A Lefschetz Fixed Point Formula for Singular Arithmetic Schemes with Smooth Generic Fibres

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Abstract. In this article, we consider singular equivariant arithmetic schemes whose generic fibres are smooth. For such schemes, we prove a relative fixed point formula of Lefschetz type in the context of Arakelov geometry. This formula is an analog, in the arithmetic case, of the Lefschetz formula proved by R. W. Thomason in [31]. In particular, our result implies a fixed point formula which was conjectured by V. Maillot and D. Rössler in [25].

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1 Introduction

It is the aim of this article to prove a singular Lefschetz fixed point formula for some schemes which admit the actions of a diagonalisable group scheme, in the context of Arakelov geometry. We first roughly describe the history of the study of such Lefschetz fixed point formulae and relative Lefschetz-Riemann-Roch problems.

Let k be an algebraically closed field and let n be an integer which is prime to the characteristic of k. A projective $k-$variety X which admits an automorphism g of order n will be called an equivariant variety. An equivariant coherent sheaf on X is a coherent sheaf F on X together with a homomorphism $\varphi : g^*F \to F$. It is clear that this homomorphism induces a family of endomorphisms $H^i(\varphi)$ on cohomology spaces $H^i(X, F)$.

A classical Lefschetz fixed point formula is to give an expression of the alternating sum of the traces of $H^i(\varphi)$, as a sum of the contributions from the components of the fixed point subvariety X_g. On the other hand, roughly speaking, a Lefschetz-Riemann-Roch theorem is a commutative diagram in equivariant $K-$theory which can be regarded as a Grothendieck type generalization of the Lefschetz fixed point formula. Indeed, when we choose the base variety in such a commutative diagram to be a point, we will get the ordinary Lefschetz fixed point formula. If X is nonsingular, P. Donovan has proved such a theorem in [12] by using the results and some of the methods of the paper of A. Borel and J. P. Serre on the Grothendieck-Riemann-Roch theorem (cf. [10]). In [1], P. Baum, W. Fulton and G. Quart generalized Donovan's theorem to singular varieties, the key step of their proof heavily relies on an elegant method called the deformation to the normal cone. Denote by $G_0(X,g)$ (resp. $K_0(X,g)$) the Quillen's algebraic K-group associated to the category of equivariant coherent sheaves (resp. vector bundles of finite rank) on X, then $K_0(\mathrm{Pt},g)$ is isomorphic to the group ring $\mathbb{Z}[k]$ and $G_0(X,g)$ (resp. $K_0(X,g)$) has a natural $K_0(\mathrm{Pt},g)$-module (resp. $K_0(\mathrm{Pt},g)$-algebra) structure. Let f be an equivariantly projective morphism between two equivariant varieties X and Y, then it is possible to define a push-forward morphism f_* from $G_0(X,g)$ to $G_0(Y,g)$ in a rather standard way. Let $\mathcal{R}$ be any flat $K_0(\mathrm{Pt},g)$-algebra in which $1-\zeta$ is invertible for each non-trivial n-th root of unity ζ in k. The main result of Baum, Fulton and Quart reads: there exists a family of group homomorphisms $L.$ between K-groups making the following diagram

$$\begin{tikzcd} G_0(X,g) \arrow[r, "L."] \arrow[d, "f_*"'] & G_0(X_g,g)\otimes_{\mathbb{Z}[k]}\mathcal{R} \arrow[d, "{f_g}_*"] \\ G_0(Y,g) \arrow[r, "L."] & G_0(Y_g,g)\otimes_{\mathbb{Z}[k]}\mathcal{R} \end{tikzcd}$$

commutative. If Z is a nonsingular equivariant variety such that there exists an equivariant closed immersion from X to Z, then for every equivariant coherent sheaf E on X the homomorphism $L.$ is exactly given by the formula

$$L.(E)=\lambda_{-1}^{-1}(N_{Z/Z_g}^\vee)\cdot\sum_j(-1)^j\mathrm{Tor}^j_{\mathcal{O}_Z}(i_*E,\mathcal{O}_{Z_g})$$

where N_{Z/Z_g} stands for the normal bundle of Z_g in Z and $\lambda_{-1}(N_{Z/Z_g}^\vee):=\sum(-1)^j\wedge^j N_{Z/Z_g}^\vee$.

We would like to indicate that one can use the same method so called the deformation to the normal cone to extend Baum, Fulton and Quart's result to general scheme case where X and Y are Noetherian, separated schemes endowed with projective actions of the diagonalisable group scheme μ_n associated to $\mathbb{Z}/n\mathbb{Z}$. Here by a μ_n-action on X we understand a morphism $m_X:\mu_n\times X\to X$ which statisfies some compatibility properties. Denote by p_X the projection from $\mu_n\times X$ to X. For a coherent $\mathcal{O}_X$-module E on X, a μ_n-action on E we mean an isomorphism $m_E : p_X^*E \to m_X^*E$ which satisfies certain associativity properties. We refer to [20] and [21, Section 2] for the group scheme action theory we are talking about.

In [31], R. W. Thomason used another way to generalize Baum, Fulton and Quart's result to the scheme case, and he removed the condition of projectivity. The strategy Thomason followed was to use Quillen's localization sequence for higher equivariant K-groups to prove an algebraic concentration theorem. Let D be an integral Noetherian ring, and let μ_n be the diagonalisable group scheme over D associated to $\mathbb{Z}/n\mathbb{Z}$. Denote the ring $K_0(D,\mu_n) \cong K_0(D)[T]/(1-T^n)$ by $R(\mu_n)$. We consider the prime ideal ρ in $R(\mu_n)$ with which the intersection of $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$ is exactly the kernel of the canonical morphism $\mathbb{Z}[T]/(1-T^n) \to \mathbb{Z}[T]/(\Phi_n)$ where Φ_n stands for the n-th cyclotomic polynomial (cf. [31, Lem. 1.6]). By construction the elements $1-T^k$ for $k=1,\ldots,n-1$ are not contained in ρ. Let X be a μ_n-equivariant scheme over D, then $G_0(X,\mu_n)$ (resp. $K_0(X,\mu_n)$) has a natural $R(\mu_n)$-module (resp. $R(\mu_n)$-algebra) structure. Denote by i the inclusion from X_{μ_n} to X. The algebraic concentration theorem reads: there exists a natural group homomorphism i_* from $G_0(X_{\mu_n},\mu_n)_\rho$ to $G_0(X,\mu_n)_\rho$ which is an isomorphism. Moreover, if X is regular, the inverse map of i_* is given by $\lambda_{-1}^{-1}(N_{X/X_{\mu_n}}^\vee)\cdot i^*$ where $N_{X/X_{\mu_n}}$ is the normal bundle of X_{μ_n} in X. This concentration theorem can be used to prove a singular Lefschetz fixed point formula which is an extension of Baum, Fulton and Quart's result in general scheme case. Thomason's approach has nothing to do with the construction of the deformation to the normal cone, and the localization he used is slightly weaker than Baum, Fulton and Quart's in the sense that the complement of the ideal ρ in $R(\mu_n)$ is not the smallest algebra in which the elements $1-T^k$ $(k=1,\ldots,n-1)$ are invertible. If one exactly chooses $\mathcal{R}$ to be the complement of the ideal ρ in $R(\mu_n)$, then these two localizations are equal to each other.

In [21], K. Köhler and D. Rössler generalized the regular case of Baum, Fulton and Quart's result to Arakelov geometry. To every regular μ_n-equivariant arithmetic scheme X, they associate an equivariant arithmetic K_0-group $\widehat{K_0}(X,\mu_n)$ which contains some smooth form class on $X_{\mu_n}(\mathbb{C})$ as analytic datum. Such an equivariant arithmetic K_0-group has a ring structure so that it is also an $R(\mu_n)$-algebra. Let $\overline{N}_{X/X_{\mu_n}}$ be the normal bundle with respect to the regular immersion $X_{\mu_n} \hookrightarrow X$ which is endowed with the quotient metric induced by a chosen Kähler metric of $X(\mathbb{C})$, then the main theorem in [21] reads: the element $\lambda_{-1}(\overline{N}_{X/X_{\mu_n}}^\vee)$ is invertible in $\widehat{K_0}(X_{\mu_n},\mu_n)\otimes_{R(\mu_n)}\mathcal{R}$ and we have the following commutative diagram

$$\begin{tikzcd}
\widehat{K_0}(X,\mu_n) \arrow[r, "\Lambda_R(f)^{-1}\cdot\tau"] \arrow[d, "f_*"'] & \widehat{K_0}(X_{\mu_n},\mu_n)\otimes_{R(\mu_n)}\mathcal{R} \arrow[d, "f_{\mu_n *}"] \\
\widehat{K_0}(D,\mu_n) \arrow[r, "\iota"] & \widehat{K_0}(D,\mu_n)\otimes_{R(\mu_n)}\mathcal{R}
\end{tikzcd}$$

where $\Lambda_R(f) := \lambda_{-1}(\overline{N}^{\vee}_{X/X_{\mu_n}}) \cdot (1 + R_g(N_{X/X_{\mu_n}}))$, τ stands for the restriction map and ι is the natural morphism from a ring or a module to its localization which sends an element e to $\frac{e}{1}$. Here $R_g(\cdot)$ is the equivariant R-genus, the definition of the two push-forward morphisms f_* and $f_{\mu_n *}$ involves an important analytic datum which is called the equivariant analytic torsion. The strategy Köhler and Rössler followed to prove such an arithmetic Lefschetz-Riemann-Roch theorem was to use the construction of the deformation to the normal cone to prove an analog of this theorem for equivariant closed immersions. After that, they decompose the morphism f to a closed immersion h from X to some projective space $\mathbb{P}^r_D$ followed by a smooth morphism p from $\mathbb{P}^r_D$ to $\mathrm{Spec}(D)$. Then the theorem in general situation follows from an argument of investigating the behavior of the error term under the morphisms h and p.

Provided X. Ma's extension of equivariant analytic torsion to higher equivariant analytic torsion form, it was conjectured by Köhler and Rössler in [22] that an analog of [21, Theorem 4.4] in relative setting holds. We have already proved this conjecture in [29]. Our method is similar to Thomason's, we first show that there exists an arithmetic concentration theorem in Arakelov geometry and then deduce from it the relative Lefschetz fixed point formula. The same as Thomason's approach, our method has nothing to do with the construction of the deformation to the normal cone, but unfortunately it only works for regular arithmetic schemes.

One may naturally asked that whether it is possible to construct a more general arithmetic $\widehat{G_0}$-theory and prove a relative Lefschetz fixed point formula for singular arithmetic schemes which is entirely an analog of Thomason's singular Lefschetz formula in Arakelov geometry. The answer is Yes, and this is what we have done in this article. To do this, one needs a $\widehat{G_0}$-theoretic vanishing theorem which can be viewed as an extension of Köhler and Rössler's fixed point formula for closed immersions to the singular case. The proof of such a vanishing theorem occupies a lot of space in this article. Let X and Y be two singular equivariant arithmetic schemes with smooth generic fibres, and let $f : X \to Y$ be an equivariant morphism which is smooth on the complex numbers. Assume that the μ_n-action on Y is trivial and f can be decomposed to be $h \circ i$ where i is an equivariant closed immersion from X to some regular arithmetic scheme Z and $h : Z \to Y$ is equivariant and smooth on the complex numbers. Let $\overline{\eta}$ be an equivariant hermitian sheaf on X. Referring to Section 6.1 for the explanations of various notations, we announce that our main theorem in this article is the following equality which holds in $\widehat{G_0}(Y, \mu_n, \emptyset)_\rho$:

$$\begin{aligned} f_*(\overline{\eta}) =& f^Z_{\mu_n *}(i^*_{\mu_n}(\lambda^{-1}_{-1}(\overline{N}^{\vee}_{Z/Z_{\mu_n}})) \cdot \sum_k (-1)^k \mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\overline{\eta}, \overline{\mathcal{O}}_{Z_{\mu_n}})) \\ &+ \int_{X_g/Y} \mathrm{Td}(Tf_g, \omega^X)\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g(\overline{\mathcal{F}}, \omega^X)\mathrm{Td}^{-1}_g(\overline{F}) \\ &- \int_{X_g/Y} \mathrm{Td}_g(Tf)\mathrm{ch}_g(\eta)R_g(N_{X/X_g}) \end{aligned}$$

$$+\int_{X_g/Y}\widetilde{\mathrm{Td}}(Tf_g,\omega^X,\omega^Z_X)\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g(\overline{N}_{Z/Z_g})\mathrm{Td}_g^{-1}(\overline{F}).$$

The structure of this article is as follows. In Section 2, we recall some differential-geometric facts for the convenience of the reader. In Section 3, we formulate and prove a vanishing theorem for equivariant closed immersions in a purely analytic setting. In Section 4, we define the arithmtic G_0-groups with respect to fixed wave front sets which are necessary for our later arguments. In Section 5 and Section 6, we formulate and prove the arithmetic concentration theorem and the relative Lefschetz fixed point formula for singular arithmetic schemes.

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2 DIFFERENTIAL-GEOMETRIC PRELIMINARIES

2.1 EQUIVARIANT CHERN-WEIL THEORY

Let G be a compact Lie group and let M be a compact complex manifold which admits a holomorphic G-action. By an equivariant hermitian vector bundle on M, we understand a hermitian vector bundle on M which admits a G-action compatible with the G-structure of M and whose metric is G-invariant. Let $g\in G$ be an automorphism of M, we shall denote by $M_g=\{x\in M\mid g\cdot x=x\}$ the fixed point submanifold. M_g is also a compact complex manifold.

Now let $\overline{E}$ be an equivariant hermitian vector bundle on M, it is well known that the restriction of $\overline{E}$ to M_g splits as a direct sum

$$\overline{E}\mid_{M_g}=\bigoplus_{\zeta\in S^1}\overline{E}_\zeta$$

where the equivariant structure g^E of E acts on $\overline{E}_\zeta$ as multiplication by ζ. We often write $\overline{E}_g$ for $\overline{E}_1$ and call it the 0-degree part of $\overline{E}\mid_{M_g}$. As usual, $A^{p,q}(M)$ stands for the space of (p,q)-forms $\Gamma^\infty(M,\Lambda^pT^{*(1,0)}M\wedge\Lambda^qT^{*(0,1)}M)$, we define

$$\widetilde{A}(M)=\bigoplus_{p=0}^{\mathrm{dim}M}(A^{p,p}(M)/(\mathrm{Im}\partial+\mathrm{Im}\overline{\partial})).$$

We denote by $\Omega^{\overline{E}_\zeta}\in A^{1,1}(M_g)$ the curvature form associated to $\overline{E}_\zeta$. Let $(\phi_\zeta)_{\zeta\in S^1}$ be a family of $\mathbf{GL}(\mathbb{C})$-invariant formal power series such that $\phi_\zeta\in\mathbb{C}[[\mathbf{gl}_{\mathrm{rk}E_\zeta}(\mathbb{C})]]$ where $\mathrm{rk}E_\zeta$ stands for the rank of E_ζ which is a locally constant function on M_g. Moreover, let $\phi\in\mathbb{C}[[\bigoplus_{\zeta\in S^1}\mathbb{C}]]$ be any formal power series. We have the following definition.

Definition 2.1. The way to associate a smooth form to an equivariant hermitian vector bundle $\overline{E}$ by setting

$$\phi_g(\overline{E}) := \phi((\phi_\zeta(-\frac{\Omega^{\overline{E}_\zeta}}{2\pi i}))_{\zeta\in S^1})$$

is called an g-equivariant Chern-Weil theory associated to $(\phi_\zeta)_{\zeta\in S^1}$ and ϕ. The class of $\phi_g(\overline{E})$ in $\widetilde{A}(M_g)$ is independent of the metric.

Write dd^c for the differential operator $\frac{\overline{\partial}\partial}{2\pi i}$. The theory of equivariant secondary characteristic classes is described in the following theorem.

Theorem 2.2. *To every short exact sequence $\overline{\varepsilon} : 0 \to \overline{E}' \to \overline{E} \to \overline{E}'' \to 0$ of equivariant hermitian vector bundles on M, there is a unique way to attach a class $\widetilde{\phi}_g(\overline{\varepsilon}) \in \widetilde{A}(M_g)$ which satisfies the following three conditions:*
(i). $\widetilde{\phi}_g(\overline{\varepsilon})$ satisfies the differential equation

$$\mathrm{dd}^c\widetilde{\phi}_g(\overline{\varepsilon}) = \phi_g(\overline{E}' \oplus \overline{E}'') - \phi_g(\overline{E});$$

(ii). for every equivariant holomorphic map $f : M' \to M$, $\widetilde{\phi}_g(f^*\overline{\varepsilon}) = f_g^*\widetilde{\phi}_g(\overline{\varepsilon})$;
(iii). $\widetilde{\phi}_g(\overline{\varepsilon}) = 0$ if $\overline{\varepsilon}$ is equivariantly and orthogonally split.

Proof. This is [21, Theorem 3.4]. □

We now give some examples of equivariant character forms and their corresponding secondary characteristic classes.

Example 2.3. (i). The equivariant Chern character form $\mathrm{ch}_g(\overline{E}) := \sum_{\zeta\in S^1}\zeta\mathrm{ch}(\overline{E}_\zeta)$.
(ii). The equivariant Todd form $\mathrm{Td}_g(\overline{E}) := \frac{c_{\mathrm{rk}E_g}(\overline{E}_g)}{\mathrm{ch}_g(\sum_{j=0}^{\mathrm{rk}E}(-1)^j\wedge^j\overline{E}^\vee)}$. As in [18, Thm. 10.1.1] one can show that

$$\mathrm{Td}_g(\overline{E}) = \mathrm{Td}(\overline{E}_g)\prod_{\zeta\neq 1}\det(\frac{1}{1-\zeta^{-1}e^{\frac{\Omega^{\overline{E}_\zeta}}{2\pi i}}}).$$

(iii). Let $\overline{\varepsilon} : 0 \to \overline{E}' \to \overline{E} \to \overline{E}'' \to 0$ be a short exact sequence of hermitian vector bundles. The secondary Bott-Chern characteristic class is given by $\widetilde{\mathrm{ch}}_g(\overline{\varepsilon}) = \sum_{\zeta\in S^1}\zeta\widetilde{\mathrm{ch}}(\overline{\varepsilon}_\zeta)$.
(iv). If the equivariant structure g^ε has the eigenvalues $\zeta_1, \cdots, \zeta_m$, then the secondary Todd class is given by

$$\widetilde{\mathrm{Td}}_g(\overline{\varepsilon}) = \sum_{i=1}^m\prod_{j=1}^{i-1}\mathrm{Td}_g(\overline{E}_{\zeta_j})\cdot\widetilde{\mathrm{Td}}(\overline{\varepsilon}_{\zeta_i})\cdot\prod_{j=i+1}^m\mathrm{Td}_g(\overline{E}'_{\zeta_j}+\overline{E}''_{\zeta_j}).$$

Remark 2.4. One can use Theorem 2.2 to give a proof of the statements (iii) and (iv) in the examples above.

Let E be an equivariant hermitian vector bundle with two different hermitian metrics h_1 and h_2, we shall write $\widetilde{\phi}_g(E, h_1, h_2)$ for the equivariant secondary characteristic class associated to the exact sequence

$$0 \to (E, h_1) \to (E, h_2) \to 0 \to 0$$

where the map from (E, h_1) to (E, h_2) is the identity map.

2.2 Equivariant analytic torsion forms

In [7], J.-M. Bismut and K. Köhler extended the Ray-Singer analytic torsion to the higher analytic torsion form T for a holomorphic submersion. The purpose of making such an extension is that the differential equation on $\mathrm{dd}^c T$ gives a refinement of the Grothendieck-Riemann-Roch theorem. Later, in his article [23], X. Ma generalized J.-M. Bismut and K. Köhler's results to the equivariant case. In this subsection, we shall briefly recall Ma's construction of the equivariant analytic torsion form. This construction is not very important for understanding the rest of this article, but the equivariant analytic torsion form itself will be used to define a reasonable push-forward morphism between equivariant arithmetic G_0-groups.

We first fix some notations and assumptions. Let $f : M \to B$ be a proper holomorphic submersion of complex manifolds, and let TM, TB be the holomorphic tangent bundle on M, B. Denote by J^{Tf} the complex structure on the real relative tangent bundle $T_{\mathbb{R}} f$, and assume that h^{Tf} is a hermitian metric on Tf which induces a Riemannian metric g^{Tf}. Let $T^H M$ be a vector subbundle of TM such that $TM = T^H M \oplus Tf$, the following definition of Kähler fibration was given in [4, Def. 1.4].

Definition 2.5. The triple $(f, h^{Tf}, T^H M)$ is said to define a Kähler fibration if there exists a smooth real $(1,1)-$form ω which satisfies the following three conditions:

(i). ω is closed;

(ii). $T^H_{\mathbb{R}} M$ and $T_{\mathbb{R}} f$ are orthogonal with respect to ω;

(iii). if $X, Y \in T_{\mathbb{R}} f$, then $\omega(X, Y) = \langle X, J^{Tf} Y\rangle_{g^{Tf}}$.

It was shown in [4, Thm. 1.5 and 1.7] that for a given Kähler fibration, the form ω is unique up to addition of a form $f^*\eta$ where η is a real, closed $(1,1)$-form on B. Moreover, for any real, closed $(1,1)$-form ω on M such that the bilinear map $X, Y \in T_{\mathbb{R}} f \mapsto \omega(J^{Tf} X, Y) \in \mathbb{R}$ defines a Riemannian metric and hence a hermitian product h^{Tf} on Tf, we can define a Kähler fibration whose associated $(1,1)$-form is ω. In particular, for a given f, a Kähler metric on M defines a Kähler fibration if we choose $T^H M$ to be the orthogonal complement of Tf in TM and ω to be the Kähler form associated to this metric.

We now recall the Bismut superconnection of a Kähler fibration. Let (ξ, h^ξ) be a hermitian complex vector bundle on M. Let ∇^{Tf}, ∇^ξ be the holomorphic hermitian connections on (Tf, h^{Tf}) and (ξ, h^ξ). Let $\nabla^{\Lambda(T^{*(0,1)}f)}$ be the connection induced by ∇^{Tf} on $\Lambda(T^{*(0,1)}f)$. Then we may define a connection on $\Lambda(T^{*(0,1)}f)\otimes\xi$ by setting

$$\nabla^{\Lambda(T^{*(0,1)}f)\otimes\xi} = \nabla^{\Lambda(T^{*(0,1)}f)}\otimes 1 + 1\otimes\nabla^\xi.$$

Let E be the infinite-dimensional bundle on B whose fibre at each point $b\in B$ consists of the C^∞ sections of $\Lambda(T^{*(0,1)}f)\otimes\xi\mid_{f^{-1}b}$. This bundle E is a smooth $\mathbb{Z}$-graded bundle. We define a connection ∇^E on E as follows. If $U\in T_{\mathbb{R}}B$, let U^H be the lift of U in $T_{\mathbb{R}}^HM$ so that $f_*U^H = U$. Then for every smooth section s of E over B, we set

$$\nabla_U^E s = \nabla_{U^H}^{\Lambda(T^{*(0,1)}f)\otimes\xi} s.$$

For $b\in B$, let $\overline{\partial}^{Z_b}$ be the Dolbeault operator acting on E_b, and let $\overline{\partial}^{Z_b*}$ be its formal adjoint with respect to the canonical hermitian product on E_b (cf. [23, 1.2]). Let $C(T_{\mathbb{R}}f)$ be the Clifford algebra of $(T_{\mathbb{R}}f, h^{Tf})$, then the bundle $\Lambda(T^{*(0,1)}f)\otimes\xi$ has a natural $C(T_{\mathbb{R}}f)$-Clifford module structure. Actually, if $U\in Tf$, let $U'\in T^{*(0,1)}f$ correspond to U defined by $U'(\cdot) = h^{Tf}(U,\cdot)$, then for $U, V\in Tf$ we set

$$c(U) = \sqrt{2}U'\wedge,\quad c(\overline{V}) = -\sqrt{2}i_{\overline{V}}$$

where $i_{(\cdot)}$ is the contraction operator (cf. [9, Definition 1.6]). Moreover, if $U, V\in T_{\mathbb{R}}B$, we set $T(U^H, V^H) = -P^{Tf}[U^H, V^H]$ where P^{Tf} stands for the canonical projection from TM to Tf.

Definition 2.6. Let $e_1,\dots,e_{2m}$ be a basis of $T_{\mathbb{R}}B$, and let $e^1,\dots,e^{2m}$ be the dual basis of $T_{\mathbb{R}}^*B$. Then the element

$$c(T) = \frac{1}{2}\sum_{1\le\alpha,\beta\le 2m} e^\alpha\wedge e^\beta\widehat{\otimes}c(T(e_\alpha^H, e_\beta^H))$$

is a section of $(f^*\Lambda(T_{\mathbb{R}}^*B)\widehat{\otimes}\mathrm{End}(\Lambda(T^{*(0,1)}f)\otimes\xi))^{\mathrm{odd}}$.

Definition 2.7. For $u>0$, the Bismut superconnection on E is the differential operator

$$B_u = \nabla^E + \sqrt{u}(\overline{\partial}^Z + \overline{\partial}^{Z*}) - \frac{1}{2\sqrt{2u}}c(T)$$

on $f^*(\Lambda(T_{\mathbb{R}}^*B))\widehat{\otimes}(\Lambda(T^{*(0,1)}f)\otimes\xi)$.

Definition 2.8. Let N_V be the number operator on $\Lambda(T^{*(0,1)}f)\otimes\xi$ and on E, namely N_V acts as multiplication by p on $\Lambda^p(T^{*(0,1)}f)\otimes\xi$. For $U, V\in T_{\mathbb{R}}B$, set $\omega^{H\overline{H}}(U,V) = \omega^M(U^H, V^H)$ where ω^M is the closed form in the definition of Kähler fibration. Furthermore, for $u>0$, set $N_u = N_V + \frac{i\omega^{H\overline{H}}}{u}$.

We now turn to the equivariant case. Let G be a compact Lie group, we shall assume that all complex manifolds, hermitian vector bundles and holomorphic morphisms considered above are $G-$equivariant and all metrics are G-invariant. We will additionally assume that the direct images $R^k f_*\xi$ are all locally free so that the G-equivariant coherent sheaf $R^{\cdot} f_*\xi$ is locally free and hence a G-equivariant vector bundle over B. [23, 1.2] gives a G-invariant hermitian metric (the L^2-metric) $h^{R^{\cdot} f_*\xi}$ on the vector bundle $R^{\cdot} f_*\xi$.
For $g \in G$, let $M_g = \{x \in M \mid g{\cdot}x = x\}$ and $B_g = \{b \in B \mid g{\cdot}b = b\}$ be the fixed point submanifolds, then f induces a holomorphic submersion $f_g : M_g \to B_g$. Let Φ be the homomorphism $\alpha \mapsto (2i\pi)^{-\deg\alpha/2}$ of $\Lambda^{\mathrm{even}}(T^*_{\mathbb{R}}B)$ into itself. We put

$$\mathrm{ch}_g(R^{\cdot} f_*\xi, h^{R^{\cdot} f_*\xi}) = \sum_{k=0}^{\dim M - \dim B} (-1)^k \mathrm{ch}_g(R^k f_*\xi, h^{R^k f_*\xi})$$

and

$$\mathrm{ch}'_g(R^{\cdot} f_*\xi, h^{R^{\cdot} f_*\xi}) = \sum_{k=0}^{\dim M - \dim B} (-1)^k k \mathrm{ch}_g(R^k f_*\xi, h^{R^k f_*\xi}).$$

Definition 2.9. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, let

$$\zeta_1(s) = -\frac{1}{\Gamma(s)} \int_0^1 u^{s-1}(\Phi \mathrm{Tr_s}[gN_u \exp(-B_u^2)] - \mathrm{ch}'_g(R^{\cdot} f_*\xi, h^{R^{\cdot} f_*\xi}))\mathrm{d}u$$

and similarly for $s \in \mathbb{C}$ with $\mathrm{Re}(s) < \frac{1}{2}$, let

$$\zeta_2(s) = -\frac{1}{\Gamma(s)} \int_1^\infty u^{s-1}(\Phi \mathrm{Tr_s}[gN_u \exp(-B_u^2)] - \mathrm{ch}'_g(R^{\cdot} f_*\xi, h^{R^{\cdot} f_*\xi}))\mathrm{d}u.$$

X. Ma has proved that $\zeta_1(s)$ extends to a holomorphic function of $s \in \mathbb{C}$ near $s = 0$ and $\zeta_2(s)$ is a holomorphic function of s.

