The Cohomology Rings of Moduli Stacks of Principal Bundles over Curves

Dedicated with gratitude
to the memory of Eckart Viehweg

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Received: October 12, 2009
Revised: March 1, 2010

Communicated by Thomas Peternell

Abstract. We prove that the cohomology of the moduli stack of G-bundles on a smooth projective curve is freely generated by the Atiyah–Bott classes in arbitrary characteristic. The main technical tool needed is the construction of coarse moduli spaces for bundles with parabolic structure in arbitrary characteristic. Using these spaces we show that the cohomology of the moduli stack is pure and satisfies base-change for curves defined over a discrete valuation ring. Thereby we get an algebraic proof of the theorem of Atiyah and Bott and conversely this can be used to give a geometric proof of the fact that the Tamagawa number of a Chevalley group is the number of connected components of the moduli stack of principal bundles.

2010 Mathematics Subject Classification: 14D23, 14F20, 14D20
Keywords and Phrases: Principal bundle, parabolic bundle, moduli space, stack, cohomology, Tamagawa number.

Contents

1 Introduction

Atiyah and Bott [1] proved that for any semisimple group G the cohomology ring of the moduli stack Bun_G of principal G-bundles on a Riemann surface C is freely generated by the Künneth components of the characteristic classes of the universal bundle on $\mathrm{Bun}_G \times C$. (Of course, in their article, this was expressed in terms of equivariant cohomology instead of the cohomology of a stack. The formulation in terms of stacks can also be found in Teleman's article [42].) The argument of Harder and Narasimhan [19] suggests that the result should also hold for curves over finite fields.

The original aim of this article was to give an algebraic proof of the result of Atiyah and Bott in positive characteristics. In the case of $G = \mathrm{GL}_n$ this was suggested by G. Harder, given as a Diploma thesis to the first author [21] (see [10] for a different approach). For general G we have to use the recent constructions of coarse moduli spaces in arbitrary characteristics [15]. The results of Behrend ([4], [5]) prove the Lefschetz trace formula for the moduli stack Bun_G over finite fields. However, purity of the cohomology groups is not so clear. One also has to check that the universal classes generate a sufficiently large subring. To prove purity, we embed the cohomology of the stack into the cohomology of a projective variety. This enables us to argue in two ways: either we use the known calculations of the Tamagawa number to prove the theorem with algebraic methods over finite fields (Theorem 3.3.5), or we use the projective variety to apply base change (Corollary 3.3.4) and deduce the general result from the known one in characteristic 0. This in turn gives a calculation of the Tamagawa number (Corollary 3.1.3) and thus provides a geometric proof of Harder's conjecture that the Tamagawa number should be the number of connected components of the moduli stack of principal G-bundles in this situation (see also the introduction to [8]). In order to make this argument precise the formalism of the six operations for sheaves on Artin stacks recently constructed by Laszlo and Olsson [29] is applied.

As pointed out by Neumann and Stuhler in [33], the computation of the cohomology ring over finite fields also gives an explicit description of the action of the Frobenius endomorphism of the moduli stack on the cohomology of the stack, even if the geometry of this action is quite mysterious.

As explained above, the main new ingredient in our approach is the purity of the cohomology and the proof of a base change theorem for the cohomology of Bun_G. The idea to prove these results is to embed the cohomology of Bun_G into the cohomology of the stack of principal G-bundles together with flags at a finite set of points of the curve ("flagged principal bundles"). On this stack one can find a line bundle, such that the open subset of stable bundles has a complement of high codimension. Furthermore, there exists a projective coarse moduli space for stable flagged principal bundles. The existence of coarse moduli spaces for flagged principal bundles in arbitrary characteristic is demonstrated in the second part of this article. So here we use Geometric Invariant Theory in order to obtain a result for the moduli stack, whereas one usually argues in the other direction.

Our main theorem is:

Theorem. *Assume that C is a curve over a field k. Then the cohomology of the connected components Bun_G^ϑ of Bun_G is freely generated by the canonical classes, i.e.,*

$$H^\star\big(\mathrm{Bun}^\vartheta_{G,\overline{k}}, \overline{\mathbb{Q}}_\ell\big) = \overline{\mathbb{Q}}_\ell[a_1,\dots,a_r] \otimes \bigwedge^\star [b_i^j]_{i=1,\dots r, j=1,\dots,2g} \otimes \overline{\mathbb{Q}}_\ell[f_1,\dots,f_r].$$

(The canonical classes are obtained from the Künneth components of the universal principal bundle on $\mathrm{Bun}_G \times C$, see Section 3.1.)

As remarked above, the main technical ingredient is the construction of proper coarse moduli spaces for flagged principal bundles in positive characteristic. It is contained in the second part of this paper and might be of independent interest. Let us therefore give a statement of this result as well.

Since there are different definitions of parabolic bundles in the literature, we have used the term flagged principal bundles instead. The precise definition is as follows. Let $\underline{x} = (x_i)_{i=1,\dots,b}$ be a finite set of distinct k-rational points of C, and let $\underline{P} = (P_i)_{i=1,\dots,b}$ be a tuple of parabolic subgroups of G. A *principal G-bundle with a flagging of type* $(\underline{x}, \underline{P})$ is a tuple $(\mathscr{P}, \underline{s})$ that consists of a principal G-bundle $\mathscr{P}$ on C and a tuple $\underline{s} = (s_1, \dots, s_b)$ of sections $s_i\colon \{x_i\} \to (\mathscr{P} \times_C \{x_i\})/P_i$, i.e., s_i is a reduction of the structure group of $\mathscr{P} \times_C \{x_i\}$ to P_i, $i = 1, \dots, b$.

In Section 4 we introduce a semistability concept for such bundles. It depends on a parameter $\underline{a}$ which, as in the case of parabolic vector bundles, has to satisfy a certain admissibility condition. Using this notion we show:

Theorem. *For any type $(\underline{x}, \underline{P})$ of flaggings and any admissible stability parameter $\underline{a}$, there exists a projective coarse moduli space $\mathscr{M}(\underline{x}, \underline{P})^{\underline{a}\text{-ss}}$ of $\underline{a}$-semistable flagged principal G-bundles.*

Finally one should note that it is well known that one can use the computation of the cohomology of the moduli stack and the splitting of the Gysin sequence for the Harder–Narasimhan stratification of Bun_G (as in [1], this holds in arbitrary characteristic) to calculate the cohomology of the moduli stack of semistable bundles. If the connected component $\mathrm{Bun}_G^{\vartheta}$ is such that there are no properly semistable bundles, this gives a computation of the cohomology of the coarse moduli space (as in the proof of Corollary 3.3.2).

2 Preliminaries

In this section we collect some well known results on the moduli stacks Bun_G and their cohomology.

2.1 Basic Properties of the Moduli Stack of Principal Bundles

Let C be a smooth, projective curve of genus g over the (locally noetherian) scheme S. It would be reasonable to assume that C is a curve over a field, but since we want to be able to transport our results from characteristic p to characteristic 0, we will finally need some base ring.
Let G/S be a reductive group of rank r. Denote by Bun_G the moduli stack of principal G-bundles over C, i.e., for a scheme $X \to S$, the X-valued points of Bun_G are defined as

$$\mathrm{Bun}_G(X) := \text{Category of principal } G\text{-bundles over } C \times X.$$

Recall the following basic fact which is proved in [4], Proposition 4.4.6 and Corollary 4.5.2.

Proposition 2.1.1. *The stack Bun_G is an algebraic stack, locally of finite type and smooth of relative dimension $(g-1)\dim G$ over S.*

Furthermore the connected components of Bun_G are known ([14], Proposition 5, [24]). (In the first reference, the result is stated only for simply connected groups, but the proof gives the result in the general case.)

Proposition 2.1.2. *If $S = \mathrm{Spec}(\overline{k})$, or if G is a split reductive group, then the connected components of Bun_G are in natural bijection to $\pi_1(G)$.*

Remark 2.1.3. The stack Bun_G is smooth (2.1.1). Therefore, its connected components are also irreducible.

2.2 Behrend's Trace Formula

Let us now assume that $S = \mathrm{Spec}(k)$ is the spectrum of a field. In the following, we will write $\mathrm{Bun}^{\vartheta}_{G,\overline{k}}$ with $\vartheta \in \pi_1(G_{\overline{k}})$ for the corresponding connected component of $\mathrm{Bun}_{G,\overline{k}}$.

Since the stack $\mathrm{Bun}^{\vartheta}_{G}$ is only locally of finite type, we define its ℓ-adic cohomology as the limit of the cohomologies of all open substacks of finite type:

$$H^\star(\mathrm{Bun}^{\vartheta}_{G}, \overline{\mathbb{Q}}_\ell) := \lim_{\substack{U \subset \mathrm{Bun}_G \\ \text{open, fin. type}}} H^\star(U, \overline{\mathbb{Q}}_\ell).$$

Remark 2.2.1. The basic reference for stacks and their cohomology is [30]. The general formalism of cohomology has been developed in the articles by Laszlo and Olsson [29]. Behrend in [7] also constructed all the functors that we will use. In particular, we will compute cohomology groups with respect to the lisse-étale topology. To simplify the statement of our main theorem we will use $\overline{\mathbb{Q}}_\ell$ coefficients, because we want to chose generators of the cohomology ring that are eigenvectors for the Frobenius action.

By semi-purity, which is recalled below, the cohomology of Bun_G in degrees $< 2i$ is equal to the cohomology of $U \subset \mathrm{Bun}_G$, if the codimension of the complement of U is at least i:

Lemma 2.2.2 (Semi-purity). *Let X be a smooth stack of finite type and $U \overset{j}{\hookrightarrow} X$ an open substack with complement $Z := X \setminus U \overset{i}{\hookrightarrow} X$. Then,*

$$H^\star(X, \overline{\mathbb{Q}}_\ell) \cong H^\star(U, \overline{\mathbb{Q}}_\ell) \quad \textit{for } \star < 2\,\mathrm{codim}(Z).$$

Proof. As usual, this can be deduced from the corresponding statement for schemes. For schemes instead of stacks, this follows from the long exact sequence for cohomology with compact support,

$$\cdots \longrightarrow H^\star_c(U, \overline{\mathbb{Q}}_\ell) \longrightarrow H^\star_c(X, \overline{\mathbb{Q}}_\ell) \longrightarrow H^\star_c(Z, \overline{\mathbb{Q}}_\ell) \longrightarrow \cdots,$$

the vanishing of $H^\star_c(Z, \overline{\mathbb{Q}}_\ell)$ for $\star > 2\dim Z$, and Poincaré duality,

$$H_c^{2\dim U - \star}(U, \overline{\mathbb{Q}}_\ell) \cong H^\star\big(U, \overline{\mathbb{Q}}_\ell(\dim U)\big)^\vee.$$

Now, if $X_0 \twoheadrightarrow X$ is a smooth atlas of the stack X, and $X_n := X_0 \times_X X_0 \times_X \cdots \times_X X_0$, then there is a spectral sequence:

$$H^p(X_q, \overline{\mathbb{Q}}_\ell) \Rightarrow H^{p+q}(X, \overline{\mathbb{Q}}_\ell).$$

Since the codimension is preserved under smooth pull-backs, for any $U \subset X$, we get the atlas $U_0 := U \times_X X_0 \to U$, and the induced embeddings $U_q \to X_q$ have complements of codimension $\mathrm{codim}(U)$. Therefore we can apply the lemma in the case of schemes to the morphism of spectral sequences

$$H^p(U_q, \overline{\mathbb{Q}}_\ell) \to H^p(X_q, \overline{\mathbb{Q}}_\ell)$$

to prove our claim. ∎

Remark 2.2.3. The same argument applies to the higher direct image sheaves in the relative situation $X \to S$, if X is smooth over S and $U \subset X$ is of codimension i in every fiber.

