Duality and Integrability on Contact Fano Manifolds

Jarosław Buczyński [1]

Received: February 2 2010
Revised: June 8, 2010

Communicated by Thomas Peternell

Abstract. We address the problem of classification of contact Fano manifolds. It is conjectured that every such manifold is necessarily homogeneous. We prove that the Killing form, the Lie algebra grading and parts of the Lie bracket can be read from geometry of an arbitrary contact manifold. Minimal rational curves on contact manifolds (or contact lines) and their chains are the essential ingredients for our constructions.

2010 Mathematics Subject Classification: Primary: 14M17; Secondary: 53C26, 14M20, 14J45;
Keywords and Phrases: complex contact manifold, Fano variety, minimal rational curves, adjoint variety, Killing form, Lie bracket, Lie algebra grading

1 Introduction

In this article we are interested in the classification of contact Fano manifolds. We review the relevant definitions in §2. So far the only known examples of contact Fano manifolds are obtained as follows. For a simple Lie group G consider its adjoint action on $\mathbb{P}(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G. This action has a unique closed orbit X and this X has a natural contact structure. In this situation X is called a *projectivised minimal nilpotent orbit*, or the *adjoint variety* of G. By the duality determined by the Killing form, equivalently we can consider the coadjoint action of G on $\mathbb{P}(\mathfrak{g}^*)$ and X is isomorphic to the unique closed orbit in $\mathbb{P}(\mathfrak{g}^*)$.

[1] Dedicated in memory of Marcin Hauzer.

Conjecture 1.1 (LeBrun, Salamon). *If X is a Fano complex contact manifold, then X is the adjoint variety of a simple Lie group G.*

This problem originated with a problem in Riemannian geometry. In [Ber55] a list of all possible holonomy groups of simply connected Riemannian manifolds is given. The existence problem for all the cases has been solved locally. Also compact non-homogeneous examples with most of the possible holonomy groups were constructed with the unique exception of the quaternion-Kähler manifolds. It is conjectured that the compact quaternion-Kähler manifolds must be homogeneous (see [LeB95] and references therein).

Conjecture 1.2 (LeBrun, Salamon). *Let M be a compact quaternion-Kähler manifold. Then M is a homogeneous symmetric space (more precisely, it is one of the Wolf spaces — see [Wol65]).*

The relation between the conjectures is given by the construction of a twistor space. The S^2-bundle of complex structures on tangent spaces to a quaternion-Kähler manifold M is called the twistor space of M. If M is compact, it has positive scalar curvature, and then the twistor space X has a natural complex structure and is a contact Fano manifold with a Kähler-Einstein metric. In particular, the twistor space of a Wolf space is an adjoint variety. Hence Conjecture 1.1 implies Conjecture 1.2. Conversely, if X is a contact Fano manifold with Kähler-Einstein metric, then it is a twistor space of a quaternion-Kähler manifold — see [LeB95].

In order to study the non-homogeneous contact manifolds (potentially non-existent) it is natural to assume $\operatorname{Pic} X \simeq \mathbb{Z}$ and further that X is not isomorphic to a projective space. This only exludes the adjoint varieties of types A and C (see §2 for more details).

With this assumption, we take a closer look at three pieces of the homogeneous structure on adjoint varieties: the Killing form B on $\mathfrak{g}$, the Lie algebra grading $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ (see [LM02, §6.1] and references therein) and a part of the Lie bracket on $\mathfrak{g}$. Understanding the underlying geometry allows us to define the appropriate generalisations of these notions on arbitrary contact Fano manifolds.

An essential building block for our constructions is the notion of a *contact line* (or simply *line*) on X. These contact lines were studied by Kebekus [Keb01], [Keb05] and Wiśniewski [Wiś00]. Also they are an instance of minimal rational curves, which are studied extensively. The geometry of contact lines was the original motivation to study Legendrian subvarieties in projective space (see [Bucz09] for an overview and many details). We briefly review the subject of lines on contact Fano manifolds in §3.1.

The key ingedient is the construction of a family of divisors D_x parametrised by points $x \in X$ (see §3.3). These divisors are swept by pairs of intersecting contact lines, one of which passes through x. In other words, set theoretically D_x is the set of points of X, which can be joined with x using at most 2 intersecting contact lines. The idea to study these loci comes from Wiśniewski [Wiś00] where he observed, that (under an additional minor assumption) these loci contain some non-trivial divisorial components and he studied the intersection numbers of certain curves on X with the divisorial components. Here we prove all the components of D_x are divisorial and draw conclusions from that observation going into a different direction than those of [Wiś00].

Theorem 1.3. *Let X be a contact Fano manifold with $\operatorname{Pic} X \simeq \mathbb{Z}$ and assume X is not isomorphic to a projective space. Then the locus $D_x \subset X$ swept by the pairs of intersecting contact lines, one of which passes through $x \in X$ is of pure codimension 1 and thus D_x determines a divisor on X. Let $\langle D\rangle \subset H^0(\mathcal{O}(D_x))$ be the linear system spanned by these divisors. Let $\phi\colon X \to \mathbb{P}\langle D\rangle^*$ be the map determined by the linear system $\langle D\rangle$ and let $\psi\colon X \to \mathbb{P}\langle D\rangle$ be the map $x \mapsto D_x$. Then:*

(i) both ϕ and ψ are regular maps.

(ii) there exists a unique up to scalar non-degenerate bilinear form B on $\langle D\rangle$, which determines an isomorphism $\mathbb{P}\langle D\rangle^* \simeq \mathbb{P}\langle D\rangle$ making the following diagram commutative:

$$\begin{tikzcd} & \mathbb{P}\langle D\rangle^* \arrow[dd, "\simeq"] \\ X \arrow[ur, "\phi"] \arrow[dr, "\psi"] & \\ & \mathbb{P}\langle D\rangle. \end{tikzcd}$$

(iii) The bilinear form B is either symmetric or skew-symmetric.

(iv) If $X \subset \mathbb{P}(\mathfrak{g}^*)$ is the adjoint variety of simple Lie group G, then $\langle D\rangle = \mathfrak{g}$ and B is the Killing form on $\mathfrak{g}$.

With the notation of the theorem, after fixing a pair of general points $x, w \in X$ there are certain natural linear subspaces of $\langle D\rangle$, which we denote $\langle D\rangle_{-2}$, $\langle D\rangle_{-1}$, $\langle D\rangle_0$, $\langle D\rangle_1$ and $\langle D\rangle_2$ (see §5 for details).