Definition 2.10. The smooth form $T_g(\omega^M, h^\xi) := \frac{\partial}{\partial s}(\zeta_1 + \zeta_2)(0)$ on B_g is called the equivariant analytic torsion form.

Theorem 2.11. *The form $T_g(\omega^M, h^\xi)$ lies in $\bigoplus_{p\geq 0} A^{p,p}(B_g)$ and satisfies the following differential equation*

$$\mathrm{dd}^c T_g(\omega^M, h^\xi) = \mathrm{ch}_g(R^{\cdot} f_*\xi, h^{R^{\cdot} f_*\xi}) - \int_{M_g/B_g} \mathrm{Td}_g(Tf, h^{Tf}) \mathrm{ch}_g(\xi, h^\xi).$$

Here $A^{p,p}(B_g)$ stands for the space of smooth forms on B_g of type (p,p).

Proof. This is [23, Theorem 2.12]. □

We define a secondary characteristic class

$$\widetilde{\mathrm{ch}}_g(R^{\cdot} f_*\xi, h^{R^{\cdot} f_*\xi}, h'^{R^{\cdot} f_*\xi}) := \sum_{k=0}^{\dim M - \dim B} (-1)^k \widetilde{\mathrm{ch}}_g(R^k f_*\xi, h^{R^k f_*\xi}, h'^{R^k f_*\xi})$$

such that it satisfies the following differential equation

$$\mathrm{dd}^c\widetilde{\mathrm{ch}}_g(R^{\cdot}f_*\xi, h^{R^{\cdot}f_*\xi}, h'^{R^{\cdot}f_*\xi}) = \mathrm{ch}_g(R^{\cdot}f_*\xi, h^{R^{\cdot}f_*\xi}) - \mathrm{ch}_g(R^{\cdot}f_*\xi, h'^{R^{\cdot}f_*\xi}),$$

then the anomaly formula can be described as follows.

Theorem 2.12. *(Anomaly formula) Let ω' be the form associated to another Kähler fibration for $f : M \to B$. Let h'^{Tf} be the metric on Tf in this new fibration and let h'^{ξ} be another metric on ξ. The following identity holds in $\widetilde{A}(B_g) := \bigoplus_{p\geq 0}(A^{p,p}(B_g)/(\mathrm{Im}\partial + \mathrm{Im}\overline{\partial}))$:*

$$\begin{aligned} T_g(\omega^M, h^{\xi}) - T_g(\omega'^M, h'^{\xi}) =& \widetilde{\mathrm{ch}}_g(R^{\cdot}f_*\xi, h^{R^{\cdot}f_*\xi}, h'^{R^{\cdot}f_*\xi}) \\ &- \int_{M_g/B_g}[\widetilde{\mathrm{Td}}_g(Tf, h^{Tf}, h'^{Tf})\mathrm{ch}_g(\xi, h^{\xi}) \\ &+ \mathrm{Td}_g(Tf, h'^{Tf})\widetilde{\mathrm{ch}}_g(\xi, h^{\xi}, h'^{\xi})]. \end{aligned}$$

In particular, the class of $T_g(\omega^M, h^{\xi})$ in $\widetilde{A}(B_g)$ only depends on (h^{Tf}, h^{ξ}).

Proof. This is [23, Theorem 2.13]. □

2.3 Equivariant Bott-Chern singular currents

The Bott-Chern singular current was defined by J.-M. Bismut, H. Gillet and C. Soulé in [5] in order to generalize the usual Bott-Chern secondary characteristic class to the case where one considers the resolutions of hermitian vector bundles associated to the closed immersions of complex manifolds. In [2], J.-M. Bismut generalized this topic to the equivariant case. We shall recall Bismut's construction of the equivariant Bott-Chern singular current in this subsection. Similar to the equivariant analytic torsion form, the construction itself is not very important for understanding our later arguments, we just recall it for the convenience of the reader. Bismut's construction was realized via some current valued zeta function which involves the supertraces of Quillen's superconnections. This is similar to the non-equivariant case.

As before, let g be the automorphism corresponding to an element in a compact Lie group G. Let $i : Y \to X$ be an equivariant closed immersion of G-equivariant Kähler manifolds, and let $\overline{\eta}$ be an equivariant hermitian vector bundle on Y. Assume that $\overline{\xi}.$ is a complex (of homological type) of equivariant hermitian vector bundles on X which provides a resolution of $i_*\overline{\eta}$. We denote the differential of the complex $\xi.$ by v. Note that $\xi.$ is acyclic outside Y and the homology sheaves of its restriction to Y are locally free and hence they are all vector bundles. We write $H_n = \mathcal{H}_n(\xi.\mid_Y)$ and define a $\mathbb{Z}$-graded bundle $H = \bigoplus_n H_n$. For each $y \in Y$ and $u \in TX_y$, we denote by $\partial_u v(y)$ the derivative of v at y in the direction u in any given holomorphic trivialization of $\xi.$ near y. Then the map $\partial_u v(y)$ acts on H_y as a chain map, and this action only depends on the image z of u in N_y where N stands for the normal bundle of $i(Y)$ in X. So we get a chain complex of holomorphic vector bundles $(H, \partial_z v)$.

Let π be the projection from the normal bundle N to Y, then we have a canonical identification of $\mathbb{Z}$-graded chain complexes

$$(\pi^*H, \partial_z v) \cong (\pi^*(\wedge N^\vee \otimes \eta), i_z).$$

For this, one can see [3, Section I. b]. Moreover, such an identification is an identification of G-bundles which induces a family of canonical isomorphisms $\gamma_n : H_n \cong \wedge^n N^\vee \otimes \eta$. Another way to describe these canonical isomorphisms γ_n is applying [13, Exp. VII, Lemma 2.4 and Proposition 2.5]. These two constructions coincide because they are both locally, on a suitable open covering $\{U_j\}_{j\in J}$, determined by any complex morphism over the identity map of $\eta\mid_{U_j}$ from $(\xi.\mid_{U_j}, v)$ to the minimal resolution of $\eta\mid_{U_j}$ (e.g. the Koszul resolution). The advantage of using the construction given in [13] is that it remains valid for arithmetic varieties over any base instead of the complex numbers. Later in [2], for the use of normalization, J.-M. Bismut considered the automorphism of $N^\vee$ defined by multiplying a constant $-\sqrt{-1}$, it induces an isomorphism of chain complexes

$$(\pi^*(\wedge N^\vee \otimes \eta), i_z) \cong (\pi^*(\wedge N^\vee \otimes \eta), \sqrt{-1}i_z)$$

and hence

$$(\pi^*H, \partial_z v) \cong (\pi^*(\wedge N^\vee \otimes \eta), \sqrt{-1}i_z).$$

This identification induces a family of isomorphisms $\widetilde{\gamma_n} : H_n \cong \wedge^n N^\vee \otimes \eta$. By finite dimensional Hodge theory, for each $y \in Y$, there is a canonical isomorphism

$$H_y \cong \{f \in \xi._y \mid vf = 0, v^*f = 0\}$$

where v^* is the dual of v with respect to the metrics on $\xi.$. This means that H can be regarded as a smooth $\mathbb{Z}$-graded G-equivariant subbundle of ξ so that it carries an induced G-invariant metric. On the other hand, we endow $\wedge N^\vee \otimes \eta$ with the metric induced from $\overline{N}$ and $\overline{\eta}$. J.-M. Bismut introduced the following definition.

Definition 2.13. We say that the metrics on the complex of equivariant hermitian vector bundles $\overline{\xi}.$ satisfy Bismut assumption (A) if the identification $(\pi^*H, \partial_z v) \cong (\pi^*(\wedge N^\vee \otimes \eta), \sqrt{-1}i_z)$ also identifies the metrics, it is equivalent to the condition that the identification $(\pi^*H, \partial_z v) \cong (\pi^*(\wedge N^\vee \otimes \eta), i_z)$ identifies the metrics.

Proposition 2.14. *There always exist G-invariant metrics on $\xi.$ which satisfy Bismut assumption (A) with respect to the equivariant hermitian vector bundles $\overline{N}$ and $\overline{\eta}$.*

Proof. This is [2, Proposition 3.5]. □

From now on we always suppose that the metrics on a resolution satisfy Bismut assumption (A). Let ∇^ξ be the canonical hermitian holomorphic connection on $\xi.$, then for each $u > 0$, we may define a G-invariant superconnection

$$C_u := \nabla^\xi + \sqrt{u}(v + v^*)$$

on the $\mathbb{Z}_2$-graded vector bundle ξ. Moreover, let Φ be the map $\alpha \in \wedge(T^*_{\mathbb{R}}X_g) \to (2\pi i)^{-\deg\alpha/2}\alpha \in \wedge(T^*_{\mathbb{R}}X_g)$ and denote

$$(\mathrm{Td}_g^{-1})'(\overline{N}) := \frac{\partial}{\partial b}\mid_{b=0} (\mathrm{Td}_g(b\cdot \mathrm{Id} - \frac{\Omega^{\overline{N}}}{2\pi i})^{-1})$$

where $\Omega^{\overline{N}}$ is the curvature form associated to $\overline{N}$. We recall as follows the construction of the equivariant singular current given in [2, Section VI].

LEMMA 2.15. *Let N_H be the number operator on the complex ξ. i.e. it acts on ξ_j as multiplication by j, then for $s \in \mathbb{C}$ and $0 < \mathrm{Re}(s) < \frac{1}{2}$, the current valued zeta function*

$$Z_g(\overline{\xi}.)(s) := \frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}[\Phi\mathrm{Tr_s}(N_H g\exp(-C_u^2)) + (\mathrm{Td}_g^{-1})'(\overline{N})\mathrm{ch}_g(\overline{\eta})\delta_{Y_g}]\mathrm{d}u$$

is well-defined on X_g and it has a meromorphic continuation to the complex plane which is holomorphic at $s = 0$.

DEFINITION 2.16. The equivariant singular Bott-Chern current on X_g associated to the resolution $\overline{\xi}.$ is defined as

$$T_g(\overline{\xi}.) := \frac{\partial}{\partial s}\mid_{s=0} Z_g(\overline{\xi}.)(s).$$

THEOREM 2.17. *The current $T_g(\overline{\xi}.)$ is a sum of (p,p)-currents and it satisfies the differential equation*

$$\mathrm{dd}^c T_g(\overline{\xi}.) = i_{g*}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N}) - \sum_k (-1)^k \mathrm{ch}_g(\overline{\xi}_k).$$

Moreover, the wave front set of $T_g(\overline{\xi}.)$ is contained in $N^\vee_{g,\mathbb{R}}$ where $N^\vee_{g,\mathbb{R}}$ stands for the underlying real bundle of the dual of N_g.

Proof. This follows from [2, Theorem 6.7, Remark 6.8]. □

Finally, we recall a theorem concerning the relationship of equivariant Bott-Chern singular currents involved in a double complex. This theorem will be used to show that our definition of a general embedding morphism in equivariant arithmetic G_0-theory is reasonable.

THEOREM 2.18. *Let*

$$\overline{\chi}: \quad 0 \to \overline{\eta}_n \to \cdots \to \overline{\eta}_1 \to \overline{\eta}_0 \to 0$$

be an exact sequence of equivariant hermitian vector bundles on Y. Assume that we have the following double complex consisting of resolutions of $i_*\overline{\chi}$ such that all rows are exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & \overline{\xi}_{n,\cdot} \arrow[r] \arrow[d] & \cdots \arrow[r] & \overline{\xi}_{1,\cdot} \arrow[r] \arrow[d] & \overline{\xi}_{0,\cdot} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & i_*\overline{\eta}_n \arrow[r] & \cdots \arrow[r] & i_*\overline{\eta}_1 \arrow[r] & i_*\overline{\eta}_0 \arrow[r] & 0.
\end{tikzcd}$$

For each k, we write $\overline{\varepsilon}_k$ for the exact sequence

$$0 \to \overline{\xi}_{n,k} \to \cdots \to \overline{\xi}_{1,k} \to \overline{\xi}_{0,k} \to 0.$$

Then we have the following equality in $\widetilde{\mathcal{U}}(X_g) := \bigoplus_{p\geq 0}(D^{p,p}(X_g)/(\mathrm{Im}\partial + \mathrm{Im}\overline{\partial}))$

$$\sum_{j=0}^{n}(-1)^j T_g(\overline{\xi}_{j,\cdot}) = i_{g*}\frac{\widetilde{\mathrm{ch}}_g(\overline{\chi})}{\mathrm{Td}_g(\overline{N})} - \sum_k (-1)^k \widetilde{\mathrm{ch}}_g(\overline{\varepsilon}_k).$$

Here $D^{p,p}(X_g)$ stands for the space of currents on X_g of type (p,p).

Proof. This is [21, Theorem 3.14]. □

2.4 Bismut-Ma's immersion formula

In this subsection, we shall recall Bismut-Ma's immersion formula which reflects the behaviour of the equivariant analytic torsion forms of a Kähler fibration under composition of an immersion and a submersion. By translating to the equivariant arithmetic G_0-theoretic language, such a formula can be used to measure, in arithmetic G_0-theory, the difference between a push-forward morphism and the composition formed as an embedding morphism followed by a push-forward morphism. Although Bismut-Ma's immersion formula plays a very important role in our arguments, we shall not recall its proof since it is rather long and technical.

Let $i : Y \to X$ be an equivariant closed immersion of G-equivariant Kähler manifolds. Let S be a complex manifold with the trivial G-action, and let $f : Y \to S$, $l : X \to S$ be two equivariant holomorphic submersions such that $f = l \circ i$. Assume that $\overline{\eta}$ is an equivariant hermitian vector bundle on Y and $\overline{\xi}.$ provides a resolution of $i_*\overline{\eta}$ on X whose metrics satisfy Bismut assumption (A). Let ω^Y, ω^X be the real, closed and G-invariant $(1,1)$-forms on Y, X which induce the Kähler fibrations with respect to f and l respectively. We additionally assume that ω^Y is the pull-back of ω^X so that the Kähler metric on Y is induced by the Kähler metric on X. As before, denote by N the normal bundle of $i(Y)$ in X. Consider the following exact sequence

$$\overline{\mathcal{N}}: \quad 0 \to \overline{Tf} \to \overline{Tl}\mid_Y \to \overline{N} \to 0$$

where N is endowed with the quotient metric, we shall write $\widetilde{\mathrm{Td}}_g(\overline{Tf}, \overline{Tl}\mid_Y)$ for $\widetilde{\mathrm{Td}}_g(\overline{\mathcal{N}})$ the equivariant Bott-Chern secondary characteristic class associated to $\overline{\mathcal{N}}$. It satisfies the following differential equation

$$\mathrm{dd}^c \widetilde{\mathrm{Td}}_g(\overline{Tf}, \overline{Tl}\mid_Y) = \mathrm{Td}_g(Tf, h^{Tf})\mathrm{Td}_g(\overline{N}) - \mathrm{Td}_g(Tl\mid_Y, h^{Tl}).$$

For simplicity, we shall suppose that in the resolution $\xi.$, ξ_j are all $l-$acyclic and moreover η is $f-$acyclic. By an easy argument of long exact sequence, we have the following exact sequence

$$\Xi: \quad 0 \to l_*(\xi_m) \to l_*(\xi_{m-1}) \to \ldots \to l_*(\xi_0) \to f_*\eta \to 0.$$

By the semi-continuity theorem, all the elements in the exact sequence above are vector bundles. In this case, we recall the definition of the L^2-metrics on direct images precisely as follows. We just take f_*h^η as an example. Note that the semi-continuity theorem implies that the natural map

$$(R^0 f_*\eta)_s \to H^0(Y_s, \eta\mid_{Y_s})$$

is an isomorphism for every point $s \in S$ where Y_s stands for the fibre over s. We may endow $H^0(Y_s, \eta\mid_{Y_s})$ with a L^2-metric given by the formula

$$< u, v >_{L^2} := \frac{1}{(2\pi)^{d_s}} \int_{Y_s} h^\eta(u, v) \frac{{\omega^Y}^{d_s}}{d_s!}$$

where d_s is the complex dimension of the fibre Y_s. It can be shown that these metrics depend on s in a C^∞ manner (cf. [9, p.278]) and hence define a hermitian metric on $f_*\eta$. We shall denote it by f_*h^η.

In order to understand the statement of Bismut-Ma's immersion formula, we still have to recall an important concept defined by J.-M. Bismut, the equivariant R-genus. Let W be a G-equivariant complex manifold, and let $\overline{E}$ be an equivariant hermitian vector bundle on W. For $\zeta \in S^1$ and $s > 1$ consider the zeta function

$$L(\zeta, s) = \sum_{k=1}^{\infty} \frac{\zeta^k}{k^s}$$

and its meromorphic continuation to the whole complex plane. Define the formal power series in x

$$\widetilde{R}(\zeta, x) := \sum_{n=0}^{\infty} \big(\frac{\partial L}{\partial s}(\zeta, -n) + L(\zeta, -n) \sum_{j=1}^{n} \frac{1}{2j} \big) \frac{x^n}{n!}.$$

DEFINITION 2.19. The Bismut equivariant R-genus of an equivariant hermitian vector bundle $\overline{E}$ with $\overline{E}\mid_{X_g} = \sum_\zeta \overline{E}_\zeta$ is defined as

$$R_g(\overline{E}) := \sum_{\zeta \in S^1} \big(\mathrm{Tr}\widetilde{R}(\zeta, -\frac{\Omega^{\overline{E}_\zeta}}{2\pi i}) - \mathrm{Tr}\widetilde{R}(1/\zeta, \frac{\Omega^{\overline{E}_\zeta}}{2\pi i}) \big)$$

where $\Omega^{\overline{E}_\zeta}$ is the curvature form associated to $\overline{E}_\zeta$. Actually, the class of $R_g(\overline{E})$ in $\widetilde{A}(X_g)$ is independent of the metric and we just write $R_g(E)$ for it. Furthermore, the class $R_g(\cdot)$ is additive.

THEOREM 2.20. *(Immersion formula) Let notations and assumptions be as* *above. Then the equality*

$$\begin{aligned}&\sum_{i=0}^{m}(-1)^i T_g(\omega^X,h^{\xi_i})-T_g(\omega^Y,h^\eta)+\widetilde{\mathrm{ch}}_g(\Xi,h^{L^2})\\ =&\int_{X_g/S}\mathrm{Td}_g(Tl,h^{Tl})T_g(\overline{\xi}.)+\int_{Y_g/S}\frac{\widetilde{\mathrm{Td}}_g(\overline{Tf},\overline{Tl}\mid_Y)}{\mathrm{Td}_g(\overline{N})}\mathrm{ch}_g(\overline{\eta})\\ &+\int_{X_g/S}\mathrm{Td}_g(Tl)R_g(Tl)\sum_{i=0}^{m}(-1)^i\mathrm{ch}_g(\xi_i)-\int_{Y_g/S}\mathrm{Td}_g(Tf)R_g(Tf)\mathrm{ch}_g(\eta)\end{aligned}$$

holds in $\widetilde{A}(S)$.

Proof. This is the combination of [8, Theorem 0.1 and 0.2], the main theorems in that paper. □

3 A vanishing theorem for equivariant closed immersions

3.1 The statement

By a projective manifold we shall understand a compact complex manifold which is projective algebraic, that means a projective manifold is the complex analytic space $X(\mathbb{C})$ associated to a smooth projective variety X over $\mathbb{C}$ (cf. [17, Appendix B]). Let μ_n be the diagonalisable group variety over $\mathbb{C}$ associated to $\mathbb{Z}/n\mathbb{Z}$. We say X is μ_n-equivariant if it admits a μ_n-projective action, this means the associated projective manifold $X(\mathbb{C})$ admits an action by the group of complex n-th roots of unity. Denote by X_{μ_n} the fixed point subscheme of X, by GAGA principle, $X_{\mu_n}(\mathbb{C})$ is equal to $X(\mathbb{C})_g$ where g is the automorphism on $X(\mathbb{C})$ corresponding to a fixed primitive n-th root of unity. If no confusion arises, we shall not distinguish between X and $X(\mathbb{C})$ as well as X_{μ_n} and X_g. Since the classical arguments of locally free resolutions may not be compatible with the equivariant setting, we summarize some crucial facts we need as follows.

(i). Every equivariant coherent sheaf on an equivariantly projective scheme is an equivariant quotient of an equivariant locally free coherent sheaf.

(ii). Every equivariant coherent sheaf on an equivariantly projective scheme admits an equivariant locally free resolution. It is finite if the equivariant scheme is regular.

(iii). An exact sequence of equivariant coherent sheaves on an equivariantly projective scheme admits an exact sequence of equivariant locally free resolutions.

(iv). Any two equivariant locally free resolutions of an equivariant coherent sheaf on an equivariantly projective scheme can be dominated by a third one.

Now let $i: Y\to X$ be a μ_n-equivariant closed immersion of projective manifolds with normal bundle N. Let S be a projective manifold with the trivial μ_n-action and let $h: X\to S$ be an equivariant holomorphic submersion whose restriction $f : Y \to S$ is an equivariant holomorphic submersion. According to our assumptions, we may define a Kähler fibration with respect to h by choosing a $\mu_n(\mathbb{C})$-invariant Kähler form ω^X on X. By restricting ω^X to Y we obtain a Kähler fibration with respect to f. The same thing goes to $h_g : X_g \to S$ and $f_g : Y_g \to S$. Let $\overline{\eta}$ be an equivariant hermitian holomorphic vector bundle on Y, assume that $(\overline{\xi}.,v)$ is a complex of equivariant hermitian vector bundles on X which provides a resolution of $i_*\overline{\eta}$, whose metrics satisfy Bismut assumption (A).

Write N_g for the 0-degree part of $N\mid_{Y_g}$ which is isomorphic to the normal bundle of $i_g(Y_g)$ in X_g and denote by F the orthogonal complement of N_g. According to [13, Exp. VII, Lemma 2.4 and Proposition 2.5] we know that there exists a canonical isomorphism from the homology sheaf $H(\xi.\mid_{X_g})$ to $i_{g*}(\wedge^{\cdot}F^\vee\otimes\eta\mid_{Y_g})$ which is equivariant. Then the restriction of $(\xi.,v)$ to X_g can always split into a series of short exact sequences in the following way:

$$(*):\quad 0\to \mathrm{Im}\to \mathrm{Ker}\to i_{g*}(\wedge^{\cdot}F^\vee\otimes\eta\mid_{Y_g})\to 0$$

and

$$(**):\quad 0\to \mathrm{Ker}\to \xi.\mid_{X_g}\to \mathrm{Im}\to 0.$$

Suppose that $\wedge^{\cdot}F^\vee\otimes\eta\mid_{Y_g}$ and $\xi.\mid_{X_g}$ are all acyclic (higher direct images vanish). Then according to an easy argument of long exact sequence, these short exact sequences $(*)$ and $(**)$ induce a series of short exact sequences of direct images:

$$\mathcal{H}(*):\quad 0\to R^0h_{g*}(\mathrm{Im})\to R^0h_{g*}(\mathrm{Ker})\to R^0f_{g*}(\wedge^{\cdot}F^\vee\otimes\eta\mid_{Y_g})\to 0$$

and

$$\mathcal{H}(**):\quad 0\to R^0h_{g*}(\mathrm{Ker})\to R^0h_{g*}(\xi.\mid_{X_g})\to R^0h_{g*}(\mathrm{Im})\to 0.$$

By semi-continuity theorem, all elements in the exact sequences above are vector bundles. We endow $R^0h_{g*}(\xi.\mid_{X_g})$ and $R^0f_{g*}(\wedge^{\cdot}F^\vee\otimes\eta\mid_{Y_g})$ with the L^2-metrics which are induced by the metrics on $\xi.$, η and F. Here the normal bundle N admits the quotient metric induced from the exact sequence

$$0\to Tf\to Th\mid_Y\to N\to 0$$

and the bundle F admits the metric induced by the metric on N. Moreover, we endow $R^0h_{g*}(\mathrm{Im})$ and $R^0h_{g*}(\mathrm{Ker})$ with the metrics induced by the L^2-metrics of $R^0h_{g*}(\xi.\mid_{X_g})$ so that $\mathcal{H}(*)$ and $\mathcal{H}(**)$ become short exact sequences of equivariant hermitian vector bundles. Denote by $\widetilde{\mathrm{ch}}_g(\overline{\xi}.,\overline{\eta})$ the alternating sum of the equivariant secondary Bott-Chern characteristic classes of $\mathcal{H}(*)$ and $\mathcal{H}(**)$ such that it satisfies the following differential equation

$$\mathrm{dd}^c\widetilde{\mathrm{ch}}_g(\overline{\xi}.,\overline{\eta})=\sum_j(-1)^j\mathrm{ch}_g(R^0f_{g*}(\wedge^j\overline{F}^\vee\otimes\overline{\eta}\mid_{Y_g})) \\ -\sum_j(-1)^j\mathrm{ch}_g(R^0h_{g*}(\overline{\xi}_j\mid_{X_g})).$$

Now the difference

$$\begin{aligned}\delta(i,\overline{\eta},\overline{\xi}.) :=& \widetilde{\mathrm{ch}}_g(\overline{\xi}.,\overline{\eta}) - \sum_k (-1)^k T_g(\omega^{Y_g}, h^{\wedge^k F^\vee \otimes \eta|_{Y_g}}) \\ &+ \sum_k (-1)^k T_g(\omega^{X_g}, h^{\xi_k|_{X_g}}) - \int_{X_g/S} T_g(\overline{\xi.})\mathrm{Td}(\overline{Th_g}) \\ &- \int_{Y_g/S} \mathrm{Td}(Tf_g)\mathrm{Td}_g^{-1}(F)\mathrm{ch}_g(\eta)R(N_g) \\ &- \int_{Y_g/S} \mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N})\widetilde{\mathrm{Td}}(\overline{Tf_g},\overline{Th_g}\mid_{Y_g})\end{aligned}$$

makes sense and it is an element in $\bigoplus_{p\geq 0} A^{p,p}(S)/(\mathrm{Im}\partial + \mathrm{Im}\overline{\partial})$. Here the symbols $T_g(\cdot)$ in the summations stand for analytic torsion forms introduced in Section 2.1, the symbol $T_g(\overline{\xi.})$ in the integral is the equivariant Bott-Chern singular current introduced in Section 2.2.

The vanishing theorem for equivariant closed immersions can be formulated as the following.

Theorem 3.1. *Let $i: Y \to X$ be an equivariant closed immersion of projective manifolds, and let S be a projective manifold with the trivial μ_n-action. Assume that we are given two equivariant holomorphic submersions $f: Y \to S$ and $h: X \to S$ such that $f = h \circ i$. Then X admits an equivariant hermitian very ample invertible sheaf $\overline{\mathcal{L}}$ relative to the morphism h, and for any equivariant hermitian resolution $0 \to \overline{\xi}_m \to \cdots \overline{\xi}_1 \to \overline{\xi}_0 \to i_*\overline{\eta} \to 0$ we have*

$$\delta(i, \overline{\eta} \otimes i^*\overline{\mathcal{L}}^{\otimes n}, \overline{\xi}. \otimes \overline{\mathcal{L}}^{\otimes n}) = 0 \qquad \textit{for} \quad n \gg 0.$$

Here the metrics on the resolution are supposed to satisfy Bismut assumption (A).

3.2 Deformation to the normal cone

To prove the vanishing theorem for closed immersions, we use a geometric construction called the deformation to the normal cone which allows us to deform a resolution of hermitian vector bundle associated to a closed immersion of projective manifolds to a simpler one. The δ-difference of this new simpler resolution is much easier to compute.

Let $i: Y \hookrightarrow X$ be a closed immersion of projective manifolds with normal bundle $N_{X/Y}$. For a vector bundle E on X or Y, the notation $\mathbb{P}(E)$ will stand for the projective space bundle $\mathrm{Proj}(\mathrm{Sym}(E^\vee))$.

Definition 3.2. The deformation to the normal cone $W(i)$ of the immersion i is the blowing up of $X \times \mathbb{P}^1$ along $Y \times \{\infty\}$. We shall just write W for $W(i)$ if there is no confusion about the immersion.