Behrend proved ([4], [5]) that, if C is a curve defined over a finite field k, the Lefschetz trace formula holds for the stack Bun_G.

Theorem 2.2.4 (Behrend). *Let C be a smooth, projective curve over the finite field $k = \mathbb{F}_q$ and G a semisimple group over k. Let* Frob *denote the arithmetic Frobenius acting on $H^\star(\mathrm{Bun}_{G,\overline{k}}, \overline{\mathbb{Q}}_\ell)$. Then, we have*

$$q^{\dim(\mathrm{Bun}_G)} \sum_{i \geq 0} (-1)^i \operatorname{tr}\big(\mathrm{Frob}, H^i(\mathrm{Bun}_{G,\overline{k}}, \overline{\mathbb{Q}}_\ell)\big) = \sum_{x \in \mathrm{Bun}_G(\mathbb{F}_q)} \frac{1}{\# \operatorname{Aut}(x)(\mathbb{F}_q)}.$$

As in [19], a result of Siegel allows us to calculate the right hand side of the formula. To state it, we first recall a theorem of Steinberg.

Proposition 2.2.5 (Steinberg). *Let G be a semisimple group over $k = \mathbb{F}_q$. There are integers $d_1, \dots, d_r$ and roots of unity $\epsilon_1, \dots, \epsilon_r$ such that:*

- $\#G(\mathbb{F}_q) = q^{\dim G} \prod_{i=1}^r (1 - \epsilon_i q^{-d_i})$
- *Let BG be the classifying stack of principal G-bundles. Then, $H^\star(BG_{\overline{k}}, \overline{\mathbb{Q}}_\ell) = \overline{\mathbb{Q}}_\ell[c_1, \dots, c_r]$ with $c_i \in H^{2d_i}(BG, \overline{\mathbb{Q}}_\ell)$ and* $\mathrm{Frob}(c_i) = \epsilon_i q^{-d_i}$.

The second part is of course not stated in this form in Steinberg's book [40], but one only has to recall the argument from topology. First the theorem holds for tori, since $H^\star(\mathbb{G}_m, \overline{\mathbb{Q}}_\ell) = \overline{\mathbb{Q}}_\ell[x]$. For a maximal torus T contained in the Borel subgroup $B \subset G$, the map $BT \to BB$ induces an isomorphism in cohomology: since the fibers are isomorphic to BU where $U \cong \mathbb{A}^n$ is the unipotent radical of B, they have no higher cohomology. The fibers of the map $BB \to BG$ are isomorphic to the flag manifold G/B. Thus the map $\mathrm{ind}_T^G \colon BT \to BG$ induces an injection $\mathrm{ind}_T^{G,\star} \colon H^\star(BG, \overline{\mathbb{Q}}_\ell) \hookrightarrow H^\star(BT, \overline{\mathbb{Q}}_\ell)$ which lies in the part invariant under the Weyl group. For dimensional reasons—since we already stated the trace formula, this follows most easily from $1/(\#G(\mathbb{F}_q)) = q^{-\dim G} \sum \operatorname{tr}(\mathrm{Frob}, H^i(BG_{\overline{k}}, \overline{\mathbb{Q}}_\ell))$ and the fact that the d_i are the degrees of the homogeneous generators in $H^\star(BT_{\overline{k}}, \overline{\mathbb{Q}}_\ell)^W$—it must then be isomorphic to the invariant ring.
With the notations from Steinberg's theorem we can state a theorem of Siegel. A nice reference for the theorem is [26], Section 3. In this article, you can also find a short reminder on the Tamagawa number $\tau(G)$.

Theorem 2.2.6 (Siegel's formula). *Let $G_{/\mathbb{F}_q}$ be a semisimple group, and denote* *by α_j the eigenvalues of the geometric Frobenius on $H^1(C_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$. Then,*

$$\sum_{x \in \mathrm{Bun}_G(\mathbb{F}_q)} \frac{1}{\#\,\mathrm{Aut}(x)(\mathbb{F}_q)} = \tau(G) \prod_{p \in C} \frac{1}{\mathrm{vol}(G(\mathscr{O}_p))}$$
$$= \tau(G) q^{(g-1)\dim G} \prod_{i=1}^{\mathrm{rk}\,G} \frac{\prod_{j=1}^{2g}(1-\epsilon_i \alpha_j q^{-d_i})}{(1-\epsilon_i q^{-d_i})(1-\epsilon_i q^{(1-d_i)})}.$$

3 The Cohomology of Bun_G

Our next aim is to recall from [1] the construction of the canonical classes in the cohomology ring of Bun_G and to prove that these generate a free subalgebra over any field. We will then explain how to deduce our main theorem from the purity of the cohomology of Bun_G which will occupy the rest of this article.

3.1 The Subring Generated by the Atiyah–Bott Classes

Fix $\vartheta \in \pi_0(\mathrm{Bun}_G) = \pi_1(G)$. The universal principal G-bundle $\mathscr{P}_{\mathrm{univ}}$ on $\mathrm{Bun}_G^\vartheta \times C$ defines a map $f\colon \mathrm{Bun}_G^\vartheta \times C \to BG$. The characteristic classes of $\mathscr{P}_{\mathrm{univ}}$ are defined as $c_i(\mathscr{P}_{\mathrm{univ}}) := f^\star c_i$ where the c_i are, as in Proposition 2.2.5, the standard generators of the cohomology ring of BG.
Note that the Künneth theorem for stacks can be deduced from the corresponding result for schemes using the spectral sequence computing the cohomology of the stack from the cohomology of an atlas as in Lemma 2.2.2.
We choose a basis $(\gamma_i)_{i=1,\dots,2g}$ of $H^1(C, \overline{\mathbb{Q}}_\ell)$. In the case that C is defined over a finite field k, we choose the γ_i as eigenvectors for the geometric Frobenius of eigenvalue α_i. The Künneth decomposition of $c_i(\mathscr{P}_{\mathrm{univ}})$ is therefore of the form:

$$c_i(\mathscr{P}_{\mathrm{univ}}) =: a_i \otimes 1 + \sum_{j=1}^{2g} b_i^j \otimes \gamma_j + f_i \otimes [\mathrm{pt}].$$

Note that $d_i > 1$, because we assume that G is semisimple. Thus, the f_i are not constant. Of course, these classes depend on ϑ, but we don't want to include this dependence in our notation.

Proposition 3.1.1. *The classes (a_i, b_i^j, f_i) generate a free graded subalgebra of the cohomology ring $H^\star(\mathrm{Bun}_{G,\overline{k}}^\vartheta, \overline{\mathbb{Q}}_\ell)$, i.e., there is an inclusion:*

$$\mathrm{can}\colon \overline{\mathbb{Q}}_\ell[a_1,\dots,a_r] \otimes \bigwedge^\star [b_i^j]_{i=1,\dots r, j=1,\dots,2g} \otimes \overline{\mathbb{Q}}_\ell[f_1,\dots,f_r] \hookrightarrow H^\star(\mathrm{Bun}_{G,\overline{k}}^\vartheta, \overline{\mathbb{Q}}_\ell).$$

If k is a finite field, then the classes a_i, b_i^j, f_i are eigenvectors for the action of the arithmetic Frobenius with eigenvalues, $q^{-d_i}, q^{-d_i}\alpha_j, q^{1-d_i}$ respectively.

Proof. Denote by $\mathrm{Can}^\star \subset H^\star(\mathrm{Bun}_G^\vartheta, \overline{\mathbb{Q}}_\ell)$ the subring generated by the classes (a_i, b_i^j, f_i).

Note first that the analog of the theorem holds for $G = \mathbb{G}_m$. In this case, $\mathrm{Bun}_{\mathbb{G}_m}$ is the disjoint union of the stacks $\mathrm{Bun}^d_{\mathbb{G}_m}$ classifying line bundles of degree d. There is the $\mathbb{G}_m$-gerbe $\mathrm{Bun}^d_{\mathbb{G}_m} \to \mathrm{Pic}^d_C$ which is trivial over any field over which C has a rational point, because in this case Pic^d_C is a fine moduli space for line bundles together with a trivialization at a fixed rational point p. Forgetting the trivialization at p corresponds to taking the quotient of Pic by the trivial $\mathbb{G}_m$-action. Thus, $\mathrm{Bun}^d_{\mathbb{G}_m} \cong \mathrm{Pic}^d \times B\mathbb{G}_m$ and the cohomology of this stack is $H^\star(\mathrm{Pic}^d, \overline{\mathbb{Q}}_\ell) \otimes \overline{\mathbb{Q}}_\ell[c_1]$. Here, the first factor is the exterior algebra generated by the Künneth components of the Poincaré bundle.

Let $T \subset G$ be a maximal torus and fix an isomorphism $T \cong \mathbb{G}_m^r$ in order to apply the result for $\mathbb{G}_m$. Then, $X^\star(T)^\vee \cong \mathbb{Z}^r$. Recall furthermore that the G-bundle induced from a T-bundle of degree $\underline{k} \in \mathbb{Z}^r \cong X^\star(T)^\vee$ lies in Bun^ϑ_G, if and only if $\underline{k} \equiv \vartheta \in X^\star(T)^\vee/\Lambda^\vee$. We denote this coset by $\mathbb{Z}^r_\vartheta$.

Write $H^\star(BT_{\overline{k}}, \overline{\mathbb{Q}}_\ell) \cong \overline{\mathbb{Q}}_\ell[x_1, \dots x_r]$ and, for every degree $\underline{k} \in \mathbb{Z}^r$, denote by $A_i, B_i^j \in H^\star(\mathrm{Bun}^{\underline{k}}_T, \overline{\mathbb{Q}}_\ell)$ the Künneth components of the Chern classes of the universal T-bundle. Note that, since $\Lambda^\vee \subset \mathbb{Z}^r$ has finite index, we have the injective map

$$\overline{\mathbb{Q}}_\ell[A_1, \dots, A_r] \otimes \bigwedge^\star [B_i^j]_{i=1,\dots r, j=1,\dots,2g} \otimes \overline{\mathbb{Q}}_\ell[K_1, \dots, K_r] \hookrightarrow \prod_{\underline{k} \in \mathbb{Z}^r_\vartheta} H^\star(\mathrm{Bun}^{\underline{k}}_T, \overline{\mathbb{Q}}_\ell)$$

defined by $K_i \mapsto (k_i)_{\underline{k} \in \mathbb{Z}^r_\vartheta}$ where k_i is considered as an element of $H^0(\mathrm{Bun}^{\underline{k}}_T, \overline{\mathbb{Q}}_\ell) = \overline{\mathbb{Q}}_\ell$.

