On Reductions of Families of Crystalline Galois Representations

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Abstract. Let K_f be the finite unramified extension of $\mathbb{Q}_p$ of degree f and E any finite large enough coefficient field containing K_f. We construct analytic families of étale (φ, Γ)-modules which give rise to families of crystalline E-representations of the absolute Galois group G_{K_f} of K_f. For any irreducible effective two-dimensional crystalline E-representation of G_{K_f} with labeled Hodge-Tate weights $\{0, -k_i\}_{\tau_i}$ induced from a crystalline character of $G_{K_{2f}}$, we construct an infinite family of crystalline E-representations of G_{K_f} of the same Hodge-Tate type which contains it. As an application, we compute the semisimplified mod p reductions of the members of each such family.

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1 INTRODUCTION

Let p be a prime number and $\bar{\mathbb{Q}}_p$ a fixed algebraic closure of $\mathbb{Q}_p$. Let N be a positive integer and $g = \sum\limits_{n\geq 1} a_n q^n$ a newform of weight $k \geq 2$ over $\Gamma_1(N)$ with character ψ. The complex coefficients a_n are algebraic over $\mathbb{Q}$ and may be viewed as elements of $\bar{\mathbb{Q}}_p$ after fixing embeddings $\bar{\mathbb{Q}} \to \mathbb{C}$ and $\bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$. By work of Eichler-Shimura when $k = 2$ and Deligne when $k > 2$, there exists a continuous irreducible two-dimensional p-adic representation $\rho_g : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$ attached to g. If $l \nmid pN$, then ρ_g is unramified at l and $\det(X - \rho_g(\mathrm{Frob}_l)) = X^2 - a_l X + \psi(l)\, l^{k-1}$, where Frob_l is any choice of an arithmetic Frobenius at l. The contraction of the maximal ideal of the ring of integers of $\bar{\mathbb{Q}}_p$ via an embedding $\bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$ gives rise to the choice of a place of $\bar{\mathbb{Q}}$ above p, and the decomposition group D_p at this place is isomorphic to the local Galois group $G_{\mathbb{Q}_p}$ via the same embedding. The local representation

$$\rho_{g,p} : G_{\mathbb{Q}_p} \longrightarrow \mathrm{GL}_2(\bar{\mathbb{Q}}_p),$$

obtained by restricting ρ_g to D_p, is de Rham with Hodge-Tate weights $\{0, k-1\}$ ([Tsu99]). If $p \nmid N$ the representation $\rho_{g,p}$ is crystalline and the characteristic polynomial of Frobenius of the weakly admissible filtered φ-module $\mathbb{D}_{k,a_p} := \mathbb{D}_{\mathrm{cris}}(\rho_{g,p})$ attached to $\rho_{g,p}$ by Fontaine is $X^2 - a_p X + \psi(p)\, p^{k-1}$ ([Fal89] and [Sc90]). The roots of Frobenius are distinct if $k = 2$ and conjecturally distinct if $k \geq 3$ (see [CE98]). In this case, weak admissibility imposes a unique up to isomorphism choice of the filtration of $\mathbb{D}_{k,a_p}$, and the isomorphism class of the crystalline representation $\rho_{g,p}$ is completely determined by the characteristic polynomial of Frobenius of $\mathbb{D}_{k,a_p}$. The mod p reduction $\bar{\rho}_{g,p} : G_{\mathbb{Q}_p} \longrightarrow \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ of the local representation $\rho_{g,p}$ is well defined up to semisimplification and plays a role in the proof of Serre's modularity conjecture, now a theorem of Khare and Wintenberger [KW09a], [KW09b], which states that any irreducible continuous odd Galois representation $\rho : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ is similar to a representation of the form $\bar{\rho}_g$ for a certain newform g which should occur in level $N(\rho)$, an integer prime-to-p, and weight $\kappa(\rho) \geq 2$, which Serre explicitly defined in [Ser87]. If $\rho_{g,p}$ is crystalline, the semisimplified mod p reduction $\bar{\rho}_{g,p}$ has been given concrete descriptions in certain cases by work of Berger-Li-Zhu [BLZ04] combined with work of Breuil [Bre03], which extended previous results of Deligne, Fontaine, Serre and Edixhoven, and more recently by Buzzard-Gee [BG09] using the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. For a more detailed account and the shape of these reductions see [Ber10, §5.2].

Recall that (up to unramified twist) all irreducible two-dimensional crystalline representations of $G_{\mathbb{Q}_p}$ with fixed Hodge-Tate weights in the range $[0;\ p]$ have the same irreducible mod p reduction. Reductions of crystalline representations of $G_{\mathbb{Q}_{p^f}} := \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_{p^f})$ with $f \neq 1$, where $\mathbb{Q}_{p^f}$ is the unramified extension of $\mathbb{Q}_p$ of degree f, are more complicated. For example, in the simpler case where $f = 2$, there exist irreducible two-dimensional crystalline representation of $G_{\mathbb{Q}_{p^2}}$ with Hodge-Tate weights in the range $[0;\ p-1]$ sharing the same characteristic polynomial and filtration, with distinct irreducible or reducible reductions (cf. Proposition 6.22).

The purpose of this paper is to extend the constructions of [BLZ04] to two-dimensional crystalline representations of $G_{\mathbb{Q}_{p^f}}$, and to compute the semisimplified mod p reductions of the crystalline representations constructed. The strategy for computing reductions is to fit irreducible representations of G_{K_f} which are not induced from crystalline characters of $G_{K_{2f}}$ into families of representations of the same Hodge-Tate type and with the same mod p reduction, which contain some member which is either reducible or irreducible induced.

Serre's conjecture has been recently generalized by Buzzard, Diamond and Jarvis [BDJ] for irreducible totally odd two-dimensional $\bar{\mathbb{F}}_p$-representations of the absolute Galois group of any totally real field unramified at p, and has subsequently been extended by Schein [Sch08] to cases where p is odd and tamely ramified in F. Crystalline representations of the absolute Galois group of finite unramified extensions of $\mathbb{Q}_p$ arise naturally in this context of the conjecture of Buzzard, Diamond and Jarvis, and their modulo p reductions are crucial for the weight part of this conjecture (see [BDJ, §3]).

Let F be a totally real number field of degree $d > 1$, and let $I = \{\tau_1, ..., \tau_d\}$ be the set of real embeddings of F. Let $\mathbf{k} = (k_{\tau_1}, k_{\tau_2}, ..., k_{\tau_d}, w) \in \mathbb{N}_{\geq 1}^{d+1}$ with $k_{\tau_i} \equiv w \mathrm{mod}\ 2$. We denote by $\mathcal{O}$ the ring of integers of F and we let $\mathfrak{n} \neq 0$ be an ideal of $\mathcal{O}$. The space $\mathrm{S}_{\mathbf{k}}(\mathrm{U}_1(\mathfrak{n}))$ of Hilbert modular cusp forms of level $\mathfrak{n}$ and weight $\mathbf{k}$ is a finite dimensional complex vector space endowed with actions of Hecke operators $\mathrm{T}_{\mathfrak{q}}$ indexed by the nonzero ideals $\mathfrak{q}$ of $\mathcal{O}$ (for the precise definitions see [Tay89]). Let $0 \neq g \in \mathrm{S}_{\mathbf{k}}(\mathrm{U}_1(\mathfrak{n}))$ be an eigenform for all the $\mathrm{T}_{\mathfrak{q}}$, and fix embeddings $\bar{\mathbb{Q}} \to \mathbb{C}$ and $\bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$. By constructions of Rogawski-Tunnell [RT83], Ohta [Oht84], Carayol [Car86], Blasius-Rogawski [BR89], Taylor [Tay89], and Jarvis [Jar97], one can attach to g a continuous Galois representation $\rho_g : G_F \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$, where G_F is the absolute Galois group of the totally real field F. Fixing an isomorphism between the residue field of $\bar{\mathbb{Q}}_p$ with $\bar{\mathbb{F}}_p$, the mod p reduction $\bar{\rho}_g : G_F \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ is well defined up to semisimplification. A continuous representation $\rho : G_F \longrightarrow \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ is called modular if $\rho \sim \bar{\rho}_g$ for some Hilbert modular eigenform g. Conjecturally, every irreducible totally odd continuous Galois representation $\rho : G_F \longrightarrow \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ is modular ([BDJ]). We now assume that $k_{\tau_i} \geq 2$ for all i. We fix an isomorphism

$\bar{\mathbb{Q}}_p \overset{i}{\simeq} \mathbb{C}$ and an algebraic closure $\bar{F}$ of F. For each prime ideal $\mathfrak{p}$ of $\mathcal{O}$ lying above p we denote by $F_{\mathfrak{p}}$ the completion of F at $\mathfrak{p}$, and we fix an algebraic closure $\bar{F}_{\mathfrak{p}}$ of $F_{\mathfrak{p}}$ and an F-embedding $\bar{F} \hookrightarrow \bar{F}_{\mathfrak{p}}$. These determine a choice of a decomposition group $D_{\mathfrak{p}} \subset G_F$ an isomorphism $D_{\mathfrak{p}} \simeq G_{F_{\mathfrak{p}}}$. For each embedding $\tau : F_{\mathfrak{p}} \to \bar{\mathbb{Q}}_p$, let k_τ be the weight of g corresponding to the embedding $\tau_{|F} : F \to \bar{\mathbb{Q}}_p \overset{i}{\simeq} \mathbb{C}$. By works of Blasius-Rogawski [BR93], Saito [Sai09], Skinner [Ski09], and T. Liu [Liu09], the local representation $\rho_{g,F_{\mathfrak{p}}} : G_{F_{\mathfrak{p}}} \longrightarrow \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$, obtained by restricting ρ_g to the decomposition subgroup $G_{F_{\mathfrak{p}}}$, is de Rham with labeled Hodge-Tate weights $\{\frac{k-k_\tau}{2}, \frac{k+k_\tau-2}{2}\}_{\tau:F_{\mathfrak{p}}\to\bar{\mathbb{Q}}_p}$, where $k = \max\{k_{\tau_i}\}$. This has also been proved by Kisin [Kis08, Theorem 4.3] under the assumption that $\rho_{g,F_{\mathfrak{p}}}$ is residually irreducible. If p is odd, unramified in F and prime to $\mathfrak{n}$, then $\rho_{g,F_{\mathfrak{p}}}$ is crystalline by works of Breuil [Bre99, Théorème 1(1)] and Berger [Ber04a, Théorème IV.2.1].

In the newform case, assuming that $\rho_{g,p}$ is crystalline, the weight of g and the eigenvalue of the Hecke operator T_p on g completely determine the structure of the filtered φ-module $\mathbb{D}_{\mathrm{cris}}(\rho_{g,p})$. In the Hilbert modular newform case, assuming that $\rho_{g,F_{\mathfrak{p}}}$ is crystalline, the structure of $\mathbb{D}_{\mathrm{cris}}(\rho_{g,F_{\mathfrak{p}}})$ is more complicated and the characteristic polynomial of Frobenius and the labeled Hodge-Tate weights do not suffice to completely determine its structure. The filtration of $\mathbb{D}_{\mathrm{cris}}(\rho_{g,F_{\mathfrak{p}}})$ is generally unknown, and, even worse, the characteristic polynomial of Frobenius and the filtration are not enough to determine the structure of the filtered φ-module $\mathbb{D}_{\mathrm{cris}}(\rho_{g,F_{\mathfrak{p}}})$. In this case, the isomorphism class is (roughly) determined by an extra parameter in $\left(\bar{\mathbb{Q}}_p^\times\right)^{f_{\mathfrak{p}}-1}$ (for a precise statement see [Dou10, §§6, 7]). As a consequence, if $f_{\mathfrak{p}} \geq 2$ there exist infinite families of non-isomorphic, irreducible two-dimensional crystalline representations of $G_{\mathbb{Q}_{p^{f_{\mathfrak{p}}}}}$ sharing the same characteristic polynomial and filtration.

For higher-dimensional crystalline E-representations of $G_{\mathbb{Q}_{p^f}}$, we mention that even in the simpler case of three-dimensional crystalline representations of $G_{\mathbb{Q}_p}$, there exist non-isomorphic Frobenius-semisimple crystalline representations sharing the same characteristic polynomial and filtration, with the same mod p reductions with respect to appropriately chosen Galois-stable $\mathcal{O}_E$-lattices. This follows by applying the constructions of §4 to the higher-dimensional case, and a proof is not included in this paper.

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1.1 Preliminaries and statement of results

Throughout this paper p will be a fixed prime number, $K_f = \mathbb{Q}_{p^f}$ the finite unramified extension of $\mathbb{Q}_p$ of degree f, and E a finite large enough extension of

K_f with maximal ideal $\mathfrak{m}_E$ and residue field k_E. We simply write K whenever the degree over $\mathbb{Q}_p$ plays no role. We denote by σ_K the absolute Frobenius of K. We fix once and for all an embedding $K \overset{\tau_0}{\hookrightarrow} E$ and we let $\tau_j = \tau_0 \circ \sigma_K^j$ for all $j = 0, 1, ..., f-1$. We fix the f-tuple of embeddings $\mid \tau \mid := (\tau_0, \tau_1, ..., \tau_{f-1})$ and we denote $E^{|\tau|} := \prod_{\tau : K \hookrightarrow E} E$. The map $\xi : E \otimes K \to E^{|\tau|}$ with $\xi_K(x \otimes y) = (x\tau(y))_\tau$ and the embeddings ordered as above is a ring isomorphism. The ring automorphism $1_E \otimes \sigma_K : E \otimes K \to E \otimes K$ transforms via ξ to the automorphism $\varphi : E^{|\tau|} \to E^{|\tau|}$ with $\varphi(x_0, x_1, ..., x_{f-1}) = (x_1, ..., x_{f-1}, x_0)$. We denote by $e_j = (0, ..., 1, ..., 0)$ the idempotent of $E^{|\tau|}$ where the 1 occurs in the τ_j-th coordinate for each $j \in \{0, 1, ..., f-1\}$.

It is well-known (see for instance [BM02, Lemme 2.2.1.1]) that every continuous representation $\rho : G_K \to \mathrm{GL}_n(\bar{\mathbb{Q}}_p)$ is defined over some finite extension of $\mathbb{Q}_p$. Let $\rho : G_K \to \mathrm{GL}_E(V)$ be a continuous E-linear representation. Recall that $\mathbb{D}_{\mathrm{cris}}(V) = (\mathbb{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K}$, where $\mathbb{B}_{\mathrm{cris}}$ is the ring constructed by Fontaine in [Fon88], is a filtered φ-module over K with E-coefficients, and V is crystalline if and only if $\mathbb{D}_{\mathrm{cris}}(V)$ is free over $E \otimes K$ of rank $\dim_E V$. One can easily prove that V is crystalline as an E-linear representation of G_K if and only if it is crystalline as a $\mathbb{Q}_p$-linear representation of G_K (cf. [CDT99] appendix B). We may therefore extend E whenever appropriate without affecting crystallinity. By a variant of the fundamental theorem of Colmez and Fontaine ([CF00], Théorème A) for nontrivial coefficients, the functor $V \mapsto \mathbb{D}_{\mathrm{cris}}(V)$ is an equivalence of categories from the category of crystalline E-linear representations of G_K to the category of weakly admissible filtered φ-modules $(\mathbb{D}, \varphi)$ over K with E-coefficients (see [BM02], §3). Such a filtered module $\mathbb{D}$ is a module over $E \otimes K$ and may be viewed as a module over $E^{|\tau|}$ via the ring isomorphism ξ defined above. Its Frobenius endomorphism is bijective and semilinear with respect to the automorphism φ of $E^{|\tau|}$. For each embedding τ_i of K into E we define $\mathbb{D}_i := e_i\mathbb{D}$. We have the decomposition $\mathbb{D} = \bigoplus_{i=0}^{f-1} \mathbb{D}_i$, and we filter each component $\mathbb{D}_i$ by setting $\mathrm{Fil}^{\mathrm{j}}\mathbb{D}_i := e_i \mathrm{Fil}^{\mathrm{j}}\mathbb{D}$. An integer j is called a labeled Hodge-Tate weight with respect to the embedding τ_i of K in E if and only if $e_i\mathrm{Fil}^{-\mathrm{j}}\mathbb{D} \neq e_i\mathrm{Fil}^{-\mathrm{j}+1}\mathbb{D}$ and is counted with multiplicity $\dim_E \left(e_i\mathrm{Fil}^{-\mathrm{j}}\mathbb{D}/e_i\mathrm{Fil}^{-\mathrm{j}+1}\mathbb{D}\right)$. Since the Frobenius endomorphism of $\mathbb{D}$ restricts to an E-linear isomorphism from $\mathbb{D}_i$ to $\mathbb{D}_{i-1}$ for all i, the components $\mathbb{D}_i$ are equidimensional over E. As a consequence, there are $n = \mathrm{rank}_{E \otimes K}(\mathbb{D})$ labeled Hodge-Tate weights for each embedding, counting multiplicities. The labeled Hodge-Tate weights of $\mathbb{D}$ are by definition the f-tuple of multisets $(W_i)_{\tau_i}$, where each such multiset W_i contains n integers, the opposites of the jumps of the filtration of $\mathbb{D}_i$. For crystalline characters we usually write $(-k_0, -k_1, ..., -k_{f-1})$ instead of $\{-k_i\}_{\tau_i}$. The characteristic polynomial of a crystalline E-linear representation of G_K is the characteristic polynomial of the $E^{|\tau|}$-linear map φ^f, where $(\mathbb{D}, \varphi)$ is the weakly admissible filtered φ-module corresponding to it by Fontaine's functor.

Definition 1.1. *A filtered φ-module $(\mathbb{D}, \varphi)$ is called F-semisimple, non-F-*

semisimple, or F-scalar if the $E^{|\tau|}$-linear map φ^f has the corresponding property.

We may twist $\mathbb{D}$ by some appropriate rank one weakly admissible filtered φ-module (see Proposition 3.5) and assume that $W_i = \{-w_{in-1} \leq ... \leq -w_{i2} \leq -w_{i1} \leq 0\}$ for all $i = 0, 1, ..., f-1$, for some non-negative integers w_{ij}. The Hodge-Tate weights of a crystalline representation V are the opposites of the jumps of the filtration of $\mathbb{D}_{\mathrm{cris}}(V)$. If they are all non-positive, the crystalline representation is called effective or positive. To avoid trivialities, throughout the paper we assume that at least one labeled Hodge-Tate weight is strictly negative.

Notation 1.2. *Let k_i be nonnegative integers which we call weights. Assume that after ordering them and omitting possibly repeated weights we get $w_0 < w_1 < ... < w_{t-1}$, where w_0 is the smallest weight, w_1 the second smallest weight,..., and w_{t-1}, for some $1 \leq t \leq f$, is the largest weight. The largest weight w_{t-1} will be usually denoted by k. For convenience we define $w_{-1} = 0$. Let $I_0 = \{0, 1, ..., f-1\}$ and $I_0^+ = \{i \in I_0 : k_i > 0\}$. For $j = 1, 2, ..., t-1$ we let $I_j = \{i \in I_0 : k_i > w_{j-1}\}$ and for $j = t$ we define $I_t = \varnothing$. Let $f^+ = \left|I_0^+\right|$ be the number of strictly positive weights.*
For each subset $J \subset I_0$ we write $f_J = \sum\limits_{i \in J} e_i$ and $E^{|\tau_J|} = f_J \cdot E^{|\tau|}$. We may visualize the sets $E^{|\tau_{I_j}|}$ as follows: $E^{|\tau_{I_0}|}$ is the Cartesian product E^f. Starting with $E^{|\tau_{I_0}|}$, we obtain $E^{|\tau_{I_1}|}$ by killing the coordinates where the smallest weight occurs i.e. by killing the i-th coordinate for all i with $k_i = w_0$. We obtain $E^{|\tau_{I_2}|}$ by further killing the coordinates where the second smallest weight w_1 occurs and so on.
For any vector $\vec{x} \in E^{|\tau|}$ we denote by x_i its i-th coordinate and by $J_{\vec{x}}$ its support $\{i \in I_0 : x_i \neq 0\}$. We define as norm of $\vec{x}$ with respect to φ the vector $\mathrm{Nm}_\varphi(\vec{x}) := \prod\limits_{i=0}^{f-1} \varphi^i(\vec{x})$ and we write $\mathrm{v}_p(\mathrm{Nm}_\varphi(\vec{x})) := \mathrm{v}_p\left(\prod\limits_{i=0}^{f-1} x_i\right)$, where v_p is the normalized p-adic valuation of $\bar{\mathbb{Q}}_p$. If ℓ is an integer we write $\vec{\ell} = (\ell, \ell, ..., \ell)$ and $\mathrm{v}_p(\vec{x}) > \vec{\ell}$ (resp. if $\mathrm{v}_p(\vec{x}) \geq \vec{\ell}$) if and only if $\mathrm{v}_p(x_i) > \ell$ (resp. $v_p(x_i) \geq \ell$) for all i. Finally, for any matrix $A \in M_n(E^{|\tau|})$ we define as its φ-norm the matrix $\mathrm{Nm}_\varphi(A) := A\varphi(A) \cdots \varphi^{f-1}(A)$, with φ acting on each entry of A.

In §3 we construct the effective crystalline characters of G_{K_f}. More precisely, for $i = 0, 1, ..., f-1$ we construct E-characters χ_i of G_{K_f} with labeled Hodge-Tate weights $-e_{i+1} = (0, ..., -1, ...0)$ with the -1 appearing in the $(i+1)$-place for all i, and we show that any crystalline E-character of G_{K_f} with labeled Hodge-Tate weights $\{-k_i\}_{\tau_i}$ can be written uniquely in the form $\chi = \eta \cdot \chi_0^{k_1} \cdot \chi_1^{k_2} \cdot \cdots \cdot \chi_{f-2}^{k_{f-1}} \cdot \chi_{f-1}^{k_0}$ for some unramified character η of G_{K_f}. In the same section we prove the following.

Theorem 1.3. *Let $\{\ell_i, \ell_{i+f}\} = \{0, k_i\}$, where the k_i, $i = 0, 1, ..., f-1$ are nonnegative integers. Let f^+ be the number of strictly positive k_i and assume that $f^+ \geq 1$.*

(i) The crystalline character $\chi_{\vec{\ell}} = \chi_0^{\ell_1} \cdot \chi_1^{\ell_2} \cdot \cdots \cdot \chi_{2f-2}^{\ell_{2f-1}} \cdot \chi_{2f-1}^{\ell_0}$ of $G_{K_{2f}}$ has labeled Hodge-Tate weights $(-\ell_0, -\ell_1, ..., -\ell_{2f-1})$ and does not extend to G_{K_f}. The induced representation $\mathrm{Ind}_{K_{2f}}^{K_f}(\chi_{\vec{\ell}})$ is irreducible and crystalline with labeled Hodge-Tate weights $\{0, -k_i\}_{\tau_i}$.

(ii) Let V be an irreducible two-dimensional crystalline E-representation of G_{K_f} with labeled Hodge-Tate weights $\{0, -k_i\}_{\tau_i}$, whose restriction to $G_{K_{2f}}$ is reducible. There exist an unramified character η of G_{K_f} and nonnegative integers m_i, $i = 0, 1, ..., 2f-1$, with $\{m_i, m_{i+f}\} = \{0, k_i\}$ for all $i = 0, 1, ..., f-1$, such that

$$V \simeq \eta \otimes \mathrm{Ind}_{K_{2f}}^{K_f}\left(\chi_0^{m_1} \cdot \chi_1^{m_2} \cdot \cdots \cdot \chi_{2f-2}^{m_{2f-1}} \cdot \chi_{2f-1}^{m_0}\right).$$

(iii) $\mathrm{Ind}_{K_{2f}}^{K_f}(\chi_{\vec{\ell}}) \simeq \mathrm{Ind}_{K_{2f}}^{K_f}(\chi_{\vec{m}})$ if and only if $\chi_{\vec{\ell}} = \chi_{\vec{m}}$ or $\chi_{\vec{\ell}}^{\sigma} = \chi_{\vec{m}}$, where $\chi_{\vec{\ell}}^{\sigma} = \chi_0^{\ell_1'} \cdot \chi_1^{\ell_2'} \cdot \cdots \cdot \chi_{2f-2}^{\ell_{2f-1}'} \cdot \chi_{2f-1}^{\ell_f'}$, with $\ell_i' = \ell_{i+f}$ and indices viewed modulo $2f$.

(iv) Up to twist by some unramified character, there exist precisely 2^{f^+-1} distinct isomorphism classes of irreducible two-dimensional crystalline E-representations of G_{K_f} with labeled Hodge-Tate weights $\{0, -k_i\}_{\tau_i}$, induced from crystalline characters of $G_{K_{2f}}$.

Next, we turn our attention to generically irreducible families of two-dimensional crystalline E-representations of G_{K_f}. For any irreducible effective two-dimensional crystalline E-representation of G_{K_f} with labeled Hodge-Tate weights $\{0, -k_i\}_{\tau_i}$ which is induced from a crystalline character of $G_{K_{2f}}$, we construct an infinite family of crystalline E-representations of G_{K_f} of the same Hodge-Tate type which contains it. The members of each of these families have the same semisimplified mod p reductions which we explicitly compute.
Let $V_{\vec{\ell}} = \mathrm{Ind}_{K_{2f}}^{K_f}\left(\chi_0^{\ell_1} \cdot \chi_1^{\ell_2} \cdot \cdots \cdot \chi_{2f-2}^{\ell_{2f-1}} \cdot \chi_{2f-1}^{\ell_0}\right)$, where $\{\ell_i, \ell_{i+f}\} = \{0, k_i\}$ for all $i = 0, 1, ..., f-1$, and assume that at least one k_i is strictly positive. Theorem 1.3 asserts that $V_{\vec{\ell}}$ is irreducible and crystalline with labeled Hodge-Tate weights $\{0, -k_i\}_{\tau_i}$. We describe the members of the family containing $V_{\vec{\ell}}$ in terms of their corresponding by the Colmez-Fontaine theorem weakly admissible filtered φ-modules.

Definition 1.4. *We define the following four types of matrices*

$$t_1\colon \begin{pmatrix} p^{k_i} & 0 \\ X_i & 1 \end{pmatrix}, \; t_2\colon \begin{pmatrix} X_i & 1 \\ p^{k_i} & 0 \end{pmatrix}, \; t_3\colon \begin{pmatrix} 1 & X_i \\ 0 & p^{k_i} \end{pmatrix}, \; t_4: \begin{pmatrix} 0 & p^{k_i} \\ 1 & X_i \end{pmatrix},$$

where the X_i are indeterminates. Let $k = \max\{k_i,\ i = 0, 1, ..., f-1\}$ and let

$$m := \begin{cases} \lfloor \frac{k-1}{p-1} \rfloor & \text{if } k \geq p \textit{ and } k_i \neq p \textit{ for some } i, \\ 0 & \text{if } k \leq p-1 \textit{ or } k_i = p \textit{ for all } i. \end{cases}$$

Let $P(\overrightarrow{X}) = (P_1(X_1), P_2(X_2), ..., P_f(X_f))$ be a matrix whose coordinates $P_j(X_j)$ are matrices of type $1, 2, 3$ or 4. To each such f-tuple we attach a type-vector $\vec{i} \in \{1, 2, 3, 4\}^f$, where for any $j = 1, 2, ..., f$, the j-th coordinate of $\vec{i}$ is defined to be the type of the matrix P_j. We write $P(\overrightarrow{X}) = P^{\vec{i}}(\overrightarrow{X})$. The set of all f-tuples of matrices of type $1, 2, 3, 4$ will be denoted by $\mathcal{P}$. There is no loss to assume that the first $f-1$ coordinates of $P(\overrightarrow{X})$ are of type 1 or 2 (see Remark 6.13) and unless otherwise stated we always assume so. Matrices of type t_1 or t_3 are called of odd type while matrices of type t_2 or t_4 are called of even type.

For any vector $\vec{a} = (\alpha_1, \alpha_2, ..., \alpha_f) \in (p^m \mathfrak{m}_E)^f$ we obtain a matrix

$$P^{\vec{i}}(\vec{\alpha}) = (P_1(\alpha_1), P_2(\alpha_2), ..., P_f(\alpha_f))$$

by evaluating each indeterminate X_i at α_i. We view indices of f-tuples mod f, so $P_f = P_0$. To construct the family containing $V_{\vec{\ell}}$ we choose the types of the matrices P_i as follows:
(1) If $\ell_1 = 0$, $P_1 = t_2$;
(2) If $\ell_1 = k_1 > 0$, $P_1 = t_1$.
For $i = 2, 3, ..., f-1$ we choose the type of the matrix P_i as follows:
(1) If $\ell_i = 0$, then:

- If an even number of coordinates of $(P_1, P_2, ..., P_{i-1})$ is of even type, $P_i = t_2$;
- If an odd number of coordinates of $(P_1, P_2, ..., P_{i-1})$ is of even type, $P_i = t_1$.

(2) If $\ell_i = k_i > 0$, then:

- If an even number of coordinates of $(P_1, P_2, ..., P_{i-1})$ is of even type, $P_i = t_1$;
- If an odd number of coordinates of $(P_1, P_2, ..., P_{i-1})$ is of even type, $P_i = t_2$.

Finally, we choose the type of the matrix P_0 as follows:
(1) If $\ell_0 = 0$, then:

- If an even number of coordinates of $(P_1, P_2, ..., P_{f-1})$ is of even type, $P_0 = t_4$;

- If an odd number of coordinates of $(P_1, P_2, ..., P_{f-1})$ is of even type, $P_0 = t_3$.

(2) If $\ell_0 = k_i > 0$, then:

- If an even number of coordinates of $(P_1, P_2, ..., P_{f-1})$ is of even type, $P_0 = t_2$;
- If an odd number of coordinates of $(P_1, P_2, ..., P_{f-1})$ is of even type, $P_0 = t_1$.

We define families of rank two filtered φ-modules $\left(\mathbb{D}_{\vec{k}}^{\vec{i}}(\vec{\alpha}), \varphi\right)$ over $E^{|\tau|}$ by equipping $\mathbb{D}_{\vec{k}}^{\vec{i}}(\vec{\alpha}) = E^{|\tau|}\eta_1 \bigoplus E^{|\tau|}\eta_2$ with the Frobenius endomorphism defined by $(\varphi(\eta_1), \varphi(\eta_2)) = (\eta_1, \eta_2) P^{\vec{i}}(\vec{\alpha})$ and the filtration

$$\mathrm{Fil}^{\mathrm{j}}(\mathbb{D}_{\vec{k}}^{\vec{i}}(\vec{\alpha})) = \begin{cases} E^{|\tau|}\eta_1 \bigoplus E^{|\tau|}\eta_2 & \text{if } \ j \leq 0, \\ E^{|\tau_{I_0}|}(\vec{x}\eta_1 + \vec{y}\eta_2) & \text{if } \ 1 \leq j \leq w_0, \\ E^{|\tau_{I_1}|}(\vec{x}\eta_1 + \vec{y}\eta_2) & \text{if } \ 1 + w_0 \leq j \leq w_1, \\ \quad\cdots\cdots & \\ E^{|\tau_{I_{t-1}}|}(\vec{x}\eta_1 + \vec{y}\eta_2) & \text{if } \ 1 + w_{t-2} \leq j \leq w_{t-1}, \\ \qquad 0 & \text{if } \ j \geq 1 + w_{t-1}, \end{cases} \tag{1.1}$$

where $\vec{x} = (x_0, x_1, ..., x_{f-1})$ and $\vec{y} = (y_0, y_1, ..., y_{f-1})$, with

$$(x_i, y_i) = \begin{cases} (1, -\alpha_i) \text{ if } P_i \text{ has type 1 or 2,} \\ (-\alpha_i, 1) \text{ if } P_i \text{ has type 3 or 4.} \end{cases} \tag{1.2}$$

Theorem 1.5. *Let $\vec{i}$ be the type-vector attached to the f-tuple $(P_1, P_2, ..., P_f)$ defined above. For any $\vec{\alpha} \in (p^m \mathfrak{m}_E)^f$,*

(i) The filtered φ-module $\mathbb{D}_{\vec{k}}^{\vec{i}}(\vec{\alpha})$ is weakly admissible and corresponds to a two-dimensional crystalline E-representations $V_{\vec{k}}^{\vec{i}}(\vec{\alpha})$ of G_{K_f} with labeled Hodge-Tate weights $\{0, -k_i\}_{\tau_i}$;

(ii) $V_{\vec{k}}^{\vec{i}}(\vec{0}) = \mathrm{Ind}_{K_{2f}}^{K_f}\left(\chi_0^{\ell_1} \cdot \chi_1^{\ell_2} \cdot \cdots \cdot \chi_{2f-2}^{\ell_{2f-1}} \cdot \chi_{2f-1}^{\ell_0}\right)$;

(iii) $\overline{V}_{\vec{k}}^{\vec{i}}(\vec{\alpha}) = \overline{V}_{\vec{k}}^{\vec{i}}(\vec{0})$;

(iv) $\left(\overline{V}_{\vec{k}}^{\vec{i}}(\vec{\alpha})_{|I_{K_f}}\right)^{s.s.} = \omega_{2f,\bar{\tau}_0}^{\beta} \bigoplus \omega_{2f,\bar{\tau}_0}^{p^f\beta}$, *where* $\beta = -\sum_{i=0}^{2f-1} p^i \ell_i$;

(v) The residual representation $\overline{V}_{\vec{k}}^{\vec{i}}(\vec{\alpha})$ is irreducible if and only if $1 + p^f \nmid \beta$;

(vi) Any irreducible member of the family $\left\{V_{\vec{k}}^{\vec{i}}(\vec{\alpha}),\ \vec{\alpha} \in (p^m \mathfrak{m}_E)^f\right\}$, other than $V_{\vec{k}}^{\vec{i}}(\vec{0})$, is non-induced.

Notice that in the cases where $1+p^f \nmid \beta$, all the members of the family $\left\{V_{\vec{k}}^{\vec{i}}(\vec{\alpha}),\ \vec{\alpha}\in (p^m\mathfrak{m}_E)^f\right\}$ are forced to be irreducible. Next, we compute the semisimplified reduction of any reducible two-dimensional crystalline E-representation of G_{K_f}. After enlarging E if necessary, any reducible rank two weakly admissible filtered φ-module $\mathbb{D}$ over $E^{|\tau|}$ with labeled Hodge-Tate weights $\{0,-k_i\}_{\tau_i}$ contains an ordered basis $\underline{\eta}=(\eta_1,\eta_2)$ in which the matrix of Frobenius takes the form $\mathrm{Mat}_{\underline{\eta}}(\varphi)=\begin{pmatrix}\vec{\alpha} & \vec{0}\\ \vec{*} & \vec{\delta}\end{pmatrix}$ such that $\mathbb{D}_2=\left(E^{|\tau|}\right)\eta_2$ is a φ-stable weakly admissible submodule (see Proposition 6.4). The filtration of $\mathbb{D}$ in such a basis $\underline{\eta}$ has the form

$$\mathrm{Fil}^{\mathrm{j}}(\mathbb{D})=\begin{cases} E^{|\tau|}\eta_1\bigoplus E^{|\tau|}\eta_2 & \text{if } j\le 0,\\ E^{|\tau_{I_0}|}\left(\vec{x}\eta_1+\vec{y}\eta_2\right) & \text{if } 1\le j\le w_0,\\ E^{|\tau_{I_1}|}\left(\vec{x}\eta_1+\vec{y}\eta_2\right) & \text{if } 1+w_0\le j\le w_1,\\ \cdots\cdots & \\ E^{|\tau_{I_{t-1}}|}\left(\vec{x}\eta_1+\vec{y}\eta_2\right) & \text{if } 1+w_{t-2}\le j\le w_{t-1},\\ 0 & \text{if } j\ge 1+w_{t-1},\end{cases}$$

for some vectors $\vec{x},\vec{y}\in E^{|\tau|}$ with $(x_i,y_i)\neq(0,0)$ for all i. For each $i\in I_0$, let

$$m_i=\begin{cases}0 & \text{if } x_i\neq 0\ ,\\ k_i & \text{if } x_i=0.\end{cases}$$

Theorem 1.6. *Let V be any reducible two-dimensional crystalline E-representation of G_{K_f} with labeled Hodge-Tate weights $\{0,-k_i\}_{\tau_i}$ corresponding to the weakly admissible filtered φ-module $\mathbb{D}$ as above.*

(i) There exist unramified characters η_i of G_{K_f} such that

$$V\simeq\begin{pmatrix}\psi_1 & *\\ 0 & \psi_2\end{pmatrix},$$

where $\psi_1=\eta_1\cdot\chi_0^{m_1}\cdot\cdots\cdot\chi_{f-2}^{m_{f-1}}\cdot\chi_{f-1}^{m_0}$ and $\psi_2=\eta_2\cdot\chi_0^{k_1-m_1}\cdot\chi_1^{k_2-m_2}\cdot\cdots\cdot\chi_{f-2}^{k_{f-1}-m_{f-1}}\cdot\chi_{f-1}^{k_0-m_0}$;

(ii) $\left(\overline{V}_{|I_K}\right)^{s.s.}=\omega_{f,\bar{\tau}_0}^{\beta_1}\bigoplus\omega_{f,\bar{\tau}_0}^{\beta_2}$, where $\beta_1=-\sum\limits_{i=0}^{f-1}m_ip^i$ and $\beta_2=\sum\limits_{i=0}^{f-1}(m_i-k_i)\,p^i$.