There are too many geometric objects and morphisms appearing in the construction of the deformation to the normal cone, we have to fix various notations in a clear way. We denote by p_X (resp. p_Y) the projection $X \times \mathbb{P}^1 \to X$ (resp. $Y \times \mathbb{P}^1 \to Y$) and by π the blow-down map $W \to X \times \mathbb{P}^1$. We also denote by q_X (resp. q_Y) the projection $X \times \mathbb{P}^1 \to \mathbb{P}^1$ (resp. $Y \times \mathbb{P}^1 \to \mathbb{P}^1$) and by q_W the composition $q_X \circ \pi$. It is well known that the map q_W is flat and for $t \in \mathbb{P}^1$, we have

$$q_W^{-1}(t) \cong \begin{cases} X \times \{t\}, & \text{if } t \neq \infty, \\ P \cup \widetilde{X}, & \text{if } t = \infty, \end{cases}$$

where $\widetilde{X}$ is isomorphic to the blowing up of X along Y and P is isomorphic to the projective completion of $N_{X/Y}$ i.e. the projective space bundle $\mathbb{P}(N_{X/Y} \oplus \mathcal{O}_Y)$. Denote the canonical projection from $\mathbb{P}(N_{X/Y} \oplus \mathcal{O}_Y)$ to Y by π_P, then the morphism $\mathcal{O}_Y \to N_{X/Y} \oplus \mathcal{O}_Y$ induces a canonical section $i_\infty : Y \hookrightarrow \mathbb{P}(N_{X/Y} \oplus \mathcal{O}_Y)$ which is called the zero section embedding. Moreover, let $j : Y \times \mathbb{P}^1 \to W$ be the canonical closed immersion induced by $i \times \mathrm{Id}$, then the component $\widetilde{X}$ doesn't meet $j(Y \times \mathbb{P}^1)$ and the intersection of $j(Y \times \mathbb{P}^1)$ with P is exactly the image of Y under the section i_∞.

The advantage of the construction of the deformation to the normal cone is that we may control the rational equivalence class of the fibres $q_W^{-1}(t)$. More precisely, in the language of line bundles, we have the isomorphisms $\mathcal{O}(X) \cong \mathcal{O}(P + \widetilde{X}) \cong \mathcal{O}(P) \otimes \mathcal{O}(\widetilde{X})$ which is an immediate consequence of the isomorphism $\mathcal{O}(0) \cong \mathcal{O}(\infty)$ on $\mathbb{P}^1$.

On $P = \mathbb{P}(N_{X/Y} \oplus \mathcal{O}_Y)$, there exists a tautological exact sequence

$$0 \to \mathcal{O}(-1) \to \pi_P^*(N_{X/Y} \oplus \mathcal{O}_Y) \to Q \to 0$$

where Q is the tautological quotient bundle. This exact sequence and the inclusion $\mathcal{O}_P \to \pi_P^*(N_{X/Y} \oplus \mathcal{O}_Y)$ induce a section $\sigma : \mathcal{O}_P \to Q$ which vanishes along the zero section $i_\infty(Y)$. By duality we get a morphism $Q^\vee \to \mathcal{O}_P$, and this morphism induces the following exact sequence

$$0 \to \wedge^n Q^\vee \to \cdots \to \wedge^2 Q^\vee \to Q^\vee \to \mathcal{O}_P \to i_{\infty *}\mathcal{O}_Y \to 0$$

where n is the rank of Q. Note that i_∞ is a section of π_P i.e. $\pi_P \circ i_\infty = \mathrm{Id}$, the projection formula implies the following definition.

Definition 3.3. For any vector bundle η on Y, the following complex of vector bundles

$$0 \to \wedge^n Q^\vee \otimes \pi_P^*\eta \to \cdots \to \wedge^2 Q^\vee \otimes \pi_P^*\eta \to Q^\vee \otimes \pi_P^*\eta \to \pi_P^*\eta \to 0$$

provides a resolution of $i_{\infty *}\eta$ on P. This complex is called the Koszul resolution of $i_{\infty *}\eta$ and will be denoted by $\kappa(\eta, N_{X/Y})$. If the normal bundle $N_{X/Y}$ admits some hermitian metric, then the tautological exact sequence induces a hermitian metric on Q. If, moreover, the bundle η also admits a hermitian metric, then the Koszul resolution is a complex of hermitian vector bundles and will be denoted by $\overline{\kappa}(\overline{\eta}, \overline{N}_{X/Y})$.

Now, assume that X is a μ_n-equivariant projective manifold and E is an equivariant locally free sheaf on X. Then according to [20, (1.4) and (1.5)], $\mathbb{P}(E)$ admits a canonical μ_n-equivariant structure such that the projection map $\mathbb{P}(E) \to X$ is equivariant and the canonical bundle $\mathcal{O}(1)$ admits an equivariant structure. Moreover, let $Y \to X$ be an equivariant closed immersion of projective manifolds, according to [20, (1.6)] the action of μ_n on X can be extended to the blowing up $\mathrm{Bl}_Y X$ such that the blow-down map is equivariant and the canonical bundle $\mathcal{O}(1)$ admits an equivariant structure. So by endowing $\mathbb{P}^1$ with the trivial μ_n-action, the construction of the deformation to the normal cone described above is compatible with the equivariant setting.
For the use of our later arguments, the Kähler metric chosen on W should be well controlled on the fibres of the deformation. For this purpose, it is necessary to introduce the following definition.

Definition 3.4. (Rössler) A metric h on W is said to be normal to the deformation if
(a). it is invariant and Kähler;
(b). the restriction $h\mid_{j_{g*}(Y_g\times\mathbb{P}^1)}$ is a product $h'\times h''$, where h' is a Kähler metric on Y_g and h'' is a Kähler metric on $\mathbb{P}^1$;
(c). the intersections of $X\times\{0\}$ with $j_*(Y\times\mathbb{P}^1)$ and of P with $j_*(Y\times\mathbb{P}^1)$ are orthogonal at the fixed points.

Lemma 3.5. *For any μ_n-invariant Kähler metric h^X on X which induces an invariant Kähler metric h^Y on Y, there exists a metric h^W on W which is normal to the deformation and the restriction of h^W to $X\cong X\times\{0\}$ (resp. $Y\cong Y\times\{\infty\}$) is exactly h^X (resp. h^Y). Moreover, we may require that the hermitian normal bundles $\overline{N}_{Y\times\mathbb{P}^1/Y\times\{0\}}$ and $\overline{N}_{Y\times\mathbb{P}^1/Y\times\{\infty\}}$ are both isometric to the trivial bundles with trivial metrics.*

Proof. The existence of the metric which is normal to the deformation is the content of [21, Lemma 6.13] and [28, Lemma 6.14], such a metric is constructed via the Grassmannian graph construction. Roughly speaking, according to another description of the deformation to the normal cone via the Grassmannian graph construction, we have an embedding $W\to X\times\mathbb{P}^r\times\mathbb{P}^1$ and the metric h^W is the μ_n-average of the restriction of a product metric on $X\times\mathbb{P}^r\times\mathbb{P}^1$ (cf. [28, Lemma 6.14]). When we endow X in the product with the metric h^X, the requirements on restrictions are automatically satisfied since h^X is μ_n-invariant. To fulfill the requirements on hermitian normal bundles, we may just choose the Fubini-Study metric on $\mathbb{P}^1$. $\square$

We summarize some very important results about the application of the deformation to the normal cone as follows. Their proofs can be found in [21, Section 2 and 6.2].

Theorem 3.6. *Let $i: Y\to X$ be an equivariant closed immersion of equivariant projective manifolds, and let $W=W(i)$ be the deformation to the normal cone of i. Assume that $\overline{\eta}$ is an equivariant hermitian vector bundle on Y. Then*

(i). there exists an equivariant hermitian resolution of $j_*p_Y^*(\overline{\eta})$ on W, whose metrics satisfy Bismut assumption (A) and whose restriction to $\widetilde{X}$ is equivariantly and orthogonally split;
(ii). the natural morphism from the deformation to the normal cone $W(i_g)$ to the fixed point submanifold $W(i)_g$ is a closed immersion, this closed immersion induces the closed immersions $\mathbb{P}(N_{X_g/Y_g}\oplus\mathcal{O}_{Y_g})\to\mathbb{P}(N_{X/Y}\oplus\mathcal{O}_Y)_g$ and $\widetilde{X_g}\to\widetilde{X}_g$;
(iii). the fixed point submanifold of $\mathbb{P}(N_{X/Y}\oplus\mathcal{O}_Y)$ is $\mathbb{P}(N_{X_g/Y_g}\oplus\mathcal{O}_{Y_g})\coprod_{\zeta\neq 1}\mathbb{P}((N_{X/Y})_\zeta)$;
(iv). the closed immersion $i_{\infty,g}$ factors through $\mathbb{P}(N_{X_g/Y_g}\oplus\mathcal{O}_{Y_g})$ and the other components $\mathbb{P}((N_{X/Y})_\zeta)$ don't meet Y. Hence the complex $\kappa(\mathcal{O}_Y,N_{X/Y})_g$, obtained by taking the 0-degree part of the Koszul resolution, provides a resolution of $\mathcal{O}_{Y_g}$ on $\mathbb{P}(N_{X/Y}\oplus\mathcal{O}_Y)_g$.

3.3 Proof of the vanishing theorem

We shall first prove the first part of the vanishing theorem for closed immersions i.e. the existence of an equivariant hermitian very ample invertible sheaf on X which is relative to the morphism $h:X\to S$. Generally speaking, such an invertible sheaf can be constructed rather easily since X admits a μ_n-projective action and the μ_n-action on S is supposed to be trivial, but for the whole proof of the vanishing theorem we would like to construct a special one which is the pull-back of some equivariant hermitian very ample invertible sheaf on $W(i)$ under the identification $X\cong X\times\{0\}$. Our starting point is the following.

Definition 3.7. Let M be a μ_n-projective manifold, and let $\mathbb{P}^n_M$ be some relative projective space over M. A μ_n-action on $\mathbb{P}^n_M$ arising from some μ_n-action on the free sheaf $\mathcal{O}_M^{\oplus n+1}$ via the functorial properties of the Proj symbol will be called a global μ_n-action.

The advantage of considering global μ_n-action is that on a projective space which admits a global μ_n-action the twisted line bundle $\mathcal{O}(1)$ is naturally μ_n-equivariant.

Lemma 3.8. *The morphism $h:X\to S$ factors though some relative projective space $\mathbb{P}^r_S$ which admits a global μ_n-action.*

Proof. By assumption, X admits a μ_n-projective action. Then [21, Lemma 2.4 and 2.5] imply that there exists an equivariant closed immersion from X to some projective space $\mathbb{P}^r$ endowed with a global action. By using the universal property of fibre product, we obtain a morphism from X to $\mathbb{P}^r_S=S\times\mathbb{P}^r$ which is equivariant. Moreover, this morphism is clearly a closed immersion. Since the action on S is trivial, the induced action on the fibre product $S\times\mathbb{P}^r$ is still global. So we are done. $\square$

Lemma 3.9. *Let $l:W(i)\to S$ be the composition $h\circ p_X\circ\pi$. Then $W(i)$ admits an equivariant very ample invertible sheaf $\mathcal{L}$ which is relative to l.*

Proof. By Lemma 3.8, $h : X \to S$ factors through some relative projective space $\mathbb{P}_S^r$ which admits a global μ_n-action. So X admits an equivariant very ample invertible sheaf relative to h. Since the μ_n-action on S is supposed to be trivial, $\mathbb{P}_X^1 = X \times \mathbb{P}^1 \cong X \times_S \mathbb{P}_S^1$ also admits an equivariant very ample invertible sheaf relative to the morphism $h \circ p_X$ which is denoted by $\mathcal{G}$. Moreover, by construction, $W(i)$ admits a very ample invertible sheaf $\mathcal{O}_W(1) \otimes \pi^*\mathcal{G}^{\otimes b}$ for some $b \geq 0$ which is relative to the blow-down map π (cf. [17, II. Proposition 7.10]). Assume that $\mathbb{P}_X^1 \times_S \mathbb{P}_S^m$ is the relative projective space associated to $\mathcal{O}_W(1) \otimes \pi^*\mathcal{G}^{\otimes b}$, and that $\mathbb{P}_S^n$ is the relative projective space associated to $\mathcal{G}$. Then the very ample invertible sheave on $\mathbb{P}_X^1 \times_S \mathbb{P}_S^m$ with respect to the embedding

$$\mathbb{P}_X^1 \times_S \mathbb{P}_S^m \hookrightarrow \mathbb{P}_S^n \times_S \mathbb{P}_S^m$$

can be written as $\mathcal{G} \boxtimes \mathcal{O}_{\mathbb{P}_S^m}(1)$ whose restriction to $W(i)$ is equal to $\mathcal{O}_W(1) \otimes \pi^*\mathcal{G}^{\otimes b+1}$. Therefore, $\mathcal{O}_W(1) \otimes \pi^*\mathcal{G}^{\otimes b+1}$ is a very ample invertible sheaf on $W(i)$ relative to $l : W(i) \to S$, this invertible sheaf is clearly equivariant. □

From now on, we shall fix the equivariant very ample invertible sheaf $\mathcal{L}$ constructed in Lemma 3.9. We also fix a μ_n-invariant hermitian metric on $\mathcal{L}$, note that this metric always exists according to an argument of partition of unity. When we deal with the tensor product of a coherent sheaf $\mathcal{F}$ with some power $\mathcal{L}^{\otimes n}$, we just write it as $\mathcal{F}(n)$ for simplicity. Before we give the proof of the rest of the vanishing theorem, we shall recall the concept of equivariant standard complex and some technical results.

DEFINITION 3.10. Let S be a projective manifold and let $\overline{\xi}.$ be a bounded complex of hermitian vector bundles on S. We say $\overline{\xi}.$ is a standard complex if the homology sheaves of $\overline{\xi}.$ are all locally free and they are endowed with some hermitian metrics. We shall write a standard complex as $(\overline{\xi}., h^H)$ to emphasize the choice of the metrics on the homology sheaves.

DEFINITION 3.11. Let S be an equivariant projective manifold. An equivariant standard complex on S is a bounded complex of equivariant hermitian vector bundles on S whose restriction to S_g is standard and the metrics on the homology sheaves are g-invariant. Again we shall write an equivariant standard complex as $(\overline{\xi}., h^H)$ to emphasize the choice of the metrics on the homology sheaves.

Due to [29, Theorem 5.9], to every equivariant standard complex $(\overline{\xi}., h^H)$ on an equivariant projective manifold S, there is a unique axiomatical way to associate an element $\widetilde{\mathrm{ch}}_g(\overline{\xi}., h^H)$ in $\bigoplus_{p\geq 0} A^{p,p}(S_g)/(\mathrm{Im}\partial + \mathrm{Im}\overline{\partial})$ which satisfies the differential equation

$$\mathrm{dd}^c\widetilde{\mathrm{ch}}_g(\overline{\xi}., h^H) = \sum_j (-1)^j \mathrm{ch}_g(H_j(\overline{\xi}. \mid_{S_g})) - \sum_j (-1)^j \mathrm{ch}_g(\overline{\xi}_j).$$

Let $0 \to \overline{\xi}.' \to \overline{\xi}. \to \overline{\xi}.'' \to 0$ be a short exact sequence of equivariant standard complexes on S. Then by restricting to the fixed point submanifold S_g, we get a short exact sequence of standard complexes $0\to\overline{\xi.}'\mid_{S_g}\to\overline{\xi.}\mid_{S_g}\to\overline{\xi.}''\mid_{S_g}\to 0$. Hence we obtain a long exact sequence of homology sheaves of these three standard complexes. We shall make a stronger assumption. Suppose that for any $j\geq 0$, we have short exact sequence $0\to H_j(\overline{\xi.}'\mid_{S_g})\to H_j(\overline{\xi.}\mid_{S_g})\to H_j(\overline{\xi.}''\mid_{S_g})\to 0$ which is denoted by $\overline{\chi}_j$. Moreover, for any $j\geq 0$, denote by $\overline{\varepsilon}_j$ the short exact sequence $0\to\overline{\xi}_j'\to\overline{\xi}_j\to\overline{\xi}_j''\to 0$.

Lemma 3.12. *Let notations and assumptions be as above. The identity*

$$\widetilde{\mathrm{ch}}_g(\overline{\xi.}',h^H)-\widetilde{\mathrm{ch}}_g(\overline{\xi.},h^H)+\widetilde{\mathrm{ch}}_g(\overline{\xi.}'',h^H)=\sum(-1)^j\widetilde{\mathrm{ch}}_g(\overline{\chi}_j)-\sum(-1)^j\widetilde{\mathrm{ch}}_g(\overline{\varepsilon}_j)$$

holds in $\bigoplus_{p\geq 0}A^{p,p}(S_g)/(\mathrm{Im}\partial+\mathrm{Im}\overline{\partial})$.

Proof. On S_g, every equivariant standard complex $(\overline{\xi.},h^H)$ splits into a series of short exact sequences of equivariant hermitian vector bundles in the following way

$$0\to\overline{\mathrm{Im}}\to\overline{\mathrm{Ker}}\to\overline{H.}\to 0$$

and

$$0\to\overline{\mathrm{Ker}}\to\overline{\xi.}\mid_{S_g}\to\overline{\mathrm{Im}}\to 0.$$

According to the argument given after [29, Remark 5.10], $\widetilde{\mathrm{ch}}_g(\overline{\xi.},h^H)$ is equal to the alternating sum of the equivariant Bott-Chern secondary characteristic classes of the short exact sequences above. Now since we have supposed that $0\to H_j(\overline{\xi.}'\mid_{S_g})\to H_j(\overline{\xi.}\mid_{S_g})\to H_j(\overline{\xi.}''\mid_{S_g})\to 0$ are all exact, by using Snake lemma, we know that $0\to\mathrm{Im}(\overline{\xi.}'\mid_{S_g})\to\mathrm{Im}(\overline{\xi.}\mid_{S_g})\to\mathrm{Im}(\overline{\xi.}''\mid_{S_g})\to 0$ and $0\to\mathrm{Ker}(\overline{\xi.}'\mid_{S_g})\to\mathrm{Ker}(\overline{\xi.}\mid_{S_g})\to\mathrm{Ker}(\overline{\xi.}''\mid_{S_g})\to 0$ are also all exact sequences. Then the identity in the statement of this lemma immediately follows from the construction of $\widetilde{\mathrm{ch}}_g(\overline{\xi.},h^H)$ and the additivity property of the equivariant Bott-Chern secondary characteristic classes. $\square$

Corollary 3.13. *Let* $0\to\overline{\xi.}^{(m)}\to\cdots\to\overline{\xi.}^{(1)}\to\overline{\xi.}^{(0)}\to 0$ *be an exact sequence of equivariant standard complexes on* S *such that for every* $j\geq 0$, $0\to H_j(\overline{\xi.}^{(m)}\mid_{S_g})\to\cdots\to H_j(\overline{\xi.}^{(1)}\mid_{S_g})\to H_j(\overline{\xi.}^{(0)}\mid_{S_g})\to 0$ *is exact. Then the identity*

$$\sum_{k=0}^{m}(-1)^k\widetilde{\mathrm{ch}}_g(\overline{\xi.}^{(k)},h^H)=\sum(-1)^j\widetilde{\mathrm{ch}}_g(\overline{\chi}_j)-\sum(-1)^j\widetilde{\mathrm{ch}}_g(\overline{\varepsilon}_j)$$

holds in $\bigoplus_{p\geq 0}A^{p,p}(S_g)/(\mathrm{Im}\partial+\mathrm{Im}\overline{\partial})$.

Proof. We claim that for every $1\leq k\leq m$, the kernel of the complex morphism $\overline{\xi.}^{(k)}\to\overline{\xi.}^{(k-1)}$ is still an equivariant standard complex on S. It is clear that we only need to prove this for $k=1$. Firstly, the kernel of $\overline{\xi.}^{(1)}\to\overline{\xi.}^{(0)}$ is a complex of equivariant hermitian vector bundles, let's denote it by $\overline{K}$. By restricting to

S_g and using an argument of long exact sequence, we know that the homology sheaves of $\overline{K}\mid_{S_g}$ are all equivariant hermitian vector bundles since for any $j\geq 0$ the bundle morphism $H_j(\overline{\xi.}^{(1)}\mid_{S_g})\to H_j(\overline{\xi.}^{(0)}\mid_{S_g})$ is already surjective. Therefore, the assumption of exactness on homologies implies that we can split $0\to\overline{\xi.}^{(m)}\to\cdots\to\overline{\xi.}^{(1)}\to\overline{\xi.}^{(0)}\to 0$ into a series of short exact sequences of equivariant standard complexes, so the identity in the statement of this corollary follows from Lemma 3.12. □

REMARK 3.14. A generalized version of Corollary 3.13, in which the exact sequence of (equivariant) standard complexes is replaced by an (equivariant) double standard complex was obtained in Xiaonan Ma's Ph.D thesis (cf. [24]) where more discussions concerning spectral sequences were involved. Anyway, for arithmetical reason, we only need these special versions as in Lemma 3.12 and Corollary 3.13.

Now we turn back to our proof of the vanishing theorem. As before, let $W=W(i)$ be the deformation to the normal cone associated to an equivariant closed immersion of projective manifolds $i:Y\to X$. For simplicity, denote by P_g^0 the projective space bundle $\mathbb{P}(N_{X_g/Y_g}\oplus\mathcal{O}_{Y_g})$. Moreover, given an invariant Kähler metric on X, we fix an invariant Kähler metric on W which is constructed in Lemma 3.5. In this situation, all normal bundles appearing in the construction of the deformation to the normal cone will be endowed with the quotient metrics. We recall the following lemma.

LEMMA 3.15. *Over $W(i_g)$, there are hermitian metrics on $\mathcal{O}(X_g)$, $\mathcal{O}(P_g^0)$ and $\mathcal{O}(\widetilde{X_g})$ such that the isometry $\overline{\mathcal{O}}(X_g)\cong\overline{\mathcal{O}}(P_g^0)\otimes\overline{\mathcal{O}}(\widetilde{X_g})$ holds and such that the restriction of $\overline{\mathcal{O}}(X_g)$ to X_g yields the metric of $N_{W(i_g)/X_g}$, the restriction of $\overline{\mathcal{O}}(\widetilde{X_g})$ to $\widetilde{X_g}$ yields the metric of $N_{W(i_g)/\widetilde{X_g}}$ and the restriction of $\overline{\mathcal{O}}(P_g^0)$ to P_g^0 induces the metric of $N_{W(i_g)/P_g^0}$.*

Proof. This is [21, Lemma 6.15]. □

DEFINITION 3.16. Let $\overline{\eta}$ be an equivariant hermitian vector bundle on Y, we say that a resolution

$$\overline{\Xi}:\quad 0\to\overline{\widetilde{\xi}}_m\to\cdots\to\overline{\widetilde{\xi}}_0\to j_*p_Y^*(\overline{\eta})\to 0$$

satisfies the condition (T) if
(i). the metrics on $\widetilde{\xi}.$ satisfy Bismut assumption (A);
(ii). the restriction of $\overline{\Xi}$ to $\widetilde{X}$ is an equivariantly and orthogonally split exact sequence;
(iii). the restrictions of $\overline{\Xi}_\nabla$ to $W(i_g)$, X_g, P_g^0, $\widetilde{X_g}$ and $P_g^0\cap\widetilde{X_g}$ are complexes with l-acyclic elements and l-acyclic homologies, here $\overline{\Xi}_\nabla$ is the complex of hermitian vector bundles obtained by omitting the last term $j_*p_Y^*(\overline{\eta})$ in $\overline{\Xi}$;

(iv). the tensor products $\overline{\Xi}_\nabla\mid_{W(i_g)}\otimes\overline{\mathcal{O}}(-X_g)$, $\overline{\Xi}_\nabla\mid_{W(i_g)}\otimes\overline{\mathcal{O}}(-P_g^0)$ and $\overline{\Xi}_\nabla\mid_{W(i_g)}\otimes\overline{\mathcal{O}}(-\widetilde{X_g})$ are complexes with l-acyclic elements and l-acyclic homologies.

From Theorem 3.6 (i), we already know that there always exists a resolution of $j_*p_Y^*(\overline{\eta})$ which satisfies the conditions (i) and (ii) in Definition 3.16. Let $\overline{\Xi}$ be such a resolution, we have the following.

PROPOSITION 3.17. *For $n\gg 0$, $\overline{\Xi}(n)$ satisfies the condition (T).*

Proof. The reason is that $W(i_g)$, X_g, P_g^0, $\widetilde{X_g}$ and $P_g^0\cap\widetilde{X_g}$ are all closed submanifolds of W. □

It is well known that both two squares in the following deformation diagram

$$\begin{tikzcd}
Y\times\{0\} \arrow[r, "s_0"] \arrow[d, "i"'] & Y\times\mathbb{P}^1 \arrow[d, "j"] & Y\times\{\infty\} \arrow[l, "s_\infty"'] \arrow[d, "i_\infty"] \\
X\times\{0\} \arrow[r] & W & \mathbb{P}(N_{X/Y}\oplus N_{\mathbb{P}^1/\infty}) \arrow[l]
\end{tikzcd}$$

are Tor-independent. Moreover, according to our choices of the Kähler metrics, we may identify $Y\times\{0\}$ with Y, $X\times\{0\}$ with X, $Y\times\{\infty\}$ with Y and $\mathbb{P}(N_{X/Y}\oplus N_{\mathbb{P}^1/\infty})$ with $P=\mathbb{P}(N_{X/Y}\oplus\mathcal{O}_Y)$. So if $\overline{\Xi}$ is a resolution of $j_*p_Y^*(\overline{\eta})$ on W, then the restriction of $\overline{\Xi}$ to X (resp. P) provides a resolution of $i_*\overline{\eta}$ (resp. $i_{\infty*}\overline{\eta}$). The following theorem is the kernel of the whole proof of the vanishing theorem.