Recall that the induced map $H^\star(BG_{\overline{k}}, \overline{\mathbb{Q}}_\ell) \to H^\star(BT_{\overline{k}}, \overline{\mathbb{Q}}_\ell) \cong \overline{\mathbb{Q}}_\ell[x_1, \dots, x_{\mathrm{rk}\, G}]$ is given by $c_i \mapsto \sigma_i(x_1, \dots, x_r)$ where σ_i is a homogeneous polynomial of degree d_i. Therefore, we can calculate the image of the canonical classes under the map

$$\begin{aligned} H^\star(\mathrm{Bun}^\vartheta_G, \overline{\mathbb{Q}}_\ell) \otimes H^\star(C, \overline{\mathbb{Q}}_\ell) &\to H^\star(\mathrm{Bun}_T, \overline{\mathbb{Q}}_\ell) \otimes H^\star(C, \overline{\mathbb{Q}}_\ell) \\ &\cong \prod_{\underline{k} \in \mathbb{Z}^r_\vartheta} H^\star(\mathrm{Bun}^{\underline{k}}_T, \overline{\mathbb{Q}}_\ell) \otimes H^\star(C, \overline{\mathbb{Q}}_\ell) \end{aligned}$$

which respects the Künneth decomposition. It is given by

$$c_i(\mathscr{P}_{\mathrm{univ}}) \mapsto \prod_{\underline{k} \in \mathbb{Z}^r_\vartheta} \sigma_i\Big(A_1 \otimes 1 + \sum_{j=1}^{2g} B_1^j \otimes \gamma_j + k_1 \otimes [\mathrm{pt}], \dots\Big).$$

The Künneth decomposition of this class is

$$\begin{aligned}\sigma_i\Big(A_1\otimes 1+\sum_{j=1}^{2g} B_1^j\otimes\gamma_j+k_1\otimes[\mathrm{pt}],\dots\Big)&\\ &=\sigma_i(A_1,\dots,A_r)\otimes 1\\ &\quad+\sum_{j=1}^{2g}\Big(\sum_{m=1}^{r}(\partial_m\sigma_i)(A_1,\dots,A_r)\Big)B_m^j\otimes\gamma_j\\ &\quad+\sum_{m=1}^{r}(\partial_m\sigma_i)(A_1,\dots,A_{\mathrm{rk}\,G})k_m\otimes[\mathrm{pt}]\\ &\quad+\sum B_i^jB_{i'}^{j\prime}\cdot P_{j,j'}(A_1,\dots,A_{\mathrm{rk}\,G})\otimes[\mathrm{pt}],\end{aligned}$$

where the $P_{j,j'}$ are some polynomials. In particular, we see that the above map factors through the subring

$$\overline{\mathbb{Q}}_\ell[A_1,\dots,A_r]\otimes\bigwedge^{\star}[B_i^j]_{i=1,\dots r,\ j=1,\dots,2g}\otimes\overline{\mathbb{Q}}_\ell[K_1,\dots,K_r]\hookrightarrow\prod_{\underline{k}\in\mathbb{Z}^r_\vartheta}H^\star(\mathrm{Bun}_T^{\underline{k}},\overline{\mathbb{Q}}_\ell)$$

defined above. We already know that the elements $\sigma_i(A_1,\dots,A_{\mathrm{rk}\,G})$ are algebraically independent in $H^\star(\mathrm{Bun}_{T,\overline{k}},\overline{\mathbb{Q}}_\ell)$. In particular, since the map $\mathbb{A}^{\mathrm{rk}\,G}\to\mathbb{A}^{\mathrm{rk}\,G}\cong(\mathbb{A}^{\mathrm{rk}\,G}/W)$ defined by the polynomials σ_i is generically a Galois covering with Galois group W, we also know that the derivatives $\partial\sigma_i$ are linearly independent. This shows our claim. $\square$

Remark 3.1.2. In the proof above, we have only used the fact that $H^\star(\mathrm{Pic}^0_C,\mathbb{Q}_\ell)\cong\bigwedge^\star H^1(C,\mathbb{Q}_\ell)$. Thus, one might note that the proof shows that for any smooth, projective variety X the analogous classes a_i,b_i^j,f_i^k, where f_i^k are the Künneth components corresponding to a basis of $\mathrm{NS}(X)_{\mathbb{Q}}$, generate a free subalgebra of the cohomology of the moduli stack of principal bundles on X.

In the following, we will denote the graded subring constructed above by $\mathrm{Can}^\star$. Of course, we want to show that $\mathrm{Can}^\star$ is indeed the whole cohomology ring of $\mathrm{Bun}^\vartheta_{G,\overline{k}}$.

Corollary 3.1.3. *Let k be a finite field and let $G_{/k}$ be a semisimple group. If $H^\star(\mathrm{Bun}^\vartheta_{G,\overline{k}},\overline{\mathbb{Q}}_\ell)$ is generated by the canonical classes for all ϑ, then the Tamagawa number $\tau(G)$ satisfies $\tau(G)=\dim H^0(\mathrm{Bun}_G,\overline{\mathbb{Q}}_\ell)=\#\pi_0(\mathrm{Bun}_G)$. Conversely, if the cohomology of Bun_G is pure and the Tamagawa number fulfills $\tau(G)=\#\pi_0(\mathrm{Bun}_G)$, then $H^\star(\mathrm{Bun}_G,\overline{\mathbb{Q}}_\ell)=\mathrm{Can}^\star$.*

Proof. For the graded ring Can^i generated by the canonical classes, we know that

$$\sum_{i=0}^{\infty}(-1)^i\,\mathrm{tr}(\mathrm{Frob},\mathrm{Can}^i)=\frac{\prod_{i=1}^r\prod_{j=1}^{2g}(1-\epsilon_i\alpha_jq^{-d_i})}{\prod_{i=1}^r(1-\epsilon_iq^{-d_i})(1-\epsilon_iq^{1-d_i})}.$$

Comparing this with Siegel's formula, we get the first claim.
Furthermore we know that the Zeta function of Bun_G converges and is equal to

$$\begin{aligned} Z(\mathrm{Bun}_G, t) &= \exp\Big(\sum_{i=1}^{\infty} \#\mathrm{Bun}_G(\mathbb{F}_{q^n})\frac{t^i}{i}\Big) \\ &= \prod_{i=0}^{\infty} \det\Big(1 - \mathrm{Frob}\cdot q^{\dim(\mathrm{Bun}_G)}\cdot t, H^i(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)\Big)^{(-1)^{i+1}}. \end{aligned}$$

Now, since such a product expansion of an analytic function is unique and the eigenvalues of Frob on H^i have absolute value $q^{i/2}$, there can be no cancellations. Thus, the Poincaré series of the cohomology ring can be read off the Zeta function. □

3.2 The Main Results on Moduli Spaces of Flagged Principal Bundles

Let $\underline{x} = (x_i)_{i=1,\dots,b}$ be a finite set of distinct k-rational points of C, and let $\underline{P} = (P_i)_{i=1,\dots,b}$ be a tuple of parabolic subgroups of G. A *principal G-bundle with a flagging of type* $(\underline{x}, \underline{P})$ is a tuple $(\mathscr{P}, \underline{s})$ that consists of a principal G-bundle $\mathscr{P}$ on C and a tuple $\underline{s} = (s_1, ..., s_b)$ of sections $s_i\colon \{x_i\} \to (\mathscr{P} \times_C \{x_i\})/P_i$, i.e., s_i is a reduction of the structure group of $\mathscr{P} \times_C \{x_i\}$ to P_i, $i = 1, ..., b$.

Remark 3.2.1. For $G = \mathrm{GL}_r(k)$, parabolic subgroups correspond to flags of quotients of k^r, so that a flagged principal $\mathrm{GL}_r(k)$-bundle may be identified with a vector bundle $\mathscr{E}$ together with flags of quotients $\mathscr{E}_{x_i} \twoheadrightarrow V_{j,i}$, $j = 1, ..., t_i$, $i = 1, ..., b$, of the fibers of $\mathscr{E}$ at x_i, $i = 1, ..., b$. (A "flag of quotients" means of course that $K_{1,i} \subsetneq \cdots \subseteq K_{t_i,i}$, $K_{j,i} := \ker(\mathscr{E}_{x_i} \twoheadrightarrow V_{j,i})$, $j = 1, ..., t_i$, $i = 1, ..., b$.) These objects were introduced by Mehta and Seshadri [31] and called *quasi-parabolic vector bundles*. We had to chose a different name, because the notion of a *parabolic principal bundle* has been used differently in [2]. The same objects that we are looking at have also been considered in [9] and [43].

Lemma 3.2.2. *Fix a type* $(\underline{x}, \underline{P})$ *as in the definition.*
i) *The principal G-bundles with a flagging of type* $(\underline{x}, \underline{P})$ *form the smooth algebraic stack* $\mathrm{Bun}_{G,\underline{x},\underline{P}}$.
ii) *The forgetful map* $\mathrm{Bun}_{G,\underline{x},\underline{P}} \to \mathrm{Bun}_G$ *is a locally trivial bundle whose fibers are isomorphic to* $\prod_{i=1}^{s}(G/P_i)$.
iii) *The cohomology algebra* $H^\star(\mathrm{Bun}_{G,\underline{x},\underline{P}}, \overline{\mathbb{Q}}_\ell)$ *is a free module over* $H^\star(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ *with a basis of pure cohomology classes. The same holds for all open substacks of* Bun_G *and their preimages in* $\mathrm{Bun}_{G,\underline{x},\underline{P}}$.

Proof. The first parts are easy, because for a G-bundle $\mathscr{P} \to T \times C$ the space $\prod_i(\mathscr{P}|_{T\times x_i})/P_i \to T$ parameterizes flaggings of $\mathscr{P}$ at $T \times \underline{x}$. This is a $\prod_{i=1}^{s}(G/P_i)$ bundle over T. The last part follows from the second by the theorem of Leray–Hirsch: the flagging of the universal bundle at x_i defines a P_i-bundle over $\mathrm{Bun}_{G,\underline{x},\underline{P}}$ and thus a map $\mathrm{Bun}_{G,\underline{x},\underline{P}} \to BP_i$. But the map $G/P_i \to BP_i$ induces a surjection on cohomology, and thus the pull back of the universal classes in $H^\star(BP_i, \overline{\mathbb{Q}}_\ell)$ to $H^\star(\mathrm{Bun}_{G,\underline{x}}, \overline{\mathbb{Q}}_\ell)$ generate the cohomology of all the fibers of $\mathrm{Bun}_{G,\underline{x}} \to \mathrm{Bun}_G$. $\square$

In Section 4, we will introduce a notion of $\underline{a}$-stability for flagged principal bundles depending on some parameter $\underline{a}$. As in the case of vector bundles, we will define a coprimality condition for $\underline{a}$ (see 4.2.1) as well as some admissibility condition (following Remark 4.1.5).

In $\mathrm{Bun}_{G,\underline{x},\underline{P}}$ there are open substacks $\mathrm{Bun}^{\underline{a}\text{-(s)s}}_{G,\underline{x},\underline{P}}$ of $\underline{a}$-(semi)stable flagged principal G-bundles of type $(\underline{x}, \underline{P})$. Our main results on the coarse moduli spaces of these substacks are collected in the following theorem.

Theorem 3.2.3. i) *For any type $(\underline{x}, \underline{P})$ and any admissible stability parameter $\underline{a}$, there exists a projective coarse moduli space $\mathscr{M}(\underline{x}, \underline{P})^{\underline{a}\text{-ss}}$ for $\underline{a}$-semistable flagged principal G-bundles of type $(\underline{x}, \underline{P})$.*

ii) *If $\underline{a}$ is of coprime type, then the notions of $\underline{a}$-semi stability and $\underline{a}$-stability coincide. In this case, $\mathrm{Bun}^{\underline{a}\text{-s}}_{G,\underline{x},\underline{P}}$ is a proper, smooth quotient-stack with finite stabilizer groups.*

iii) *For any substack $U \subset \mathrm{Bun}_G$ of finite type and any $i > 0$, there exist $s > 0$, a type $(\underline{x}, \underline{P})$, and an admissible stability parameter $\underline{a}$ of coprime type, such that U lies in the image of the map $\mathrm{Bun}^{\underline{a}\text{-s}}_{G,\underline{x},\underline{P}} \to \mathrm{Bun}_G$ and such that the subset of $\underline{a}$-unstable bundles is of codimension $> i$ in $\mathrm{Bun}_{G,\underline{x},\underline{P}}$.*

The proof of this theorem takes up the largest part of this article. We will prove the existence of the coarse moduli spaces in Section 5. The projectivity then follows from our semistable reduction theorem 4.4.1. The last two parts of the theorem are much easier. We will prove them in Section 4.