The computation of the semisimplified mod p reduction of a reducible two-dimensional crystalline representation is easy and does not require the construction of the Wach module (see §2.1 for the definition) corresponding to some G_{K_f}-stable lattice contained in it. Computing the non-semisimplified mod p reduction of a two-dimensional crystalline representations with reducible

reduction is an interesting problem not pursued in this paper. For results of this flavour for $K=\mathbb{Q}_{p^2}$, see [CD09].
Up to twist by some unramified character, any split-reducible two-dimensional crystalline E-representations of G_{K_f} with labeled Hodge-Tate weights $\{0,-k_i\}_{\tau_i}$ is of the form

$$V_{\vec{\ell},\vec{\ell}'}(\eta)=\eta\cdot\chi_0^{\ell_1}\cdot\chi_1^{\ell_2}\cdot\dots\cdot\chi_{f-2}^{\ell_{f-1}}\cdot\chi_{f-1}^{\ell_0}\bigoplus\chi_0^{\ell_1'}\cdot\chi_1^{\ell_2'}\cdot\dots\cdot\chi_{f-2}^{\ell_{f-1}'}\cdot\chi_{f-1}^{\ell_0'},$$

for some unramified character η and some nonnegative integers ℓ_i and ℓ_i' such that $\{\ell_i,\ell_i'\}=\{0,k_i\}$ for all i. In Theorem 1.5 we showed that each irreducible representation of G_{K_f} induced from some crystalline character of $G_{K_{2f}}$ belongs to an infinite family of crystalline representations of the same Hodge-Tate types with the same mod p reductions. In the next theorem we prove the same for any split-reducible, non-ordinary two-dimensional crystalline E-representation of G_{K_f}. We list the weakly admissible filtered φ-modules corresponding to these families. In order to construct the infinite family containing $V_{\vec{\ell},\vec{\ell}'}(\eta)$, we define a matrix $P^{\vec{i}}(\overrightarrow{X})\in\mathcal{P}$ by choosing the $(f-1)$-tuple $(P_1,P_2,...,P_{f-1})$ as in Theorem 1.5. If $\eta=\eta_c$ is the unramified character which maps the geometric Frobenius Frob_{K_f} of G_{K_f} to c, we replace the entry p^{k_0} in the definition of the matrix P_0 by cp^{k_0}. The type of the matrix P_0 is chosen as follows:
(1) If $\ell_0=0$, then:

- If an even number of coordinates of $(P_1,P_2,...,P_{f-1})$ is of even type, $P_0=t_3$;
- If an odd number of coordinates of $(P_1,P_2,...,P_{f-1})$ is of even type, $P_0=t_4$.

(2) If $\ell_0=k_0>0$, then:

- If an even number of coordinates of $(P_1,P_2,...,P_{f-1})$ is of even type, $P_0=t_1$;
- If an odd number of coordinates of $(P_1,P_2,...,P_{f-1})$ is of even type, $P_0=t_2$.

Using the matrices $P^{\vec{i}}(\overrightarrow{X})$ we define families of two-dimensional crystalline E-representations $\left\{V_{\vec{k}}^{\vec{i}}(\vec{\alpha}),\ \vec{\alpha}\in(p^m\mathfrak{m}_E)^f\right\}$ of G_{K_f} as in Theorem 1.5 and prove the following.

Theorem 1.7. *Let $\vec{i}$ be the type-vector attached to the f-tuple $(P_1,P_2,...,P_f)$ defined above.*

(i) There exists some unramified character μ such that $V_{\vec{k}}^{\vec{i}}(\vec{0})\simeq\mu\otimes V_{\vec{\ell},\vec{\ell}'}(\eta)$;

(ii) Assume that $\vec{\ell}\neq\vec{0}$ and $\vec{\ell}'\neq\vec{0}$. For any $\vec{\alpha}\in(p^m\mathfrak{m}_E)^f$, $\overline{V}_{\vec{k}}^{\vec{i}}(\vec{\alpha})\simeq\overline{V}_{\vec{k}}^{\vec{i}}(\vec{0})$;

(iii) $\overline{V}_{\vec{\ell},\vec{\ell}'}(\eta)_{|I_{K_f}} = \omega_{f,\bar{\tau}_0}^{\beta} \bigoplus \omega_{f,\bar{\tau}_0}^{\beta'}$, *where* $\beta = -\sum_{i=0}^{f-1} \ell_i p^i$ *and* $\beta' = -\sum_{i=0}^{f-1} \ell'_i p^i$.

A family as in Theorem 1.7 can contain simultaneously split and non-split reducible, as well as irreducible crystalline representations. For example, in the family $\left\{V_{\vec{k}}^{(1,3)}(\vec{\alpha}),\ \vec{\alpha} \in (p^m \mathfrak{m}_E)^2\right\}$, the representation $V_{\vec{k}}^{(1,3)}(\vec{\alpha})$ is split-reducible if and only if $\vec{\alpha} = \vec{0}$, non-split-reducible if and only if precisely one of the coordinates α_i of $\vec{\alpha}$ is zero, and irreducible if and only if $\alpha_0\alpha_1 \neq 0$ (cf. Proposition 6.21). The families of Wach modules which give rise to $V_{\vec{k}}^{(1,3)}(\vec{\alpha})$ contain infinite sub-families of non-split reducible Wach modules which can be used to compute the non-semisimplified mod p reduction of the corresponding crystalline representations with respect to G_{K_f}-stable $\mathcal{O}_E$-lattices. Some reducible two-dimensional crystalline representations with labeled Hodge-Tate weights $\{0, -k_i\}_{\tau_i}$ are easily recognized by looking at their trace of Frobenius. More precisely, if $\mathrm{Tr}\left(\varphi^f\right) \in \mathcal{O}_E^{\times}$, then the representation is reducible (cf. Proposition 6.5), with the converse being false.

2 Overview of the theory

2.1 Étale (φ, Γ)-modules and Wach modules

The general theory of (φ, Γ)-modules works for arbitrary finite extensions K of $\mathbb{Q}_p$. However, a theory of Wach modules, which is our main tool and which we briefly recall in this section, currently exists only when K is unramified over $\mathbb{Q}_p$. We temporarily allow K to be any finite extension of $\mathbb{Q}_p$; we will go back to assume that K is unramified after Theorem 2.2. Let $K_n = K(\zeta_{p^n})$, where ζ_{p^n} is a primitive p^n-th root of unity inside $\bar{\mathbb{Q}}_p$, and let $K_\infty = \cup_{n\geq 1} K_n$. Let $\chi : G_K \to \mathbb{Z}_p^{\times}$ be the cyclotomic character, and let $H_K = \ker \chi = \mathrm{Gal}(\bar{\mathbb{Q}}_p/K_\infty)$ and $\Gamma_K = G_K/H_K = \mathrm{Gal}(K_\infty/K)$. Fontaine ([Fon90]) has constructed topological rings $\mathbb{A}$ and $\mathbb{B}$ endowed with continuous commuting Frobenius φ and $G_{\mathbb{Q}_p}$-actions. Unless otherwise stated and whenever applicable, continuity means continuity with respect to the topologies induced by the weak topologies of the rings $\mathbb{A}$ and $\mathbb{B}$. Let $\mathbb{A}_K = \mathbb{A}^{H_K}$ and $\mathbb{B}_K = \mathbb{B}^{H_K}$, and define $\mathbb{A}_{K,E} := \mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{A}_K$ and $\mathbb{B}_{K,E} := E \otimes_{\mathbb{Q}_p} \mathbb{B}_K$. The actions of φ and Γ_K extend to $\mathbb{A}_{K,E}$ and $\mathbb{B}_{K,E}$ by $\mathcal{O}_E$ (resp. E, k_E)-linearity, and one easily sees that $\mathbb{A}_{K,E} = \mathbb{A}_E^{H_K}$ and $\mathbb{B}_{K,E} = \mathbb{B}_E^{H_K}$.

Definition 2.1. *A (φ, Γ)-module over $\mathbb{A}_{K,E}$ (resp. $\mathbb{B}_{K,E}$) is an $\mathbb{A}_{K,E}$-module of finite type (resp. a free $\mathbb{B}_{K,E}$-module of finite type) endowed with a semilinear and continuous action of Γ_K, and with a semilinear map φ which commutes with the action of Γ_K. A (φ, Γ)-module M over $\mathbb{A}_{K,E}$ is called étale if $\varphi^*(M) = M$, where $\varphi^*(M)$ is the $\mathbb{A}_{K,E}$-module generated by the set $\varphi(M)$. A (φ, Γ)-module M over $\mathbb{B}_{K,E}$ is called étale if it contains a basis $(e_1, ..., e_d)$ over $\mathbb{B}_{K,E}$ such that $(\varphi(e_1), ..., \varphi(e_d)) = (e_1, ..., e_d)A$ for some matrix $A \in \mathrm{GL}_d(\mathbb{A}_{K,E})$.*

If V is a continuous E-linear representation of G_K, we equip the $\mathbb{B}_{K,E}$-module $\mathbb{D}(V) := (\mathbb{B}_E \otimes_E V)^{H_K}$ with a Frobenius endomorphism φ defined by $\varphi(b \otimes v) := \varphi(b) \otimes v$, where φ on the right hand side is the Frobenius of $\mathbb{B}_E$, and with an action of Γ_K given by $\bar{g}(b \otimes v) := gb \otimes gv$ for any $g \in G_K$. This Γ_K-action commutes with φ and is continuous. Moreover, $\mathbb{D}(V)$ is an étale (φ, Γ)-module over $\mathbb{B}_{K,E}$. Conversely, if D is an étale (φ, Γ)-module over $\mathbb{B}_{K,E}$, let $\mathbb{V}(D) := \left(\mathbb{B}_E \otimes_{\mathbb{B}_{K,E}} D\right)^{\varphi=1}$, where $\varphi(b \otimes d) := \varphi(b) \otimes \varphi(d)$. The E-vector space $\mathbb{V}(D)$ is finite dimensional and is equipped with a continuous E-linear G_K-action given by $g(b \otimes d) := gb \otimes \bar{g}d$. We have the following fundamental theorem of Fontaine.

Theorem 2.2. *[Fon90]*

(i) There is an equivalence of categories between continuous E-linear representations of G_K and étale (φ, Γ)-modules over $\mathbb{B}_{K,E}$ given by

$$\mathbb{D} : \mathrm{Rep}_E(G_K) \to \mathcal{M}od^{\acute{e}t}_{(\varphi,\Gamma)}(\mathbb{B}_{K,E}) : V \longmapsto \mathbb{D}(V) := (\mathbb{B}_E \otimes_E V)^{H_K},$$

with quasi-inverse functor

$$\mathbb{V} : \mathcal{M}od^{\acute{e}t}_{(\varphi,\Gamma)}(\mathbb{B}_{K,E}) \to \mathrm{Rep}_E(G_K) : D \longmapsto \mathbb{V}(D) := \left(\mathbb{B}_E \otimes_{\mathbb{B}_{K,E}} D\right)^{\varphi=1}.$$

(ii) There is an equivalence of categories between continuous $\mathcal{O}_E$-linear representations of G_K and étale (φ, Γ)-modules over $\mathbb{A}_{K,E}$ given by

$$\mathbb{D} : \mathrm{Rep}_{\mathcal{O}_E}(G_K) \to \mathcal{M}od^{\acute{e}t}_{(\varphi,\Gamma)}(\mathbb{A}_{K,E}) : \mathrm{T} \longmapsto \mathbb{D}(T) := (\mathbb{A}_E \otimes_{\mathcal{O}_E} \mathrm{T})^{H_K},$$

with quasi-inverse functor

$$\mathbb{T} : \mathcal{M}od^{\acute{e}t}_{(\varphi,\Gamma)}(\mathbb{A}_{K,E}) \to \mathrm{Rep}_{\mathcal{O}_E}(G_K) : D \longmapsto \mathbb{T}(D) := \left(\mathbb{A}_E \otimes_{\mathbb{A}_{K,E}} D\right)^{\varphi=1}.$$

We return to assume that K is unramified over $\mathbb{Q}_p$. In this case $\mathbb{A}_K$ has the form $\mathbb{A}_K = \{\sum_{n=-\infty}^{\infty} \alpha_n \pi_K^n : \alpha_n \in \mathcal{O}_K \text{ and } \lim_{n \to -\infty} \alpha_n = 0\}$ for some element π_K which can be thought of as a formal variable. The Frobenius endomorphism φ of $\mathbb{A}_K$ extends the absolute Frobenius of $\mathcal{O}_K$ and is such that $\varphi(\pi_K) = (1+\pi_K)^p - 1$. The Γ_K-action of $\mathbb{A}_K$ is $\mathcal{O}_K$-linear, commutes with Frobenius, and is such that $\gamma(\pi_K) = (1+\pi_K)^{\chi(\gamma)} - 1$ for all $\gamma \in \Gamma_K$. For simplicity we write π instead of π_K. The ring $\mathbb{A}_K$ is local with maximal ideal (p), fraction field $\mathbb{B}_K = \mathbb{A}_K[\frac{1}{p}]$, and residue field $\mathbb{E}_K := k_K((\pi))$, where k_K is the residue field of K. The rings $\mathbb{A}_K$, $\mathbb{A}_{K,E}$, $\mathbb{B}_K$ and $\mathbb{B}_{K,E}$ contain the subrings $\mathbb{A}_K^+ = \mathcal{O}_K[[\pi]]$, $\mathbb{A}_{K,E}^+ := \mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{A}_K^+$, $\mathbb{B}_K^+ = \mathbb{A}_K^+[\frac{1}{p}]$ and $\mathbb{B}_{K,E}^+ := E \otimes_{\mathbb{Q}_p} \mathbb{B}_K^+$ respectively, and these subrings are equipped with the restrictions of the φ and the Γ_K-actions of the rings containing them. There is a ring isomorphism

$$\xi : \mathbb{A}_{K,E}^+ \to \prod_{\tau : K \hookrightarrow E} \mathcal{O}_E[[\pi]] \tag{2.1}$$

given by $\xi(a\otimes b) = (a\tau_0(b), a\tau_1(b), ..., a\tau_{f-1}(b))$, where $\tau_i\left(\sum_{n=0}^{\infty}\beta_n\pi^n\right) = \sum_{n=0}^{\infty}\tau_i(\beta_n)\pi^n$ for all $b=\sum_{n=0}^{\infty}\beta_n\pi^n \in \mathbb{A}_K^+$. The ring $\mathcal{O}_E[[\pi]]^{|\tau|} := \prod_{\tau:K\hookrightarrow E}\mathcal{O}_E[[\pi]]$ is equipped via ξ with commuting $\mathcal{O}_E$-linear actions of φ and Γ_K given by

$$\varphi(\alpha_0(\pi),\alpha_1(\pi),...,\alpha_{f-1}(\pi)) = (\alpha_1(\varphi(\pi)),...,\alpha_{f-1}(\varphi(\pi)),\alpha_0(\varphi(\pi))) \tag{2.2}$$

$$\text{and } \gamma(\alpha_0(\pi),\alpha_1(\pi),...,\alpha_{f-1}(\pi)) = (\alpha_0(\gamma\pi),\alpha_1(\gamma\pi),...,\alpha_{f-1}(\gamma\pi)) \tag{2.3}$$

for all $\gamma\in\Gamma_K$.

DEFINITION 2.3. *Suppose $k\geq 0$. A Wach module over $\mathbb{A}_{K,E}^+$ (resp. $\mathbb{B}_{K,E}^+$) with weights in $[-k;\ 0]$ is a free $\mathbb{A}_{K,E}^+$-module (resp. $\mathbb{B}_{K,E}^+$-module) N of finite rank, endowed with an action of Γ_K which becomes trivial modulo π, and also with a Frobenius map φ which commutes with the action of Γ_K and such that $\varphi(N)\subset N$ and $N/\varphi^*(N)$ is killed by q^k, where $q:=\varphi(\pi)/\pi$.*

A natural question is to determine the types of étale (φ,Γ)-modules which correspond to crystalline representations via Fontaine's functor. An answer is given by the following theorem of Berger who built on previous work of Wach [Wac96], [Wac97] and Colmez [Col99].

THEOREM 2.4. *[Ber04a]*

(i) An E-linear representation V of G_K is crystalline with Hodge-Tate weights in $[-k;\ 0]$ if and only if $\mathbb{D}(V)$ contains a unique Wach module $\mathbb{N}(V)$ of rank $\dim_E V$ with weights in $[-k;\ 0]$. The functor $V\mapsto\mathbb{N}(V)$ defines an equivalence of categories between crystalline representations of G_K and Wach modules over $\mathbb{B}_{K,E}^+$, compatible with tensor products, duality and exact sequences.

(ii) For a given crystalline E-representation V, the map $\mathrm{T}\mapsto\mathbb{N}(\mathrm{T}):=\mathbb{N}(V)\cap\mathbb{D}(\mathrm{T})$ induces a bijection between G_K-stable, $\mathcal{O}_E$-lattices of V and Wach modules over $\mathbb{A}_{K,E}^+$ which are $\mathbb{A}_{K,E}^+$-lattices contained in $\mathbb{N}(V)$. Moreover $\mathbb{D}(\mathrm{T})=\mathbb{A}_{K,E}\otimes_{\mathbb{A}_{K,E}^+}\mathbb{N}(\mathrm{T})$.

(iii) If V is a crystalline E-representation of G_K, and if we endow $\mathbb{N}(V)$ with the filtration $\mathrm{Fil}^{\mathrm{i}}\mathbb{N}(V)=\{x\in\mathbb{N}(V)|\varphi(x)\in q^i\mathbb{N}(V)\}$, then we have an isomorphism

$$\mathbb{D}_{\mathrm{cris}}(V)\to\mathbb{N}(V)/\pi\mathbb{N}(V)$$

of filtered φ-modules over $E^{|\tau|}$ (with the induced filtration on $\mathbb{N}(V)/\pi\mathbb{N}(V)$).

In view of Theorems 2.2 and 2.4, constructing the Wach module $\mathbb{N}(T)$ of a G_K-stable $\mathcal{O}_E$-lattice T in a crystalline representation V amounts to explicitly constructing the crystalline representation. Indeed, we have

$$V\simeq E\otimes_{\mathcal{O}_E}\left(\mathbb{A}_{K,E}\otimes_{\mathbb{A}_{K,E}^+}\mathbb{N}(\mathrm{T})\right)^{\varphi=1}.$$

An obvious advantage of using Wach modules is that instead of working with the more complicated rings $\mathbb{A}_{K,E}$ and $\mathbb{B}_{K,E}$, one works with the simpler ones $\mathbb{A}^+_{K,E}$ and $\mathbb{B}^+_{K,E}$.

2.2 Wach modules of restricted representations

In this section we relate the Wach module of an effective n-dimensional effective crystalline E-representation V_{K_f} of G_{K_f} to the Wach module of its restriction $V_{K_{df}}$ to $G_{K_{df}}$.

Proposition 2.5. *(i) The Wach module associated to the representation $V_{K_{df}}$ is given by*

$$\mathbb{N}(V_{K_{df}}) = \mathbb{B}^+_{K_{df},E} \otimes_{\mathbb{B}^+_{K_f,E}} \mathbb{N}(V_{K_f}),$$

where $\mathbb{N}(V_{K_f})$ is the Wach module associated to V_{K_f}.

(ii) If T_{K_f} is a G_{K_f}-stable $\mathcal{O}_E$-lattice in V_f associated to the Wach-module $\mathbb{N}(\mathrm{T}_{K_f})$, then V_{df} contains some $G_{K_{df}}$-stable $\mathcal{O}_E$-lattice $\mathrm{T}_{K_{df}}$ whose associated Wach module is

$$\mathbb{N}(\mathrm{T}_{K_{df}}) = \mathbb{A}^+_{K_{df},E} \otimes_{\mathbb{A}^+_{K_f,E}} \mathbb{N}(\mathrm{T}_{K_f}).$$

Proof. (i) Since $\mathbb{N}(V_{K_f})$ is a free $\mathbb{B}^+_{K_f,E}$-module of rank $\dim_E V$ contained in $\mathbb{D}(V_{K_f})$, $N := \mathbb{B}^+_{K_{df},E} \otimes_{\mathbb{B}^+_{K_f,E}} \mathbb{N}(V_{K_f})$ is a free $\mathbb{B}^+_{K_{df},E}$-module of rank $\dim_E V$ contained in $\mathbb{D}(V_{K_{df}}) \supseteq \mathbb{D}(V_{K_f})$. Moreover, N is endowed with an action of $\Gamma_{K_{df}}$ which becomes trivial modulo π, and also with a Frobenius map φ which commutes with the action of $\Gamma_{K_{df}}$ and such that $\varphi(N) \subset N$ and $N/\varphi^*(N)$ is killed by q^k. Hence $N = \mathbb{N}(V_{K_{df}})$ by the uniqueness part of Theorem 2.4(i). Part (ii) follows immediately from Theorem 2.4(ii) since $\mathbb{A}^+_{K_{df},E} \otimes_{\mathbb{A}^+_{K_f,E}} \mathbb{N}(\mathrm{T}_{K_f})$ is an $\mathbb{A}^+_{K_{df},E}$-lattice in $\mathbb{N}(V_{K_{df}})$. $\square$

We fix once and for all an embedding $\tau^0_{K_{df}} : K_{df} \hookrightarrow E$ and we let $\tau^j_{K_{df}} = \tau^0_{K_{df}} \circ \sigma^j_{K_{df}}$ for $j = 0, 1, ..., df-1$, where $\sigma_{K_{df}}$ is the absolute Frobenius of K_{df}. We fix the df-tuple of embeddings $\mid \tau_{K_{df}} \mid := (\tau^0_{K_{df}}, \tau^1_{K_{df}}, ..., \tau^{df-1}_{K_{df}})$. We adjust the notation of §1.1 for the embeddings of K_f into E to the relative situation considered in this section. Let ι be the natural inclusion of K_f into K_{df}, in the sense that $\iota \circ \sigma_{K_f} = \sigma_{K_{df}} \circ \iota$, where σ_{K_f} is the absolute Frobenius of K_f. This induces a natural inclusion of $\mathbb{A}^+_K$ to $\mathbb{A}^+_{K_{df}}$ which we also denote by ι. Let $\tau^j_{K_f} := \tau^0_{K_{df}} \circ \iota \circ \sigma^j_{K_f}$ for $j = 0, 1, ..., f-1$. We fix the f-tuple of embeddings $\mid \tau_{K_f} \mid := (\tau^0_{K_f}, \tau^1_{K_f}, ..., \tau^{f-1}_{K_f})$. Since the restriction of $\sigma_{K_{df}}$ to K_f is σ_{K_f}, we obtain the following commutative diagram

$$\begin{array}{ccc} \mathbb{A}^+_{K_f,E} & \xrightarrow{\xi_{\mathrm{K_f}}} & \mathcal{O}_E^{|\tau_{K_f}|}[[\pi]] \\ {\scriptstyle 1_{\mathcal{O}_{\mathrm{E}}}\otimes\iota}\downarrow & & \downarrow{\scriptstyle\theta} \\ \mathbb{A}^+_{K_{df},E} & \xrightarrow{\xi_{\mathrm{K_{df}}}} & \mathcal{O}_E^{|\tau_{K_{df}}|}[[\pi]] \end{array}$$

where θ is the ring homomorphism defined by

$$\theta(\alpha_0,\alpha_1,...,\alpha_{f-1}) = \underbrace{(\alpha_0,\alpha_1,...,\alpha_{f-1},\alpha_0,\alpha_1,...,\alpha_{f-1},...,\alpha_0,\alpha_1,...,\alpha_{f-1})}_{d\text{-times}}$$
$$=: (\alpha_0,\alpha_1,...,\alpha_{f-1})^{\otimes d}.$$

For any matrix $A \in M_n\left(\mathcal{O}_E^{|\tau_{K_f}|}[[\pi]]\right)$ we denote by $A^{\otimes d}$ the matrix obtained by replacing each entry $\vec{\alpha}$ of A by $\vec{\alpha}^{\otimes d}$. A similar commutative diagram is obtained by replacing $\mathbb{A}^+_K$ by $\mathbb{B}^+_K$ and $\mathcal{O}_E^{|\tau_K|}[[\pi]]$ by $\mathcal{O}_E^{|\tau_K|}[[\pi]][\frac{1}{p}]$. The following proposition follows easily from the discussion above.

PROPOSITION 2.6. *Let V_{K_f}, $V_{K_{df}}$, T_{K_f}, and $\mathrm{T}_{K_{df}}$ be as in Proposition 2.5.*

(i) If the Wach module $\mathbb{N}(V_{K_f})$ of V_{K_f} is defined by the actions of φ and Γ_{K_f} given by $(\varphi(\eta_1),\varphi(\eta_2),...,\varphi(\eta_n)) = \underline{\eta}\cdot\Pi_{K_f}$ and $(\gamma(\eta_1),\gamma(\eta_2),...,\gamma(\eta_n)) = \underline{\eta}\cdot G^\gamma_{K_f}$ for all $\gamma\in\Gamma_{K_f}$ for some ordered basis $\underline{\eta} = (\eta_1,\eta_2,...,\eta_n)$, then the Wach module $\mathbb{N}(V_{K_{df}})$ of $V_{K_{df}}$ is defined by $(\varphi(\eta'_1),\varphi(\eta'_2),...,\varphi(\eta'_n)) = \underline{\eta'}\cdot\Pi_{K_{df}}$ and $(\gamma(\eta'_1),\gamma(\eta'_2),...,\gamma(\eta'_n)) = \underline{\eta'}\cdot G^\gamma_{K_{df}}$ for all $\gamma\in\Gamma_{K_{df}}$, where $\Pi_{K_{df}} = (\Pi_{K_f})^{\otimes d}$ and $G^\gamma_{K_{df}} = \left(G^\gamma_{K_f}\right)^{\otimes d}$ for all $\gamma\in\Gamma_{K_{df}}$, for some ordered basis $\underline{\eta'}$ of $\mathbb{N}(V_{K_{df}})$.

(ii) If the Wach module $\mathbb{N}(\mathrm{T}_{K_f})$ of T_{K_f} is defined by the actions of φ and Γ_{K_f} given by $(\varphi(\eta_1),\varphi(\eta_2),...,\varphi(\eta_n)) = \underline{\eta}\cdot\Pi_{K_f}$ and $(\gamma(\eta_1),\gamma(\eta_2),...,\gamma(\eta_n)) = \underline{\eta}\cdot G^\gamma_{K_f}$ for all $\gamma\in\Gamma_{K_f}$ for some ordered basis $\underline{\eta} = (\eta_1,\eta_2,...,\eta_n)$, then the Wach module $\mathbb{N}(\mathrm{T}_{K_{df}})$ of $\mathrm{T}_{K_{df}}$ is defined by $(\varphi(\eta'_1),\varphi(\eta'_2),...,\varphi(\eta'_n)) = \underline{\eta'}\cdot\Pi_{K_{df}}$ and $(\gamma(\eta'_1),\gamma(\eta'_2),...,\gamma(\eta'_n)) = \underline{\eta'}\cdot G^\gamma_{K_{df}}$ for all $\gamma\in\Gamma_{K_{df}}$, where $\Pi_{K_{df}} = (\Pi_{K_f})^{\otimes d}$ and $G^\gamma_{K_{df}} = \left(G^\gamma_{K_f}\right)^{\otimes d}$ for all $\gamma\in\Gamma_{K_{df}}$, for some ordered basis $\underline{\eta'}$ of $\mathbb{N}(V_{K_{df}})$.

COROLLARY 2.7. *If V_{K_f} is a two-dimensional effective crystalline E-representation of G_{K_f} with labeled Hodge-Tate weights $\{0,-k_i\}_{\tau_i}$, $i = 0,1,...,f-1$, then $V_{K_{df}}$ is an effective crystalline E-representation of $G_{K_{df}}$ with labeled Hodge-Tate weights $\{0,-k_i\}_{\tau_i}$, $i = 0,1,...,df-1$, with $k_j = k_j$ for all $i,j = 0,1,...,df-1$ with $i\equiv j \bmod f$.*

Proof. By Proposition 2.6 there exist ordered bases $\underline{\eta}$ and $\underline{\eta}'$ of $\mathbb{N}(V_{K_f})$ and $\mathbb{N}(V_{K_{df}})$ respectively, such that $\varphi\left(\underline{\eta}\right) = \underline{\eta} \cdot \Pi_{K_f}$, $\gamma\left(\underline{\eta}\right) = \underline{\eta} \cdot G^{\gamma}_{K_f}$ for all $\gamma \in \Gamma_{K_f}$ and $\varphi\left(\underline{\eta}'\right) = \underline{\eta}' \cdot \left(\Pi_{K_f}\right)^{\otimes d}$, $\gamma\left(\underline{\eta}'\right) = \underline{\eta}' \cdot \left(G^{\gamma}_{K_f}\right)^{\otimes d}$ for all $\gamma \in \Gamma_{K_{df}}$. By Theorem 2.4, $x \in \mathrm{Fil}^{\mathrm{j}}\left(\mathbb{N}(V_{K_f})\right)$ if and only if $\varphi\left(x\right) \in q^j \mathbb{N}(V_{K_f})$, from which it follows that $\mathrm{Fil}^{\mathrm{j}}\left(\mathbb{N}(V_{K_{df}})\right) = \left(\mathrm{Fil}^{\mathrm{j}}\left(\mathbb{N}(V_{K_f})\right)\right)^{\otimes d}$ for all j. By Theorem 2.4, $\mathbb{D}(V_{K_f}) \simeq \mathbb{N}(V_{K_f})/\pi\mathbb{N}(V_{K_f})$ as filtered φ-modules over $E^{|\tau_{K_f}|}$. This implies that $\mathrm{Fil}^{\mathrm{j}}\left(\mathbb{D}(V_{K_{df}})\right) = \left(\mathrm{Fil}^{\mathrm{j}}\left(\mathbb{D}(V_{K_f})\right)\right)^{\otimes d}$ for all j and the corollary follows. □

3 Effective Wach modules of rank one

In this section we construct the rank one Wach modules over $\mathcal{O}_E[[\pi]]^{|\tau|}$ with labeled Hodge-Tate weights $\{-k_i\}_{\tau_i}$.

Definition 3.1. *Recall that $q = \frac{\varphi(\pi)}{\pi}$ where $\varphi\left(\pi\right) = (1+\pi)^p - 1$. We define $q_1 = q$ and $q_n = \varphi^{n-1}\left(q\right)$ for all $n \geq 1$. Let $\lambda_f = \prod\limits_{n=0}^{\infty}\left(\frac{q_{nf+1}}{p}\right)$. For each $\gamma \in \Gamma_K$, we define $\lambda_{f,\gamma} = \frac{\lambda_f}{\gamma\lambda_f}$.*

Lemma 3.2. *For each $\gamma \in \Gamma_K$, the functions λ_f and $\lambda_{f,\gamma} \in \mathbb{Q}_p[[\pi]]$ have the following properties:*

(i) $\lambda_f(0) = 1$;

(ii) $\lambda_{f,\gamma} \in 1 + \pi\mathbb{Z}_p\left[[\pi]\right].$

Proof. (i) This is clear since $\frac{q_n(0)}{p} = 1$ for all $n \geq 1$. (ii) One can easily check that $\frac{q}{\gamma q} \in 1 + \pi\mathbb{Z}_p\left[[\pi]\right].$ From this we deduce that $\lambda_{f,\gamma} \in 1 + \pi\mathbb{Z}_p\left[[\pi]\right].$ □

Consider the rank one module $\mathbb{N}_{\vec{k},c} = \left(\mathcal{O}_E[[\pi]]^{|\tau|}\right)\eta$ equipped with the semilinear φ and Γ_K-actions defined by $\varphi(\eta) = (c \cdot q^{k_1}, q^{k_2}, ..., q^{k_{f-1}}, q^{k_0})\eta$ and $\gamma(\eta)=$ $(g^{\gamma}_1(\pi), g^{\gamma}_2(\pi), ...g^{\gamma}_{f-1}(\pi), g^{\gamma}_0(\pi))\eta$ for all $\gamma \in \Gamma_K$, where $c \in \mathcal{O}_E^{\times}$. We want to define the functions $g_i(\pi) = g^{\gamma}_i(\pi) \in \mathcal{O}_E[[\pi]]$ appropriately to make $\mathbb{N}_{\vec{k},c}$ a Wach module over $\mathcal{O}_E[[\pi]]^{|\tau|}$. The actions of φ and γ should commute and a short computation shows that g_0 should satisfy the equation

$$\varphi^f\left(g_0\right) = g_0\left(\frac{\gamma q}{q}\right)^{k_0}\varphi(\frac{\gamma q}{q})^{k_1}\cdots\varphi^{f-1}(\frac{\gamma q}{q})^{k_{f-1}}. \tag{3.1}$$

Lemma 3.3. *Equation 3.1 has a unique $\equiv 1 \bmod \pi$ solution in $\mathbb{Z}_p[[\pi]]$ given by*

$$g_0 = \lambda^{k_0}_{f,\gamma}\varphi(\lambda_{f,\gamma})^{k_1}\varphi^2(\lambda_{f,\gamma})^{k_2}\cdots\varphi^{f-1}(\lambda_{f,\gamma})^{k_{f-1}}.$$

Proof. Notice that $\varphi^f(\lambda_f)=\frac{\lambda_f}{(\frac{q}{p})}$ and $\varphi^f(\gamma\lambda_f)=\frac{\gamma\lambda_f}{(\frac{\gamma q}{p})}$, hence $\lambda_{f,\gamma}=\frac{\lambda_f}{\gamma\lambda_f}$ solves the equation $\varphi^f(u)=u\left(\frac{\gamma q}{q}\right)$. It is straightforward to check that

$$g_0=\lambda_{f,\gamma}^{k_0}\varphi(\lambda_{f,\gamma})^{k_1}\varphi^2(\lambda_{f,\gamma})^{k_2}\cdots\varphi^{f-1}(\lambda_{f,\gamma})^{k_{f-1}}$$

is a solution of equation 3.1. By Lemma 3.2, $g_0\equiv 1 \bmod \pi$. If g_0 and g_0' are two solutions of equation 3.1 congruent to $1 \bmod \pi$, then $(\frac{g_0'}{g_0})\in\mathbb{Z}_p[[\pi]]$ is fixed by φ^f and is congruent to $1 \bmod \pi$, hence equals 1. ☐

Commutativity of φ with the Γ_K-actions implies that

$$g_1=(\frac{q}{\gamma q})^{k_1}\varphi(\frac{q}{\gamma q})^{k_2}\cdots\varphi^{f-2}(\frac{q}{\gamma q})^{k_{f-1}}\varphi^{f-1}(\lambda_{f,\gamma})^{k_0}\varphi^f(\lambda_{f,\gamma})^{k_1}\cdots\varphi^{2f-2}(\lambda_{f,\gamma})^{k_{f-1}},$$

$$\cdots\cdots$$

$$g_{f-2}=(\frac{q}{\gamma q})^{k_{f-2}}\varphi(\frac{q}{\gamma q})^{k_{f-1}}\varphi^2(\lambda_{f,\gamma})^{k_0}\varphi^3(\lambda_{f,\gamma})^{k_1}\cdots\varphi^{f+1}(\lambda_{f,\gamma})^{k_{f-1}},$$

$$g_{f-1}=(\frac{q}{\gamma q})^{k_{f-1}}\varphi(\lambda_{f,\gamma})^{k_0}\varphi^2(\lambda_{f,\gamma})^{k_1}\varphi^3(\lambda_{f,\gamma})^{k_2}\cdots\varphi^f(\lambda_{f,\gamma})^{k_{f-1}},$$

and Lemma 3.2 implies that $g_i\equiv 1 \bmod \pi$ for all i.