THEOREM 3.18. *(Deformation theorem) Let $\overline{\Xi}$ be a resolution of $j_*p_Y^*(\overline{\eta})$ on W which satisfies the condition (T), then we have $\delta(\overline{\Xi}\mid_X)=\delta(\overline{\Xi}\mid_P)$.*

Proof. Consider the following tensor product of $\overline{\Xi}_\nabla\mid_{W(i_g)}$ with the Koszul resolution associated to the immersion $X_g\hookrightarrow W(i_g)$

$$0\to\overline{\Xi}_\nabla\mid_{W(i_g)}\otimes\overline{\mathcal{O}}(-X_g)\to\overline{\Xi}_\nabla\mid_{W(i_g)}\otimes\overline{\mathcal{O}}_{W(i_g)}\to\overline{\Xi}_\nabla\mid_{W(i_g)}\otimes i_{X_g*}\overline{\mathcal{O}}_{X_g}\to 0.$$

We have to caution the reader that here the tensor product is not the usual tensor product of two complexes, precisely our resulting sequence is a double complex and we don't take its total complex. Since we have assumed that $\overline{\Xi}$ satisfies the condition (T), this tensor product induces a short exact sequence of equivariant standard complexes on S by taking direct images. For $j\geq 0$, its j-th row is the following short exact sequence

$$\overline{\varepsilon}_j:0\to R^0l^0_{g*}(\overline{\mathcal{O}}(-X_g)\otimes\overline{\overline{\xi}}_j\mid_{W(i_g)})\to R^0l^0_{g*}(\overline{\overline{\xi}}_j\mid_{W(i_g)})\to R^0h_{g*}(\overline{\overline{\xi}}_j\mid_{X_g})\to 0$$

where l_g^0 is the composition of the inclusion $W(i_g)\hookrightarrow W$ with the morphism l. Note that the j-th homology of $\overline{\Xi}_\nabla\mid_{W(i_g)}$ is equal to $j_{g*}(\wedge^j\overline{\widetilde{F}}^\vee\otimes p_{Y_g}^*\overline{\eta}\mid_{Y_g})\mid_{W(i_g)}$ where $\widetilde{F}$ is the non-zero degree part of the normal bundle associated to the immersion j. Actually j_g factors through $j_g^0 : Y_g \times \mathbb{P}^1 \hookrightarrow W(i_g)$, then the j-th homology of $\overline{\Xi}_\nabla \mid_{W(i_g)}$ can be rewritten as $j^0_{g*}(\wedge^j \overline{\widetilde{F}}^\vee \otimes p^*_{Y_g}\overline{\eta} \mid_{Y_g})$. Write $Y_{g,0} := Y_g \times \{0\}$ for simplicity. Using the fact that $j_g^{0*}\mathcal{O}(-X_g)$ is isomorphic to $\mathcal{O}(-Y_{g,0})$, we deduce from the short exact sequence

$$\begin{aligned}0 \to j^0_{g*}(\overline{\mathcal{O}}(-Y_{g,0}) \otimes \wedge^j \overline{\widetilde{F}}^\vee \otimes p^*_{Y_g}\overline{\eta} \mid_{Y_g}) \to j^0_{g*}(\overline{\mathcal{O}}_{Y_g\times\mathbb{P}^1} \otimes \wedge^j \overline{\widetilde{F}}^\vee \otimes p^*_{Y_g}\overline{\eta} \mid_{Y_g})\\ \to j^0_{g*}(i_{Y_g*}\overline{\mathcal{O}}_{Y_g} \otimes \wedge^j \overline{\widetilde{F}}^\vee \otimes p^*_{Y_g}\overline{\eta} \mid_{Y_g}) \to 0\end{aligned}$$

that the j-th homologies of the induced short exact sequence of equivariant standard complexes form a short exact sequence

$$\begin{aligned}\overline{\chi}_j : 0 \to R^0u_{g*}(\overline{\mathcal{O}}(-Y_{g,0}) \otimes \wedge^j \overline{\widetilde{F}}^\vee \otimes p^*_{Y_g}\overline{\eta} \mid_{Y_g}) \to R^0u_{g*}(\wedge^j \overline{\widetilde{F}}^\vee \otimes p^*_{Y_g}\overline{\eta} \mid_{Y_g})\\ \to R^0f_{g*}(\wedge^j \overline{F}^\vee \otimes \overline{\eta} \mid_{Y_g}) \to 0\end{aligned}$$

where u_g is the composition of the inclusion $Y_g \times \mathbb{P}^1 \hookrightarrow W(i_g)$ with the morphism l_g^0.
The main idea of this proof is that the equivariant Bott-Chern secondary characteristic class of the quotient term of the induced short exact sequence of equivariant standard complexes is nothing but $\widetilde{\mathrm{ch}}_g(\overline{\Xi}_\nabla \mid_X, h^H)$ which appears in the expression of $\delta(\overline{\Xi} \mid_X)$ and the equivariant secondary characteristic classes of $\overline{\chi}_j, \overline{\varepsilon}_j$ can be computed by Bismut-Ma's immersion formula.
Precisely, denote by g_{X_g} the Euler-Green current associated to X_g which was constructed by Bismut, Gillet and Soulé in [6, Section 3. (f)], it satisfies the differential equation $\mathrm{dd}^c g_{X_g} = \delta_{X_g} - c_1(\overline{\mathcal{O}}(X_g))$. We write $\mathrm{Td}(\overline{X_g})$ for $\mathrm{Td}(\overline{\mathcal{O}}(X_g))$, [6, Theorem 3.17] implies that $\mathrm{Td}^{-1}(\overline{X_g})g_{X_g}$ is equal to the singular Bott-Chern current of the Koszul resolution associated to $X_g \hookrightarrow W(i_g)$ modulo $\mathrm{Im}\partial + \mathrm{Im}\overline{\partial}$. Moreover, write $\overline{\xi.}$ for the restriction $\overline{\Xi}_\nabla \mid_X$. Then for any $j \geq 0$, we compute

$$\begin{aligned}\widetilde{\mathrm{ch}}_g(\overline{\varepsilon}_j) =& T_g(\omega^{X_g}, h^{\xi_j|_{X_g}}) - T_g(\omega^{W(i_g)}, h^{\widetilde{\xi}_j|_{W(i_g)}})\\ &+ T_g(\omega^{W(i_g)}, h^{\mathcal{O}(-X_g)\otimes\widetilde{\xi}_j|_{W(i_g)}})\\ &+ \int_{W(i_g)/S} \mathrm{ch}_g(\overline{\widetilde{\xi}}_j)\mathrm{Td}(\overline{Tl^0_g})\mathrm{Td}^{-1}(\overline{X_g})g_{X_g}\\ &+ \int_{X_g/S} \mathrm{ch}_g(\overline{\xi}_j)\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/X_g})\widetilde{\mathrm{Td}}(\overline{Th_g}, \overline{Tl^0_g} \mid_{X_g})\\ &+ \int_{X_g/S} \mathrm{ch}_g(\xi_j)R(N_{W(i_g)/X_g})\mathrm{Td}(Th_g).\end{aligned}$$

Here, one should note that to simplify the last two terms in the right-hand side of Bismut-Ma's immersion formula, we have used an Atiyah-Segal-Singer type formula for immersion

$$i_{g*}(\mathrm{Td}_g^{-1}(N)\mathrm{ch}_g(x)) = \mathrm{ch}_g(i_*(x)).$$

This formula is the content of [21, Theorem 6.16]. Similarly, for any $j \geq 0$, we compute

$$\begin{aligned}\widetilde{\mathrm{ch}}_g(\overline{\chi}_j) =& T_g(\omega^{Y_g}, h^{\wedge^j F^\vee \otimes \eta|_{Y_g}}) - T_g(\omega^{Y_g\times\mathbb{P}^1}, h^{\wedge^j \widetilde{F}^\vee \otimes p_{Y_g}^*\eta|_{Y_g}}) \\ &+ T_g(\omega^{Y_g\times\mathbb{P}^1}, h^{\mathcal{O}(-Y_{g,0})\otimes\wedge^j \widetilde{F}^\vee \otimes p_{Y_g}^*\eta|_{Y_g}}) \\ &+ \int_{Y_g\times\mathbb{P}^1/S} \mathrm{ch}_g(\wedge^j \overline{\widetilde{F}}^\vee \otimes p_{Y_g}^*\overline{\eta}\mid_{Y_g})\mathrm{Td}(\overline{Tu_g})\mathrm{Td}^{-1}(\overline{Y_{g,0}})g_{Y_{g,0}} \\ &+ \int_{Y_g/S} \mathrm{ch_g}(\wedge^j \overline{F}^\vee \otimes \overline{\eta}\mid_{Y_g})\mathrm{Td}^{-1}(\overline{N}_{Y_g\times\mathbb{P}^1/Y_{g,0}})\widetilde{\mathrm{Td}}(\overline{Tf_g}, \overline{Tu_g}\mid_{Y_{g,0}}) \\ &+ \int_{Y_g/S} \mathrm{ch}_g(\wedge^j F^\vee \otimes \eta\mid_{Y_g})R(N_{Y_g\times\mathbb{P}^1/Y_{g,0}})\mathrm{Td}(Tf_g).\end{aligned}$$

Denote by $\overline{\Omega}(W(i_g))$ (resp. $\overline{\Omega}(-X_g)$) the middle (resp. sub) term of the induced short exact sequence of equivariant standard complexes. According to Lemma 3.12, we have

$$\begin{aligned}&\widetilde{\mathrm{ch}}_g(\overline{\Xi}_\nabla\mid_X, h^H) - \widetilde{\mathrm{ch}}_g(\overline{\Omega}(W(i_g)), h^H) + \widetilde{\mathrm{ch}}_g(\overline{\Omega}(-X_g), h^H) \\ =& \sum(-1)^j\widetilde{\mathrm{ch}}_g(\overline{\chi}_j) - \sum(-1)^j\widetilde{\mathrm{ch}}_g(\overline{\varepsilon}_j) \\ =& \sum(-1)^j T_g(\omega^{Y_g}, h^{\wedge^j F^\vee\otimes\eta|_{Y_g}}) - \sum(-1)^j T_g(\omega^{Y_g\times\mathbb{P}^1}, h^{\wedge^j \widetilde{F}^\vee\otimes p_{Y_g}^*\eta|_{Y_g}}) \\ &+ \sum(-1)^j T_g(\omega^{Y_g\times\mathbb{P}^1}, h^{\mathcal{O}(-Y_{g,0})\otimes\wedge^j\widetilde{F}^\vee\otimes p_{Y_g}^*\eta|_{Y_g}}) \\ &+ \int_{Y_g\times\mathbb{P}^1/S}\sum(-1)^j\mathrm{ch}_g(\wedge^j\overline{\widetilde{F}}^\vee\otimes p_{Y_g}^*\overline{\eta}\mid_{Y_g})\mathrm{Td}(\overline{Tu_g})\mathrm{Td}^{-1}(\overline{Y_{g,0}})g_{Y_{g,0}} \\ &+ \int_{Y_g/S}\sum(-1)^j\mathrm{ch_g}(\wedge^j\overline{F}^\vee\otimes\overline{\eta}\mid_{Y_g})\mathrm{Td}^{-1}(\overline{N}_{Y_g\times\mathbb{P}^1/Y_{g,0}})\widetilde{\mathrm{Td}}(\overline{Tf_g}, \overline{Tu_g}\mid_{Y_{g,0}}) \\ &+ \int_{Y_g/S}\sum(-1)^j\mathrm{ch}_g(\wedge^j F^\vee\otimes\eta\mid_{Y_g})R(N_{Y_g\times\mathbb{P}^1/Y_{g,0}})\mathrm{Td}(Tf_g) \\ &- \sum(-1)^j T_g(\omega^{X_g}, h^{\xi_j|_{X_g}}) + \sum(-1)^j T_g(\omega^{W(i_g)}, h^{\widetilde{\xi}_j|_{W(i_g)}}) \\ &- \sum(-1)^j T_g(\omega^{W(i_g)}, h^{\mathcal{O}(-X_g)\otimes\widetilde{\xi}_j|_{W(i_g)}}) \\ &- \int_{W(i_g)/S}\sum(-1)^j\mathrm{ch}_g(\overline{\widetilde{\xi}}_j)\mathrm{Td}(\overline{Tl_g^0})\mathrm{Td}^{-1}(\overline{X_g})g_{X_g} \\ &- \int_{X_g/S}\sum(-1)^j\mathrm{ch_g}(\overline{\xi}_j)\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/X_g})\widetilde{\mathrm{Td}}(\overline{Th_g}, \overline{Tl_g^0}\mid_{X_g}) \\ &- \int_{X_g/S}\sum(-1)^j\mathrm{ch}_g(\xi_j)R(N_{W(i_g)/X_g})\mathrm{Td}(Th_g). \qquad (1)\end{aligned}$$

Similarly, we consider the tensor products of $\overline{\Xi}_\nabla\mid_{W(i_g)}$ with the following three Koszul resolutions

$$0 \to \overline{\mathcal{O}}(-P_g^0) \to \overline{\mathcal{O}}_{W(i_g)} \to i_{P_g^0 *}\overline{\mathcal{O}}_{P_g^0} \to 0,$$

$$0 \to \overline{\mathcal{O}}(-\widetilde{X_g}) \to \overline{\mathcal{O}}_{W(i_g)} \to i_{\widetilde{X_g}*}\overline{\mathcal{O}}_{\widetilde{X_g}} \to 0,$$

and

$$\begin{aligned} 0 \to \overline{\mathcal{O}}(-\widetilde{X_g}) \otimes \overline{\mathcal{O}}(-P_g^0) \to \overline{\mathcal{O}}(-\widetilde{X_g}) \oplus \overline{\mathcal{O}}(-P_g^0) \to \overline{\mathcal{O}}_{W(i_g)} \\ \to i_{\widetilde{X_g}\cap P_g^0 *}\overline{\mathcal{O}}_{\widetilde{X_g}\cap P_g^0} \to 0. \end{aligned}$$

We shall still denote by $\overline{\chi.}$ (resp. $\overline{\varepsilon.}$) the exact sequences consisting of homologies (resp. elements) in the induced exact sequences of equivariant standard complexes.

For the first one, denote by $g_{P_g^0}$ the Euler-Green current associated to P_g^0 and write $\overline{\xi.}^\infty$ for the restriction $\overline{\Xi}_\nabla \mid_P$. Moreover, denote by $\overline{\Omega}(-P_g^0)$ the sub term of the induced short exact sequence of equivariant standard complexes and denote by b_g the composition of the inclusion $P_g^0 \hookrightarrow W(i_g)$ with the morphism l_g^0. According to Lemma 3.12, we have

$$\begin{aligned}
&\widetilde{\mathrm{ch}}_g(\overline{\Xi}_\nabla \mid_{P_g^0}, h^H) - \widetilde{\mathrm{ch}}_g(\overline{\Omega}(W(i_g)), h^H) + \widetilde{\mathrm{ch}}_g(\overline{\Omega}(-P_g^0), h^H) \\
=& \sum (-1)^j \widetilde{\mathrm{ch}}_g(\overline{\chi}_j) - \sum (-1)^j \widetilde{\mathrm{ch}}_g(\overline{\varepsilon}_j) \\
=& \sum (-1)^j T_g(\omega^{Y_g}, h^{\wedge^j F_\infty^\vee \otimes \eta|_{Y_g}}) - \sum (-1)^j T_g(\omega^{Y_g\times\mathbb{P}^1}, h^{\wedge^j \widetilde{F}^\vee \otimes p_{Y_g}^*\eta|_{Y_g}}) \\
&+ \sum (-1)^j T_g(\omega^{Y_g\times\mathbb{P}^1}, h^{\mathcal{O}(-Y_{g,\infty})\otimes\wedge^j \widetilde{F}^\vee \otimes p_{Y_g}^*\eta|_{Y_g}}) \\
&+ \int_{Y_g\times\mathbb{P}^1/S} \sum (-1)^j \mathrm{ch}_g(\wedge^j \overline{\widetilde{F}}^\vee \otimes p_{Y_g}^*\overline{\eta} \mid_{Y_g}) \mathrm{Td}(\overline{Tu_g}) \mathrm{Td}^{-1}(\overline{Y_{g,\infty}}) g_{Y_{g,\infty}} \\
&+ \int_{Y_g/S} \{ \sum (-1)^j \mathrm{ch}_g(\wedge^j \overline{F}_\infty^\vee \otimes \overline{\eta} \mid_{Y_g}) \mathrm{Td}^{-1}(\overline{N}_{Y_g\times\mathbb{P}^1/Y_{g,\infty}}) \\
&\qquad \cdot \widetilde{\mathrm{Td}}(\overline{Tf_g}, \overline{Tu_g} \mid_{Y_{g,\infty}}) \} \\
&+ \int_{Y_g/S} \sum (-1)^j \mathrm{ch}_g(\wedge^j F_\infty^\vee \otimes \eta \mid_{Y_g}) R(N_{Y_g\times\mathbb{P}^1/Y_{g,\infty}}) \mathrm{Td}(Tf_g) \\
&- \sum (-1)^j T_g(\omega^{P_g^0}, h^{\xi_j^\infty|_{P_g^0}}) + \sum (-1)^j T_g(\omega^{W(i_g)}, h^{\widetilde{\xi}_j|_{W(i_g)}}) \\
&- \sum (-1)^j T_g(\omega^{W(i_g)}, h^{\mathcal{O}(-P_g^0)\otimes\widetilde{\xi}_j|_{W(i_g)}}) \\
&- \int_{W(i_g)/S} \sum (-1)^j \mathrm{ch}_g(\overline{\widetilde{\xi}}_j) \mathrm{Td}(\overline{Tl_g^0}) \mathrm{Td}^{-1}(\overline{P_g^0}) g_{P_g^0} \\
&- \int_{P_g^0/S} \sum (-1)^j \mathrm{ch}_g(\overline{\xi}_j^\infty) \mathrm{Td}^{-1}(\overline{N}_{W(i_g)/P_g^0}) \widetilde{\mathrm{Td}}(\overline{Tb_g}, \overline{Tl_g^0} \mid_{P_g^0}) \\
&- \int_{P_g^0/S} \sum (-1)^j \mathrm{ch}_g(\xi_j^\infty) R(N_{W(i_g)/P_g^0}) \mathrm{Td}(Tb_g) \qquad (2)
\end{aligned}$$

where $\overline{F}_\infty$ is the non-zero degree part of the hermitian normal bundle $\overline{N}_\infty$ associated to i_∞.

For the second one, denote by $g_{\widetilde{X_g}}$ the Euler-Green current associated to $\widetilde{X_g}$ and denote by $\overline{\Omega}(-\widetilde{X_g})$ the sub term of the induced short exact sequence of equivariant standard complexes. Since the restriction of $\overline{\Xi}$ to the component $\widetilde{X}$ is equivariantly and orthogonally split, we know that $\widetilde{\mathrm{ch}}_g(\overline{\Xi}\mid_{\widetilde{X_g}},h^H)$ is equal to 0 and the summation $\sum(-1)^j\mathrm{ch}_g(\overline{\widetilde{\xi}_j})$ vanishes on $\widetilde{X_g}$. Using again Lemma 3.12, we obtain

$$\begin{aligned}
&-\widetilde{\mathrm{ch}}_g(\overline{\Omega}(W(i_g)),h^H)+\widetilde{\mathrm{ch}}_g(\overline{\Omega}(-\widetilde{X_g}),h^H)\\
=&\sum(-1)^j\widetilde{\mathrm{ch}}_g(\overline{\chi}_j)-\sum(-1)^j\widetilde{\mathrm{ch}}_g(\overline{\varepsilon}_j)\\
=&-\sum(-1)^jT_g(\omega^{Y_g\times\mathbb{P}^1},h^{\wedge^j\widetilde{F}^\vee\otimes p_{Y_g}^*\eta|_{Y_g}})\\
&+\sum(-1)^jT_g(\omega^{Y_g\times\mathbb{P}^1},h^{j_g^{0*}\mathcal{O}(-\widetilde{X_g})\otimes\wedge^j\widetilde{F}^\vee\otimes p_{Y_g}^*\eta|_{Y_g}})\\
&-\int_{Y_g\times\mathbb{P}^1/S}\{\sum(-1)^j\mathrm{ch}_g(\wedge^j\overline{\widetilde{F}}^\vee\otimes p_{Y_g}^*\overline{\eta}\mid_{Y_g})\mathrm{Td}(\overline{Tu_g})\\
&\qquad\qquad\cdot\widetilde{\mathrm{ch}}(j_g^{0*}\overline{\mathcal{O}}(-\widetilde{X_g}),\overline{O}_{Y_g\times\mathbb{P}^1})\}\\
&+\sum(-1)^jT_g(\omega^{W(i_g)},h^{\widetilde{\xi}_j|_{W(i_g)}})\\
&-\sum(-1)^jT_g(\omega^{W(i_g)},h^{\mathcal{O}(-\widetilde{X_g})\otimes\widetilde{\xi}_j|_{W(i_g)}})\\
&-\int_{W(i_g)/S}\sum(-1)^j\mathrm{ch}_g(\overline{\widetilde{\xi}}_j)\mathrm{Td}(\overline{Tl_g^0})\mathrm{Td}^{-1}(\overline{\widetilde{X_g}})g_{\widetilde{X_g}}.
\end{aligned}\tag{3}$$

Here the element $\widetilde{\mathrm{ch}}(j_g^{0*}\overline{\mathcal{O}}(-\widetilde{X_g}),\overline{O}_{Y_g\times\mathbb{P}^1})$ is the equivariant secondary characteristic class of the following short exact sequence

$$0\to0\to j_g^{0*}\overline{\mathcal{O}}(-\widetilde{X_g})\to\overline{O}_{Y_g\times\mathbb{P}^1}\to0.$$

We now consider the last one. This is also a Koszul resolution because $\widetilde{X_g}$ and P_g^0 intersect transversally. By [6, Theorem 3.20], the Euler-Green current associated to $\widetilde{X_g}\cap P_g^0$ is the current $c_1(\overline{\mathcal{O}}(P_g^0))g_{\widetilde{X_g}}+\delta_{\widetilde{X_g}}g_{P_g^0}$. Then, by using the isometry $\overline{\mathcal{O}}(X_g)\cong\overline{\mathcal{O}}(P_g^0)\otimes\overline{\mathcal{O}}(\widetilde{X_g})$ and Corollary 3.13, we get

$$\begin{aligned}
&-\widetilde{\mathrm{ch}}_g(\overline{\Omega}(W(i_g)),h^H)+\widetilde{\mathrm{ch}}_g(\overline{\Omega}(-\widetilde{X_g}),h^H)\\
&\qquad\qquad+\widetilde{\mathrm{ch}}_g(\overline{\Omega}(-P_g^0),h^H)-\widetilde{\mathrm{ch}}_g(\overline{\Omega}(-X_g),h^H)\\
=&\sum(-1)^j\widetilde{\mathrm{ch}}_g(\overline{\chi}_j)-\sum(-1)^j\widetilde{\mathrm{ch}}_g(\overline{\varepsilon}_j)\\
=&-\sum(-1)^jT_g(\omega^{Y_g\times\mathbb{P}^1},h^{\wedge^j\widetilde{F}^\vee\otimes p_{Y_g}^*\eta|_{Y_g}})\\
&+\sum(-1)^jT_g(\omega^{Y_g\times\mathbb{P}^1},h^{j_g^{0*}\mathcal{O}(-\widetilde{X_g})\otimes\wedge^j\widetilde{F}^\vee\otimes p_{Y_g}^*\eta|_{Y_g}})\\
&+\sum(-1)^jT_g(\omega^{Y_g\times\mathbb{P}^1},h^{\mathcal{O}(-Y_{g,\infty})\otimes\wedge^j\widetilde{F}^\vee\otimes p_{Y_g}^*\eta|_{Y_g}})
\end{aligned}$$

$$
\begin{aligned}
&- \sum(-1)^j T_g(\omega^{Y_g\times\mathbb{P}^1}, h^{\mathcal{O}(-Y_{g,0})\otimes\wedge^j\widetilde{F}^\vee\otimes p_{Y_g}^*\eta|_{Y_g}})\\
&- \int_{Y_g\times\mathbb{P}^1/S}\sum(-1)^j\mathrm{ch}_g(\wedge^j\overline{\widetilde{F}}^\vee\otimes p_{Y_g}^*\overline{\eta}\mid_{Y_g})\mathrm{Td}(\overline{Tu_g})\widetilde{\mathrm{ch}}(\overline{\Theta})\\
&+ \sum(-1)^j T_g(\omega^{W(i_g)}, h^{\widetilde{\xi}_j|_{W(i_g)}})\\
&- \sum(-1)^j T_g(\omega^{W(i_g)}, h^{\mathcal{O}(-\widetilde{X_g})\otimes\widetilde{\xi}_j|_{W(i_g)}})\\
&- \sum(-1)^j T_g(\omega^{W(i_g)}, h^{\mathcal{O}(-P_g^0)\otimes\widetilde{\xi}_j|_{W(i_g)}})\\
&+ \sum(-1)^j T_g(\omega^{W(i_g)}, h^{\mathcal{O}(-X_g)\otimes\widetilde{\xi}_j|_{W(i_g)}})\\
&- \int_{W(i_g)/S}\{\sum(-1)^j\mathrm{ch}_g(\overline{\widetilde{\xi}}_j)\mathrm{Td}(\overline{Tl_g^0})\mathrm{Td}^{-1}(\overline{\widetilde{X_g}})\mathrm{Td}^{-1}(\overline{P_g^0})\\
&\qquad\qquad\qquad\qquad\cdot[c_1(\overline{\mathcal{O}}(P_g^0))g_{\widetilde{X_g}}+\delta_{\widetilde{X_g}}g_{P_g^0}]\}.
\end{aligned}
\tag{4}
$$

Here the element $\widetilde{\mathrm{ch}}(\overline{\Theta})$ is the equivariant secondary characteristic class of the following short exact sequence

$$\overline{\Theta}:\quad 0\to\overline{\mathcal{O}}(-Y_{g,0})\to j_g^{0*}\overline{\mathcal{O}}(-\widetilde{X_g})\oplus\overline{\mathcal{O}}(-Y_{g,\infty})\to\overline{O}_{Y_g\times\mathbb{P}^1}\to 0.$$

Since $s_0: Y\times\{0\}\to Y\times\mathbb{P}^1$ and $s_\infty: Y\times\{\infty\}\to Y\times\mathbb{P}^1$ are sections of smooth morphism, the normal sequences

$$0\to\overline{Tf_g}\to\overline{Tu_g}\mid_{Y_{g,0}}\to\overline{N}_{Y_g\times\mathbb{P}^1/Y_{g,0}}\to 0$$

and

$$0\to\overline{Tf_g}\to\overline{Tu_g}\mid_{Y_{g,\infty}}\to\overline{N}_{Y_g\times\mathbb{P}^1/Y_{g,\infty}}\to 0$$

are orthogonally split so that $\widetilde{\mathrm{Td}}(\overline{Tf_g},\overline{Tu_g}\mid_{Y_{g,0}})$ and $\widetilde{\mathrm{Td}}(\overline{Tf_g},\overline{Tu_g}\mid_{Y_{g,\infty}})$ are both equal to 0. Moreover, the normal bundles $N_{Y_g\times\mathbb{P}^1/Y_{g,0}}$ and $N_{Y_g\times\mathbb{P}^1/Y_{g,\infty}}$ are isomorphic to trivial bundles so that $R(N_{Y_g\times\mathbb{P}^1/Y_{g,0}})$ and $R(N_{Y_g\times\mathbb{P}^1/Y_{g,\infty}})$ are both equal to 0. Furthermore, we may drop all the terms where an integral is taken over $\widetilde{X_g}$ because $\sum(-1)^j\mathrm{ch}_g(\overline{\widetilde{\xi}}_j)$ vanishes on $\widetilde{X_g}$.
Now, we compute (1)−(2)−(3)+(4) which is

$$
\begin{aligned}
&\widetilde{\mathrm{ch}}_g(\overline{\Xi}_\nabla\mid_X, h^H)-\widetilde{\mathrm{ch}}_g(\overline{\Xi}_\nabla\mid_{P_g^0}, h^H)+\sum(-1)^jT_g(\omega^{X_g}, h^{\xi_j|_{X_g}})\\
&\quad-\sum(-1)^jT_g(\omega^{P_g^0}, h^{\xi_j^\infty|_{P_g^0}})-\sum(-1)^jT_g(\omega^{Y_g}, h^{\wedge^jF^\vee\otimes\eta|_{Y_g}})\\
&\qquad\qquad\qquad\qquad+\sum(-1)^jT_g(\omega^{Y_g}, h^{\wedge^jF_\infty^\vee\otimes\eta|_{Y_g}})\\
&=\int_{Y_g\times\mathbb{P}^1/S}\{\sum(-1)^j\mathrm{ch}_g(\wedge^j\overline{\widetilde{F}}^\vee\otimes p_{Y_g}^*\overline{\eta}\mid_{Y_g})\mathrm{Td}(\overline{Tu_g})\cdot[\mathrm{Td}^{-1}(\overline{Y_{g,0}})g_{Y_{g,0}}\\
&\qquad-\mathrm{Td}^{-1}(\overline{Y_{g,\infty}})g_{Y_{g,\infty}}+\widetilde{\mathrm{ch}}(j_g^{0*}\overline{\mathcal{O}}(-\widetilde{X}_g),\overline{O}_{Y_g\times\mathbb{P}^1})-\widetilde{\mathrm{ch}}_g(\overline{\Theta})]\}
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{W(i_g)/S}\sum(-1)^j\mathrm{ch}_g(\overline{\overline{\widetilde{\xi}}}_j)\mathrm{Td}(\overline{Tl^0_g})\mathrm{Td}^{-1}(\overline{X_g})g_{X_g}\\
&-\int_{X_g/S}\sum(-1)^j\mathrm{ch}_{\mathrm{g}}(\overline{\xi}_j)\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/X_g})\widetilde{\mathrm{Td}}(\overline{Th_g},\overline{Tl^0_g}\mid_{X_g})\\
&-\int_{X_g/S}\sum(-1)^j\mathrm{ch}_g(\xi_j)R(N_{W(i_g)/X_g})\mathrm{Td}(Th_g)\\
&+\int_{W(i_g)/S}\sum(-1)^j\mathrm{ch}_g(\overline{\overline{\widetilde{\xi}}}_j)\mathrm{Td}(\overline{Tl^0_g})\mathrm{Td}^{-1}(\overline{P^0_g})g_{P^0_g}\\
&+\int_{P^0_g/S}\sum(-1)^j\mathrm{ch}_{\mathrm{g}}(\overline{\xi}^\infty_j)\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/P^0_g})\widetilde{\mathrm{Td}}(\overline{Tb_g},\overline{Tl^0_g}\mid_{P^0_g})\\
&+\int_{P^0_g/S}\sum(-1)^j\mathrm{ch}_g(\xi^\infty_j)R(N_{W(i_g)/P^0_g})\mathrm{Td}(Tb_g)\\
&+\int_{W(i_g)/S}\sum(-1)^j\mathrm{ch}_g(\overline{\overline{\widetilde{\xi}}}_j)\mathrm{Td}(\overline{Tl^0_g})\mathrm{Td}^{-1}(\overline{\widetilde{X}_g})g_{\widetilde{X}_g}\\
&-\int_{W(i_g)/S}\sum(-1)^j\mathrm{ch}_g(\overline{\overline{\widetilde{\xi}}}_j)\mathrm{Td}(\overline{Tl^0_g})\mathrm{Td}^{-1}(\overline{\widetilde{X}_g})\mathrm{Td}^{-1}(\overline{P^0_g})c_1(\overline{\mathcal{O}}(P^0_g))g_{\widetilde{X}_g}.
\end{aligned}
$$