Remark 3.2.4. For simplicity, we have stated Theorem 3.2.3 only for curves defined over a field. In order to prove our base change theorem, we will need the result in the case that C is a smooth, projective family of curves with geometrically reduced, connected fibers, defined over an integral ring R, finitely generated over $\mathbb{Z}$, and G a semisimple Chevalley group over R.

Seshadri proved in [39] (Theorem 4, p. 269) that GIT-quotients can be constructed for families over R. Further, the parameter spaces constructed in Section 5 are given by quot schemes which exist over base schemes, and, in Section 5.6, we finally need a Poincaré bundle on the relative Picard scheme. A Poincaré bundle exists, if the family $C \longrightarrow \mathrm{Spec}(R)$ has a section. This certainly holds after an étale extension of R. Hence, the first assertion still holds after an étale extension of R.

Except for the properness assertion for the stack of stable flagged principal bundles which is Lemma 3.3.1, the last two parts of the theorem carry over to this situation without modification.

We will come back to the issue of the base ring in Remarks 5.2.4, 5.3.3, and 5.5.4.

Before we proceed with the proof of the theorem, we want to deduce our main application.

3.3 Purity of $H^\star(\mathrm{Bun}_G)$

Assume that k is a finite field. Since all open substacks of finite type of Bun_G can be written as $[X/\mathrm{GL}_N]$ where X is a smooth variety, we know that the eigenvalues λ_i of the (arithmetic) Frobenius on $H^i(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ satisfy $|\lambda_i| \leq q^{-\frac{i}{2}}$ [8]. To prove equality, i.e., to prove that the cohomology is pure, we cannot rely on such a general argument. But, using the results on coarse moduli spaces, we can show that for all i the cohomology $H^i(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ occurs as a direct summand in the i-th cohomology of a projective variety, parameterizing stable flagged principal bundles.

Lemma 3.3.1. *Assume that R is a field or a discrete valuation ring with quotient field K. Let $G_{/R}$ be a reductive group, acting on the projective scheme $\overline{X}_R$ and $\mathscr{L}$ a G-linearized ample line bundle on $\overline{X}_R$, such that all points of $X := \overline{X}^{\mathrm{ss}}_{\mathscr{L}}$ are stable with respect to the chosen linearization. Then, the quotient stack $[X/G]$ is separated and the map $[X/G] \to X/\!\!/G$ is proper.*

Proof. If R is a field, we can apply GIT ([32] Corollary 2.5), saying that the map $G \times X \to X \times X$ is proper. Therefore, the diagonal $[X/G] \to [X/G] \times [X/G]$ is universally closed, i.e., $[X/G]$ is separated.
We claim that we may prove the separatednedness of the map $[X/G] \to X/\!\!/G$ over a discrete valuation ring R in the same manner. To show the lifting criterion for properness for the group action, we assume that we are given $x_1, x_2 \in X(R)$ and $g \in G(K)$, such that $g.x_1 = x_2$. We have to show that $g \in G(R)$. We may (after possibly replacing R by a finite extension as in [32], Appendix to Chapter 2.A) apply the Iwahori decomposition to write $g = g_0 z g_0'$ with $g_0, g_0' \in G(R)$ and $z \in T(K)$ for a maximal torus $T \subset G$. Thus, we have reduced the problem to the case that $g = z \in T(K)$. Choose a local parameter $\pi \in R$. Multiplying with an element of $T(R)$, we may further assume that there is a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to T$, such that $z = \lambda(\pi)$. Assume that λ is non-trivial. Now, embed $\overline{X}_R \subset \mathbb{P}(V)$ into a projective space and decompose $V = \sum_{i\in\mathbb{Z}} V_i$ into the eigenspaces of λ. Write $x_1 = \sum_{i\in\mathbb{Z}} v_i$ and $x_2 = \sum_{i\in\mathbb{Z}} w_i$ as sums of eigenvectors for λ. Since the reduction $\overline{x}_1$ of $x_1 \mod \pi$ is stable, there must be indices $i_- < 0 < i_+$ with $\overline{v}_{i_-} \neq 0 \neq \overline{v}_{i_+}$. The analogous condition holds for $\overline{x}_2$. But, one readily checks that $x_2 = z.x_1$ implies $\overline{w}_i = 0$, for $i > 0$, a contradiction.
Now, for algebraically closed fields K, the map $[X/G] \to X/\!\!/G$ induces a bijection on isomorphism classes of K-points. Thus, since we already know separatedness, it is sufficient to show that given a discrete valuation ring R and a point $\overline{x} \in X/\!\!/G(R)$, then we can find an extension R' of R, such that $\overline{x}$ lifts to a point $x \in X(R')$ and thus to a point in $[X/G]$. Let K be the quotient field of R, $\eta \in X$ a point lying over the generic point of $\overline{x}$. Then, the closure of $G \times \eta \subset X$ is a G-invariant subset. Since $X/\!\!/G$ is a good quotient, its image

is closed and contains $\overline{x}$. Thus, the orbit of η specializes to a point lying over the closed point of $\overline{x}$, and we can find $x \in X(R')$ as claimed. □

Corollary 3.3.2. *Assume that C is a smooth projective curve, defined over the finite field k. If $\underline{a}$ is of coprime type, then $H^\star(\mathrm{Bun}_{G,\underline{x},\underline{P}}^{\underline{a}\text{-s}}, \overline{\mathbb{Q}}_\ell)$ is pure.*

Proof. The stack $\mathrm{Bun}_{G,\underline{x},\underline{P}}^{\underline{a}\text{-s}}$ of $\underline{a}$-stable flagged principal G-bundles of type $(\underline{x},\underline{P})$ is a smooth quotient stack. Therefore, its i-th cohomology is of weight $\geq i$. This is proved in [8], Theorem 5.21. (Observe the different conventions for the Frobenius map.) Furthemore, by the definition of stability, all automorphism groups of stable parabolic bundles are finite. In particular, by the preceding lemma, the map $p\colon \mathrm{Bun}_{G,\underline{x},\underline{P}}^{\underline{a}\text{-s}} \to \mathscr{M}(\underline{x},\underline{P})^{\underline{a}\text{-s}}$ is proper. In order to prove that $Rp_\star\mathbb{Q}_\ell \cong \mathbb{Q}_\ell$, it is therefore sufficient to compare the stalks of these sheaves ([34], Theorem 1.3). But the fibers are quotients of $\mathrm{Spec}(K)$ by finite group schemes. Thus, for rational coefficients, the higher cohomology of the fibers vanishes. In particular, p induces an isomorphism on cohomology. Since the scheme $\mathscr{M}(\underline{x},\underline{P})^{\underline{a}\text{-s}}$ is proper (Theorem 3.2.3), its i-th cohomology is of weight $\leq i$, by Deligne's theorem ([12], Théorème I), □

Remark 3.3.3. i) So far, we have treated the moduli spaces only over algebraically closed fields. Of course, they will be defined over a finite extension of $\mathbb{F}_q$. (In fact, as the construction of the moduli spaces will reveal, they will be defined over the same field as the points in the tuple $\underline{x}$.) If we replace $\mathbb{F}_q$ by a finite extension, the new Frobenius is a power of the original Frobenius. The purity statement is obviously not affected, because it concerns only the absolute values of the eigenvalues of the Frobenius map.
ii) The moduli space $\mathscr{M}(\underline{x},\underline{P})^{\underline{a}\text{-s}}$ will, in general, have finite quotient singularities. Therefore, we could obtain both estimates for the weights from the coarse moduli space.

Corollary 3.3.4. *Suppose R is of finite type over $\mathbb{Z}$, regular, and of dimension at most 1, and let $C_{/R}$ be a smooth projective curve and G a split semisimple group scheme over R. Then, the cohomology of $\mathrm{Bun}_G \to \mathrm{Spec}(R)$ is locally constant over $\mathrm{Spec}(R)$.*

Proof. By Theorem 3.2.3, iii), we know that, for fixed i, the i-th cohomology sheaf of Bun_G is a direct summand of the corresponding sheaf of $\mathrm{Bun}_{G,\underline{x},\underline{P}}^{\underline{a}\text{-s}}$ for suitable type $(\underline{x},\underline{P})$ and suitable stability parameter $\underline{a}$. Further, by Lemma 3.3.1, the map $p\colon \mathrm{Bun}_{G,\underline{x},\underline{P}}^{\underline{a}\text{-s}} \to \mathscr{M}(\underline{x},\underline{P})^{\underline{a}\text{-s}}$ is proper. Since the coarse moduli space is proper as well, we can again apply Olsson's base change theorem ([34], Theorem 1.3) to the proper map $\pi : \mathrm{Bun}_{G,\underline{x},\underline{P}}^{\underline{a}\text{-s}} \to \mathscr{M}(\underline{x},\underline{P})^{\underline{a}\text{-s}} \to \mathrm{Spec}(R)$. In particular, the fibers of $\mathbb{R}\pi_\star\mathbb{Q}_\ell$ compute the cohomology of the fibers of π.
Moreover, the stack $\mathscr{X} := \mathrm{Bun}_{G,\underline{x},\underline{P}}^{\underline{a}\text{-s}}$ is smooth. Thus, we may use local acyclicity of smooth maps as in [11], Chapitre V. To see that this holds for stacks, let us recall the argument. We may suppose that the base $S = \mathrm{Spec}(R)$ is strictly

henselian. Denote by $\overline{\eta}$ the spectrum of an algebraic closure of the generic point of S and let s denote the special point of S. We have a cartesian diagram

$$\begin{tikzcd}
\mathscr{X}_{\overline{\eta}} \arrow[r, "\epsilon'"] \arrow[d] & \mathscr{X} \arrow[d, "f"] & \mathscr{X}_s \arrow[l, "i'"'] \arrow[d] \\
\overline{\eta} \arrow[r, "\epsilon"] & S & \{s\}. \arrow[l, "i"']
\end{tikzcd}$$

Now, $\mathbb{R}\epsilon'_\star\mathbb{Q}_\ell \cong f^\star\mathbb{R}\epsilon_\star\mathbb{Q}_\ell$, because this holds for any smooth covering $U \to \mathscr{X}$ and $i'^*\mathbb{R}\epsilon_\star\mathbb{Q}_\ell = \mathbb{Q}_\ell$. Thus, using the above calculation and proper base change for the last equality, we find:

$$H^\star(\mathscr{X}_{\overline{\eta}}, \mathbb{Q}_\ell) \cong H^\star(\mathscr{X}, \mathbb{R}\epsilon'_\star\mathbb{Q}_\ell) \cong H^\star(\mathscr{X}_s, \mathbb{Q}_\ell).$$

This settles the claim. □

We may now derive our main result.