PROPOSITION 3.4. *We equip $\mathbb{N}_{\vec{k},c}=\left(\mathcal{O}_E[[\pi]]^{|\tau|}\right)\eta$ with semilinear φ and Γ_K-actions defined by $\varphi(\eta)=(c\cdot q^{k_1},q^{k_2},...,q^{k_{f-1}},q^{k_0})\eta$ and $\gamma(\eta)=(g_1^{\gamma}(\pi),g_2^{\gamma}(\pi),...g_{f-1}^{\gamma}(\pi),g_0^{\gamma}(\pi))\eta$ for the $g_i(\pi)=g_i^{\gamma}(\pi)$ defined above, where $c\in\mathcal{O}_E^{\times}$. The module $\mathbb{N}_{\vec{k},c}$ is a Wach module over $\mathcal{O}_E[[\pi]]^{|\tau|}$ with labeled Hodge-Tate weights $\{-k_i\}_{\tau_i}$. Moreover, $\mathbb{D}_{\vec{k},c}\simeq E^{|\tau|}\bigotimes_{\mathcal{O}_E^{|\tau|}}\left(\mathbb{N}_{\vec{k},c}/\pi\mathbb{N}_{\vec{k},c}\right)$ as filtered φ-modules over $E^{|\tau|}$, where $\mathbb{D}_{\vec{k},c}=\left(E^{|\tau|}\right)\eta$ is the filtered φ-module with Frobenius endomorphism $\varphi(\eta)=(c\cdot p^{k_1},p^{k_2},...,p^{k_{f-1}},p^{k_0})\eta$ and filtration*

$$\mathrm{Fil}^{\mathrm{j}}(\mathbb{D}_{\vec{k},c})=\begin{cases}E^{|\tau_{I_0}|}\eta & \text{if } j\leq w_0,\\ E^{|\tau_{I_1}|}\eta & \text{if } 1+w_0\leq j\leq w_1,\\ \cdots\cdots & \\ E^{|\tau_{I_{t-1}}|}\eta & \text{if } 1+w_{t-2}\leq j\leq w_{t-1},\\ 0 & \text{if } j\geq 1+w_{t-1}.\end{cases}$$

Proof. (i) To prove that Γ_K acts on $\mathbb{N}_{\vec{k},c}$, it suffices to prove that $g_i^{\gamma_1\gamma_2}(\pi)=g_i^{\gamma_1}\gamma_1(g_i^{\gamma_2})$ for all $\gamma_1,\gamma_2\in\Gamma_K$ and $i\in I_0$. This follows immediately from the cocycle relations

$$\frac{q}{\gamma_1\gamma_2(q)}=\frac{q}{\gamma_1(q)}\gamma_1\left(\frac{q}{\gamma_2(q)}\right)\text{ and }\lambda_{f,\gamma_1\gamma_2}=\lambda_{f,\gamma_1}\gamma_1(\lambda_{f,\gamma_2}),$$

and the definition of the $g_i^{\gamma}(\pi)$. Since $g_i^{\gamma}(\pi)\equiv 1 \bmod \pi$ for all $i\in I_0$, the action of Γ_K on $\mathbb{N}_{\vec{k},c}/\pi\mathbb{N}_{\vec{k},c}$ is trivial. (ii) Let $k=\max\{k_0,k_1,...,k_{f-1}\}$ and let $\varphi^*(\mathbb{N}_{\vec{k},c})$ be the $\mathcal{O}_E[[\pi]]^{|\tau|}$-linear span of the set $\varphi(\mathbb{N}_{\vec{k},c})$. Let $c_1=c^{-1}$ and $c_i=1$ if

$i \neq 1$. Since $q^k \eta = \sum_{i=0}^{f-1} \left(q^{k-k_i} c_i e_i\right) \varphi(\eta) \in \varphi^*(\mathbb{N}_{\vec{k},c})$, it follows that q^k kills $\mathbb{N}_{\vec{k},c}/\varphi^*(\mathbb{N}_{\vec{k},c})$. *(iii)* To compute the filtration of $\mathbb{N}_{\vec{k},c}$, we use the fact that $q^j \mid \varphi(x)$ if and only if $\pi^j \mid x$ for any $x \in \mathcal{O}_E[[\pi]]$. Let $x = (x_0, x_1, ..., x_{f-1})\eta \in \mathbb{N}_{\vec{k},c}$. By Theorem 2.4, $x \in \mathrm{Fil}^{\mathrm{j}} \mathbb{N}_{\vec{k},c}$ if and only if $\varphi(x) \in q^j \mathbb{N}_{\vec{k},c}$ or equivalently $q^j \mid \varphi(x_i) q^{k_i}$ for all $i \in I_0$. If $j \leq k_i$ there are no restrictions on the x_i, whereas if $j > k_i$ this is equivalent to $x_i \equiv 0 \mathrm{mod}\ \pi^{j-k_i}$. Therefore,

$$e_i \mathrm{Fil}^{\mathrm{j}} \mathbb{N}_{\vec{k},c} = \begin{cases} e_i \mathbb{N}_{\vec{k},c} & \text{if } j \leq k_i, \\ e_i \pi^{j-k_i} \mathcal{O}_E[[\pi]]\eta & \text{if } j \geq 1 + k_i. \end{cases}$$

This implies that

$$E^{|\tau|} \bigotimes_{\mathcal{O}_E^{|\tau|}} e_i \mathrm{Fil}^{\mathrm{j}} \left(\mathbb{N}_{\vec{k},c}/\pi \mathbb{N}_{\vec{k},c}\right) = \begin{cases} e_i E^{|\tau|} \overline{\eta} & \text{if } j \leq k_i, \\ 0 & \text{if } j \geq 1 + k_i. \end{cases}$$

For the filtration, we have

$$E^{|\tau|} \bigotimes_{\mathcal{O}_E^{|\tau|}} \mathrm{Fil}^{\mathrm{j}} \left(\mathbb{N}_{\vec{k},c}/\pi \mathbb{N}_{\vec{k},c}\right) = \bigoplus_{i=0}^{f-1} \left(E^{|\tau|} \bigotimes_{\mathcal{O}_E^{|\tau|}} e_i \mathrm{Fil}^{\mathrm{j}} (\mathbb{N}_{\vec{k},c}/\pi \mathbb{N}_{\vec{k},c}) \right).$$

Recall from Notation 1.2 that after ordering the weights k_i and omitting possibly repeated weights we get $w_0 < w_1 < ... < w_{t-1}$. By the formulas above,

$$\mathrm{Fil}^{\mathrm{j}}(\mathbb{D}_{\vec{k},c}) = \begin{cases} E^{|\tau|} \left(\sum_{i \in I_0} e_i \right) \eta & \text{if } \ j \leq w_0, \\ E^{|\tau|} \left(\sum_{\{i \in I_0 : k_i > w_0\}} e_i \right) \eta & \text{if } \ 1 + w_0 \leq j \leq w_1, \\ E^{|\tau|} \left(\sum_{\{i \in I_0 : k_i > w_1\}} e_i \right) \eta & \text{if } \ 1 + w_1 \leq j \leq w_2 \\ \cdots\cdots & \\ E^{|\tau|} \left(\sum_{\{i \in I_0 : k_i > w_{t-2}\}} e_i \right) \eta & \text{if } 1 + w_{t-2} \leq j \leq w_{t-1}, \\ 0 & \text{if } j \geq 1 + w_{t-1}. \end{cases}$$

The formula for the filtration follows immediately, recalling that $I_j = \{i \in I_0 : k_i > w_{j-1}\}$ for each $j = 1, 2, ..., t-1$, and $E^{|\tau_{I_r}|} := E^f \left(\sum_{i \in I_r} e_i \right)$ for each $r = 0, 1, ..., t-1$. The isomorphism of filtered φ-modules is obvious. □

Proposition 3.5. *Let $k_0, k_1, ..., k_{f-1}$ be arbitrary integers.*

(i) The weakly admissible rank one filtered φ-modules over $E^{|\tau|}$ with labeled Hodge-Tate weights $\{-k_i\}_{\tau_i}$ are of the form $\mathbb{D}_{\vec{k},\vec{\alpha}} = \left(E^{|\tau|}\right)\eta$, with $\varphi(\eta) = (\alpha_0, \alpha_1, ..., \alpha_{f-1})\eta$ for some $\vec{\alpha} = (\alpha_0, \alpha_1, ..., \alpha_{f-1}) \in (E^{\times})^{|\tau|}$ such that $\mathrm{v_p}(\mathrm{Nm}_{\varphi}(\vec{\alpha})) = \sum\limits_{i \in I_0} k_i$ and

$$\mathrm{Fil^j}(\mathbb{D}_{\vec{k},\vec{\alpha}}) = \begin{cases} E^{|\tau_{I_0}|}\eta & \text{if } j \leq w_0, \\ E^{|\tau_{I_1}|}\eta & \text{if } \ 1 + w_0 \leq j \leq w_1, \\ \quad \cdots\cdots & \\ E^{|\tau_{I_{t-1}}|}\eta & \text{if } 1 + w_{t-2} \leq j \leq w_{t-1}, \\ \ \ 0 & \text{if } j \geq 1 + w_{t-1}. \end{cases}$$

(ii) The filtered φ-modules $\mathbb{D}_{\vec{k},\vec{\alpha}}$ and $\mathbb{D}_{\vec{v},\vec{\beta}}$ are isomorphic if and only if $\vec{k} = \vec{v}$ and $\mathrm{Nm}_{\varphi}(\vec{\alpha}) = \mathrm{Nm}_{\varphi}(\vec{\beta})$.

Proof. Follows easily arguing as in [Dou10], §§4 and 6. □

Corollary 3.6. *All the effective crystalline E-characters of G_K are those constructed in Proposition 3.4.*

Let $c \in \mathcal{O}_E^{\times}$ and $\vec{k} = (-k_1, -k_2, ..., -k_{f-1}, -k_0)$. We denote by $\chi_{c,\vec{k}}$ the crystalline character of G_K corresponding to the Wach module $\mathbb{N}_{\vec{k},c} = \left(\mathcal{O}_E[[\pi]]^{|\tau|}\right)\eta$ with φ action defined by $\varphi(e) = (c \cdot q^{k_1}, q^{k_2}, ..., q^{k_{f-1}}, q^{k_0})\eta$ and the unique commuting with it Γ_K-action defined in Proposition 3.4. When $c = 1$ we simply write $\chi_{\vec{k}}$. By Proposition 3.4 the crystalline character $\chi_i := \chi_{e_i}$ has labeled Hodge-Tate weights $-e_{i+1}$ for all i. By taking tensor products we see that $\chi_{c,\vec{k}} = \chi_{c,\vec{0}} \cdot \chi_0^{k_1} \cdot \chi_1^{k_2} \cdot \cdots \cdot \chi_{f-2}^{k_{f-1}} \cdot \chi_{f-1}^{k_0}$. As usual, we denote by Frob_p be the geometric Frobenius of $G_{\mathbb{Q}_p}$ and by Frob_K the geometric Frobenius of G_K. We have the following.

Lemma 3.7. *(i) The unramified character of G_{K_f} which maps Frob_{K_f} to c equals $\chi_{c,\vec{0}}$ for any $c \in \mathcal{O}_E^{\times}$;*

(ii) For any $i = 0, 1, ..., f-1$, $(\chi_i)_{|G_{K_{2f}}} = \chi_i \cdot \chi_{i+f}$, where the character on the left hand side is a character of G_{K_f} and the characters on the right hand side are characters of $G_{K_{2f}}$;

(iii) If χ is a crystalline character of G_{K_f} with labeled Hodge-Tate weights $\{-k_i\}_{\tau_i}$, $i = 0, 1, ..., f-1$, its restriction to $G_{K_{2f}}$ has labeled weights $\{-k_i\}_{\tau_i}$, $i = 0, 1, ..., 2f-1$, with $k_{i+f} = k_i$ for all $i = 0, 1, ..., f-1$;

(iv) If χ and ψ are crystalline characters of G_{K_f}, then $\chi_{|G_{K_{df}}} = \psi_{|G_{K_{df}}}$ if and only if $\chi = \eta \cdot \psi$, where η is an unramified character of G_{K_f} which maps Frob_{K_f} to a d-th root of unity.

Proof. (i) Let $\sqrt[f]{c}$ be any choice of an f-th root of c in E. The filtered φ-module with trivial filtration and $\varphi(e) = \sqrt[f]{c} \cdot e$ corresponds to the unramified character η of $G_{\mathbb{Q}_p}$ which maps Frob_p to $\sqrt[f]{c}$. Since the $\mathrm{Frob}_{K_f} = (\mathrm{Frob}_p)^f_{|K_f}$, the restriction of η_c of η to K_f maps Frob_{K_f} to c. By Proposition 2.6 the rank one filtered φ-module corresponding to the unramified character η_c has trivial filtration and Frobenius $\varphi(e) = (\sqrt[f]{c}, \sqrt[f]{c}, ..., \sqrt[f]{c})\, e$, and by Proposition 3.5(ii) the latter is isomorphic to the rank one filtered φ-module with trivial filtration and $\varphi(e) = (c, 1, ...1)\, e$. Part (ii) follows from the definition of the characters χ_i and Proposition 2.6. Part (iii) follows immediately from part (ii). For part (iv) it suffices to prove that any crystalline character η of G_{K_f} with trivial restriction to $G_{K_{df}}$ is an unramified character of G_{K_f} which maps Frob_{K_f} to a d-th root of unity. The restriction of η to $G_{K_{df}}$ has all its labeled Hodge-Tate weights equal to zero, and by Corollary 2.7 so does η. By part (i) η is an unramified character of G_{K_f} which maps Frob_{K_f} to some constant, say c. The restriction of η to $G_{K_{df}}$ is trivial and maps $\mathrm{Frob}_{K_{df}} = \left(\mathrm{Frob}_{K_f}\right)^d_{|K_{df}}$ to c^d, therefore c is a d-th root of unity and part (iv) follows. □

Let χ be any E-character of G_K, and let $h \in G_{\mathbb{Q}_p}$. Since K is unramified over $\mathbb{Q}_p$, it is h-stable and the character χ^h with $\chi^h(g) := \chi\left(hgh^{-1}\right)$ is well defined. We have $h_{|K} =: \sigma_K^{n(h)}$ for a unique integer $n(h)$ modulo f. We denote by $\mathrm{T}(\chi)$ the rank one $\mathcal{O}_E$-representation of G_K defined by $\gamma e = \chi(\gamma)\, e$ for any basis element e and any $\gamma \in G_K$.

Lemma 3.8. *Let χ be the crystalline character corresponding to the Wach module defined in Proposition 3.4, and let $h \in G_{\mathbb{Q}_p}$. Let $\eta_1 = \left(\bar{h}^{-1}_{|K}\right) \cdot \eta$. The rank one module $\mathbb{N}^h := \left(\mathcal{O}_E[[\pi]]^{|\tau|}\right)\eta_1$ endowed with semilinear Frobenius and Γ_K-actions defined by*

$$\begin{aligned}
\varphi(\eta_1) &= \left(c \cdot q^{k_{f+1-n(h)}},\ q^{k_{f+2-n(h)}}, ...,\ q^{k_{2f-n(h)}}\right)\eta_1 \text{ and}\\
\gamma(\eta_1) &= \left(g^{h\gamma h^{-1}}_{f+1+n(h^{-1})},\ g^{h\gamma h^{-1}}_{f+2-n(h)}, ...,\ g^{h\gamma h^{-1}}_{2f-1-n(h)},\ g^{h\gamma h^{-1}}_{2f-n(h)}\right)\eta_1,
\end{aligned}$$

where the indices are viewed modulo f, is a Wach module whose corresponding crystalline character is χ^h.

Proof. It is trivial to check that $\mathbb{N}^h$ with the above defined actions is a Wach module. By Theorems 2.2 and 2.4, $\mathrm{T}(\chi) \simeq \left(\mathbb{A}_{K,E} \otimes_{\mathbb{A}^+_{K,E}} \mathbb{N}(\mathrm{T}(\chi))\right)^{\varphi=1}$, hence there exists some $\alpha \in \mathbb{A}_{K,E}$ such that $\varphi(\alpha \otimes \eta) = \alpha \otimes \eta$ and $\gamma(\alpha \otimes \eta) = \chi(\gamma)(\alpha \otimes \eta)$ for all $\gamma \in G_K$. This is equivalent to

$$\varphi(\alpha) \cdot \xi^{-1}\left(c \cdot q^{k_1}, q^{k_2}, ..., q^{k_0}\right) \otimes \eta = \alpha \otimes \eta \text{ and} \tag{3.2}$$

$$\gamma(\alpha) \cdot \xi^{-1}\left(g_1^{\gamma}, g_2^{\gamma}, ...g_{f-1}^{\gamma}, g_0^{\gamma}\right) \otimes \eta = \chi(\gamma)\, \alpha \otimes \eta \tag{3.3}$$

for all $\gamma \in G_K$, where ξ is the isomorphism defined in formula 2.1. A little computation shows that for any $(x_0, x_1, ..., x_{f-1}) \in \mathcal{O}_E[[\pi]]^{|\tau|}$,

$$h^{-1}\left(\xi^{-1}(x_0, x_1, ..., x_{f-1})\right) = \xi^{-1}\left(x_{f-n(h)}, x_{f+1-n(h)}, ..., x_{2f-1-n(h)}\right). \tag{3.4}$$

Let $\alpha_1 := h^{-1}\alpha \in \mathbb{A}_{K,E}$. We show that $\varphi(\alpha_1 \otimes \eta_1) = \alpha_1 \otimes \eta_1$ and $\gamma(\alpha_1 \otimes \eta_1) = \chi^h(\gamma)(\alpha_1 \otimes \eta_1)$ for all $\gamma \in G_K$. Indeed,

$$\begin{aligned}
\varphi(\alpha_1 \otimes \eta_1) &= \varphi\left(h^{-1}\alpha\right) \otimes \varphi(\eta_1) \\
&= h^{-1}\varphi(\alpha) \cdot \xi^{-1}\left(c \cdot q^{k_{f+1-n(h)}},\ q^{k_{f+2-n(h)}}\ , ...,\ q^{k_{2f-n(h)}}\right) \otimes \eta_1 \\
&\overset{3.4}{=} h^{-1}\varphi(\alpha) \cdot h^{-1}\xi^{-1}\left(c \cdot q^{k_1}, q^{k_2}, ..., q^{k_{f-1}}, q^{k_0}\right) \otimes h^{-1}\eta \\
&\overset{3.2}{=} h^{-1}(\alpha \otimes \eta) = \alpha_1 \otimes \eta_1.
\end{aligned}$$

Also,

$$\begin{aligned}
\gamma(\alpha_1 \otimes \eta_1) &= \gamma\left(h^{-1}\alpha\right) \cdot \xi^{-1}\left(g_{f+1-n(h)}^{h\gamma h^{-1}},\ g_{f+2-n(h)}^{h\gamma h^{-1}}, ...,\ g_{2f-1-n(h)}^{h\gamma h^{-1}},\ g_{2f-n(h)}^{h\gamma h^{-1}}\right) \otimes \eta_1 \\
&\overset{3.4}{=} h^{-1}\left(h\gamma h^{-1}\alpha \cdot \xi^{-1}\left(g_1^{h\gamma h^{-1}},\ g_2^{h\gamma h^{-1}}, ...,\ g_{f-1}^{h\gamma h^{-1}},\ g_f^{h\gamma h^{-1}}\right) \otimes \eta\right) \\
&\overset{3.3}{=} h^{-1}\left(\chi\left(h\gamma h^{-1}\right)\alpha \otimes \eta\right) = \chi^h(\gamma)(\alpha_1 \otimes \eta_1)
\end{aligned}$$

for all $\gamma \in G_K$. By Theorems 2.2 and 2.4, it follows that the crystalline character which corresponds to $\mathbb{N}^h$ is χ^h. $\square$

COROLLARY 3.9. *If χ is a crystalline E-characters of G_K with labeled Hodge-Tate weights $\{-k_i\}_{\tau_i}$, the character χ^h is crystalline with labeled Hodge-Tate weights $\{-\ell_i\}_{\tau_i}$, where $\ell_i = k_{f+i-n(h)}$ for all i, with the indices $f+i-n(h)$ viewed modulo f.*

COROLLARY 3.10. *The representation*

$$V_{K_f} \simeq \mathrm{Ind}_{K_{2f}}^{K_f}\left(\chi_0^{k_1} \cdot \chi_1^{k_2} \cdot \cdots \cdot \chi_{2f-2}^{k_{2f-1}} \cdot \chi_{2f-1}^{k_0}\right)$$

is crystalline. Moreover, V_{K_f} is irreducible if and only if $k_i \neq k_{i+f}$ for some $i \in \{0,1,...,f-1\}$.

Proof. Since $V_{K_{2f}}$ is crystalline, V_{K_f} is crystalline. The corollary follows from Mackey's irreducibility criterion and Corollary 3.9. $\square$

PROPOSITION 3.11. *Let V_K be an irreducible two-dimensional crystalline E-representation of G_{K_f} with labeled Hodge-Tate weights $\{0,-k_i\}_{\tau_i}$, whose restriction to $G_{K_{2f}}$ is reducible. There exist some unramified character η of G_{K_f} and some nonnegative integers $\ell_i,\ i=0,1,...,2f-1$ with $\{\ell_i, \ell_{i+f}\} = \{0, k_i\}$ for all $i = 0,1,...,f-1$ and $\ell_i \neq \ell_{i+f}$ for some $i \in \{0,1,...,f-1\}$, such that*

$$V_{K_f} \simeq \eta \otimes \mathrm{Ind}_{K_{2f}}^{K_f}\left(\chi_0^{\ell_1} \cdot \chi_1^{\ell_2} \cdot \cdots \cdot \chi_{2f-2}^{\ell_{2f-1}} \cdot \chi_{2f-1}^{\ell_0}\right).$$

Proof. Let χ be a constituent of $V_{K_{2f}}$. By Corollary 3.6, $\chi = \chi_c \cdot \chi_0^{\ell_1} \cdot \chi_1^{\ell_2} \cdot \dots \cdot \chi_{2f-2}^{\ell_{2f-1}} \cdot \chi_{2f-1}^{\ell_0}$ for some $c \in \mathcal{O}_E^{\times}$ and some integers ℓ_i. Let η be the unramified character of G_{K_f} which maps Frob_{K_f} to $\sqrt[2]{c}$. Arguing as in the proof of Lemma 3.7(i) we see that the restriction of η to $G_{K_{2f}}$ is χ_c, hence $\chi_0^{\ell_1} \cdot \chi_1^{\ell_2} \cdot \dots \cdot \chi_{2f-2}^{\ell_{2f-1}} \cdot \chi_{2f-2}^{\ell_0}$ is a constituent of $\left(\eta^{-1} \otimes V_{K_f}\right)_{|K_{2f}}$. Since $\eta^{-1} \otimes V_{K_f}$ is irreducible,

$$\eta^{-1} \otimes V_{K_f} \simeq \mathrm{Ind}_{K_{2f}}^{K_f} \left(\chi_0^{\ell_1} \cdot \chi_1^{\ell_2} \cdot \dots \cdot \chi_{2f-2}^{\ell_{2f-1}} \cdot \chi_{2f-1}^{\ell_0}\right)$$

by Frobenius reciprocity. By Mackey's formula and Corollary 3.9,

$$V_{K_{2f}} \simeq \left(\chi_c \cdot \chi_0^{\ell_1} \cdot \chi_1^{\ell_2} \cdot \dots \cdot \chi_{2f-2}^{\ell_{2f-1}} \cdot \chi_{2f-1}^{\ell_0}\right) \bigoplus \\ \bigoplus \left(\chi_c \cdot \chi_0^{\ell_{1+f}} \cdot \chi_1^{\ell_{2+f}} \cdot \dots \cdot \chi_{2f-2}^{\ell_{3f-1}} \cdot \chi_{2f-1}^{\ell_{3f}}\right),$$

where the indices of the exponents of the second summand are viewed modulo $2f$. By Corollary 2.7, the restricted representation $V_{K_{2f}}$ has labeled Hodge-Tate weights $\{0, -k_i\}_{\tau_i}$, $i = 0, 1, 2, ..., 2f-1$, where $k_{i+f} = k_i$ for all $i = 0, 1, ..., f-1$. The labeled Hodge-Tate weights of the direct sum of characters in formula 3 with respect to the embedding τ_i of K_{2f} to E are $\{-\ell_i, -\ell_{i+f}\}$ for all $i = 0, 1, 2, ..., 2f-1$, with the indices $i+f$ viewed modulo $2f$. Therefore $\{\ell_i, \ell_{i+f}\} = \{0, k_i\}$ for all $i = 0, 1, ..., f-1$. The rest of the proposition follows from Corollary 3.10. □

Proposition 3.12. *Up to twist by some unramified character, there exist precisely 2^{f^+-1} distinct isomorphism classes of irreducible crystalline two-dimensional E-representations of G_{K_f} with labeled Hodge-Tate weights $\{0, -k_i\}_{\tau_i}$, whose restriction to $G_{K_{2f}}$ is reducible.*

Proof. In Proposition 3.11, notice that $\ell_{i+f} = k_i - \ell_l$ for all $i = 0, 1, ..., f-1$. The corollary follows since $\mathrm{Ind}_{K_{2f}}^{K_f}(\chi) \simeq \mathrm{Ind}_{K_{2f}}^{K_f}(\psi)$ if and only if $\{\chi, \chi^h\} = \{\psi, \psi^h\}$, where h is any element in $G_{\mathbb{Q}_p}$ lifting a generator of $\mathrm{Gal}\left(K_{2f}/K_f\right)$. □

4 Families of effective Wach modules of arbitrary weight and rank

We extend the method used by Berger-Li-Zhu in [BLZ04] for two-dimensional crystalline representations of $G_{\mathbb{Q}_p}$, in order to construct families of Wach modules of effective crystalline representations of G_K of arbitrary rank. Fixing a basis, we need to exhibit matrices Π and G_γ such that $\Pi\varphi(G_\gamma) = G_\gamma \gamma(\Pi)$ for all $\gamma \in \Gamma_K$, with some additional properties imposed by Theorem 2.4. In the two-dimensional case, for representations of $G_{\mathbb{Q}_p}$ and for a suitable basis, it is trivial to write down such a matrix Π assuming that the valuation of the trace of Frobenius of the corresponding filtered φ-module is suitably large, and the

main difficulty is in constructing a Γ_K-action which commutes with Π. When $K \neq \mathbb{Q}_p$, finding a matrix Π which gives rise to a prescribed weakly admissible filtration seems to be already hard, even in the two-dimensional case. Assuming that such a matrix Π is available, it is usually very hard to explicitly write down the matrices G_γ. There are exceptions to this, for example some split-reducible two-dimensional crystalline representations. In the general case, instead of explicitly writing down the matrices G_γ we prove that such matrices exist using a successive approximation argument.
Let $\mathcal{S} = \{X_i\ ;\ i = 0, 1, ..., m-1\}$ be a set of indeterminates, were $m \geq 1$ is any integer. We extend the actions of φ and Γ_K defined in equations 2.2 and 2.3 on the ring $\mathcal{O}_E[[\pi]]^{|\tau|} := \prod\limits_{\tau : K \hookrightarrow E} \mathcal{O}_E[[\pi]]$ to an action on $\mathcal{O}_E[[\pi, \mathcal{S}]]^{|\tau|} := \prod\limits_{\tau : K \hookrightarrow E} \mathcal{O}_E[[\pi, \mathcal{S}]]$, by letting φ and Γ_K act trivially on each indeterminate X_i. We let φ and Γ_K act on the matrices of $M_n^{\mathcal{S}} := M_n(\mathcal{O}_E[[\pi, \mathcal{S}]]^{|\tau|})$ entry-wise for any integer $n \geq 2$. For any integer $s \geq 0$, we write $\vec{\pi}^s = (\pi^s, \pi^s, ..., \pi^s)$, and for any $\alpha \in \mathcal{O}_E[[\pi, \mathcal{S}]]$ and any vector $\vec{r} = (r_0, r_1, ..., r_{f-1})$ with nonnegative integer coordinates we write $\alpha^{\vec{r}} = (\alpha^{r_0}, \alpha^{r_1}, ..., \alpha^{r_{f-1}})$. As usual, we assume that k_i are nonnegative integers and we write $k := w_{t-1} = \max\{k_0, k_1, ..., k_{f-1}\}$. Let $\ell \geq k$ be any fixed integer. We start our constructions with the following lemma.

LEMMA 4.1. *Let $\Pi_i = \Pi_i(\mathcal{S}),\ i = 0, 1, ..., f-1$ be matrices in $M_n(\mathcal{O}_E[[\pi, \mathcal{S}]])$ such that $\det(\Pi_i) = c_i q^{k_i}$, with $c_i \in \mathcal{O}_E[[\pi]]^\times$. We denote by $\Pi(\mathcal{S})$ the matrix $(\Pi_1, \Pi_2, ..., \Pi_{f-1}, \Pi_0)$ and view it as an element of $M_n^{\mathcal{S}}$ via the natural isomorphism $M_n^{\mathcal{S}} \simeq M_n(\mathcal{O}_E[[\pi, \mathcal{S}]])^{|\tau|}$. We denote by $P_i = P_i(\mathcal{S})$ the reduction of $\Pi_i \mathrm{mod}\ \pi$ for all i. Assume that for each $\gamma \in \Gamma_K$ there exists a matrix $G_\gamma^{(\ell)} = G_\gamma^{(\ell)}(\mathcal{S}) \in M_n^{\mathcal{S}}$ such that:*

1. *$G_\gamma^{(\ell)}(\mathcal{S}) \equiv \overrightarrow{Id} \mathrm{mod}\ \vec{\pi}^\ell$;*
2. *$G_\gamma^{(\ell)}(\mathcal{S}) - \Pi(\mathcal{S})\varphi(G_\gamma^{(\ell)}(\mathcal{S}))\gamma(\Pi(\mathcal{S})^{-1}) \in \vec{\pi}^\ell M_n^{\mathcal{S}}$;*
3. *There is no nonzero matrix $H \in M_n(\mathcal{O}_E[[\mathcal{S}]]^{|\tau|})$ such that $HU = p^{ft}UH$ for some $t > 0$, where $U = \mathrm{Nm}_\varphi(P)$ and $P = P(\mathcal{S}) = (P_1, P_2, ..., P_{f-1}, P_0)$;*
4. *For each $s \geq \ell + 1$ the operator*
$$H \longmapsto H - Q_f H(p^{f(s-1)} Q_f^{-1}) : M_n(\mathcal{O}_E[[\mathcal{S}]]) \longrightarrow M_n(\mathcal{O}_E[[\mathcal{S}]]), \quad (4.1)$$
where $Q_f = P_1 P_2 \cdots P_{f-1} P_0$, is surjective.

Then for each $\gamma \in \Gamma_K$ there exists a unique matrix $G_\gamma(\mathcal{S}) \in M_n^{\mathcal{S}}$ such that

(i) $G_\gamma(\mathcal{S}) \equiv \overrightarrow{Id} \mathrm{mod}\ \vec{\pi}$ *and*

(ii) $\Pi(\mathcal{S})\varphi(G_\gamma(\mathcal{S})) = G_\gamma(\mathcal{S})\gamma(\Pi(\mathcal{S}))$.

Proof. Uniqueness*:* Suppose that both the matrices $G_\gamma(\mathcal{S})$ and $G'_\gamma(\mathcal{S})$ satisfy the conclusions of the lemma and let $H = G'_\gamma(\mathcal{S})G_\gamma(\mathcal{S})^{-1}$. We easily see that $H \in \vec{Id} + \vec{\pi} M_n^{\mathcal{S}}$ and $H\Pi(\mathcal{S}) = \Pi(\mathcal{S})\varphi(H)$. We'll show that $H = \vec{Id}$. We write $H = \vec{Id} + \pi^t H_t + \cdots$, where $H_t \in M_n(\mathcal{O}_E[[\mathcal{S}]]^{|\tau|})$ for some $t \geq 1$ and $\Pi(\mathcal{S}) = P + \pi P^{(1)} + \pi^2 P^{(2)} + \cdots$, and we will show that $H_t = 0$. Since $H\Pi(\mathcal{S}) = \Pi(\mathcal{S})\varphi(H)$, we have $(H - \vec{Id})\Pi(\mathcal{S}) = \Pi(\mathcal{S})\varphi(H - \vec{Id})$. We divide both sides of this equation by π^t using that $\varphi(\pi) = q\pi$, and reduce mod π. Since $q \equiv p \bmod \pi$, this gives $H_t P = p^t P\varphi(H_t)$ which implies that $H_t U = p^{ft} U \varphi^f(H_t)$, where $U = \mathrm{Nm}_\varphi(P)$. Since φ acts trivially on X_i and $\mathcal{O}_E$, the map φ^f acts trivially on $M_n(\mathcal{O}_E[[\mathcal{S}]]^{|\tau|})$. Therefore $H_t U = p^{ft} U H_t$ and $H_t = 0$ by assumption (iii) of the lemma.