Theorem 1.4. *If $X \subset \mathbb{P}(\mathfrak{g}^*)$ is the adjoint variety of a simple Lie group G with $\operatorname{Pic} X \simeq Z$ and X not isomorphic to a projective space, then there exists a choice of a maximal torus of G and a choice of order of roots of $\mathfrak{g}$, such that $\langle D\rangle_i = \mathfrak{g}_i$ for every $i \in \{-2, -1, 0, 1, 2\}$, where $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is the Lie algebra grading of $\mathfrak{g}$.*

Finally, if X is the adjoint variety of G, then there is a rational map

$$[\cdot,\cdot]\colon X \times X \dashrightarrow \mathbb{P}(\mathfrak{g}),$$

which is the Lie bracket on $\mathfrak{g}$ (up to projectivisation). Also there is a divisor $D \subset X \times X$, such that for general $(x, z) \in D$ the Lie bracket $[x, z]$ is in X. We recover this bracket restricted to D for general contact manifolds:

Theorem 1.5. *For X and D_x and in Theorem 1.3, let $D \subset X \times X$ be the divisor consisting of pairs $(x, z) \in X \times X$, such that $z \in D_x$. There exists a rational map $[\cdot,\cdot]^D : D \dashrightarrow X$, such that $[x, z]^D = y$, where y is an intersection point of a pair of contact lines that join x and z. In particular, this intersection point y and the pair of lines are unique for general pair $(x, z) \in D$. Moreover, if X is the adjoint variety of a simple Lie group G, then $[\cdot,\cdot]^D$ is the restricion of the Lie bracket.*

In §2 we introduce and motivate our assumptions and notation.
In §3 we review the notion of contact lines and their properties. We continue by studying certain types of loci swept by those lines and calculate their dimensions. In particular we prove there Theorem 3.6, which is a part of results summarised in Theorem 1.3. We also study the tangent bundle to D_x as a subspace of TX.
In §4 we study the duality of maps ϕ and ψ introduced in Theorem 1.3 together with the consequences of this duality. This section is culminated with the proof of Theorem 1.3.
In §5 we generalise the Lie algebra grading to arbitrary contact manifolds and prove Theorem 1.4.
In §6 we prove that certain lines are integrable with respect to a special distribution on D_x and we apply this to prove Theorem 1.5.

Acknowledgements

The author is supported by Marie Curie International Outgoing Fellowship.
The author would like to thank the following people for many enlightening discussions: Jun-Muk Hwang, Stefan Kebekus, Joseph M. Landsberg, Jarosław Wiśniewski and Fyodor L. Zak. Comments of Laurent Manivel helped to improve the presentation in the paper.

2 Preliminaries

Throughout the paper all our projectivisations $\mathbb{P}$ are naive. This means, if V is a vector space, then $\mathbb{P}V = (V \setminus 0)/\mathbb{C}^*$, and similarly for vector bundles.
A complex manifold X of dimension $2n + 1$ is *contact* if there exists a vector subbundle $F \subset TX$ of rank $2n$ fitting into an exact sequence:

$$0 \to F \to TX \xrightarrow{\theta} L \to 0$$

such that the derivative $\mathrm{d}\theta \in H^0(\bigwedge^2 F^* \otimes L)$ of the twisted form $\theta \in H^0(T^*X \otimes L)$ is nowhere degenerate. In particular, $\mathrm{d}\theta_x$ is a symplectic form on the fibre of contact distribution F_x. See [Bucz09, §E.3 and Chapter C] and references therein for an overview of the subject.
A projective manifold X is Fano, if its anticanonical divisor $K_X{}^* = \bigwedge^{\dim X} TX$ is ample.

If X is a projective contact manifold, then by Theorem of Kebekus, Peternell, Sommese and Wiśniewski [KPSW00] combined with a result by Demailly [Dem02], X is either a projectivisation of a cotangent bundle to a smooth projective manifold or X is a contact Fano manifold, with $\operatorname{Pic} X \simeq \mathbb{Z}$. In the second case, since $K_X \simeq (L^*)^{\otimes(n+1)}$, by [KO73], either $X \simeq \mathbb{P}^{2n+1}$ or $\operatorname{Pic} X = \mathbb{Z} \cdot [L]$. Here we are interested in the case $X \not\simeq \mathbb{P}^{2n+1}$. Thus our assumption spelled out below only exclude some well understood cases (the projectivised cotangent bundles and the projective space) and they agree with the assumptions of Theorems 1.3, 1.4 and 1.5.

Notation 2.1. Throughout the paper X denotes a contact Fano manifold with $\operatorname{Pic} X$ generated by the class of L, where $L = TX/F$ and $F \subset TX$ is the contact distribution on X. We also assume $\dim X = 2n+1$.

From Theorem of Ye [Ye94] it follows that $n \geq 2$.
We will also consider the homogeneous examples of contact manifolds (i.e. the adjoint varieties). Thus we fix notation for the Lie group and its Lie algebra.

Notation 2.2. Throughout the paper G denotes a simple complex Lie group, not of types A or C (i.e. not isomorphic to SL_n nor Sp_{2n} nor their discrete quotients). Further $\mathfrak{g}$ is the Lie algebra of G. Thus $\mathfrak{g}$ is one of $\mathfrak{so}_n$ (types B and D), or one of the exceptional Lie algebras $\mathfrak{g}_2$, $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$ or $\mathfrak{e}_8$.

The contact structure on $\mathbb{P}^{2n-1} = \mathbb{P}(\mathbb{C}^{2n})$ is determined by a symplectic form ω on $\mathbb{C}^{2n}$. The precise relation between the contact and symplectic structures is decribed for instance in [Bucz09, §E.1] (see also [LeB95, Ex. 2.1]). In particular, for all $x \in X$, the projectivisation of a fibre of the contact distribution $\mathbb{P}F_x$ comes with a natural contact structure.
Let M be a projective contact manifold (in our case $M = X$ with X as in Notation 2.1 or $M = \mathbb{P}^{2n-1}$). A subvariety $Z \subset M$ is *Legendrian*, if for all smooth points $z \in Z$ the tangent space $T_z Z$ is contained in the contact distribution of M and Z is of pure dimension $\frac{1}{2}(\dim M - 1)$.
Recall from [Har95, Lecture 20] or [Mum99, III.§3,§4] the notion of *tangent cone*. For a subvariety $Z \subset X$, and a point $x \in Z$ let $\tau_x Z \subset T_x X$ be the tangent cone of Z at x. In this article we will only need the following elementary properties of the tangent cone:

- $\tau_x Z$ is an affine cone (i.e. it is invariant under the standard action of $\mathbb{C}^*$ on $T_x X$).
- $\dim_x Z = \dim \tau_x Z$ and thus if Z is irreducible, then $\dim Z = \dim \tau_x Z$.
- If $x \in Z_1 \subset Z_2 \subset X$, then $\tau_x Z_1 \subset \tau_x Z_2$.
- If Z is smooth at x, then $\tau_x Z = T_x Z$.

Since $\tau_x Z$ is a cone, let $\mathbb{P}\tau_x Z \subset \mathbb{P}T_x X$ be the corresponding projective variety.

3 Loci swept out by lines

A rational curve $l \subset X$ is a *contact line* (or simply a *line*) if $\deg L|_l = 1$.
Let $RatCurves^n(X)$ be the normalised scheme parametrising rational curves on X, as in [Kol96, II.2.11]. Let $Lines(X) \subset RatCurves^n(X)$ be the subscheme parametrising lines. Then every component of $Lines(X)$ is a minimal component of X in the sense of [HM04]. We fix $\mathcal{H} \neq \emptyset$ a union of some irreducible components of $Lines(X)$.
By a slight abuse of notation, from now on we say l is a *(contact) line* if and only if $l \in \mathcal{H}$. For simplicity, the reader may choose to restrict his attention to one of the extreme cases: either to the case $\mathcal{H} = Lines(X)$ (and thus be consistent with [Wiś00] and the first sentence of this section) or to the case where $\mathcal{H}$ is one of the irreducible components of $Lines(X)$ (and thus be consistent with [Keb01, Keb05]). In general it is expected that $Lines(X)$ (with X as in Notation 2.1) is irreducible and all the cases are the same.

3.1 Legendrian varieties swept by lines

We denote by $C_x \subset X$ the locus of contact lines through $x \in X$. Let $\mathscr{C}_x := \mathbb{P}\tau_x C_x \subset \mathbb{P}(TX)$. Note that with our assumptions both C_x and $\mathscr{C}_x$ are closed subsets of X or $\mathbb{P}(T_x X)$ respectively.
The following theorem briefly summarises results of [Keb05] and earlier:

Theorem 3.1. *With X as in Notation 2.1 let $x \in X$ be any point. Then:*

(i) There exist lines through x, in particular C_x and $\mathscr{C}_x$ are non-empty.

