An Inverse K-Theory Functor

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Abstract. Thomason showed that the K-theory of symmetric monoidal categories models all connective spectra. This paper describes a new construction of a permutative category from a Γ-space, which is then used to re-prove Thomason's theorem and a non-completed variant.

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1. Introduction

In [15], Segal described a functor from (small) symmetric monoidal categories to infinite loop spaces, or equivalently, connective spectra. This functor is often called the K-theory functor: When applied to the symmetric monoidal category of finite rank projective modules over a ring R, the resulting spectrum is Quillen's algebraic K-theory of R. A natural question is then which connective spectra arise as the K-theory of symmetric monoidal categories? Thomason answered this question in [18], showing that every connective spectrum is the K-theory of a symmetric monoidal category; moreover, he showed that the K-theory functor is an equivalence between an appropriately defined stable homotopy category of symmetric monoidal categories and the stable homotopy category of connective spectra.

This paper provides a new proof of Thomason's theorem by constructing a new homotopy inverse to Segal's K-theory functor. As a model for the category of infinite loop spaces, we work with Γ-spaces, following the usual conventions of [1, 4]: We understand a Γ-space to be a functor X from Γ^{op} (finite based sets) to based simplicial sets such that $X(\mathbf{0}) = *$. A Γ-space has an associated spectrum [15, §1] (or [4, §4] with these conventions), and a map of Γ-spaces $X \to Y$ is called a *stable equivalence* when it induces a stable equivalence of

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the associated spectra. We understand the stable homotopy category of Γ-spaces to be the homotopy category obtained by formally inverting the stable equivalences. The foundational theorem of Segal [15, 3.4], [4, 5.8] is that the stable homotopy category of Γ-spaces is equivalent to the stable category of connective spectra.

On the other side, the category of small symmetric monoidal categories admits a number of variants, all of which have equivalent stable homotopy categories. We discuss some of these variants in Section 3 below. For definiteness, we state the main theorem in terms of the category of small permutative categories and strict maps: The objects are the small permutative categories, i.e., those symmetric monoidal categories with strictly associative and unital product, and the maps are the functors that strictly preserve the product, unit, and symmetry. Segal [15, §2] constructed K-theory as a composite functor $K' = N \circ \mathcal{K}'$ from permutative categories to Γ-spaces, where $\mathcal{K}'$ is a functor from permutative categories to Γ-categories, and N is the nerve construction applied objectwise to a Γ-category to obtain a Γ-space. We actually use a slightly different but weakly equivalent functor $K = N \circ \mathcal{K}$ described in Section 3. A *stable equivalence* of permutative categories is defined to be a map that induces a stable equivalence on K-theory Γ-spaces. (We review an equivalent more intrinsic homological definition of stable equivalence in Proposition 3.8 below.) We understand the stable homotopy category of small permutative categories to be the homotopy category obtained from the category of small permutative categories by formally inverting the stable equivalences.

In Section 4, we construct a functor P from Γ-spaces to small permutative categories. Like K, we construct P as a composite functor $P = \mathcal{P} \circ \mathcal{S}$, with $\mathcal{P}$ a functor from Γ-categories to permutative categories and $\mathcal{S}$ a functor from simplicial sets to categories applied objectwise. The functor $\mathcal{S}$ is the left adjoint of the Quillen equivalence between the category of small categories and the category of simplicial sets from [7, 17]; the right adjoint is $\mathrm{Ex}^2 N$, where Ex is Kan's right adjoint to the subdivision functor Sd. As we review in Section 2, we have natural transformations

$$N\,\mathcal{S}\,X \longleftarrow \mathrm{Sd}^2\,X \longrightarrow X \qquad \text{and} \qquad \mathcal{S}\,N\mathcal{X} \longrightarrow \mathcal{X} \tag{1.1}$$

which are always weak equivalences, where we understand a weak equivalence of categories as a functor that induces a weak equivalences on nerves. The functor $\mathcal{P}$ from Γ-categories to permutative categories is a certain Grothendieck construction (homotopy colimit)

$$\mathcal{P}(\mathcal{X}) = \mathcal{A} \textstyle\int A\mathcal{X},$$

we describe in detail in Section 4. In brief, $\mathcal{A}$ is a category whose objects are the sequences of positive integers $\vec{m} = (m_1, \ldots, m_r)$ including the empty sequence, and whose morphisms are generated by permuting the sequence, maps of finite (unbased) sets, and partitioning; for a Γ-category X, we get a (strict) functor $A\mathcal{X}$ from $\mathcal{A}$ to the category of small categories satisfying

$$A\mathcal{X}(m_1, \ldots, m_r) = \mathcal{X}(\mathbf{m}_1) \times \cdots \times \mathcal{X}(\mathbf{m}_r)$$

and $A\mathcal{X}() = \mathcal{X}(\mathbf{0}) = *$ (the category with a unique object $*$ and unique morphism). The concatenation of sequences induces the permutative product on $\mathcal{P}\mathcal{X}$ with $*$ in $A\mathcal{X}()$ as the unit. In Section 4, we construct a natural transformation of permutative categories and natural transformations of Γ-categories

$$(1.2)\qquad \mathcal{P}\mathcal{K}\mathcal{C} \longrightarrow \mathcal{C} \qquad \text{and} \qquad \mathcal{X} \longleftarrow \mathcal{W}\mathcal{X} \longrightarrow \mathcal{K}\mathcal{P}\mathcal{X},$$

where $\mathcal{W}$ is a certain functor from Γ-categories to itself (Definition 4.8). In Section 5, we show that these natural transformations are natural stable equivalences, which then proves the following theorem, the main theorem of the paper.

Theorem 1.3. *The functor P from Γ-spaces to small permutative categories preserves stable equivalences. It induces an equivalence between the stable homotopy category of Γ-spaces and the stable homotopy category of permutative categories, inverse to Segal's K-theory functor.*

The arguments actually prove a "non-group-completed" version of this theorem. To explain this, recall that a Γ-space X is called *special* [4, p. 95] when the canonical map $X(\mathbf{a} \vee \mathbf{b}) \to X(\mathbf{a}) \times X(\mathbf{b})$ is a weak equivalence for any finite based sets $\mathbf{a}$ and $\mathbf{b}$; we define a special Γ-category analogously. Note that because of the weak equivalences in (1.1), a Γ-category $\mathcal{X}$ is special if and only if the Γ-space $N\mathcal{X}$ is special, and a Γ-space X is special if and only if the Γ-category $\mathcal{S}X$ is special. For special Γ-spaces, the associated spectrum is an Ω-spectrum after the zeroth space [15, 1.4]; the associated infinite loop space is the group completion of $X(\mathbf{1})$. For any permutative category $\mathcal{C}$, $K\mathcal{C}$ is a special Γ-space and $\mathcal{K}\mathcal{C}$ is a special Γ-category. We show in Corollary 5.5 that the natural transformation $\mathcal{P}\mathcal{K}\mathcal{C} \to \mathcal{C}$ of (1.2) is a weak equivalence for any permutative category $\mathcal{C}$, and we show in Theorem 4.10 and Corollary 5.6 that the natural transformations $W\mathcal{X} \to \mathcal{X}$ and $W\mathcal{X} \to \mathcal{K}\mathcal{P}\mathcal{X}$ of (1.2) are (objectwise) weak equivalences for any special Γ-category $\mathcal{X}$. We obtain the following theorem.

Theorem 1.4. *The following homotopy categories are equivalent:*

(i) *The homotopy category obtained from the category of small permutative categories by inverting the weak equivalences.*
(ii) *The homotopy category obtained from the subcategory of special Γ-spaces by inverting the objectwise weak equivalences.*

Theorem 1.3 implies that for an arbitrary Γ-space X, the Γ-space KPX is a special Γ-space stably equivalent to X. A construction analogous to $\mathcal{P}$ on the simplicial set level produces such a special Γ-space more directly: For a Γ-space X, we get a functor AX from $\mathcal{A}$ to based simplicial sets with

$$AX(m_1, \ldots, m_r) = X(\mathbf{m}_1) \times \cdots \times X(\mathbf{m}_r)$$

and $AX() = X(\mathbf{0}) = *$. Define

$$EX = \operatorname{hocolim}_{\mathcal{A}} AX.$$

Using the Γ-spaces $X(\mathbf{n} \wedge (-))$, we obtain a Γ-space $E^\Gamma X$,

$$E^\Gamma X(\mathbf{n}) = E(X(\mathbf{n} \wedge (-)))/N\mathcal{A},$$

with $EX \to E^\Gamma X(\mathbf{1})$ a weak equivalence. The inclusion of $X(\mathbf{1})$ as $AX(1)$ provides a natural transformation of simplicial sets $X(\mathbf{1}) \to EX$ and of Γ-spaces $X \to E^\Gamma X$. In Section 6, we prove the following theorems about these constructions.

THEOREM 1.5. *For any Γ-space X, the Γ-space $E^\Gamma X$ is special and the natural map $X \to E^\Gamma X$ is a stable equivalence. If X is special, then the natural map $X \to E^\Gamma X$ is an objectwise weak equivalence.*

THEOREM 1.6. *For any Γ-space X, the simplicial set EX has the natural structure of an E_∞ space over the Barratt-Eccles operad (and in particular the structure of a monoid).*

The previous two theorems functorially produce two additional infinite loop spaces from the Γ-space X, the infinite loop space of the spectrum associated to $E^\Gamma X$ and the group completion of EX. Since the map $X \to E^\Gamma X$ is a stable equivalence, it induces a stable equivalence of the associated spectra and hence the associated infinite loop spaces. The celebrated theorem of May and Thomason [12] then identifies the group completion of EX.

COROLLARY 1.7. *For any Γ-space X, the group completion of the E_∞ space EX is equivalent to the infinite loop space associated to X.*

As a consequence of Theorem 1.5, the Γ-space $E^\Gamma X$ is homotopy initial among maps from X to a special Γ-space. Theorem 1.6 then identifies $EX \simeq E^\Gamma X(\mathbf{1})$ as a reasonable candidate for the (non-completed) E_∞ space of X. Motivated by Theorem 1.4, we propose the following definition.

DEFINITION 1.8. We say a map of Γ-spaces $X \to Y$ is a *pre-stable equivalence* when the map $EX \to EY$ is a weak equivalence.

With this definition we obtain the equivalence of the last three homotopy categories in the following theorem from the theorems above. We have included the first category for easy comparison with other non-completed theories of E_∞ spaces; we prove the equivalence in Section 6.