Existence: Fix a $\gamma \in \Gamma_K$. By assumptions (i) and (ii) of the lemma, there exists a matrix $G_\gamma^{(\ell)} \in \vec{Id} + \vec{\pi}^\ell\, M_n^{\mathcal{S}}$ such that

$$G_\gamma^{(\ell)} - \Pi(\mathcal{S})\varphi(G_\gamma^{(\ell)})\gamma(\Pi(\mathcal{S})^{-1}) = \vec{\pi}^\ell R^{(\ell)}$$

for some matrix $R^{(\ell)} = R^{(\ell)}(\gamma) \in M_n^{\mathcal{S}}$. We shall prove that for each $s \geq \ell + 1$ there exist matrices $R^{(s)} = R^{(s)}(\gamma) \in M_n^{\mathcal{S}}$ and $G_\gamma^{(s)} \in M_n^{\mathcal{S}}$ such that $G_\gamma^{(s)} \equiv G_\gamma^{(s-1)} \bmod \vec{\pi}^{s-1} M_n^{\mathcal{S}}$ and $G_\gamma^{(s)} - \Pi(\mathcal{S})\varphi(G_\gamma^{(s)})\gamma(\Pi(\mathcal{S})^{-1}) = \vec{\pi}^s R^{(s)}$. Let $G_\gamma^{(s)} = G_\gamma^{(s-1)} + \vec{\pi}^{s-1} H^{(s)}$, where $H^{(s)} \in M_n(\mathcal{O}_E[[\mathcal{S}]]^{|\tau|})$ and write $R^{(s)} = \bar{R}^{(s)} + \vec{\pi} \cdot C$ with $C \in M_n^{\mathcal{S}}$. We need

$$\left(G_\gamma^{(s-1)} + \vec{\pi}^{(s-1)} H^{(s)}\right) - \Pi(\mathcal{S}) \left(\varphi(G_\gamma^{(s-1)}) + \varphi\,(\vec{\pi})^{(s-1)} \varphi\left(H^{(s)}\right)\right) \gamma(\Pi(\mathcal{S})^{-1}) \in \vec{\pi}^s M_n^{\mathcal{S}},$$

or equivalently

$$\begin{aligned} &G_\gamma^{(s-1)} - \Pi(\mathcal{S})\varphi(G_\gamma^{(s-1)})\gamma(\Pi(\mathcal{S})^{-1}) + \vec{\pi}^{(s-1)} H^{(s)} - \\ &- (\vec{q\pi})^{(s-1)} \Pi(\mathcal{S})\varphi\left(H^{(s)}\right)\gamma(\Pi(\mathcal{S})^{-1}) \in \vec{\pi}^s M_n^{\mathcal{S}}. \end{aligned}$$

The latter is equivalent to

$$\vec{\pi}^{(s-1)} R^{(s-1)} + \vec{\pi}^{(s-1)} H^{(s)} - (\vec{q\pi})^{(s-1)} \Pi(\mathcal{S})\varphi\left(H^{(s)}\right)\gamma(\Pi(\mathcal{S})^{-1}) \in \vec{\pi}^s M_n^{\mathcal{S}},$$

which is in turn equivalent to

$$H^{(s)} - \vec{q}^{(s-1)} \Pi(\mathcal{S})\varphi\left(H^{(s)}\right)\gamma(\Pi(\mathcal{S})^{-1}) \equiv -R^{(s-1)} \bmod \vec{\pi} M_n^{\mathcal{S}}.$$

This holds if and only if

$$H^{(s)} - \vec{p}^{(s-1)} P\left(\mathcal{S}\right)\varphi\left(H^{(s)}\right) P\left(\mathcal{S}\right)^{-1} = -\bar{R}^{(s-1)}. \tag{4.2}$$

Notice that $\vec{p}^{(s-1)} P(\mathcal{S})^{-1} \in M_n(\mathcal{O}_E[[\mathcal{S}]]^{|\tau|})$ since $s - 1 \geq \ell \geq k = \max\{k_0, k_1, ..., k_{f-1}\}$. We write

$$H^{(s)} = \left(H_1^{(s)}, H_2^{(s)}, ..., H_{f-1}^{(s)}, H_0^{(s)}\right)$$

and
$$-\bar{R}^{(s-1)} = \left(\bar{R}_1^{(s-1)}, \bar{R}_2^{(s-1)}, ..., \bar{R}_{f-1}^{(s-1)}, \bar{R}_0^{(s-1)}\right).$$

Equation 4.2 is equivalent to the system of equations in $M_n\left(\mathcal{O}_E\left[[\mathcal{S}]\right]\right)$

$$H_i^{(s)} - P_i \cdot H_{i+1}^{(s)} \cdot \left(p^{s-1}P_i^{-1}\right) = \bar{R}_i^{(s-1)}, \tag{4.3}$$

where $i = 1, 2, ..., f$, with indices viewed mod f. These imply that

$$H_1^{(s)} - Q_f H_1^{(s)}(p^{f(s-1)}Q_f^{-1}) = \bar{R}_1^{(s-1)} + Q_1\bar{R}_2^{(s-1)}(p^{(s-1)}Q_1^{-1}) + \\ + Q_2\bar{R}_3^{(s-1)}(p^{2(s-1)}Q_2^{-1}) + \cdots + Q_{f-1}\bar{R}_0^{(s-1)}(p^{(s-1)(f-1)}Q_{f-1}^{-1}),$$

where $Q_i = P_1 \cdots P_i$ for all $i = 1, 2, ..., f$. From equations 4.3 we see that the matrices $H_i^{(s)}$, $i = 2, 3, ..., f$, are uniquely determined by the matrix $H_1^{(s)}$, so it suffices to prove that the operator defined in formula 4.1 contains

$$A = \bar{R}_1^{(s-1)} + Q_1\bar{R}_2^{(s-1)}(p^{(s-1)}Q_1^{-1}) + Q_2\bar{R}_3^{(s-1)}(p^{2(s-1)}Q_2^{-1}) + \cdots \\ + Q_{f-1}\bar{R}_0^{(s-1)}(p^{(s-1)(f-1)}Q_{f-1}^{-1})$$

in its image. Since $p^{i(s-1)}Q_i^{-1} \in M_n\left(\mathcal{O}_E\left[[\mathcal{S}]\right]\right)$ for all i, this is true by assumption (iv) of the lemma. We define $G_\gamma(\mathcal{S}) = \lim\limits_{s\to\infty} G_\gamma^{(s)}(\mathcal{S})$ and the proof is complete. □

Let $\widetilde{M_n}$ be the ring $M_n\left(\mathcal{O}_E\left[[\mathcal{S}]\right]\right)/I$ where I is the ideal of $M_n\left(\mathcal{O}_E\left[[\mathcal{S}]\right]\right)$ generated by the set $\{p \cdot Id,\ X_i \cdot Id :\ X_i \in \mathcal{S}\}$. We use the notation of Lemma 4.1 and its proof, and we are interested in the image of the operator $\overline{H} \mapsto \overline{H - Q_f H(p^{f\ell}Q_f^{-1})} : \widetilde{M_n} \to \widetilde{M_n}$ where bar denotes reduction modulo I.

Proposition 4.2. *If the operator*

$$\overline{H} \mapsto \overline{H - Q_f H(p^{f\ell}Q_f^{-1})} : \widetilde{M_n} \to \widetilde{M_n} \tag{4.4}$$

is surjective, then for each $s \geq \ell + 1$ the operator defined in formula 4.1 is surjective.

Proof. Case (i). $s \geq k+2$. In this case $f(s-1) - \sum\limits_{i=0}^{f-1} k_i \geq f\left(s-1-k\right) \geq f \geq 1$. Since $Q_f^{-1} = P_0^{-1}P_{f-1}^{-1}P_{f-2}^{-1}...P_1^{-1}$ and $\det(P_i) = \bar{c}_i p^{k_i}$, it follows that $p^{f(s-1)}Q_f^{-1} \in pM_n(\mathcal{O}_E[[\mathcal{S}]])$. Let B be any matrix in $M_n\left(\mathcal{O}_E\left[[\mathcal{S}]\right]\right)$. We write

$$B = B - Q_f B\left(p^{f(s-1)}Q_f^{-1}\right) + pB_1$$

for some matrix $B_1 \in M_n\left(\mathcal{O}_E\left[[\mathcal{S}]\right]\right)$. Similarly,

$$B_1 = B_1 - Q_f B_1\left(p^{f(s-1)}Q_f^{-1}\right) + pB_2$$

for some matrix $B_2 \in M_n(\mathcal{O}_E[[\mathcal{S}]])$ and

$$B = (B + pB_1) - Q_f(B + pB_1)\left(p^{f(s-1)}Q_f^{-1}\right) + p^2 B_2.$$

Continuing in the same fashion we get

$$B = \left(\sum_{i=0}^{N} p^i B_i\right) - Q_f\left(\sum_{i=0}^{N} p^i B_i\right)\left(p^{f(s-1)}Q_f^{-1}\right) + p^{N+1}B_{N+1}$$

for some matrix $B_{N+1} \in M_n(\mathcal{O}_E[[\mathcal{S}]])$ with $B_0 = B$. Let $H = \sum_{i=0}^{\infty} p^i B_i$. Then $H \in M_n(\mathcal{O}_E[[\mathcal{S}]])$ and $B = H - Q_f H\left(p^{f(s-1)}Q_f^{-1}\right)$.

Case (ii). $\ell = k$ and $s = k+1$. We reduce modulo the ideal I defined before Proposition 4.2. Let A be any element of $M_n(\mathcal{O}_E[[\mathcal{S}]])$. The operator

$$\overline{H} \longmapsto \overline{H - Q_f H\left(p^{f\ell}Q_f^{-1}\right)} : \widetilde{M_n} \to \widetilde{M_n}$$

contains $\bar{A} = A \bmod I$ in its image by the assumption of the lemma. Let $A = A_0 - Q_f A_0\left(p^{f\ell}Q_f^{-1}\right) \bmod I$ for some matrix $A_0 \in M_n(\mathcal{O}_E[[\mathcal{S}]])$. We write

$$A = A_0 - Q_f A_0\left(p^{f\ell}Q_f^{-1}\right) + pB_m + X_0 B_0 + \cdots + X_{m-1}B_{m-1}$$

for matrices $B_i \in M_n(\mathcal{O}_E[[\mathcal{S}]])$. Similarly $B_i = B_i^0 - Q_f B_i^0\left(p^{f\ell}Q_f^{-1}\right) \bmod I$ for matrices $B_i^0 \in M_n(\mathcal{O}_E[[\mathcal{S}]])$ and for all i. Then

$$\begin{aligned} A = A_0 - Q_f A_0\left(p^{f\ell}Q_f^{-1}\right) + pB_m^0 - Q_f\left(pB_m^0\right)\left(p^{f\ell}Q_f^{-1}\right) + \\ + X_0 B_1^0 - Q_f\left(X_0 B_1^0\right)\left(p^{f\ell}Q_f^{-1}\right) + \\ + \cdots + X_{m-1}B_{m-1}^0 - Q_f\left(X_{m-1}B_{f-1}^0\right)\left(p^{f\ell}Q_f^{-1}\right) \bmod I^2, \end{aligned}$$

therefore

$$\begin{aligned} A = {} & (A_0 + pB_m^0 + X_0 B_1^0 + \cdots + X_{m-1}B_{m-1}^0) - \\ & - Q_f(A_0 + pB_m^0 + X_0 B_1^0 + \cdots + X_{f-1}B_{m-1}^0)\left(p^{f\ell}Q_f^{-1}\right) \bmod I^2. \end{aligned}$$

By induction, $A = H - Q_f H\left(p^{f\ell}Q_f^{-1}\right)$ for some $H \in M_n(\mathcal{O}_E[[\mathcal{S}]])$. □

The surjectivity assumption of Proposition 4.2 is usually trivial to check thanks to the following proposition.

Proposition 4.3. *Assume that $\ell > k$ or $\ell = k$ and the weights k_i are not all equal. Then the operator defined in formula 4.4 is surjective.*

Proof. The proposition follows immediately because $\det Q_f = \bar{c}p^{k_1+k_2+\cdots+k_f}$, where $\bar{c} = \bar{c}_1\bar{c}_2\cdots\bar{c}_f$, since $f\ell > k_1 + \cdots + k_f$ and $p \in I$. □

The following lemma summarizes the results of this section. We use the notation of Lemma 4.1.

LEMMA 4.4. *Let $\ell \geq k$ be a fixed integer. We assume that for each $\gamma \in \Gamma_K$ there exists a matrix $G_\gamma^{(\ell)} = G_\gamma^{(\ell)}(\mathcal{S}) \in M_n^{\mathcal{S}}$ such that:*

1. $G_\gamma^{(\ell)}(\mathcal{S}) \equiv \overrightarrow{Id} \bmod \vec{\pi}^\ell$;
2. $G_\gamma^{(\ell)}(\mathcal{S}) - \Pi(\mathcal{S})\varphi(G_\gamma^{(\ell)}(\mathcal{S}))\gamma(\Pi(\mathcal{S})^{-1}) \in \vec{\pi}^\ell M_n^{\mathcal{S}}$;
3. *There is no nonzero matrix $H \in M_n(\mathcal{O}_E[[\mathcal{S}]]^{|\tau|})$ such that $HU = p^{ft}UH$ for some $t > 0$;*
4. *If $\ell = k$ and $k = k_i$ for all i, we additionally assume that the operator*
$$\overline{H} \mapsto \overline{H - Q_f H(p^{f\ell}Q_f^{-1})} : \widetilde{M_n} \to \widetilde{M_n}$$
is surjective.

Then for each $\gamma \in \Gamma_K$ there exists a unique matrix $G_\gamma(\mathcal{S}) \in M_n^{\mathcal{S}}$ such that

(i) $G_\gamma(\mathcal{S}) \equiv \overrightarrow{Id} \bmod \vec{\pi}$, *and*

(ii) $\Pi(\mathcal{S})\varphi(G_\gamma(\mathcal{S})) = G_\gamma(\mathcal{S})\gamma(\Pi(\mathcal{S}))$.

For any vector $\vec{a} = (a_0, a_1, ..., a_{f-1}) \in \mathfrak{m}_E^{|\mathcal{S}|}$ we denote by $\Pi(\vec{a}) = (\Pi_1(a_1), \Pi_2(a_2), ..., \Pi_{f-1}(a_{f-1}), \Pi_0(a_0))$ the matrix obtained from $\Pi(\mathcal{S}) = (\Pi_1(X_1), \Pi_2(X_2), ..., \Pi_{f-1}(X_{f-1}), \Pi_0(X_0))$ by substituting $a_i \in \mathfrak{m}_E$ in each indeterminate X_i of $\Pi_i(X_i)$.

PROPOSITION 4.5. *For any $\vec{a} = (a_0, a_1, ..., a_{f-1}) \in \mathfrak{m}_E^{|\mathcal{S}|}$ and any $\gamma_1, \gamma_2, \gamma \in \Gamma_K$, the following equations hold:*

(i) $G_{\gamma_1\gamma_2}(\vec{a}) = G_{\gamma_1}(\vec{a})\gamma_1(G_{\gamma_2}(\vec{a}))$ *and*

(ii) $\Pi(\vec{a})\varphi(G_\gamma(\vec{a})) = G_\gamma(\vec{a})\gamma(\Pi(\vec{a}))$.

Proof. Both matrices $G_{\gamma_1\gamma_2}(\mathcal{S})$ and $G_{\gamma_1}(\mathcal{S})\gamma_1(G_{\gamma_2}(\mathcal{S}))$ are $\equiv \overrightarrow{Id} \bmod \vec{\pi}$ and are solutions in A of the equation $\Pi(\mathcal{S})\varphi(A) = A\gamma(\Pi(\mathcal{S}))$. They are equal by the uniqueness part of Lemma 4.1. The second equation follows from part (ii) of the same lemma. □

For any vector $\vec{a} \in \mathfrak{m}_E^{|\mathcal{S}|}$ we equip the module $\mathbb{N}(\vec{a}) = \bigoplus_{i=1}^{n} \left(\mathcal{O}_E[[\pi]]^{|\tau|}\right)\eta_i$ with semilinear φ and Γ_K-actions defined by $(\varphi(\eta_1), \varphi(\eta_2), ..., \varphi(\eta_n)) = (\eta_1, \eta_2, ..., \eta_n)\Pi(\vec{a})$ and $(\gamma(\eta_1), \gamma(\eta_2), ..., \gamma(\eta_n)) = (\eta_1, \eta_2, ..., \eta_n)G_\gamma(\vec{a})$ for any $\gamma \in \Gamma_K$. Proposition 4.5 implies that $(\gamma_1\gamma_2)x = \gamma_1(\gamma_2 x)$ and $\varphi(\gamma x) = \gamma(\varphi(x))$ for all $x \in \mathbb{N}(\vec{a})$ and $\gamma, \gamma_1, \gamma_2 \in \Gamma_K$. Since $G_\gamma(\vec{a}) \equiv \overrightarrow{Id} \bmod \vec{\pi}$, it follows that Γ_K acts trivially on $\mathbb{N}(\vec{a})/\pi\mathbb{N}(\vec{a})$.

Proposition 4.6. *For any* $\vec{a} \in \mathfrak{m}_E^{|\mathcal{S}|}$, *the module* $\mathbb{N}(\vec{a})$ *equipped with the* φ *and* Γ_K*-actions defined above is a Wach module over* $\mathcal{O}_E[[\pi]]^{|\tau|}$ *corresponding (by Theorem 2.4) to some* G_K*-stable* $\mathcal{O}_E$*-lattice inside some* n*-dimensional crystalline* E*-representation of* G_K *with Hodge-Tate weights in* $[-k;\ 0]$.

Proof. The only thing left to prove is that $q^k\mathbb{N}(\vec{a}) \subset \varphi^*(\mathbb{N}(\vec{a}))$. Since $\det(\Pi_i) = c_i q^{k_i}$ we have $\det \Pi(\vec{a}) = (c_1 q^{k_1}, c_2 q^{k_2}, ..., c_0 q^{k_0})$ and

$$\begin{aligned}
& (q^k\eta_1, q^k\eta_2, ..., q^k\eta_n) \\
= \ & (\eta_1, \eta_2, ..., \eta_n) \det \Pi(\vec{a}) \left(c_1^{-1}q^{k-k_1}, c_2^{-1}q^{k-k_2}, ..., c_0^{-1}q^{k-k_0}\right) \\
= \ & (\eta_1, \eta_2, ..., \eta_n) \left(\Pi(\vec{a}) \cdot \mathrm{adj}\,(\Pi(\vec{a}))\right) \left(c_1^{-1}q^{k-k_1}, c_2^{-1}q^{k-k_2}, ..., c_0^{-1}q^{k-k_0}\right) \\
= \ & (\varphi(\eta_1), \varphi(\eta_2), ..., \varphi(\eta_n)) \cdot (\mathrm{adj}\Pi(\vec{a})) \left(c_1^{-1}q^{k-k_1}, c_2^{-1}q^{k-k_2}, ..., c_0^{-1}q^{k-k_0}\right).
\end{aligned}$$

Hence $(q^k\eta_1, q^k\eta_2, ..., q^k\eta_n) \in \varphi^*(\mathbb{N}(\vec{a}))$ and $q^k\mathbb{N}(\vec{a}) \subset \varphi^*(\mathbb{N}(\vec{a}))$. ∎

We proceed to prove the main theorem concerning the modulo p reductions of the crystalline representations corresponding to the families of Wach modules constructed in Proposition 4.6. By reduction modulo p we mean reduction modulo the maximal ideal $\mathfrak{m}_E$ of the ring of integers of the coefficient field E. If T is a G_K-stable $\mathcal{O}_E$-lattice in some E-linear representation V of G_K, we denote by $\overline{V} = k_E \bigotimes_{\mathcal{O}_E} \mathrm{T}$ the reduction of V modulo p, where k_E is the residue field of $\mathcal{O}_E$. The reduction $\overline{V}$ depends on the choice of the lattice T, and a theorem of Brauer and Nesbitt asserts that the semisimplification

$$\overline{V}^{s.s.} = \left(k_E \bigotimes_{\mathcal{O}_E} \mathrm{T}\right)^{s.s.}$$

is independent of T. Instead of the precise statement "there exist G_K-stable $\mathcal{O}_E$-lattices T_V and T_W inside the E-linear representation V and W of G_K respectively, such that $k_E \bigotimes_{\mathcal{O}_E} \mathrm{T}_V \simeq k_E \bigotimes_{\mathcal{O}_E} \mathrm{T}_W$", we abuse notation and write $\overline{V} \simeq \overline{W}$. For each $\vec{a} \in \mathfrak{m}_E^{|\mathcal{S}|}$, let $V(\vec{a}) = E \otimes_{\mathcal{O}_E} \mathrm{T}(\vec{a})$, where $\mathrm{T}(\vec{a}) = \mathbb{T}(\mathbb{D}(\vec{a}))$, and $\mathbb{D}(\vec{a}) = \mathbb{A}_{K,E} \bigotimes_{\mathbb{A}_{K,E}^+} \mathbb{N}(\vec{a})$. The representations $V(\vec{a})$ are n-dimensional crystalline E-representations of G_K with Hodge-Tate weights in $[-k;\ 0]$. Concerning their mod p reductions, we have the following theorem.

Theorem 4.7. *For any* $\vec{a} \in \mathfrak{m}_E^{|\mathcal{S}|}$, *the isomorphism* $\overline{V}(\vec{a}) \simeq \overline{V}(\vec{0})$ *holds.*

Proof. We prove that the k_E-linear representations $k_E \bigotimes_{\mathcal{O}_E} \mathrm{T}(\vec{a})$ and $k_E \bigotimes_{\mathcal{O}_E} \mathrm{T}(\vec{0})$ of G_K are isomorphic. Since $\Pi(\mathcal{S})$ and $G_\gamma(\mathcal{S}) \in M_n^{\mathcal{S}}$, we have $G_\gamma(\vec{a}) \equiv G_\gamma(\vec{0}) \mathrm{mod}\ \mathfrak{m}_E$ and $\Pi(\vec{a}) \equiv \Pi(\vec{0}) \mathrm{mod}\ \mathfrak{m}_E$. As (φ, Γ_K)-modules over $k_E((\pi))^{|\tau|}$, $\mathbb{D}(\vec{a})/\mathfrak{m}_E\mathbb{D}(\vec{a}) \simeq \mathbb{D}(\vec{0})/\mathfrak{m}_E\mathbb{D}(\vec{0})$. Hence $\mathbb{T}\left(\mathbb{D}(\vec{a})/\mathfrak{m}_E\mathbb{D}(\vec{a})\right) \simeq \mathbb{T}\left(\mathbb{D}(\vec{0})/\mathfrak{m}_E\mathbb{D}(\vec{0})\right)$, where $\mathbb{T}$ is Fontaine's functor on representations mod $\mathfrak{m}_E$. Since Fontaine's functor is exact, $\mathbb{T}\left(\mathbb{D}(\vec{a})/\mathfrak{m}_E\mathbb{D}(\vec{a})\right) \simeq \mathrm{T}(\vec{a})/\mathfrak{m}_E\mathrm{T}(\vec{a})$ and $\mathrm{T}(\vec{a})/\mathfrak{m}_E\mathrm{T}(\vec{a}) \simeq \mathrm{T}(\vec{0})/\mathfrak{m}_E\mathrm{T}(\vec{0})$. ∎

5 Families of two-dimensional crystalline representations

The main difficulty in applying Lemma 4.4 is in constructing the matrices $G_\gamma^{(\ell)}(\mathcal{S})$ which satisfy conditions (1) and (2). Conditions (3) and (4) are usually easy to check. Throughout this section we retain the notations of Lemma 4.4. We denote by E_{ij} the 2×2 matrix with 1 in the (i,j)-entry and 0 everywhere else. Recall that $E_{ij}\cdot E_{kl}=\delta_{jk}\cdot E_{il}$, where δ is the Kronecker delta function. Also recall our assumption that at least one of the weights k_i is strictly positive.

Proposition 5.1. *The operator $\overline{H}\mapsto\overline{H-Q_f H(p^{f\ell}Q_f^{-1})}:\widetilde{M}_2\to\widetilde{M}_2$ is surjective, unless $\ell=k,\ k=k_i$ for all i and $\bar{Q}_f\in\{E_{11},E_{22}\}$.*

Proof. It is straightforward to check that $\bar{Q}_f=E_{ij}$ for some $i,j\in\{1,2\}$ and

$$p^{k\ell}Q_f^{-1}\mathrm{mod}\ I=\begin{cases} E_{22} & \text{if } \bar{Q}_f=E_{11},\\ E_{11} & \text{if } \bar{Q}_f=E_{22},\\ -E_{12} & \text{if } \bar{Q}_f=E_{12},\\ -E_{21} & \text{if } \bar{Q}_f=E_{21}.\end{cases}$$

If $\bar{Q}_f=E_{11}$ (respectively E_{22}), the image is the set of matrices with zero $(1,2)$ (respectively $(2,1)$) entry, while if $\bar{Q}_f=E_{12}$ or $\bar{Q}_f=E_{21}$ the operator becomes

$$\begin{pmatrix} h_{11} & h_{12}\\ h_{21} & h_{22}\end{pmatrix}\longmapsto\begin{pmatrix} h_{11} & h_{12}+h_{21}\\ h_{21} & h_{22}\end{pmatrix}$$

and

$$\begin{pmatrix} h_{11} & h_{12}\\ h_{21} & h_{22}\end{pmatrix}\longmapsto\begin{pmatrix} h_{11} & h_{12}\\ h_{21}+h_{12} & h_{22}\end{pmatrix}$$

respectively and is clearly surjective. The proposition follows from Proposition 4.3. ∎

Lemma 5.2. *If the matrix $Q_f=P_1P_2\cdots P_{f-1}P_f$ (with $P_f=P_0$) does not have eigenvalues which are a scalar multiple of each other, then the matrix $U=\mathrm{Nm}_\varphi(P)$, where $P=(P_1,P_2,...,P_{f-1},P_0)$, satisfies condition (3) of Lemma 4.4.*

Proof. Let $H\in M_n(\mathcal{O}_E[[\mathcal{S}]]^{|\tau|})$ be a nonzero matrix such that $HU=p^{ft}UH$ for some $t>0$. We write $H=(H_1,H_2,...,H_f)$ and $U=(U_1,U_2,...,U_f)$. Since $P\cdot\varphi(U)\cdot P^{-1}=U$, we have $P_iU_{i+1}P_i^{-1}=U_i$ for all i. Since $Q_f=U_1$, none of the U_i has eigenvalues which are a scalar multiple of each other. If H is invertible then $U_1=Q_f$ has eigenvalues with quotient p^{ft} which contradicts the assumption of the lemma. If H is nonzero and not invertible, there exists an index i such that $H_iU_i=p^{ft}\,U_iH_i$ and $\mathrm{rank}\,(H_i)=1$. There also exists an invertible matrix B such that

$$BH_iB^{-1}=\begin{pmatrix}\alpha_{11} & 0\\ \alpha_{21} & 0\end{pmatrix}$$

with $(\alpha_{11}, \alpha_{21}) \neq (0,0)$. Let $\Gamma = BU_iB^{-1}$ and write $\Gamma = (\gamma_{ij})$. The equation $H_iU_i = p^{ft}U_iH_i$ is equivalent to $p^{ft}\Gamma BH_iB^{-1} = BH_iB^{-1}\Gamma$ which implies that $\gamma_{12} = 0$ and $p^{ft}\gamma_{11}\alpha_{11} = \alpha_{11}\gamma_{11}$. If $\alpha_{11} \neq 0$, then $\gamma_{11} = 0$ a contradiction since Γ is invertible. If $\alpha_{11} = 0$, then $p^{ft}\alpha_{21}\gamma_{22} = \alpha_{21}\gamma_{11}$ and $p^{ft}\gamma_{22} = \gamma_{11}$ (since $\alpha_{21} \neq 0$). Since $\gamma_{12} = 0$, the latter implies that Γ has two eigenvalues with quotient p^{ft}. This in turn implies that U_i and its conjugate $Q_f = U_1$ have eigenvalues with quotient p^{ft} and contradicts the assumption of the lemma. Hence $H = 0$. □

In the two-dimensional case, instead of checking condition (3) of Lemma 4.4 it is often more convenient to use following corollary.

Corollary 5.3. *If $\mathrm{Tr}\,(Q_f) \notin \bar{\mathbb{Q}}_p$, then the matrix $U = \mathrm{Nm}_\varphi(P)$ satisfies condition (3) of Lemma 4.4.*

Proof. Since the determinant of Q_f is a nonzero scalar, the eigenvalues of Q_f are a scalar multiple of each other if and only if $\mathrm{Tr}\,(Q_f)$ is a scalar. □

5.1 Families of rank two Wach modules

We now apply Lemma 4.4 for matrices Π_i as in the following definition.

Definition 5.4. *For a fixed integer $\ell \geq k = \max\{k_0, k_1, ..., k_{f-1}\}$ we define matrices of the following four types*

$$t_1\colon \begin{pmatrix} c_iq^{k_i} & 0 \\ X_i\varphi(z_i) & 1 \end{pmatrix}, \quad t_2\colon \begin{pmatrix} X_i\varphi(z_i) & 1 \\ c_iq^{k_i} & 0 \end{pmatrix},$$

$$t_3\colon \begin{pmatrix} 1 & X_i\varphi(z_i) \\ 0 & c_iq^{k_i} \end{pmatrix}, \quad t_4\colon \begin{pmatrix} 0 & c_iq^{k_i} \\ 1 & X_i\varphi(z_i) \end{pmatrix},$$

where X_i is an indeterminate, $c_i \in \mathcal{O}_E$, and z_i is a polynomial of degree $\leq \ell - 1$ in $\mathbb{Z}_p[\pi]$ such that $z_i \equiv p^{m_\ell} \bmod \pi$, where $m_\ell := \lfloor \frac{\ell-1}{p-1} \rfloor$. Matrices of type t_1 or t_3 are called of odd type while matrices of type t_2 or t_4 are called of even type. We write $\Pi^{\vec{i}}(S) = (\Pi_1(X_1), \Pi_2(X_2), ..., \Pi_{f-1}(X_{f-1}), \Pi_0(X_0))$ with $\vec{i} = (i_1, i_2, ..., i_{f-1}, i_0)$ the vector in $\{1,2,3,4\}^f$ whose j-th coordinate i_j is the type of the matrix Π_j for all $j \in I_0$. We call $\vec{i}$ the type-vector attached to the matrix f-tuple $\Pi^{\vec{i}}(S)$.

The polynomials z_i appearing in the entries of the matrices Π_i will be defined shortly. We will also define functions $x_i^\gamma,\, y_i^\gamma \in 1 + \pi\mathbb{Z}_p[[\pi]]$ such that

$$G_\gamma^{(\ell)} - \Pi(\mathcal{S})\varphi(G_\gamma^{(\ell)})\gamma(\Pi(\mathcal{S})^{-1}) \in \vec{\pi}^\ell M_2^{\mathcal{S}}$$

for all $\gamma \in \Gamma_K$, where

$$G_\gamma^{(\ell)} = \mathrm{diag}\left(\left(x_0^\gamma, x_1^\gamma, ..., x_{f-1}^\gamma\right), \left(y_0^\gamma, y_1^\gamma, ..., y_{f-1}^\gamma\right)\right).$$

Let

$$\Pi(\mathcal{S})=\begin{pmatrix}(\alpha_1,\alpha_2,...\alpha_{f-1},\alpha_0) & (\beta_1,\beta_2,...,\beta_{f-1},\beta_0)\\ (\gamma_1,\gamma_2,...,\gamma_{f-1},\gamma_0) & (\delta_1,\delta_2,..., \delta_{f-1},\delta_0)\end{pmatrix} \text{ with } \begin{pmatrix}\alpha_i & \beta_i\\ \gamma_i & \delta_i\end{pmatrix}$$

belonging to the set

$$\left\{\begin{pmatrix}c_iq^{k_i} & 0\\ X_i\varphi(z_i) & 1\end{pmatrix},\begin{pmatrix}X_i\varphi(z_i) & 1\\ c_iq^{k_i} & 0\end{pmatrix},\begin{pmatrix}1 & X_i\varphi(z_i)\\ 0 & c_iq^{k_i}\end{pmatrix},\begin{pmatrix}0 & c_iq^{k_i}\\ 1 & X_i\varphi(z_i)\end{pmatrix}\right\}.$$

For each $i=1,2,...,f$ we demand that all of the elements

$$x^{\gamma}_{i-1}-\frac{\alpha_i\varphi\left(x^{\gamma}_i\right)(\gamma\delta_i)-\beta_i\varphi\left(y^{\gamma}_i\right)(\gamma\gamma_i)}{\varepsilon_i\left(\gamma q\right)^{k_i}},\ \frac{\beta_i\varphi\left(y^{\gamma}_i\right)(\gamma\alpha_i)-\alpha_i\varphi\left(x^{\gamma}_i\right)(\gamma\beta_i)}{\varepsilon_i\left(\gamma q\right)^{k_i}}, \tag{5.1}$$

$$y^{\gamma}_{i-1}-\frac{\delta_i\varphi\left(y^{\gamma}_i\right)(\gamma\alpha_i)-\gamma_i\varphi\left(x^{\gamma}_i\right)(\gamma\beta_i)}{\varepsilon_i\left(\gamma q\right)^{k_i}},\ \frac{\gamma_i\varphi\left(x^{\gamma}_i\right)(\gamma\delta_i)-\delta_i\varphi\left(y^{\gamma}_i\right)(\gamma\gamma_i)}{\varepsilon_i\left(\gamma q\right)^{k_i}} \tag{5.2}$$

of $\mathcal{O}_E[[\pi,X_0,...,X_{f-1}]][q^{-1}]$ which belong to $\mathcal{O}_E[[\pi]][q^{-1}]$ are zero, and those which contain an indeterminate belong to $\pi^{\ell}\mathcal{O}_E[[\pi,X_0,...,X_{f-1}]]$, where in the formulas above $\varepsilon_i=1$ if Π_i has type 1 or 3 and $\varepsilon_i=-1$ if Π_i has type 2 or 4. As usual lower indices are viewed modulo f.