Denote by i_X (resp. i_P) the inclusion from X to $W(i)$ (resp. P to $W(i)$). We may use the Atiyah-Segal-Singer type formula for immersions and the projection formula in cohomology to compute

$$
\begin{aligned}
&i_{X_g *}\big(\sum(-1)^j\mathrm{ch}_g(\xi_j)R(N_{W(i_g)/X_g})\mathrm{Td}(Th_g)\big)\\
=&i_{X_g *}\big(R(N_{W(i_g)/X_g})\mathrm{Td}(Th_g)i_{g*}(\mathrm{Td}_g^{-1}(N_{X/Y})\mathrm{ch}_g(\eta))\big)\\
=&(i_{X_g}\circ i_g)_*\big(R(N_{W(i_g)/X_g})\mathrm{Td}(Th_g)\mathrm{Td}_g^{-1}(N_{X/Y})\mathrm{ch}_g(\eta)\big).
\end{aligned}
$$

Note that the restriction of $N_{W(i_g)/X_g}$ to Y_g is trivial so that the last expression vanishes. An entirely analogous reasoning implies that

$$
i_{P_g *}\big(\sum(-1)^j\mathrm{ch}_g(\xi^\infty_j)R(N_{W(i_g)/P^0_g})\mathrm{Td}(Tb_g)\big)=0.
$$

Thus, we are left with the equality

$$
\begin{aligned}
&\widetilde{\mathrm{ch}}_g(\overline{\Xi}_\nabla\mid_X,h^H)-\widetilde{\mathrm{ch}}_g(\overline{\Xi}_\nabla\mid_{P^0_g},h^H)+\sum(-1)^jT_g(\omega^{X_g},h^{\xi_j\mid_{X_g}})\\
&-\sum(-1)^jT_g(\omega^{P^0_g},h^{\xi^\infty_j\mid_{P^0_g}})-\sum(-1)^jT_g(\omega^{Y_g},h^{\wedge^jF^\vee\otimes\eta\mid_{Y_g}})\\
&\qquad\qquad+\sum(-1)^jT_g(\omega^{Y_g},h^{\wedge^jF^\vee_\infty\otimes\eta\mid_{Y_g}})\\
=&\int_{Y_g\times\mathbb{P}^1/S}\{\sum(-1)^j\mathrm{ch}_g(\wedge^j\overline{\overline{\widetilde{F}}}^\vee\otimes p^*_{Y_g}\overline{\eta}\mid_{Y_g})\mathrm{Td}(\overline{Tu_g})\cdot[\mathrm{Td}^{-1}(\overline{Y_{g,0}})g_{Y_{g,0}}\\
&\qquad-\mathrm{Td}^{-1}(\overline{Y_{g,\infty}})g_{Y_{g,\infty}}+\widetilde{\mathrm{ch}}(j_g^{0*}\overline{\mathcal{O}}(-\widetilde{X}_g),\overline{O_{Y_g\times\mathbb{P}^1}})-\widetilde{\mathrm{ch}}_g(\overline{\Theta})]\}\\
&-\int_{W(i_g)/S}\{\sum(-1)^j\mathrm{ch}_g(\overline{\overline{\widetilde{\xi}}}_j)\mathrm{Td}(\overline{Tl^0_g})\cdot[\mathrm{Td}^{-1}(\overline{X_g})g_{X_g}-\mathrm{Td}^{-1}(\overline{P^0_g})g_{P^0_g}
\end{aligned}
$$

$$- \mathrm{Td}^{-1}(\overline{\widetilde{X_g}})g_{\widetilde{X_g}} + \mathrm{Td}^{-1}(\overline{\widetilde{X_g}})\mathrm{Td}^{-1}(\overline{P_g^0})c_1(\overline{\mathcal{O}}(P_g^0))g_{\widetilde{X_g}}]\}$$

$$- \int_{X_g/S}\sum(-1)^j\mathrm{ch_g}(\overline{\xi}_j)\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/X_g})\widetilde{\mathrm{Td}}(\overline{Th_g},\overline{Tl_g^0}\mid_{X_g})$$

$$+ \int_{P_g^0/S}\sum(-1)^j\mathrm{ch_g}(\overline{\xi}_j^\infty)\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/P_g^0})\widetilde{\mathrm{Td}}(\overline{Tb_g},\overline{Tl_g^0}\mid_{P_g^0}).$$

Using the differential equation which $T_g(\overline{\widetilde{\xi}.})$ satisfies, we compute

$$\begin{aligned}
&- \int_{W(i_g)/S}\{\sum(-1)^j\mathrm{ch}_g(\overline{\widetilde{\xi}}_j)\mathrm{Td}(\overline{Tl_g^0})\cdot[\mathrm{Td}^{-1}(\overline{X_g})g_{X_g} - \mathrm{Td}^{-1}(\overline{P_g^0})g_{P_g^0}\\
&\qquad - \mathrm{Td}^{-1}(\overline{\widetilde{X_g}})g_{\widetilde{X_g}} + \mathrm{Td}^{-1}(\overline{\widetilde{X_g}})\mathrm{Td}^{-1}(\overline{P_g^0})c_1(\overline{\mathcal{O}}(P_g^0))g_{\widetilde{X_g}}]\}\\
&= \int_{W(i_g)/S}\{\mathrm{Td}(\overline{Tl_g^0})T_g(\overline{\widetilde{\xi}.})\cdot[\mathrm{Td}^{-1}(\overline{X_g})\delta_{X_g} - \mathrm{Td}^{-1}(\overline{P_g^0})\delta_{P_g^0}\\
&\qquad - \mathrm{Td}^{-1}(\overline{\widetilde{X_g}})\delta_{\widetilde{X_g}} + \mathrm{Td}^{-1}(\overline{\widetilde{X_g}})\mathrm{Td}^{-1}(\overline{P_g^0})c_1(\overline{\mathcal{O}}(P_g^0))\delta_{\widetilde{X_g}}]\}\\
&- \int_{W(i_g)/S}\{\mathrm{Td}(\overline{Tl_g^0})\mathrm{ch}_g(p_Y^*\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N}_{W/Y\times\mathbb{P}^1})\delta_{Y_g\times\mathbb{P}^1}\cdot[\mathrm{Td}^{-1}(\overline{X_g})g_{X_g}\\
&- \mathrm{Td}^{-1}(\overline{P_g^0})g_{P_g^0} - \mathrm{Td}^{-1}(\overline{\widetilde{X_g}})g_{\widetilde{X_g}} + \mathrm{Td}^{-1}(\overline{\widetilde{X_g}})\mathrm{Td}^{-1}(\overline{P_g^0})c_1(\overline{\mathcal{O}}(P_g^0))g_{\widetilde{X_g}}]\}.
\end{aligned} \tag{5}$$

Here we have used the equation

$$\begin{aligned}
\mathrm{Td}^{-1}(\overline{X_g})c_1(\overline{\mathcal{O}}(X_g)) - \mathrm{Td}^{-1}(\overline{P_g^0})c_1(\overline{\mathcal{O}}(P_g^0)) - \mathrm{Td}^{-1}(\overline{\widetilde{X_g}})c_1(\overline{\mathcal{O}}(\widetilde{X_g}))\\
+\mathrm{Td}^{-1}(\overline{\widetilde{X_g}})\mathrm{Td}^{-1}(\overline{P_g^0})c_1(\overline{\mathcal{O}}(P_g^0))c_1(\overline{\mathcal{O}}(\widetilde{X_g})) = 0
\end{aligned} \tag{6}$$

which is [21, (23)].

Again using the fact that $\overline{\widetilde{\xi}.}$ is equivariantly and orthogonally split on $\widetilde{X}$, the first integral in the right-hand side of (5) is equal to

$$\int_{X_g/S}\mathrm{Td}(\overline{Tl_g^0})T_g(\overline{\xi.})\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/X_g}) - \int_{P_g^0/S}\mathrm{Td}(\overline{Tl_g^0})T_g(\overline{\xi.}^\infty)\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/P_g^0}).$$

According to the normal sequence $0\to\overline{Th_g}\to\overline{Tl_g^0}\mid_{X_g}\to\overline{N}_{W(i_g)/X_g}\to 0$, we may write

$$\mathrm{Td}(\overline{Tl_g^0}) = \mathrm{Td}(\overline{Th_g})\mathrm{Td}(\overline{N}_{W(i_g)/X_g}) - \mathrm{dd}^c\widetilde{\mathrm{Td}}(\overline{Th_g},\overline{Tl_g^0}\mid_{X_g}).$$

So we get

$$\begin{aligned}
&\int_{X_g/S}\mathrm{Td}(\overline{Tl_g^0})T_g(\overline{\xi.})\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/X_g})\\
=&\int_{X_g/S}\mathrm{Td}(\overline{Th_g})T_g(\overline{\xi.})\\
&-\int_{X_g/S}\widetilde{\mathrm{Td}}(\overline{Th_g},\overline{Tl_g^0}\mid_{X_g})\delta_{Y_g}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N})\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/X_g})\\
&+\int_{X_g/S}\sum(-1)^j\mathrm{ch_g}(\overline{\xi}_j)\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/X_g})\widetilde{\mathrm{Td}}(\overline{Th_g},\overline{Tl_g^0}\mid_{X_g}).
\end{aligned}$$

Similarly we have

$$\begin{aligned}
&\int_{P_g^0/S}\mathrm{Td}(\overline{Tl_g^0})T_g(\overline{\xi.}^{\infty})\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/P_g^0})\\
=&\int_{P_g^0/S}\mathrm{Td}(\overline{Tb_g})T_g(\overline{\xi.}^{\infty})\\
&-\int_{P_g^0/S}\widetilde{\mathrm{Td}}(\overline{Tb_g},\overline{Tl_g^0}\mid_{P_g^0})\delta_{Y_g}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N}_\infty)\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/P_g^0})\\
&+\int_{P_g^0/S}\sum(-1)^j\mathrm{ch_g}(\overline{\xi}_j^{\infty})\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/P_g^0})\widetilde{\mathrm{Td}}(\overline{Tb_g},\overline{Tl_g^0}\mid_{P_g^0}).
\end{aligned}$$

Note that the normal sequence of $\overline{Th_g}$ in $\overline{Tl_g^0}$ (resp. $\overline{Tb_g}$ in $\overline{Tl_g^0}$) is orthogonally split on $Y_g\times\{0\}$ (resp. $Y_g\times\{\infty\}$), then $\widetilde{\mathrm{Td}}(\overline{Th_g},\overline{Tl_g^0}\mid_{X_g})\delta_{Y_g}$ and $\widetilde{\mathrm{Td}}(\overline{Tb_g},\overline{Tl_g^0}\mid_{P_g^0})\delta_{Y_g}$ are both equal to 0. Combining these computations above we get

$$\begin{aligned}
&\int_{X_g/S}\mathrm{Td}(\overline{Tl_g^0})T_g(\overline{\xi.})\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/X_g})\\
&\qquad\qquad-\int_{P_g^0/S}\mathrm{Td}(\overline{Tl_g^0})T_g(\overline{\xi.}^{\infty})\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/P_g^0})\\
=&\int_{X_g/S}\mathrm{Td}(\overline{Th_g})T_g(\overline{\xi.})\\
&+\int_{X_g/S}\sum(-1)^j\mathrm{ch_g}(\overline{\xi}_j)\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/X_g})\widetilde{\mathrm{Td}}(\overline{Th_g},\overline{Tl_g^0}\mid_{X_g})\\
&-\int_{P_g^0/S}\mathrm{Td}(\overline{Tb_g})T_g(\overline{\xi.}^{\infty})\\
&-\int_{P_g^0/S}\sum(-1)^j\mathrm{ch_g}(\overline{\xi}_j^{\infty})\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/P_g^0})\widetilde{\mathrm{Td}}(\overline{Tb_g},\overline{Tl_g^0}\mid_{P_g^0}).
\end{aligned}\tag{7}$$

We now compute the second integral in the right-hand side of (5). According to the normal sequence

$$0 \to \overline{Tu_g} \to \overline{Tl_g^0} \mid_{Y_g\times\mathbb{P}^1} \to \overline{N}_{W(i_g)/Y_g\times\mathbb{P}^1} \to 0,$$

we may write

$$\mathrm{Td}(\overline{Tl_g^0}) = \mathrm{Td}(\overline{Tu_g})\mathrm{Td}(\overline{N}_{W(i_g)/Y_g\times\mathbb{P}^1}) - \mathrm{dd}^c\widetilde{\mathrm{Td}}(\overline{Tu_g}, \overline{Tl_g^0} \mid_{Y_g\times\mathbb{P}^1}).$$

Hence

$$\begin{aligned}
&-\int_{W(i_g)/S}\{\mathrm{Td}(\overline{Tl_g^0})\mathrm{ch}_g(p_Y^*\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N}_{W/Y\times\mathbb{P}^1})\delta_{Y_g\times\mathbb{P}^1}\cdot[\mathrm{Td}^{-1}(\overline{X_g})g_{X_g}\\
&-\mathrm{Td}^{-1}(\overline{P_g^0})g_{P_g^0} - \mathrm{Td}^{-1}(\overline{\widetilde{X_g}})g_{\widetilde{X_g}} + \mathrm{Td}^{-1}(\overline{\widetilde{X_g}})\mathrm{Td}^{-1}(\overline{P_g^0})c_1(\overline{\mathcal{O}}(P_g^0))g_{\widetilde{X_g}}]\}\\
=&-\int_{Y_g\times\mathbb{P}^1/S}\{\mathrm{Td}(\overline{Tu_g})\mathrm{ch}_g(p_Y^*\overline{\eta})\mathrm{Td}_g^{-1}(\overline{\overline{F}})\cdot j_g^{0*}[\mathrm{Td}^{-1}(\overline{X_g})g_{X_g}\\
&-\mathrm{Td}^{-1}(\overline{P_g^0})g_{P_g^0} - \mathrm{Td}^{-1}(\overline{\widetilde{X_g}})g_{\widetilde{X_g}} + \mathrm{Td}^{-1}(\overline{\widetilde{X_g}})\mathrm{Td}^{-1}(\overline{P_g^0})c_1(\overline{\mathcal{O}}(P_g^0))g_{\widetilde{X_g}}]\}\\
&+\int_{Y_g\times\mathbb{P}^1/S}\{\widetilde{\mathrm{Td}}(\overline{Tu_g}, \overline{Tl_g^0} \mid_{Y_g\times\mathbb{P}^1})\mathrm{ch}_g(p_Y^*\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N}_{W/Y\times\mathbb{P}^1})\\
&\cdot[\mathrm{Td}^{-1}(\overline{X_g})(\delta_{X_g} - c_1(\overline{\mathcal{O}}(X_g))) - \mathrm{Td}^{-1}(\overline{P_g^0})(\delta_{P_g^0} - c_1(\overline{\mathcal{O}}(P_g^0)))\\
&-\mathrm{Td}^{-1}(\overline{\widetilde{X_g}})(\delta_{\widetilde{X_g}} - c_1(\overline{\mathcal{O}}(\widetilde{X_g})))\\
&+\mathrm{Td}^{-1}(\overline{\widetilde{X_g}})\mathrm{Td}^{-1}(\overline{P_g^0})c_1(\overline{\mathcal{O}}(P_g^0))(\delta_{\widetilde{X_g}} - c_1(\overline{\mathcal{O}}(\widetilde{X_g})))]\}.
\end{aligned}$$

By our choices of the metrics, we have $\mathrm{Td}_g^{-1}(\overline{N}_{W/Y\times\mathbb{P}^1}) \mid_{Y_{g,0}} = \mathrm{Td}_g^{-1}(\overline{N})$, $\mathrm{Td}(\overline{X_g}) \mid_{Y_{g,0}} = 1$ and $\mathrm{Td}_g^{-1}(\overline{N}_{W/Y\times\mathbb{P}^1}) \mid_{Y_{g,\infty}} = \mathrm{Td}_g^{-1}(\overline{N}_\infty)$, $\mathrm{Td}(\overline{P_g^0}) \mid_{Y_{g,\infty}} = 1$. Furthermore, by replacing all tangent bundles by relative tangent bundles, on can carry through the proof given in [21, P. 378-379] to show that

$$\widetilde{\mathrm{Td}}(\overline{Tu_g}, \overline{Tl_g^0} \mid_{Y_g\times\mathbb{P}^1}) \mid_{Y_{g,0}} = \widetilde{\mathrm{Td}}(\overline{Tf_g}, \overline{Th_g} \mid_{Y_g})$$

and

$$\widetilde{\mathrm{Td}}(\overline{Tu_g}, \overline{Tl_g^0} \mid_{Y_g\times\mathbb{P}^1}) \mid_{Y_{g,\infty}} = \widetilde{\mathrm{Td}}(\overline{Tf_g}, \overline{Tb_g} \mid_{Y_g}).$$

So combining with the equation (6), we get

$$\begin{aligned}
&-\int_{W(i_g)/S}\{\mathrm{Td}(\overline{Tl_g^0})\mathrm{ch}_g(p_Y^*\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N}_{W/Y\times\mathbb{P}^1})\delta_{Y_g\times\mathbb{P}^1}\cdot[\mathrm{Td}^{-1}(\overline{X_g})g_{X_g}\\
&-\mathrm{Td}^{-1}(\overline{P_g^0})g_{P_g^0} - \mathrm{Td}^{-1}(\overline{\widetilde{X_g}})g_{\widetilde{X_g}} + \mathrm{Td}^{-1}(\overline{\widetilde{X_g}})\mathrm{Td}^{-1}(\overline{P_g^0})c_1(\overline{\mathcal{O}}(P_g^0))g_{\widetilde{X_g}}]\}\\
=&-\int_{Y_g\times\mathbb{P}^1/S}\{\mathrm{Td}(\overline{Tu_g})\mathrm{ch}_g(p_Y^*\overline{\eta})\mathrm{Td}_g^{-1}(\overline{\overline{F}})\cdot j_g^{0*}[\mathrm{Td}^{-1}(\overline{X_g})g_{X_g}\\
&-\mathrm{Td}^{-1}(\overline{P_g^0})g_{P_g^0} - \mathrm{Td}^{-1}(\overline{\widetilde{X_g}})g_{\widetilde{X_g}} + \mathrm{Td}^{-1}(\overline{\widetilde{X_g}})\mathrm{Td}^{-1}(\overline{P_g^0})c_1(\overline{\mathcal{O}}(P_g^0))g_{\widetilde{X_g}}]\}
\end{aligned}$$

$$+\int_{Y_g/S}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N})\widetilde{\mathrm{Td}}(\overline{Tf_g},\overline{Th_g}\mid_{Y_g})$$
$$-\int_{Y_g/S}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N}_\infty)\widetilde{\mathrm{Td}}(\overline{Tf_g},\overline{Tb_g}\mid_{Y_g}). \tag{8}$$

At last, using the fact that the intersections in the deformation diagram are transversal and the fact that $j_g^0(Y_g\times\mathbb{P}^1)$ has no intersection with $\widetilde{X_g}$, we can compute

$$\begin{aligned}\int_{Y_g\times\mathbb{P}^1/S}&\{\sum(-1)^j\mathrm{ch}_g(\wedge^j\overline{\widetilde{F}}^\vee\otimes p_{Y_g}^*\overline{\eta}\mid_{Y_g})\mathrm{Td}(\overline{Tu_g})\cdot[\mathrm{Td}^{-1}(\overline{Y_{g,0}})g_{Y_{g,0}}\\ &\qquad -\mathrm{Td}^{-1}(\overline{Y_{g,\infty}})g_{Y_{g,\infty}}+\widetilde{\mathrm{ch}}(j_g^{0*}\overline{\mathcal{O}}(-\widetilde{X_g}),\overline{O}_{Y_g\times\mathbb{P}^1})-\widetilde{\mathrm{ch}}_g(\overline{\Theta})]\}\\ =\int_{Y_g\times\mathbb{P}^1/S}&\{\mathrm{Td}(\overline{Tu_g})\mathrm{ch}_g(p_Y^*\overline{\eta})\mathrm{Td}_g^{-1}(\overline{\widetilde{F}})\cdot j_g^{0*}[\mathrm{Td}^{-1}(\overline{X_g})g_{X_g}-\mathrm{Td}^{-1}(\overline{P_g^0})g_{P_g^0}\\ &\qquad -\mathrm{Td}^{-1}(\overline{\widetilde{X_g}})g_{\widetilde{X_g}}+\mathrm{Td}^{-1}(\overline{\widetilde{X_g}})\mathrm{Td}^{-1}(\overline{P_g^0})c_1(\overline{\mathcal{O}}(P_g^0))g_{\widetilde{X_g}}]\}.\end{aligned} \tag{9}$$

Gathering (5), (7), (8) and (9) we finally get

$$\begin{aligned}&\widetilde{\mathrm{ch}}_g(\Xi_\nabla\mid_X,h^H)-\widetilde{\mathrm{ch}}_g(\Xi_\nabla\mid_{P_g^0},h^H)+\sum(-1)^jT_g(\omega^{X_g},h^{\xi_j\mid_{X_g}})\\ &-\sum(-1)^jT_g(\omega^{P_g^0},h^{\xi_j^\infty\mid_{P_g^0}})-\sum(-1)^jT_g(\omega^{Y_g},h^{\wedge^jF^\vee\otimes\eta\mid_{Y_g}})\\ &\qquad+\sum(-1)^jT_g(\omega^{Y_g},h^{\wedge^jF_\infty^\vee\otimes\eta\mid_{Y_g}})\\ =&\int_{X_g/S}\mathrm{Td}(\overline{Th_g})T_g(\overline{\xi.})-\int_{P_g^0/S}\mathrm{Td}(\overline{Tb_g})T_g(\overline{\xi.}^\infty)\\ &+\int_{Y_g/S}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N})\widetilde{\mathrm{Td}}(\overline{Tf_g},\overline{Th_g}\mid_{Y_g})\\ &-\int_{Y_g/S}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N}_\infty)\widetilde{\mathrm{Td}}(\overline{Tf_g},\overline{Tb_g}\mid_{Y_g}).\end{aligned} \tag{10}$$

On the other hand, by definition, we have

$$\begin{aligned}\delta(\Xi\mid_P):=&\widetilde{\mathrm{ch}}_g(\overline{\xi.}^\infty,\overline{\eta})-\sum_k(-1)^kT_g(\omega^{Y_g},h^{\wedge^kF_\infty^\vee\otimes\eta\mid_{Y_g}})\\ &+\sum_k(-1)^kT_g(\omega^{P_g},h^{\xi_k^\infty\mid_{P_g}})\\ &-\int_{Y_g/S}\mathrm{Td}(Tf_g)\mathrm{Td}_g^{-1}(F)\mathrm{ch}_g(\eta)R(N_g)\\ &-\int_{P_g/S}T_g(\overline{\xi.}^\infty)\mathrm{Td}(\overline{Tb'_g})\end{aligned}$$

$$-\int_{Y_g/S}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N}_\infty)\widetilde{\mathrm{Td}}(\overline{Tf_g},\overline{Tb'_g}\mid_{Y_g})$$

where $b' : P \to S$ is the composition of the inclusion $P \hookrightarrow W(i)$ and the morphism l. Note that P_g^0 is an open and closed submanifold of P_g and $\overline{\xi.}^\infty$ is orthogonally split on the other components since they all belong to $\widetilde{X}_g$, then we can rewrite $\delta(\overline{\Xi}\mid_P)$ as

$$\begin{aligned}\delta(\overline{\Xi}\mid_P) =&\widetilde{\mathrm{ch}}_g(\overline{\Xi}_\nabla\mid_{P_g^0},h^H)-\sum_k(-1)^kT_g(\omega^{Y_g},h^{\wedge^kF_\infty^\vee\otimes\eta|_{Y_g}})\\ &+\sum_k(-1)^kT_g(\omega^{P_g^0},h^{\xi_k^\infty|_{P_g^0}})\\ &-\int_{Y_g/S}\mathrm{Td}(Tf_g)\mathrm{Td}_g^{-1}(F)\mathrm{ch}_g(\eta)R(N_g)\\ &-\int_{P_g^0/S}T_g(\overline{\xi.}^\infty)\mathrm{Td}(\overline{Tb_g})\\ &-\int_{Y_g/S}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N}_\infty)\widetilde{\mathrm{Td}}(\overline{Tf_g},\overline{Tb_g}\mid_{Y_g}).\end{aligned}$$

Comparing with the definition of $\delta(\overline{\Xi}\mid_X)$, the equality (10) implies that

$$\delta(\overline{\Xi}\mid_X)-\delta(\overline{\Xi}\mid_P)=0$$

which completes the proof of this deformation theorem. □

Now we consider the zero section imbedding $i_\infty : Y \to P = \mathbb{P}(N_\infty \oplus \mathcal{O}_Y)$. Here we use the fact that N_∞ is isomorphic to $N_{X/Y}$, we caution the reader that this is not necessarily an isometry since $\overline{N}_\infty$ carries the quotient metric induced by the Kähler metric on P but $N_{X/Y}$ carries the quotient metric induced by the Kähler metric on X. We recall that on P we have a tautological exact sequence

$$0\to\mathcal{O}(-1)\to\pi_P^*(N_\infty\oplus\mathcal{O}_Y)\to Q\to 0.$$

The equivariant section $\sigma : \mathcal{O}_P \to \pi_P^*(N_\infty \oplus \mathcal{O}_Y) \to Q$ induces the following Koszul resolution

$$0\to\wedge^{\mathrm{rk}Q}Q^\vee\to\cdots\to Q^\vee\to\mathcal{O}_P\to i_{\infty *}\mathcal{O}_Y\to 0.$$

Since σ is equivariant, the image of $\mathcal{O}_{P_g}$ under $\sigma\mid_{P_g}$ is contained in Q_g. This means that $\sigma\mid_{P_g}$ induces a Koszul resolution on P_g of the following form

$$0\to\wedge^{\mathrm{rk}Q_g}Q_g^\vee\to\cdots\to Q_g^\vee\to\mathcal{O}_{P_g}\to i_{\infty,g_*}\mathcal{O}_{Y_g}\to 0.$$

PROPOSITION 3.19. *Let $\overline{\kappa} := \overline{\kappa}(\overline{\eta},\overline{N}_\infty)$ be a hermitian Koszul resolution on P defined in Definition 3.3. Then for $n \gg 0$, we have $\delta(\overline{\kappa}(n)) = 0$.*

Proof. Denote the non-zero degree part of $Q\mid_{P_g}$ by $Q_\perp$, then we have the following isometry

$$\wedge^i\overline{Q}^\vee\mid_{P_g}=\wedge^i(\overline{Q}_g^\vee\oplus\overline{Q}_\perp^\vee)\cong\bigoplus_{t+s=i}(\wedge^t\overline{Q}_g^\vee\otimes\wedge^s\overline{Q}_\perp^\vee).$$