THEOREM 1.9. *The following homotopy categories are equivalent:*

(i) *The homotopy category obtained from the category of E_∞ spaces over the Barratt-Eccles operad (in simplicial sets) by inverting the weak equivalences.*
(ii) *The homotopy category obtained from the category of Γ-spaces by inverting the pre-stable equivalences.*
(iii) *The homotopy category obtained from the subcategory of special Γ-spaces by inverting the objectwise weak equivalences.*
(iv) *The homotopy category obtained from the category of small permutative categories by inverting the weak equivalences.*

The previous theorem provides a homotopy theory for permutative categories and Γ-spaces before group completion, which now allows the construction of "spectral monoid rings" associated to Γ-spaces. For a topological monoid M, the suspension spectrum $\Sigma^\infty_+ M$ has the structure of an associative S-algebra (A_∞ ring spectrum) with M providing the multiplicative structure. The spectral monoid ring is a stable homotopy theory refinement of the monoid ring $\mathbb{Z}[\pi_0 M]$, which is $\pi_0 \Sigma^\infty_+ M$ or $\pi_0^S M$. For a Γ-space X, we can use EX in place of M and $\Sigma^\infty_+ EX$ is an E_∞ ring spectrum with the addition on EX providing the multiplication on $\Sigma^\infty_+ EX$. The spectral group ring of the associated infinite loop space, $\Sigma^\infty_+ \Omega^\infty X$, is the localization of $\Sigma^\infty_+ EX$ with respect the multiplicative monoid $\pi_0 EX \subset \pi_0^S EX$. Spectral monoid rings and algebras arise in the construction of twisted generalized cohomology theories (as explained, for example, in [3, 2.5] and [2]), and the localization $\Sigma^\infty_+ EX \to \Sigma^\infty_+ \Omega^\infty X$, specifically, plays a role in current work in extending notions of log geometry to derived algebraic geometry and stable homotopy theory (see the lecture notes by Rognes on log geometry available at [14]).

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2. Review of Γ-categories and Γ-spaces

This section briefly reviews the equivalence between the homotopy theory of Γ-spaces and of Γ-categories. We begin by introducing the notation used throughout the paper.

Notation 2.1. We denote by $\underline{n}$ the finite set $\{1, \dots, n\}$ and $\mathbf{n}$ the finite based set $\{0, 1, \dots, n\}$, with zero as base-point. We write $\mathcal{N}$ for the category with objects the finite sets $\underline{n}$ for $n \geq 0$ (with $\underline{0}$ the empty set) and morphisms the maps of sets. We write $\mathcal{F}$ for the category with objects the finite based sets $\mathbf{n}$ for $n \geq 0$ and morphisms the based maps of based sets.

We typically regard a Γ-space or Γ-category as a functor from $\mathcal{F}$ to simplicial sets or categories rather than from the whole category of finite based sets.

Definition 2.2. A Γ-*space* is a functor X from $\mathcal{F}$ to simplicial sets with $X(\mathbf{0}) = *$. A map of Γ-spaces is a natural transformation of functors from $\mathcal{F}$ to simplicial sets. A Γ-*category* is a functor $\mathcal{X}$ from $\mathcal{F}$ to the category of small categories with $\mathcal{X}(\mathbf{0}) = *$, the category with the unique object $*$ and the unique morphism id_*. A map of Γ-categories is a natural transformation of functors from $\mathcal{F}$ to small categories.

We emphasize that $\mathcal{X}$ must be a strict functor to small categories: For $\phi\colon \mathbf{m} \to \mathbf{n}$ and $\psi\colon \mathbf{n} \to \mathbf{p}$, the functors $\mathcal{X}(\psi \circ \phi)$ and $\mathcal{X}(\psi) \circ \mathcal{X}(\phi)$ must be equal (and not just naturally isomorphic). A map of Γ-categories $f\colon \mathcal{X} \to \mathcal{Y}$ consists of a sequence of functors $f_n\colon \mathcal{X}(\mathbf{n}) \to \mathcal{Y}(\mathbf{n})$ such that for every map $\phi\colon \mathbf{m} \to \mathbf{n}$ in $\mathcal{F}$, the diagram

$$\begin{tikzcd} \mathcal{X}(\mathbf{m}) \arrow[r, "f_m"] \arrow[d, "\mathcal{X}(\phi)"'] & \mathcal{Y}(\mathbf{m}) \arrow[d, "\mathcal{Y}(\phi)"] \\ \mathcal{X}(\mathbf{n}) \arrow[r, "f_n"'] & \mathcal{Y}(\mathbf{n}) \end{tikzcd}$$

commutes strictly. In particular, applying the nerve functor objectwise to a Γ-category then produces a Γ-space. We use this in defining the "strict" homotopy theory of Γ-categories.

DEFINITION 2.3. A map $X \to Y$ of Γ-spaces is a *weak equivalence* if each map $X(\mathbf{n}) \to Y(\mathbf{n})$ is a weak equivalence of simplicial sets. A map of Γ-categories $X \to Y$ is a *weak equivalence* if the induced map $N\mathcal{X} \to N\mathcal{Y}$ is a weak equivalence of Γ-spaces.

More important than weak equivalence is the notion of stable equivalence, which for our purposes is best understood in terms of very special Γ-spaces. A Γ-space X is *special* when for each n the canonical map

$$X(\mathbf{n}) \longrightarrow X(\mathbf{1}) \times \cdots \times X(\mathbf{1}) = X(\mathbf{1})^{\times n}$$

is a weak equivalence. This canonical map is induced by the *indicator* maps $\mathbf{n} \to \mathbf{1}$ which send all but one of the non-zero element of $\mathbf{n}$ to 0. For a special Γ-space, $\pi_0 X(\mathbf{1})$ is an abelian monoid under the operation

$$\pi_0 X(\mathbf{1}) \times \pi_0 X(\mathbf{1}) \cong \pi_0 X(\mathbf{2}) \longrightarrow \pi_0 X(\mathbf{1})$$

induced by the map $\mathbf{2} \to \mathbf{1}$ sending both non-basepoint elements of $\mathbf{2}$ to the non-basepoint element of $\mathbf{1}$. A special Γ-space is *very special* when the monoid $\pi_0 X(\mathbf{1})$ is a group.

DEFINITION 2.4. A map of Γ-spaces $f\colon X \to Y$ is a *stable equivalence* when for every very special Γ-space Z, the map $f^*\colon [Y, Z] \to [X, Z]$ is a bijection, where $[-, -]$ denotes maps in the homotopy category obtained by formally inverting the weak equivalences. A map of Γ-categories $\mathcal{X} \to \mathcal{Y}$ is a *stable equivalence* when the induced map $N\mathcal{X} \to N\mathcal{Y}$ of Γ-spaces is a stable equivalence.

Equivalently, a map of Γ-spaces is a stable equivalence if and only if it induces a weak equivalence of associated spectra [4, 5.1,5.8].

In order to compare the homotopy theory of Γ-spaces and Γ-categories, we use the Fritsch-Latch-Thomason Quillen equivalence of the category of simplicial sets and the category of small categories [7, 17]. We call a map in the category of small categories a weak equivalence if it induces a weak equivalence on nerves. The nerve functor has a left adjoint "categorization functor" c, which generally does not behave well homotopically. However, $c \circ \mathrm{Sd}^2$ preserves weak equivalences, where $\mathrm{Sd}^2 = \mathrm{Sd} \circ \mathrm{Sd}$ is the second subdivision functor [8, §7].

The functor $\mathrm{Ex}^2 N$ is right adjoint to $c\,\mathrm{Sd}^2$, and for any simplicial set X and any category $\mathcal{C}$, the unit and counit of the adjunction,

$$X \longrightarrow \mathrm{Ex}^2 Nc\,\mathrm{Sd}^2 X \qquad \text{and} \qquad c\,\mathrm{Sd}^2 \mathrm{Ex}^2 N\mathcal{C} \longrightarrow \mathcal{C},$$

are always weak equivalences. Since the natural map $X \to \mathrm{Ex}^2 X$ is always a weak equivalence and the diagrams

$$\begin{tikzcd} & \mathrm{Sd}^2 X \arrow[r] \arrow[d, "\sim"'] & Nc\,\mathrm{Sd}^2 X \arrow[d, "\sim"] \\ X \arrow[r, "\sim"] \arrow[rr, bend right, "\sim"'] & \mathrm{Ex}^2 \mathrm{Sd}^2 X \arrow[r] & \mathrm{Ex}^2 Nc\,\mathrm{Sd}^2 X \end{tikzcd} \qquad \begin{tikzcd} c\,\mathrm{Sd}^2 N\mathcal{C} \arrow[d, "\sim"] & \\ c\,\mathrm{Sd}^2 \mathrm{Ex}^2 N\mathcal{C} \arrow[r, "\sim"] & \mathcal{C} \end{tikzcd}$$

commute, we have natural weak equivalences

$$Nc\,\mathrm{Sd}^2 X \longleftarrow \mathrm{Sd}^2 X \longrightarrow X \qquad \text{and} \qquad c\,\mathrm{Sd}^2 N\mathcal{C} \longrightarrow \mathcal{C}. \tag{2.5}$$

The functor Sd^2 takes the one-point simplicial set $*$ to an isomorphic simplicial set; replacing Sd^2 by an isomorphic functor if necessary, we can arrange that $\mathrm{Sd}^2 * = *$.

Definition 2.6. Let $\mathcal{S}$ be the functor from Γ-spaces to Γ-categories obtained by applying $c\,\mathrm{Sd}^2$ objectwise.

We then obtain the natural weak equivalences of Γ-spaces and Γ-categories (1.1) from (2.5). Inverting weak equivalences or stable equivalences, we get equivalences of homotopy categories.

Proposition 2.7. *The functors N and $\mathcal{S}$ induce inverse equivalences between the homotopy categories of Γ-spaces and Γ-categories obtained by inverting the weak equivalences.*

Proposition 2.8. *The functors N and $\mathcal{S}$ induce inverse equivalences between the homotopy categories of Γ-spaces and Γ-categories obtained by inverting the stable equivalences.*

Since both the weak equivalences and stable equivalences of Γ-spaces provide the weak equivalences in model structures (see, for example, [4]), the homotopy categories in the previous propositions are isomorphic to categories with small hom sets.

3. Review of the K-theory functor

This section reviews Segal's K-theory functor from symmetric monoidal categories to Γ-spaces and some variants of this functor. All the material in this section is well-known to experts, and most can be found in [5, 10, 11, 18]. We include it here to refer to specific details, for completeness, and to make this paper more self-contained.

For a small symmetric monoidal category $\mathcal{C}$, we typically denote the symmetric monoidal product as $\Box$ and the unit as u. We construct a Γ-category $\mathcal{KC}$ as follows.

Construction 3.1. Let $\mathcal{KC}(\mathbf{0}) = *$ the category with a unique object $*$ and the identity map. For $\mathbf{n}$ in $\mathcal{F}$ with $n > 0$, we define the category $\mathcal{KC}(\mathbf{n})$ to have as objects the collections $(x_I, f_{I,J})$ where

- For each subset I of $\underline{n} = \{1, \ldots, n\}$, x_I is an object of $\mathcal{C}$, and
- For each pair of disjoint subsets I, J of $\underline{n}$,
$$f_{I,J} \colon x_{I\cup J} \longrightarrow x_I \mathbin{\Box} x_J$$
is a map in $\mathcal{C}$

such that

- When I is the empty set $\emptyset$, $x_I = u$ and $f_{I,J}$ is the inverse of the unit isomorphism.
- $f_{I,J} = \gamma \circ f_{J,I}$ where γ is the symmetry isomorphism $x_J \Box x_I \cong x_I \Box x_J$.
- Whenever I_1, I_2, and I_3 are mutually disjoint, the diagram
$$\begin{tikzcd} x_{I_1\cup I_2\cup I_3} \arrow[r, "f_{I_1, I_2\cup I_3}"] \arrow[d, "f_{I_1\cup I_2, I_3}"'] & x_{I_1} \Box x_{I_2\cup I_3} \arrow[d, "\mathrm{id}\,\Box f_{I_2,I_3}"] \\ x_{I_1\cup I_2} \Box x_{I_3} \arrow[r, "f_{I_1,I_2}\Box\mathrm{id}"'] & x_{I_1} \Box x_{I_2} \Box x_{I_3} \end{tikzcd}$$
commutes (where we have omitted notation for the associativity isomorphism in $\mathcal{C}$). We write f_{I_1,I_2,I_3} for the common composite into a fixed association.