PROPOSITION 5.5. *For each i, equations 5.1 and 5.2 imply that*

$$x^{\gamma}_{i-1}=\left(\frac{q}{\gamma q}\right)^{\ell_i}\varphi\left(w^{\gamma}_i\right)\ \textit{and}\ y^{\gamma}_{i-1}=\left(\frac{q}{\gamma q}\right)^{\ell'_i}\varphi\left(\left(w^{\gamma}_i\right)'\right), \tag{5.3}$$

with $\ell_i\in\{0,k_i\},\ w^{\gamma}_i\in\{x^{\gamma}_i,y^{\gamma}_i\},\ \ell'_i=k_i-\ell_i,$ *and* $\left(w^{\gamma}_i\right)'=\begin{cases}x^{\gamma}_i \text{ if } w^{\gamma}_i=y^{\gamma}_i,\\ y^{\gamma}_i \text{ if } w^{\gamma}_i=x^{\gamma}_i.\end{cases}$

Proof. If Π_i is of type 1, then $\beta_i=0$, $\alpha_i=c_iq^{k_i}$ and $\delta_i=1$. We must have $q^{k_i}\varphi\left(x^{\gamma}_i\right)=x^{\gamma}_{i-1}\left(\gamma q\right)^{k_i}$ and $\varphi\left(y^{\gamma}_i\right)=y^{\gamma}_{i-1}$. The proposition holds with $\ell_i=k_i$, $w^{\gamma}_i=x^{\gamma}_i$, $\ell'_i=0$, and $\left(w^{\gamma}_i\right)'=y^{\gamma}_i$. The cases where Π_i is of type $2,3,$ or 4 are identical. $\square$

From Proposition 5.5 it follows that

$$x^{\gamma}_0=\left(\prod_{i=0}^{f-1}\varphi^i\left(\frac{q}{\gamma q}\right)^{s_i}\right)\varphi^f\left(z^{\gamma}_f\right)\text{ and } y^{\gamma}_0=\left(\prod_{i=0}^{f-1}\varphi^i\left(\frac{q}{\gamma q}\right)^{s'_i}\right)\varphi^f\left(\left(z^{\gamma}_f\right)'\right), \tag{5.4}$$

with $s'_i,s_i\in\{\ell_i,\ell'_i\}$. If $z^{\gamma}_f=x^{\gamma}_0$, then $\left(z^{\gamma}_f\right)'=y^{\gamma}_0$, and by Lemma 3.3 equations 5.4 have unique $\equiv 1 \bmod \pi$ solutions given by

$$x^{\gamma}_0=\prod_{i=0}^{f-1}\varphi^i\left(\lambda_{f,\gamma}\right)^{s_i}\text{ and } y^{\gamma}_0=\prod_{i=0}^{f-1}\varphi^i\left(\lambda_{f,\gamma}\right)^{s'_i}. \tag{5.5}$$

If If $z_f^\gamma = y_0^\gamma$, then $\left(z_f^\gamma\right)' = x_0^\gamma$ and equations 5.4 imply that

$$x_0^\gamma = \prod_{i=0}^{f-1} \left(\varphi^i \left(\frac{q}{\gamma q}\right)^{s_i} \cdot \varphi^{i+f} \left(\frac{q}{\gamma q}\right)^{s_i'} \right) \varphi^{2f} \left(x_0^\gamma\right), \tag{5.6}$$

$$y_0^\gamma = \prod_{i=0}^{f-1} \left(\varphi^i \left(\frac{q}{\gamma q}\right)^{s_i'} \cdot \varphi^{i+f} \left(\frac{q}{\gamma q}\right)^{s_i} \right) \varphi^{2f} \left(y_0^\gamma\right), \tag{5.7}$$

which by Lemma 3.3 have unique $\equiv 1 \bmod \pi$ solutions given by

$$x_0^\gamma = \prod_{i=0}^{f-1} \left(\varphi^i \left(\lambda_{2f,\gamma}\right)^{s_i} \cdot \varphi^{i+f} \left(\lambda_{2f,\gamma}\right)^{s_i'} \right), \tag{5.8}$$

$$y_0^\gamma = \prod_{i=0}^{f-1} \left(\varphi^i \left(\lambda_{2f,\gamma}\right)^{s_i'} \cdot \varphi^{i+f} \left(\lambda_{2f,\gamma}\right)^{s_i'} \right). \tag{5.9}$$

Equations 5.3 for $i = f$ give the unique $\equiv 1 \bmod \pi$ solutions for x_{f-1}^γ and y_{f-1}^γ, and continuing for $i = f-1, f-2, ..., 2$, we get the unique $\equiv 1 \bmod \pi$ solutions for x_i^γ and y_i^γ. We now define the polynomials z_i so that for each $\gamma \in \Gamma_K$, the matrix $G_\gamma^{(\ell)} \equiv \vec{Id} \bmod \vec{\pi}$ satisfies the congruence $G_\gamma^{(\ell)} - \Pi(\mathcal{S})\varphi(G_\gamma^{(\ell)})\gamma(\Pi(\mathcal{S})^{-1}) \in \vec{\pi}^\ell M_2^{\mathcal{S}}$.

Lemma 5.6. *Let $\mathcal{R} = \{\sum_{i\geq 0} a_i \pi^i \in \mathbb{Q}_p[[\pi]] : \mathrm{v_p}(a_i) + \frac{i}{p-1} \geq 0$ for all $i \geq 0\}$. The set $\mathcal{R}$ endowed with the addition and the multiplication of $\mathbb{Q}_p[[\pi]]$ is a subring of $\mathbb{Q}_p[[\pi]]$ which is stable under the φ and the Γ_K-actions. Moreover,*

(i) $(\frac{q_n}{p})^{\pm 1} \in \mathcal{R}$ for all $n \geq 1$ and $(\lambda_f)^{\pm 1} \in \mathcal{R}$ for all $f \geq 1$;

(ii) Let $b = cp^N b^*$, where $c \in \mathcal{O}_E^\times$, $n \in \mathbb{Z}$, and $b^* \in \mathcal{R} \setminus \{0\}$ is such that $\frac{b^*}{\gamma b^*} \in 1 + \pi \mathbb{Z}_p[[\pi]]$ for all $\gamma \in \Gamma_K$. If $\ell \geq 1$ is a fixed integer, there exists some polynomial $z = z(\ell, b) \in \mathbb{Z}_p[\pi]$ with $\deg_\pi z \leq \ell - 1$ and $z \equiv p^{m_\ell} \bmod \pi$, where $m_\ell = \lfloor \frac{\ell-1}{p-1} \rfloor$, such that $z - \gamma z \frac{b}{\gamma b} \in \pi^\ell \mathbb{Z}_p[[\pi]]$ for all $\gamma \in \Gamma_K$.

Proof. We notice that the coefficients a_i of π^i in $\frac{q}{p}$ are such that $\mathrm{v_p}(a_i) + \frac{i}{p-1} \geq 0$ for all $i = 0, 1, ...$ Motivated by this we consider the set $\mathcal{R}$ of all functions of $\mathbb{Q}_p[[\pi]]$ with the same property. This is a ring with the obvious operations, stable under φ and Γ_K. One easily checks that $\left(\frac{p}{q}\right)^{\pm 1} \in \mathcal{R}$ and therefore $\left(\frac{q_n}{p}\right)^{\pm 1} \in \mathcal{R}$ for all $n \geq 1$ from which (i) follows easily. (ii) Since Γ_K acts trivially on $\mathcal{O}_E^\times$ we may replace b by $c^{-1}b$ and assume that $c = 1$. We write $b = p^n b^*$. Let $p^m b = z + a$, where $a \in \pi^\ell \mathbb{Q}_p[[\pi]]$ and $\deg_\pi z \leq \ell - 1$, for integer m which will be chosen large enough so that $z \in \mathbb{Z}_p[\pi]$. Let $z = \sum_{j=0}^{\ell-1} z_j \pi^j$. Since

$p^{m+n}b^* = z + a$ and $b^* \in \mathcal{R}$, we have $\mathrm{v_p}(z_j) - m - n + \frac{j}{p-1} \geq 0$ for all $j \geq 0$. We need $\mathrm{v_p}(z_j) > -1$ for all $j = 0, 1, ..., \ell - 1$ and it suffices to have $m + n - \frac{\ell-1}{p-1} > -1$. We choose $m = \lfloor \frac{\ell-1}{p-1} \rfloor - n$. Then $z \in \mathbb{Z}_p[\pi]$, $\deg_\pi z \leq \ell - 1$ and $z \equiv p^{m+n} = p^{m_\ell} \mathrm{mod}\ \pi$, For any $\gamma \in \Gamma_K$, $z - \gamma z \frac{b}{\gamma b} = p^m b - a - b\gamma(b^{-1})(p^m(\gamma b) - \gamma a) = b\gamma(b^{-1})\gamma a - a \in \pi^\ell \mathbb{Q}_p[[\pi]]$. Since $z \in \mathbb{Z}_p[\pi]$ and $b\gamma(b^{-1}) \in 1 + \pi\mathbb{Z}_p[[\pi]]$, we have $z - \gamma z \frac{b}{\gamma b} \in \pi^\ell \mathbb{Z}_p[[\pi]] = \mathbb{Z}_p[[\pi]] \cap \pi^\ell \mathbb{Q}_p[[\pi]]$ for all $\gamma \in \Gamma_K$. $\square$

LEMMA 5.7. *For any $\gamma \in \Gamma_K$ and $i \in I_0$,*

(i) $x_i^\gamma, y_i^\gamma \in 1 + \pi\mathbb{Z}_p[[\pi]]$;

(ii) $x_i^\gamma = \frac{a_i}{\gamma a_i}$ *and* $y_i^\gamma = \frac{b_i}{\gamma b_i}$ *for some* a_i *and* b_i *with* $(a_i)^{\pm 1}$ *and* $(b_i)^{\pm 1} \in \mathcal{R}$.

Proof. (i) is clear by the definition of the x_i^γ, y_i^γ and Lemma 3.2. (ii) Let $i = 0$. If $z_f^\gamma = x_0^\gamma$, by equation 5.5 we have $x_0^\gamma = \frac{a_0}{\gamma a_0}$, where $a_0 = \prod_{i=0}^{f-1} \varphi^i(\lambda_f)^{s_i} \in \mathcal{R}$. Since $(\lambda_f)^{\pm 1} \in \mathcal{R}$ and $\mathcal{R}$ is φ-stable, $(a_0)^{\pm 1} \in \mathcal{R}$. If $z_f^\gamma = y_0^\gamma$, by equation 5.8 we have $x_0^\gamma = \frac{a_0}{\gamma a_0}$, where $a_0 = \prod_{i=0}^{f-1} \left(\varphi^i(\lambda_f)^{s_i} \varphi^{i+f}(\lambda_f)^{s_i'} \right) \in \mathcal{R}$, therefore $(a_0)^{\pm 1} \in \mathcal{R}$. The proof for y_0^γ and $(b_i)^{\pm 1}$ is similar. For x_{f-1}^γ, notice that $x_{f-1}^\gamma = \left(\frac{q}{\gamma q}\right)^{\ell_i} \varphi(w_i^\gamma) = \frac{\gamma\left(\varphi(c_0)\left(\frac{q}{p}\right)^{\ell_f}\right)}{\varphi(c_0)\left(\frac{q}{p}\right)^{\ell_f}}$ with $c_0 \in \{a_0, b_0\}$. Since $(a_0)^{\pm 1}, (b_0)^{\pm 1} \in \mathcal{R}$, it follows that $x_{f-1}^\gamma \in \mathcal{R}$. Since $\left(\varphi(c_0)\left(\frac{q}{p}\right)^{\ell_f}\right)^{\pm 1} \in \mathcal{R}$, it follows that $(a_{f-1})^{\pm 1} \in \mathcal{R}$. Similarly y_{f-1}^γ and $(b_{f-1})^{\pm 1} \in \mathcal{R}$. The lemma follows by induction. $\square$

To define the polynomials z_i we will also need the following lemma.

LEMMA 5.8. *If $\alpha \in \pi^\ell \mathcal{O}_E[[\pi]]$ and $0 \leq k \leq \ell$ is an integer, then $\frac{\varphi(\alpha)}{(\gamma q)^k} \in \pi^\ell \mathcal{O}_E[[\pi]]$.*

Proof. Since $\frac{\gamma q}{q} \equiv 1 \mathrm{mod}\ \pi$, it suffices to prove that $\frac{\varphi(\alpha)}{q^k} \in \pi^\ell \mathcal{O}_E[[\pi]]$. Let $\alpha = \pi^\ell \beta$ for some $\beta \in \mathcal{O}_E[[\pi]]$. We have $\varphi\left(\frac{\alpha}{\pi^k}\right) = \varphi(\pi)^{\ell-k} \varphi(\beta) = q^{\ell-k}\pi^{\ell-k}\varphi(\beta)$. Hence $\frac{\varphi(\alpha)}{q^k} = \pi^k \varphi\left(\frac{\alpha}{\pi^k}\right) = \pi^k q^{\ell-k}\pi^{\ell-k}\varphi(\beta) = \pi^\ell q^{\ell-k}\varphi(\beta) \in \pi^\ell \mathcal{O}_E[[\pi]]$. $\square$

PROPOSITION 5.9. *Let $k = \max\{k_0, k_1, ..., k_{f-1}\}$, let $\ell \geq k$ be a fixed integer and let $m_\ell = \lfloor \frac{\ell-1}{p-1} \rfloor$. There exist polynomials $z_i \in \mathbb{Z}_p[\pi]$ with $\deg_\pi z_i \leq \ell - 1$ such that $z_i \equiv p^{m_\ell} \mathrm{mod}\ \pi$ with the following properties:*

(i) $G_\gamma^{(\ell)} \equiv \overrightarrow{Id} \mathrm{mod}\ \vec{\pi}$;

(ii) $G_\gamma^{(\ell)} - \Pi(\mathcal{S})\varphi(G_\gamma^{(\ell)})\gamma(\Pi(\mathcal{S})^{-1}) \in \vec{\pi}^\ell M_2^{\mathcal{S}}$ *for each* $\gamma \in \Gamma_K$.

Proof. Suppose that P_i is of type 2 and $\alpha_i = X_i\varphi(z_i)$ for some polynomial z_i to be defined. Then $\beta_i = 1$ and $\beta_i\varphi(y_i^\gamma) = x_{i-1}^\gamma(\gamma\beta_i)$ implies that $x_{i-1}^\gamma = \varphi(y_i^\gamma)$. We need

$$X_i\left(\varphi(z_i)\varphi(x_i^\gamma) - x_{i-1}^\gamma\varphi(\gamma z_i)\right)\frac{1}{(\gamma q)^{k_i}} \in \pi^\ell\mathcal{O}_E[[\pi, X_0, ..., X_{f-1}]]$$

for all $\gamma\in\Gamma_K$. By Lemma 5.8 it suffices to define z_i so that $z_ix_i^\gamma - y_i^\gamma\gamma z_i \in \pi^\ell\mathcal{O}_E[[\pi]]$ for all $\gamma\in\Gamma_K$. Since $x_i^\gamma \in 1+\pi\mathbb{Z}_p[[\pi]]$ for all $\gamma\in\Gamma_K$, this is equivalent to $z_i - \frac{y_i^\gamma}{x_i^\gamma}\gamma z_i \in \pi^\ell\mathcal{O}_E[[\pi]]$. By Lemma 5.7 we have $\frac{y_i^\gamma}{x_i^\gamma} = \frac{b}{\gamma b}$, where $b = a_i(b_i)^{-1}\in\mathcal{R}$. Since $\frac{y_i^\gamma}{x_i^\gamma}\in 1+\pi\mathbb{Z}_p[[\pi]]$, the existence of the z_i follows from Lemmata 5.6 and 5.7. The proof for P_i of type 1, 3 and 4 is identical. ☐

PROPOSITION 5.10. *If* $\alpha(\pi) = \sum_{n=0}^{\infty}\alpha_n\pi^n \in \mathbb{Q}_p[[\pi]]$ *is such that* $\mathrm{v_p}(\alpha_i)\geq 0$ *for all* $i = 0, 1, 2, ..., p-2$ *and* $\mathrm{v_p}(\alpha_{p-1}) > -1$, *then the first* $p-1$ *coefficients of* $\alpha(\pi)^p$ *are in* $\mathbb{Z}_p$. *In particular, the first* $p-1$ *coefficients of the* p*-th power of any element of* $\mathcal{R}$ *are in* $\mathbb{Z}_p$.

Proof. Follows easily using the binomial expansion. ☐

PROPOSITION 5.11. *If* $k_i = p$ *for all* i, *then there exist polynomials* $z_i\in\mathbb{Z}_p[\pi]$ *with* $\deg_\pi z_i \leq p-1$ *such that* $z_i \equiv 1 \bmod \pi$, *and such that* $G_\gamma^{(p)} - \Pi(\mathcal{S})\varphi(G_\gamma^{(p)})\gamma(\Pi(\mathcal{S})^{-1}) \in \vec{\pi}^p M_2^{\mathcal{S}}$ *for any* $\gamma\in\Gamma_K$.

Proof. Assume that P_i is of type 2 and let x_0^γ and y_0^γ be as in the proof of Proposition 5.9. First we notice that the exponents s_i and s_i' in formulas 5.5 or 5.8 and 5.9 for the x_0^γ and y_0^γ are either 0 or p. With the notation of Lemma 5.7 we have $\frac{y_0^\gamma}{x_0^\gamma} = c_0\left(\gamma c_0^{-1}\right)$, where $c_0 = a_0^{-1}b_0$. The formulas for a_0^{-1} and b_0 in the proof of Lemma 5.7 imply that they are both p-th powers of elements of $\mathcal{R}$. From the same formulas and Lemma 3.2 it follows that $a_0^{-1}(0) = b_0(0) = 1$. By Lemma 5.10, $c_0 = z_0 + a$ for some polynomial $z_0\in\mathbb{Z}_p[\pi]$ of degree $\leq p-1$ and constant term 1 and some $a\in\pi^p\mathbb{Q}_p[[\pi]]$. For any $\gamma\in\Gamma_K$, $z_0 - \frac{y_0^\gamma}{x_0^\gamma}\gamma z_0 = c_0 - a - c_0\left(\gamma c_0^{-1}\right)(\gamma c_0 - \gamma a) = c_0\left(\gamma c_0^{-1}\right)\gamma a - a \in \pi^p\mathbb{Q}_p[[\pi]]$. Since $\frac{y_0^\gamma}{x_0^\gamma}\in 1+\pi\mathbb{Z}_p[[\pi]]$ and $z_0\in\mathbb{Z}_p[\pi]$, $z_0 - \frac{y_0^\gamma}{x_0^\gamma}\gamma z_0 \in \mathbb{Z}_p[[\pi]]\cap\pi^p\mathbb{Q}_p[[\pi]] = \pi^p\mathbb{Z}_p[[\pi]]$. The proof for the other z_i is similar, using formulas 5.3 and noticing that $\left(\frac{q}{\gamma q}\right)^{\pm 1}\in 1+\pi\mathbb{Z}_p[[\pi]]$. The proof for P_i of type 1, 3 or 4 is identical. ☐

REMARK 5.12. *If* $k_i = p$ *for all* i, *then there exist polynomials* $z_i\in\mathbb{Z}_p[\pi]$ *with* $\deg_\pi z_i\leq p-1$ *and* $z_i\equiv 1 \bmod \pi$ *which satisfy properties (i) and (ii) of Proposition 5.9. This follows immediately from Proposition 5.11.*

Next, we explicitly determine when $\mathrm{Tr}(Q_f)\notin\bar{\mathbb{Q}}_p$. We first need some definitions.

Definition 5.13. *(i) We define* C_1 *to be the set of* f*-tuples* $(P_1, P_2, ..., P_f)$ *where the types of the matrices* P_i *are chosen as follows:* $P_1 \in \{t_2, t_3\}$*. For* $i = 2, 3, ..., f-1$, $P_i \in \{t_2, t_3\}$ *if an even number of coordinates of* $(P_1, P_2, ..., P_{i-1})$ *is of even type, and* $P_i \in \{t_1, t_4\}$ *if an odd number of coordinates of* $(P_1, P_2, ..., P_{i-1})$ *is of even type. Finally,* $P_0 = t_3$ *if an even number of coordinates of* $(P_1, P_2, ..., P_{f-1})$ *is of even type, and* $P_0 = t_4$ *otherwise.*

(ii) We define C_2 *to be the set of* f*-tuples* $(P_1, P_2, ..., P_f)$ *where the types of the matrices* P_i *are chosen as follows:* $P_1 \in \{t_1, t_4\}$*. For* $i = 2, 3, ..., f-1$, $P_i \in \{t_1, t_4\}$ *if an even number of coordinates of* $(P_1, P_2, ..., P_{i-1})$ *is of even type, and* $P_i \in \{t_2, t_3\}$ *if an odd number of coordinates of* $(P_1, P_2, ..., P_{i-1})$ *is of even type. Finally,* $P_0 = t_1$ *if an even number of coordinates of* $(P_1, P_2, ..., P_{f-1})$ *is of even type, and* $P_0 = t_2$ *otherwise.*

In Definition 5.13 the type of the matrix P_0 has been chosen so that an even number of coordinates of the f-tuple $(P_1, P_2, ..., P_{f-1}, P_0)$ is of even type.

Definition 5.14. *(i) We define* C_1^* *to be the set of* f*-tuples* $(P_1, P_2, ..., P_f)$ *where the types of the matrices* P_i *are chosen as follows:* $P_1 \in \{t_2, t_3\}$*. For* $i = 2, 3, ..., f-1$, $P_i \in \{t_2, t_3\}$ *if an even number of coordinates of* $(P_1, P_2, ..., P_{i-1})$ *is of even type, and* $P_i \in \{t_1, t_4\}$ *if an odd number of coordinates of* $(P_1, P_2, ..., P_{i-1})$ *is of even type. Finally,* $P_0 = t_2$ *if an even number of coordinates of* $(P_1, P_2, ..., P_{f-1})$ *is of even type, and* $P_0 = t_1$ *otherwise.*

(ii) We define C_2^* *to be the set of* f*-tuples* $(P_1, P_2, ..., P_f)$ *where the types of the matrices* P_i *are chosen as follows:* $P_1 \in \{t_1, t_4\}$*. For* $i = 2, 3, ..., f-1$, $P_i \in \{t_1, t_4\}$ *if an even number of coordinates of* $(P_1, P_2, ..., P_{i-1})$ *is of even type, and* $P_i \in \{t_2, t_3\}$ *if an odd number of coordinates of* $(P_1, P_2, ..., P_{i-1})$ *is of even type. Finally,* $P_0 = t_4$ *if an even number of coordinates of* $(P_1, P_2, ..., P_{f-1})$ *is of even type, and* $P_0 = t_3$ *otherwise.*

In Definition 5.14 the type of the matrix P_0 has been chosen so that an odd number of coordinates of the f-tuple $(P_1, P_2, ..., P_{f-1}, P_0)$ is of even type.

Lemma 5.15. *Assume that* $f \geq 2$ *and, as before, let* $Q_f = P_1 P_2 \cdots P_f$*.*

(i) If $(P_1, P_2, ..., P_f) \in C_1^*$*, then* $Q_f = \begin{pmatrix} \alpha & \beta \\ \gamma & 0 \end{pmatrix}$ *with* β, γ *nonconstant polynomials in* $X_1, X_2, ..., X_f$ *(with* $X_f = X_0$*), linearly independent over* $\bar{\mathbb{Q}}_p$*, and* α *nonscalar.*

(ii) If $(P_1, P_2, ..., P_f) \in C_2^*$*, then* $Q_f = \begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix}$ *with* β, γ *nonconstant polynomials in* $X_1, X_2, ..., X_f$*, linearly independent over* $\bar{\mathbb{Q}}_p$*, and* δ *nonscalar.*

(iii) *If* $(P_1, P_2, ..., P_f) \in C_1$, *then* $Q_f = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$ *with* β *a nonzero polynomial in* $X_1, X_2, ..., X_f$, *and* α, δ *nonzero scalars.*

(iv) *If* $(P_1, P_2, ..., P_f) \in C_2$, *then* $Q_f = \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}$ *with* γ *a nonzero polynomial in* $X_1, X_2, ..., X_f$, *and* α, δ *nonzero scalars.*

Proof. Follows easily by induction on f. □

LEMMA 5.16. *Assume that* $f \geq 2$.

(i) *If an odd number of coordinates of* $(P_1, P_2, ..., P_f)$ *is of even type, then* Q_f *has one of the following forms:*

(a) $Q_f = \begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix}$ *with* β, γ *nonconstant polynomials in* $X_1, X_2, ..., X_f$, *linearly independent over* $\bar{\mathbb{Q}}_p$, *and* δ *nonscalar. This case occurs if and only if* $(P_1, P_2, ..., P_f) \in C_2^*$.

(b) $Q_f = \begin{pmatrix} \alpha & \beta \\ \gamma & 0 \end{pmatrix}$ *with* β, γ *nonconstant polynomials in* $X_1, X_2, ..., X_f$, *linearly independent over* $\bar{\mathbb{Q}}_p$, *and* α *nonscalar. This case occurs if and only if* $(P_1, P_2, ..., P_f) \in C_1^*$.

(c) *In any other case,* $Q_f = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ *with* β, γ *nonconstant polynomials in* $X_1, X_2, ..., X_f$, *linearly independent over* $\bar{\mathbb{Q}}_p$, $\alpha\delta \neq 0$, *and* $\mathrm{Tr}\,(Q_f)$ *nonscalar.*

(ii) *If an even number of coordinates of* $(P_1, P_2, ..., P_f)$ *is of even type, then* Q_f *has one of the following forms:*

(d) $Q_f = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$ *with* β *a nonzero polynomial in* $X_1, X_2, ..., X_f$, *and* α, δ *nonzero scalars. This case occurs if and only if* $(P_1, P_2, ..., P_f) \in C_1$.

(e) $Q_f = \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}$ *with* γ *a nonzero polynomial in* $X_1, X_2, ..., X_f$, *and* α, δ *nonzero scalars. This case occurs if and only if* $(P_1, P_2, ..., P_f) \in C_2$.

(f) *In any other case,* $Q_f = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ *with* β, γ *nonconstant polynomials in* $X_1, X_2, ..., X_f$, *linearly independent over* $\bar{\mathbb{Q}}_p$, $\alpha\gamma \neq 0$ *and* $\mathrm{Tr}\,(Q_f)$ *is nonscalar.*

Proof. By induction on f. If $f=2$ the proof of the lemma is by a direct computation. Suppose $f\geq 3$. Case (i). An odd number of coordinates of $(P_1,P_2,...,P_f)$ is of even type.
(a) If an odd number of coordinates of $(P_1,P_2,...,P_{f-1})$ is of even type, then $P_0\in\{t_1,t_3\}$. We have the following three subcases:
(α) $Q_{f-1}=\begin{pmatrix}0&\beta\\ \gamma&\delta\end{pmatrix}$ with β,γ nonconstant polynomials in $X_1,X_2,,,.X_{f-1}$, linearly independent over $\bar{\mathbb{Q}}_p$, and δ nonscalar. If $P_0=t_1$, then Q_f is as in case (c), and by Lemma 5.15 $(P_1,P_2,...,P_f)\notin C_1^*\cup C_2^*$. If $P_0=t_3$, then Q_f is as in case (a). By the inductive hypothesis $(P_1,P_2,...,P_{f-1})\in C_2^*$, and since $P_0=t_3$, $(P_1,P_2,...,P_f)\in C_2^*$.
(β) $Q_{f-1}=\begin{pmatrix}\alpha&\beta\\ \gamma&0\end{pmatrix}$ with β,γ nonconstant polynomials in $X_1,X_2,,,.X_{f-1}$, linearly independent over $\bar{\mathbb{Q}}_p$, and α nonscalar. If $P_0=t_1$, then Q_f is as in case (b). By the inductive hypothesis $(P_1,P_2,...,P_{f-1})\in C_1^*$, and since $P_0=t_1$, $(P_1,P_2,...,P_f)\in C_1^*$. If $P_0=t_3$, then Q_f is as in case (c), and by Lemma 5.15 $(P_1,P_2,...,P_f)\notin C_1^*\cup C_2^*$.
(γ) $Q_{f-1}=\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}$ with β,γ nonconstant polynomials in $X_1,X_2,,,.X_{f-1}$, linearly independent over $\bar{\mathbb{Q}}_p$, $\alpha\delta\neq 0$, and $\mathrm{Tr}\,(Q_f)$ nonscalar. If $P_0\in\{t_1,t_3\}$ then Q_f is as in case (c), and by Lemma 5.15 $(P_1,P_2,...,P_f)\notin C_1^*\cup C_2^*$.
(b) If an even number of coordinates of $(P_1,P_2,...,P_{f-1})$ is of even type, then $P_0\in\{t_2,t_4\}$. We have the following three subcases:
(α) $Q_{f-1}=\begin{pmatrix}\alpha&\beta\\ 0&\delta\end{pmatrix}$ with β a nonzero polynomial in $X_1,X_2,,,.X_{f-1}$, and α,δ nonzero scalars. If $P_0=t_2$, then Q_f is as in case (b). Since $(P_1,P_2,...,P_{f-1})\in C_1$ and $P_0=t_2$, $(P_1,P_2,...,P_f)\in C_1^*$. If $P_0=t_4$, then Q_f is as in case (c), and by Lemma 5.15 $(P_1,P_2,...,P_f)\notin C_1^*\cup C_2^*$.
(β) $Q_{f-1}=\begin{pmatrix}\alpha&0\\ \gamma&\delta\end{pmatrix}$ with γ a nonzero polynomial in $X_1,X_2,,,.X_{f-1}$, and α,δ nonzero scalars. If $P_0=t_2$, then Q_f is as in case (c), and by Lemma 5.15 $(P_1,P_2,...,P_f)\notin C_1^*\cup C_2^*$. If $P_0=t_4$, then Q_f is as in case (a). Since $(P_1,P_2,...,P_{f-1})\in C_2$ and $P_0=t_4$, $(P_1,P_2,...,P_f)\in C_2^*$.
(γ) $Q_{f-1}=\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}$ with β,γ nonconstant polynomials in $X_1,X_2,,,.X_f$, linearly independent over $\bar{\mathbb{Q}}_p$, $\alpha\gamma\neq 0$ and $\mathrm{Tr}\,(Q_f)$ is nonscalar. Then $(P_1,P_2,...,P_{f-1})\notin C_1\cup C_2$. If $P_0\in\{t_2,t_4\}$, then Q_f is as in case (c), and by Lemma 5.15 $(P_1,P_2,...,P_f)\notin C_1^*\cup C_2^*$.
Case (ii). An even number of coordinates of $(P_1,P_2,...,P_f)$ is of even type. The rest of the lemma is proved by a case-by-case analysis similar to that of Case (i). ☐

COROLLARY 5.17. $\mathrm{Tr}\,(Q_f)\in\bar{\mathbb{Q}}_p$ *if and only if* $(P_1,P_2,...,P_{f-1},P_0)\in C_1\cup C_2$.

REMARK 5.18. *If* $(P_1,P_2,...,P_f)\in C_1\cup C_2$, *the filtered* φ*-modules* $\mathbb{D}_{\vec{k}}^{\vec{i}}\,(\vec{0})$

are weakly admissible and the corresponding crystalline representation is split-reducible and ordinary (see §6.3). The filtered φ-modules $\mathbb{D}_{\vec{k}}^{\vec{i}}\ (\vec{\alpha})$ make sense for any $\vec{\alpha} \in \mathcal{O}_E^f$. One can check by induction that $\mathrm{Tr}\left(\varphi^f\right) = 1 + p^{\sum_{i=0}^{f-1} k_i}$*, therefore whenever $\mathbb{D}_{\vec{k}}^{\vec{i}}\ (\vec{\alpha})$ is weakly admissible the corresponding crystalline representation is reducible (see Proposition 6.5). Since we have not constructed the Wach modules which give rise to these filtered modules, weak admissibility is not automatic and has to be checked.*

We now turn our attention to condition (iv) of Lemma 4.4. By Proposition 5.1 the problematic cases are those with $\ell = k$, all the weights k_i equal and $Q_f \in \{E_{11}, E_{22}\}$. We have the following.

LEMMA 5.19. *If $\bar{Q}_f = E_{11}$ then $(P_1, ..., P_f) \in C_1$; (ii) If $\bar{Q}_f = E_{22}$, then $(P_1, ..., P_f) \in C_2$.*

Proof. By induction on f. First, we notice that

$$P \bmod (p \cdot Id,\ X_i \cdot Id) = \begin{cases} E_{22} \text{ if } P = t_1, \\ E_{12} \text{ if } P = t_2, \\ E_{11} \text{ if } P = t_3, \\ E_{21} \text{ if } P = t_4. \end{cases}$$

Suppose that $\bar{Q}_f = E_{11}$ and $f = 2$. Then $P_1 \in \{t_2, t_3\}$. If $P_1 = t_2$ then $P_0 = t_4$ and if $P_1 = t_3$ then $P_0 = t_3$. Suppose $\bar{Q}_f = E_{11}$ and $f > 2$. Then $\bar{Q}_{f-1} = E_{11}$ and $P_f = t_3$ or $\bar{Q}_{f-1} = E_{12}$ and $P_f = t_4$. In the first case the inductive hypothesis implies that $(P_1, P_2, ..., P_{f-1}) \in C_1$ and $P_f = t_3$. If an even number of coordinates of $(P_1, P_2, ..., P_{f-2})$ is of even type, then $P_{f-1} = t_3$. In this case an even number of coordinates of $(P_1, P_2, ..., P_{f-1})$ is of even type and $P_f = t_3$, hence $(P_1, ..., P_f) \in C_1$. If an odd number of coordinates of $(P_1, P_2, ..., P_{f-2})$ is of even type, then $P_{f-1} = t_4$. In this case an even number of coordinates of $(P_1, P_2, ..., P_{f-1})$ is of even type and $P_f = t_3$, hence $(P_1, ..., P_f) \in C_1$. Now assume that $\bar{Q}_{f-1} = E_{12}$ and $P_f = t_4$. This implies that either $\bar{Q}_{f-2} = E_{12}$, $P_{f-1} = t_4$ and $P_f = t_4$ which is absurd since in this case $\bar{Q}_f = 0$, or $\bar{Q}_{f-2} = E_{11}$, $P_{f-1} = t_2$ and $P_f = t_4$. If $f = 3$, then $P_1 = t_3$, $P_2 = t_2$, $P_3 = t_4$ and the lemma holds. If $f \geq 4$ and an even number of coordinates $(P_1, P_2, ..., P_{f-3})$ is of even type, then $P_{f-2} = t_3$, $P_{f-1} = t_2$ and $P_f = t_4$. Then an odd number of coordinates $(P_1, P_2, ..., P_{f-1})$ is of even type and $P_f = t_4$, hence $(P_1, ..., P_f) \in C_1$. If an odd number of coordinates $(P_1, P_2, ..., P_{f-3})$ is of even type, then $P_{f-2} = t_4$, $P_{f-1} = t_2$ and $P_f = t_4$. Then an odd number of coordinates $(P_1, P_2, ..., P_{f-1})$ is of even type and $P_f = t_4$, hence $(P_1, ..., P_f) \in C_1$. Part (ii) is proved similarly. □

COROLLARY 5.20. *If $(P_1, P_2, ..., P_f) \in \mathcal{P}$ and $\mathrm{Tr}\,(Q_f) \notin \bar{\mathbb{Q}}_p$, then the operator*

$$\overline{H} \mapsto \overline{H - Q_f H(p^{f\ell} Q_f^{-1})} : \widetilde{M}_2 \to \widetilde{M}_2$$

is surjective.

5.2 CORRESPONDING FAMILIES OF RANK TWO FILTERED φ-MODULES

Let $\Pi^{\vec{i}}(\mathcal{S}) = (\Pi_1(X_1), \Pi_2(X_2), ..., \Pi_{f-1}(X_{f-1}), \Pi_0(X_0))$ with $\vec{i} \in \{1,2,3,4\}^f$ and matrices $\Pi_i(X_i)$ as in Definition 5.4. The definition of the Π_i and $P_i = \Pi_i \mathrm{mod}\ \pi$ depends on the choice of the z_i in Proposition 5.9 and therefore on ℓ. For the rest of the paper we assume that $(P_1, P_2, ..., P_0) \notin C_1 \cup C_2$ and we choose $\ell = k = \max\{k_0, k_1, ..., k_{f-1}\}$.

PROPOSITION 5.21. *For any $\gamma \in \Gamma_K$, there exists a unique matrix $G_\gamma(\mathcal{S}) = G_\gamma(\mathcal{S}) \in M_2^{\mathcal{S}}$ such that:*

(i) $G_\gamma(\mathcal{S}) \equiv \overrightarrow{Id} \mathrm{mod}\ \vec{\pi}$;

(ii) $\Pi^{\vec{i}}(\mathcal{S})\varphi(G_\gamma(\mathcal{S})) = G_\gamma(\mathcal{S})\gamma(\Pi^{\vec{i}}(\mathcal{S}))$.