Theorem 3.3.5. *Assume that C is a curve over the field k. Then, the cohomology of* Bun_G *is freely generated by the canonical classes, i.e.,*

$$H^\star\big(\mathrm{Bun}^\vartheta_{G,\overline{k}}, \overline{\mathbb{Q}}_\ell\big) = \overline{\mathbb{Q}}_\ell[a_1, \ldots, a_r] \otimes \bigwedge^\star [b_i^j]_{i=1,\ldots,r, j=1,\ldots,2g} \otimes \overline{\mathbb{Q}}_\ell[f_1, \ldots, f_r].$$

Proof. First method. One can deduce the result from the theorem of Atiyah and Bott. By the base change corollary above, knowing the theorem for $k = \mathbb{C}$ implies the claim over an arbitrary algebraically closed field. For $k = \mathbb{C}$ Atiyah and Bott proved the result. Namely they constructed a continuous atlas $X \to \mathrm{Bun}_G$, where X is contractible and Bun_G is the quotient of X by the action of an infinite dimensional group $\mathscr{G}$. In the article of Atiyah and Bott the equivariant cohomology of X with respect to this group action is computed. However, the spectral sequence computing equivariant cohomology from the cohomology of $\mathscr{G}$ coincides with the sequence computing the cohomology of Bun_G from the atlas $X \to \mathrm{Bun}_G$.
Second method. By the base change corollary 3.3.4, it is sufficient to prove the claim in the case that C is defined over a finite field k. We have just seen (Corollary 3.3.2) that in this case the cohomology of Bun_G is pure. Furthermore, Harder proved [18] that $\tau(G) = 1$ for semisimple simply connected groups and Ono showed how to deduce $\tau(G) = \#\pi_1(G)$ for arbitrary semisimple groups (see [8], §6). Thus, we can apply Corollary 3.1.3 to Siegel's formula and Behrend's trace formula. □

Remark 3.3.6. For $G = \mathrm{SL}_n(k)$ (or $G = \mathrm{GL}_n(k)$), one can use Beauville's trick [3] which shows that the cohomology of $\mathrm{Bun}^{\underline{a}\text{-s}}_{\mathrm{SL}_n,\underline{x},\underline{P}}$ is generated by the classes constructed in Remark 3.2.2. This gives a direct proof of the theorem.

4 Semistability for Flagged Principal Bundles

In this section, we introduce the parameter dependent notion of semistability for flagged principal bundles. After discussing its basic features, including the important fact that any principal bundle can be turned into a stable flagged principal bundle for a suitable type and a suitable stability parameter, we apply Behrend's formalism of complementary polyhedra to derive the Harder–Narasimhan reduction for semistable flagged principal bundles. We conclude with a proof of the semistable reduction theorem for flagged principal bundles, generalizing the arguments from [22] and [23].

4.1 Definition of Semistability

We want to define a notion of semistability for flagged principal bundles. For an algebraic group P let us denote by $X^\star(P) := \mathrm{Hom}(P, \mathbb{G}_m)$ the group of characters and by $X^\star(P)^\vee_{\mathbb{Q}} := \mathrm{Hom}(X^\star(P), \mathbb{Q})$ the rational cocharacters. The notion of semistability will depend on parameters a_i varying over the sets

$$X^\star(P_i)^\vee_{\mathbb{Q},+} := \left\{ a \in X^\star(P_i)^\vee_{\mathbb{Q}} \,\middle|\, \begin{array}{l} \text{for all parabolic subgroups } P' \supset P_i \\ a(\det_{P'} \otimes \det_{P_i}^{-1}) < 0 \end{array} \right\},$$

$i = 1, ..., b$. (Since $\mathrm{Bun}_{G,\underline{x},\underline{P}} \to \mathrm{Bun}_G$ is a locally trivial fibration with fiber $\prod_{i=1}^s G/P_i$, we see that the Picard group of $\mathrm{Bun}_{G,\underline{x},\underline{P}}$ is a free $\mathbb{Z}$-module generated by $\mathrm{Pic}(\mathrm{Bun}_G) \cong \mathbb{Z}$ and $\prod_{i=1}^s X^\star(P_i)$. Therefore the notion of semistability should depend on an element in $X^\star(P_i)^+$. Since this has a canonical basis, the dual appears in our definition.) To state this in terms closer to Geometric Invariant Theory, note that the pairing of characters and one-parameter subgroups of a parabolic subgroup of G is invariant under conjugation. Therefore, conjugacy classes of rational one-parameter subgroups of P_i are given by $X^\star(P_i)^\vee_{\mathbb{Q}}$, $i = 1, ..., b$. A one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ defines the parabolic subgroup

$$P(\lambda) := P_G(\lambda) = \left\{ g \in G \,\middle|\, \lim_{z \to 0} \lambda(z) g \lambda(z)^{-1} \text{ exists in } G \right\}.$$

For later purposes, we also introduce

$$Q_G(\lambda) := P_G(-\lambda) = \left\{ g \in G \,\middle|\, \lim_{z \to \infty} \lambda(z) g \lambda(z)^{-1} \text{ exists in } G \right\}.$$

Example 4.1.1. Any one-parameter subgroup $\lambda\colon \mathbb{G}_m(k) \longrightarrow \mathrm{GL}(V)$ defines a set of weights $\gamma_1 < \cdots < \gamma_{t+1}$ and a decomposition

$$V = \bigoplus_{l=1}^{t+1} V^l \text{ with } V^l := \left\{ v \in V \,\middle|\, \lambda(z)(v) = z^{\gamma_l} \cdot v,\ \forall z \in \mathbb{G}_m(k) \right\},\ l = 1, ..., t+1,$$

into eigenspaces. We derive the flag

$$V_\bullet(\lambda) : \{0\} \subsetneq V_1 := V^1 \subsetneq V_2 := V^1 \oplus V^2 \subsetneq \cdots \subsetneq V_t := V^1 \oplus \cdots \oplus V^t \subsetneq V.$$

Note that the group $Q_{\mathrm{GL}(V)}(\lambda)$ is the stabilizer of the flag $V_\bullet(\lambda)$. As an additional datum, we define the tuple $\beta_\bullet(\lambda) = (\beta_1, ..., \beta_t)$ with $\beta_l := (\gamma_{l+1} - \gamma_l)/\dim(V)$, $l = 1, ..., t$. The pair $(V_\bullet(\lambda), \beta_\bullet(\lambda))$ is the *weighted flag of* λ.

Since $P(\lambda) = P(n\lambda)$ for all $n \in \mathbb{N}$, the group $P(\lambda)$ is also well defined for rational one-parameter subgroups, and it only depends on the conjugacy class of λ in $P(\lambda)$. Finally, writing G as a product of root groups, we see that $\lambda \in X_\star(P_i)_{\mathbb{Q}}$ defines an element $\lambda \in X^\star(P_i)^\vee_{\mathbb{Q},+}$, if and only if $P_i = P(\lambda)$. It will often be convenient for us to view $a_i \in X^\star(P_i)^\vee_{\mathbb{Q},+}$ as a rational one-parameter subgroup of G which we will denote by the same symbol.

Remark 4.1.2. i) Let $(\mathscr{P}, \underline{s})$ be a flagged principal G-bundle and $\mathscr{P}_{x_i,P_i}$ the P_i-torsor over x_i defined by s_i, $i = 1, ..., b$. Denote further $P_{s_i} := \mathrm{Aut}_{P_i}(\mathscr{P}_{x_i,P_i}) \subset \mathrm{Aut}_G(\mathscr{P}_{x_i})$ the corresponding parabolic subgroup. Any (P_i-equivariant) trivialization $\mathscr{P}_{x_i,P_i} \cong P_i$ defines an isomorphism $P_{s_i} \cong P_i$. This isomorphism is canonical up to inner automorphisms of P_i, so that we obtain canonical isomorphisms $X^\star(P_i)_{\mathbb{Q}} \cong X^\star(P_{s_i})_{\mathbb{Q}}$ and $X^\star(P_i)^\vee_{\mathbb{Q},+} \cong X^\star(P_{s_i})^\vee_{\mathbb{Q},+}$, $i = 1, ..., b$. Given $a_i \in X^\star(P_i)^\vee_{\mathbb{Q},+}$ we will denote the corresponding element in $X^\star(P_{s_i})^\vee_{\mathbb{Q},+}$ by a_{s_i}. The "one-parameter subgroup" a_{s_i} is well-defined only up to conjugation in P_{s_i}. If we choose a maximal torus $T \subset P_{s_i}$, we may assume that a_{s_i} is a one-parameter subgroup of T. As such it is well-defined.
ii) Likewise, if a parabolic subgroup Q of G, a character χ of Q, and a reduction $\mathscr{P}_Q$ of $\mathscr{P}$ to Q are given, then we get in each point x_i a parabolic subgroup Q_i in $\mathrm{Aut}(\mathscr{P}_{x_i})$ and a character χ_{s_i} of that parabolic subgroup, $i = 1, ..., b$.
iii) Any two parabolic subgroups P and Q of G share a maximal torus, and all common maximal tori are conjugate in $Q \cap P$. Let $Q_i \subset \mathrm{Aut}(\mathscr{P}_{x_i})$ be a parabolic subgroup, $i = 1, ..., b$. By our previous remarks, we may assume that a_{s_i} is a subgroup of $Q_i \cap P_{s_i}$. Then, for any i and any character $\chi_i \in X^\star(Q_i)$, the value of the pairing $\langle \chi_i, a_{s_i} \rangle$ is well-defined.

These remarks also show the following.

Lemma 4.1.3. *Let $Q, P \subset G$ be parabolic subgroups, $a \in X^\star(P)^\vee_{\mathbb{Q},+}$, and $\chi \in X^\star(Q)$ a dominant character. Denote by ${}^g\chi = \chi(g^{-1} \cdot \_ \cdot g)$ the corresponding character of gQg^{-1}. Then, the value of the function*

$$\begin{aligned} G &\longrightarrow \mathbb{Q} \\ g &\longmapsto \langle {}^g\chi, a \rangle \end{aligned}$$

at an element of G depends only on the image of that element in $Q\backslash G/P$.

Example 4.1.4. Using the notations of the above lemma, assume that $P = B$ is a Borel subgroup and assume that Q contains B. Choose a maximal torus $T \subset B$, denote by Δ_P and Δ_Q the roots of P and Q, respectively, and by W and W_Q the Weyl groups of G and $Q/R_u(Q)$, respectively. Then, the double coset $Q\backslash G/P$ is in bijection to $W_Q\backslash W$ and, by Bruhat decomposition, we know that $QwP/P \subset G/P$ lies in the closure of $Qw'P/P$ only if all roots of wQw^{-1} which do not lie in Δ_P are contained in $\Delta_{w'Qw'^{-1}}$. Now, since $a \in X^\star(P)^\vee_{\mathbb{Q},+}$,

we know that $\langle \alpha, a\rangle < 0$ occurs precisely for the roots $\alpha \notin \varDelta_P$. Thus, we find $\langle {}^w\chi, a\rangle \geq \langle {}^{w'}\chi, a\rangle$, whenever QwP lies in the closure of $Qw'P$ and equality implies that the double cosets coincide.
In particular the largest value of $\langle {}^w\chi, a\rangle$ is obtained for $w = 1$ and the most negative one for the longest element of W.