A morphism g in $\mathcal{KC}(\mathbf{n})$ from $(x_I, f_{I,J})$ to $(x'_I, f'_{I,J})$ consists of maps $h_I \colon x_I \to x'_I$ in $\mathcal{C}$ for all I such that $h_\emptyset$ is the identity and the diagram
$$\begin{tikzcd} x_{I\cup J} \arrow[r, "h_{I\cup J}"] \arrow[d, "f_{I,J}"'] & x'_{I\cup J} \arrow[d, "f'_{I,J}"] \\ x_I \Box x_J \arrow[r, "h_I \Box h_J"'] & x'_I \Box x'_J \end{tikzcd}$$
commutes for all disjoint I, J.

The categories $\mathcal{KC}(\mathbf{n})$ assemble into a Γ-category as follows. For $\phi \colon \mathbf{m} \to \mathbf{n}$ in $\mathcal{F}$ and $X = (x_I, f_{I,J})$ in $\mathcal{KC}(\mathbf{m})$, define $\phi_* X = Y = (y_I, g_{I,J})$ where
$$y_I = x_{\phi^{-1}(I)} \qquad \text{and} \qquad g_{I,J} = f_{\phi^{-1}(I),\phi^{-1}(J)}$$
(replacing $*$ with u or vice-versa if $\mathbf{m}$ or $\mathbf{n}$ is $\mathbf{0}$), and likewise on maps. We obtain a Γ-space by applying the nerve functor to each category $\mathcal{KC}(\mathbf{n})$.

Definition 3.2. For a symmetric monoidal category $\mathcal{C}$, $K\mathcal{C}$ is the Γ-space $K\mathcal{C}(\mathbf{n}) = N\mathcal{KC}(\mathbf{n})$.

In terms of functoriality, $\mathcal{K}$ is obviously functorial in *strict* maps of symmetric monoidal categories, i.e., functors $F \colon \mathcal{C} \to \mathcal{D}$ that strictly preserve the product $\Box$, the unit object and isomorphism, and the associativity and symmetry isomorphisms. In fact, $\mathcal{K}$ extends to a functor on the *strictly unital op-lax* maps: An *op-lax* map of symmetric monoidal categories consists of a functor $F\colon \mathcal{C} \to \mathcal{D}$, a natural transformation

$$\lambda\colon F(x \mathbin{\Box} y) \longrightarrow F(x) \mathbin{\Box} F(y),$$

and a natural transformation $\epsilon\colon F(u) \to u$ such that the following unit, symmetry, and associativity diagrams commute,

$$\begin{tikzcd}
F(u \mathbin{\Box} y) \arrow[r, "\lambda"] \arrow[d, "F(\eta)"'] & F(u) \mathbin{\Box} F(y) \arrow[d, "\epsilon"] \\
F(y) & u \mathbin{\Box} F(y) \arrow[l, "\eta"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F(x \mathbin{\Box} y) \arrow[r, "F(\gamma)"] \arrow[d, "\lambda"'] & F(y \mathbin{\Box} x) \arrow[d, "\lambda"] \\
F(x) \mathbin{\Box} F(y) \arrow[r, "\gamma"'] & F(y) \mathbin{\Box} F(x)
\end{tikzcd}$$

$$\begin{tikzcd}
 & F(x \mathbin{\Box} y) \mathbin{\Box} F(z) \arrow[dr, "\lambda"] & \\
F((x \mathbin{\Box} y) \mathbin{\Box} z) \arrow[ur, "\lambda"] \arrow[d, "F(\alpha)"'] & & (F(x) \mathbin{\Box} F(y)) \mathbin{\Box} F(z) \arrow[d, "\alpha"] \\
F(x \mathbin{\Box} (y \mathbin{\Box} z)) \arrow[dr, "\lambda"'] & & F(x) \mathbin{\Box} (F(y) \mathbin{\Box} F(z)) \\
 & F(x) \mathbin{\Box} F(y \mathbin{\Box} z) \arrow[ur, "\lambda"'] &
\end{tikzcd}$$

where η, γ, and α denote the unit, symmetry and associativity isomorphisms, respectively. An op-lax map is *strictly unital* when the unit map ϵ is the identity (i.e., F strictly preserves the unit object). A strictly unital op-lax map $\mathcal{C} \to \mathcal{D}$ induces a map of Γ-categories $\mathcal{KC} \to \mathcal{KD}$ sending $(x_I, f_{I,J})$ in $\mathcal{KC}(\mathbf{n})$ to $(F(x_I), \lambda \circ F(f_{I,J}))$ in $\mathcal{KD}(\mathbf{n})$, and likewise for morphisms.
We will need the following additional structure in Section 5. Recall that an op-lax natural transformation $\nu\colon F \to G$ between op-lax maps is a natural transformation such that the following diagrams commute.

$$\begin{tikzcd}
F(x \mathbin{\Box} y) \arrow[r, "\nu"] \arrow[d, "\lambda"'] & G(x \mathbin{\Box} y) \arrow[d, "\lambda"] \\
F(x) \mathbin{\Box} F(y) \arrow[r, "\nu \Box \nu"'] & G(x) \mathbin{\Box} G(y)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(u) \arrow[rr, "\nu"] \arrow[dr, "\epsilon"'] & & G(u) \arrow[dl, "\epsilon"] \\
 & u &
\end{tikzcd}$$

An op-lax natural transformation between strictly unital op-lax maps induces a natural transformation between the induced maps of Γ-categories, compatible with the Γ-structure. We summarize the discussion of the previous paragraphs in the following proposition.

Proposition 3.3. *$\mathcal{K}$ and K are functors from the category of small symmetric monoidal categories and strictly unital op-lax maps to the category of Γ-categories and the category of Γ-spaces, respectively. An op-lax natural transformation induces a natural transformation on $\mathcal{K}$ and a homotopy on K between the induced maps.*

In particular, by restricting to the subcategory consisting of the permutative categories and the strict maps, we get the functors $\mathcal{K}$ and K in the statements of the theorems in the introduction. These functors admit several variants, which extend to different variants of the category of symmetric monoidal categories.

Variant 3.4. In the construction of $\mathcal{K}$, we can require the maps $f_{I,J}$ to be isomorphisms. This is Segal's original K-theory functor as described for example in [11]. The natural domain of this functor is the category of small symmetric monoidal categories and strictly unital strong maps; these are the strictly unital op-lax maps where λ is an isomorphism.

Variant 3.5. In the construction of $\mathcal{K}$, we can require the maps $f_{I,J}$ to go the other direction, i.e.,

$$f_{I,J}\colon x_I \mathbin{\Box} x_J \longrightarrow x_{I\cup J}.$$

This is the functor called Segal K-theory in [5]; its natural domain is the category of small symmetric monoidal categories and strictly unital lax maps. A lax map $\mathcal{C} \to \mathcal{D}$ consists of a functor and natural transformations

$$\lambda\colon F(x) \mathbin{\Box} F(y) \longrightarrow F(x \mathbin{\Box} y) \qquad \text{and} \qquad \epsilon\colon u \longrightarrow F(u)$$

making the evident unit, symmetry, and associativity diagrams commute. Put another way, (F, λ, ϵ) defines an op-lax map $\mathcal{C} \to \mathcal{D}$ if and only if $(F^{\mathrm{op}}, \lambda, \epsilon)$ defines a lax map $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$.

We also have variants which loosen the unit condition, but the constructions occur most naturally by way of functors between different categories of small symmetric monoidal categories. The inclusion of the category with strictly unital op-lax maps into the category with op-lax maps has a left adjoint U. Concretely, $U\mathcal{C}$ has objects the objects of $\mathcal{C}$ plus a new disjoint object v. Morphisms in $U\mathcal{C}$ between objects of $\mathcal{C}$ are just the morphisms in $\mathcal{C}$, and morphisms to and from v are defined by

$$U\mathcal{C}(x, v) = \mathcal{C}(x, u), \qquad U\mathcal{C}(v, x) = \emptyset, \qquad U\mathcal{C}(v, v) = \{\mathrm{id}_v\},$$

for x an object of $\mathcal{C}$ and u the unit in $\mathcal{C}$. We obtain a symmetric monoidal product on $U\mathcal{C}$ from the symmetric monoidal product on $\mathcal{C}$ with v chosen to be a strict unit (i.e., $v \mathbin{\Box} x = x$ for all x in $U\mathcal{C}$); the inclusion of $\mathcal{C}$ in $U\mathcal{C}$ is then op-lax monoidal and the functor $U\mathcal{C} \to \mathcal{C}$ sending v to u is strictly unital op-lax (in fact, strict). Then $\mathcal{K} \circ U$ defines a functor from the category of small symmetric monoidal categories and op-lax maps to Γ-categories.

To compare these variants and to understand K, we construct weak equivalences

$$p_n\colon \mathcal{K}\mathcal{C}(\mathbf{n}) \longrightarrow \mathcal{C} \times \cdots \times \mathcal{C} = \mathcal{C}^{\times n}. \tag{3.6}$$

Define p_n to be the functor sends the object $(x_I, f_{I,J})$ of $\mathcal{KC}(n)$ to the object $(x_{\{1\}}, \ldots, x_{\{n\}})$ of $\mathcal{C}^{\times n}$, and likewise for maps. This functor has a right adjoint q_n that sends $(y_1, \ldots, y_n)$ to the system $(x_I, f_{I,J})$ with

$$x_{\{i_1,\ldots,i_r\}} = (\cdots(y_{i_1} \,\square\, y_{i_2}) \,\square\, \cdots) \,\square\, y_{i_r}$$

for $i_1 < \cdots < i_r$ and the maps $f_{I,J}$ induced by the associativity and symmetry isomorphisms. In the case $n = 1$, these are inverse isomorphisms of categories, and under these isomorphisms, (3.6) is induced by the indicator maps $\mathbf{n} \to \mathbf{1}$. Because an adjunction induces inverse homotopy equivalences on nerves, this proves the following proposition.

Proposition 3.7. *For any small symmetric monoidal category $\mathcal{C}$, $K\mathcal{C}$ is a special Γ-space with $K\mathcal{C}(\mathbf{1})$ isomorphic to $N\mathcal{C}$.*

Similar observations apply to the variant functors above. We have natural transformations relating the strictly unital strong construction to both the strictly unital lax and op-lax constructions. It follows that the K-theory Γ-spaces obtained are naturally weakly equivalent. Likewise, the constructions with the weakened units map to the constructions with strict units. Since the map $U\mathcal{C} \to \mathcal{C}$ induces a homotopy equivalence on nerves, these natural transformations induce natural weak equivalences of Γ-spaces.

Recall that we say that a functor between small categories is a *weak equivalence* when it induces a weak equivalence on nerves. As a consequence of the previous proposition, the K-theory functor preserves weak equivalences. As in the introduction, we say that a strictly unital op-lax map of symmetric monoidal categories is a *stable equivalence* if it induces a stable equivalence on K-theory Γ-spaces, or equivalently, if it induces a weak equivalence on the group completion of the nerves. Quillen's homological criterion to identify the group completion [13] then applies to give an intrinsic characterization of the stable equivalences.