Proof. Conditions (1) and (2) of Lemma 4.4 are satisfied by Proposition 5.9. Condition (3) of Lemma 4.4 is satisfied by the assumption that $(P_1, P_2, ..., P_0) \notin C_1 \cup C_2$ and Corollaries 5.3 and 5.17. Finally, condition (4) of Lemma 4.4 is satisfied by Proposition 5.1 and Lemma 5.19. The proposition follows from Lemma 4.4. □

For any $\vec{a} \in \mathfrak{m}_E^f$ we equip the module $\mathbb{N}_{\vec{k}}^{\vec{i}}(\vec{a}) = \left(\mathcal{O}_E[[\pi]]^{|\tau|}\right)\eta_1 \bigoplus \left(\mathcal{O}_E[[\pi]]^{|\tau|}\right)\eta_2$ with semilinear φ and Γ_K-actions defined as in Proposition 4.6. For any $\vec{a} \in \mathfrak{m}_E^f$ we consider the matrices of $\mathrm{GL}_2\left(E^{|\tau|}\right)$ obtained from the matrices $P^{\vec{i}}(\vec{a}) = (P_1(X_1), P_2(X_2), ..., P_{f-1}(X_{f-1}), P_0(X_0))$ by substituting $X_j = a_j$ in $P_j(X_j)$. We define families of filtered φ-modules $\mathbb{D}_{\vec{k}}^{\vec{i}}(\vec{a}) = \left(E^{|\tau|}\right)\eta_1 \bigoplus \left(E^{|\tau|}\right)\eta_2$ with Frobenius endomorphisms given by $(\varphi(\eta_1), \varphi(\eta_2)) = (\eta_1, \eta_2)P^{\vec{i}}(\vec{a})$, and filtrations given by

$$\mathrm{Fil}^{\mathrm{j}}(\mathbb{D}_{\vec{k}}^{\vec{i}}(\vec{a})) = \begin{cases} E^{|\tau|}\eta_1 \bigoplus E^{|\tau|}\eta_2 & \text{if } j \leq 0, \\ E^{|\tau_{I_0}|}\left(\vec{x}\eta_1 + \vec{y}\eta_2\right) & \text{if } 1 \leq j \leq w_0, \\ E^{|\tau_{I_1}|}\left(\vec{x}\eta_1 + \vec{y}\eta_2\right) & \text{if } 1 + w_0 \leq j \leq w_1, \\ \quad\cdots\cdots & \\ E^{|\tau_{I_{t-1}}|}\left(\vec{x}\eta_1 + \vec{y}\eta_2\right) & \text{if } 1 + w_{t-2} \leq j \leq w_{t-1}, \\ \qquad 0 & \text{if } j \geq 1 + w_{t-1}, \end{cases} \tag{5.10}$$

where $\vec{x} = (x_0, x_1, ..., x_{f-1})$ and $\vec{y} = (y_0, y_1, ..., y_{f-1})$ with

$$(x_i, y_i) = \begin{cases} (1, -\alpha_i) & \text{if } P_i \text{ has type 1 or 2,} \\ (-\alpha_i, 1) & \text{if } P_i \text{ has type 3 or 4,} \end{cases} \tag{5.11}$$

and $\alpha_i = a_i z_i(0)$ for all i. Since $\ell = k$, Remark 5.12 implies that $\alpha_i \in p^m \mathfrak{m}_E$ for all i, where

$$m := \begin{cases} \lfloor \frac{k-1}{p-1} \rfloor & \text{if } k \geq p \text{ and } k_i \neq p \text{ for some } i, \\ 0 & \text{if } k \leq p-1 \text{ or } k_i = p \text{ for all } i. \end{cases}$$

PROPOSITION 5.22. *For any* $\vec{a} \in \mathfrak{m}_E^f$ *the filtered* φ*-modules* $(\mathbb{D}_{\vec{k}}^{\vec{i}}(\vec{a}), \varphi)$ *defined above are weakly admissible and* $\mathbb{D}_{\vec{k}}^{\vec{i}}(\vec{a}) \simeq E^{|\tau|} \bigotimes_{\mathcal{O}_E^{|\tau|}} (\mathbb{N}_{\vec{k}}^{\vec{i}}(\vec{a})/\pi\mathbb{N}_{\vec{k}}^{\vec{i}}(\vec{a}))$ *as filtered* φ*-modules over* $E^{|\tau|}$.

Proof. By Theorem 2.4, $\vec{x}\eta_1 + \vec{y}\eta_2 \in \mathrm{Fil}^{\mathrm{j}}(\mathbb{N}_{\vec{k}}^{\vec{i}}(\vec{a}))$ if and only if $\varphi(\vec{x})\varphi(\eta_1)+\varphi(\vec{y})\varphi(\eta_2) \in q^j\mathbb{N}_{\vec{k}}^{\vec{i}}(\vec{a})$ or equivalently

$$e_i\varphi(\vec{x})\varphi(\eta_1) + e_i\varphi(\vec{y})\varphi(\eta_2) \in q^j e_i\mathbb{N}_{\vec{k}}^{\vec{i}}(\vec{a}) \text{ for all } i \in I_0, \tag{5.12}$$

with the idempotents e_i as in §1.1. We fix some $i \in I_0$ and calculate in the case where Π_i is of type 2. Then $\Pi_i(a_i) = \begin{pmatrix} 0 & c_i q^{k_i} \\ 1 & a_i\varphi(z_i) \end{pmatrix}$ and equation 5.12 is equivalent to $\begin{cases} q^j \mid \varphi(y_i)q^{k_i}, \\ q^j \mid \varphi(x_i + y_i a_i z_i). \end{cases}$ We use that $q^j \mid \varphi(x)$ if and only if $\pi^j \mid x$ for any $x \in \mathcal{O}_E[[\pi]]$. If $j \geq 1 + k_i$, then $x_i, y_i \equiv 0 \mathrm{mod}\ \pi$. If $1 \leq j \leq k_i$, the system above is equivalent to $\pi^j \mid x_i + y_i a_i z_i$. Since $a_i z_i \equiv \alpha_i \mathrm{mod}\ \pi$,

$$e_i\vec{x}\eta_1 + e_i\vec{y}\eta_2 + \pi\mathbb{N}_{\vec{k}}^{\vec{i}}(\vec{a}) = \begin{cases} \alpha_i\bar{y}_i e_i\eta_1 + \bar{y}_i e_i\eta_2 + \pi\mathbb{N}_{\vec{k}}^{\vec{i}}(\vec{a}) & \text{if } 1 \leq j \leq k_i, \\ 0 & \text{if } j \geq k_i \end{cases}$$

where $\bar{y}_i = y_i \mathrm{mod}\ \pi$ can be any element of $\mathcal{O}_E$. Since $\mathrm{Fil}^0(\mathbb{N}_{\vec{k}}^{\vec{i}}(\vec{a})/\pi\mathbb{N}_{\vec{k}}^{\vec{i}}(\vec{a})) = (\mathcal{O}_E^{|\tau|})\eta_1 \bigoplus (\mathcal{O}_E^{|\tau|})\eta_2$, we get

$$e_i\mathrm{Fil}^{\mathrm{j}}(\mathbb{N}_{\vec{k}}^{\vec{i}}(\vec{a})/\pi\mathbb{N}_{\vec{k}}^{\vec{i}}(\vec{a})) = \begin{cases} e_i(\mathcal{O}_E^{|\tau|})\eta_1 \bigoplus e_i(\mathcal{O}_E^{|\tau|})\eta_2 & \text{if } j \leq 0, \\ e_i(\mathcal{O}_E^{|\tau|})(\vec{x}^i\eta_1 + \vec{y}^i\eta_2) & \text{if } 1 \leq j \leq k_i, \\ 0 & \text{if } j \geq 1 + k_i, \end{cases}$$

with $e_i\vec{x}^i = (0, ..., x_i, ..., 0)$, $e_i\vec{y}^i = (0, ..., y_i, ..., 0)$ and $(x_i, y_i) = (-\alpha_i, 1)$. Calculating for the other choices of $\Pi_i(a_i)$ we see that for all $i \in I_0$, (x_i, y_i) is as in formula 5.10. Since $\mathrm{Fil}^{\mathrm{j}}(\mathbb{N}_{\vec{k}}^{\vec{i}}(\vec{a})/\pi\mathbb{N}_{\vec{k}}^{\vec{i}}(\vec{a})) = \bigoplus_{i=0}^{f-1} e_i\mathrm{Fil}^{\mathrm{j}}(\mathbb{N}_{\vec{k}}^{\vec{i}}(\vec{a})/\pi\mathbb{N}_{\vec{k}}^{\vec{i}}(\vec{a}))$, arguing as in the proof of Proposition 3.4 we get

$$\mathrm{Fil}^{\mathrm{j}}(\mathbb{N}_{\vec{k}}^{\vec{i}}(\vec{a})/\pi\mathbb{N}_{\vec{k}}^{\vec{i}}(\vec{a})) = \begin{cases} (\mathcal{O}_E^{|\tau|})\eta_1 \bigoplus (\mathcal{O}_E^{|\tau|})\eta_2 & \text{if } j \leq 0, \\ (\mathcal{O}_E^{|\tau|})f_{I_0}(\vec{x}\eta_1 + \vec{y}\eta_2) & \text{if } 1 \leq j \leq w_0, \\ (\mathcal{O}_E^{|\tau|})f_{I_1}(\vec{x}\eta_1 + \vec{y}\eta_2) & \text{if } 1 + w_0 \leq j \leq w_1, \\ \qquad \cdots\cdots & \\ (\mathcal{O}_E^{|\tau|})f_{I_{t-1}}(\vec{x}\eta_1 + \vec{y}\eta_2) & \text{if } 1 + w_{t-2} \leq j \leq w_{t-1}, \\ 0 & \text{if } j \geq 1 + w_{t-1}, \end{cases}$$

with $\vec{x} = (x_0, x_1, ..., x_{f-1})$ and $\vec{y} = (y_0, y_1, ..., y_{f-1})$ and (x_i, y_i) as in formula 5.10. The isomorphism $\mathbb{D}_{\vec{k}}^{\vec{i}}(\vec{a}) \simeq E^{|\tau|} \bigotimes_{\mathcal{O}_E^{|\tau|}} (\mathbb{N}_{\vec{k}}^{\vec{i}}(\vec{a})/\pi\mathbb{N}_{\vec{k}}^{\vec{i}}(\vec{a}))$ is now obvious. $\square$

The crystalline representation corresponding to $\mathbb{D}_{\vec{k}}^{\vec{i}}(\vec{a})$ is denoted by $V_{\vec{k},\vec{a}}^{\vec{i}}$.

6 REDUCTIONS OF CRYSTALLINE REPRESENTATIONS

In this section we explicitly compute the semisimplified modulo p reductions of the families of crystalline representations constructed in §5. We will need the following lemma.

LEMMA 6.1. *Let F be any field, G any group and H any finite index subgroup. Let V be an irreducible finite-dimensional FG-module whose restriction to H contains some FH-submodule W with $\dim_F V = [G : H] \dim_F W$. Then $V \simeq \mathrm{Ind}_H^G (W)$.*

Proof. By Frobenius reciprocity there exists some nonzero $\alpha \in \mathrm{Hom}_{FG}(\mathrm{Ind}_H^G (W), V)$. It is an isomorphism because V is irreducible and $\mathrm{Ind}_H^G (W)$ and V have the same dimension over F. □

We start with the reductions of crystalline characters and reducible two-dimensional crystalline representations of G_K. The embeddings τ_i of K_f into E fixed in the introduction induce embeddings of residue fields $k_{K_f} \xrightarrow{\bar{\tau}_i} k_E$. The level f fundamental characters $\omega_{f,\bar{\tau}_i}$ of I_{K_f} are defined by composing the embeddings $\bar{\tau}_i$ with the homomorphism $I_{K_f} \to k_{K_f}^{\times}$ obtained from local class field theory, with uniformizers corresponding to geometric Frobenius elements. We recall the following lemma which follows immediately from [BDJ, Lemma 3.8], where the χ_i are as in §3.

LEMMA 6.2. *(i) $(\bar{\chi}_i)_{|I_{K_f}} = \omega_{f,\bar{\tau}_{i+1}}^{-1}$ for $i = 0, 1, ..., f-1$; (ii) $\omega_{f,\bar{\tau}_i} = \omega_{f,\bar{\tau}_0}^{p^i}$ for all i; (iii) $\omega_{2f,\bar{\tau}_0}^{1+p^f} = \omega_{f,\bar{\tau}_0}$; (iv) $\omega = \prod_{i \in I_0} \omega_{f,\bar{\tau}_i}$, where ω is the cyclotomic character modulo $\mathfrak{m}_E$.*

Our next goal is to compute the determinant of a two-dimensional crystalline representations in terms of its labeled Hodge-Tate weights. To do this, we will need some facts about weakly admissible filtered φ-modules which we briefly recall. For the missing details we refer to [Dou10]. We remark that similar results for odd p have been obtained by Imai in [Ima09].

PROPOSITION 6.3. *Let $(\mathbb{D}, \varphi)$ be a rank two F-semisimple, nonscalar filtered φ-module over $E^{|\tau|}$ with labeled Hodge-Tate weights $\{0, -k_i\}_{\tau_i}$. After enlarging E if necessary, there exists an ordered basis $\underline{\eta}$ of $\mathbb{D}$ over $E^{|\tau|}$ with respect to which the matrix of Frobenius takes the form $\mathrm{Mat}_{\underline{\eta}}(\varphi) = \mathrm{diag}(\vec{\alpha}, \vec{\delta})$ for some vectors $\vec{\alpha}, \vec{\delta} \in (E^{\times})^{|\tau|}$ with $\mathrm{Nm}_{\varphi}(\vec{\alpha}) \neq \mathrm{Nm}_{\varphi}(\vec{\delta})$. The filtration in the same basis has the form of formula 5.10 for some vectors $\vec{x}, \vec{y} \in E^{|\tau|}$ with $(x_i, y_i) \neq (0, 0)$ for all $i \in I_0$. We call such a basis $\underline{\eta}$ a standard basis of $(\mathbb{D}, \varphi)$. The Frobenius-fixed submodules are 0, $\mathbb{D}$, $\mathbb{D}_1 := \left(E^{|\tau|}\right) \eta_1$ and $\mathbb{D}_2 := \left(E^{|\tau|}\right) \eta_2$. The module $\mathbb{D}$*

is weakly admissible if and only if

$$(1)\ \mathrm{v_p}(\mathrm{Nm}_{\varphi}(\vec{\alpha})\mathrm{Nm}_{\varphi}(\vec{\delta})) = \sum_{i\in I_0} k_i;$$
$$(2)\ \mathrm{v_p}(\mathrm{Nm}_{\varphi}(\vec{\alpha})) \geq \sum_{\{i\in I_0:\ y_i=0\}} k_i;$$
$$(3)\ \mathrm{v_p}(\mathrm{Nm}_{\varphi}(\vec{\delta})) \geq \sum_{\{i\in I_0:\ x_i=0\}} k_i.$$

Assuming that $\mathbb{D}$ is weakly admissible,

(i) The filtered φ-module $\mathbb{D}$ is irreducible if and only if both the inequalities (2) *and* (3) *above are strict;*

(ii) The filtered φ-module $\mathbb{D}$ is split-reducible if and only if both inequalities (2) *and* (3) *are equalities, or equivalently $I_0^+ \cap J_{\vec{x}} \cap J_{\vec{y}} = \varnothing$. In this case, the only nontrivial weakly admissible submodules are $\mathbb{D}_i$, $i = 1, 2$, and we have $\mathbb{D} = \mathbb{D}_1 \bigoplus \mathbb{D}_2$;*

(iii) In any other case the filtered φ-module $\mathbb{D}$ is reducible, non-split.

In the Proposition above, if $\mathrm{v_p}(\mathrm{Nm}_{\varphi}(\vec{\alpha})) = \sum_{\{i\in I_0:\ y_i=0\}} k_i$, the only nontrivial weakly admissible submodule is $(\mathbb{D}_1, \varphi)$. If $\mathrm{v_p}(\mathrm{Nm}_{\varphi}(\vec{\delta})) = \sum_{\{i\in I_0:\ x_i=0\}} k_i$, the only nontrivial weakly admissible submodule is $(\mathbb{D}_2, \varphi)$. If $(\mathbb{D}, \varphi)$ is not F-semisimple, after extending E if necessary, there exists an ordered basis $\underline{\eta} = (\eta_1, \eta_2)$ of $\mathbb{D}$ over $E^{|\tau|}$ with respect to which the matrix of Frobenius takes the form

$$\mathrm{Mat}_{\underline{\eta}}(\varphi) = \begin{pmatrix} \vec{\alpha} & \vec{0} \\ \vec{\gamma} & \vec{\alpha} \end{pmatrix}$$

for some vectors $\vec{\alpha} \in (E^{\times})^{|\tau|}$ and $\vec{\gamma} \in E$ (see [Dou10, §2.1]). The filtration in this basis has the shape of formula 5.10. The filtered φ-module $(\mathbb{D}, \varphi)$ is weakly admissible if any only if $2 \cdot \mathrm{v_p}(\mathrm{Nm}_{\varphi}(\vec{\alpha})) = \sum_{i\in I_0} k_i$ and $\mathrm{v_p}(\mathrm{Nm}_{\varphi}(\vec{\alpha})) \geq \sum_{\{i\in I_0: x_i=0\}} k_i$. The corresponding crystalline representation is irreducible if and only if the latter inequality is strict and reducible, non-split otherwise. In this case, the only φ-stable weakly admissible submodule is $(\mathbb{D}_2, \varphi)$ (see also [Dou10, § 5.4]). If $(\mathbb{D}, \varphi)$ is F-scalar, there exists an ordered basis $\underline{\eta} = (\eta_1, \eta_2)$ of $\mathbb{D}$ over $E^{|\tau|}$ with respect to which $\mathrm{Mat}_{\underline{\eta}}(\varphi) = \mathrm{diag}\left(\alpha \cdot \vec{1}, \alpha \cdot \vec{1}\right)$ for some $\alpha \in E^{\times}$ and the filtration is as in formula 5.10. The only φ-stable submodules are the $\mathbb{D}_i$, $i = 1, 2$ and $\mathbb{D}(c) = \left(E^{|\tau|}\right)(\eta_1 + c \cdot \vec{1} \cdot \eta_2)$ for any $c \in E^{\times}$ (cf. [Dou10, §5.3]). To summarize, we have the following.

Proposition 6.4. *Let $(\mathbb{D},\varphi)$ be a reducible weakly admissible rank two filtered φ-module over $E^{|\tau|}$ with labeled Hodge-Tate weights $\{0,-k_i\}_{\tau_i}$. After enlarging E if necessary, there exists an ordered basis $\underline{\eta}=(\eta_1,\eta_2)$ of $\mathbb{D}$ over $E^{|\tau|}$ with respect to which the matrix of Frobenius takes the form* $\mathrm{Mat}_{\underline{\eta}}(\varphi)=\begin{pmatrix}\vec{\alpha} & \vec{0}\\ * & \vec{\delta}\end{pmatrix}$ *and is such that $\mathbb{D}_2=\left(E^{|\tau|}\right)\eta_2$ is a φ-stable, weakly admissible submodule.*

The following propositions which will be used in §§ 6.2 and 6.3.

Proposition 6.5. *A rank two weakly admissible effective filtered φ-module $(\mathbb{D},\varphi)$ with labeled Hodge-Tate weights $\{-k_i,0\}_{\tau_i}$ and $\mathrm{v_p}\left(\mathrm{Tr}(\varphi^f)\right)=0$ is reducible.*

Proof. If $\mathbb{D}$ is F-semisimple and nonscalar, see [Dou10, Corollary 7.2]. Suppose that this is not the case. Since we assume that $k_i>0$ for at least one i, for any F-scalar or non-F-semisimple filtered φ-module with labeled weights $\{-k_i,0\}_{\tau_i}$, $\mathrm{v_p}\left(\mathrm{Tr}(\varphi^f)\right)\neq 0$. Indeed, in this case there exists an ordered basis $\underline{\eta}$ of $\mathbb{D}$ over $E^{|\tau|}$ with respect to which the matrix of Frobenius takes the form

$$\mathrm{Mat}_{\underline{\eta}}(\varphi)=\begin{pmatrix}\vec{\alpha} & \vec{0}\\ \vec{\gamma} & \vec{\alpha}\end{pmatrix}$$

for some vectors $\vec{\alpha}\in(E^{\times})^{|\tau|}$ and $\vec{\gamma}\in E$ (see [Dou10, §2.1]). Weak admissibility implies that $2\cdot\mathrm{v_p}(\mathrm{Nm}_{\varphi}(\vec{\alpha}))=\sum\limits_{i\in I_0}k_i>0$ (see [Dou10, Propositions 4.3 and 4.4]), therefore $\mathrm{v_p}\left(\mathrm{Tr}(\varphi^f)\right)=\mathrm{v_p}\left(2\cdot\mathrm{Nm}_{\varphi}(\vec{\alpha})\right)>0$. ∎

The following lemma allows us to compute determinants of two-dimensional crystalline representations in terms of their labeled Hodge-Tate weights.

Lemma 6.6. *If $(\mathbb{D},\varphi)$ is a rank two weakly admissible filtered φ-module over K with E-coefficients and labeled Hodge-Tate weights $\{0,-k_i\}_{\tau_i}$, then $(\wedge^2_{E\otimes K}\mathbb{D},\wedge^2_{E\otimes K}\varphi)$ is weakly admissible with labeled Hodge-Tate weights $\{-k_i\}_{\tau_i}$.*

Proof. Let $\underline{\eta}=(\eta_1,\eta_2)$ be a standard basis of $(\mathbb{D},\varphi)$ such that $\mathrm{Mat}_{\underline{\eta}}(\varphi)$ is as in Proposition 6.4 and $\mathrm{Fil^j}\mathbb{D}$ as in Formula 5.10. Clearly $(\wedge^2\varphi)(\eta_1\wedge\eta_2)=\vec{\alpha}\cdot\vec{\delta}(\eta_1\wedge\eta_2)$. Since $\mathrm{Fil^j}(\mathbb{D}\wedge\mathbb{D})=\sum\limits_{j_1+j_2=j}(\mathrm{Fil^{j_1}}\mathbb{D}\wedge_{E\otimes K}\mathrm{Fil^{j_2}}\mathbb{D})$ and $J_{\vec{x}}\cup J_{\vec{y}}=I_0$, a simple computation yields

$$\mathrm{Fil^j}(\mathbb{D}\wedge\mathbb{D})=\begin{cases} E^{|\tau_{I_0}|}(\eta_1\wedge\eta_2) & \text{if } j\leq w_0,\\ E^{|\tau_{I_1}|}(\eta_1\wedge\eta_2) & \text{if } 1+w_0\leq j\leq w_1,\\ \qquad\cdots\cdots & \\ E^{|\tau_{I_{t-1}}|}(\eta_1\wedge\eta_2) & \text{if } 1+w_{t-2}\leq j\leq w_{t-1},\\ 0 & \text{if } j\geq 1+w_{t-1},\end{cases}$$

from which the statement about the labeled Hodge-Tate weights follows immediately. Weak admissibility is clear. ∎

Corollary 6.7. *If V is the crystalline representation corresponding to $\mathbb{D}$, then*

$$\det V \simeq \eta \cdot \chi_{e_0}^{k_1} \cdot \chi_{e_1}^{k_2} \cdot \dots \cdot \chi_{e_{f-2}}^{k_{f-1}} \cdot \chi_{e_{f-1}}^{k_0} \text{ and } (\det \overline{V})_{|I_K} \simeq \omega_{f,\bar{\tau}_0}^{\alpha},$$

where η is an unramified character of G_K and $\alpha = -\sum_{i=0}^{f-1} p^i k_i$.

Proof. By Proposition 3.4 and Lemma 6.6 the crystalline character $\det V \otimes \left(\chi_{e_0}^{k_1} \cdot \chi_{e_1}^{k_2} \cdot \dots \cdot \chi_{e_{f-2}}^{k_{f-1}} \cdot \chi_{e_{f-1}}^{k_0}\right)^{-1}$ has labeled Hodge-Tate weights $\{0\}_{\tau_i}$. If the corresponding filtered φ-module has Frobenius endomorphism $\varphi(\eta) = \vec{a}\cdot\eta$, then by Proposition 3.5 $\mathrm{Nm}_{\varphi}(\vec{a}) = c \cdot \vec{1}$ for some $c \in E^{\times}$ with $\mathrm{v_p}(c) = 0$. Lemma 3.7 implies that $\det V \otimes \left(\chi_{e_0}^{k_1} \cdot \chi_{e_1}^{k_2} \cdot \dots \cdot \chi_{e_{f-2}}^{k_{f-1}} \cdot \chi_{e_{f-1}}^{k_0}\right)^{-1}$ is the unramified character of G_K which maps Frob_K to c. The rest of the corollary follows from Lemma 6.2. □

We recall the following well-known proposition in which the field K has absolute inertia degree f and is not assumed to be unramified over $\mathbb{Q}_p$.

Proposition 6.8. *[Bre07, Prop. 2.7] Let $\bar{\rho} : G_K \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ be a continuous representation. Then*

$$\bar{\rho}_{|I_K} \simeq \begin{pmatrix} \omega_{2f}^{m} & * \\ 0 & \omega_{2f}^{mp^f} \end{pmatrix}$$

for some integer m. The representation $\bar{\rho}$ is irreducible if and only $1 + p^f \nmid m$, and in this case $* = 0$.

Corollary 6.9. *Let χ be a crystalline character of $G_{K_{2f}}$ with labeled Hodge-Tate weights $\{-k_i\}_{\tau_i}$, where the k_i are arbitrary integers for all $i = 0, 1, ..., 2f-1$, and let $V = \mathrm{Ind}_{K_{2f}}^{K_f}(\chi)$. The residual representation $\overline{V}$ is irreducible if and only if $1 + p^f \nmid \sum_{i=0}^{2f-1} p^i k_i$.*

Proof. Follows immediately from Lemma 6.2 and Proposition 6.8. □

6.1 Reductions of reducible two-dimensional crystalline representations

In this section we compute the semisimplified modulo p reduction of any reducible two-dimensional crystalline representation of G_{K_f}.

Lemma 6.10. *Let $k_0, k_1, ..., k_{f-1}$ be arbitrary integers and let*

$$\mathrm{Fil}^{\mathrm{j}}\mathbb{D} = \begin{cases} E^{|\tau_{I_0}|}\eta & \text{if } j \le w_0, \\ E^{|\tau_{I_1}|}\eta & \text{if } 1 + w_0 \le j \le w_1, \\ \quad \cdots\cdots & \\ E^{|\tau_{I_{t-1}}|}\eta & \text{if } 1 + w_{t-2} \le j \le w_{t-1}, \\ 0 & \text{if } j \ge 1 + w_{t-1}. \end{cases} \tag{6.1}$$

For each $i \in I_0$,

$$e_i \mathrm{Fil}^{\mathrm{j}} \mathbb{D} = \begin{cases} e_i E^{|\tau_{I_0}|}\eta & \text{if } j \leq k_i, \\ 0 & \text{if } r \geq 1 + k_i. \end{cases}$$

Proof. Let $k_i = w_r$ for some $r \in \{1, ..., t-1\}$. Since $w_r > w_{r-1}$ we have $i \in I_r$ from the definition of I_r. Similarly, since $k_i = w_r$ we have $i \notin I_{r+1}$. The same is clear for $r = 0$. Hence $e_i f_{I_r} = e_i$ and $e_i f_{I_{r+1}} = 0$ for all r. Multiplying formula 6.1 by e_i, we get

$$e_i \mathrm{Fil}^{\mathrm{j}} \mathbb{D} = \begin{cases} e_i E^{|\tau_{I_0}|}\eta & \text{if } j \leq w_r, \\ 0 & \text{if } r \geq 1 + w_r. \end{cases}$$

□

Let $\mathbb{D}$ be as in Proposition 6.4 and let $\mathrm{Mat}_{\underline{\eta}}(\varphi) = \begin{pmatrix} \vec{\alpha} & \vec{0} \\ * & \vec{\delta} \end{pmatrix}$. The filtration is as in formula 5.10 for some vectors $\vec{x}, \vec{y} \in E^{|\tau|}$. By Proposition 2.10 in [Dou10] (or by a direct computation),

$$\mathrm{Fil}^{\mathrm{j}}(\mathbb{D}_2) = \mathbb{D}_2 \cap \mathrm{Fil}^{\mathrm{j}}\mathbb{D} = \begin{cases} \mathbb{D}_2 & \text{if } \ j \leq 0, \\ E^{|\tau_{I_{0,\vec{x}}}|}\eta_2 & \text{if } \ 1 \leq j \leq w_0, \\ \quad \cdots\cdots & \\ E^{|\tau_{I_{t-1,\vec{x}}}|}\eta_2 & \text{if } \ 1 + w_{t-2} \leq j \leq w_{t-1}, \\ 0 & \text{if } \ j \geq 1 + w_{t-1}, \end{cases}$$

where $I_{r,\vec{x}} = I_r \cap J'_{\vec{x}} = \{i \in I_r : x_i = 0\}$. Let $\vec{\delta} = (\delta_0, \delta_1, ..., \delta_{f-1})$. By Lemma 6.10,

$$e_i \mathrm{Fil}^{\mathrm{j}}(\mathbb{D}_2) = \begin{cases} e_i E^{|\tau|}\eta_2 & \text{if } j \leq 0 \\ e_i E^{|\tau|} f_{J'_{\vec{x}}}\eta_2 & \text{if } 1 \leq j \leq k_i, \\ 0 & \text{if } j \geq 1 + k_i, \end{cases}$$

therefore the labeled Hodge-Tate weight of $\mathbb{D}_2$ with respect to the embedding τ_i is

$$m_i = \begin{cases} 0 & \text{if } x_i \neq 0 \ , \\ k_i & \text{if } x_i = 0, \end{cases}$$

and $(\mathbb{D}_2, \varphi_2)$ corresponds to the effective crystalline character $\chi_{c,\vec{0}} \cdot \chi_{e_{f-1}}^{m_0} \cdot \chi_{e_0}^{m_1} \cdot \cdots \cdot \chi_{e_{f-2}}^{m_{f-1}}$, where $c = \left(\prod\limits_{i \in I_0} \delta_i \right) \cdot p^{-\sum\limits_{i \in I_0} k_i}$. The following theorem follows immediately from Corollary 6.7.

THEOREM 6.11. *(i)*

$$V \simeq \begin{pmatrix} \psi_1 & * \\ 0 & \psi_2 \end{pmatrix},$$

where $\psi_1 = \eta_1 \cdot \chi_{e_{f-1}}^{m_0} \cdot \chi_{e_0}^{m_1} \cdot \cdots \cdot \chi_{e_{f-2}}^{m_{f-1}}$ *and* $\psi_2 = \eta_2 \cdot \chi_{e_0}^{k_1 - m_1} \cdot \chi_{e_1}^{k_2 - m_2} \cdot \cdots \cdot \chi_{e_{f-2}}^{k_{f-1} - m_{f-1}} \cdot \chi_{e_{f-1}}^{k_0 - m_0}$, *where* η_i *are unramified characters of* G_{K_f}.

(ii)

$$\left(\overline{V}_{|I_K}\right)^{s.s.} = \omega_{f,\bar{\tau}_0}^{\alpha_1} \oplus \omega_{f,\bar{\tau}_0}^{\alpha_2},$$

where $\alpha_1 = -\sum_{i=0}^{f-1} m_i p^i$ *and* $\alpha_2 = \sum_{i=0}^{f-1} (m_i - k_i)\, p^i$.

Notice that for an ordered basis is in Proposition 6.4, $\left(\overline{V}_{|I_{K_f}}\right)^{s.s.}$ only depends on the filtration with respect to that basis.

6.2 Proof of theorem 1.5

Let $\{\ell_i, \ell_{i+f}\} = \{0, k_i\}$ for $i = 0, 1, ..., f-1$ and assume that at least one k_i is strictly positive. In this section we construct infinite families of crystalline representations of Hodge-Tate type $\{0, -k_i\}_{\tau_i}$ which contain the irreducible representations $V_{\vec{\ell}} = \mathrm{Ind}_{G_{K_{2f}}}^{G_{K_f}} \left(\chi_{e_0}^{\ell_1} \cdot \chi_{e_1}^{\ell_2} \cdot \cdots \cdot \chi_{e_{2f-2}}^{\ell_{2f-1}} \cdot \chi_{e_{2f-1}}^{\ell_0}\right)$ of Proposition 3.11, and have the same mod p reductions with $V_{\vec{\ell}}$. We choose f-tuples of matrices $(\Pi_1, \Pi_2, ..., \Pi_f)$ (with $\Pi_f = \Pi_0$), where the types of the matrices Π_i (recall Definition 5.4) are chosen as follows:
(1) If $\ell_1 = 0$, $\Pi_1 \in \{t_2, t_3\}$;
(2) If $\ell_1 = k_1 > 0$, $\Pi_1 \in \{t_1, t_4\}$.
For $i = 2, 3, ..., f-1$, we choose the type of the matrix Π_i as follows:
(1) If $\ell_i = 0$, then:

- If an even number of coordinates of $(\Pi_1, \Pi_2, ..., \Pi_{i-1})$ is of even type, $\Pi_i \in \{t_2, t_3\}$;
- If an odd number of coordinates of $(\Pi_1, \Pi_2, ..., \Pi_{i-1})$ is of even type, $\Pi_i \in \{t_1, t_4\}$.

(2) If $\ell_i = k_i > 0$, then:

- If an even number of coordinates of $(\Pi_1, \Pi_2, ..., \Pi_{i-1})$ is of even type, $\Pi_i \in \{t_1, t_4\}$;
- If an odd number of coordinates of $(\Pi_1, \Pi_2, ..., \Pi_{i-1})$ is of even type, $\Pi_i \in \{t_2, t_3\}$.

Finally, we choose the type of the matrix Π_0 as follows:
(1) If $\ell_0 = 0$, then:

- If an even number of coordinates of $(\Pi_1, \Pi_2, ..., \Pi_{f-1})$ is of even type, $\Pi_0 = t_4$;
- If an odd number of coordinates of $(\Pi_1, \Pi_2, ..., \Pi_{f-1})$ is of even type, $\Pi_0 = t_3$.

(2) If $\ell_0 = k_0 > 0$, then:

- If an even number of coordinates of $(\Pi_1, \Pi_2, ..., \Pi_{f-1})$ is of even type, $\Pi_0 = t_2$;
- If an odd number of coordinates of $(\Pi_1, \Pi_2, ..., \Pi_{f-1})$ is of even type, $\Pi_0 = t_1$.

Notice that from the choice of Π_0, an odd number of coordinates of $(\Pi_1, \Pi_2, ..., \Pi_f)$ is of even type. Let $\vec{i} = (i_1, i_2, ..., i_0) \in \{1,2,3,4\}^f$ be the type-vector attached to $(\Pi_1, \Pi_2, ..., \Pi_f)$. For the matrices Π_i, we assume that $c_i = 1$ for all i. Let $P_i = \Pi_i \mathrm{mod}\ \pi$ for each i and notice that from the choice of the matrices Π_i it follows that $(P_1, P_2, ..., P_f) \notin C_1 \cup C_2$. The type of P_i is defined to be the type of Π_i. For any $\vec{a} \in \mathfrak{m}_E^f$ we consider the families of crystalline E-representations $V_{\vec{k}}^{\vec{i}}(\vec{a})$ of G_{K_f} with labeled Hodge-Tate weights $\{0, -k_i\}_{\tau_i}$ constructed in §5.2. We prove the following.