(ii) C_x is Legendrian in X and $\mathscr{C}_x \subset \mathbb{P}(F_x)$ and $\mathscr{C}_x$ is Legendrian in $\mathbb{P}(F_x)$.

(iii) If in addition x is a general point of X, then $\mathscr{C}_x$ is smooth and each irreducible component of $\mathscr{C}_x$ is linearly non-degenerate in $\mathbb{P}(F_x)$. Further C_x is isomorphic to the projective cone over $\mathscr{C}_x \subset \mathbb{P}(F_x)$, i.e. $C_x \simeq \tilde{\mathscr{C}}_x \subset \mathbb{P}(F_x \oplus \mathbb{C})$, in such a way that lines through x are mapped bijectively onto the generators of the cone and restriction of L to C_x via this isomorphism is identified with the restriction of $\mathcal{O}_{\mathbb{P}(F_x \oplus \mathbb{C})}(1)$ to $\tilde{\mathscr{C}}_x$. In particular all lines through x are smooth and two different lines intersecting at x will not intersect anywhere else, nor they will share a tangent direction.

Proof. Part (i) is proved in [Keb01, §2.3].
The proof of (ii) is essentially contained in [KPSW00, Prop. 2.9]. Explicit statements are in [Keb01, Prop. 4.1] for C_x and in [Wiś00, Lemma 5] for $\mathscr{C}_x$. Also [HM99] may claim the authorship of this observation, since the proof in the homogeneous case is no different than in the general case.
Assume $x \in X$ is a general point. The statements of (iii) are basically [Keb05, Thm 1.1], which however assumes (in the statement) that $\mathcal{H}$ is irreducible. This is never used in the proof, with the exception of the argument for the irreducibility of C_x — see however Remark 3.2. Thus $\mathscr{C}_x$ is smooth and C_x is

isomorphic to the cone over $\mathscr{C}_x$ as claimed. Each irreducible component $\mathscr{C}_x$ is non-degenerate in $\mathbb{P}F_x$ by [Keb01, Thm 4.4] — again the statement is only for $\mathscr{C}_x$, not for its components, however the proof stays correct in this more general setup. In particular, [Keb01, Lemma 4.3] implies that C_x polarised by $L|_{C_x}$ is not isomorphic with a linear subspace with polarised by $\mathcal{O}(1)$. Thus the other results of this theorem give alternate (but more complicated) proof of that generalised non-degeneracy.

□

Remark 3.2. Note that (assuming $\mathcal{H}$ is irreducible) Kebekus [Keb05] also stated that C_x and $\mathscr{C}_x$ are irreducible for general x. However it was observed by Kebekus himself together with the author that there is a gap in the proof. This gap is on page 234 in Step 2 of proof of Proposition 3.2 where Kebekus claims to construct "a well defined family of cycles" parametrised by a divisor D^0. This is not necessarily a well defined family of cycles: Condition (3.10.4) in [Kol96, §I.3.10] is not necessarily satisfied if D^0 is not normal and there seem to be no reason to expect that D^0 is normal. As a consequence the map $\Phi\colon D^0 \to \mathrm{Chow}(X)$ is not necessarily regular at non-normal points of D^0 and it might contract some curves.

Let us define:

$$\begin{aligned} &C^2 \subset X \times X \\ &C^2 := \{(x,y) \mid y \in C_x\}\,, \end{aligned}$$

i.e. this is the locus of those pairs (x,y), which are both on the same contact line. Again this locus is a closed subset of $X \times X$.
Analogously, define:

$$C^3 := C^2 \times_X C^2$$

so that:

$$\begin{aligned} &C^3 \subset X \times X \times X \\ &C^3 := \{(x,y,z) \mid y \in C_x, z \in C_y\}\,. \end{aligned}$$

Finally, for $x \in X$ we also define C^2_x:

$$\begin{aligned} &C^2_x \subset X \times X \simeq \{x\} \times X \times X \\ &C^2_x := \{(y,z) \mid y \in C_x, z \in C_y\}\,, \end{aligned}$$

with the scheme structure of the fibre of C^3 under the projection on the first coordinate. Since for all $x \in X$ all irreducible components of C_x are of dimension n (see Theorem 3.1) we conclude:

Proposition 3.3. *All C^2, C^2_x, C^3 are projective subschemes, they are all of pure dimension, and their dimensions are:*

- $\dim C^2 = 3n + 1$.

- $\dim C^2_x = 2n$.
- $\dim C^3 = 4n + 1$.

□

3.2 Joins and secants of Legendrian subvarieties

For subvarieties $Y_1, Y_2 \subset \mathbb{P}^N$ recall that their *join* $Y_1 * Y_2$ is the closure of the locus of lines between points $y_1 \in Y_1$ and $y_2 \in Y_2$. Note that the expected dimension of $Y_1 * Y_2$ is $\dim Y_1 + \dim Y_2 + 1$. We are only concerned with two special cases: either Y_1 and Y_2 are disjoint or $Y_1 = Y_2$.

Lemma 3.4. *If $Y_1, Y_2 \subset \mathbb{P}^N$ are two disjoint subvarieties of dimensions $k-1$ and $N-k$ respectively, then their join $Y_1 * Y_2$ fills out the ambient space, i.e. this join is of expected dimension.*

Proof. Let $p \in \mathbb{P}^N$ be a general point and consider the projection $\pi : \mathbb{P}^N \dashrightarrow \mathbb{P}^{N-1}$ away from p. Let $Z_i = \pi(Y_i)$ for $i = 1, 2$. Since p is general, $\dim Z_i = \dim Y_i$ and thus $Z_1 \cap Z_2$ is non-empty. Let $q \in Z_1 \cap Z_2$ be any point. The preimage $\pi^{-1}(q)$ is a line in $\mathbb{P}^N$ intersecting both Y_1 and Y_2 and passing through p.

□

Recall, that the special case of join is when $Y = Y_1 = Y_2$ and $\sigma_2(Y) := Y * Y$ is the *secant variety* of Y.

Proposition 3.5.
- *Let $Y \subset \mathbb{P}^{2n-1}$ be an irreducible linearly non-degenerate Legendrian variety. Then $\sigma_2(Y) = \mathbb{P}^{2n-1}$.*
- *Let $Y_1, Y_2 \subset \mathbb{P}^{2n-1}$ be two disjoint Legendrian subvarieties. Then $Y_1 * Y_2 = \mathbb{P}^{2n-1}$.*

Proof. If Y is irreducible, then this is proved in the course of proof of Prop. 17(2) in [LM07].
If Y_1 and Y_2 are disjoint, then the result follows from Lemma 3.4.

□

3.3 Divisors swept by broken lines

Following the idea of Wiśniewski [Wiś00] we introduce the locus of broken lines (or reducible conics, or chains of 2 lines) through x:

$$D_x := \bigcup_{y \in C_x} C_y.$$

Note that D_x is a closed subset of X as it can be interpreted as the image of projective variety $C^2_x \subset X \times X$ under a proper map, which is the projection onto the last coordinate. By analogy to the case of lines consider also:

$$D^2 \subset X \times X$$
$$D^2 := \{(x,z) \mid \exists_{y \in C_x} \text{ s.t. } z \in C_y\},$$

i.e. D^2 is the projection of C^3 onto first and third coordinates. Thus again D^2 is a closed subset of the product. Set theoretically D_x is the fibre over x of (either of) the projection $D^2 \to X$ and if we consider D^2 as a reduced scheme, then we can assign to D_x the scheme structure of the fibre.
It follows immediately from the above discussion and Proposition 3.3, that every component of D_x has dimension at most $2n$ and every component of D^2 has dimension at most $4n+1$. In fact the equality holds.