Consider the following complex of equivariant hermitian vector bundles on P_g

$$0\to\wedge^{\mathrm{rk}Q_g}\overline{Q}_g^\vee\otimes(\wedge^k\overline{Q}_\perp^\vee\otimes\pi_{P_g}^*\overline{\eta}\mid_{Y_g})\to\cdots\to\overline{Q}_g^\vee\otimes(\wedge^k\overline{Q}_\perp^\vee\otimes\pi_{P_g}^*\overline{\eta}\mid_{Y_g})$$
$$\to\wedge^k\overline{Q}_\perp^\vee\otimes\pi_{P_g}^*\overline{\eta}\mid_{Y_g}\to 0$$

which provides a resolution of $i_{\infty,g_*}(\wedge^k\overline{F}_\infty^\vee\otimes\overline{\eta}\mid_{Y_g})$ where F_∞, as before, is the non-zero degree part of the normal bundle N_∞ associated to i_∞. We denote this resolution by $\overline{\kappa}^{(k)}$, then according to the arguments given before this proposition we have a decomposition of complexes $\overline{\kappa}_\nabla\mid_{P_g}\cong\bigoplus_{k\geq 0}\overline{\kappa}_\nabla^{(k)}[-k]$ where $\overline{\kappa}_\nabla^{(k)}[-k]$ is obtained from $\overline{\kappa}_\nabla^{(k)}$ by shifting degree. Replacing $\overline{\kappa}$ by $\overline{\kappa}(n)$ for big enough n, we may assume that all elements in $\overline{\kappa}$ and $\overline{\kappa}^{(k)}$ are acyclic. Therefore, by Bisumt-Ma's immersion formula we have the following equality

$$\begin{aligned}\widetilde{\mathrm{ch}}_g(b'_{g_*}\overline{\kappa}^{(k)})=&T_g(\omega^{Y_g},h^{\wedge^kF_\infty^\vee\otimes\eta\mid_{Y_g}})-\sum_{i=0}^{\mathrm{rk}Q_g}(-1)^iT_g(\omega^{P_g},h^{\wedge^iQ_g^\vee\otimes\wedge^kQ_\perp^\vee\otimes\pi_{P_g}^*\eta\mid_{Y_g}})\\&+\int_{Y_g/S}\mathrm{ch}_g(\wedge^kF_\infty^\vee\otimes\eta\mid_{Y_g})R(N_{\infty,g})\mathrm{Td}(Tf_g)\\&+\int_{P_g/S}\mathrm{Td}(\overline{Tb'_g})T_g(\overline{\kappa}^{(k)})\\&+\int_{Y_g/S}\mathrm{ch}_g(\wedge^k\overline{F}_\infty^\vee\otimes\overline{\eta}\mid_{Y_g})\mathrm{Td}^{-1}(\overline{N}_{\infty,g})\widetilde{\mathrm{Td}}(\overline{Tf_g},\overline{Tb'_g}\mid_{Y_g}).\end{aligned}$$

It is easily seen from the decomposition $\overline{\kappa}_\nabla\mid_{P_g}=\bigoplus_{k\geq 0}\overline{\kappa}_\nabla^{(k)}[-k]$ that the secondary characteristic class $\widetilde{\mathrm{ch}}_g(\overline{\kappa})$ appearing in the definition of $\delta(\overline{\kappa})$ is exactly $\sum(-1)^k\widetilde{\mathrm{ch}}_g(b'_{g_*}\overline{\kappa}^{(k)})$. So taking the alternating sum of both two sides of the equality above and using the fact that equivariant analytic torsion form is additive for direct sum of acyclic bundles, we know that to prove $\delta(\overline{\kappa})=0$, we are left to show that $\sum(-1)^kT_g(\overline{\kappa}^{(k)})$ is equal to $T_g(\overline{\kappa})$. In fact, by using [21, Lemma 3.15], we have modulo $\mathrm{Im}\partial+\mathrm{Im}\overline{\partial}$

$$\begin{aligned}\sum(-1)^kT_g(\overline{\kappa}^{(k)})&=\sum(-1)^k\mathrm{ch}_g(\wedge^k\overline{Q}_\perp^\vee)\mathrm{ch}_g(\pi_{P_g}^*\overline{\eta}\mid_{Y_g})T_g(\overline{\wedge^\cdot Q_g^\vee})\\&=\mathrm{Td}_g^{-1}(\overline{Q}_\perp)\mathrm{ch}_g(\pi_{P_g}^*\overline{\eta}\mid_{Y_g})T_g(\overline{\wedge^\cdot Q_g^\vee})\\&=\mathrm{Td}_g^{-1}(\overline{Q})\mathrm{ch}_g(\pi_{P_g}^*\overline{\eta}\mid_{Y_g})T_g(\overline{\wedge^\cdot Q_g^\vee})\mathrm{Td}(\overline{Q}_g)\\&=\mathrm{ch}_g(\pi_{P_g}^*\overline{\eta}\mid_{Y_g})T_g(\overline{\wedge^\cdot Q^\vee})\\&=T_g(\overline{\kappa}).\end{aligned}$$

So we are done. ∎

It's now ready to finish the proof of the vanishing theorem for equivariant closed immersions. Let $\overline{\eta}$ be an equivariant hermitian vector bundle on Y, assume that

$$\overline{\Psi}: \quad 0 \to \overline{\xi}_m \to \cdots \to \overline{\xi}_1 \to \overline{\xi}_0 \to i_*\overline{\eta} \to 0$$

is a resolution of $i_*\overline{\eta}$ by equivariant hermitian vector bundles on X which satisfies Bismut assumption (A). We need to prove that for $n \gg 0$, $\delta(\overline{\Psi}(n)) = 0$.

Proof. (of Theorem 3.1) We first construct a resolution of $p_Y^*\overline{\eta}$ on $W(i)$ as

$$\overline{\Xi}: \quad 0 \to \overline{\widetilde{\xi}}_m \to \cdots \to \overline{\widetilde{\xi}}_0 \to \overline{\widetilde{\xi}}_0 \to j_*p_Y^*(\overline{\eta}) \to 0$$

which satisfies the condition (i) and (ii) in Definition 3.16. Then the restriction of $\overline{\Xi}$ to X (resp. P) provides a resolution of $i_*\overline{\eta}$ (resp. $i_{\infty *}\overline{\eta}$). Over X, we can find a third resolution $\overline{\Phi}$ of $i_*\overline{\eta}$ which dominates $\overline{\Psi}$ and $\overline{\Xi}\mid_X$. Namely we get short exact sequences of exact sequences

$$0 \to \overline{\mathrm{Ker}}(n) \to \overline{\Phi}(n) \to \overline{\Psi}(n) \to 0$$

and

$$0 \to \overline{\mathrm{Ker}}'(n) \to \overline{\Phi}(n) \to \overline{\Xi}(n)\mid_X \to 0.$$

Then after omitting $i_*\overline{\eta}$ their restrictions to X_g become two exact sequences of complexes. Since $n \gg 0$ we may assume that all elements and homologies in the induced double complexes are acyclic, so that by taking direct images we get two exact sequences of equivariant standard complexes on S. These two short exact sequences of equivariant standard complexes clearly satisfy the assumptions in Lemma 3.12. Therefore, using Lemma 3.12, Bismut-Ma's immersion formula and the double complex formula of equivariant Bott-Chern singular currents (cf. Theorem 2.18), we obtain that

$$\begin{aligned}
&\widetilde{\mathrm{ch}}_g(\overline{\Psi}(n)) - \widetilde{\mathrm{ch}}_g(\overline{\Phi}(n)) + \widetilde{\mathrm{ch}}_g(\overline{\mathrm{Ker}}(n)) \\
&\qquad + T_g(\omega^{X_g}, h^{\Psi(n)_\triangledown}) - T_g(\omega^{X_g}, h^{\Phi(n)_\triangledown}) + T_g(\omega^{X_g}, h^{\mathrm{Ker}(n)_\triangledown}) \\
&= \int_{X_g/S} [T_g(\overline{\Psi}(n)_\triangledown) - T_g(\overline{\Phi}(n)_\triangledown) + T_g(\overline{\mathrm{Ker}}(n)_\triangledown)] \cdot \mathrm{Td}(\overline{Th_g})
\end{aligned}$$

which implies that

$$\delta(\overline{\Phi}(n)) = \delta(\overline{\Psi}(n)) + \delta(\overline{\mathrm{Ker}}(n)).$$

By applying Bismut-Ma's immersion formula to the case where the immersion is the identity map and $\overline{\eta}$ is equal to the zero bundle, we get $\delta(\overline{\mathrm{Ker}}(n)) = 0$ so that $\delta(\overline{\Phi}(n)) = \delta(\overline{\Psi}(n))$. Similarly, we have $\delta(\overline{\Phi}(n)) = \delta(\overline{\Xi}(n)\mid_X)$ and hence $\delta(\overline{\Psi}(n)) = \delta(\overline{\Xi}(n)\mid_X)$. An entirely analogous reasoning implies that $\delta(\overline{\kappa}(n)) = \delta(\overline{\Xi}(n)\mid_P)$. Then the vanishing of $\delta(\overline{\Psi}(n))$ follows from Theorem 3.18 and Proposition 3.19. $\square$

4 Equivariant arithmetic Grothendieck groups with fixed wave front sets

By an arithmetic ring D we understand a regular, excellent, Noetherian integral ring, together with a finite set $\mathcal{S}$ of embeddings $D \hookrightarrow \mathbb{C}$, which is invariant under a conjugate-linear involution F_∞ (cf. [15, Def. 3.1.1]). Denote by μ_n the diagonalisable group scheme over D associated to $\mathbb{Z}/n\mathbb{Z}$. A μ_n-equivariant arithmetic scheme over D is a Noetherian scheme of finite type, endowed with a μ_n-projective action over D (cf. [21, Section 2]). Let X be a μ_n-equivariant arithmetic scheme whose generic fibre is smooth, then $X(\mathbb{C})$, the set of complex points of the variety $\coprod_{\sigma\in\mathcal{S}} X\times_D\mathbb{C}$, is a disjoint union of projective manifolds. This manifold admits an action of the group of complex n-th roots of unity and an anti-holomorphic involution induced by F_∞ which is still denoted by F_∞. It was shown in [31, Prop. 3.1] that if X is regular, then the fixed point subscheme X_{μ_n} is also regular. Fix a primitive n-th root of unity ζ_n and denote its corresponding holomorphic automorphism on $X(\mathbb{C})$ by g, by GAGA principle we have a natural isomorphism $X_{\mu_n}(\mathbb{C})\cong X(\mathbb{C})_g$.

Definition 4.1. An equivariant hermitian sheaf (resp. vector bundle) $\overline{E}$ on X is a coherent sheaf (resp. vector bundle) E on X, assumed locally free on $X(\mathbb{C})$, endowed with a μ_n-action which lifts the action of μ_n on X and a hermitian metric h on the associated bundle $E_{\mathbb{C}}$, which is invariant under F_∞ and g.

Remark 4.2. Let $\overline{E}$ be an equivariant hermitian sheaf (resp. vector bundle) on X, the restriction of $\overline{E}$ to the fixed point subscheme X_{μ_n} has a natural $\mathbb{Z}/n\mathbb{Z}$-grading structure $\overline{E}\mid_{X_{\mu_n}}\cong\oplus_{k\in\mathbb{Z}/n\mathbb{Z}}\overline{E}_k$. We shall often write $\overline{E}_{\mu_n}$ for $\overline{E}_0$. It is clear that the associated bundle of $\overline{E}_{\mu_n}$ over $X(\mathbb{C})$ is exactly equal to $\overline{E}_g$.

Over a complex manifold M, we may consider the current space which is the continuous dual of the space of smooth complex valued differential forms (cf. [27, Chapter IX]). The wave front set $\mathrm{WF}(\omega)$ of a current ω over M is a closed conical subset of the cotangent bundle $T^*_{\mathbb{R}}M_0:=T^*_{\mathbb{R}}M\backslash\{0\}$. This conical subset measures the singularities of ω, actually the projection of $\mathrm{WF}(\omega)$ in M is equal to the singular locus of the support of ω. It also allows us to define certain products and pull-backs of currents. We refer to [19, Chapter VIII] for the definition and various properties of wave front set.

Now let X be a μ_n-equivariant arithmetic scheme with smooth generic fibre and let S be a conical subset of $T^*_{\mathbb{R}}X(\mathbb{C})_{g,0}$, denote by $D^{p,p}(X(\mathbb{C})_g,S)$ the set of currents ω of type (p,p) on $X(\mathbb{C})_g$ which satisfy $F_\infty^*\omega=(-1)^p\omega$ and whose wave front sets are contained in S, we shall write $\widetilde{\mathcal{U}}(X_{\mu_n},S)$ for the current class

$$\widetilde{\mathcal{U}}(X(\mathbb{C})_g,S):=\bigoplus_{p\geq 0}(D^{p,p}(X(\mathbb{C})_g,S)/(\mathrm{Im}\partial+\mathrm{Im}\overline{\partial})).$$

Let $\overline{E}$ be an equivariant hermitian sheaf or vector bundle on X. Following the same notations and definitions as in [21, Section 3], we write $\mathrm{ch}_g(\overline{E})$ for the equivariant Chern character form $\mathrm{ch}_g((E_{\mathbb{C}},h))$ associated to the hermitian holomorphic vector bundle $(E_{\mathbb{C}},h)$ on $X(\mathbb{C})$. Similarly, we have the equivariant Todd form $\mathrm{Td}_g(\overline{E})$. Furthermore, let $\overline{\varepsilon}: 0\to \overline{E}'\to\overline{E}\to\overline{E}''\to 0$ be an exact sequence of equivariant hermitian sheaves or vector bundles on X, we can associate to it an equivariant Bott-Chern secondary characteristic class $\widetilde{\mathrm{ch}}_g(\overline{\varepsilon})\in\widetilde{\mathcal{U}}(X_{\mu_n},\emptyset)$ which satisfies the differential equation

$$\mathrm{dd}^c\widetilde{\mathrm{ch}}_g(\overline{\varepsilon})=\mathrm{ch}_g(\overline{E}')-\mathrm{ch}_g(\overline{E})+\mathrm{ch}_g(\overline{E}'').$$

Definition 4.3. Let X be a μ_n-equivariant arithmetic scheme with smooth generic fibre and let S be a conical subset of $T^*_{\mathbb{R}}X(\mathbb{C})_{g,0}$, we define the equivariant arithmetic Grothendieck group $\widehat{G_0}(X,\mu_n,S)$ (resp. $\widehat{K_0}(X,\mu_n,S)$) with respect to X and S as the free abelian group generated by the elements of $\widetilde{\mathcal{U}}(X_{\mu_n},S)$ and by the equivariant isometry classes of equivariant hermitian sheaves (resp. vector bundles) on X, together with the relations
(i). for every exact sequence $\overline{\varepsilon}$ as above, $\widetilde{\mathrm{ch}}_g(\overline{\varepsilon})=\overline{E}'-\overline{E}+\overline{E}''$;
(ii). if $\alpha\in\widetilde{\mathcal{U}}(X_{\mu_n},S)$ is the sum of two elements α' and α'' in $\widetilde{\mathcal{U}}(X_{\mu_n},S)$, then the equality $\alpha=\alpha'+\alpha''$ holds in $\widehat{G_0}(X,\mu_n,S)$ (resp. $\widehat{K_0}(X,\mu_n,S)$).

Remark 4.4. (i). When $S'\subset S$, [30, Theorem 3.9 (ii)] implies that the natural map from $\widetilde{\mathcal{U}}(X_{\mu_n},S')$ to $\widetilde{\mathcal{U}}(X_{\mu_n},S)$ is injective. So the first generating relation in Definition 4.3 does make sense.
(ii). When X is regular, one can carry out the proof of [21, Proposition 4.2] to show that the natural morphism from $\widehat{K_0}(X,\mu_n,S)$ to $\widehat{G_0}(X,\mu_n,S)$ is an isomorphism.
(iii). The definition of the equivariant arithmetic Grothendieck group implies that there are exact sequences

$$\widetilde{\mathcal{U}}(X_{\mu_n},S)\xrightarrow{a}\widehat{G_0}(X,\mu_n,S)\xrightarrow{\pi}G_0(X,\mu_n)\longrightarrow 0$$

and

$$\widetilde{\mathcal{U}}(X_{\mu_n},S)\xrightarrow{a}\widehat{K_0}(X,\mu_n,S)\xrightarrow{\pi}K_0(X,\mu_n)\longrightarrow 0$$

where a is the natural map which sends $\alpha\in\widetilde{\mathcal{U}}(X_{\mu_n},S)$ to the class of α in $\widehat{G_0}(X,\mu_n,S)$ (resp. $\widehat{K_0}(X,\mu_n,S)$) and π is the forgetful map. Here the group $G_0(X,\mu_n)$ is the Grothendieck group of μ_n-equivariant coherent sheaves which are locally free on $X(\mathbb{C})$, by a theorem of Quillen (cf. [26, Thm. 3 Cor. 1]) we know that it is isomorphic to the ordinary Grothendieck group of μ_n-equivariant coherent sheaves.

In [29, Section 3], we have introduced the ring structure of $\widehat{K_0}(X,\mu_n,\emptyset)$. Since we may have a well-defined product of two currents if their wave front sets have no intersection, and the wave front set is invariant under the operation of multiplying a smooth current, we know that the Grothendieck group $\widehat{K_0}(X,\mu_n,S)$ has a $\widehat{K_0}(X,\mu_n,\emptyset)$-module structure. The same thing goes to $\widehat{G_0}(X,\mu_n,S)$.

Furthermore, consider the isomorphism $R(\mu_n) \cong K_0(D)[T]/(1-T^n)$. Let $\overline{I}$ be the μ_n-equivariant hermitian D-module whose term of degree 1 is D endowed with the trivial metric and whose other terms are 0. Then we may make $\widehat{K_0}(D,\mu_n,\emptyset)$ an $R(\mu_n)$-algebra under the ring morphism which sends T to $\overline{I}$. By doing pull-backs, we may endow every arithmetic Grothendieck group we defined before with an $R(\mu_n)$-module structure. Notice finally that there is a well-defined map from $\widehat{G_0}(X,\mu_n,\emptyset)$ (resp. $\widehat{K_0}(X,\mu_n,\emptyset)$) to the space of complex closed differential forms, which is defined by the formula $\mathrm{ch}_g(\overline{E}+\alpha) := \mathrm{ch}_g(\overline{E}) + \mathrm{dd}^c\alpha$ where $\overline{E}$ is an equivariant hermitian sheaf (resp. vector bundle) and $\alpha \in \widetilde{\mathcal{U}}(X_{\mu_n},\emptyset)$.

Now we investigate the wave front set of a current after doing push-forward. Let f be a holomorphic map of compact complex manifolds, we may define a push-forward f_* on current space which is the dual map of the pull-back of smooth forms. When f is smooth, the push-forward f_* extends the integration of smooth forms over the fibre. Assume that we are given a smooth morphism $f: U \to V$ of compact complex manifolds, then f_* induces a current K over the product space $V \times U$ defined as

$$K(\alpha\otimes\beta) = (f_*\beta)(\alpha)$$

where α and β are smooth forms over V and U respectively. Define

$$M = \{(v,u)\in V\times U \mid f(u)=v\}$$

which is a submanifold in $V\times U$. From the fact that $f_*\beta$ is just the integration of smooth forms over the fibre, it is easily seen that the current $K \in D^*(V\times U)$ is exactly the object $\mathrm{d}S_M$ in [19, Theorem 8.1.5]. Then by that theorem, the wave front set of K is equal to

$$\mathrm{WF}(K) = \{(v,u,\xi,-f^*(\xi)) \in T^*_{\mathbb{R}}V \times T^*_{\mathbb{R}}U \mid f(u)=v, \xi\neq 0\}.$$

Let S be a conical subset of $T^*_{\mathbb{R}}U_0$, we fix some notations as follows.

$$\begin{aligned}\mathrm{WF}(K)_V =& \{(v,\xi)\in T^*_{\mathbb{R}}V_0 \mid \exists u\in U, (v,u,\xi,0)\in \mathrm{WF}(K)\}\\ \mathrm{WF}'(K)_U =& \{(u,\eta)\in T^*_{\mathbb{R}}U_0 \mid \exists v\in V, (v,u,0,-\eta)\in \mathrm{WF}(K)\}\\ \mathrm{WF}'(K)_V\circ S =& \{(v,\xi)\in T^*_{\mathbb{R}}V_0 \mid \exists (u,\eta)\in S, (v,u,\xi,-\eta)\in \mathrm{WF}(K)\}.\end{aligned}$$

Theorem 4.5. *Let notations and assumptions be as above. Assume that ω is a current over U whose wave front set is contained in S with $S\cap \mathrm{WF}'(K)_U=\emptyset$, then the wave front set of $f_*\omega$ is contained in*

$$S' := \mathrm{WF}(K)_V \cup \mathrm{WF}'(K)\circ S.$$

Proof. This follows from [19, Theorem 8.2.12 and 8.2.13]. □

Remark 4.6. (i) In our situation, the condition $S\cap \mathrm{WF}'(K)_U=\emptyset$ is always satisfied because by definition we have $\mathrm{WF}'(K)_U=\emptyset$.

(ii). In our situation, S' is always equal to $\mathrm{WF}'(K)\circ S$ because $\mathrm{WF}(K)_V=\emptyset$.
(iii). If S is the empty set, then S' is also empty. This is compatible with the push-forward of smooth forms.
(iv). Assume that the restriction of f to a closed submanifold W is also smooth. Denote by $N_{U/W}$ the normal bundle of W in U. If $S=N^\vee_{U/W,\mathbb{R}}\setminus\{0\}$, then $S'=\emptyset$.

We now turn to the arithmetic case. Let X,Y be two μ_n-equivariant arithmetic schemes with smooth generic fibres, and let $f:X\to Y$ be an equivariant morphism over D which is smooth on the complex numbers. Fix a $\mu_n(\mathbb{C})$-invariant Kähler metric on $X(\mathbb{C})$ so that we get a Kähler fibration with respect to the holomorphic submersion $f_{\mathbb{C}}:X(\mathbb{C})\to Y(\mathbb{C})$. Let $\overline{E}$ be an f-acyclic μ_n-equivariant hermitian sheaf on X, we know that the direct image f_*E is locally free on $Y(\mathbb{C})$ and it can be endowed with a natural equivariant structure and the L^2-metric. Let $\widehat{G_0}^{\mathrm{ac}}(X,\mu_n,S)$ be the group generated by f-acyclic equivariant hermitian sheaves on X and the elements of $\widetilde{\mathcal{U}}(X_{\mu_n},S)$, with the same relations as in Definition 4.3. A theorem of Quillen (cf. [26, Cor.3 P. 111]) for the algebraic analogs of these groups implies that the natural map $\widehat{G_0}^{\mathrm{ac}}(X,\mu_n,S)\to\widehat{G_0}(X,\mu_n,S)$ is an isomorphism. So the following definition does make sense.

Definition 4.7. Let notations and assumptions be as above. The push-forward morphism $f_*:\widehat{G_0}(X,\mu_n,S)\to\widehat{G_0}(Y,\mu_n,S')$ is defined in the following way.
(i). For every f-acyclic μ_n-equivariant hermitian sheaf $\overline{E}$ on X, $f_*\overline{E}=(f_*E,f_*h^E)-T_g(\omega^X,h^E)$.
(ii). For every element $\alpha\in\widetilde{\mathcal{U}}(X_{\mu_n},S)$, $f_*\alpha=\int_{X_g/Y_g}\mathrm{Td}_g(Tf,h^{Tf})\alpha\in\widetilde{\mathcal{U}}(Y_{\mu_n},S')$.

Remark 4.8. If Y is regular, by Remark 4.4 (ii) we know that $\widehat{K_0}(Y,\mu_n,S')$ is naturally isomorphic to $\widehat{G_0}(Y,\mu_n,S')$ so that (f_*E,f_*h^E) admits a finite equivariant hermitian resolution; if the morphism f is flat and Y is reduced, then (f_*E,f_*h^E) is locally free when E is so. Therefore in both two cases above, one can also define a reasonable push-forward morphism $f_*:\widehat{K_0}(X,\mu_n,S)\to\widehat{K_0}(Y,\mu_n,S')$.

Theorem 4.9. *The push-forward morphism f_* is a well-defined group homomorphism.*

Proof. The argument is the same as in the proof of [29, Theorem 6.2]. □

Lemma 4.10. *(Projection formula) For any elements $y\in\widehat{K_0}(Y,\mu_n,\emptyset)$ and $x\in\widehat{G_0}(X,\mu_n,S)$, the identity $f_*(f^*y\cdot x)=y\cdot f_*x$ holds in $\widehat{G_0}(Y,\mu_n,S')$.*

Proof. Assume that $y=\overline{E}$ is an equivariant hermitian vector bundle and $x=\overline{F}$ is an f-acyclic equivariant hermitian sheaf, then $f^*y\cdot x=f^*\overline{E}\otimes\overline{F}$. By projection formula for direct images and the definition of L^2-metric, we know

that $f_*(f^*\overline{E}\otimes\overline{F})$ is isometric to $\overline{E}\otimes f_*\overline{F}$. Moreover, concerning the analytic torsion form, we have $T_g(\omega^X, h^{f^*E\otimes F}) = \mathrm{ch}_g(\overline{E})T_g(\omega^X, h^F)$. So the projection formula $f_*(f^*y\cdot x) = y\cdot f_*x$ holds in this case.
Assume that $y=\overline{E}$ is an equivariant hermitian vector bundle and $x=\alpha$ is represented by some singular current. We write f_g^* and f_{g*} for the pull-back and push-forward of currents respectively, then

$$\begin{aligned}f_*(f^*y\cdot x) =& f_*(f_g^*\mathrm{ch}_g(\overline{E})\alpha) = f_{g*}(f_g^*\mathrm{ch}_g(\overline{E})\alpha\mathrm{Td}_g(\overline{Tf}))\\ =&\mathrm{ch}_g(\overline{E})f_{g*}(\alpha\mathrm{Td}_g(\overline{Tf}))\\ =&\mathrm{ch}_g(\overline{E})\int_{X_g/Y_g}\alpha\mathrm{Td}_g(\overline{Tf}) = y\cdot f_*x.\end{aligned}$$

Here we have used an extension of projection formula of smooth forms $p_*(p^*\alpha_1\wedge\alpha_2) = \alpha_1\wedge p_*\alpha_2$ (cf. [14, Prop. IX p. 303]) to the case where the second variable α_2 is replaced by a singular current. The fact that this extension is valid follows from the definition of p_* and the definition of the product of smooth form and singular current.
Assume that $y=\beta$ is represented by some smooth form and $x=\overline{E}$ is an f-acyclic hermitian sheaf, then

$$\begin{aligned}f_*(f^*y\cdot x) =& f_*(f_g^*(\beta)\mathrm{ch}_g(\overline{F})) = f_{g*}(f_g^*(\beta)\mathrm{ch}_g(\overline{F})\mathrm{Td}_g(\overline{Tf}))\\ =&\beta f_{g*}(\mathrm{ch}_g(\overline{E})\mathrm{Td}_g(\overline{Tf}))\\ =&\beta\int_{X_g/Y_g}\mathrm{ch}_g(\overline{E})\mathrm{Td}_g(\overline{Tf}) = \beta(\mathrm{ch}_g(\overline{f_*F}) - \mathrm{dd}^cT_g(\omega^X, h^F))\end{aligned}$$

which is exactly $y\cdot f_*x$.
Finally, assume that $y=\beta$ is represented by some smooth form and $x=\alpha$ is represented by some singular current, then

$$\begin{aligned}f_*(f^*y\cdot x) =& f_*(f_g^*(\beta)\mathrm{dd}^c\alpha) = f_{g*}(f_g^*(\beta)\mathrm{dd}^c\alpha\mathrm{Td}_g(\overline{Tf}))\\ =&\beta\mathrm{dd}^cf_{g*}(\alpha\mathrm{Td}_g(\overline{Tf}))\end{aligned}$$

which is also equal to $y\cdot f_*x$.
Since f_* and f^* are both group homomorphisms, we may conclude the projection formula by linear extension. □

Remark 4.11. Lemma 4.10 implies that f_* is a homomorphism of $R(\mu_n)$-modules, and hence it induces a push-forward morphism after taking localization.

To end this section, we recall an important lemma which will be used frequently in our later arguments.

