birational geometry of terminal quartic 3-folds I. Am. J. Math. 126 (2004), No. 4, 739–761.

[11] A. Corti, A. Pukhlikov, M. Reid: Fano 3-fold hypersurfaces. Explicit birational geometry of 3-folds, 175–258, London Math. Soc. Lecture Note Ser., 281, Cambridge Univ. Press, Cambridge, 2000.

[12] M.M. Grinenko: Mori structures on a Fano threefold of index 2 and degree 1. Tr. Mat. Inst. Steklova 246 (2004), Algebr. Geom. Metody, Svyazi i Prilozh., 116–141; translation in Proc. Steklov Inst. Math. 2004, no. 3 (246), 103–128.

[13] M.M. Grinenko: Birational automorphisms of a three-dimensional double cone. Mat. Sb. 189 (1998), no. 7, 37–52; translation in Sb. Math. 189 (1998), no. 7-8, 991–1007.

[14] J. Hausen: Cox rings and combinatorics II. Mosc. Math. J., 8 (2008), 711–757.

[15] J. Hausen, H. Süß: The Cox ring of an algebraic variety with torus action. Adv. Math. 225 (2010), no. 2, 977 -1012.

[16] A.R. Iano-Fletcher: Working with weighted complete intersections. Explicit birational geometry of 3-folds, 101 -173, London Math. Soc. Lecture Note Ser., 281, Cambridge Univ. Press, Cambridge, 2000.

[17] M.-N. Ishida: Graded factorial rings of dimension 3 of a restricted type. J. Math. Kyoto Univ. 17 (1977), no. 3, 441–456.

[18] V.A. Iskovskih: Fano threefolds. II. Izv. Akad. Nauk SSSR Ser. Mat. 42 (1978), no. 3, 506- 549.

[19] V.A. Iskovskih: Birational automorphisms of three-dimensional algebraic varieties. Current problems in mathematics, Vol. 12 (Russian), pp. 159–236, 239 (loose errata), VINITI, Moscow, 1979.

[20] V.A. Iskovskih, Yu.I. Manin: Three-dimensional quartics and counterexamples to the Lüroth problem. Mat. Sb. (N.S.) 86 (128) (1971), 140–166.

[21] J.M. Johnson, J. Kollár: Fano hypersurfaces in weighted projective 4-spaces. Experiment. Math. 10 (2001), no. 1, 151–158.

[22] F. Knop, H Kraft, T. Vust: The Picard group of a G-variety. In: Algebraic Transformation Groups and Invariant Theory, DMV Seminar, Band 3, Birkhäuser.

[23] S. Mori: Graded factorial domains. Japan J. Math. 3 (1977), no. 2, 223–238.

[24] L. Petersen, H. Süß: Torus invariant divisors. Preprint, arXiv:0811.0517 (2008), to appear in Israel J. Math.

[25] A.V. Pukhlikov: Birational automorphisms of a three-dimensional quartic with a simple singularity. Mat. Sb. (N.S.) 135(177) (1988), no. 4, 472–496, 559; translation in Math. USSR-Sb. 63 (1989), no. 2, 457–482.

[26] A.V. Pukhlikov: Birational isomorphisms of four-dimensional quintics. Invent. Math. 87, no. 2 (1987), 303–329.

[27] K. Watanabe, M. Watanabe: The classification of Fano 3-folds with torus embeddings. Tokyo J. Math. 5, 37–48 (1982).

[28] H. Süß: Canonical divisors on T-varieties. Preprint, arXiv:0811.0626v1.

[29] A.S. Tikhomirov: The intermediate Jacobian of double $\mathbf{P}^3$ that is branched in a quartic. Izv. Akad. Nauk SSSR Ser. Mat. 44 (1980), no. 6, 1329–1377, 1439.

[30] Voisin, Claire: Sur la jacobienne intermédiaire du double solide d'indice deux. Duke Math. J. 57 (1988), no. 2, 629–646.

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