Theorem 3.6. *Let $x \in X$ be any point. Then the locus D_x is of pure codimension* 1.

Proof. Assume first that $x \in X$ is a general point. Recall, that $C^2_x \subset X \times X$ has two projections:

$$\begin{tikzcd} C^2_x \arrow[r, two heads, "\pi_2"] \arrow[d, "\pi_1"] & D_x \\ C_x & \end{tikzcd}$$

Fix $(D_x)^\bullet$ to be an irreducible component of D_x. Then $(D_x)^\bullet$ is dominated by some component $(C^2_x)^\bullet$ of C^2_x. Dimension of $(C^2_x)^\bullet$ is equal to $2n$ by Proposition 3.3.
For $y \in C_x$ the fiber ${\pi_1}^{-1}(y) \subset C^2_x$ is equal to $\{y\} \times C_y$. In particular, by Theorem 3.1(ii) the fibers of π_1 have constant dimension n. Thus $(C^2_x)^\bullet$ is mapped onto an irreducible component $(C_x)^\bullet$ of C_x. Finally, let C' be an irreducible component of the preimage ${\pi_1}^{-1}(x)$ which is contained in $(C^2_x)^\bullet$. Note that C' can be identified with an irreducible component of C_x, because ${\pi_1}^{-1}(x) = \{x\} \times C_x$.
We claim that the projectivised tangent cone $\mathbb{P}\tau_x(D_x)^\bullet$ contains the join of two tangent cones

$$(\mathbb{P}\tau_x C') * (\mathbb{P}\tau_x (C_x)^\bullet) \subset \mathbb{P}F_x \subset \mathbb{P}T_x X.$$

The proof of the claim is a baby version of [HK05, Thm 3.11]. There however Hwang and Kebekus assume C_x is irreducible and thus their results do not neccessarily apply directly here. Let l_0 be a general line through x contained in C' and let l be a general line through x contained in $(C_x)^\bullet$. To prove the claim it is enough to show there exists a surface $S \subset D_x$ containing l_0 and l which is smooth at x, since in such a case $T_x S \subset \tau_x D_x$ and $\mathbb{P}T_x S$ is the line between $\mathbb{P}T_x l$ and $\mathbb{P}T_x l_0$.

We obtain S by varying l_0. Consider $\mathcal{H}_l \subset \mathcal{H}$ the parameter space for lines on X, which intersect l. By Theorem 3.1(iii) the space $\mathcal{H}_l$ comes with a projection $\xi\colon \mathcal{H}_l \dashrightarrow l$, which maps $l' \in \mathcal{H}_l$ to the intersection point of l and l', and which is well defined on an open subset containg all lines through x.
By generality of our choices, l_0 is a smooth point of $\mathcal{H}_l$ and ξ is submersive at l_0. In the neighbourhood of l_0 choose a curve $A \subset \mathcal{H}_l$ smooth at l_0 for which $\xi|_A$ is submersive at l_0. Then the locus in X of lines which are in A sweeps a surface $S \subset X$, which is smooth at x, contains l_0, and contains an open subset of l around x. Thus the claim is proved and:

$$(\mathbb{P}\tau_x C') * (\mathbb{P}\tau_x (C_x)^\bullet) \subset \mathbb{P}\tau_x (D_x)^\bullet \tag{3.7}$$

Now we claim that $F_x \subset \tau_x D_x$. For this purpose we separate two cases.
In the first case $C' = (C_x)^\bullet$. Then $\mathbb{P}\tau_x C'$ is non-degenerate by Theorem 3.1 and thus

$$(\mathbb{P}\tau_x C') * (\mathbb{P}\tau_x (C_x)^\bullet) = \sigma_2(\mathbb{P}\tau_x C') = \mathbb{P}(F_x)$$

by Proposition 3.5. Combining with (3.7) we obtain the claim.
In the second case C' and $(C_x)^\bullet$ are different components of C_x. Then by generality of x and by Theorem 3.1, the two tangent cones $(\mathbb{P}\tau_x C')$ and $(\mathbb{P}\tau_x (C_x)^\bullet)$ are disjoint. Thus again

$$(\mathbb{P}\tau_x C') * (\mathbb{P}\tau_x (C_x)^\bullet) = \mathbb{P}(F_x)$$

by Proposition 3.5. Combining with (3.7) we obtain the claim.
Thus in any case for a general $x \in X$, every component of D_x has dimension at least $2n$. The dimension can only jump up at special points when one has a fibration, thus also at special points every component of D_x has dimension at least $2n$. Earlier we observed that $\dim D_x \leq 2n$, thus the theorem is proved. $\square$

PROPOSITION 3.8. *If X is the adjoint variety of G, and $x \in X$, then D_x is the hyperplane section of $X \subset \mathbb{P}(\mathfrak{g})$ perpendicular to x via the Killing form.*

PROOF. Let $X = G/P$, where P is the parabolic subgroup preserving x. Notice, that D_x must be reduced (because D is reduced and D_x is a general fibre of D). Also D_x is P-invariant, because the set of lines is G invariant and D_x is determined by x and the geometry of lines on X. We claim, there is a unique P-invariant reduced divisor on X, and thus it must be the hyperplane section as in the statment of proposition.
So let Δ be a P-invariant divisor linearly equivalent to L^k for some $k \geq 0$. Also let ρ_Δ be a section of L^k which determines Δ. The module of sections $H^0(L^k)$ is an irreducible G-module by Borel-Weil theorem (see [Ser95]), with some highest weight ω. Since the Lie algebra $\mathfrak{p}$ of P contains all positive root spaces, by [FH91, Prop. 14.13] there is a unique 1-dimensional $\mathfrak{p}$-invariant submodule of $H^0(L^k)$, it is the highest weight space $H^0(L^k)_\omega$. So $\rho_\Delta \in H^0(L^k)_\omega$ and Δ is unique.

The hyperplane section of $X \subset \mathbb{P}(\mathfrak{g})$ perpendicular to x via the Killing form is a divisor in $|L|$, and it is P-invariant, and so are its multiples in $|L^k|$. So by the uniqueness Δ must be equal to k times this hyperplane section. Thus Δ is reduced if and only $k = 1$ and so D_x is the hyperplane section.