Lemma 4.12. *([21, Lemma 4.5]) Let X be a regular μ_n-equivariant arithmetic scheme and let $\overline{E}$ be an equivariant hermitian vector bundle on X_{μ_n} such that $\overline{E}_{\mu_n}=0$. Then the element $\lambda_{-1}(\overline{E})$ is invertible in $\widehat{K_0}(X_{\mu_n},\mu_n,S)_\rho$.*

5 Arithmetic concentration theorem

In this section, we shall prove the arithmetic concentration theorem which is an analog of Thomason's result in Arakelov geometry. Let X be a μ_n-equivariant arithmetic scheme with smooth generic fibre, we consider a special closed immersion $i : X_{\mu_n} \hookrightarrow X$ where X_{μ_n} is the fixed point subscheme of X. We shall first construct a well-defined group homomorphism i_* between equivariant arithmetic G_0-groups as in the algebraic case. To construct i_*, some analytic datum, which is the equivariant Bott-Chern singular current, should be involved. Precisely speaking, let $\overline{\eta}$ be a μ_n-equivariant hermitian sheaf on X_{μ_n} and let $\overline{\xi}.$ be a bounded complex of μ_n-equivariant hermitian sheaves which provides a resolution of $i_*\overline{\eta}$ on X. Such a resolution always exists since the generic fibre of X is supposed to be smooth. Then we may have an equivariant Bott-Chern singular current $T_g(\overline{\xi}.) \in \widetilde{\mathcal{U}}(X_{\mu_n})$. Note that on the complex numbers the 0-degree part of the normal bundle $N := N_{X/X_g}$ vanishes (cf. [21, Prop. 2.12]) so that the wave front set of $T_g(\overline{\xi}.)$ is the empty set. This fact means that the following definition does make sense.

Definition 5.1. Let notations and assumptions be as above. The embedding morphism
$$i_* : \quad \widehat{G_0}(X_{\mu_n}, \mu_n, S) \to \widehat{G_0}(X, \mu_n, S)$$
is defined in the following way.
(i). For every μ_n-equivariant hermitian sheaf $\overline{\eta}$ on X_{μ_n}, suppose that $\overline{\xi}.$ is a resolution of $i_*\overline{\eta}$ on X whose metrics satisfy Bismut assumption (A), $i_*[\overline{\eta}] = \sum_k (-1)^k[\overline{\xi}_k] + T_g(\overline{\xi}.)$.
(ii). For every $\alpha \in \widetilde{\mathcal{U}}(X_{\mu_n}, S)$, $i_*\alpha = \alpha \mathrm{Td}_g^{-1}(\overline{N})$.

Theorem 5.2. *The embedding morphism i_* is a well-defined group homomorphism.*

Proof. The argument is the same as in the proof of [29, Theorem 5.2]. □

Lemma 5.3. *(Projection formula) For any elements $x \in \widehat{K_0}(X, \mu_n, \emptyset)$ and $y \in \widehat{G_0}(X_{\mu_n}, \mu_n, S)$, the identity $i_*(i^*x \cdot y) = x \cdot i_*y$ holds in $\widehat{G_0}(X, \mu_n, S)$.*

Proof. Assume that $x = \overline{E}$ is an equivariant hermitian vector bundle and $y = \overline{F}$ is an equivariant hermitian sheaf. Let $\overline{\xi}.$ be a resolution of $i_*\overline{F}$ on X, then $\overline{E} \otimes \overline{\xi}.$ provides a resolution of $i_*(i^*\overline{E} \otimes \overline{F})$. By definition we have
$$i_*(i^*x \cdot y) = \sum (-1)^k[\overline{\xi}_k \otimes \overline{E}] + \mathrm{ch}_g(\overline{E})T_g(\overline{\xi}.)$$
which is exactly $x \cdot i_*y$. Assume that $x = \alpha$ is represented by some smooth form and $y = \overline{F}$ is an equivariant hermitian sheaf. Again let $\overline{\xi}.$ be a resolution of $i_*\overline{F}$ on X, then
$$i_*(i^*x \cdot y) = \alpha \mathrm{Td}_g^{-1}(\overline{N}_{X/X_g})\mathrm{ch}_g(\overline{F}) = \alpha[\mathrm{dd}^c T_g(\overline{\xi}.) + \sum (-1)^k \mathrm{ch}_g(\overline{\xi}_k)]$$

which is exactly $x \cdot i_* y$. Now assume that $x = \overline{E}$ is an equivariant hermitian vector bundle and $y = \alpha$ is represented by some singular current, then

$$i_*(i^* x \cdot y) = i_*(\mathrm{ch}_g(\overline{E})\alpha) = \mathrm{ch}_g(\overline{E})\alpha \mathrm{Td}_g^{-1}(\overline{N}_{X/X_g})$$

which is exactly $x \cdot i_* y$. Finally, if x is represented by some smooth form and y is represented by some singular current then their product is well-defined and $i_*(i^* x \cdot y)$ is obviously equal to $x \cdot i_* y$. Note that i_* and i^* are group homomorphisms, so we may conclude the projection formula from its correctness on generators. This completes the proof. □

REMARK 5.4. Lemma 5.3 implies that i_* is even a homomorphism of $R(\mu_n)$-modules so that it induces a homomorphism between arithmetic G_0-groups after taking localization.

With Remark 5.4, we may formulate the arithmetic concentration theorem as follows.

THEOREM 5.5. *The embedding morphism* $i_* : \widehat{K_0}(X_{\mu_n}, \mu_n, S)_\rho \to \widehat{K_0}(X, \mu_n, S)_\rho$ *is an isomorphism if X is regular. In this case, the inverse morphism of i_* is given by* $\lambda_{-1}^{-1}(\overline{N}_{X/X_{\mu_n}}^\vee) \cdot i^*$ *where* $N_{X/X_{\mu_n}}$ *is the normal bundle of* $i(X_{\mu_n})$ *in* X.

Before we give the proof of this concentration theorem, we recall a crucial lemma as follows.

LEMMA 5.6. *Let $\overline{\eta}$ be an equivariant hermitian vector bundle on X_{μ_n}. Assume that $\overline{\xi.}$ is an equivariant hermitian resolution of $i_*\overline{\eta}$ on X whose metrics satisfy Bismut assumption (A). Then the equality*

$$\lambda_{-1}(\overline{N}_{X/X_{\mu_n}}^\vee) \cdot \overline{\eta} - \sum_j (-1)^j i^*(\overline{\xi}_j) = T_g(\overline{\xi}.)$$

holds in the group $\widehat{K_0}(X_{\mu_n}, \mu_n, S)$.

Proof. This is [29, Lemma 5.13]. □

Proof. (of Theorem 5.5) Denote by U the complement of X_{μ_n} in X, then $j : U \hookrightarrow X$ is a μ_n-equivariant open subscheme of X whose fixed point set is empty. We consider the following double complex

$$\begin{tikzcd}
\widetilde{\mathcal{U}}(X_{\mu_n},S)_\rho \arrow[r, "i_*"] \arrow[d, "a"] & \widetilde{\mathcal{U}}(X_{\mu_n},S)_\rho \arrow[r, "j^*"] \arrow[d, "a"] & \widetilde{\mathcal{U}}(U_{\mu_n},\emptyset)_\rho \arrow[r] \arrow[d, "a"] & 0 \\
\widehat{K_0}(X_{\mu_n},\mu_n,S)_\rho \arrow[r, "i_*"] \arrow[d, "\pi"] & \widehat{K_0}(X,\mu_n,S)_\rho \arrow[r, "j^*"] \arrow[d, "\pi"] & \widehat{K_0}(U,\mu_n,\emptyset)_\rho \arrow[r] \arrow[d, "\pi"] & 0 \\
K_0(X_{\mu_n},\mu_n)_\rho \arrow[r, "i_*"] \arrow[d] & K_0(X,\mu_n)_\rho \arrow[r, "j^*"] \arrow[d] & K_0(U,\mu_n)_\rho \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 &
\end{tikzcd}$$

whose first and second columns are both exact sequences according to Remark 4.4 (iii). For the third column, $K_0(U,\mu_n)_\rho$ is equal to 0 by [31, (2.1.3)], $\widetilde{\mathcal{U}}(U_{\mu_n},\emptyset)_\rho$ is also equal to 0 since U_{μ_n} is empty. Then from Remark 4.4 (iii) we know that $\widehat{K_0}(U,\mu_n,\emptyset)_\rho$ is equal to 0. We claim that $i_*:\widehat{K_0}(X_{\mu_n},\mu_n,S)_\rho\to\widehat{K_0}(X,\mu_n,S)_\rho$ is surjective. Indeed, for any element $x\in\widehat{K_0}(X,\mu_n,S)_\rho$ we may find an element $y\in\widehat{K_0}(X_{\mu_n},\mu_n,S)_\rho$ such that $i_*\pi(y)=\pi(x)$ because the third line is exact. This means $x-i_*(y)$ is in the kernel of π, so there exists an element $\alpha\in\widetilde{\mathcal{U}}(X_{\mu_n},S)_\rho$ such that $\alpha=x-i_*(y)$ in $\widehat{K_0}(X,\mu_n,S)_\rho$. Set $\beta=\alpha\mathrm{Td}_g(\overline{N})$, we get $i_*(y+\beta)=i_*(y)+\alpha=x$ in $\widehat{K_0}(X,\mu_n,S)_\rho$. Hence, i_* is surjective.

We now prove that the embedding morphism $i_*:\widehat{K_0}(X_{\mu_n},\mu_n,S)_\rho\to\widehat{K_0}(X,\mu_n,S)_\rho$ is really an isomorphism by constructing its inverse morphism. Let ω be an element in $\widetilde{\mathcal{U}}(X_{\mu_n},S)$, by definition we have

$$\begin{aligned}
\lambda_{-1}^{-1}(\overline{N}_{X/X_{\mu_n}}^\vee)\cdot i^*i_*(\omega) =&\lambda_{-1}^{-1}(\overline{N}_{X/X_{\mu_n}}^\vee)\cdot\omega\mathrm{Td}_g^{-1}(\overline{N}_{X/X_g})\\
=&\mathrm{ch}_g(\lambda_{-1}^{-1}(\overline{N}_{X/X_g}^\vee))\omega\mathrm{Td}_g^{-1}(\overline{N}_{X/X_g})\\
=&\omega.
\end{aligned}$$

Let $\overline{\eta}$ be an equivariant hermitian vector bundle on X_{μ_n} and assume that $\overline{\xi.}$ is an equivariant hermitian resolution of $i_*\overline{\eta}$ on X whose metrics satisfy Bismut assumption (A), then by the definition of the embedding morphism i_* and Lemma 5.6 we have

$$\lambda_{-1}^{-1}(\overline{N}_{X/X_{\mu_n}}^\vee)\cdot i^*i_*(\overline{\eta}) =\lambda_{-1}^{-1}(\overline{N}_{X/X_{\mu_n}}^\vee)\cdot i^*(\sum_k(-1)^k\overline{\xi}_k+T_g(\overline{\xi.}))=\overline{\eta}.$$

So the inverse morphism of i_* is of the form $\lambda_{-1}^{-1}(\overline{N}_{X/X_{\mu_n}}^\vee)\cdot i^*$ and we are done. $\square$

6 A Lefschetz fixed point formula for singular arithmetic schemes with smooth generic fibres

6.1 The statement

We formulate in this subsection the statement of our main theorem, a singular Lefschetz fixed point formula for equivariant arithmetic schemes with smooth generic fibres. Its proof will be given in next two subsections. Let $f : X \to Y$ be a μ_n-equivariant morphism between two arithmetic schemes with smooth generic fibres, which is smooth on the complex numbers. This morphism f is automatically projective and hence proper, according to the definition of equivariant arithmetic scheme. Suppose that f factors through some regular equivariant arithmetic scheme Z. More precisely, our assumption is that there exist an equivariant closed immersion $i : X \hookrightarrow Z$ and an equivariant morphism $h : Z \to Y$ such that $f = h \circ i$ and h is also smooth on the complex numbers. Moreover, we shall assume that the μ_n-action on Y is trivial.

Let η be an equivariant coherent sheaf on X, then there exists a bounded complex of equivariant vector bundles which provides a resolution of $i_*\eta$ on Z because Z is regular. Since any two equivariant resolution of $i_*\eta$ can be dominated by a third one, the symbol $\mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\eta, \mathcal{O}_{Z_{\mu_n}})$ does make sense.

We choose arbitrary μ_n-invariant Kähler forms ω^Z and ω^X on $Z(\mathbb{C})$ and $X(\mathbb{C})$ respectively, the Kähler form ω^X is not necessarily the Kähler form induced by ω^Z. The Kähler form on $X(\mathbb{C})$ induced by ω^Z will be denoted by ω^Z_X. Denote by N the normal bundle of $i_{\mathbb{C}}(X(\mathbb{C}))$ in $Z(\mathbb{C})$, we endow it with the quotient metric provided that $TX(\mathbb{C})$ carries the Kähler metric corresponding to ω^Z_X. Let $\overline{F}$ be the non-zero degree part of $\overline{N}$, then by [13, Exp. VII, Lem. 2.4 and Prop. 2.5] for any equivariant hermitian sheaf $\overline{\eta}$ on X there exists a canonical isomorphism on X_g

$$\mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\eta, \mathcal{O}_{Z_{\mu_n}})_{\mathbb{C}} \cong \wedge^k F^\vee \otimes \eta_{\mathbb{C}} \mid_{X_g}$$

which is equivariant. This means we may endow $\mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\eta, \mathcal{O}_{Z_{\mu_n}})_{\mathbb{C}}$ with a hermitian metric induced by the metrics on F and η so that it becomes an equivariant hermitian sheaf on X_{μ_n}. Moreover, we know that the hermitian vector bundle $\overline{F}$ fits the following exact sequence

$$(\overline{\mathcal{F}}, \omega^X): \quad 0 \to \overline{N}_{X/X_g} \to \overline{N}_{Z/Z_g} \to \overline{F} \to 0$$

where N_{Z/Z_g} admits the quotient metric associated to ω^Z and N_{X/X_g} admits the quotient metric associated to ω^X. Similarly, we shall denote by $(\overline{\mathcal{F}}, \omega^Z_X)$ the hermitian exact sequence $\overline{\mathcal{F}}$ whose metric on N_{X/X_g} is induced by ω^Z_X.

The push-forward homomorphism from the arithmetic G_0-group $\widehat{G_0}(X, \mu_n, \emptyset)$ to $\widehat{G_0}(Y, \mu_n, \emptyset)$ with respect to the Kähler form ω^X is denoted by f_* as usual. The push-forward homomorphism from $\widehat{G_0}(X_{\mu_n}, \mu_n, \emptyset)$ to $\widehat{G_0}(Y, \mu_n, \emptyset)$ with respect to the Kähler form ω^Z_X will be denoted by $f^Z_{\mu_n *}$.

Write $\widetilde{\mathrm{Td}}(Tf_g,\omega^X,\omega^Z_X)$ for the secondary characteristic class of the exact sequence

$$0 \longrightarrow (Tf_g,\omega^X) \xrightarrow{\mathrm{Id}} (Tf_g,\omega^Z_X) \longrightarrow 0 \longrightarrow 0$$

where the middle term carries the metric induced by ω^Z_X and the sub term carries the metric induced by ω^X. Then the singular Lefschetz fixed point formula for equivariant arithmetic schemes with smooth generic fibres can be formulated as follows.

Theorem 6.1. *Let notations and assumptions be as above. Then for any equivariant hermitian sheaf $\overline{\eta}$ on X, the equality*

$$\begin{aligned} f_*(\overline{\eta}) =& f^Z_{\mu_n *}(i^*_{\mu_n}(\lambda^{-1}_{-1}(\overline{N}^\vee_{Z/Z_{\mu_n}}))\cdot\sum_k(-1)^k\mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\overline{\eta},\overline{\mathcal{O}}_{Z_{\mu_n}})) \\ &+\int_{X_g/Y}\mathrm{Td}(Tf_g,\omega^X)\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g(\overline{\mathcal{F}},\omega^X)\mathrm{Td}^{-1}_g(\overline{F}) \\ &-\int_{X_g/Y}\mathrm{Td}_g(Tf)\mathrm{ch}_g(\eta)R_g(N_{X/X_g}) \\ &+\int_{X_g/Y}\widetilde{\mathrm{Td}}(Tf_g,\omega^X,\omega^Z_X)\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g(\overline{N}_{Z/Z_g})\mathrm{Td}^{-1}_g(\overline{F}) \end{aligned}$$

holds in the group $\widehat{G_0}(Y,\mu_n,\emptyset)_\rho$.

Remark 6.2. This arithmetic Lefschetz fixed point formula was inspired by [31, Théorème 3.5].

6.2 Equivariant arithmetic G_0-theoretic vanishing theorem

The central actor in the proof of Theorem 6.1 is the following vanishing theorem in equivariant arithmetic G_0-theory, which can be viewed as a translation of Theorem 3.1.

Theorem 6.3. *Let notations and assumptions be as in last subsection. Let $\overline{\eta}$ be an equivariant hermitian sheaf on X, and let*

$$\overline{\Psi}:\quad 0\to\overline{\xi}_m\to\cdots\to\overline{\xi}_1\to\overline{\xi}_0\to i_*\overline{\eta}\to 0$$

be a resolution of $i_*\overline{\eta}$ by equivariant hermitian vector bundles on Z. Denote by $h_{\mu_n *}$ the push-forward homomorphism from $\widehat{K_0}(Z_{\mu_n},\mu_n,N^\vee_{g,\mathbb{R}}\setminus\{0\})$ to $\widehat{G_0}(Y,\mu_n,\emptyset)$ with respect to the Kähler form ω^Z. Then the formula

$$\begin{aligned} &f^Z_{\mu_n *}(\sum(-1)^k\mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\overline{\eta},\overline{\mathcal{O}}_{Z_{\mu_n}}))-h_{\mu_n *}(\sum(-1)^k(\overline{\xi}_k\mid_{Z_{\mu_n}})) \\ =&\int_{Z_g/Y}T_g(\overline{\xi.})\mathrm{Td}(\overline{Th_g})+\int_{X_g/Y}\mathrm{Td}(Tf_g)\mathrm{Td}^{-1}_g(F)\mathrm{ch}_g(\eta)R(N_g) \\ &+\int_{X_g/Y}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}^{-1}_g(\overline{N})\widetilde{\mathrm{Td}}((Tf_g,\omega^Z_X),\overline{Th_g}\mid_{X_g}) \end{aligned}$$

holds in $\widehat{G_0}(Y,\mu_n,\emptyset)$.

Proof. Following the same arguments given in the proof of Lemma 3.9, we may show that the deformation to the normal cone $W(i)$ admits an equivariant hermitian very ample invertible sheaf $\overline{\mathcal{L}}$ which is relative to the morphism $l:W(i)\to Y$. By Theorem 3.1 and the fact that $\mathcal{L}$ is very ample, we conclude that there exists an integer $k_0>0$ such that for $n\geq k_0$, $\mathcal{L}^{\otimes n}$ is l-acyclic and $\delta(\overline{\Psi}(n)_{\mathbb{C}})=0$. Then l factors through an equivariant projective space bundle $\mathbb{P}(\mathcal{E})$ where $\mathcal{E}$ is locally free of rank $r+1$ on Y and $l_*\mathcal{L}^{\otimes k_0}$ is an equivariant quotient of $\mathcal{E}$. Denote by $p:\mathbb{P}(\mathcal{E})\to Y$ the canonical projection. On $P:=\mathbb{P}(\mathcal{E})$, we have a canonical exact sequence

$$\mathcal{H}:\quad 0\to\mathcal{O}_P\to p^*(\mathcal{E}^\vee)(1)\to\cdots\to p^*(\wedge^{r+1}\mathcal{E}^\vee)(r+1)\to 0.$$

Restricting this sequence to Z, we obtain an exact sequence of exact sequences

$$0\to\Psi\to\Psi\otimes h^*(\mathcal{E}^\vee)(1)\to\cdots\to\Psi\otimes h^*(\wedge^{r+1}\mathcal{E}^\vee)(r+1)\to 0.$$

Endow $\mathcal{E}$ with any $\mu_n(\mathbb{C})$-invariant hermitian metric. We claim that the assumption that Theorem 6.3 holds for $\overline{\Psi}\otimes h^*(\wedge^n\overline{\mathcal{E}}^\vee)(n)$ with $n\geq 1$ implies that it holds for $\overline{\Psi}$. In fact, since $\mathcal{H}$ is an exact sequence of flat modules, for any $k\geq 0$ we have the following exact sequence on X_{μ_n}

$$\begin{aligned}0\to\mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\overline{\eta},\overline{\mathcal{O}}_{Z_{\mu_n}})\to&\mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\overline{\eta},\overline{\mathcal{O}}_{Z_{\mu_n}})\otimes f_{\mu_n}^*(\overline{\mathcal{E}}^\vee)(1)\to\cdots\\ \to&\mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\overline{\eta},\overline{\mathcal{O}}_{Z_{\mu_n}})\otimes f_{\mu_n}^*(\wedge^{r+1}\overline{\mathcal{E}}^\vee)(r+1)\to 0.\end{aligned}$$

We compute

$$\begin{aligned}&f^Z_{\mu_n*}(\mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\overline{\eta},\overline{\mathcal{O}}_{Z_{\mu_n}}))\\ =&f^Z_{\mu_n*}(-\sum_{j=1}^{r+1}(-1)^j\mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\overline{\eta},\overline{\mathcal{O}}_{Z_{\mu_n}})\otimes f_{\mu_n}^*(\wedge^j\overline{\mathcal{E}}^\vee)(j))\\ &+\int_{X_g/Y}\mathrm{Td}(Tf_g,\omega_X^Z)\mathrm{ch}_g(\wedge^k\overline{F}^\vee)\mathrm{ch}_g(\overline{\eta})(-1)^{r+1}\widetilde{\mathrm{ch}}_g(\overline{\mathcal{H}})\end{aligned}$$

and

$$\begin{aligned}&\sum_{k=0}^m(-1)^k h_{\mu_n*}(\overline{\xi}_k\mid_{Z_{\mu_n}})\\ =&\sum_{k=0}^m(-1)^k h_{\mu_n*}(-\sum_{j=1}^{r+1}(-1)^j\overline{\xi}_k\mid_{Z_{\mu_n}}\otimes h_{\mu_n}^*(\wedge^j\overline{\mathcal{E}}^\vee)(j))\\ &+\sum_{k=0}^m(-1)^k\int_{Z_g/Y}\mathrm{Td}(\overline{Th_g})\mathrm{ch}_g(\overline{\xi}_k)(-1)^{r+1}\widetilde{\mathrm{ch}}_g(\overline{\mathcal{H}}).\end{aligned}$$

Moreover, we have

$$\begin{aligned}&\int_{X_g/Y}\mathrm{Td}(Tf_g)\mathrm{ch}_g(\mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\eta,\mathcal{O}_{Z_{\mu_n}}))R(N_g)\\ =&\int_{X_g/Y}-\sum_{j=1}^{r+1}(-1)^j\mathrm{ch}_g(\mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\eta,\mathcal{O}_{Z_{\mu_n}})\otimes f^*_{\mu_n}(\wedge^j\mathcal{E}^\vee)(j))R(N_g)\mathrm{Td}(Tf_g)\end{aligned}$$

and

$$\begin{aligned}&\int_{Z_g/Y}T_g(\overline{\xi.})\mathrm{Td}(\overline{Th_g})\\ =&\int_{Z_g/Y}\mathrm{Td}(\overline{Th_g})\{\delta_{X_g}\mathrm{Td}_g^{-1}(\overline{N})\mathrm{ch}_g(\overline{\eta})(-1)^{r+1}\widetilde{\mathrm{ch}}_g(\overline{\mathcal{H}})\\ &-\sum_{k=0}^{m}(-1)^k\mathrm{ch}_g(\overline{\xi}_k)(-1)^{r+1}\widetilde{\mathrm{ch}}_g(\overline{\mathcal{H}})-\sum_{j=1}^{r+1}(-1)^jT_g(\overline{\xi.})\mathrm{ch}_g(h^*_{\mu_n}(\wedge^j\overline{\mathcal{E}}^\vee)(j))\}\end{aligned}$$

by the double complex formula of equivariant Bott-Chern singular currents. At last, we also have

$$\begin{aligned}&\int_{X_g/Y}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N})\widetilde{\mathrm{Td}}((Tf_g,\omega_X^Z),\overline{Th_g}\mid_{X_g})\\ =&\int_{X_g/Y}\{\mathrm{dd}^c(-1)^{r+1}\widetilde{\mathrm{ch}}_g(\overline{\mathcal{H}})\mathrm{ch}_g(\overline{\eta})-\sum_{j=1}^{r+1}(-1)^j\mathrm{ch}_g(\overline{\eta}\otimes f^*(\wedge^j\overline{\mathcal{E}}^\vee)(j))\}\\ &\qquad\cdot\mathrm{Td}_g^{-1}(\overline{N})\widetilde{\mathrm{Td}}((Tf_g,\omega_X^Z),\overline{Th_g}\mid_{X_g})\\ =&-\int_{X_g/Y}(-1)^{r+1}\widetilde{\mathrm{ch}}_g(\overline{\mathcal{H}})\mathrm{ch}_g(\overline{\eta})\cdot\{\mathrm{Td}_g^{-1}(\overline{N})\mathrm{Td}(\overline{Th_g})\\ &\qquad-\mathrm{Td}(Tf_g,\omega_X^Z)\mathrm{Td}_g^{-1}(\overline{F})\}\\ &-\int_{X_g/Y}\sum_{j=1}^{r+1}(-1)^j\mathrm{ch}_g(\overline{\eta}\otimes f^*(\wedge^j\overline{\mathcal{E}}^\vee)(j))\mathrm{Td}_g^{-1}(\overline{N})\widetilde{\mathrm{Td}}((Tf_g,\omega_X^Z),\overline{Th_g}\mid_{X_g}).\end{aligned}$$

Gathering all these computations above and using our assumption, we get

$$\begin{aligned}&f^Z_{\mu_n *}(\sum(-1)^k\mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\overline{\eta},\overline{\mathcal{O}}_{Z_{\mu_n}}))-h_{\mu_n *}(\sum(-1)^k\overline{\xi}_k\mid_{Z_{\mu_n}})\\ &-\int_{Z_g/Y}T_g(\overline{\xi.})\mathrm{Td}(\overline{Th_g})-\int_{X_g/Y}\mathrm{Td}(Tf_g)\mathrm{Td}_g^{-1}(F)\mathrm{ch}_g(\eta)R(N_g)\\ &\qquad-\int_{X_g/Y}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N})\widetilde{\mathrm{Td}}((Tf_g,\omega_X^Z),\overline{Th_g}\mid_{X_g})\\ =&\int_{X_g/Y}\mathrm{Td}(Tf_g,\omega_X^Z)\mathrm{Td}_g^{-1}(\overline{F})\mathrm{ch}_g(\overline{\eta})(-1)^{r+1}\widetilde{\mathrm{ch}}_g(\overline{\mathcal{H}})\end{aligned}$$

$$
\begin{aligned}
&-\sum_{k=0}^{m}(-1)^k\int_{Z_g/Y}\mathrm{Td}(\overline{Th_g})\mathrm{ch}_g(\overline{\xi}_k)(-1)^{r+1}\widetilde{\mathrm{ch}}_g(\overline{\mathcal{H}})\\
&-\int_{X_g/Y}\mathrm{Td}(\overline{Th_g})\mathrm{Td}_g^{-1}(\overline{N})\mathrm{ch}_g(\overline{\eta})(-1)^{r+1}\widetilde{\mathrm{ch}}_g(\overline{\mathcal{H}})\\
&+\sum_{k=0}^{m}(-1)^k\int_{Z_g/Y}\mathrm{Td}(\overline{Th_g})\mathrm{ch}_g(\overline{\xi}_k)(-1)^{r+1}\widetilde{\mathrm{ch}}_g(\overline{\mathcal{H}})\\
&+\int_{X_g/Y}(-1)^{r+1}\widetilde{\mathrm{ch}}_g(\overline{\mathcal{H}})\mathrm{ch}_g(\overline{\eta})\{\mathrm{Td}_g^{-1}(\overline{N})\mathrm{Td}(\overline{Th_g})\\
&\qquad\qquad -\mathrm{Td}(Tf_g,\omega_X^Z)\mathrm{Td}_g^{-1}(\overline{F})\}
\end{aligned}
$$

which vanishes. This ends the proof of our claim.