Proposition 3.8. *A map of symmetric monoidal categories $\mathcal{C} \to \mathcal{D}$ is a stable equivalence if and only if it induces an isomorphism of localized homology rings*

$$H_*\mathcal{C}[(\pi_0\mathcal{C})^{-1}] \longrightarrow H_*\mathcal{D}[(\pi_0\mathcal{D})^{-1}]$$

obtained by inverting the multiplicative monoids $\pi_0\mathcal{C} \subset H_0\mathcal{C}$ and $\pi_0\mathcal{D} \subset H_0\mathcal{D}$.

Although not needed in what follows, for completeness of exposition, we offer the following well-known observation on the homotopy theory of the various categories of symmetric monoidal categories. Recall that we say that a functor between small categories is a *weak equivalence* when it induces a weak equivalence on nerves. The following theorem can be proved using the methods of [10, 4.2] and [6, 4.2].

Theorem 3.9. *The following homotopy categories are equivalent:*

(i) *The homotopy category obtained from the category of small permutative categories and strict maps by inverting the weak equivalences.*

(ii) *The homotopy category obtained from the category of small symmetric monoidal categories and strict maps by inverting the weak equivalences.*
(iii) *The homotopy category obtained from the category of small symmetric monoidal categories and strictly unital strong maps by inverting the weak equivalences.*
(iv) *The homotopy category obtained from the category of small symmetric monoidal categories and strictly unital op-lax maps by inverting the weak equivalences.*
(v) *The homotopy category obtained from the category of small symmetric monoidal categories and strictly unital lax maps by inverting the weak equivalences.*

4. Construction of the inverse K-theory functor

In this section we construct the inverse K-theory functor $\mathcal{P}$ as a Grothendieck construction (or homotopy colimit) over a category $\mathcal{A}$ described below. We construct the natural transformations displayed in (1.2) relating the composites $\mathcal{KP}$ and $\mathcal{PK}$ with the identity. This section contains only the constructions; we postpone almost all homotopical analysis to the next section.

We begin with the construction of the category $\mathcal{A}$. As indicated in the introduction, we define the objects of $\mathcal{A}$ to consist of the sequences of positive integers $(n_1, \ldots, n_s)$ for all $s \geq 0$, with $s = 0$ corresponding to the empty sequence $()$. We think of each n_i as the finite (unbased) set $\underline{n}_i$, and we define the maps in $\mathcal{A}$ to be the maps generated by maps of finite sets, permutations in the sequence, and partitioning $\underline{n}_i$ into subsets. We make this precise in the following definition.

Definition 4.1. For $\vec{m} = (m_1, \ldots, m_r)$ and $\vec{n} = (n_1, \ldots, n_s)$ with $r, s > 0$, we define the morphisms $\mathcal{A}(\vec{m}, \vec{n})$ to be the subset of the maps of finite (unbased) sets

$$\underline{m}_1 \amalg \cdots \amalg \underline{m}_r \longrightarrow \underline{n}_1 \amalg \cdots \amalg \underline{n}_s$$

satisfying the property that the inverse image of each subset $\underline{n}_j$ is either empty or contained in a single $\underline{m}_i$ (depending on j). For the object $()$, we define $\mathcal{A}((), \vec{n})$ consist of a single point for all $\vec{n}$ in $\mathcal{A}$ and we define $\mathcal{A}(\vec{m}, ())$ to be empty for $\vec{m} \neq ()$.

For a Γ-category $\mathcal{X}$, let $A\mathcal{X}() = \mathcal{X}(\mathbf{0})$ and

$$A\mathcal{X}(n_1, \ldots, n_s) = \mathcal{X}(\mathbf{n}_1) \times \cdots \times \mathcal{X}(\mathbf{n}_s).$$

For a map $\phi \colon \vec{m} \to \vec{n}$ in $\mathcal{A}$, define

$$A\phi \colon \mathcal{X}(\mathbf{m}_1) \times \cdots \times \mathcal{X}(\mathbf{m}_r) \longrightarrow \mathcal{X}(\mathbf{n}_1) \times \cdots \times \mathcal{X}(\mathbf{n}_s)$$

as follows. If $s = 0$, then $r = 0$ and ϕ is the identity, and we take $A\phi$ to be the identity. If $r = 0$ and $s > 0$, we take $A\phi$ to be the map $\mathcal{X}(\mathbf{0}) \to \mathcal{X}(\mathbf{n}_j)$ on each coordinate. If $r > 0$, then by definition, for each j, the subset $\underline{n}_j$ of

$$\underline{n}_1 \amalg \cdots \amalg \underline{n}_s$$

has inverse image either empty or contained in a single $\underline{m}_i$ for some i; if the inverse image is non-empty, then ϕ restricts to a map of unbased sets $\underline{m}_i \to \underline{n}_j$, which we extend to a map of based sets $\mathbf{m}_i \to \mathbf{n}_j$ that is the identity on the basepoint 0. In this case, we define $A\phi$ on the j-th coordinate to be the composite of the projection

$$\mathcal{X}(\mathbf{m}_1) \times \cdots \times \mathcal{X}(\mathbf{m}_r) \longrightarrow \mathcal{X}(\mathbf{m}_i)$$

and the map $\mathcal{X}(\mathbf{m}_i) \to \mathcal{X}(\mathbf{n}_j)$ induced by the restriction of ϕ. In the case when the inverse image of $\underline{n}_j$ is empty, we define $A\phi$ on the j-th coordinate to be the composite of the projection

$$\mathcal{X}(\mathbf{m}_1) \times \cdots \times \mathcal{X}(\mathbf{m}_r) \longrightarrow * = \mathcal{X}(\mathbf{0})$$

and the map $\mathcal{X}(\mathbf{0}) \to \mathcal{X}(\mathbf{n}_j)$. An easy check gives the following observation.

PROPOSITION 4.2. *A is a functor from the category of small Γ-categories to the category of functors from $\mathcal{A}$ to the category of small categories.*

We can now define the functors $\mathcal{P}$ and P, at least on the level of functors to small categories.

DEFINITION 4.3. Let $\mathcal{P}\mathcal{X} = \mathcal{A} \int A\mathcal{X}$. Let $P = \mathcal{P} \circ \mathcal{S}$.

More concretely, the category $\mathcal{P}\mathcal{X}$ has as objects the disjoint union of the objects of $A\mathcal{X}(\vec{n})$ where $\vec{n}$ varies over the objects of $\mathcal{A}$. For $x \in A\mathcal{X}(\vec{m})$ and $y \in A\mathcal{X}(\vec{n})$, a map in $\mathcal{P}\mathcal{X}$ from x to y consists of a map $\phi\colon \vec{m} \to \vec{n}$ in $\mathcal{A}$ together with a map $\phi_* x \to y$ in $A\mathcal{X}(\vec{n})$, where $\phi_* = A\phi$ is the functor $A\mathcal{X}(\vec{m}) \to A\mathcal{X}(\vec{n})$ above.

Variant 4.4. We can regard $A\mathcal{X}$ as a contravariant functor on $\mathcal{A}^{\mathrm{op}}$ and form the contravariant Grothendieck construction $\mathcal{P}^{\mathrm{lax}}\mathcal{X} = \mathcal{A}^{\mathrm{op}} \int A\mathcal{X}$. This has the same objects as $\mathcal{P}\mathcal{X}$ but for $x \in A\mathcal{X}(\vec{m})$ and $y \in A\mathcal{X}(\vec{n})$, a map in $\mathcal{P}^{\mathrm{lax}}\mathcal{X}$ from x to y consists of a map $\phi\colon \vec{n} \to \vec{m}$ in $\mathcal{A}$ together with a map $x \to \phi_* y$. This functor is better adapted to the category of symmetric monoidal categories and strictly unital lax maps. All results and constructions in this paper admit analogues for $\mathcal{P}^{\mathrm{lax}}$, replacing "op-lax" with "lax" in the work below.

The category $\mathcal{A}$ has the structure of a permutative category under concatenation of sequences, with the empty sequence as the unit and the symmetry morphisms induced by permuting elements in the sequences. The category of small categories is symmetric monoidal under cartesian product and the functor $A\mathcal{X}\colon \mathcal{A} \to \mathbf{Cat}$ associated to a Γ-category $\mathcal{X}$ is a strong symmetric monoidal functor. For formal reasons, then the Grothendieck construction $\mathcal{P}\mathcal{X}$ naturally obtains the structure of a symmetric monoidal category; we can describe this structure concretely as follows. For any object x in $A\mathcal{X}(\vec{m})$, we can write $x = (x_1, \ldots, x_r)$ for objects x_i in $\mathcal{X}(\mathbf{m}_i)$; then for y in $A\mathcal{X}(\vec{n})$,

$$x \mathbin{\Box} y = (x_1, \ldots, x_r, y_1, \ldots, y_s) \in \operatorname{Ob} A\mathcal{X}(m_1, \ldots, m_r, n_1, \ldots, n_s),$$

where we understand the unique object of $A\mathcal{X}()$ as a strict unit. The product on maps admits an analogous description. This concrete description makes it clear that $\mathcal{PX}$ is in fact a permutative category. Moreover, a map $\mathcal{X} \to \mathcal{Y}$ of Γ-categories induces a strict map of permutative categories $\mathcal{PX} \to \mathcal{PY}$. We obtain the following theorem.

THEOREM 4.5. *$\mathcal{P}$ defines a functor from the category of Γ-categories to the category of permutative categories and strict maps.*

Next we construct the natural transformations of (1.2). Starting with a symmetric monoidal category $\mathcal{C}$, we construct the map $\mathcal{PKC} \to \mathcal{C}$ using the homotopy colimit property of the Grothendieck construction. Specifically, we construct functors $\alpha_{\vec{m}}$ from $A\mathcal{PKC}(\vec{m})$ to $\mathcal{C}$ and suitably compatible natural transformations for the maps in $\mathcal{A}$.

For each $\vec{m}$ in $\mathcal{A}$, define the functor

$$\alpha_{\vec{m}} \colon A\mathcal{KC}(\vec{m}) = \mathcal{KC}(\mathbf{m}_1) \times \cdots \times \mathcal{KC}(\mathbf{m}_r) \longrightarrow \mathcal{C}$$

to take the object $\vec{X} = (X_1, \ldots, X_r)$ to

$$(\cdots(x^1_{\underline{m}_1} \,\square\, x^2_{\underline{m}_2}) \,\square \cdots) \,\square\, x^r_{\underline{m}_r}$$

where $X_i = (x^i_I, f_{I,J})$ for $I \subset \underline{m}_i$, and likewise for maps in $\mathcal{KC}(\vec{m})$. For $\vec{m} = ()$, we understand $\alpha_{()}$ to include the category $A\mathcal{KC}() = *$ in $\mathcal{C}$ as the unit u and the identity on u.