□

3.4 Tangent bundles restricted to lines

Let l be a line through a general point $y \in X$. Recall from [Keb05, Fact 2.3] that:

$$\begin{aligned} TX|_l &\simeq \mathcal{O}_l(2) \oplus \mathcal{O}_l(1)^{n-1} \oplus {\mathcal{O}_l}^{n-1} \oplus {\mathcal{O}_l}^2 \\ F|_l &\simeq \mathcal{O}_l(2) \oplus \mathcal{O}_l(1)^{n-1} \oplus {\mathcal{O}_l}^{n-1} \oplus \mathcal{O}_l(-1) \\ Tl &\simeq \mathcal{O}_l(2) \end{aligned}$$

and for general $z \in l$:

$$TC_z|_{l \setminus \{z\}} \simeq \mathcal{O}_l(2) \oplus \mathcal{O}_l(1)^{n-1}.$$

If $x \in X$ is a general point and $y \in C_x$ is a general point of any of the irreducible components of C_x and l is a line through y, then we want to express $TD_x|_l$ in terms of those splittings. In a neighbourhood of l the divisor D_x is swept by deformations l_t of $l = l_0$ such that l_t intersects C_x. By the standard deformation theory argument taking derivative of l_t by t at a point $z \in l$, we obtain that:

$$T_z D_x \supset \left\{ s(z) \in T_z X \mid \exists s \in H^0(TX|_l) \text{ s.t. } s(y) \in T_y C_x \right\} \tag{3.9}$$

Moreover, at a general point z we have equality in (3.9). If we mod out $TX|_l$ by the rank n positive bundle $(TX|_l)^{>0} := \mathcal{O}_l(2) \oplus \mathcal{O}_l(1)^{n-1}$, then we are left with a trivial bundle ${\mathcal{O}_l}^{n+1}$. Thus, since by Theorem 3.6 the dimension of $T_z D_x = 2n$ for general $z \in l$, the vector space $T_y C_x$ must be transversal to $(TX|_l)^{>0}$ at y. In particular, if $z \neq y$, then dimension of the right hand side in (3.9) is $2n$ and thus (3.9) is an equality for each point $z \in l$, such that z is a smooth point of D_x.
We conclude:

Proposition 3.10. *Let $x \in X$ be a general point and $y \in C_x$ be a general point of any of the irreducible components of C_x and l be any line through y. Then there exists a subbundle $\Gamma \subset TX|_l$ such that:*

$$\begin{aligned} \Gamma &= \mathcal{O}_l(2) \oplus \mathcal{O}_l(1)^{n-1} \oplus {\mathcal{O}_l}^n, \\ \Gamma \cap F|_l &= \mathcal{O}_l(2) \oplus \mathcal{O}_l(1)^{n-1} \oplus {\mathcal{O}_l}^{n-1} = (F|_l)^{\geq 0} \end{aligned}$$

and if $z \in l$ is a smooth point of D_x, then $T_z D_x = \Gamma_z$.

□

4 Duality

An effective divisor Δ on X is an element of divisor group (and thus a positive integral combination of codimension 1 subvarieties of X) and also a point in the projective space $\mathbb{P}(H^0\mathcal{O}_X(\Delta))$ or a hyperplane in $\mathbb{P}(H^0\mathcal{O}_X(\Delta)^*)$. In this section we will constantly interchange these three interpretations of Δ. In order to avoid confusion we will write:

- Δ^{div} to mean the divisor on X;
- $\Delta^{\mathbb{P}}$ to mean the point in $\mathbb{P}(H^0\mathcal{O}_X(\Delta))$ or in a fixed linear subsystem.
- $\Delta^{\mathbb{P}\perp}$ to mean the hyperplane in $\mathbb{P}(H^0\mathcal{O}_X(\Delta)^*)$ or in dual of the fixed subsystem.

In §3.3 we have defined $D \subset X \times X$, which we now view as a family of divisors on X parametrised by X. Since the Picard group of X is discrete and X is smooth and connected, it follows that all the divisors D_x are linearly equivalent. Thus let $E \simeq L^{\otimes k}$ be the line bundle $\mathcal{O}_X(D_x)$. Consider the following vector space $\langle D\rangle \subset H^0(E)$:

$$\langle D\rangle := \operatorname{span}\{s_x : x \in X\} \text{ where } s_x \text{ is a section of } E \text{ vanishing on } D_x.$$

Hence $\mathbb{P}\langle D\rangle$ is the linear system spanned by all the D_x.
Further, consider the map

$$\phi : X \to \mathbb{P}\langle D\rangle^*$$

determined by the linear system $\langle D\rangle$, i.e. mapping point $x \in X$ to the hyperplane in $\mathbb{P}\langle D\rangle$ consisting of all divisors containing x.

Remark 4.1. Note that ϕ is regular, since for every $x \in X$ there exists $w \in X$, such that $x \notin D_w$ (or equivalently, $w \notin D_x$).

Since E is ample, it must intersect every curve in X and hence ϕ does not contract any curve. Therefore ϕ is finite to one.

Proposition 4.2. *If X is an adjoint variety, then $k = 1$, i.e. $E \simeq L$. If $k = 1$ and the automorphism group of X is reductive, then X is isomorphic to an adjoint variety.*

Proof. If X is the adjoint variety of G, and $x \in X$, then D_x is the hyperplane section of $X \subset \mathbb{P}(\mathfrak{g})$ by Proposition 3.8.
If $k = 1$ and the automorphism group of X is reductive, since ϕ is finite to one, we can apply Beauville Theorem [Bea98]. Thus X is isomorphic to an adjoint variety.

□

4.1 Dual map

In algebraic geometry it is standard to consider maps determined by linear systems (such as ϕ defined above). However in our situation, we also have another map determined by the family of divisors D. Namely:

$$\begin{aligned} \psi\colon X &\to \mathbb{P}\langle D\rangle \\ x &\mapsto {D_x}^{\mathbb{P}}. \end{aligned}$$

So let $\mathcal{S} \subset \mathcal{O}_X \otimes \langle D\rangle^* \simeq X \times \langle D\rangle^*$ be the pullback under ϕ of the universal hyperplane bundle, i.e. the corank 1 subbundle such that the fibre of $\mathcal{S}$ over x is ${D_x}^{\mathbb{P}\perp} \subset \langle D\rangle^*$. We note that $\mathbb{P}(\mathcal{S})$ is both a projective space bundle on X and also it is a divisor on $X \times \mathbb{P}\langle D\rangle^*$. Also $D = (\mathrm{id}_X \times\phi)^*\mathbb{P}(\mathcal{S})$ as divisors.
We can also consider the line bundle dual to the cokernel of $\mathcal{S} \to \mathcal{O}_X \otimes \langle D\rangle^*$, i.e. the subbundle $\mathcal{S}^{\perp} \subset \mathcal{O}_X \otimes \langle D\rangle$. This line subbundle determines section $X \to X \times \mathbb{P}\langle D\rangle$, where $x \mapsto (x, {D_x}^{\mathbb{P}})$. So ψ is the composition of the section and the projection:

$$X \to X \times \mathbb{P}\langle D\rangle \to \mathbb{P}\langle D\rangle.$$

Every map to a projective space is determined by some linear system. We claim the ψ is determined by $\langle D\rangle$, precisely the system that defines ϕ and thus that there is a natural linear isomorphism between $\mathbb{P}\langle D\rangle$ and $\mathbb{P}\langle D\rangle^*$.

Proposition 4.3. *We have $\psi^*\mathcal{O}_{\mathbb{P}\langle D\rangle}(1) \simeq E$ and the linear system cut out by hyperplanes*

$$\psi^* H^0\left(\mathcal{O}_{\mathbb{P}\langle D\rangle}(1)\right) := \left\{\psi^* s : s \in \langle D\rangle^*\right\} \subset H^0(E)$$

is equal to $\langle D\rangle$.