Fix $\underline{a} \in \prod_{i=1}^b X^\star(P_i)^\vee_{\mathbb{Q},+}$. Using Remark 4.1.2 and Lemma 4.1.3, we define the *$\underline{a}$-parabolic degree* (*of the reduction $\mathscr{P}_Q$ of $\mathscr{P}$*) as the function

$$\begin{array}{rcl} \underline{\underline{a}\text{-deg}}(\mathscr{P}_Q) : X^\star(Q) & \longrightarrow & \mathbb{Q} \\ \chi & \longmapsto & \deg\big(\mathscr{P}_Q(\chi)\big) + \displaystyle\sum_{i=1}^b \langle \chi_{s_i}, a_{s_i}\rangle. \end{array}$$

(As usual, $\mathscr{P}_Q(\chi)$ is the line bundle on C that is associated with the principal Q-bundle $\mathscr{P}_Q$ and the character $\chi\colon Q \longrightarrow \mathbb{G}_m(k)$.) We write $\underline{a}\text{-deg}(\mathscr{P}_Q) := \underline{\underline{a}\text{-deg}}(\mathscr{P}_Q)(\det_Q)$ where $\det_Q$ is the character defined by the determinant of the adjoint representation of Q.
A flagged principal G-bundle $(\mathscr{P}, \underline{s})$ is called *$\underline{a}$-(semi)stable*, if for any parabolic subgroup $Q \subset G$ and any reduction $\mathscr{P}_Q$ of $\mathscr{P}$ to Q, the condition

$$\underline{a}\text{-deg}(\mathscr{P}_Q)(\leq)0$$

is verified. Here the standard notation $(\leq)$ means that for stable bundles we require a strict inequality, whereas for semistable bundles $\leq$ is allowed.
The *$\underline{a}$-parabolic degree of instability of* $(\mathscr{P}, \underline{s})$ is set to be

$$\begin{aligned} \mathrm{ideg}_{\underline{a}}(\mathscr{P}, \underline{s}) \quad := \quad \max\Big\{ & \underline{a}\text{-deg}(\mathscr{P}_Q) \,\Big|\, Q \subset G \text{ a parabolic subgroup} \\ & \text{and } \mathscr{P}_Q \text{ a reduction of } \mathscr{P} \text{ to } Q \Big\}. \end{aligned}$$

Remark 4.1.5. i) Let Q be a maximal parabolic subgroup of G. Then, all dominant characters on Q are positive rational multiples of the corresponding *fundamental weight*. Thus, they are also positive rational multiples of the character $\det_Q$. If Q is an arbitrary parabolic subgroup and χ is a dominant character on it, then one finds maximal parabolic subgroups $Q_1, ..., Q_T$ that contain it and such that χ is a positive rational linear combination of the characters $\det_{Q_1}$,...,$\det_{Q_T}$ (viewed as characters of Q). Therefore, a flagged principal G-bundle $(\mathscr{P}, \underline{s})$ is $\underline{a}$-semistable, if and only if for any parabolic subgroup Q, any reduction $\mathscr{P}_Q$ of $\mathscr{P}$ to Q, and any dominant character $\chi \in X^\star(Q)$, we have $\underline{\underline{a}\text{-deg}}(\mathscr{P}_Q)(\chi) \leq 0$. Or, equivalently, we may use anti-dominant characters χ and require $\underline{\underline{a}\text{-deg}}(\mathscr{P}_Q)(\chi) \geq 0$. (We have used the version with dominant characters, because this allows us to adapt Behrend's existence proof of the canonical reduction ([4], [6]) more easily. For our GIT computations below, the formulation with anti-dominant characters seems better suited.)
ii) From our observations in i), we also infer that it suffices to test semistability for maximal parabolic subgroups.

iii) The $\underline{a}$-parabolic degree of instability is finite, because the degree of instability is finite and the values of $\langle \chi_{s_i}, a_{s_i} \rangle$, $i = 1, ..., b$, are bounded for every fixed $\underline{a}$, and only finitely many χ occur.

An element $a_i \in X^\star(P_i)^\vee_{\mathbb{Q},+}$ is called *admissible*, if for some maximal torus $T \subset P_i$, such that a_i factors through T, we have $|\langle \alpha, a_i \rangle| < \frac{1}{2}$ for all roots α. Note that this does not depend on the choice of T, because all maximal tori are conjugate over k and conjugation permutes the roots. The stability parameter $\underline{a}$ is called *admissible*, if a_i is admissible for $i = 1, ..., b$.

4.2 General Remarks on Semistability

As in the case of vector bundles, the notions of $\underline{a}$-semistability and $\underline{a}$-stability will coincide, if $\underline{a}$ satisfies some coprimality condition. In the following lemma, we will also allow real stability parameters $\underline{a} \in \bigoplus_{i=1}^b X^\star(P_i)^\vee_{\mathbb{R}}$ in order to define a nice chamber decomposition. Clearly, $\underline{a}$-(semi)stability may also be defined for such parameters.

Lemma 4.2.1. *Fix the type $(\underline{x}, \underline{P})$. For every parabolic subgroup $Q \in G$ and every $d \in \mathbb{Z}$, we introduce the* wall

$$W_{Q,d} := \left\{ \underline{a} \in \bigoplus_{i=1}^b X^\star(P_i)^\vee_{\mathbb{R}} \mid \sum_{i=1}^b \langle \det_Q, a_i \rangle = d \right\}.$$

Then, the following properties are satisfied:
i) *For every bounded subset $A \subset X^\star(P_i)^\vee_{\mathbb{R}}$, there are only finitely many walls $W_{Q,d}$ with $W_{Q,d} \cap A \neq \varnothing$.*
ii) *If one of the groups P_i is a Borel subgroup, then $W_{Q,d}$ is for all parabolic subgroups Q and all integers d a proper subset of codimension 1 or empty.*
iii) *If*

$$\underline{a} \notin \bigcup_{Q \subset G \text{ parabolic},\ d \in \mathbb{Z}} W_{Q,d},$$

then every $\underline{a}$-semistable bundle is $\underline{a}$-stable.
iv) *If the stability parameters $\underline{a}$ and $\underline{a}'$ lie in the same connected component of*

$$\bigoplus_{i=1}^b X^\star(P_i)^\vee_{\mathbb{R}} \setminus \bigcup_{Q \subset G \text{ parabolic},\ d \in \mathbb{Z}} W_{Q,d},$$

then the notions of $\underline{a}$-(semi)stability and $\underline{a}'$-(semi)stability coincide.
v) *Let $\mathscr{C}$ be a connected component of $\bigoplus_{i=1}^b X^\star(P_i)^\vee_{\mathbb{R}} \setminus \bigcup_{Q \subset G \text{ parabolic},\ d \in \mathbb{Z}} W_{Q,d}$. If $\underline{a} \in \mathscr{C}$ and $\underline{a}' \in \overline{\mathscr{C}}$, then every $\underline{a}'$-stable bundle is $\underline{a}$-stable and every $\underline{a}$-semistable bundle is $\underline{a}'$-semistable.*

A stability parameter $\underline{a}$ satisfying the condition stated in iii) of the lemma is said to be *of coprime type*.

Proof. Let $\mathfrak{c}$ be a conjugacy class of parabolic subgroups in G and $Q_{\mathfrak{c}}$ a representative of $\mathfrak{c}$. For a parabolic subgroup Q in the class $\mathfrak{c}$ and $i \in \{1, ..., b\}$, the number $\langle \det_Q, a_i \rangle$ depends only on the class of Q in $Q_{\mathfrak{c}} \backslash G/P_i$. This was shown in Lemma 4.1.3. Since there are only finitely many conjugacy classes of parabolic subgroups and any set of the form $Q\backslash G/P$, P, Q parabolic subgroups of G, is finite, there are only finitely many functions of the form

$$\underline{a} \longmapsto \sum_{i=1}^{b} \langle \det_Q, a_i \rangle$$

on $\bigoplus_{i=1}^b X^\star(P_i)^\vee_{\mathbb{R}}$, and any bounded set A is "hit" by only finitely many walls. The second part is easy, because, for a Borel subgroup, one has $X^\star(B) = X^\star(T)$, so that $\langle \det_Q, . \rangle$ cannot vanish identically on $X^\star(B)^\vee_{\mathbb{R}}$.
For a properly semistable flagged principal G-bundle $(\mathscr{P}, \underline{s})$, there are a parabolic subgroup Q and a reduction $\mathscr{P}_Q$ of $\mathscr{P}$ to Q, such that $\sum_{i=1}^b \langle \det_Q, a_i \rangle = -\deg(\mathscr{P}_Q) \in \mathbb{Z}$. This immediately yields iii) and also proves the last two statements. $\square$

Proposition 4.2.2. *Fix a connected component* Bun_G^ϑ *of* Bun_G *and a Borel subgroup* $B \subset G$. *Then, for all* $h \in \mathbb{Z}$, *there exists a number* $b_0 \in \mathbb{N}$, *such that, for any* $b > b_0$, *and any collection* $\underline{x} = (x_1, ..., x_b)$ *of distinct* k*-rational points on* C, *there is an admissible stability parameter* $\underline{a}^b \in \prod_{i=1}^b X^\star(B)^\vee_{\mathbb{Q},+}$ *of coprime type with the following property: for every principal* G*-bundle* $\mathscr{P}$ *with degree of instability* $\leq h$, *there exists a flagging* $\underline{s}$ *with* $s_i\colon \{x_i\} \to \mathscr{P}_{|\{x_i\}}/B$, $i = 1, ..., b$, *such that* $(\mathscr{P}, \underline{s})$ *is an* $\underline{a}^b$*-stable flagged principal* G*-bundle of type* $(\underline{x}^b, (B, ..., B))$.

Proof. Part v) of Lemma 4.2.1 shows that we may replace any stability parameter by one of coprime type, while enlarging the set of stable bundles. So we do not have to worry about the coprimality condition on $\underline{a}$.
Let $\mathrm{Bun}_G^{\vartheta;\leq h}$ be the stack of principal G-bundles of instability degree $\leq h$. This is an open substack of finite type of Bun_G [4]. Choose $a \in X^\star(B)^\vee_{\mathbb{Q},+}$, such that for all parabolic subgroups $Q \subset G$ one has either $\langle \det_Q, a \rangle > 0$ or $\langle \det_Q, a \rangle < -2h$. Such a choice is possible by Lemma 4.2.1, ii): we can find $a' \in X^\star(B)_{\mathbb{Q}}$, such that the finitely many values $\langle \det_Q, a' \rangle$ are all non-zero. Multiplying a' with a sufficiently large constant, we find a. Set

$$D := \max\big\{ \langle \det_Q, a \rangle \,|\, Q \subset G \text{ a parabolic subgroup} \big\}.$$

Note that this is a positive number.
Next, choose a sequence $(x_n)_{n \geq 1}$ of distinct points in $C(k)$, set $\underline{x}^b := (x_1, ..., x_b)$, and consider, for $b \in \mathbb{N}$, the stability parameter $\underline{a}^b := (a/b, \ldots, a/b)$. It will be admissible for $b \gg 0$.

Observation. *Let* $\mathscr{P}$ *be a principal* G*-bundle,* $Q \subset G$ *a parabolic subgroup, and* $\mathscr{P}_Q$ *a reduction of* $\mathscr{P}$ *to* Q, *such that* $\deg(\mathscr{P}_Q) < -D$. *Then, for any*

b and any choice of sections $s_i\colon \{x_i\} \to \mathscr{P}_{|\{x_i\}}/B$, $i = 1, ..., b$, we have $\underline{a}^b\text{-}\deg(\mathscr{P}_Q) < 0$.