For a map ϕ in $\mathcal{A}$ from $\vec{m}$ to $\vec{n}$, define

$$\alpha_\phi \colon \alpha_{\vec{m}}(y_1, \ldots, y_r) \longrightarrow \alpha_{\vec{n}}(\phi_*(y_1, \ldots, y_r))$$

to be the map induced by the associativity, symmetry, and inverse unit isomorphisms in $\mathcal{C}$ and the maps $f^i_{I_1,\ldots,I_k}$ in y_i: if ϕ sends $\underline{m}_i$ into $\underline{n}_{j_1}, \ldots, \underline{n}_{j_t}$, then composing maps $f^i_{I,J}$ in y_i gives a well-defined map

$$f_{I_1,\ldots,I_t} \colon x^i_{\underline{m}_i} \longrightarrow (\cdots(x^i_{I_1} \,\square\, x^i_{I_2}) \,\square \cdots) \,\square\, x^i_{I_t}$$

where I_k is the subset of $\underline{m}_i$ landing in $\underline{n}_{j_k}$. The map α_ϕ is a natural transformation of functors from $\alpha_{\vec{m}}$ to $\alpha_{\vec{n}} \circ \phi_*$. Moreover, given a map ψ from $\vec{n}$ to $\vec{p}$ in $\mathcal{A}$, the following diagram commutes

$$\begin{tikzcd}
\alpha_{\vec{m}}\vec{X} \arrow[r, "\alpha_\phi"] \arrow[d, "\alpha_{\psi\circ\phi}"'] & \alpha_{\vec{n}}(\phi_*\vec{X}) \arrow[d, "\alpha_\psi"] \\
\alpha_{\vec{p}}((\psi\circ\phi)_*\vec{X}) \arrow[r, phantom, "="] & \alpha_{\vec{p}}(\psi_*\phi_*\vec{X})
\end{tikzcd}$$

for any $\vec{X}$ in $A\mathcal{PKC}(\vec{m})$.

DEFINITION 4.6. Let $\alpha \colon \mathcal{PKC} \to \mathcal{C}$ be the functor $\mathcal{A} \int AX \to \mathcal{KC}$ that sends $\vec{X}$ in $A\mathcal{KC}(\vec{m})$ to $\alpha_{\vec{m}}\vec{X}$ and sends the map

$$\phi \colon \vec{m} \longrightarrow \vec{n}, \quad f \colon \phi_*\vec{X} \longrightarrow \vec{Y}$$

to the map $\alpha_{\vec{n}}(f) \circ \alpha_\phi$ in $\mathcal{C}$.

Examining the construction of α and the symmetric monoidal structure on $\mathcal{PKC}$, we obtain the following theorem.

Theorem 4.7. *The functor $\alpha\colon \mathcal{PKC} \to \mathcal{C}$ satisfies the following properties.*

(i) *α is a strictly unital strong map of symmetric monoidal categories.*
(ii) *α is natural up to natural transformation in strictly unital op-lax maps.*
(iii) *α is a strict map when $\mathcal{C}$ is a permutative category.*
(iv) *α is natural in strict maps.*

The meaning of (ii) and (iv) is that for a strictly unital op-lax map $F\colon \mathcal{C} \to \mathcal{D}$, the diagram

$$\begin{tikzcd} \mathcal{PKC} \arrow[r] \arrow[d, "\alpha"'] & \mathcal{PKD} \arrow[d, "\alpha"] \\ \mathcal{C} \arrow[r] & \mathcal{D} \end{tikzcd}$$

commutes up to natural transformation, namely, the natural transformation

$$\lambda\colon F(x_1 \mathbin{\Box} \cdots \mathbin{\Box} x_r) \longrightarrow F(x_1) \mathbin{\Box} \cdots \mathbin{\Box} F(x_r)$$

which is part of the structure of the op-lax map. When $\mathcal{C} \to \mathcal{D}$ is a strict map, the diagram commutes strictly (the natural transformation is the identity).
For the remaining natural transformation in (1.2), note that for a Γ-category $\mathcal{X}$, we have a canonical inclusion $\iota\colon \mathcal{X}(\mathbf{n}) \to \mathcal{KPX}(\mathbf{n})$ sending an object x in $\mathcal{X}(\mathbf{n})$ to the object $\iota x = (x_I, f_{I,J})$ in $\mathcal{KPX}(\mathbf{n})$ with

$$x_I = \pi^I_*(x) \in \mathcal{X}(\mathbf{m}) = A\mathcal{X}(m)$$

for $m = |I|$, $I = \{i_1, \ldots, i_m\}$ with $i_1 < \cdots < i_m$, and $\pi^I\colon \mathbf{n} \to \mathbf{m}$ the map that sends i_k to k and every other element of $\mathbf{n}$ to 0. The map

$$f_{I,J}\colon x_{I\cup J} \longrightarrow x_I \mathbin{\Box} x_J = (x_I, x_J) \in A\mathcal{X}(|I|, |J|)$$

is induced by the map $(|I \cup J|) \to (|I|, |J|)$ in $\mathcal{A}$ corresponding to the partition of the ordered set $I \cup J$ into I and J. This does not fit together into a map of Γ-categories: For a map $\phi\colon \mathbf{m} \to \mathbf{n}$ in $\mathcal{F}$,

$$\iota(\phi_* x)_I = \pi^I_*(\phi_* x), \qquad \text{but} \qquad \phi_*(\iota x)_I = \pi^{\phi^{-1}I}_*(x).$$

Writing ϕ' for the map in $\mathcal{N}$ corresponding to the restriction of ϕ to the map $\phi^{-1}I \to I$, then

$$\pi^I \circ \phi = \phi' \circ \pi^{\phi^{-1}I}$$

in $\mathcal{F}$. We can interpret ϕ' as a map

$$\pi^{\phi^{-1}I}_*(x) \longrightarrow \pi^I_*(\phi_* x)$$

in $\mathcal{PX}$. These maps in turn assemble to a map

$$\omega_\phi\colon \phi_*\iota x \longrightarrow \iota\phi_* x$$

in $\mathcal{KPX}$, natural in x.

Definition 4.8. For a Γ-category $\mathcal{X}$, let $\mathcal{WX}(\mathbf{n})$ be the category whose objects consist of triples (y, x, g) with y an object of $\mathcal{KP}(\mathbf{n})$, x an object of $\mathcal{X}(\mathbf{n})$ and

$g\colon y \to \iota x$ a map in $\mathcal{KP}(\mathbf{n})$. The morphisms of $\mathcal{WX}(\mathbf{n})$ are the commuting diagrams. For $\phi\colon \mathbf{m} \to \mathbf{n}$ in $\mathcal{F}$, define

$$\mathcal{WX}(\phi)\colon \mathcal{WX}(\mathbf{m}) \longrightarrow \mathcal{WX}(\mathbf{n})$$

to be the functor that takes (y, x, g) to $(\phi_* y, \phi_* x, \omega_\phi \circ \phi_* g)$.

We note for later use that for (y, x, g) an object in $\mathcal{WX}(\mathbf{n})$, writing $y = (y_I, f_{I,J})$ with y_I in $\mathcal{PX}$, we must have each y_I in $A\mathcal{X}()$ or $A\mathcal{X}(m_I)$ for some m_I. This is because ιx_I is in $A\mathcal{X}(n)$ and maps in $\mathcal{A}$ cannot decrease the length of the sequence.
We claim that the categories $\mathcal{WX}(\mathbf{n})$ and functors $\mathcal{WX}(\phi)$ assemble into a Γ-category. For $\phi\colon \mathbf{m} \to \mathbf{n}$, $\psi\colon \mathbf{n} \to \mathbf{p}$, and $I \subset \underline{p}$, write

$$\pi^I \circ \psi \circ \phi = \psi' \circ \pi^{\psi^{-1}I} \circ \phi = \psi' \circ \phi' \circ \pi^{\phi^{-1}(\psi^{-1}I)}$$

as above, with $\psi'\colon \psi^{-1}I \to I$ and $\phi'\colon \phi^{-1}(\psi^{-1}I) \to \psi^{-1}I$ the restrictions of ψ and ϕ, using the natural order on $I \subset \underline{p}$, $\psi^{-1}I \subset \underline{n}$, and $\phi^{-1}(\psi^{-1}I) \subset \underline{m}$ to view these as maps in $\mathcal{F}$. Then $\psi' \circ \phi'\colon (\psi \circ \phi)^{-1}I \to I$ is the restriction of $\psi \circ \phi$; this is the check required to see that the diagram

$$\begin{tikzcd}
\pi_*^{(\psi\circ\phi)^{-1}I} x \arrow[rr] \arrow[dr] & & \pi_*^{\psi^{-1}I}(\phi_* x) \arrow[dl] \\
& \pi_*^I((\psi \circ \phi)_* x) &
\end{tikzcd}$$

in $\mathcal{PX}$ commutes. Examination of the structure maps $f_{I,J}$ in ιx shows that the diagram

$$\begin{tikzcd}
(\psi \circ \phi)_* \iota x \arrow[rr, "\psi_* \omega_\phi"] \arrow[dr, "\omega_{\psi\circ\phi}"'] & & \psi_* \iota(\phi_* x) \arrow[dl, "\omega_\psi"] \\
& \iota((\psi \circ \phi)_* x) &
\end{tikzcd}$$

in $\mathcal{KPX}$ commutes. This proves the following theorem.

THEOREM 4.9. *The maps $\mathcal{WX}(\phi)$ above make $\mathcal{WX}$ into a Γ-category.*

Since $\mathcal{WX}$ is natural in maps of Γ-categories $\mathcal{X}$, we can regard $\mathcal{W}$ as an endofunctor on Γ-categories. By construction, the forgetful functors $\omega\colon \mathcal{WX} \to \mathcal{X}$ and $\upsilon\colon \mathcal{WX} \to \mathcal{KPX}$ are natural transformations of endofunctors. For fixed $\mathbf{n}$, the functor $\mathcal{WX}(\mathbf{n}) \to \mathcal{X}(\mathbf{n})$ is a left adjoint: The right adjoint sends x in $\mathcal{X}(\mathbf{n})$ to $(\iota x, x, \mathrm{id}_{\iota x})$ in $\mathcal{WX}(\mathbf{n})$. It follows that ω is always a weak equivalence of Γ-categories. We summarize this in the following theorem.

THEOREM 4.10. *The maps $\upsilon\colon \mathcal{WX} \to \mathcal{KPX}$ and $\omega\colon \mathcal{WX} \to \mathcal{X}$ are natural transformations of endofunctors on Γ-categories, and ω is a weak equivalence for any $\mathcal{X}$.*

5. Proof of Theorems 1.3 and 1.4

This section provides the homotopical analysis of the functors and natural transformations constructed in the previous section. This leads directly to the proof of the main theorem, Theorem 1.3, and its non-completed variant, Theorem 1.4.

Most of the arguments hinge on the following lemma of Thomason [16]:

Lemma 5.1 (Thomason). *Let $\mathcal{A}$ be a small category and F a functor from $\mathcal{A}$ to the category of small categories. There is a natural weak equivalence of simplicial sets*

$$\operatorname{hocolim}_{\mathcal{A}} NF \longrightarrow N(\mathcal{A} \textstyle\int F).$$

The natural transformation is easy to describe. We write an object of $\mathcal{A}\int F$ as $(\vec{n}, x)$ with $\vec{n}$ an object of $\mathcal{A}$ and x an object of $F\vec{n}$, and we write a map in $\mathcal{A}\int F$ as $(\phi, f)\colon (\vec{n}, x) \to (\vec{p}, y)$ where $\phi\colon \vec{n} \to \vec{p}$ is a map in $\mathcal{A}$ and $f\colon F(\phi)(x) \to y$ is a map in $F\vec{p}$. Then a q-simplex of the nerve $N(\mathcal{A} \int F)$ is a sequence of q composable maps

$$(\vec{n}_0, x_0) \xrightarrow{(\phi_1,f_1)} (\vec{n}_1, x_1) \xrightarrow{(\phi_2,f_2)} \cdots \xrightarrow{(\phi_q,f_q)} (\vec{n}_q, x_q).$$

Likewise, a q-simplex in the homotopy colimit consists of a sequence of q composable maps in $\mathcal{A}$ together with q composable maps in $F(\vec{n}_0)$:

$$\vec{n}_0 \xrightarrow{\phi_1} \vec{n}_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_q} \vec{n}_q$$

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_q} x_q.$$

The natural transformation sends this simplex of the homotopy colimit to the simplex

$$(\vec{n}_0, x_0) \xrightarrow{(\phi_1,f_1)} (\vec{n}_1, x_1') \xrightarrow{(\phi_2,f_2')} \cdots \xrightarrow{(\phi_q,f_q')} (\vec{n}_q, x_q'),$$

where $x_k' = F(\phi_{k,\dots,1})(x_k)$ and $f_k' = F(\phi_{k-1,\dots,1})(f_k)$ for $\phi_{k,\dots,1} = \phi_k \circ \cdots \circ \phi_1$. A Quillen Theorem A style argument proves that this map is a weak equivalence [16, §1.2].