PROPOSITION 6.12. *(i)* $V_{\vec{k}}^{\vec{i}}(\vec{0}) = \mathrm{Ind}_{K_{2f}}^{K_f}\left(\chi_{e_0}^{\ell_1} \cdot \chi_{e_1}^{\ell_2} \cdot \cdots \cdot \chi_{e_{2f-2}}^{\ell_{2f-1}} \cdot \chi_{e_{2f-1}}^{\ell_0}\right)$ *and* $V_{\vec{k}}^{\vec{i}}(\vec{0})$ *is irreducible;*

(ii) For any $\vec{a} \in \mathfrak{m}_E^f$, $\overline{V}_{\vec{k}}^{\vec{i}}(\vec{a}) = \overline{V}_{\vec{k}}^{\vec{i}}(\vec{0})$*;*

(iii) For any $\vec{a} \in \mathfrak{m}_E^f$, $\left(\overline{V}_{\vec{k}}^{\vec{i}}(\vec{a})_{|I_{K_f}}\right)^{s.s.} = \omega_{2f,\bar{\tau}_0}^{\beta} \bigoplus \omega_{2f,\bar{\tau}_0}^{p^f\beta}$, *where* $\beta = -\sum_{i=0}^{2f-1} p^i \ell_i$*;*

(iv) $\overline{V}_{\vec{k}}^{\vec{i}}(\vec{a})$ *is irreducible if and only if* $1 + p^f \nmid \beta$*;*

(v) Any irreducible member of the family $\left\{V_{\vec{k}}^{\vec{i}}(\vec{a}),\ \vec{a} \in \mathfrak{m}_E^f\right\}$*, other than* $V_{\vec{k}}^{\vec{i}}(\vec{0})$*, is non-induced.*

Proof. We restrict $V_{\vec{k}}^{\vec{i}}(\vec{0})$ to $G_{K_{2f}}$. By the construction of the representation $V_{\vec{k}}^{\vec{i}}(\vec{0})$ in §5.1, there exists some G_{K_f}-stable lattice $\left(\mathrm{T}_{\vec{k}}^{\vec{i}}(\vec{0})\right)_{G_{K_f}}$ inside $V_{\vec{k}}^{\vec{i}}(\vec{0})$ whose Wach module has φ-action defined by $(\varphi(\eta_1), \varphi(\eta_2)) = (\eta_1, \eta_2)\,\Pi(\vec{0})$, where $\Pi(\vec{0}) = (\Pi_1(0), \Pi_2(0), ..., \Pi_{f-1}(0), \Pi_0(0))$. By Proposition 2.6, the Wach module of the $G_{K_{2f}}$-stable lattice $\left(\mathrm{T}_{\vec{k}}^{\vec{i}}(\vec{0})\right)_{|G_{K_{2f}}}$ inside $\left(V_{\vec{k}}^{\vec{i}}(\vec{0})\right)_{|G_{K_{2f}}}$ is defined by $(\varphi(\eta_1), \varphi(\eta_2)) = (\eta_1, \eta_2)\,\Pi(0)^{\otimes 2}$, therefore the filtered φ-module corresponding to $\left(V_{\vec{k},\vec{0}}^{\vec{i}}\right)_{|G_{K_{2f}}}$ has Frobenius endomorphism $(\varphi(\eta_1), \varphi(\eta_2)) = (\eta_1, \eta_2)\,P(\vec{0})^{\otimes 2}$. By Corollary 2.7 the restricted representation $\left(V_{\vec{k}}^{\vec{i}}(\vec{0})\right)_{|G_{K_{2f}}}$ has labeled Hodge-Tate weights $(\{0, -k_i\})_{\tau_i}$, $i =$

$0,1,...,2f-1$, with $k_{i+f}=k_i$ for $i=0,1,...,f-1$, and filtration as in formula 5.10 for some vectors $\vec{x},\vec{y}$, with the sets I_j being defined with respect to the $2f$ weights above. We prove that $\left(V_{\vec{k}}^{\vec{i}}(\vec{0})\right)_{|G_{K_{2f}}}$ is reducible and determine its irreducible constituents. First, we change the basis to diagonalize the matrix of Frobenius. We see that

$$P_i\left(0\right)=\begin{cases} R\left(\bar{\beta}_i,\bar{\gamma}_i\right), & \text{with } \{\bar{\beta}_i,\bar{\gamma}_i\}=\{1,p^{k_i}\} \text{ if } P_i \text{ has type 2 or 4,}\\ \operatorname{diag}\left(\bar{\alpha}_i,\bar{\delta}_i\right), & \text{with } \{\bar{\alpha}_i,\bar{\delta}_i\}=\{1,p^{k_i}\} \text{ if } P_i \text{ has type 1 or 3,}\end{cases}$$

where $R\left(\bar{\beta}_i,\bar{\gamma}_i\right)$ is the 2×2 matrix with $\bar{\beta}_i$ in the $(1,2)$ entry, $\bar{\gamma}_i$ in the $(2,1)$ entry, and zero on the diagonal. Let $Q_0=Id$,

$$Q_1=\begin{cases} Id \text{ if } P_1\in\{t_1,t_3\},\\ R \ \text{ if } P_1\in\{t_2,t_4\},\end{cases}$$

where $R:=R\left(1,1\right)$,

$$Q_i=\begin{cases} Id \text{ if } Q_{i-1}=Id \text{ and } P_i\in\{t_1,t_3\},\\ R \text{ if } Q_{i-1}=Id \text{ and } P_i\in\{t_2,t_4\},\\ R \text{ if } Q_{i-1}=R \text{ and } P_i\in\{t_1,t_3\},\\ Id \text{ if } Q_{i-1}=R \text{ and } P_i\in\{t_2,t_4\}\end{cases} \tag{6.2}$$

for $i=2,3,...,2f-1$. Let $Q=\left(Q_0,Q_1,...,Q_{2f-1}\right)$. By the definition of the matrices Q_i, the matrix $Q\cdot P(\vec{0})^{\otimes 2}\cdot\varphi\left(Q^{-1}\right)$ is diagonal. By induction, $Q_0=Id$ and

$$Q_i=\begin{cases} Id \text{ if an even number of coordinates}\\ \text{of } \left(P_1,P_2,...,P_i\right) \text{ is of even type,}\\ \\ R \text{ if an odd number of coordinates}\\ \text{of } \left(P_1,P_2,...,P_i\right) \text{ is of even type,}\end{cases} \tag{6.3}$$

for $i=1,2,...,2f-1$, where $P_{i+f}=P_i$ for $i=0,1,...,f-1$. We claim that for each $i=0,1,...,f-1$, $Q_i=Id$ if and only if $Q_{i+f}=R$. Indeed, for $i=0$, $Q_0=Id$. Since an odd number of coordinates of $(P_1,P_2,...,P_f)$ is of even type, $Q_f=R$. Let q_{ij}^r be the r-th coordinate of the (i,j)-entry $\vec{q}_{ij}$ of Q for each $i,j\in\{1,2\}$ and $r\in\{0,1,...,2f-1\}$. Assume that $i\in\{1,2,...,f-1\}$. From the definition of the matrices Q_i we see that

$$q_{11}^i=\begin{cases} 1 \text{ if an even number of coordinates}\\ \text{of } \left(P_1,P_2,...,P_i\right) \text{ is of even type,}\\ \\ 0 \text{ if an odd number of coordinates}\\ \text{of } \left(P_1,P_2,...,P_i\right) \text{ is of even type.}\end{cases} \tag{6.4}$$

For any $i = 0, 1, ..., f-1$ we have

$$q_{11}^{i+f} = \begin{cases} 1 \text{ if an even number of coordinates of} \\ (P_1, P_2, ..., P_f, ...P_{i+f}) \text{ is of even type,} \\ \\ 0 \text{ if an odd number of coordinates of} \\ (P_1, P_2, ..., P_f, ...P_{i+f}) \text{ is of even type.} \end{cases} \tag{6.5}$$

Since an odd number of coordinates of $(P_1, P_2, ..., P_f)$ is of even type and $P_i = P_{i+f}$ for all i, this is equivalent to

$$q_{11}^{i+f} = \begin{cases} 1 \text{ if an odd number of coordinates} \\ \text{of } (P_1, P_2, ..., P_i) \text{ is of even type,} \\ \\ 0 \text{ if an even number of coordinates} \\ \text{of } (P_1, P_2, ..., P_i) \text{ is of even type,} \end{cases} \tag{6.6}$$

which implies that $q_{11}^{i+f} = 1 - q_{11}^{i}$ for all $i = 0, 1, ..., f-1$. Similarly $q_{ij}^{r+f} = 1 - q_{ij}^{r}$ for all entries ij. Consider the ordered basis $\underline{\zeta} = (\zeta_1, \zeta_2)$ defined by $(\zeta_1, \zeta_2) := (\eta_1, \eta_2)\, Q^{-1}$. In the ordered basis $\underline{\zeta}$ the filtration is as in formula 5.10 with the vector $\vec{x}\eta_1 + \vec{y}\eta_2$ replaced by $\vec{x} \cdot (\vec{q}_{11} \cdot \zeta_1 + \vec{q}_{12} \cdot \zeta_2) + \vec{y} \cdot (\vec{q}_{12} \cdot \zeta_1 + \vec{q}_{22} \cdot \zeta_2)$. Let $\vec{z} = \vec{x} \cdot \vec{q}_{11} + \vec{y} \cdot \vec{q}_{12}$ and $\vec{w} = \vec{x} \cdot \vec{q}_{12} + \vec{y} \cdot \vec{q}_{22}$. From the definition of the matrices Q_i, the matrix of Frobenius in this new basis is the diagonal matrix

$$\operatorname{diag}\left(\vec{\lambda}, \vec{\mu}\right) := \\ \left(Q_0 \cdot P_1 \cdot Q_1^{-1}, ..., Q_{f-1} \cdot P_f \cdot Q_f^{-1}, Q_f \cdot P_{f+1} \cdot Q_{f+1}^{-1}, ..., Q_{2f-1} \cdot P_0 \cdot Q_0^{-1}\right).$$

We prove that $\mathrm{Nm}_\varphi(\vec{\lambda}) = \mathrm{Nm}_\varphi(\vec{\mu}) = p^{\sum_{i=0}^{f-1} k_i} \cdot \vec{1}$. First we see that $\lambda_i \mu_i = p^{k_i}$ for all i. Since $Q_i = Id$ if and only if $Q_{i+f} = R$, a case by case analysis for the choices of Q_i and Q_{i+1}, bearing in mind that $P_{i+f} = P_i$, implies that $Q_i \cdot P_{i+1} \cdot Q_{i+1}^{-1} = \operatorname{diag}(\lambda_{i+1}, \mu_{i+1})$ if and only if $Q_{i+f} \cdot P_{i+f+1} \cdot Q_{i+f+1}^{-1} = \operatorname{diag}(\mu_{i+1}, \lambda_{i+1})$. Therefore,

$$\begin{aligned} & \prod_{i=0}^{2f-1} \left(Q_i \cdot P_{i+1} \cdot Q_{i+1}^{-1}\right) \\ = & \prod_{i=0}^{f-1} \left(Q_i \cdot P_{i+1} \cdot Q_{i+1}^{-1}\right) \cdot \prod_{i=0}^{f-1} \left(Q_{i+f} \cdot P_i \cdot Q_{i+f+1}^{-1}\right) \\ = & \prod_{i=0}^{f-1} \operatorname{diag}(\lambda_{i+1}, \mu_{i+1}) \cdot \prod_{i=0}^{f-1} \operatorname{diag}(\mu_{i+1}, \lambda_{i+1}) = p^{\sum_{i=0}^{f-1} k_i} \cdot Id. \end{aligned}$$

Next we notice that $\vec{y} = \vec{1} - \vec{x}$ and $\vec{q}_{12} = \vec{1} - \vec{q}_{11}$, so $\vec{z} = \vec{x} \cdot \vec{q}_{11} + \left(\vec{1} - \vec{x}\right) \cdot \left(\vec{1} - \vec{q}_{11}\right) = \vec{1} + 2 \cdot \vec{x} \cdot \vec{q}_{11} - \vec{q}_{11} - \vec{x}$. Since x_i and $q_{11}^i \in \{0, 1\}$ for all i, $z_i = 0$ if and only if $q_{11}^i = 1$ and $x_i = 0$, or $q_{11}^i = 0$ and $x_i = 1$. Recall from formula 5.11 that $x_i = 0$ if and only if $P_i \in \{t_3, t_4\}$ and $x_i = 1$ if and only if $P_i \in \{t_1, t_2\}$. This combined with the definition of the matrices Q_i gives

$$z_i = 0 \Leftrightarrow \begin{cases} i = 0 \text{ and } P_0 \in \{t_3, t_4\}, \text{ or} \\ \\ i \geq 1,\ P_i \in \{t_3, t_4\} \text{ and an even number of coordinates} \\ \text{of } (P_1, P_2, ..., P_i) \text{ is of even type, or} \\ \\ i \geq 1,\ P_i \in \{t_1, t_2\} \text{ and an odd number of coordinates} \\ \text{of } (P_1, P_2, ..., P_i) \text{ is of even type.} \end{cases} \tag{6.7}$$

Similarly,

$$z_i = 1 \Leftrightarrow \begin{cases} i = 0 \text{ and } P_0 \in \{t_1, t_2\}, \text{ or} \\ \\ i \geq 1,\ P_i \in \{t_1, t_2\} \text{ and an even number of coordinates} \\ \text{of } (P_1, P_2, ..., P_i) \text{ is of even type,} \\ \\ i \geq 1,\ P_i \in \{t_3, t_4\} \text{ and an odd number of coordinates} \\ \text{of } (P_1, P_2, ..., P_i) \text{ is of even type.} \end{cases} \tag{6.8}$$

We claim that $z_{i+f} = 1 - z_i$ for all $i = 0, 1, ..., f-1$. Indeed, $z_{i+f} = 1 + 2 \cdot x_{i+f} \cdot q_{11}^{i+f} - q_{11}^{i+f} - x_{i+f}$. Since $P_i = P_{i+f}$, we have $x_i = x_{i+f}$ for all i, and since $q_{11}^{i+f} = 1 - q_{11}^f$ we get $z_{i+f} = 1 - z_i$. Since $z_i \in \{0, 1\}$ for all i,

$$\sum_{\substack{i=0 \\ z_i=0}}^{2f-1} k_i = \sum_{\substack{i=0 \\ z_i=0}}^{f-1} k_i + \sum_{\substack{i=0 \\ z_{i+f}=0}}^{f-1} k_i = \sum_{i=0}^{f-1} k_i.$$

Since $\mathrm{v_p}\left(\mathrm{Nm}_\varphi\left(\vec{\mu}\right)\right) = \sum_{i=0}^{f-1} k_i = \sum_{\substack{i=0 \\ z_i=0}}^{2f-1} k_i$, Proposition 6.3[1] implies that the representation $\left(V_{\vec{k}}^{\vec{i}}(\vec{0})\right)_{|G_{K_{2f}}}$ is reducible. If $\mathbb{D}_2 := \left(E^{|\tau_{K_{2f}}|}\right)\zeta_2$, by [Dou10, proof of Prop. 4.3] (or by a direct computation),

$$\mathrm{Fil}^{\mathrm{j}}\mathbb{D}_2 = \begin{cases} \mathbb{D}_2 & \text{if } j \leq 0, \\ \left(E^{|\tau_{K_{2f}}|} f_{I_{i,\vec{z}}}\right)\zeta_2 & \text{if } 1 + w_{i-1} \leq j \leq w_i,\ i = 0, 1, ..., t-1, \\ 0 & \text{if } j \geq 1 + w_{t-1}, \end{cases} \tag{6.9}$$

where $I_{i,\vec{z}} = I_i \cap \{j \in \{0, 1, ..., 2f-1\} : z_j = 0\}$. Let $i \in \{0, 1, ..., 2f-1\}$.

[1] F-semisimplicity is not assumed here, but the part of Proposition 6.3 used still holds.

Arguing as in Lemma 6.10 we see that

$$e_i\mathrm{Fil}^{\mathrm{j}}\mathbb{D}_2=\begin{cases} e_iE^{|\tau_{K_{2f}}|}\zeta_2 & \text{if } \ j\leq 0,\\ e_i\left(\sum\limits_{\substack{r=0\\ z_r=0}}^{2f-1} e_i\right)E^{|\tau_{K_{2f}}|}\zeta_2 & \text{if } \ 1\leq j\leq k_i,\\ 0 & \text{if } \ j\geq 1+k_i.\end{cases}$$

Hence

$$e_i\mathrm{Fil}^{\mathrm{j}}\mathbb{D}_2=\begin{cases} e_iE^{|\tau_{K_{2f}}|}\zeta_2 & \text{if } \ j\leq 0,\\ 0 & \text{if } \ j\geq 1\end{cases}$$

if $z_i=1$ and

$$e_i\mathrm{Fil}^{\mathrm{j}}\mathbb{D}_2=\begin{cases} e_iE^{|\tau_{K_{2f}}|}\zeta_2 & \text{if } \ j\leq k_i,\\ 0 & \text{if } \ j\geq 1+k_i\end{cases}$$

if $z_i=0$. The labeled Hodge-Tate weight of $\mathbb{D}_2$ with respect to the embedding τ_i of K_{2f} into E is 0 if $z_i=1$ and $-k_i$ if $z_i=0$. Next we prove that

$$z_i=\begin{cases} 0 \text{ if } \ell_i=0,\\ 1 \text{ if } \ell_i=k_i>0\end{cases}$$

for $i=0,1,...,f-1$, and

$$z_{i+f}=\begin{cases} 1 \text{ if } \ell_i=0,\\ 0 \text{ if } \ell_i=k_i>0.\end{cases}$$

Since $z_{i+f}=1-z_i$ for all $i=0,1,...,f-1$, it suffices to prove the first formula. Suppose that $\ell_1=0$. Then $P_1\in\{t_2,t_3\}$ and by formula 6.7, $z_1=0$. If $\ell_1=k_1>0$, then $P_1\in\{t_1,t_4\}$ and formula 6.7 implies that $z_1=1$. Let $i\in\{1,2,...,f-2\}$ and assume that $\ell_i=0$. If an even number of coordinates of $(P_1,P_2,...,P_{i-1})$ is of even type, then $P_i\in\{t_2,t_3\}$ and formula 6.7 implies $z_i=0$. Arguing similarly we see that if $\ell_i=k_i>0$, formula 6.8 implies $z_i=1$. Finally, assume that $i=f$ and $\ell_f=0$. If an even number of coordinates of $(P_1,P_2,...,P_{f-1})$ is of even type, then $P_0=P_f=t_4$ and formula 6.7 implies that $z_0=z_f=0$. We finish the proof by verifying similarly the remaining cases. By the formulas above, the labeled Hodge-Tate weight of $\mathbb{D}_2$ with respect to the embedding τ_i is

$$\begin{cases} -k_i \text{ if } \ell_i=0,\\ \ \ 0 \ \ \text{ if } \ell_i=k_i>0\end{cases}$$

for $i=0,1,...,f-1$ and

$$\begin{cases} -k_i \text{ if } \ell_i=k_i>0,\\ \ \ 0 \ \ \text{ if } \ell_i=0\end{cases}$$

for $i=f,f+1,...,2f-1$. Therefore the labeled Hodge-Tate weight of $\mathbb{D}_2$ with respect to the embedding τ_i is

$$\begin{cases} -(k_i-\ell_i) & \text{if } i=0,1,...,f-1\\ -\ell_i & \text{if } i=f,f+1,...,2f-1.\end{cases}$$

Since $\{\ell_i, \ell_{i+f}\} = \{0, k_i\}$ for all $i = 0, 1, ..., f-1$, the labeled Hodge-Tate weights of $\mathbb{D}_2$ are $(-\ell_0, -\ell_1, ..., -\ell_{f-1}, -\ell_f, -\ell_{f+1}, ..., -\ell_{2f-1})$. Since $\mathrm{Nm}_\varphi(\vec{\mu}) = p^{\sum_{i=0}^{f-1} k_i} \cdot \vec{1}$, Proposition 3.5 implies that the crystalline character corresponding to $\mathbb{D}_2$ is $\chi_{e_0}^{\ell_1} \cdot \chi_{e_1}^{\ell_2} \cdot \cdots \cdot \chi_{e_{2f-2}}^{\ell_{2f-1}} \cdot \chi_{e_{2f-1}}^{\ell_0}$. Suppose that $V_{\vec{k}}^{\vec{i}}(\vec{0})$ is reducible. Then there exists some irreducible constituent of $V_{\vec{k}}^{\vec{i}}(\vec{0})$ whose restriction to $G_{K_{2f}}$ is $\chi_{e_0}^{\ell_1} \cdot \chi_{e_1}^{\ell_2} \cdot \cdots \cdot \chi_{e_{2f-2}}^{\ell_{2f-1}} \cdot \chi_{e_{2f-1}}^{\ell_0}$. Since the labeled weights of the latter character are $(-\ell_0, -\ell_1, ..., -\ell_{f-1}, -\ell_f, ..., -\ell_{2f-1})$, Corollary 2.7 implies that $\ell_i = \ell_{i+f}$ for all $i = 0, 1, ..., f-1$. Since $\{\ell_i, \ell_{i+f}\} = \{0, k_i\}$ for $i = 0, 1, ..., f-1$, and since some labeled weight is strictly positive this is absurd. Hence $V_{\vec{k}}^{\vec{i}}(\vec{0})$ is irreducible and its restriction to $G_{K_{2f}}$ contains $\chi_{e_0}^{\ell_1} \cdot \chi_{e_1}^{\ell_2} \cdot \cdots \cdot \chi_{e_{2f-2}}^{\ell_{2f-1}} \cdot \chi_{e_{2f-1}}^{\ell_0}$ as an irreducible constituent. By Frobenius reciprocity,

$$V_{\vec{k}}^{\vec{i}}(\vec{0}) = \mathrm{Ind}_{K_{2f}}^{K_f}\left(\chi_{e_0}^{\ell_1} \cdot \chi_{e_1}^{\ell_2} \cdot \cdots \cdot \chi_{e_{2f-2}}^{\ell_{2f-1}} \cdot \chi_{e_{2f-1}}^{\ell_0}\right).$$

This finishes the proof of part (i). Part (ii) follows from Theorem 4.7 and parts (iii) and (4) follow from Corollary 6.9. For part (iv), notice that any irreducible induced member of $V_{\vec{k}}^{\vec{i}}(\vec{a})$ has the form $\eta_c \cdot \mathrm{Ind}_{K_{2f}}^{K_f}\left(\chi_{e_0}^{\ell_1'} \cdot \chi_{e_1}^{\ell_2'} \cdot \cdots \cdot \chi_{e_{2f-2}}^{\ell_{2f-1}'} \cdot \chi_{e_{2f-1}}^{\ell_0'}\right)$ for some unramified character η_c and some nonnegative integers with $\{\ell_i', \ell_{i+f}'\} = \{0, k_i\}$ for all i (see Proposition 3.11). All the members of $V_{\vec{k}}^{\vec{i}}(\vec{a})$ have determinant $(-1)^t p^{\sum_{i=0}^{f-1} k_i}$, where t is the number of even coordinates of $\vec{i}$. This equals the determinant of $\mathrm{Ind}_{K_{2f}}^{K_f}\left(\chi_{e_0}^{\ell_1'} \cdot \chi_{e_1}^{\ell_2'} \cdot \cdots \cdot \chi_{e_{2f-2}}^{\ell_{2f-1}'} \cdot \chi_{e_{2f-1}}^{\ell_0'}\right)$ and forces the unramified character η_c to be trivial. Hence the only irreducible induced member of our family is $V_{\vec{k}}^{\vec{i}}(\vec{0})$. □

Remark 6.13. *Let R be as in the proof of Proposition 6.12. If $\mathcal{A}$ is a set of 2×2 matrices, let $R\mathcal{A} := \{R\cdot A : A \in \mathcal{A}\}$ and $\mathcal{A}R := \{A\cdot R : A \in \mathcal{A}\}$. We write $\{t_i, t_j\}$ for a set which contains matrices of type t_i and t_j. Then $R\{t_1, t_2\} = \{t_1, t_2\}$, $R\{t_3, t_4\} = \{t_3, t_4\}$, $\{t_1, t_2\}R = \{t_3, t_4\}$ and $\{t_3, t_4\}R = \{t_1, t_2\}$. In the definition of the matrices P_i we may always assume that $P_i \in \{t_1, t_2\}$ for all $i = 1, 2, ...f-1$. Indeed, let $Q_0 = R$, and for $i = 1, 2, ..., f-1$ let*

$$Q_i = \begin{cases} Id \text{ if } P_i \in \{t_1, t_2\}, \\ R \ \text{ if } P_i \in \{t_3, t_4\}. \end{cases}$$

After changing the basis by the matrix $Q = (Q_0, Q_1, ..., Q_{f-1})$ we have $P_i \in \{t_1, t_2\}$ for all $i = 1, 2, ...f-1$. By the definition preceding Proposition 6.12, the type of the matrix $P_0 \in \{t_1, t_2, t_3, t_4\}$ is uniquely determined by $(P_1, P_2, ..., P_{f-1})$.

Theorem 6.14. *Theorem 1.5 holds.*

Proof. Follows from Proposition 6.12 and Remark 6.13. □

Example 6.15. *Let $f = 2$ and $k_i > 0$ for $i = 0, 1$. Up to twist by some unramified character, there exist two distinct isomorphism classes of irreducible two-dimensional crystalline E-representations of G_{K_2} with labeled Hodge-Tate weights $(\{0, -k_0\}, \{0, -k_1\})$ induced from crystalline characters of G_{K_4}.*
(i) If $\ell_0 = k_0$ and $\ell_1 = k_1$, then from the definition of the matrices Π_i preceding Proposition 6.12 and Remark 6.13, $(\Pi_1, \Pi_0) = (t_1, t_2)$. Let $P_i = \Pi_i \bmod \pi$. The polynomials z_i in the definition of the matrices Π_i are such that $z_i \equiv p^m \bmod \pi$, where $m := \lfloor \frac{\max\{k_0, k_1\}-1}{p-1} \rfloor$, unless $k_0 = k_1 = p$ in which case we define $m = 0$. For any $\vec{a} = (a_0, a_1) \in \mathfrak{m}_E^2$ we consider the family of crystalline representations $V_{\vec{k},\vec{a}}^{(1,2)}$ constructed in §5.2. The corresponding family of filtered φ-modules is $\left(\mathbb{D}_{\vec{k},\vec{a}}^{(1,2)}, \varphi\right)$, with $(\varphi(\eta_1), \varphi(\eta_2)) = (\eta_1, \eta_2) P^{(1,2)}(\vec{a})$, where

$$P^{(1,2)}(\vec{a}) = \begin{pmatrix} (p^{k_1}, a_0 p^m) & (0,1) \\ (a_1 p^m, p^{k_0}) & (1,0) \end{pmatrix},$$

and the filtrations are

$$\mathrm{Fil^j}(\mathbb{D}_{\vec{k},\vec{a}}^{(1,2)}) = \begin{cases} E^2\eta_1 \bigoplus E^2\eta_2 & \text{if} \quad j \leq 0, \\ E^2(\vec{x}\cdot\eta_1 + \vec{y}\cdot\eta_2) & \text{if} \quad 1 \leq j \leq w_0, \\ E^1 f_{I_1}(\vec{x}\cdot\eta_1 + \vec{y}\cdot\eta_2) & \text{if} \quad 1 + w_0 \leq j \leq w_1, \\ 0 & \text{if} \quad j \geq 1 + w_1, \end{cases} \tag{6.10}$$

with $w_0 = \min\{k_0, k_1\}$ and $w_1 = \max\{k_0, k_1\}$,

$$f_{I_1} = \begin{cases} (0,1) & \text{if } k_0 < k_1, \\ (1,0) & \text{if } k_1 < k_0, \\ (0,0) & \text{if } k_0 = k_1, \end{cases}$$

and $(\vec{x}, \vec{y}) = ((1,1), (0,0))$. We have

$$V_{\vec{k},\vec{0}}^{(1,2)} \simeq \mathrm{Ind}_{K_4}^{K_2}\left(\chi_{e_0}^{k_1} \cdot \chi_{e_3}^{k_0}\right),$$

and for any $\vec{a} \in \mathfrak{m}_E^2$,

$$\left(\left(\overline{V}_{\vec{k},\vec{a}}^{(1,2)}\right)_{|I_{K_2}}\right)^{s.s.} \simeq \omega_{4,\bar{\tau}_0}^{-(k_0+pk_1)} \bigoplus \omega_{4,\bar{\tau}_0}^{-(k_0+pk_1)p^2}.$$

Let $\alpha_i = a_i p^m$ and $A = \alpha_1 + p^{k_1}\alpha_0$. Assume that $A^2 \neq -4p^{k_0+k_1}$ and let $\varepsilon_0 \neq \varepsilon_1$ be the distinct roots of the characteristic polynomial $X^2 - A\cdot X + p^{k_0+k_1}$. Arguing as in the proof of Proposition 2.2 in [Dou10], we get the following "standard parametrization" for the family $V_{\vec{k},\vec{a}}^{(1,2)}$,

$$\varphi(\eta_1) = (1, \varepsilon_0)\,\eta_1, \;\; \varphi(\eta_2) = \left(\lambda, \frac{\varepsilon_1}{\lambda}\right)\eta_2,$$

where

$$\lambda = \lambda(\alpha_0) = \frac{\varepsilon_1}{\varepsilon_0} \cdot \frac{(\varepsilon_1 - A + p^{k_1}\alpha_0)}{(\varepsilon_0 - A + p^{k_1}\alpha_0)}$$

(notice that $\varepsilon_i \neq A - \alpha_0 p^{k_1}$), and filtrations are as in Formula 6.10 with $\vec{x} = \vec{y} = \vec{1}$.
(ii) If $\ell_0 = \ell_1 = 0$. Again, taking into account Remark 6.13, we may only consider the case $(\Pi_1, \Pi_0) = (t_2, t_3)$. For each $\vec{a} \in \mathfrak{m}_E^2$ consider the family $V_{\vec{k},\vec{a}}^{(2,3)}$ of two-dimensional crystalline E-representations of G_{K_2} with labeled Hodge-Tate weights $\{0, -k_i\}_{\tau_i}$, $i = 0, 1$. We have

$$V_{\vec{k},\vec{0}}^{(2,3)} \simeq \mathrm{Ind}_{K_4}^{K_2}\left(\chi_{e_2}^{k_1} \cdot \chi_{e_1}^{k_0}\right) \simeq \mathrm{Ind}_{K_4}^{K_2}\left(\chi_{e_0}^{k_1} \cdot \chi_{e_3}^{k_0}\right).$$

For any $\vec{a} \in \mathfrak{m}_E^2$,

$$\left(\left(\overline{V}_{\vec{k},\vec{a}}^{(2,3)}\right)_{|I_{K_2}}\right)^{s.s.} \simeq \omega_{4,\tau_0}^{-(k_0+pk_1)} \bigoplus \omega_{4,\tau_0}^{-(k_0+pk_1)p^2}.$$

Notice that the family $\left\{V_{\vec{k},\vec{a}}^{(1,2)},\ \vec{a} \in \mathfrak{m}_E^2\right\}$ of case (i) coincides with the family $\left\{V_{\vec{k},\vec{a}}^{(2,3)},\ \vec{a} \in \mathfrak{m}_E^2\right\}$, as the second family is obtained from the first one by changing the basis by the matrix $Q = (R, R)$.
(iii) Let $f = 2$, $\ell_0 = 0$ and $\ell_1 = k_1 > 0$. Then $(\Pi_1, \Pi_0) = (t_1, t_4)$. For each $\vec{a} \in \mathfrak{m}_E^2$ consider the family $V_{\vec{k},\vec{a}}^{(1,4)}$ of two-dimensional crystalline E-representations of G_{K_2} with labeled Hodge-Tate weights $\{0, -k_i\}_{\tau_i}$, $i = 0, 1$. Then

$$V_{\vec{k},\vec{0}}^{(1,4)} \simeq \mathrm{Ind}_{K_4}^{K_2}\left(\chi_{e_0}^{k_1} \cdot \chi_{e_1}^{k_0}\right),$$

and for any $\vec{a} \in \mathfrak{m}_E^2$,

$$\left(\left(\overline{V}_{\vec{k},\vec{a}}^{(1,4)}\right)_{|I_{K_2}}\right)^{s.s.} \simeq \omega_{4,\bar{\tau}_0}^{-(pk_1+p^2k_0)} \bigoplus \omega_{4,\bar{\tau}_0}^{-(pk_1+p^2k_0)p^2}.$$

Let $\alpha_i = a_i p^m$ and $A = \alpha_0 + p^{k_0}\alpha_1$. Assume that $A^2 \neq -4p^{k_0+k_1}$ and let $\varepsilon_0 \neq \varepsilon_1$ be the distinct roots of the characteristic polynomial $X^2 - A \cdot X + p^{k_0+k_1}$. Arguing as in the proof of Proposition 2.2 in [Dou10], we get the following "standard parametrization" for the family $V_{\vec{k},\vec{a}}^{(1,4)}$,

$$\varphi(\eta_1) = (1, \varepsilon_0)\,\eta_1,\ \varphi(\eta_2) = \left(\lambda, \frac{\varepsilon_1}{\lambda}\right)\eta_2,$$

where

$$\lambda = \lambda(\alpha_1) = \left(\frac{\varepsilon_1}{\varepsilon_0}\right)^2 \cdot \frac{(A - p^{k_0}\alpha_1 - \varepsilon_0)}{(A - p^{k_0}\alpha_1 - \varepsilon_1)}$$

(notice that $\varepsilon_i \neq A - \alpha_1 p^{k_0}$), and filtrations as in Formula 6.10 with $\vec{x} = \vec{y} = \vec{1}$.
(iv) If $f = 2$ $\ell_0 = k_0 > 0$ and $\ell_1 = 0$. Then $(\Pi_1, \Pi_0) = (t_2, t_1)$ and this gives the family obtained in case (iii).

Example 6.16. *If* $f=2$, $k_0>0$ *and* $k_1=0$. *Then up to unramified twist,* $\mathrm{Ind}_{K_4}^{K_2}\left(\chi_{e_3}^{k_0}\right)$ *is a unique isomorphism class of two-dimensional crystalline irreducible* E*-representations of* G_{K_2} *with labeled weights* $(\{0,-k_0\},\{0,0\})$ *induced from* G_{K_4}. *We have*

$$V_{\vec{k},\vec{0}}^{(2,3)} \simeq \mathrm{Ind}_{K_4}^{K_2}\left(\chi_{e_3}^{k_0}\right) \simeq \mathrm{Ind}_{K_4}^{K_2}\left(\chi_{e_1}^{k_0}\right),$$

and for any $\vec{a}\in\mathfrak{m}_E^2$,

$$\left(\left(\overline{V}_{\vec{k},\vec{a}}^{(2,3)}\right)_{|I_{K_2}}\right)^{s.s.} \simeq \omega_{4,\bar{\tau}_0}^{-k_0}\bigoplus\omega_{4,\bar{\tau}_0}^{-p^2k_0}.$$

Example 6.17. *Let* $f=3$, $k_i>0$ *for all* $i=0,1,2$. *Up to twist by some unramified character, there exist* 4 *distinct isomorphism classes of irreducible two-dimensional crystalline* E*-representations of* G_{K_3} *with labeled Hodge-Tate weights* $(\{0,-k_0\},\{0,-k_1\},\{0,-k_2\})$ *induced from* G_{K_6}. *One of those classes is represented by* $\mathrm{Ind}_{K_6}^{K_3}\left(\chi_{e_0}^{k_1}\cdot\chi_{e_1}^{k_2}\cdot\chi_{e_2}^{k_0}\right)$. *For the families containing it we have* $\ell_i=k_i>0$ *for all* $i=0,1,2$. *Since* $k_0>0$, $\Pi_0=t_2$ *if* $\Pi_2=t_4$ *and* $\Pi_0=t_1$ *if* $\Pi_2=t_1$. *Hence* $(\Pi_1,\Pi_2,\Pi_0)\in\{(t_4,t_4,t_2),(t_4,t_1,t_1),(t_1,t_2,t_1),(t_1,t_3,t_2)\}$. *By Remark 6.13 we may only consider the case* $(\Pi_1,\Pi_2,\Pi_0)=(t_1,t_2,t_1)$. *For any* $\vec{a}\in\mathfrak{m}_E^3$, *consider the the families* $V_{\vec{k},\vec{a}}^{(1,2,1)}$ *of two-dimensional crystalline* E*-representations of* G_{K_3} *with labeled Hodge-Tate weights* $\{0,-k_i\}_{\tau_i}$, $i=0,1,2$. *We have*

$$V_{\vec{k},\vec{0}}^{(1,2,1)} \simeq \mathrm{Ind}_{K_6}^{K_3}\left(\chi_{e_0}^{k_1}\cdot\chi_{e_1}^{k_2}\cdot\chi_{e_2}^{k_0}\right),$$

and for any $\vec{a}\in\mathfrak{m}_E^3$,

$$\left(\left(\overline{V}_{\vec{k},\vec{a}}^{(1,2,1)}\right)_{|I_{K_3}}\right)^{s.s.} \simeq \omega_{6,\bar{\tau}_0}^{-\left(k_0+pk_1+p^2k_2\right)}\bigoplus\omega_{6,\bar{\tau}_0}^{-\left(k_0+pk_1+p^2k_2\right)p^3}.$$

6.3 Proof of theorem 1.7

Let $V_{\vec{\ell},\vec{\ell}'}(\eta)=\eta\cdot\chi_{e_0}^{\ell_1}\cdot\chi_{e_1}^{\ell_2}\cdot\dots\cdot\chi_{e_{f-1}}^{\ell_0}\bigoplus\chi_{e_0}^{\ell_1'}\cdot\chi_{e_1}^{\ell_2'}\cdot\dots\cdot\chi_{e_{f-1}}^{\ell_0'}$ with $\{\ell_i,\ell_i'\}=\{0,k_i\}$ for all i, where $\eta=\eta_c$ is the unramified character of G_{K_f} which maps the geometric Frobenius Frob_{K_f} of G_{K_f} to $c\in\mathcal{O}_E^{\times}$. As usual, we assume that at least one k_i is strictly positive. We choose f-tuples of matrices $(\Pi_1,\Pi_2,...,\Pi_f)$ (with $\Pi_f=\Pi_0$) as follows:
(1) If $\ell_1=0$, $\Pi_1\in\{t_2,t_3\}$;
(2) If $\ell_1=k_1>0$, $\Pi_1\in\{t_1,t_4\}$.
For $i=2,3,...,f-1$, we choose the type of the matrix Π_i as follows:
(1) If $\ell_i=0$, then:

- If an even number of coordinates of $(\Pi_1,\Pi_2,...,\Pi_{i-1})$ is of even type, $\Pi_i\in\{t_2,t_3\}$;
- If an odd number of coordinates of $(\Pi_1,\Pi_2,...,\Pi_{i-1})$ is of even type, $\Pi_i\in\{t_1,t_4\}$.