We want to estimate the dimension of the space of $\underline{a}^b$-unstable flagged principal G-bundles $(\mathscr{P}, \underline{s})$ of type $(\underline{x}^b, (B, ..., B))$ with $\mathscr{P} \in \mathrm{Bun}_G^{\vartheta, \leq h}$. First of all, the stack

$$\mathrm{Reductions} := \left\langle \big(\mathscr{P}, \mathscr{P}_Q\big) \;\middle|\; \begin{array}{l} \mathscr{P} \in \mathrm{Bun}_G^{\vartheta,\leq h}, \\ \mathscr{P}_Q \text{ a reduction of } \mathscr{P} \text{ to the parabolic} \\ \text{subgroup } Q \text{ with } \deg(\mathscr{P}_Q) \geq -D \end{array} \right\rangle$$

is an algebraic stack of finite type: reductions of a principal G-bundle $\mathscr{P}$ to Q are given by sections of $\mathscr{P}/Q$, and $\mathscr{P}/Q$ is projective over the base. Thus, by Grothendieck's construction of the quot schemes, these sections are parametrized by a countable union of quasi-projective schemes. We may apply this to the universal bundle over $\mathrm{Bun}_G^{\vartheta,\leq h} \times C$, because locally we may use the quot schemes for any bounded family over a scheme and the resulting schemes glue, because the functor is defined over the stack. The substack of reductions of fixed degree is of finite type, because the reduction is defined by the induced vector subbundle of the adjoint bundle of rank $\dim(Q)$ and the same degree as the reduction. In any bounded family of vector bundles, the vector subbundles of given rank and degree form also a bounded family. Finally, recall that we look only at degrees between $-D$ and h.
Therefore, the fiber product

$$\mathrm{Test} := \mathrm{Reductions} \times_{\mathrm{Bun}_G^{\vartheta,\leq h}} \mathrm{Bun}_{G,\underline{x}^b}^{\vartheta,\leq h}$$

parameterizing flagged principal G-bundles of type $(\underline{x}^b, (B, ..., B))$ together with a reduction of bounded degree to a parabolic subgroup is for any $b \in \mathbb{N}$ of finite type. Consider the closed substack $\mathrm{Bad} \subset \mathrm{Test}$ given by $(\mathscr{P}, \underline{s}, \mathscr{P}_Q)$ with $\underline{a}^b\text{-}\deg(\mathscr{P}_Q) \geq 0$. We can estimate the dimension of the fibers of $\mathrm{Bad} \to \mathrm{Reductions}$ as follows: fix $\mathscr{P} \in \mathrm{Bun}_G^{\vartheta,\leq h}$, a parabolic subgroup $Q \subset G$, and a reduction $\mathscr{P}_Q$ of $\mathscr{P}$ to Q. Given b, the variety of flaggings of $\mathscr{P}$ is $\mathsf{X}_{i=1}^b \mathscr{P}_{|\{x_i\}}/B \cong (G/B)^{\times b}$. Now, for every i, the subset

$$\left\{\, s_i \in \mathscr{P}_{|\{x_i\}}/B \,|\, \langle \det_Q, a_{s_i} \rangle < 0 \,\right\} \subset \mathscr{P}_{x_i}/B$$

is non-empty and open. Denote its complement by Z_i. Now, if $\#\{i | s_i \notin Z_i\} > b \cdot (h + D)/(2h + D)$, then $(\mathscr{P}, \underline{s})$ is $\underline{a}^b$-stable: indeed, we compute

$$\begin{aligned} \underline{a}^b\text{-}\deg(\mathscr{P}_Q) &= \deg(\mathscr{P}_Q) + \sum_{i=1}^{b} \langle \det_Q, a_{s_i} \rangle \\ &< h - b \cdot \frac{h + D}{2h + D} \cdot \frac{2h}{b} + b \cdot \left(1 - \frac{h + D}{2h + D}\right) \cdot \frac{D}{b} = 0. \end{aligned}$$

Thus,

$$\dim(\mathrm{Bad}) \leq \dim(\mathrm{Reductions}) + b \cdot \dim(G/B) - b \cdot \frac{h}{2h + D}.$$

Thus, for $b \gg 0$, we see that $\dim(\mathrm{Bad}) < b \cdot \dim(G/B)$ and therefore the image of Bad in $\mathrm{Bun}_{G,\underline{x}^b}^{\vartheta,\leq h}$ cannot contain any fiber of $\mathrm{Bun}_{G,\underline{x}^b}^{\vartheta,\leq h} \to \mathrm{Bun}_G^{\vartheta,\leq h}$. $\square$

Remark 4.2.3. The proof also shows that we may make the codimension of the locus of $\underline{a}^b$-unstable flagged principal G-bundles as large as we wish.

4.3 The Canonical Reduction for Flagged Principal Bundles

Motivated by work of Harder [20], Stuhler explained in [41] how to define a notion of stability for Arakelov group schemes over curves and how to use Behrend's technique of complementary polyhedra to prove the existence of a canonical reduction to a parabolic subgroup in this situation. We only had to translate this to our special case of flagged principal G-bundles. According to Behrend, it suffices to show that the parabolic degree defined above defines a complementary polyhedron, a concept which we will recall below. All the results of this section are due to Behrend [6] (with some simplifications given by Harder and Stuhler in the above references). We only have to verify that his theory applies to our situation. Since in our case of flagged principal bundles the arguments simplify a bit, we will try to give a self-contained account.
Let $(\mathscr{P}, \underline{s})$ be a flagged principal G-bundle on C and fix a stability parameter $\underline{a}$. Let $P \subset G$ be a parabolic subgroup. A reduction $\mathscr{P}_P$ of $\mathscr{P}$ to P is called *canonical*, if

(1) $\underline{a}$-$\deg(\mathscr{P}_P) = \mathrm{ideg}_{\underline{a}}(\mathscr{P}, \underline{s})$.

(2) P is a maximal element in the set of parabolic subgroups for which there is a reduction $\mathscr{P}_P$ of degree $\mathrm{ideg}_{\underline{a}}(\mathscr{P}, \underline{s})$.

Remark 4.3.1. Let $\mathscr{P}_P$ be a canonical reduction of $\mathscr{P}$ and denote by $R_u(P)$ the unipotent radical of P. Note that by Remark 4.1.2, iii), the induced principal $(P/R_u(P))$-bundle $\mathscr{P}_P/R_u(P)$ inherits a flagging $\underline{s}'$: indeed, we may choose a representative for a_{s_i} which lies in a maximal torus of $\mathrm{Aut}(\mathscr{P})_{|\{x_i\}}$ which is contained in the intersection of the parabolic subgroup given by the flagging at x_i with the parabolic subgroup given by the canonical reduction and define the parabolic subgroup of $\mathrm{Aut}(\mathscr{P}_P/R_u(P))_{|\{x_i\}}$ associated with a_{s_i} as the flagging s_i' of $\mathscr{P}_P/R_u(P)$ at x_i, $i = 1, ..., b$. Using this, we find that $\mathscr{P}_P$ has the following properties:

(1′) $(\mathscr{P}_P/R_u(P), \underline{s}')$ is an $\underline{a}$-semistable flagged principal bundle.
This holds, because the preimage of a reduction of positive degree of $\mathscr{P}_P/R_u(P)$ would define a parabolic reduction of larger degree in $\mathscr{P}$.

(2′) For all parabolic subgroups P' containing P, one has the inequality $\underline{\underline{a}\text{-deg}}(\mathscr{P}_P)(\det_P \otimes \det_{P'}^{-1}) > 0$. In fact, by the definition of a canonical reduction, we know that $\underline{\underline{a}\text{-deg}}(\mathscr{P}_P)(\det_{P'}) = \underline{a}\text{-deg}(\mathscr{P}_{P'}) < \underline{a}\text{-deg}(\mathscr{P}_P) = \underline{\underline{a}\text{-deg}}(\mathscr{P}_P)(\det_P)$.

We can now state the analog of Behrend's theorem for flagged principal bundles:

Theorem 4.3.2. *Let $(\mathscr{P}, \underline{s})$ be a flagged principal G-bundle and $\underline{a}$ an admissible stability parameter. Then, there is a unique reduction of $\mathscr{P}$ to a parabolic subgroup $P \subset G$, satisfying the above conditions $(1')$ and $(2')$. Moreover, this is a canonical reduction of $(\mathscr{P}, \underline{s})$.*

Let us rewrite Behrend's proof in our situation. Since canonical reductions of $\mathscr{P}$ exist, only the uniqueness has to be proved. Thus, fix two parabolic subgroups P and Q of G and let $\mathscr{P}_P$ and $\mathscr{P}_Q$ be reductions of $\mathscr{P}$ to P and Q, respectively. Since any two parabolic subgroups share a maximal torus, we may assume that, locally at the generic point $\eta \in C$, there is a reduction $\mathscr{P}_{T,\eta}$ of $\mathscr{P}$ to a torus $T \subset P \cap Q$, such that $\mathscr{P}_{P,\eta} = \mathscr{P}_{T,\eta} \times^T P$ and $\mathscr{P}_{Q,\eta} = \mathscr{P}_{T,\eta} \times^T Q$ as subbundles of $\mathscr{P}$.
Note further that any reduction of the generic fiber of $\mathscr{P}$ to a parabolic subgroup canonically extends to a reduction of $\mathscr{P}$, so that $\mathscr{P}_P$ and $\mathscr{P}_Q$ are determined by $\mathscr{P}_{P,\eta}$ and $\mathscr{P}_{Q,\eta}$, respectively. We therefore fix a reduction $\mathscr{P}_{T,\eta}$. For any parabolic subgroup $T \subset P \subset G$, this defines a reduction $\mathscr{P}_P$ of $\mathscr{P}$, and we only need to study how the degree of $\mathscr{P}_P$ varies with P. Finally, given a Borel subgroup $T \subset B \subset P$, the parabolic degree $\underline{\underline{a\text{-deg}}}(\mathscr{P}_B)$ determines $\underline{\underline{a\text{-deg}}}(\mathscr{P}_P)$. Thus, like Behrend, we consider these degrees as a map:

$$\begin{array}{rcl} d\colon \left\{\, T \subset B \subset G \,|\, \text{Borel subgroup} \,\right\} & \longrightarrow & X^\star(T)^\vee \\ B & \longmapsto & \underline{\underline{a\text{-deg}}}(\mathscr{P}_B). \end{array}$$

This map is a "complementary polyhedron", i.e., it satisfies:

(P1) If B and B' are two Borel subgroups contained in the parabolic subgroup $P \subset G$, then $d(B)_{|X^\star(P)} = d(B')_{|X^\star(P)}$.

(P2) Let B and B' be two Borel subgroups, such that the simple roots of B are $I_B = \{\alpha, \alpha_1, \dots \alpha_{r-1}\}$ and $\{-\alpha\} = -I_B \cap \Delta_{B'}$. Then, $d(B)(\alpha) + d(B')(-\alpha) \leq 0$.

(P1) is clear, since both sides only depend on the reduction of $\mathscr{P}$ to P.
To see (P2), let L be a Levi subgroup of $P_\alpha := BB'$, and set $L' := P_\alpha/R_u(P_\alpha)Z(L) \cong L/Z(L)$. Then, $\mathscr{L} := \mathscr{P}_\alpha/R_u(P_\alpha)Z(L)$ is the principal L'-bundle obtained from $\mathscr{P}_{P_\alpha}$ by extension of the structure group via $P_\alpha \to L'$, and we may compute $d(B)(\alpha)$ and $d(B')(\alpha)$ from $\mathscr{L}$ and the induced reductions. Thus, by replacing G by L', we may assume that G is semisimple of rank one and that B and B' define reductions $\mathscr{L}_B$ and $\mathscr{L}_{B'}$ of $\mathscr{L}$ which are opposite at the generic point. Denote by $\mathfrak{g}$, $\mathfrak{b}$, and $\mathfrak{b}'$ the Lie algebras of G, B, and B', respectively, and by $\mathfrak{u}_\alpha$ the root space of α.
Since the reductions are opposite in the generic fiber, the composition

$$\mathscr{L}_B \times^B \mathfrak{u}_\alpha \subset \mathscr{L}_B \times^B \mathfrak{g} = \mathscr{L}_{B'} \times^{B'} \mathfrak{g} \to \mathscr{L}_{B'} \times^B \mathfrak{g}/\mathfrak{b}'$$

is non-zero, i.e., there is an injective map of line bundles $\mathscr{L}_B(\alpha) \to \mathscr{L}_{B'}(\alpha)$.
If this map is an isomorphism at x_i, then $\mathscr{L}_B$ and $\mathscr{L}_{B'}$ are opposite in this fiber. In this case, if s_i defines a reduction to either $\mathscr{L}_{B,x_i}$ or $\mathscr{L}_{B',x_i}$, then

$\langle\alpha, a_{s_i}\rangle_{\mathscr{L}_B} + \langle-\alpha, a_{s_i}\rangle_{\mathscr{L}_{B'}} = 0$, and, if the reduction is different from $\mathscr{L}_{B,x_i}$ and $\mathscr{L}_{B',x_i}$, then $\langle\alpha, a_{s_i}\rangle_{\mathscr{L}_B} = \langle-\alpha, a_{s_i}\rangle_{\mathscr{L}_{B'}} \leq 0$. (Note that by our reduction to the case of semisimple rank one, there are only two possible values for the product $\langle .,.\rangle$, by Lemma 4.1.3). If the map is not an isomorphism at x_i, then $\deg(\mathscr{L}_B(\alpha)) \leq \deg(\mathscr{L}_{B'}(\alpha)) - 1$. Thus, our claim follows again, because we have chosen $\underline{a}$ to be admissible, i.e., $2|\langle\alpha, a_i\rangle| < 1$. Altogether, we have established (P2). In the case $G = \mathrm{SL}_3$, the above properties imply that the points $d(B)$ are the corners of a hexagon whose sides are parallel to the coroots. This might motivate the following observation of Behrend. (For any $M \subset X^\star(T)^\vee$, denote by $\mathrm{conv}(M)$ the convex hull of M (in $X^\star(T)^\vee_{\mathbb{R}}$).)