Applying Thomason's lemma to the Grothendieck construction $\mathcal{A}\int A\mathcal{X}$, we get the following immediate observation.

Proposition 5.2. *$\mathcal{P}$ preserves weak equivalences.*

The following theorem provides the main homotopical result we need for the remaining arguments in this section.

Theorem 5.3. *Let $\mathcal{X}$ be a special Γ-category. Then the inclusion of $\mathcal{X}(\mathbf{1})$ in $\mathcal{P}\mathcal{X}$ is a weak equivalence.*

Proof. Recall that $\mathcal{N}$ denotes the category with objects $\underline{n} = \{1, \dots, n\}$ and morphisms the maps of sets. We have an inclusion $\eta\colon \mathcal{N} \to \mathcal{A}$ sending $\underline{0}$ to $()$ and $\underline{n}$ to (n) for $n > 0$. We have a functor $\epsilon\colon \mathcal{A} \to \mathcal{N}$ sending $\vec{n} = (n_1, \dots, n_s)$

to $\underline{n}$ with $n = n_1 + \cdots + n_s$. Let $B\mathcal{X}$ be the functor from $\mathcal{A}$ to small categories defined by

$$B\mathcal{X}(\vec{n}) = \epsilon^* \mathcal{X}(\vec{n}) = \mathcal{X}(\epsilon(\vec{n})) = \mathcal{X}(\mathbf{n}).$$

Then the maps $\mathbf{n} \to \mathbf{n}_j$ coming from the partition of n as $\vec{n}$ induce a natural transformation of functors $B\mathcal{X} \to A\mathcal{X}$. The hypothesis that $\mathcal{X}$ is special implies that this map is an objectwise weak equivalence. Now applying Thomason's lemma, it suffices to show that the inclusion of $N\mathcal{X}(\mathbf{1})$ in $\operatorname{hocolim}_{\mathcal{A}} NB\mathcal{X}$ is a weak equivalence.

Since $B\mathcal{X} = \epsilon^* \mathcal{X}$ as a functor on $\mathcal{A}$ and $\mathcal{X} = \eta^* B\mathcal{X}$ as a functor on $\mathcal{N}$, we have canonical maps

$$\operatorname{hocolim}_{\mathcal{N}} N\mathcal{X} \longrightarrow \operatorname{hocolim}_{\mathcal{A}} NB\mathcal{X} \longrightarrow \operatorname{hocolim}_{\mathcal{N}} N\mathcal{X} \tag{5.4}$$

induced by ϵ and η. The composite map on $\operatorname{hocolim}_{\mathcal{N}} N\mathcal{X}$ is induced by $\epsilon \circ \eta = \mathrm{Id}_{\mathcal{N}}$, and is therefore the identity. The composite map on $\operatorname{hocolim}_{\mathcal{A}} NB\mathcal{X}$ is induced by $\eta \circ \epsilon$. We have a natural transformation from $\eta \circ \epsilon$ to the identity functor on $\mathcal{A}$ induced by the partition maps,

$$\eta \circ \epsilon(\vec{n}) = (n) \longrightarrow (n_1, \ldots, n_s) = \vec{n}.$$

Because

$$B\mathcal{X}(\eta \circ \epsilon(\vec{n})) = \mathcal{X}(\mathbf{n}) = B\mathcal{X}(\vec{n}),$$

we get a homotopy from the composite map on $\operatorname{hocolim}_{\mathcal{A}} NB\mathcal{X}$ to the identity. In other words, we have shown that the maps in (5.4) are inverse homotopy equivalences. Since $\underline{1}$ is the final object in $\mathcal{N}$, the inclusion of $N\mathcal{X}(\mathbf{1})$ in $\operatorname{hocolim}_{\mathcal{N}} N\mathcal{X}$ is a homotopy equivalence, and it follows that the inclusion of $N\mathcal{X}(\mathbf{1})$ in $\operatorname{hocolim}_{\mathcal{A}} NB\mathcal{X}$ is a homotopy equivalence. □

When $\mathcal{X} = \mathcal{K}\mathcal{C}$ for a small symmetric monoidal category $\mathcal{C}$, we have the canonical isomorphism $\mathcal{K}\mathcal{C}(\mathbf{1}) \cong \mathcal{C}$, and the composite map

$$\mathcal{C} \cong \mathcal{K}\mathcal{C}(\mathbf{1}) \longrightarrow \mathcal{P}\mathcal{K}\mathcal{C} \longrightarrow \mathcal{C}$$

is the identity on $\mathcal{C}$. Since $\mathcal{K}\mathcal{C}$ is always a special Γ-category, we get the following corollary.

Corollary 5.5. *The natural map $\alpha\colon \mathcal{P}\mathcal{K}\mathcal{C} \to \mathcal{C}$ is always a weak equivalence.*

We also get a comparison for $\mathcal{K}\mathcal{P}\mathcal{X}$ when $\mathcal{X}$ is special.

Corollary 5.6. *If $\mathcal{X}$ is special, then $\upsilon\colon \mathcal{W}\mathcal{X} \to \mathcal{K}\mathcal{P}\mathcal{X}$ is a weak equivalence.*

Proof. Let $\mathcal{X}$ be a special Γ-category. The restriction of the map ω to the $\mathbf{1}$-categories, $\mathcal{W}\mathcal{X}(\mathbf{1}) \to \mathcal{X}(\mathbf{1})$, is an equivalence of categories, and the composite map

$$\mathcal{X}(\mathbf{1}) \longrightarrow \mathcal{W}\mathcal{X}(\mathbf{1}) \longrightarrow \mathcal{K}\mathcal{P}\mathcal{X}(\mathbf{1}) = \mathcal{P}\mathcal{X}$$

is the map in the theorem, and therefore a weak equivalence. It follows that the restriction of υ to the $\mathbf{1}$-categories is a weak equivalence. Since the map $\omega\colon \mathcal{W}\mathcal{X} \to \mathcal{X}$ is a weak equivalence, $\mathcal{W}\mathcal{X}$ is also a special Γ-category, and it follows that υ is a weak equivalence. □

Together with Proposition 2.7 and Theorem 4.10, Corollaries 5.5 and 5.6 prove Theorem 1.4. To prove Theorem 1.3, we need to see that the map υ is always a stable equivalence. For this we use the following technical lemma.

Lemma 5.7. *The diagram*

$$\begin{tikzcd}
\mathcal{WKPX} \arrow[r, "\upsilon"] \arrow[dr, "\omega"'] & \mathcal{KPKPX} \arrow[d, "\mathcal{K}\alpha"] & \mathcal{KPWX} \arrow[l, "\mathcal{KP}\upsilon"'] \arrow[dl, "\mathcal{KP}\omega"] \\
& \mathcal{KPX} &
\end{tikzcd}$$

commutes up to natural transformation of maps of Γ*-categories. All maps in the diagram are weak equivalences.*

Proof. The weak equivalence statement follows from the diagram statement since α and ω are always weak equivalences and $\mathcal{K}$ and $\mathcal{P}$ preserve weak equivalences. For the diagram statement, it suffices to show that the diagrams

$$\begin{tikzcd}
\mathcal{WKC} \arrow[r, "\upsilon"] \arrow[dr, "\omega"'] & \mathcal{KPKC} \arrow[d, "\mathcal{K}\alpha"] \\
& \mathcal{KC}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{PKPX} \arrow[d, "\alpha"'] & \mathcal{PWX} \arrow[l, "\mathcal{P}\upsilon"'] \arrow[dl, "\mathcal{P}\omega"] \\
\mathcal{PX} &
\end{tikzcd}$$

commute up to natural transformation of maps of Γ-categories (on the left) for all $\mathcal{C}$ and up to op-lax natural transformation (on the right) for all $\mathcal{X}$.

On the left, starting with an object (y, x, g) in $\mathcal{WKC}(\mathbf{n})$, the top left composite takes this to $\mathcal{K}\alpha(y)$ and the diagonal arrow takes this to x; the effect on maps in $\mathcal{WKC}(\mathbf{n})$ admits the analogous description. Since $\mathcal{K}\alpha(\iota x) = x$, $\mathcal{K}\alpha(g)$ is a map from $\mathcal{K}\alpha(y)$ to x, which is natural in $\mathcal{WKC}(\mathbf{n})$, and compatible with the Γ-structure.

On the right, consider an element $X = (X_1, \ldots, X_s)$ in $A\mathcal{WX}(\vec{n})$, where $X_i = (y^i, x^i, g^i)$ is an object in $\mathcal{WX}(\mathbf{n}_i)$. As per the remark following Definition 4.8, we can write $y^i = (y^i_I, f_{I,J})$ for y^i_I some object of $\mathcal{X}(\mathbf{m}_I)$ (thought of as $A\mathcal{X}(m)$ or $A\mathcal{X}()$) for some m_I, where I ranges over the subsets of $\underline{n}_i$. The left down composite sends X to

$$\alpha(y^1, \ldots, y^s) = (y^1_{\underline{m}_{\underline{n}_1}}, \ldots, y^s_{\underline{m}_{\underline{n}_s}})$$

since the symmetric monoidal product in $\mathcal{PX}$ is concatenation. An analogous description applies to maps of X in $\mathcal{PWX}$. The diagonal in the diagram sends X to $(x^1, \ldots, x^s)$ and we have the map

$$(g^i_{\underline{m}_{\underline{n}_i}})\colon (y^i_{\underline{m}_{\underline{n}_i}}) \longrightarrow (\iota x^i_{\underline{m}_{\underline{n}_i}}) = (x^i).$$

in $\mathcal{PX}$. This map is natural in X in $\mathcal{PWX}$ and is a strictly monoidal natural transformation. $\square$

Proof of Theorem 1.3. Given Propositions 2.7 and 2.8, Theorem 4.10, and Corollaries 5.5 and 5.6, it suffices to show that the map $\upsilon\colon \mathcal{WX} \to \mathcal{KPX}$

is always a stable equivalence. Writing $[-,-]$ for maps in the homotopy category obtained by formally inverting the weak equivalences, we need to show that

$$\upsilon^* \colon [\mathcal{KPX}, \mathcal{Z}] \longrightarrow [\mathcal{WX}, \mathcal{Z}]$$

is a bijection for every very special Γ-category $\mathcal{Z}$. Since $\mathcal{K}$ and $\mathcal{P}$ preserve weak equivalences, they induce functors on the homotopy category. Using this and the fact that υ is a weak equivalence for a special Γ-category, we get a map

$$R \colon [\mathcal{WX}, \mathcal{Z}] \longrightarrow [\mathcal{KPX}, \mathcal{Z}]$$

as follows: Given f in $[\mathcal{WX}, \mathcal{Z}]$, the map Rf in $[\mathcal{KPX}, \mathcal{Z}]$ is the composite

$$\mathcal{KPX} \xrightarrow{\mathcal{KP}\omega^{-1}} \mathcal{KPWX} \xrightarrow{\mathcal{KP}f} \mathcal{KPZ} \xrightarrow{\upsilon^{-1}} \mathcal{WZ} \xrightarrow{\omega} \mathcal{Z}.$$