(2) If $\ell_i = k_i > 0$, then:

- If an even number of coordinates of $(\Pi_1, \Pi_2, ..., \Pi_{i-1})$ is of even type, $\Pi_i \in \{t_1, t_4\}$;
- If an odd number of coordinates of $(\Pi_1, \Pi_2, ..., \Pi_{i-1})$ is of even type, $\Pi_i \in \{t_2, t_3\}$.

Finally, we choose the type of the matrix Π_0 as follows:
(1) If $\ell_0 = 0$, then:

- If an even number of coordinates of $(\Pi_1, \Pi_2, ..., \Pi_{f-1})$ is of even type, $\Pi_0 = t_3$;
- If an odd number of coordinates of $(\Pi_1, \Pi_2, ..., \Pi_{f-1})$ is of even type, $\Pi_0 = t_4$.

(2) If $\ell_0 = k_i > 0$, then:

- If an even number of coordinates of $(\Pi_1, \Pi_2, ..., \Pi_{f-1})$ is of even type, $\Pi_0 = t_1$;
- If an odd number of coordinates of $(\Pi_1, \Pi_2, ..., \Pi_{f-1})$ is of even type, $\Pi_0 = t_2$.

Notice that the type of Π_0 has been chosen so that an even number of coordinates of $(\Pi_1, \Pi_2, ..., \Pi_f)$ is of even type. We choose the units c_i appearing in the entries of the matrices Π_i in Definition 5.4 so that $c_i = 1$ for all $i = 1, 2, ..., f-1$, and $c_0 = c$. Let $\vec{i}$ be the type-vector attached to $(\Pi_1, \Pi_2, ..., \Pi_f)$. We exclude those vectors $\vec{i}$ for which $(\Pi_1, \Pi_2, ..., \Pi_f) \in C_1 \cup C_2$, which is to exclude the cases where $\vec{\ell} = \vec{0}$ or $\vec{\ell}' = \vec{0}$. For any $\vec{a} \in \mathfrak{m}_E^f$ we consider the families of crystalline E-representations $V_{\vec{k}}^{\vec{i}}(\vec{a})$ of G_{K_f} with labeled Hodge-Tate weights $\{0, -k_i\}_{\tau_i}$ constructed in §5.2.

PROPOSITION 6.18. *(i) For any type vector $\vec{i}$ chosen as above there exists some unramified character μ such that $\mu \otimes V_{\vec{k}}^{\vec{i}}(\vec{0}) \simeq V_{\vec{\ell},\vec{\ell}'}(\eta)$;*

(ii) For any $\vec{a} \in \mathfrak{m}_E^f$, $\overline{V}_{\vec{k}}^{\vec{i}}(\vec{a}) \simeq \overline{V}_{\vec{k}}^{\vec{i}}(\vec{0})$ and

$$\left(\overline{V}_{\vec{k}}^{\vec{i}}(\vec{a})\right)_{|I_{K_f}} \simeq \left(\overline{V}_{\vec{\ell},\vec{\ell}'}(\eta)\right)_{|I_{K_f}} \simeq \omega_{f,\bar{\tau}_0}^{\alpha} \oplus \omega_{f,\bar{\tau}_0}^{\alpha'},$$

where $\alpha = -\sum_{i=i}^{f-1} \ell_i p^i$ and $\alpha' = -\sum_{i=0}^{f-1} \ell'_i p^i$.

Proof. For simplicity assume that $\eta = 1$. The general case follows by choosing the unit c_0 in the definition of Π_0 appropriately. We restrict $V_{\vec{k}}^{\vec{i}}(\vec{0})$ to $G_{K_{2f}}$. By the construction of the representation $V_{\vec{k}}^{\vec{i}}(\vec{0})$ in §5.1, there exists some G_{K_f}-stable lattice $\left(\mathrm{T}_{\vec{k}}^{\vec{i}}(\vec{0})\right)_{G_{K_f}}$ inside $V_{\vec{k}}^{\vec{i}}(\vec{0})$ whose Wach module has φ-action defined by $(\varphi(\eta_1), \varphi(\eta_2)) = (\eta_1, \eta_2) \cdot \Pi(\vec{0})$. By Proposition 2.6, the Wach module of the $G_{K_{2f}}$-stable lattice $\left(\mathrm{T}_{\vec{k}}^{\vec{i}}(\vec{0})\right)_{|G_{K_{2f}}}$ inside $\left(V_{\vec{k}}^{\vec{i}}(\vec{0})\right)_{|G_{K_{2f}}}$ is defined by $(\varphi(\eta_1), \varphi(\eta_2)) = (\eta_1, \eta_2) \cdot \Pi(0)^{\otimes 2}$, therefore the filtered φ-module corresponding to $\left(V_{\vec{k}}^{\vec{i}}(\vec{0})\right)_{|G_{K_{2f}}}$ has Frobenius endomorphism $(\varphi(\eta_1), \varphi(\eta_2)) = (\eta_1, \eta_2) \cdot P(0)^{\otimes 2}$. The restricted representation $\left(V_{\vec{k}}^{\vec{i}}(\vec{0})\right)_{|G_{K_{2f}}}$ has labeled weights $(\{0, -k_i\})_{\tau_i}$, $i = 0, 1, ..., 2f-1$, with $k_{i+f} = k_i$ for $i = 0, 1, ..., f-1$, and filtration as in formula 5.10 for some vectors $\vec{x}, \vec{y}$, with the sets I_j being defined with respect to the $2f$ weights above. We prove that $\left(V_{\vec{k}}^{\vec{i}}(\vec{0})\right)_{|G_{K_{2f}}}$ is reducible and determine its irreducible constituents. First we change the basis to diagonalize the matrix of Frobenius. We define matrices Q_i as in the proof of Proposition 6.12, and we let $Q = (Q_0, Q_1, ..., Q_{2f-1})$. By the definition of the matrices Q_i, the matrix $Q \cdot P(0)^{\otimes 2} \cdot \varphi(Q^{-1})$ is diagonal. By the proof of Proposition 6.12, $Q_0 = Id$ and for $i = 1, 2, ..., 2f-1$, Q_i is as in formula 6.3. We claim that for each $i = 0, 1, ..., f-1$, $Q_i = Q_{i+f}$. Indeed, from the definition of the matrices Q_i we see that q_{11}^{i} and q_{11}^{i+f} are as in formulas 6.4 and 6.5 respectively in the proof of Proposition 6.12. Since an even number of coordinates of $(P_1, P_2, ..., P_f)$ are of even type, $q_{11}^{i+f} = q_{11}^{i}$. Similarly, $q_{ij}^{r+f} = q_{ij}^{r}$ for any entry (i, j). Consider the ordered basis $\underline{\zeta} = (\zeta_1, \zeta_2)$ defined by $(\zeta_1, \zeta_2) := (\eta_1, \eta_2) \cdot Q^{-1}$. Let $\vec{q}_{ij}$ be th (i, j)-entry of Q. In the new basis $\underline{\zeta}$ the filtration is as in formula 5.10 with the vector $\vec{x}\eta_1 + \vec{y}\eta_2$ replaced by $\vec{x} \cdot (\vec{q}_{11} \cdot \zeta_1 + \vec{q}_{12} \cdot \zeta_2) + \vec{y} \cdot (\vec{q}_{12} \cdot \zeta_1 + \vec{q}_{22} \cdot \zeta_2)$. Let $\vec{z} = \vec{x} \cdot \vec{q}_{11} + \vec{y} \cdot \vec{q}_{12}$ and $\vec{w} = \vec{x} \cdot \vec{q}_{12} + \vec{y} \cdot \vec{q}_{22}$. The matrix of Frobenius in this new basis is the diagonal matrix diag$\left(\vec{\lambda}, \vec{\mu}\right)$. Arguing as in Proposition 6.12, and taking into account that $q_{ij}^{r+f} = q_{ij}^{r}$ for all $r = 0, 1, ..., f-1$ and all entries (i, j) we see that $z_{r+f} = z_r$ for all r. From the proof of the same proposition, $z_i = 0$ if and only if $q_{11}^{i} = 1$ and $x_i = 0$ or $q_{11}^{i} = 0$ and $x_i = 1$. Formula 5.11 implies that $x_i = 0$ if and only if $P_i \in \{t_4, t_3\}$ and $x_i = 1$ if and only if $P_i \in \{t_2, t_1\}$. Since $z_i = z_{i+f}$ and $k_i = k_{i+f}$ for all $i = 0, 1, ..., f-1$,

$$\sum_{\substack{i=0 \\ z_i=0}}^{2f-1} k_i = 2 \sum_{\substack{i=0 \\ z_i=0}}^{f-1} k_i = 2 \sum_{\substack{i=0 \\ Q_i=R \\ P_i=t_1}}^{f-1} k_i + 2 \sum_{\substack{i=0 \\ Q_i=R \\ P_i=t_2}}^{f-1} k_i + 2 \sum_{\substack{i=0 \\ Q_i=Id \\ P_i=t_3}}^{f-1} k_i + 2 \sum_{\substack{i=0 \\ Q_i=Id \\ P_i=t_4}}^{f-1} k_i.$$

We now show that the $(2,2)$ entry of $\prod_{i=0}^{2f-1}\left(Q_iP_{i+1}Q_{i+1}^{-1}\right)$ is the p^n, where

$$n=2\sum_{\substack{i=0\\ Q_i=R\\ P_i=t_1}}^{f-1}k_i+2\sum_{\substack{i=0\\ Q_i=R\\ P_i=t_2}}^{f-1}k_i+2\sum_{\substack{i=0\\ Q_i=Id\\ P_i=t_3}}^{f-1}k_i+2\sum_{\substack{i=0\\ Q_i=Id\\ P_i=t_4}}^{f-1}k_i. \tag{6.11}$$

Since the matrices $Q_iP_{i+1}Q_{i+1}^{-1}$ are diagonal, and since $Q_{i+f}=Q_i$ and $P_{i+f}=P_i$ for all i,

$$\begin{aligned}\prod_{i=0}^{2f-1}\left(Q_iP_{i+1}Q_{i+1}^{-1}\right)=&\prod_{\substack{i=0\\ Q_i=Id\\ P_{i+1}=t_4}}^{f-1}\left(Q_iP_{i+1}Q_{i+1}^{-1}\right)^2\cdot\prod_{\substack{i=0\\ Q_i=Id\\ P_{i+1}=t_3}}^{f-1}\left(Q_iP_{i+1}Q_{i+1}^{-1}\right)^2\cdot\\ \prod_{\substack{i=0\\ Q_i=Id\\ P_{i+1}=t_1}}^{f-1}\left(Q_iP_{i+1}Q_{i+1}^{-1}\right)^2\cdot&\prod_{\substack{i=0\\ Q_i=Id\\ P_{i+1}=t_2}}^{f-1}\left(Q_iP_{i+1}Q_{i+1}^{-1}\right)^2\cdot\prod_{\substack{i=0\\ Q_i=R\\ P_{i+1}=t_4}}^{f-1}\left(Q_iP_{i+1}Q_{i+1}^{-1}\right)^2\cdot\\ \prod_{\substack{i=0\\ Q_i=R\\ P_{i+1}=t_3}}^{f-1}\left(Q_iP_{i+1}Q_{i+1}^{-1}\right)^2\cdot&\prod_{\substack{i=0\\ Q_i=R\\ P_{i+1}=t_1}}^{f-1}\left(Q_iP_{i+1}Q_{i+1}^{-1}\right)^2\cdot\prod_{\substack{i=0\\ Q_i=R\\ P_{i+1}=t_2}}^{f-1}\left(Q_iP_{i+1}Q_{i+1}^{-1}\right)^2.\end{aligned}$$

We notice that when $Q_i=Id$ and $P_{i+1}=t_4$, then by formula 6.2, $Q_{i+1}=R$ and $Q_iP_{i+1}Q_{i+1}^{-1}=\operatorname{diag}\left(p^{k_{i+1}},1\right)$. Therefore the product $\prod_{\substack{i=0\\ Q_i=Id\\ P_{i+1}=t_4}}^{f-1}\left(Q_iP_{i+1}Q_{i+1}^{-1}\right)$ has no contribution to the $(2,2)$ entry of $\prod_{i=0}^{2f-1}\left(Q_iP_{i+1}Q_{i+1}^{-1}\right)$. Similarly, the products

$$\prod_{\substack{i=0\\ Q_i=Id\\ P_{i+1}=t_1}}^{f-1}\left(Q_iP_{i+1}Q_{i+1}^{-1}\right),\ \prod_{\substack{i=0\\ Q_i=R\\ P_{i+1}=t_3}}^{f-1}\left(Q_iP_{i+1}Q_{i+1}^{-1}\right)\ \text{and}\ \prod_{\substack{i=0\\ Q_i=R\\ P_{i+1}=t_2}}^{f-1}\left(Q_iP_{i+1}Q_{i+1}^{-1}\right)$$

have no contribution to the $(2,2)$ entry of $\prod_{i=0}^{2f-1}\left(Q_iP_{i+1}Q_{i+1}^{-1}\right)$. We now compute the product $\prod_{\substack{i=0\\ Q_i=R\\ P_{i+1}=t_1}}^{f-1}\left(Q_iP_{i+1}Q_{i+1}^{-1}\right)$. Formula 6.2 implies that if $Q_i=R$ and $P_{i+1}=t_1$ then $Q_{i+1}=R$, therefore $Q_iP_{i+1}Q_{i+1}^{-1}=\operatorname{diag}\left(1,p^{k_{i+1}}\right)$. Again, by formula 6.2, $Q_i=R$ and $P_{i+1}=t_1$ is equivalent to $Q_{i+1}=R$ and $P_{i+1}=t_1$. Hence

$$\prod_{\substack{i=0\\Q_i=R\\P_{i+1}=t_1}}^{f-1}\left(Q_iP_{i+1}Q_{i+1}^{-1}\right)=\prod_{\substack{i=0\\Q_{i+1}=R\\P_{i+1}=t_1}}^{f-1}\left(Q_iP_{i+1}Q_{i+1}^{-1}\right)=\prod_{\substack{i=0\\Q_i=R\\P_i=t_1}}^{f-1}\operatorname{diag}\left(1,p^{k_{i+1}}\right)$$

which contributes the fourth summand of the right hand side of equation 6.11. The claim made before formula 6.11 follows arguing similarly for the remaining cases. Hence $\mathrm{v_p}\left(\mathrm{Nm}_\varphi\left(\vec{\mu}\right)\right)=\sum\limits_{\substack{i=0\\z_i=0}}^{2f-1}k_i$. Proposition 6.3 implies that $\left(V_{\vec{k},\vec{0}}^{\vec{i}}\right)_{|G_{K_{2f}}}$ is reducible and $(\mathbb{D}_2,\varphi)$ is a weakly admissible submodule, where $\mathbb{D}_2=\left(E^{2f}\right)\cdot\zeta_2$. By [Dou10, proof of Prop. 4.3] (or by a direct computation),

$$\mathrm{Fil^j}\mathbb{D}_2=\begin{cases}\mathbb{D}_2 & \text{if}\quad j\leq 0,\\ \left(E^{|\tau_{K_{2f}}|}\right)f_{I_{i,\vec{z}}}\zeta_2 & \text{if}\ \ 1+w_{i-1}\leq j\leq w_i \text{ for all } i=0,1,...,t-1,\\ 0 & \text{if}\ \ j\geq 1+w_{t-1},\end{cases}\tag{6.12}$$

where $I_{i,\vec{z}}=I_i\cap\{j\in\{0,1,...,2f-1\}:z_j=0\}$. As in the proof of Proposition 6.12, the labeled weight for the embedding τ_i is 0 if $z_i=1$ and $-k_i$ if $z_i=0$. Next, we prove that for $i=0,1,...,f-1$,

$$z_i=z_{i+f}=\begin{cases}0 \text{ if } \ell_i=0,\\ 1 \text{ if } \ell_i=k_i>0.\end{cases}\tag{6.13}$$

This is proved exactly as in Proposition 6.12, taking into account that an even number of the coordinates of $(P_1,P_2,...,P_f)$ is of even type. We have $z_i=0$ for all i if and only if $\ell_i=0$ for all i if and only if $(\Pi_1,\Pi_2,...,\Pi_f)\in C_1$ and $z_i=1$ for all i if and only if $\ell_i=k_i>0$ for all i if and only if $(\Pi_1,\Pi_2,...,\Pi_f)\in C_2$, cases excluded. Therefore neither of the summands of $V_{\vec{k}}^{\vec{i}}(\vec{0})$ is unramified. By the discussion above the labeled weights of $\mathbb{D}_2$ are $\left(-\ell_0',-\ell_1',...,-\ell_{f-1}',-\ell_0',-\ell_1',...,-\ell_{f-1}'\right)$. By formula 6.13, $\mathrm{v_p}\left(\mathrm{Nm}_\varphi\left(\vec{\mu}\right)\right)=\sum\limits_{\substack{i=0\\z_i=0}}^{2f-1}k_i=\sum\limits_{i=0}^{2f-1}\ell_i'$. By Proposition 3.5 and Lemma 3.7, the corresponding crystalline character is $\psi=\chi_{e_0}^{\ell_1'}\cdot\chi_{e_1}^{\ell_2'}\cdot\cdots\cdot\chi_{e_{f-2}}^{\ell_{f-1}'}\cdot\chi_{e_{f-1}}^{\ell_0'}\cdot\chi_{e_0}^{\ell_1'}\cdot\chi_{e_1}^{\ell_2'}\cdot\cdots\cdot\chi_{e_{f-2}}^{\ell_{f-1}'}\cdot\chi_{e_{f-1}}^{\ell_0'}$. If $V_{\vec{k}}^{\vec{i}}(\vec{0})$ is irreducible, then by Frobenius reciprocity $V_{\vec{k}}^{\vec{i}}(\vec{0})=\mathrm{Ind}_{K_{2f}}^{K_f}(\psi)$, which is absurd by Corollary 3.10. Hence $V_{\vec{k}}^{\vec{i}}(\vec{0})$ is reducible and contains an irreducible constituent which restricts to ψ. By Lemma 3.7(iv) the only choices are $\eta_{\pm1}\cdot\chi_{e_0}^{\ell_1}\cdot\chi_{e_1}^{\ell_2}\cdot\cdots\cdot\chi_{e_{f-2}}^{\ell_{f-1}}\cdot\chi_{e_{f-1}}^{\ell_0}$, and we are done after twisting by $\eta_{\mp1}$. The rest of the proposition follows as in the proof of Proposition 6.12. □

THEOREM 6.19. *Theorem 1.7 holds.*

Proof. Follows from Proposition 6.18, taking into account Remark 6.13. □

EXAMPLE 6.20. *Let $f = 2$, $\ell_0 = 0$ and $\ell_1 = k_1$. Let $(\Pi_1, \Pi_0) = (t_1, t_3)$ with $c_0 = c_1 = 1$. After possibly twisting by $\eta_{\pm 1}$ we have $V_{\vec{k}}^{(1,3)}(\vec{0}) \simeq \chi_{e_0}^{k_1} \oplus \chi_{e_1}^{k_0}$.*

In the next proposition we study closer the F-semisimple members of this family assuming that $c = 1$.

PROPOSITION 6.21. *Assume that $V_{\vec{k}}^{(1,3)}(\vec{\alpha})$ is F-semisimple.*

(i) $V_{\vec{k}}^{(1,3)}(\vec{\alpha})$ is irreducible if and only if $\alpha_0\alpha_1 \neq 0$, and is non-induced for all but finitely many such $\vec{\alpha}$;

(ii) $V_{\vec{k}}^{(1,3)}(\vec{\alpha})$ is non-split reducible if and only if precisely one of the coordinates α_i of $\vec{\alpha}$ is zero;

(iii) The families $\left\{V_{\vec{k}}^{(1,3)}((\alpha_0, 0)),\ \alpha_0 \in p^m\mathfrak{m}_E \setminus \{0\}\right\}$ and $\left\{V_{\vec{k}}^{(1,3)}((0, \alpha_1)),\ \alpha_1 \in p^m\mathfrak{m}_E \setminus \{0\}\right\}$ are disjoint;

(iv) $V_{\vec{k}}^{(1,3)}(\vec{0})$ is split-reducible.

Proof. The weakly admissible filtered φ-module corresponding to $V_{\vec{k}}^{(1,3)}(\vec{\alpha})$ has Frobenius endomorphism

$$(\varphi(\eta_1), \varphi(\eta_2)) = (\eta_1, \eta_2)\begin{pmatrix} (p^{k_1}, 1) & (0, \alpha_0) \\ (\alpha_1, 0) & (1, p^{k_0}) \end{pmatrix}$$

and filtration

$$\mathrm{Fil^j}(\mathbb{D}) = \begin{cases} (E\times E)\eta_1 \bigoplus (E\times E)\eta_2 & \text{if } j \leq 0, \\ (E\times E) f_{I_0}(\vec{x}\eta_1 + \vec{y}\eta_2) & \text{if } 1 \leq j \leq w_0, \\ (E\times E) f_{I_1}(\vec{x}\eta_1 + \vec{y}\eta_2) & \text{if } 1 + w_0 \leq j \leq w_1, \\ 0 & \text{if } j \geq 1 + w_1, \end{cases} \tag{6.14}$$

with $\vec{x} = (-\alpha_0, 1)$ and $\vec{y} = (1, -\alpha_1)$. We diagonalize the matrix of Frobenius arguing as in the proof of Proposition 2.2 in [Dou10]. The characteristic polynomial is $X^2 - \left(p^{k_0} + p^{k_1} + \alpha_0\alpha_1\right)X + p^{k_0+k_1}$, and we assume that $\left(p^{k_0} + p^{k_1} + \alpha_0\alpha_1\right)^2 \neq 4p^{k_0+k_1}$ so that its roots ε_0 and ε_1 are distinct. We have the following cases.
Case (1). $\alpha_0\alpha_1 \neq 0$. We change the basis to $\underline{\xi} = (\xi_1, \xi_2)$, where

$$\xi_1 = \left(\left(\varepsilon_0 - p^{k_1} - \alpha_0\alpha_1\right)\alpha_1,\ \frac{\alpha_0(\varepsilon_0-\varepsilon_1)\left(\varepsilon_0 - p^{k_0}\right)\left(\varepsilon_0 - p^{k_0} - \alpha_0\alpha_1\right)\left(\varepsilon_1 - p^{k_1} - \alpha_0\alpha_1\right)}{\left(2\varepsilon_0\varepsilon_1 - p^{k_0}\varepsilon_1 - p^{k_1}\varepsilon_0 - \alpha_0\alpha_1\varepsilon_1\right)\left(\varepsilon_1 - p^{k_0} - \alpha_0\alpha_1\right)}\right)\eta_1$$

$$+ \left(\left(\varepsilon_0 - p^{k_1} - \alpha_0\alpha_1\right)\alpha_1,\ \frac{\alpha_0(\varepsilon_0-\varepsilon_1)\left(\varepsilon_1 - p^{k_0}\right)\left(\varepsilon_0 - p^{k_0} - \alpha_0\alpha_1\right)\left(\varepsilon_1 - p^{k_1} - \alpha_0\alpha_1\right)}{\left(2\varepsilon_0\varepsilon_1 - p^{k_0}\varepsilon_1 - p^{k_1}\varepsilon_0 - \alpha_0\alpha_1\varepsilon_1\right)\left(\varepsilon_1 - p^{k_0} - \alpha_0\alpha_1\right)}\right)\eta_2$$

and

$$\xi_2 = \left(\left(\varepsilon_1 - p^{k_1} - \alpha_0\alpha_1\right)\left(\varepsilon_0 - p^{k_1}\right), \frac{\alpha_0^2(\varepsilon_0-\varepsilon_1)\left(\varepsilon_1-p^{k_1}-\alpha_0\alpha_1\right)\left(\varepsilon_1-p^{k_0}-\alpha_0\alpha_1\right)}{\left(2\varepsilon_0\varepsilon_1-p^{k_0}\varepsilon_1-p^{k_1}\varepsilon_0-\alpha_0\alpha_1\varepsilon_1\right)\left(\varepsilon_1-p^{k_0}-\alpha_0\alpha_1\right)}\right)\eta_1$$

$$+\left(\left(\varepsilon_1 - p^{k_1} - \alpha_0\alpha_1\right)\left(\varepsilon_1 - p^{k_1}\right), \frac{\alpha_0^2(\varepsilon_0-\varepsilon_1)\left(\varepsilon_1-p^{k_1}-\alpha_0\alpha_1\right)\left(\varepsilon_1-p^{k_0}-\alpha_0\alpha_1\right)}{\left(2\varepsilon_0\varepsilon_1-p^{k_0}\varepsilon_1-p^{k_1}\varepsilon_0-\alpha_0\alpha_1\varepsilon_1\right)\left(\varepsilon_1-p^{k_0}-\alpha_0\alpha_1\right)}\right)\eta_2$$

In the ordered basis $\underline{\xi}$ we have $\varphi(\xi_1) = (1,\ \varepsilon_0)\,\xi_1$ and $\varphi(\xi_2) = \left(\lambda,\ \frac{\varepsilon_1}{\lambda}\right)\xi_2$, where

$$\lambda=-\frac{\left(\varepsilon_0 - p^{k_1} - \alpha_0\alpha_1\right)}{\left(\varepsilon_1 - p^{k_1} - \alpha_0\alpha_1\right)}\cdot\frac{\left(\varepsilon_1 - p^{k_0} - \alpha_0\alpha_1\right)}{\left(\varepsilon_0 - p^{k_0} - \alpha_0\alpha_1\right)}\cdot\frac{\left(2\varepsilon_0\varepsilon_1 - p^{k_0}\varepsilon_1 - p^{k_1}\varepsilon_0 - \alpha_0\alpha_1\varepsilon_1\right)}{\left(2\varepsilon_0\varepsilon_1 - p^{k_0}\varepsilon_0 - p^{k_1}\varepsilon_1 - \alpha_0\alpha_1\varepsilon_0\right)},$$

and the filtration is as in formula 6.14, with $\vec{x}\eta_1 + \vec{y}\eta_2$ replaced by $\xi_1 + \xi_2$. By Proposition 6.3, the representation $V_{\vec{k}}^{(1,3)}(\,\vec{\alpha})$ is irreducible. Arguing as in the proof of Proposition 6.12(iv) we see that the representations $V_{\vec{k}}^{(1,3)}(\,\vec{\alpha})$ are non-induced with the possibility of at most finitely many exceptions.

Case (2). $\alpha_0 = 0,\ \alpha_1 \neq 0$. We argue as above and see that in the ordered basis $\underline{\xi} = (\xi_1, \xi_2)$, where

$$\xi_1 = \eta_2 \text{ and } \xi_2 = \left(1,\ \frac{p^{k_0} - p^{k_1}}{\alpha_1 p^{k_1}}\right)\eta_1 - \left(\frac{\alpha_1}{p^{k_0} - p^{k_1}},\ p^{k_0 - k_1}\right)\eta_2$$

we have $\varphi(\xi_1) = \left(1,\ p^{k_0}\right)\xi_1$ and $\varphi(\xi_2) = \left(\lambda(\alpha_1),\ \frac{p^{k_1}}{\lambda(\alpha_1)}\right)\xi_2$, with $\lambda(\alpha_1) = \alpha_1^{-1}\left(p^{k_0} - p^{k_1}\right)$. The filtration in this basis is given by formula 6.14, with $\vec{x}\eta_1 + \vec{y}\eta_2$ replaced by $\xi_1 + (0,1)\,\xi_2$. By Proposition 6.3 the representation $V_{\vec{k}}^{(1,3)}((0,\alpha_1))$ is reducible and non-split.

Case (3). $\alpha_1 = 0, \alpha_0 \neq 0$. In the ordered basis $\underline{\xi} = (\xi_1, \xi_2)$, where

$$\xi_1 = \eta_2 - \left(\frac{p^{k_1}\alpha_0}{p^{k_1} - p^{k_0}},\ \frac{\alpha_0}{p^{k_1} - p^{k_0}}\right)\eta_1 \text{ and } \xi_2 = \left(\frac{\alpha_0 p^{k_0}}{p^{k_1} - p^{k_0}},\ 1\right)\eta_1,$$

we have $\varphi(\xi_1) = \left(1,\ p^{k_0}\right)\xi_1$ and $\varphi(\xi_2) = \left(\lambda(\alpha_0),\ \frac{p^{k_1}}{\lambda(\alpha_0)}\right)\xi_2$, with $\lambda(\alpha_0) = \alpha_0^{-1}\left(p^{k_1} - p^{k_0}\right)p^{k_1 - k_0}$. The filtration in the basis ξ is given by formula 6.14, with $\vec{x}\eta_1 + \vec{y}\eta_2$ replaced by $(1,0)\,\xi_1 + \xi_2$. By Proposition 6.3, $V_{\vec{k}}^{(1,3)}((\alpha_0,0))$ is reducible, non-split. By [Dou10, Proposition 7.1] it follows that there are no isomorphisms between members of the families $\left\{V_{\vec{k}}^{(1,3)}((\alpha_0,0)),\ \alpha_0 \in p^m\mathfrak{m}_E \setminus \{0\}\right\}$ and $\left\{V_{\vec{k}}^{(1,3)}((0,\alpha_1)),\ \alpha_1 \in p^m\mathfrak{m}_E \setminus \{0\}\right\}$.

Case (4). $\alpha_0 = \alpha_1 = 0$. Then $\varphi(\eta_1) = \left(p^{k_1}, 1\right)\eta_1$ and $\varphi(\eta_2) = \left(1, p^{k_0}\right)\eta_2$, while the filtration is as in formula 6.14,with $\vec{x} = (0,1)$ and $\vec{y} = (1,0)$. Since $J_{\vec{x}} \cap J_{\vec{y}} = \varnothing$, Proposition 6.3 implies that $V_{\vec{k}}^{(1,3)}(\vec{0})$ is split-reducible. $\square$

PROPOSITION 6.22. *Let* $0 < \mathrm{v_p}(\varepsilon_i) < k_0 + k_1$ *with* $\varepsilon_0 \neq \varepsilon_1$ *such that* $\varepsilon_0\varepsilon_1 = p^{k_0+k_1}$ *and assume that* $0 \leq k_i \leq p-1$. *Define the families of filtered* φ*-modules* $\mathbb{D}(\lambda)$ *with*

$$\varphi(\eta_1) = (1, \varepsilon_0)\,\eta_1, \;\; \varphi(\eta_2) = \left(\lambda, \frac{\varepsilon_1}{\lambda}\right)\eta_2,$$

and filtrations as in formula 6.10 with $\vec{x} = \vec{y} = \vec{1}$. *These filtered modules are weakly admissible, irreducible, sharing the same characteristic polynomial and filtration. Let* $V(\lambda)$ *be the corresponding to* $\mathbb{D}(\lambda)$ *crystalline representations of* $G_{\mathbb{Q}_{p^2}}$.

(i) *If* $\lambda = \frac{\varepsilon_1}{\varepsilon_0}\left(\frac{p^{k_1}\alpha-\varepsilon_0}{p^{k_1}\alpha-\varepsilon_1}\right)$, *where* $\alpha \in m_E$, *then* $\left(\overline{V(\lambda)}_{|I_{\mathbb{Q}_{p^2}}}\right)^{ss} = \omega_{4,\bar{\tau}_0}^{-(k_0+pk_1)} \bigoplus \omega_{4,\bar{\tau}_0}^{-(k_0+pk_1)p^2}$ *and* $\overline{V(\lambda)}$ *is irreducible;*

(ii) *If* $\lambda = \left(\frac{\varepsilon_1}{\varepsilon_0}\right)^2\left(\frac{p^{k_1}\alpha-\varepsilon_1}{p^{k_1}\alpha-\varepsilon_0}\right)$, *where* $\alpha \in m_E$, *then* $\left(\overline{V(\lambda)}_{|I_{\mathbb{Q}_{p^2}}}\right)^{ss} = \omega_{4,\bar{\tau}_0}^{-\left(pk_1+p^2k_0\right)} \bigoplus \omega_{4,\bar{\tau}_0}^{-\left(pk_1+p^2k_0\right)p^2}$ *and* $\overline{V(\lambda)}$ *is irreducible;*

(iii) *If* $\lambda = 1$, *then* $\overline{V(\lambda)}$ *is reducible and* $\overline{V(\lambda)}_{|I_{\mathbb{Q}_{p^2}}} = \omega_{2,\bar{\tau}_0}^{-k_1} \bigoplus \omega_{2,\bar{\tau}_0}^{-pk_0}$.

Proof. The common characteristic polynomial is $X^2 - (\varepsilon_0 + \varepsilon_1)X + p^{k_0+k_1}$. Parts (i) and (ii) follow from Examples 6.15 (i) and (iii) using the "standard parametrization" for the families $V_{\vec{k},\vec{a}}^{(1,2)}$ and $V_{\vec{k},\vec{a}}^{(1,4)}$, and taking into account that $m = 0$ and Proposition 6.8. Part (iii) follows from Proposition 6.21(i) and a little computation to prove that if $p^{k_0} + p^{k_1} + \alpha_0\alpha_1 = \varepsilon_0 + \varepsilon_1$ and $\varepsilon_0\varepsilon_1 = p^{k_0+k_1}$, then $\lambda = 1$. ☐

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