By the construction of the projective space bundle P, we have already known that $\delta(\overline{\Psi}(n)_{\mathbb{C}})$ vanishes from $n=1$ to $n=r+1$. Moreover, according to the projection formula of higher direct images, the operation of tensoring with the element $l^*(\wedge^n\overline{\mathcal{E}}^\vee)$ doesn't change the property of l-acyclicity. Hence we also have $\delta(\overline{\Psi}\otimes h^*(\wedge^n\overline{\mathcal{E}}^\vee)(n)_{\mathbb{C}})=0$. By the generating relations and the definition of push-forward morphisms of arithmetic G_0-groups, this is equivalent to say that Theorem 6.3 holds for $\overline{\Psi}\otimes h^*(\wedge^n\overline{\mathcal{E}}^\vee)(n)$. Therefore the equality in the statement of this theorem follows from our claim before. □

COROLLARY 6.4. *Let notations and assumptions be as in Theorem 6.3, and let x be any element in $\widehat{K_0}(Z,\mu_n,\emptyset)_\rho$. Then the formula*

$$
\begin{aligned}
&f^Z_{\mu_n *}(i^*x\mid_{X_{\mu_n}}\cdot\sum(-1)^k\mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\overline{\eta},\overline{\mathcal{O}}_{Z_{\mu_n}}))\\
&\qquad\qquad -h_{\mu_n *}(x\mid_{Z_{\mu_n}}\cdot\sum(-1)^k(\overline{\xi}_k\mid_{Z_{\mu_n}}))\\
=&\int_{Z_g/Y}T_g(\overline{\xi}.)\mathrm{Td}(\overline{Th_g})\mathrm{ch}_g(x)\\
&+\int_{X_g/Y}\mathrm{Td}(Tf_g)\mathrm{Td}_g^{-1}(F)\mathrm{ch}_g(\eta)R(N_g)\mathrm{ch}_g(i^*x)\\
&+\int_{X_g/Y}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N})\mathrm{ch}_g(i^*x)\widetilde{\mathrm{Td}}((Tf_g,\omega_X^Z),\overline{Th_g}\mid_{X_g})
\end{aligned}
$$

holds in $\widehat{G_0}(Y,\mu_n,\emptyset)_\rho$.

Proof. If $x=\overline{E}$ is an equivariant hermitian vector bundle on Z, then $\overline{\xi.}\otimes\overline{E}$ provides a resolution of $i_*(\overline{\eta}\otimes i^*\overline{E})$. Hence the formula follows from Theorem 6.3 in this case. If $x=\alpha$ is represented by some smooth form, then

$$
\begin{aligned}
&f^Z_{\mu_n *}(i^*x\mid_{X_{\mu_n}}\cdot\sum(-1)^k\mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\overline{\eta},\overline{\mathcal{O}}_{Z_{\mu_n}}))\\
=&\int_{Z_g/Y}\mathrm{Td}(Tf_g,\omega_X^Z)\mathrm{Td}_g^{-1}(\overline{F})\mathrm{ch}_g(\overline{\eta})\delta_{X_g}\alpha
\end{aligned}
$$

and

$$h_{\mu_n *}(x \mid_{Z_{\mu_n}} \cdot \sum(-1)^k(\overline{\xi}_k \mid_{Z_{\mu_n}})) = \int_{Z_g/Y} \mathrm{Td}(\overline{Th_g})\alpha \sum(-1)^k \mathrm{ch}_g(\overline{\xi}_k).$$

Moreover, by the definition of $\mathrm{ch}_g(x)$ we have

$$\begin{aligned}\int_{Z_g/Y} T_g(\overline{\xi}.)\mathrm{Td}(\overline{Th_g})\mathrm{ch}_g(x) = &\int_{Z_g/Y} \mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N})\delta_{X_g}\mathrm{Td}(\overline{Th_g})\alpha \\ &- \int_{Z_g/Y} \sum(-1)^k \mathrm{ch}_g(\overline{\xi}_k)\mathrm{Td}(\overline{Th_g})\alpha\end{aligned}$$

and

$$\int_{X_g/Y} \mathrm{Td}(Tf_g)\mathrm{Td}_g^{-1}(F)\mathrm{ch}_g(\eta)R(N_g)\mathrm{ch}_g(i^*x) = 0.$$

Finally, using the definition of $\widetilde{\mathrm{Td}}$ we compute

$$\begin{aligned}&\int_{X_g/Y} \mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N})\mathrm{ch}_g(i^*x)\widetilde{\mathrm{Td}}((Tf_g,\omega_X^Z),\overline{Th_g}\mid_{X_g}) \\ =&\int_{Z_g/Y} \mathrm{Td}(Tf_g,\omega_X^Z)\mathrm{Td}_g^{-1}(\overline{F})\mathrm{ch}_g(\overline{\eta})\delta_{X_g}\alpha \\ &\qquad - \int_{Z_g/Y} \mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N})\delta_{X_g}\mathrm{Td}(\overline{Th_g})\alpha.\end{aligned}$$

Gathering all computations above, we know that the formula still holds for x which is represented by smooth form. Since both two sides are additive, we are done. □

Corollary 6.5. *Let notations and assumptions be as in Theorem 6.3, and let y be any element in $\widehat{K_0}(Z_{\mu_n},\mu_n,\emptyset)_\rho$. Then the formula*

$$\begin{aligned}&f^Z_{\mu_n *}(i^*_{\mu_n}y \cdot \sum(-1)^k \mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\overline{\eta},\overline{\mathcal{O}}_{Z_{\mu_n}})) - h_{\mu_n *}(y \cdot \sum(-1)^k(\overline{\xi}_k \mid_{Z_{\mu_n}})) \\ =&\int_{Z_g/Y} T_g(\overline{\xi}.)\mathrm{Td}(\overline{Th_g})\mathrm{ch}_g(y) \\ &+\int_{X_g/Y} \mathrm{Td}(Tf_g)\mathrm{Td}_g^{-1}(F)\mathrm{ch}_g(\eta)R(N_g)\mathrm{ch}_g(i^*_{\mu_n}y) \\ &+\int_{X_g/Y} \mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N})\mathrm{ch}_g(i^*_{\mu_n}y)\widetilde{\mathrm{Td}}((Tf_g,\omega_X^Z),\overline{Th_g}\mid_{X_g})\end{aligned}$$

holds in $\widehat{G_0}(Y,\mu_n,\emptyset)_\rho$.

Proof. Provided Corollary 6.4, it is enough to prove that for any $y \in \widehat{K_0}(Z_{\mu_n},\mu_n,\emptyset)_\rho$ there exists an element $x \in \widehat{K_0}(Z,\mu_n,\emptyset)_\rho$ such that $i_Z^*x = y$.

Here i_Z stands for the inclusion $Z_{\mu_n} \hookrightarrow Z$. Actually, set $x = i_{Z*}(\lambda_{-1}^{-1}(\overline{N}_{Z/Z_{\mu_n}}^\vee)\cdot y)$, we have

$$i_Z^* x = i_Z^* i_{Z*}(\lambda_{-1}^{-1}(\overline{N}_{Z/Z_{\mu_n}}^\vee)\cdot y) = \lambda_{-1}(\overline{N}_{Z/Z_{\mu_n}}^\vee)\cdot \lambda_{-1}^{-1}(\overline{N}_{Z/Z_{\mu_n}}^\vee)\cdot y = y.$$

This follows from our arithmetic concentration theorem. □

6.3 Proof of the fixed point formula

In this subsection, we provide a complete proof of Theorem 6.1 the singular Lefschetz fixed point formula. Before that, we need to translate Bismut-Ma's immersion formula to an arithmetic G_0-theoretic version. That's the following.

Theorem 6.6. *Let notations and assumptions be as in Section 6.1. Assume that $\overline{\eta}$ is an equivariant hermitian sheaf on X and $\overline{\xi.}$ is a bounded complex of equivariant hermitian vector bundles providing a resolution of $i_*\overline{\eta}$ on Z whose metrics satisfy Bismut assumption (A). Then the equality*

$$\begin{aligned}
f_*^Z(\overline{\eta}) - \sum_{j=0}^m (-1)^j h_*(\overline{\xi}_j) = &\int_{X_g/Y} \mathrm{ch}_g(\eta)R_g(N)\mathrm{Td}_g(Tf) \\
&+ \int_{Z_g/Y} T_g(\overline{\xi}.)\mathrm{Td}_g(\overline{Th}) \\
&+ \int_{X_g/Y} \mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g((Tf,\omega_X^Z),\overline{Th}\mid_X)\mathrm{Td}_g^{-1}(\overline{N})
\end{aligned}$$

holds in $\widehat{G_0}(Y,\mu_n,\emptyset)$.

Proof. We first suppose that η and $\xi.$ are all acyclic, then the verification follows rather directly from the generating relations of arithmetic G_0-theory. In fact

$$\begin{aligned}
f_*^Z(\overline{\eta}) - \sum_{j=0}^m (-1)^j h_*(\overline{\xi}_j) =& \overline{f_*\eta} - T_g(\omega_X^Z, h^\eta) - \Big(\sum_{j=0}^m (-1)^j(\overline{h_*\xi_j} - T_g(\omega^Z, h^{\xi_j}))\Big) \\
=& \widetilde{\mathrm{ch}}_g(h_*\Xi) - T_g(\omega_X^Z, h^\eta) + \sum_{j=0}^m (-1)^j T_g(\omega^Z, h^{\xi_j}).
\end{aligned}$$

And the right-hand side of the last equality is exactly the left-hand side of Bismut-Ma's immersion formula. We emphasize again that to simplify the right-hand side of Bismut-Ma's immersion formula, we have used an Atiyah-Segal-Singer type formula for immersion

$$i_{g*}(\mathrm{Td}_g^{-1}(N)\mathrm{ch}_g(x)) = \mathrm{ch}_g(i_*(x)).$$

To remove the condition of acyclicity, one can use the argument which is essentially the same as in the proof of Theorem 6.3. Since it doesn't use any new techniques, we omit it here. □

Definition 6.7. The inclusion $i: X \hookrightarrow Z$ induces an embedding morphism

$$i_*: \quad \widehat{G_0}(X,\mu_n,\emptyset) \to \widehat{K_0}(Z,\mu_n,N_{g,\mathbb{R}}^\vee \setminus \{0\})$$

which is defined as follows.
(i). For every μ_n-equivariant hermitian sheaf $\overline{\eta}$ on X, suppose that $\overline{\xi.}$ is a resolution of $i_*\overline{\eta}$ on Z whose metrics satisfy Bismut assumption (A), $i_*[\overline{\eta}] = \sum_k (-1)^k[\overline{\xi}_k] + T_g(\overline{\xi.})$.
(ii). For every $\alpha \in \widetilde{\mathcal{U}}(X_{\mu_n},\emptyset)$, $i_*\alpha = \alpha \mathrm{Td}_g^{-1}(\overline{N})\delta_{X_g}$.

Remark 6.8. Similar to Theorem 5.2 and Lemma 5.3, one can prove that the embedding morphism is a well-defined homomorphism of $R(\mu_n)$-modules.

Proof. (of Theorem 6.1) We first prove that this fixed point formula holds when ω^X is equal to ω_X^Z, namely the Kähler metric on $X(\mathbb{C})$ is induced by the Kähler metric on $Z(\mathbb{C})$. By Theorem 6.6 and Definition 6.7, we have the following equality

$$\begin{aligned} f_*^Z(\overline{\eta}) =& h_*i_*(\overline{\eta}) + \int_{X_g/Y} \mathrm{ch}_g(\eta)R_g(N)\mathrm{Td}_g(Tf) \\ &+ \int_{X_g/Y} \mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g((Tf,\omega_X^Z),\overline{Th}\mid_X)\mathrm{Td}_g^{-1}(\overline{N}) \end{aligned}$$

which holds in $\widehat{G_0}(Y,\mu_n,\emptyset)$. Now we claim that for any element $y \in \widehat{K_0}(Z_{\mu_n},\mu_n,N_{g,\mathbb{R}}^\vee \setminus \{0\})_\rho$, we have

$$h_{\mu_n *}(y) - h_* i_{Z*}(y) = h_{\mu_n *}(y \cdot R_g(N_{Z/Z_{\mu_n}})).$$

Since all morphisms are homomorphisms of $R(\mu_n)$-modules, we can only consider the generators of $\widehat{K_0}(Z_{\mu_n},\mu_n,N_{g,\mathbb{R}}^\vee \setminus \{0\})$. Indeed, by applying Theorem 6.6 to the closed immersion i_Z, for any equivariant hermitian vector bundle $\overline{E}$ on Z_{μ_n} we have

$$\begin{aligned} h_{\mu_n *}(\overline{E}) - h_* i_{Z*}(\overline{E}) =& \int_{Z_g/Y} \mathrm{ch}_g(E)R_g(N_{Z/Z_{\mu_n}})\mathrm{Td}_g(Th_g) \\ =& h_{\mu_n *}(\mathrm{ch}_g(E)R_g(N_{Z/Z_{\mu_n}})). \end{aligned}$$

The first equality holds because the exact sequence

$$0 \to \overline{Th_g} \to \overline{Th}\mid_{Z_g} \to \overline{N}_{Z/Z_g} \to 0$$

is orthogonally split on Z_g so that $\widetilde{\mathrm{Td}}_g(\overline{Th_g},\overline{Th}\mid_{Z_g}) = 0$. The second equality follows from [21, Lemma 7.3] and the fact that $\mathrm{ch}_g(E)R_g(N_{Z/Z_{\mu_n}})$ is dd^c-closed. On the other hand, let α be an element in $\widetilde{\mathcal{U}}(Z_{\mu_n},N_{g,\mathbb{R}}^\vee \setminus \{0\})$, we have

$$\begin{aligned} h_{\mu_n *}(\alpha) - h_* i_{Z*}(\alpha) =& \int_{Z_g/Y} \alpha \mathrm{Td}_g(\overline{Th_g}) - \int_{Z_g/Y} \alpha \mathrm{Td}_g^{-1}(\overline{N}_{Z/Z_{\mu_n}})\mathrm{Td}_g(\overline{Th}) \\ =& \int_{Z_g/Y} \alpha \mathrm{Td}_g^{-1}(\overline{N}_{Z/Z_{\mu_n}})\mathrm{dd}^c\widetilde{\mathrm{Td}}_g(\overline{Th_g},\overline{Th}\mid_{Z_g}) = 0. \end{aligned}$$

Combing these two computations, we get our claim by linear extension.

Now using arithmetic concentration theorem, we compute

$$\begin{aligned}
h_*i_*(\overline{\eta}) =& h_*i_{Z*}i_{Z*}^{-1}i_*(\overline{\eta}) \\
=& h_{\mu_n *}(i_{Z*}^{-1}i_*(\overline{\eta})\cdot(1-R_g(N_{Z/Z_{\mu_n}}))) \\
=& h_{\mu_n *}(\lambda_{-1}^{-1}(\overline{N}_{Z/Z_{\mu_n}}^\vee)\cdot i_Z{}^*i_*(\overline{\eta})\cdot(1-R_g(N_{Z/Z_{\mu_n}}))) \\
=& h_{\mu_n *}(\lambda_{-1}^{-1}(\overline{N}_{Z/Z_{\mu_n}}^\vee)\cdot\{\sum_k(-1)^k(\overline{\xi}_k\mid_{Z_{\mu_n}}) \\
& \qquad\qquad + T_g(\overline{\xi.})\}\cdot(1-R_g(N_{Z/Z_{\mu_n}}))) \\
=& h_{\mu_n *}(\lambda_{-1}^{-1}(\overline{N}_{Z/Z_{\mu_n}}^\vee)\cdot\sum_k(-1)^k(\overline{\xi}_k\mid_{Z_{\mu_n}})) \\
& + h_{\mu_n *}(\mathrm{Td}_g(\overline{N}_{Z/Z_{\mu_n}})T_g(\overline{\xi.})) \\
& - h_{\mu_n *}(\mathrm{Td}_g(N_{Z/Z_{\mu_n}})\sum_k(-1)^k\mathrm{ch}_g(\xi_k)R_g(N_{Z/Z_{\mu_n}})).
\end{aligned}$$

According to Corollary 6.5, by setting $y=\lambda_{-1}^{-1}(\overline{N}_{Z/Z_{\mu_n}}^\vee)$, we compute

$$\begin{aligned}
& h_{\mu_n *}(\lambda_{-1}^{-1}(\overline{N}_{Z/Z_{\mu_n}}^\vee)\cdot\sum_k(-1)^k(\overline{\xi}_k\mid_{Z_{\mu_n}})) \\
=& f^Z_{\mu_n *}(i_{\mu_n}^*(\lambda_{-1}^{-1}(\overline{N}_{Z/Z_{\mu_n}}^\vee))\cdot\sum(-1)^k\mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\overline{\eta},\overline{\mathcal{O}}_{Z_{\mu_n}})) \\
& - \int_{Z_g/Y}T_g(\overline{\xi.})\mathrm{Td}(\overline{Th_g})\mathrm{Td}_g(\overline{N}_{Z/Z_g}) \\
& - \int_{X_g/Y}\mathrm{Td}(Tf_g)\mathrm{Td}_g^{-1}(F)\mathrm{ch}_g(\eta)R(N_g)\mathrm{Td}_g(N_{Z/Z_g}) \\
& - \int_{X_g/Y}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N})\mathrm{Td}_g(\overline{N}_{Z/Z_g})\widetilde{\mathrm{Td}}((Tf_g,\omega_X^Z),\overline{Th_g}\mid_{X_g}) \\
=& f^Z_{\mu_n *}(i_{\mu_n}^*(\lambda_{-1}^{-1}(\overline{N}_{Z/Z_{\mu_n}}^\vee))\cdot\sum(-1)^k\mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\overline{\eta},\overline{\mathcal{O}}_{Z_{\mu_n}})) \\
& - \int_{Z_g/Y}T_g(\overline{\xi.})\mathrm{Td}_g(\overline{Th}) - \int_{X_g/Y}\mathrm{Td}_g(Tf)\mathrm{ch}_g(\eta)R(N_g) \\
& - \int_{X_g/Y}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N})\mathrm{Td}_g(\overline{N}_{Z/Z_g})\widetilde{\mathrm{Td}}((Tf_g,\omega_X^Z),\overline{Th_g}\mid_{X_g}).
\end{aligned}$$

Here we have used various relations of character forms or characteristic classes arising from the following double complex

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (Tf_g,\omega_X^Z) & \longrightarrow & \overline{Th_g} & \longrightarrow & \overline{N}_g & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (Tf,\omega_X^Z) & \longrightarrow & \overline{Th} & \longrightarrow & \overline{N} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (N_{X/X_g},\omega_X^Z) & \longrightarrow & \overline{N}_{Z/Z_g} & \longrightarrow & \overline{F} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

whose columns are all orthogonally split. Also, for this double complex, one may use Example 2.3 (iv) to compute that

$$\begin{aligned}\widetilde{\mathrm{Td}}_g((Tf,\omega_X^Z),\overline{Th}\mid_X)=&\widetilde{\mathrm{Td}}_g(\overline{\mathcal{F}},\omega_X^Z)\mathrm{Td}(\overline{N}_g)\mathrm{Td}(Tf_g,\omega_X^Z)\\ &+\widetilde{\mathrm{Td}}((Tf_g,\omega_X^Z),\overline{Th_g}\mid_{X_g})\mathrm{Td}_g(\overline{N}_{Z/Z_g}).\end{aligned}\tag{11}$$

We deduce from (11) that

$$\begin{aligned}&\int_{X_g/Y}\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g((Tf,\omega_X^Z),\overline{Th}\mid_X)\mathrm{Td}_g^{-1}(\overline{N})\\ =&\int_{X_g/Y}\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}((Tf_g,\omega_X^Z),\overline{Th_g}\mid_{X_g})\mathrm{Td}_g^{-1}(\overline{N})\mathrm{Td}_g(\overline{N}_{Z/Z_g})\\ &+\int_{X_g/Y}\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g(\overline{\mathcal{F}})\mathrm{Td}_g^{-1}(\overline{F})\mathrm{Td}(Tf_g,\omega_X^Z).\end{aligned}\tag{12}$$

Moreover, we have

$$\begin{aligned}h_{\mu_n *}(\mathrm{Td}_g(\overline{N}_{Z/Z_{\mu_n}})T_g(\overline{\xi.}))&=\int_{Z_g/Y}T_g(\overline{\xi.})\mathrm{Td}(\overline{Th_g})\mathrm{Td}_g(\overline{N}_{Z/Z_g})\\ &=\int_{Z_g/Y}T_g(\overline{\xi.})\mathrm{Td}_g(\overline{Th})\end{aligned}\tag{13}$$

and

$$\begin{aligned}&h_{\mu_n *}(\mathrm{Td}_g(N_{Z/Z_{\mu_n}})\sum_k(-1)^k\mathrm{ch}_g(\xi_k)R_g(N_{Z/Z_{\mu_n}}))\\ =&\int_{Z_g/Y}\mathrm{Td}_g(N_{Z/Z_{\mu_n}})\delta_{X_g}\mathrm{ch}_g(\eta)\mathrm{Td}_g^{-1}(N)R_g(N_{Z/Z_g})\mathrm{Td}(Th_g)\end{aligned}$$

$$
\begin{aligned}
&= \int_{X_g/Y} \mathrm{Td}_g(N_{X/X_g}) \mathrm{Td}_g(F) \mathrm{ch}_g(\eta) \mathrm{Td}_g^{-1}(N) \\
&\qquad \cdot [R_g(N_{X/X_g}) + R_g(N) - R(N_g)] \mathrm{Td}(Tf_g) \mathrm{Td}(N_g) \\
&= \int_{X_g/Y} \mathrm{Td}_g(Tf) \mathrm{ch}_g(\eta) [R_g(N_{X/X_g}) + R_g(N) - R(N_g)]. \qquad (14)
\end{aligned}
$$

Gathering (12), (13) and (14) we finally get

$$
\begin{aligned}
f_*^Z(\overline{\eta}) =& f_{\mu_n *}^Z (i_{\mu_n}^* (\lambda_{-1}^{-1}(\overline{N}_{Z/Z_{\mu_n}}^\vee)) \cdot \sum_k (-1)^k \mathrm{Tor}_{\mathcal{O}_Z}^k (i_* \overline{\eta}, \overline{\mathcal{O}}_{Z_{\mu_n}})) \\
&+ \int_{X_g/Y} \mathrm{Td}(Tf_g, \omega_X^Z) \mathrm{ch}_g(\overline{\eta}) \widetilde{\mathrm{Td}}_g(\overline{\mathcal{F}}, \omega_X^Z) \mathrm{Td}_g^{-1}(\overline{F}) \\
&- \int_{X_g/Y} \mathrm{Td}_g(Tf) \mathrm{ch}_g(\eta) R_g(N_{X/X_g})
\end{aligned}
$$

which completes the proof of Theorem 6.1 in the case where the Kähler metric on $X(\mathbb{C})$ is induced by the Kähler metric on $Z(\mathbb{C})$.

In general, in analogy with the notation $\widetilde{\mathrm{Td}}(Tf_g, \omega^X, \omega_X^Z)$, we write $\widetilde{\mathrm{Td}}_g(N_{X/X_g}, \omega^X, \omega_X^Z)$ for the secondary characteristic class of the exact sequence

$$
0 \longrightarrow N_{X/X_g} \xrightarrow{\mathrm{Id}} N_{X/X_g} \longrightarrow 0 \longrightarrow 0
$$

where the middle term carries the metric induced by ω_X^Z and the sub term carries the metric induced by ω^X. Similarly, we have the notation $\widetilde{\mathrm{Td}}_g(Tf, \omega^X, \omega_X^Z)$. Then by applying the argument in the proof of (11) to the double complex

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & (Tf_g, \omega^X) \arrow[r] \arrow[d] & (Tf_g, \omega_X^Z) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (Tf, \omega^X) \arrow[r] \arrow[d] & (Tf, \omega_X^Z) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (N_{X/X_g}, \omega^X) \arrow[r] \arrow[d] & (N_{X/X_g}, \omega_X^Z) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

We get

$$\begin{aligned}\widetilde{\mathrm{Td}}_g(Tf,\omega^X,\omega^Z_X)=&\widetilde{\mathrm{Td}}(Tf_g,\omega^X,\omega^Z_X)\mathrm{Td}_g(N_{X/X_g},\omega^Z_X)\\ &+\widetilde{\mathrm{Td}}_g(N_{X/X_g},\omega^X,\omega^Z_X)\mathrm{Td}(Tf_g,\omega^X).\end{aligned}$$

Moreover, by [30, Proposition 2.8], we obtain that

$$\widetilde{\mathrm{Td}}_g(\overline{\mathcal{F}},\omega^X)=\widetilde{\mathrm{Td}}_g(\overline{\mathcal{F}},\omega^Z_X)+\widetilde{\mathrm{Td}}_g(N_{X/X_g},\omega^X,\omega^Z_X)\mathrm{Td}_g(\overline{F}).$$

With these two comparison formulae, we can compute

$$\begin{aligned}&\int_{X_g/Y}\mathrm{Td}(Tf_g,\omega^X)\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g(\overline{\mathcal{F}},\omega^X)\mathrm{Td}_g^{-1}(\overline{F})\\
&\qquad-\int_{X_g/Y}\mathrm{Td}(Tf_g,\omega^Z_X)\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g(\overline{\mathcal{F}},\omega^Z_X)\mathrm{Td}_g^{-1}(\overline{F})\\
=&\int_{X_g/Y}\mathrm{Td}(Tf_g,\omega^X)\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g(\overline{\mathcal{F}},\omega^X)\mathrm{Td}_g^{-1}(\overline{F})\\
&-\int_{X_g/Y}\mathrm{Td}(Tf_g,\omega^X)\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g(\overline{\mathcal{F}},\omega^Z_X)\mathrm{Td}_g^{-1}(\overline{F})\\
&+\int_{X_g/Y}\mathrm{Td}(Tf_g,\omega^X)\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g(\overline{\mathcal{F}},\omega^Z_X)\mathrm{Td}_g^{-1}(\overline{F})\\
&-\int_{X_g/Y}\mathrm{Td}(Tf_g,\omega^Z_X)\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g(\overline{\mathcal{F}},\omega^Z_X)\mathrm{Td}_g^{-1}(\overline{F})\\
=&\int_{X_g/Y}\widetilde{\mathrm{Td}}(Tf_g,\omega^X,\omega^Z_X)\mathrm{ch}_g(\overline{\eta})\\
&\quad\cdot[\mathrm{Td}_g(N_{X/X_g},\omega^Z_X)\mathrm{Td}_g(\overline{F})-\mathrm{Td}_g(\overline{N}_{Z/Z_g})]\mathrm{Td}_g^{-1}(\overline{F})\\
&+\int_{X_g/Y}\mathrm{Td}(Tf_g,\omega^X)\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g(N_{X/X_g},\omega^X,\omega^Z_X)\\
=&\int_{X_g/Y}\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g(Tf,\omega^X,\omega^Z_X)\\
&-\int_{X_g/Y}\widetilde{\mathrm{Td}}(Tf_g,\omega^X,\omega^Z_X)\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g(\overline{N}_{Z/Z_g})\mathrm{Td}_g^{-1}(\overline{F}).\end{aligned}$$

At last, using [21, Lemma 7.3], we get the equality

$$f_*(\overline{\eta})-f^Z_*(\overline{\eta})=\int_{X_g/Y}\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g(Tf,\omega^X,\omega^Z_X).$$

Together with the fact that the other two terms have nothing to do with the choice of the metric ω^X, we finally obtain that

$$\begin{aligned} f_*(\overline{\eta}) = & f^Z_{\mu_n *}(i^*_{\mu_n}(\lambda^{-1}_{-1}(\overline{N}^\vee_{Z/Z_{\mu_n}})) \cdot \sum_k (-1)^k \mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\overline{\eta}, \overline{\mathcal{O}}_{Z_{\mu_n}})) \\ & + \int_{X_g/Y} \mathrm{Td}(Tf_g, \omega^X)\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g(\overline{\mathcal{F}}, \omega^X)\mathrm{Td}_g^{-1}(\overline{F}) \\ & - \int_{X_g/Y} \mathrm{Td}_g(Tf)\mathrm{ch}_g(\eta)R_g(N_{X/X_g}) \\ & + \int_{X_g/Y} \widetilde{\mathrm{Td}}(Tf_g, \omega^X, \omega^Z_X)\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g(\overline{N}_{Z/Z_g})\mathrm{Td}_g^{-1}(\overline{F}) \end{aligned}$$

which ends the proof of Theorem 6.1. □

REMARK 6.9. Let Y be an affine arithmetic scheme Spec(D), and choose ω^X to be the induced Kähler form ω^Z_X. Then the formula in Theorem 6.1 is the content of [25, Conjecture 5.1].

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