To see that the composite map on $[\mathcal{WX}, \mathcal{Z}]$ is the identity, consider the following diagram,

$$\begin{tikzcd}
\mathcal{X} & \mathcal{WX} \arrow[l, "\omega"', "\sim"] \arrow[r, "f"] & \mathcal{Z} \\
\mathcal{WX} \arrow[u, "\omega", "\sim"'] \arrow[d, "\upsilon"'] & \mathcal{WWX} \arrow[l, "\mathcal{W}\omega"', "\sim"] \arrow[r, "\mathcal{W}f"] \arrow[u, "\omega", "\sim"'] \arrow[d, "\upsilon"'] & \mathcal{WZ} \arrow[u, "\omega", "\sim"'] \arrow[d, "\upsilon"', "\sim"] \\
\mathcal{KPX} & \mathcal{KPWX} \arrow[l, "\sim"', "\mathcal{KP}\omega"] \arrow[r, "\mathcal{KP}f"'] & \mathcal{KPZ}
\end{tikzcd}$$

which commutes by naturality. We see that $\mathcal{W}\omega$ is a weak equivalence (as marked) by the two-out-of-three property since ω is always a weak equivalence. The map $R(f) \circ \upsilon$ is the composite map in the homotopy category of the part of this diagram starting from the copy of $\mathcal{WX}$ in the first column and traversing maps and inverse maps to $\mathcal{Z}$ by going down, right twice, and then up twice; this agrees with the composite map in the homotopy category obtained by going up and then right twice, $f \circ \omega^{-1} \circ \omega = f$.
On the other hand, starting with g in $[\mathcal{KPX}, \mathcal{Z}]$, then

$$R(g \circ \upsilon) = \omega \circ \upsilon^{-1} \circ \mathcal{KP}(g \circ \upsilon \circ \omega^{-1}).$$

The solid arrow part of the diagram

$$\begin{tikzcd}
\mathcal{KPX} & \mathcal{WKPX} \arrow[l, "\omega"', "\sim"] \arrow[r, "\mathcal{W}g"] \arrow[d, "\sim"', "\upsilon"] & \mathcal{WZ} \arrow[r, "\omega", "\sim"'] \arrow[d, "\sim"', "\upsilon"] & \mathcal{Z} \\
\mathcal{KPWX} \arrow[u, "\mathcal{KP}\omega"] \arrow[r, "\mathcal{KP}\upsilon"'] & \mathcal{KPKPX} \arrow[r, "\mathcal{KP}g"'] \arrow[ul, dotted, "\mathcal{K}\alpha"] & \mathcal{KPZ}
\end{tikzcd}$$

commutes and Lemma 5.7 implies that the whole diagram commutes in the homotopy category. By naturality of ω, the composite $\omega \circ \mathcal{W}g \circ \omega^{-1}$ is g, and it follows that $R(g \circ \upsilon)$ is g. $\square$

6. Special Γ-spaces and non-completed E_∞ spaces

This section explores the analogue in simplicial sets of the construction of $\mathcal{P}$ in small categories, which provides a functor E from Γ-spaces to E_∞ spaces over the Barratt-Eccles operad. This section is entirely independent from the rest of the paper and we have written it to be as self-contained as possible without being overly repetitious. We assume familiarity with Γ-spaces, but not with Γ-categories or permutative categories (except where we compare E and $\mathcal{P}$ in Proposition 6.5).

Definition 4.1 describes a category $\mathcal{A}$ whose objects are the sequences of positive integers (including the empty sequence). We think of a positive integer as a finite (unbased) set, and maps between sequences $\vec{m} = (m_1, \dots, m_r)$ and $\vec{n} = (n_1, \dots, n_s)$ are generated by permuting elements in the sequence, maps of finite sets, and partitioning finite sets. For a Γ-space X, let AX be the functor from $\mathcal{A}$ to based simplicial sets with

$$AX(\vec{n}) = X(\mathbf{n}_1) \times \cdots \times X(\mathbf{n}_s)$$

for $s > 0$ and $AX() = *$. In terms of the maps in $\mathcal{A}$, a permutation of sequences induces the corresponding permutation of factors; a map of finite unbased sets $\phi\colon \underline{n} \to \underline{p}$ induces the corresponding map $X(\phi)$ (for the corresponding $\phi\colon \mathbf{n} \to \mathbf{p}$); a partition $\underline{n} = \underline{p}_1 \amalg \cdots \amalg \underline{p}_t$ induces the map

$$X(\mathbf{n}) \longrightarrow X(\mathbf{p}_1) \times \cdots \times X(\mathbf{p}_t)$$

induced by the maps $\mathbf{n} \to \mathbf{p}_i$ that pick out the elements of the subset $\mathbf{p}_i$ and send all the other elements to the basepoint. We consider the homotopy colimit.

Definition 6.1. Let $EX = \operatorname{hocolim}_{\mathcal{A}} AX$.

It is clear from the definition that E preserves weak equivalences. The proof of the remainder of the following theorem is identical to the proof of Theorem 5.3.

Theorem 6.2. *E preserves weak equivalences. If X is a special Γ-space, then the inclusion of $X(\mathbf{1})$ in EX is a weak equivalence.*

Recall that the Barratt-Eccles operad $\mathcal{E}$ has as its n-th simplicial set $\mathcal{E}(n) = NT\Sigma_n$ the nerve of the translation category on the n-th symmetric group Σ_n, with operadic multiplication induced by block sum of permutations. For any permutation σ in Σ_n, we have a functor

$$\sigma\colon \mathcal{A}^{\times n} \longrightarrow \mathcal{A}$$

induced by permutation and concatenation:

$$\sigma(\vec{m}^1, \dots, \vec{m}^n) = (m_1^{\sigma 1}, \dots, m_{r_{\sigma 1}}^{\sigma 1}, m_1^{\sigma 2}, \dots, m_{r_{\sigma n}}^{\sigma n}).$$

Permutation induces a natural transformation

$$AX^{\times n} \longrightarrow AX$$

covering σ; we therefore get an induced map on homotopy colimits

$$\sigma_*\colon (EX)^{\times n} \longrightarrow EX.$$

For any other element $\sigma' \in \Sigma_n$, the permutation $\sigma'\sigma^{-1}$ induces a natural transformation between functors

$$\sigma, \sigma' \colon \mathcal{A}^{\times n} \longrightarrow \mathcal{A},$$

compatible with the natural transformations $AX^{\times n} \to AX$ covering them. These fit together to induce a map

$$\mathcal{E}(n) \times EX^{\times n} \cong NT\Sigma_n \times \operatorname{hocolim}_{\mathcal{A}^{\times n}} AX^{\times n} \longrightarrow EX. \tag{6.3}$$

An easy check of the definitions proves the following proposition, a restatement of Theorem 1.6.

Proposition 6.4. *The maps* (6.3) *define an action of the operad $\mathcal{E}$ on the simplicial set EX. This action is natural in maps of the Γ-space X. Thus, E defines a functor from Γ-spaces to E_∞ spaces over $\mathcal{E}$.*

To compare the functor E with the functor $\mathcal{P}$, recall that the nerve of a permutative category has the natural structure of an $\mathcal{E}$ space with the map $\sigma_* \colon N\mathcal{C}^{\times n} \to N\mathcal{C}$ (for σ in Σ_n) induced by the permutation and the permutative product. In Section 5, we reviewed the map from the homotopy colimit of the nerve to the nerve of the Grothendieck construction, which we can now interpret as a natural transformation $EN \to N\mathcal{P}$. The following proposition is clear from explicit description of the map in that section.

Proposition 6.5. *For a Γ-category $\mathcal{X}$, the canonical map $EN\mathcal{X} \to N\mathcal{P}\mathcal{X}$ is a map of $\mathcal{E}$ spaces and a weak equivalence.*

Next we define the Γ-space version of the functor EX. For this we use the Γ-spaces X_n defined by

$$X_n(\mathbf{m}) = X(\mathbf{nm}),$$

where we use lexicographical ordering to make $\mathbf{nm}$ a functor of $\mathbf{m}$ from $\mathcal{F}$ to $\mathcal{F}$. Taking advantage of the fact that $\mathbf{nm}$ is also a functor of $\mathbf{n}$, the construction $EX_{(-)}$ defines a functor from $\mathcal{F}$ to simplicial sets. However, since we require Γ-spaces to satisfy $X(\mathbf{0}) = *$, we need a reduced version.

Definition 6.6. Let $E^\Gamma X$ be the Γ-space with $E^\Gamma X(\mathbf{n})$ the based homotopy colimit in the category of based simplicial sets

$$E^\Gamma X(\mathbf{n}) = \operatorname{hocolim}^*_{\mathcal{A}} AX_n.$$

The inclusions $\eta_n \colon X_n(\mathbf{1}) \to E^\Gamma X(\mathbf{n})$ now assemble to a map of Γ-spaces $X \to E^\Gamma X$. Since $\mathcal{A}$ has an initial object (), the nerve $N\mathcal{A}$ is contractible. The map

$$EX_n = \operatorname{hocolim}_{\mathcal{A}} AX_n \longrightarrow (\operatorname{hocolim}_{\mathcal{A}} AX_n)/N\mathcal{A} = \operatorname{hocolim}^*_{\mathcal{A}} AX_n = E^\Gamma X(\mathbf{n})$$

is therefore a weak equivalence. Applying Theorem 6.2 objectwise to the map $X \to E^\Gamma X$, we get the following theorem.

Theorem 6.7. *E^Γ preserves weak equivalences. If X is a special Γ-space, then the natural map $X \to E^\Gamma X$ is a weak equivalence.*

We prove the following theorem at the end of the section.