Proof. For fixed $x \in X$ let $\phi(x)^{\perp} \subset \mathbb{P}\langle D\rangle$ be the hyperplane dual to $\phi(x) \in \mathbb{P}\langle D\rangle^*$. To prove the proposition it is enough to prove

$$\psi^*(\phi(x)^{\perp}) = {D_x}^{div}. \tag{4.4}$$

Since we have the following symmetry property of D:

$$x \in D_y \iff y \in D_x,$$

the set theoretic version of (4.4) follows easily:

$$y \in \psi^*(\phi(x)^{\perp}) \iff \psi(y) \in \phi(x)^{\perp} \iff {D_y}^{\mathbb{P}\perp} \ni \phi(x) \iff D_y \ni x.$$

However, in order to prove the equality of divisors in (4.4) we must do a bit more of gymnastics, which translates the equivalences above into local equations. The details are below.
The pull back of $\phi(x)^{\perp}$ by the projection $X \times \mathbb{P}\langle D\rangle \to \mathbb{P}\langle D\rangle$ is just $X \times \phi(x)^{\perp}$. Then the pull-back of the product by the section $X \to X \times \mathbb{P}\langle D\rangle$ associated

to $\mathcal{S}^{\perp}$ is just the subscheme of X defined by $\left\{y \in X \mid (\mathcal{S}^{\perp})_y \subset \phi(x)^{\perp}\right\}$ (locally, this is just a single equation: the spanning section of $\mathcal{S}^{\perp}$ satisfies the defining equation of $\phi(x)^{\perp}$). But this is clearly equal to the dual equation $\{y \mid \mathbb{P}(\mathcal{S}_y) \ni \phi(x)\}$. If we let ρ_x be the section

$$\rho_x \colon X \to X \times X$$
$$\rho_x(y) := (y, x)$$

then we have:

$$\psi^*(\phi(x)^{\perp}) = {\rho_x}^* \circ (\mathrm{id}_X \times \phi)^*(\mathbb{P}(\mathcal{S})) = {\rho_x}^*(D) = {D_x}^{div}$$

as claimed.

□

Thus we have a canonical linear isomorphism $f \colon \mathbb{P}\langle D\rangle^* \to \mathbb{P}\langle D\rangle$ giving rise to the following commutative diagram:

$$\begin{tikzcd} & \mathbb{P}\langle D\rangle^* \arrow[d, "\simeq"] \\ X \arrow[ur, "\phi"] \arrow[dr, "\psi"'] & \\ & \mathbb{P}\langle D\rangle. \end{tikzcd} \tag{4.5}$$

We will denote the underlying vector space isomorphism $\langle D\rangle^* \to \langle D\rangle$ (which is unique up to scalar) with the same letter f. The choice of f combined with the canonical pairing $\langle D\rangle \times \langle D\rangle^* \to \mathbb{C}$, determines a non-degenerate bilinear form $B : \langle D\rangle \times \langle D\rangle \to \mathbb{C}$, with the following property:

$$B(\phi(x), \phi(y)) = 0 \iff (x, y) \in D \iff x \in D_y \iff y \in D_x. \tag{4.6}$$

Proposition 4.7. *If X is the adjoint variety of G, then $\langle D\rangle = H^0(L) \simeq \mathfrak{g}$ and B is (up to scalar) the Killing form on $\mathfrak{g}$.*

Proof. Follows immediately from Proposition 3.8 and Equation 4.6.

□

Corollary 4.8. *$\phi(x) = \phi(y)$ if and only if $D_x = D_y$.*

Proof. It is immediate from the definition of ψ and from Diagram (4.5).

□

4.2 Symmetry

Note that B has the property that for $x \in X$,

$$B(\phi(x), \phi(x)) = 0$$

(because $x \in D_x$).

Proposition 4.9. *The bilinear form B is either symmetric or skew-symmetric.*

Proof. Consider two linear maps $\langle D\rangle \to \langle D\rangle^*$:

$$\alpha(v) := B(v,\cdot) \quad \text{and} \quad \beta(v) := B(\cdot, v).$$

If $v = \phi(x)$ for some $x \in X$, then

$$\ker\bigl(\alpha(v)\bigr) = \operatorname{span}\Bigl(\ker\bigl(\alpha(v)\bigr) \cap \phi(X)\Bigr) = \operatorname{span}\bigl(\phi(D_x)\bigr)$$

and analogously $\ker(\beta(v)) = \operatorname{span}(\phi(D_x))$. So $\ker(\alpha(v)) = \ker(\beta(v))$ and hence $\alpha(v)$ and $\beta(v)$ are proportional. Therefore there exists a function $\lambda : X \to \mathbb{C}$ such that:

$$\lambda(x)\alpha(\phi(x)) = \beta(\phi(x)).$$

So for every $x, y \in X$ we have:

$$B(\phi(x),\phi(y)) = \lambda(x)B(\phi(y),\phi(x)) = \lambda(x)\lambda(y)B(\phi(x),\phi(y))$$

and hence:

$$\forall (x,y) \in X \times X \setminus D \qquad \lambda(x)\lambda(y) = 1.$$

Taking three different points we see that λ is constant and $\lambda \equiv \pm 1$. Therefore $\pm\alpha(\phi(x)) = \beta(\phi(x))$ and by linearity this extends to $\pm\alpha = \beta$ so B is either symmetric or skew-symmetric as stated in the proposition.

$\square$

Example 4.10. If X is one of the adjoint varieties, then B is symmetric (because the Killing form is symmetric).

Remark 4.11. Consider $\mathbb{P}^{2n+1}$ with a contact structure arising from a symplectic form ω on $\mathbb{C}^{2n+2}$. Recall, that this homogeneous contact Fano manifold does not satisfy our assumptions, namely, its Picard group is not generated by the equivalence class of L — in this case $L \simeq \mathcal{O}_{\mathbb{P}^{2n+1}}(2)$. However, Wiśniewski in [Wiś00] considers also this generalised situation and defines D_x to be the divisor swept by contact conics (i.e. curves C with degree of $L|_C = 2$) tangent to the contact distribution F. Then for the projective space D_x is just the hyperplane perpendicular to x with respect to ω. And thus in this case $\langle D\rangle = H^0\left(\mathcal{O}_{\mathbb{P}^{2n+1}}(1)\right)$ and the bilinear form B defined from such family of divisors would be proportional to ω, hence skew-symmetric.

Proof of Theorem 1.3. D_x is a divisor by Theorem 3.6. ϕ is regular by Remark 4.1. ψ is regular by (4.5). The non-degenerate bilinear form B is constructed in §4.1. It is either symmetric or skew-symmetric by Proposition 4.9. In the adjoint case B is the Killing form by Proposition 4.7.

$\square$

Corollary 4.12. *If B is symmetric, then $\psi(X) \subset \mathbb{P}\langle D\rangle$ is contained in the quadric $B(v,v) = 0$.*

Corollary 4.13. *If $x \in X$, then $\psi(C_x)$ is contained in a linear subspace of dimension at most $\left\lfloor \frac{\dim\langle D\rangle}{2} \right\rfloor$.*

Proof. If $y, z \in C_x$, then $z \in D_y$, so $B(\psi(y), \psi(z)) = 0$. Therefore $\mathrm{span}(\psi(C_x))$ is an isotropic linear subspace, which cannot have dimension bigger than $\left\lfloor \frac{\dim\langle D\rangle}{2} \right\rfloor$.

□

5 Grading

Suppose $X \subset \mathbb{P}\mathfrak{g}$ is the adjoint variety of G. Assume further that a maximal torus and an order of roots in $\mathfrak{g}$ has been chosen, then $\mathfrak{g}$ has a natural grading (see [LM02, §6.1]):

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

where:

(i) $\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is the parabolic subalgebra $\mathfrak{p}$ of X.

(ii) $\mathfrak{g}_0$ is the maximal reductive subalgebra of $\mathfrak{p}$.

(iii) for all $i \in \{-2,-1,0,1,2\}$ the vector space $\mathfrak{g}_i$ is a $\mathfrak{g}_0$-module.

(iv) $\mathfrak{g}_2$ is the 1-dimensional highest root space,

(v) $\mathfrak{g}_{-2}$ is the 1-dimensional lowest root space.

(vi) The restriction of the Killing form to each $\mathfrak{g}_2 \oplus \mathfrak{g}_{-2}$, $\mathfrak{g}_1 \oplus \mathfrak{g}_{-1}$ and $\mathfrak{g}_0$ is non-degenerate, and the Killing form $B(\mathfrak{g}_i, \mathfrak{g}_j)$ is identically zero for $i \neq -j$.

(vii) The Lie bracket on $\mathfrak{g}$ respects the grading, $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ (where $\mathfrak{g}_k = 0$ for $k \notin \{-2,-1,0,1,2\}$).

In fact the grading is determined by $\mathfrak{g}_{-2}$ and $\mathfrak{g}_2$ together with the geometry of X only. So let X be as in Notation 2.1 and let x and w be two general points of X. Define the following subspaces of $\langle D\rangle$:

- $\langle D\rangle_2$ to be the 1-dimensional subspace $\psi(x)$;
- $\langle D\rangle_{-2}$ to be the 1-dimensional subspace $\psi(w)$;
- $\langle D\rangle_1$ to be the linear span of affine cone of $\psi(C_x \cap D_w)$;
- $\langle D\rangle_{-1}$ to be the linear span of affine cone of $\psi(C_w \cap D_x)$;

- $\langle D\rangle_0$ to be the vector subspace of $\langle D\rangle$, whose projectivisation is:

$$\bigcap_{y\in C_x\cup C_w} f({D_y}^{\mathbb{P}\perp})$$

In the homogeneous case this is precisely the grading of $\mathfrak{g}$.

Proof of Theorem 1.4. First note that the classes of the 1-dimensional linear subspaces $\mathfrak{g}_2$ and $\mathfrak{g}_{-2}$ are both in X (as points in $\mathbb{P}\mathfrak{g}$). Moreover, they are a pair of general points, because the action of the parabolic subgroup $P < G$ preserves $\mathfrak{g}_2$ and moves freely $\mathfrak{g}_{-2}$. This is because $\hat{T}_{[\mathfrak{g}_{-2}]}X = [\mathfrak{g}_{-2}, \mathfrak{g}] = [\mathfrak{g}_{-2}, \mathfrak{p}]$. So fix $x = [\mathfrak{g}_2]$ and $w = [\mathfrak{g}_{-2}]$. We claim the linear span of C_x (respectively C_w) is just $\mathfrak{g}_2 \oplus \mathfrak{g}_1$ (respectively $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$). To see that, we observe the lines on X through x are contained in the intersection of X and the projective tangent space $\mathbb{P}(\hat{T}_xX) \subset \mathbb{P}(\mathfrak{g})$. In fact this intersection is equal to C_x: if $y \neq x$ is a point of the intersection, then the line in $\mathbb{P}\mathfrak{g}$ through x and y intersects X with multiplicity at least 3, but X is cut out by quadrics (see for instance [Pro07, §10.6.6]), so this line must be contained in X. Also C_x is non-degenerate in $\mathbb{P}(\hat{F}_x) \subset \mathbb{P}(\hat{T}_xX)$. However $\hat{F}_x$ is a $\mathfrak{p}$-invariant hyperplane in $\mathbb{P}(\hat{T}_xX)$ and the unique $\mathfrak{p}$-invariant hyperplane in

$$\hat{T}_xX = [\mathfrak{g}, \mathfrak{g}_2] = [\mathfrak{g}_{-2}, \mathfrak{g}_2] \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

is

$$\hat{F}_x = [\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_2] = \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

Further we have seen in Proposition 3.8 that D_x (respectively D_w) is the intersection of $\mathbb{P}({\mathfrak{g}_2}^{\perp_B}) = \mathbb{P}(\mathfrak{g}_2 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1})$ and X (respectively $\mathbb{P}(\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1)$ and X). Equivalently, $f({D_x}^{\mathbb{P}\perp}) = \mathbb{P}(\mathfrak{g}_2 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1})$. Thus:

$$C_x \cap D_w = C_x \cap f({D_w}^{\mathbb{P}\perp}) = C_x \cap \mathbb{P}(\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1) = C_x \cap \mathbb{P}(\mathfrak{g}_1).$$

$C_x \cap \mathbb{P}(\mathfrak{g}_1)$ is non-degenerate in $\mathbb{P}(\mathfrak{g}_1)$, thus $\langle D\rangle_1 = \mathfrak{g}_1$ and analogously $\langle D\rangle_{-1} = \mathfrak{g}_{-1}$.
It remains to prove $\langle D\rangle_0 = \mathfrak{g}_0$.

$$\begin{aligned}
\mathbb{P}\langle D\rangle_0 &= \bigcap_{y\in C_x\cup C_w} f({D_z}^{\mathbb{P}\perp}) \\
&= (C_x \cup C_w)^{\perp_B} \\
&= \mathbb{P}(\mathfrak{g}_2 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2})^{\perp_B} \\
&= \mathbb{P}(\mathfrak{g}_0).
\end{aligned}$$

□

We also note the following lemma in the homogeneous case:

Lemma 5.1. *If X is the adjoint variety of G, then*

$$X \cap \mathbb{P}(\mathfrak{g}_1) \subset C_x$$

where x is the point of projective space corresponding to $\mathfrak{g}_2$.

Proof. Suppose $y \in X \cap \mathbb{P}\mathfrak{g}_1$ and let $l \subset \mathbb{P}\mathfrak{g}$ be the line through x and y. Note that $l \subset \mathbb{P}(\mathfrak{g}_1 \oplus \mathfrak{g}_2)$. Since $\mathfrak{g}_1 \oplus \mathfrak{g}_2 \subset [\mathfrak{g}, \mathfrak{g}_2] = \hat{T}_x X$, hence $l \cap X$ has multiplicity at least 2 at x. Thus $l \cap X$ has degree at least 3 and since X is cut out by quadrics, l is contained in X.

□

6 Cointegrable subvarieties

Definition 6.1. A subvariety $\Delta \subset X$ is *F-cointegrable* if $T_x\Delta \cap F_x \subset F_x$ is a coisotropic subspace for general point x of each irreducible component of Δ.

Note that this is equivalent to the definition given in [Bucz09, §E.4] — this follows from the local description of the symplectic form on the symplectisation of the contact manifold (see [Bucz09, (C.15)]).
Clearly, every codimension 1 subvariety of X is F-cointegrable.
Assume $\Delta \subset X$ is a subvariety of pure dimension, which is F-cointegrable and let Δ_0 be the locus where $T_x\Delta \cap F_x \subset F_x$ is a coisotropic subspace of dimension $\dim \Delta - 1$. We define the *Δ-integrable distribution* $\Delta^\perp$ to be the distribution defined over Δ_0 by:

$$\Delta^{\perp}{}_x := (T_x\Delta \cap F_x)^{\perp_{\mathrm{d}\theta}} \subset F_x$$

We say an irreducible subvariety $A \subset X$ is *Δ-integral* if $A \subset \Delta$, $A \cap \Delta_0 \neq \emptyset$, and $TA \subset \Delta^\perp$ over the smooth points of $A \cap \Delta_0$.

Lemma 6.2. *Let A_1 and A_2 be two irreducible Δ-integral subvarieties. Assume $\dim A_1 = \dim A_2 = \operatorname{codim}_X \Delta$. Then either $A_1 = A_2$ or $A_1 \cap A_2 \subset \Delta \setminus \Delta_0$.*

□

Theorem 6.3. *Consider a general point $x \in X$. Then:*

(i) D_x (as reduced, but possibly not irreducible subvariety of X) is F-cointegrable.

(ii) For general y in any of the irreducible components of C_x all lines through y are D_x-integral.