Theorem 6.8. *For a* Γ*-space* X*,* $E^\Gamma X$ *is a special* Γ*-space.*

Finally, we need one further variant of this construction. Let AEX be the functor from $\mathcal{A}$ to simplicial sets with

$$AEX(\vec{n}) = EX_{n_1} \times \cdots \times EX_{n_s}$$

and $AE() = EX_0 = E*$: Although $EX_{(-)}$ is not a Γ-space, it is an $\mathcal{F}$-space (functor from $\mathcal{F}$ to simplicial sets), and this is all that is needed for the construction of the functor AEX. Let E^2X be the simplicial set

$$E^2X = \operatorname{hocolim}_{\mathcal{A}} AEX$$

(homotopy colimit in the category of unbased simplicial sets). The map of $\mathcal{F}$-spaces $EX_{(-)} \to E^\Gamma X(-)$ induces a weak equivalence $AEX \to AE^\Gamma X$ and a weak equivalence $E^2X \to E(E^\Gamma X)$.
The advantage of E^2X over $E(E^\Gamma X)$ is that we can construct a map $E^2X \to EX$ as follows. For each $\vec{m}$ in $\mathcal{A}$, we have a map

$$\begin{aligned} AEX(\vec{m}) = EX_{m_1} \times \cdots \times EX_{m_r} &\cong \operatorname{hocolim}_{\mathcal{A}^{\times r}}(AX_{m_1} \times \cdots \times AX_{m_r}) \\ &\longrightarrow \operatorname{hocolim}_{\mathcal{A}} AX = EX \end{aligned}$$

induced by the functor $\rho_{\vec{m}} \colon \mathcal{A}^{\times r} \to \mathcal{A}$, defined by

$$\rho_{\vec{m}} \colon (\vec{n}_1, \ldots, \vec{n}_r) \mapsto (m_1 n_{1,1}, \ldots, m_1 n_{1,s_1}, m_2 n_{2,1}, \ldots, m_r n_{r,s_r})$$

(where $\vec{n}_i = (n_{i,1}, \ldots, n_{i,s_i})$), together with the canonical isomorphism

$$AX_{m_1}(\vec{n}_1) \times \cdots \times AX_{m_r}(\vec{n}_r) \cong AX(\rho_{\vec{m}}(\vec{n}_1, \ldots, \vec{n}_r))$$

covering $\rho_{\vec{m}}$. These maps are compatible with maps $\vec{m}$ in $\mathcal{A}$, and so induce a map $\alpha \colon E^2X \to EX$. The technical fact about this map we need is the following lemma, which is an easy check of the construction.

Lemma 6.9. *The diagram*

$$\begin{tikzcd} EX \arrow[r, "\eta"] \arrow[d, "E\eta"'] \arrow[dr, "\mathrm{id}"] & E^2X \arrow[d, "\alpha"] \\ E^2X \arrow[r, "\alpha"'] & EX \end{tikzcd}$$

commutes where η *is induced by the inclusion of* EX *as* $AEX(1)$ *and* $E\eta$ *is induced by the inclusion of* AX *in* AEX*.*

Applying the lemma to X_n, we get a commuting diagram of $\mathcal{F}$-spaces. We can turn this into a diagram of Γ-spaces by taking the quotient by $EX_0 = E*$ or $E^2X_0 = E^2*$ at each spot. We then get a commutative diagram of Γ-spaces

$$\begin{tikzcd} E^\Gamma X \arrow[r, "\eta"] \arrow[d, "E\eta"'] \arrow[dr, "\mathrm{id}"] & E^2X_{(-)}/E^2* \arrow[d, "\alpha"] \\ E^2X_{(-)}/E^2* \arrow[r, "\alpha"'] & E^\Gamma X. \end{tikzcd}$$

We note that E^2* is contractible, and since $E^\Gamma X$ is special, η is weak equivalence. It follows that all maps in the diagram are weak equivalences. Since both

$$\eta\colon E^\Gamma X \longrightarrow E^\Gamma E^\Gamma X \qquad \text{and} \qquad E^\Gamma\eta\colon E^\Gamma X \longrightarrow E^\Gamma E^\Gamma X$$

factor through the corresponding map $E^\Gamma X \to E^2 X/E^2*$, we get the following proposition.

PROPOSITION 6.10. *The maps η and $E^\Gamma\eta$ from $E^\Gamma X$ to $E^\Gamma E^\Gamma X$ coincide in the strict homotopy category of Γ-spaces, i.e., the homotopy category obtained by formally inverting the objectwise weak equivalences.*

We use this observation to prove the following theorem, which together with Theorems 6.7 and 6.8 imply Theorem 1.5.

THEOREM 6.11. *For any Γ-space X, $\eta\colon X \to E^\Gamma X$ is a stable equivalence. Moreover η is the initial map from X to a special Γ-space in the strict homotopy category of Γ-spaces.*

Proof. We need to show that for any special Γ-space Z, the map η induces a bijection $[E^\Gamma X, Z] \to [X, Z]$ where $[-,-]$ denotes maps in the strict homotopy category of Γ-spaces. Since E^Γ preserves weak equivalences, it induces a functor on the strict homotopy category. Given a map g in $[X, Z]$, $E^\Gamma g$ is a map in $[E^\Gamma X, E^\Gamma Z]$, and since $\eta\colon Z \to EZ$ is a weak equivalence, we can compose with the map η^{-1} in the strict homotopy category to get an element $Rg = \eta^{-1}\circ E^\Gamma g$ in $[E^\Gamma X, Z]$. By naturality of η, R is a retraction. By examination of the solid arrow commuting diagram

$$\begin{tikzcd}
X \arrow[r, "\eta"] & E^\Gamma X \arrow[r, "g"] \arrow[d, "\eta", "\sim"'] & Z \arrow[d, "\eta", "\sim"'] \\
E^\Gamma X \arrow[ur, dotted, "\mathrm{id}"] \arrow[r, "E^\Gamma\eta"'] & E^\Gamma E^\Gamma X \arrow[r, "E^\Gamma g"'] & E^\Gamma Z
\end{tikzcd}$$

and applying the previous proposition, we see that R is a bijection. □

The previous theorem also provides the final piece for the proof of Theorem 1.9.

Proof of Theorem 1.9. The equivalence of (iii) and (iv) is Theorem 1.4 proved in the last section. The previous theorem proves the equivalence of (ii) and (iii), and [12, 1.8] (and the argument for [9, 1.1]) prove the equivalence of (i) and (iii). □

We close with the proof of Theorem 6.8. We thank Irene Sami for help putting together this argument.

Proof of Theorem 6.8. It suffices to show that for every $j > 0$, the map

$$EX_{j+1} \longrightarrow EX_j \times EX \tag{6.12}$$

is a weak equivalence. Using EX_j in place of $E^{\Gamma}X(\mathbf{j})$ has the advantage that we can write $EX_j \times EX$ as a homotopy colimit:

$$EX_j \times EX \cong \mathrm{hocolim}_{\mathcal{A}\times\mathcal{A}}(AX_j \times AX).$$

For clarity in formulas that follow, we will use brackets $[m]$ rather than bold $\mathbf{m}$ to denote finite based sets.

The map (6.12) is induced by the diagonal functor $\mathcal{A} \to \mathcal{A}\times\mathcal{A}$ and the natural transformation

$$AX_{j+1}(\vec{m}) \longrightarrow AX_j(\vec{m}) \times AX(\vec{m}).$$

We get a map

$$EX_j \times EX \longrightarrow EX_{j+1} \tag{6.13}$$

induced by the concatenation functor $\mathcal{A}\times\mathcal{A} \to \mathcal{A}$ and the natural transformation

$$AX_j(\vec{m}) \times AX(\vec{n}) \longrightarrow AX_{j+1}(\vec{m} \,\Box\, \vec{n})$$

(where $\Box$ denotes concatenation), sending

$$X([jm_1]) \times \cdots \times X([jm_r]) \longrightarrow X([(j+1)m_1]) \times \cdots \times X([(j+1)m_r])$$

by the map induced by the inclusion of $[j]$ in $[j+1]$, and the map

$$X([n_1]) \times \cdots \times X([n_s]) \longrightarrow X([(j+1)n_1]) \times \cdots \times X([(j+1)n_s])$$

induced by including the non-basepoint element 1 of $[1]$ as the element $j+1$ of $[j+1]$. We show that (6.12) and 6.13 are inverse generalized simplicial homotopy equivalences.

First we show that the composite on E_{j+1} is (generalized simplicial) homotopic to the identity. We denote the composite on E_{j+1} as (D,d). It is induced by the functor $D\colon \mathcal{A} \to \mathcal{A}$ that sends $\vec{m}$ to the concatenation $\vec{m} \,\Box\, \vec{m}$ and the natural transformation

$$\begin{aligned} d\colon X_{j+1}(m_i) &= \\ &X([(j+1)m_i]) \longrightarrow X([(j+1)m_i]) \times X([(j+1)m_i]) = X_{j+1}(m_i, m_i) \end{aligned}$$

induced in the first factor by sending the element $j+1$ of $[j+1]$ to the basepoint and induced in the second factor by sending the elements $1, \ldots, j$ of $[j+1]$ to the basepoint.

We construct a new map (H,h) from E_{n+1} to itself and simplicial homotopies from (H,h) to (D,d) and from (H,h) to the identity as follows. Let H be the functor $\mathcal{A} \to \mathcal{A}$ that sends $(m_1, \ldots, m_r)$ to $((j+1)m_1, \ldots, (j+1)m_r)$ and let

$$\begin{aligned} h\colon AX_{j+1}(\vec{m}) &= AX((j+1)m_1, \ldots, (j+1)m_r) \\ &\longrightarrow AX((j+1)^2 m_1, \ldots, (j+1)^2 m_r) = AX_{j+1}(H\vec{m}) \end{aligned}$$

be the natural transformation induced by the diagonal map in $\mathcal{F}$ from $[j+1]$ to $[(j+1)^2]$; then the functor H and natural transformation h induce a map (H,h) from EX_{j+1} to itself.

We have a natural transformation ϕ from H to D formed by concatenation and permutation from the maps

$$((j+1)m_i) \longrightarrow (m_i, m_i)$$

in $\mathcal{A}$ sending collapsing the first j copies of $\underline{m}_i$ to the first $\underline{m}_i$ by the codiagonal map and sending the last copy of $\underline{m}_i$ onto the second $\underline{m}_i$. The composite map

$$AX_{j+1}(\vec{m}) \xrightarrow{h} AX_{j+1}(H\vec{m}) \xrightarrow{AX_{j+1}(\phi)} AX_{j+1}(D\vec{m})$$

is d. Thus, the natural transformation ϕ induces a homotopy between the maps (H,h) and (D,d) on the homotopy colimit E_{j+1}.
Likewise, we have a natural transformation ψ from H to the identity induced by the maps $((j+1)m_i) \to (m_i)$ in $\mathcal{A}$ that collapse the $j+1$ copies of m_i by the codiagonal. Since the composite

$$AX_{j+1}(\vec{m}) \xrightarrow{h} AX_{j+1}(H\vec{m}) \xrightarrow{AX_{j+1}(\psi)} AX_{j+1}(\vec{m})$$

is the identity, it follows that ψ induces a homotopy from H to the identity on EX_{j+1}. This constructs the generalized simplicial homotopy equivalence between the composite map and the identity map on EX_{j+1}.
The argument for the other composite is easier: The composite on $EX_j \times EX$ is induced by the functor D_2 from $\mathcal{A} \times \mathcal{A}$ to itself

$$D_2(\vec{m}, \vec{n}) = (\vec{m} \,\Box\, \vec{n}, \vec{m} \,\Box\, \vec{n})$$

and the natural transformation

$$d_2 \colon AX_j(\vec{m}) \times AX(\vec{n}) \longrightarrow AX_j(\vec{m} \,\Box\, \vec{n}) \times AX(\vec{m} \,\Box\, \vec{n})$$

induced on the first factor by the inclusion of $\vec{m}$ in $\vec{m} \,\Box\, \vec{n}$ and on the second factor by the inclusion of $\vec{n}$ in $\vec{m} \,\Box\, \vec{n}$. Since these maps

$$(\vec{m}, \vec{n}) \longrightarrow (\vec{m} \,\Box\, \vec{n}, \vec{m} \,\Box\, \vec{n})$$

assemble to a natural transformation in $\mathcal{A} \times \mathcal{A}$ from the identity functor to D_2, they induce a homotopy on $EX_j \times EX$ between the identity and the map induced by D_2, d_2. $\Box$

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