(iii) For general z in any of the irreducible components of D_x the intersection $C_x \cap C_z$ is a unique point and the chain of two lines connecting x to z is unique.

Proof. Part (i) is immediate, since D_x is a divisor, by Theorem 3.6.
To prove part (ii) let l be a line through y. Then by Proposition 3.10:

$$T_z D_x \cap F_z = (F|_l)^{\geq 0}$$

and for general $z \in l$ we have $(T_z D_x \cap F_z)^{\perp_{d\theta_z}} \subset F_z$ is the $\mathcal{O}(2)$ part, i.e. the part tangent to l. So l is D_x-integral as claimed.
To prove (iii), let $U \subset X$ be an open dense subset of points $u \in X$ where two different lines through u do not share the tangent direction and do not meet in any other point. Note that since x is a general point, $x \in U$ and thus each irreducible component of C_x and D_x intersects U. Thus generality of z implies that $z \in U$ and thus each irreducible component of C_z and D_z intersects U. Also $C_x \cap C_z$ intersects U. So fix $y \in C_x \cap C_z \cap U$.
By (ii) and Lemma 6.2 the line l_z through z which intersects C_x is unique. In the same way let l_x be the unique line through x intersecting C_z. Thus

$$C_x \cap C_z = l_x \cap l_z.$$

In particular, $y \in l_x \cap l_z$. But since $y \in U$ the intersection $l_x \cap l_z$ is just one point and therefore:

$$C_x \cap C_z = \{y\}.$$

□

As a consequence of part (iii) of the theorem the surjective map $\pi_{13} \colon C^3 \to D$ is birational. Thus consider the inverse rational map $D \dashrightarrow C^3$ and compose it with the projection on the middle coordinate $\pi_2 : C^3 \to X$. We define the composition to be the *bracket map*:

$$[\cdot,\cdot]^D \colon D \dashrightarrow C^3 \xrightarrow{\pi_2} X.$$

In this setting, for $(x,z) \in D$, one has $[x,z]^D = y = C_x \cap C_z$, whenever the intersection is just one point.

Theorem 6.4. *If X is the adjoint variety of G, then the bracket map defined above agrees with the Lie bracket on $\mathfrak{g}$, in the following sense: Let $\xi, \zeta \in \mathfrak{g}$ and set $\eta := [\xi,\zeta]$ (the Lie bracket on $\mathfrak{g}$). Denote by x, y and z the projective classes in $\mathbb{P}\mathfrak{g}$ of ξ, η and ζ respectively. If $x \in D_z$ and $\eta \neq 0$, then the bracket map satisfies $[x,z]^D = y$.*

Proof. It is enough to prove the statement for a general pair $(x,z) \in D$. Suppose further $w \in C_z$ is a general point. Then the pair $(x,w) \in X \times X$ is a general pair. Thus by Theorem 1.4, we may assume $\xi \in \mathfrak{g}_2$ and $\zeta \in \mathfrak{g}_{-1}$. The restriction of the Lie bracket to $[\xi, \mathfrak{g}_{-1}]$ determines an isomorphism $\mathfrak{g}_{-1} \to \mathfrak{g}_1$ of $\mathfrak{g}_0$-modules. In particular the minimal orbit $X \cap \mathbb{P}\mathfrak{g}_{-1}$ is mapped onto $X \cap \mathbb{P}\mathfrak{g}_1$ under this isomorphism. In particular $y \in X \cap \mathbb{P}\mathfrak{g}_1 \subset C_x$ (see Lemma 5.1). Analogously $y \in C_z$, so $y \in C_x \cap C_z$.

□

References

[Bea98] Arnaud Beauville. Fano contact manifolds and nilpotent orbits. *Comment. Math. Helv.*, 73(4):566–583, 1998.

[Ber55] Marcel Berger. Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés riemanniennes. *Bull. Soc. Math. France*, 83:279–330, 1955.

[Bucz09] Jarosław Buczyński. Algebraic Legendrian varieties. *Dissertationes Math. (Rozprawy Mat.)*, 467:86, 2009. PhD thesis, Institute of Mathematics, Warsaw University, 2008.

[Dem02] Jean-Pierre Demailly. On the Frobenius integrability of certain holomorphic p-forms. In *Complex geometry (Göttingen, 2000)*, pages 93–98. Springer, Berlin, 2002.

[FH91] William Fulton and Joe Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[Har95] Joe Harris. *Algebraic geometry*, volume 133 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. A first course, Corrected reprint of the 1992 original.

[HK05] Jun-Muk Hwang and Stefan Kebekus. Geometry of chains of minimal rational curves. *J. Reine Angew. Math.*, 584:173–194, 2005.

[HM99] Jun-Muk Hwang and Ngaiming Mok. Varieties of minimal rational tangents on uniruled projective manifolds. In *Several complex variables (Berkeley, CA, 1995–1996)*, volume 37 of *Math. Sci. Res. Inst. Publ.*, pages 351–389. Cambridge Univ. Press, Cambridge, 1999.

[HM04] Jun-Muk Hwang and Ngaiming Mok. Birationality of the tangent map for minimal rational curves. *Asian J. Math.*, 8(1):51–63, 2004.

[Keb01] Stefan Kebekus. Lines on contact manifolds. *J. Reine Angew. Math.*, 539:167–177, 2001.

[Keb05] Stefan Kebekus. Lines on complex contact manifolds. II. *Compos. Math.*, 141(1):227–252, 2005.

[KO73] Shoshichi Kobayashi and Takushiro Ochiai. Characterizations of complex projective spaces and hyperquadrics. *J. Math. Kyoto Univ.*, 13:31–47, 1973.

[Kol96] János Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 1996.

[KPSW00] Stefan Kebekus, Thomas Peternell, Andrew J. Sommese, and Jarosław A. Wiśniewski. Projective contact manifolds. *Invent. Math.*, 142(1):1–15, 2000.

[LeB95] Claude LeBrun. Fano manifolds, contact structures, and quaternionic geometry. *Internat. J. Math.*, 6(3):419–437, 1995.

[LM02] Joseph M. Landsberg and Laurent Manivel. Construction and classification of complex simple Lie algebras via projective geometry. *Selecta Math. (N.S.)*, 8(1):137–159, 2002.

[LM07] Joseph M. Landsberg and Laurent Manivel. Legendrian varieties. *Asian J. Math.*, 11(3):341–359, 2007.

[Mum99] David Mumford. *The red book of varieties and schemes*, volume 1358 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, expanded edition, 1999. Includes the Michigan lectures (1974) on curves and their Jacobians, With contributions by Enrico Arbarello.

[Pro07] Claudio Procesi. *Lie groups*. Universitext. Springer, New York, 2007. An approach through invariants and representations.

[Ser95] Jean-Pierre Serre. Représentations linéaires et espaces homogènes kählériens des groupes de Lie compacts (d'après Armand Borel et André Weil). In *Séminaire Bourbaki, Vol. 2*, pages Exp. No. 100, 447–454. Soc. Math. France, Paris, 1995.

[Wiś00] Jarosław A. Wiśniewski. Lines and conics on fano contact manifolds. http://www.mimuw.edu.pl/~jarekw/, 2000.

[Wol65] Joseph A. Wolf. Complex homogeneous contact manifolds and quaternionic symmetric spaces. *J. Math. Mech.*, 14:1033–1047, 1965.

[Ye94] Yun-Gang Ye. A note on complex projective threefolds admitting holomorphic contact structures. *Invent. Math.*, 115(2):311–314, 1994.

Jarosław Buczyński
Institut Fourier
100 rue des maths, BP 74
38402 St Martin d'Hères
jabu@mimuw.edu.pl
France