Lemma 4.3.3 ([6], Lemma 2.5). *With the above notation, we have*

$$\mathrm{conv}\big(\{\, d(B) \,|\, T \subset B \,\}\big) = \bigcap_{\substack{P \supset T \\ P \text{ max. par.}}} \Big\{ x \in X^\star(T)^\vee \,|\, x(\det_P) \geq \underline{\underline{a}\text{-}\deg}(\mathscr{P}_P)(\det_P) \Big\}.$$

In particular, if $(\mathscr{P}, \underline{s})$ is semistable, then this convex set contains 0.

Note that, for a maximal parabolic subgroup P, the space $X^\star(P)_{\mathbb{Q}}$ is one dimensional, so that in the above we might replace $\det_P$ by any dominant character $\lambda \in X^\star(P)_{\mathbb{Q}}$.

Proof. Again, given a parabolic subgroup $P \supset T$, denote by Δ_P the set of roots of P and, given a Borel subgroup $B \supset T$, by I_B the set of positive simple roots.
To prove the inclusion "$\subset$", we fix P and show that $d(B)(\det_P) \geq d(P)(\det_P)$. If $B \subset P$, then this holds by definition. Otherwise, let $-\alpha_0 \in I_B \setminus \Delta_P$ be a simple root of B which is not a root of P, so that $\alpha_0 \in \Delta_P$. Let B' be the Borel subgroup that differs from B by α_0, and let P_{α_0} be the parabolic subgroup generated by BB'. If we show that $\det_P = \lambda_{\alpha_0} + m\alpha_0$, with $\lambda_{\alpha_0} \in X^\star(P_{\alpha_0})_{\mathbb{Q}}$ and $m \geq 0$, then, by the properties (P1) and (P2) of d, we see that

$$\begin{aligned} d(B)(\det_P) &= d(B)(\lambda_{\alpha_0}) + md(B)(\alpha_0) \\ &\geq d(B')(\lambda_{\alpha_0}) + md(B')(\alpha_0) = d(B')(\det_P). \end{aligned}$$

Iterating this procedure, we finally arrive at the case $B \subset P$.
Let $(.,.)$ be a W-invariant scalar product on $X^\star(T)_{\mathbb{Q}}$. Define $\alpha_0^\vee$, such that the reflection s_{α_0} is given as $\lambda \mapsto \lambda - (\lambda, \alpha_0^\vee)\alpha_0$. Then, we need to show that $(\det_P, \alpha_0^\vee) \geq 0$. Recall that $\det_P = \sum_{\alpha \in \Delta_P} \alpha$. For a root $\alpha \in \Delta_P$ with $(\alpha, \alpha_0^\vee) < 0$, we know that $s_{\alpha_0}(\alpha) \in \Delta_P$, because $\alpha_0, \alpha \in \Delta_P$, and $(s_{\alpha_0}(\alpha), \alpha_0^\vee) = -(\alpha, \alpha_0^\vee)$. Thus, our assertion is trivial.
To prove the other inclusion, Behrend proceeds by induction on the rank of G. The claim holds, if $X^\star(T)$ is one dimensional. Let $P \supset T$ be a maximal parabolic subgroup with Levi subgroup L. Then, the polyhedron for the associated Levi bundle is given by

$$\begin{aligned} &\mathrm{conv}\big(\{\, d(B) \,|\, T \subset B \subset P \,\}\big) \\ &\quad \subset \big\{\, \varphi \in X^\star(T)^\vee_{\mathbb{Q}} \,|\, \varphi(\det_P) = \underline{a}\text{-}\deg(\mathscr{P}_P)(\det_P) \big\} \cong X^\star(T/Z(L))^\vee_{\mathbb{Q}}. \end{aligned}$$

Now, in the first step of the proof, we have seen that, for any Borel subgroup $B \supset T$, either $d(B)(\det_P) > \underline{a}\text{-}\deg(\mathscr{P}_P))$ or $d(B) = d(B')$ for some Borel subgroup $B' \subset P$. Thus,

$$\begin{aligned}\operatorname{conv}\big(\{\, d(B) \,|\, T \subset B \,\}\big) \cap \{\, \varphi \,|\, \varphi(\det_P) = \underline{a}\text{-}\deg(\mathscr{P}_P) \,\}\\ = \operatorname{conv}\big(\{\, d(B) \,|\, T \subset B \subset P\}\big).\end{aligned}$$

This shows that the $d(B)$ also span the intersection of the halfspaces. □

Again, fix a scalar product $(.,.)$ on $X^\star(T)^\vee_{\mathbb{Q}}$ which is invariant under the action of the Weyl group of G. Then, Behrend's theorem follows immediately from:

Proposition 4.3.4 ([6], Proposition 3.13). *Let $\mathscr{P}_Q$ be a reduction of $\mathscr{P}$ satisfying $(1')$ and $(2')$, and let $\mathscr{P}_{T,\eta}$ be a reduction of $\mathscr{P}_Q$ to T at the generic point of C. Then, $\mathscr{P}_Q$ is also defined as the reduction to the parabolic subgroup associated with the rational one-parameter subgroup of least distance to the origin in* $\operatorname{conv}(\{\, d(B) \,|\, T \subset B \,\})$.

Proof. Again, let $Q \subset G$ be the parabolic subgroup corresponding to the reduction $\mathscr{P}_Q$, and let L be a Levi subgroup of Q. The intersection

$$\bigcap_{\substack{P \supset Q\\ P \text{ max. parabolic}}} \{\, x \in X^\star(T)^\vee_{\mathbb{Q}} \,|\, x(\det_P) = \underline{a}\text{-}\deg(\mathscr{P}_P) \,\} \cap X^\star(Z(L))^\vee_{\mathbb{Q}}$$

contains only one point, call it y_Q. Indeed, $X^\star(Q)_{\mathbb{Q}} \cong X^\star(Z(L))_{\mathbb{Q}}$ and, if $P_i \supset Q$, $i = 1, \ldots, m$, are the maximal parabolic subgroups containing Q, then $(\det_{P_i})_{i=1,\ldots,m}$ is a basis for $X^\star(Q)_{\mathbb{Q}}$.
Claim 1: Under the identification $X^\star(T)^\vee \cong X_\star(T)$, the parabolic subgroup defined by $y_Q \in X_\star(T)$ is Q.
First, $y_Q \in X^\star(Z(L))^\vee$ implies that $y_Q \in X_\star(Z(L))_{\mathbb{Q}}$. Furthermore, since the characters $\det_{P_i}$, $i = 1, \ldots, m$, form a basis of $X^\star(Q)_{\mathbb{Q}}$, we have $y_Q(\det_P) = \underline{\underline{a}\text{-}\deg}(\mathscr{P}_Q)(\det_P)$, for all maximal parabolic subgroups $P \supset Q$. Therefore, property $(2')$ of $\mathscr{P}_Q$ implies that the parabolic subgroup associated with y_Q is Q (compare the comments before Remark 4.1.2).
Claim 2: $y_Q \in \operatorname{conv}(\{\, d(B) \,|\, T \subset B \subset Q \,\}) \subset \operatorname{conv}(\{\, d(B) \,|\, T \subset B \,\})$.
We have the exact sequence

$$X^\star(Z(L))^\vee_{\mathbb{Q}} \longrightarrow X^\star(T)^\vee_{\mathbb{Q}} \xrightarrow{\pi} X^\star(T/Z(L))^\vee,$$

and $\pi(\operatorname{conv}\{\, d(B) \,|\, T \subset B \subset Q \,\})$ is the polyhedron of the Levi bundle $\mathscr{P}_Q/R_u(Q)$, which is semistable by assumption. In particular, $0 \in \pi(\operatorname{conv}\{\, d(B) \,|\, T \subset B \subset Q \,\})$ (Lemma 4.3.3). Thus, $\operatorname{conv}(\{\, d(B) \,|\, T \subset B \subset Q \,\}) \cap X^\star(Z(L))^\vee \neq \varnothing$, and y_Q is the only point that can be contained in this intersection.
Claim 3: Under the identification $X^\star(T)^\vee_{\mathbb{R}} \cong X^\star(T)_{\mathbb{R}}$ given by the W-invariant scalar product $(.,.)$, we have $y_Q = \sum_{i=1}^m n_i \det_{P_i}$ with $n_i > 0$, $i = 1, \ldots, m$.

First, $X^\star(Q)_\mathbb{R} \cong X^\star(Z(L))_\mathbb{R}$ is the intersection of the subspaces invariant under the reflections s_{α_i}, for $\alpha_i \in I_B \setminus I_Q$, i.e., $X^\star(Q)_\mathbb{R} = (\bigoplus_{\alpha_i\in I_B\setminus I_Q} \mathbb{R}\alpha_i)^\perp$. In particular, $X^\star(Z(L))^\vee_\mathbb{R}$ is the subspace that is invariant under the Weyl group W_L of L.
Let $B \subset Q$ be a Borel subgroup, α_i a simple root of B for which $-\alpha_i$ is not a root of Q, and $P_i^{\min}$ the parabolic subgroup obtained from Q by adding the root α_i, $i = 1, ..., m$. Define $\widetilde{\alpha}_i := \det_Q \otimes \det^{-1}_{P_i^{\min}} \in X^\star(Q)$, $i = 1, ..., m$. Then, $\widetilde{\alpha}_i = l\alpha_i + \sum_{\beta\in I_B\setminus I_Q} l_\beta\beta \in X^\star(Q)$ with $l > 0, l_\beta \geq 0$, $i = 1, ..., m$. Therefore, $\widetilde{\alpha}_i$ is the l-fold multiple of the orthogonal projection of α_i to $X^\star(Q)$, $i = 1, ..., m$. Moreover, $\det_{P_i}$ is invariant under the reflection s_α, for $\alpha \in I_B \setminus \{\alpha_i\}$, $i = 1, ..., m$. Since $\widetilde{\alpha}_i$ and $\det_{P_i}$ are both positive linear combinations of the simple roots, we find that $(\det_{P_j}, \widetilde{\alpha}_k) = c_j\delta_{jk}$ with $c_j > 0$, $j, k = 1, ..., m$. Now, $y_{Q|X^\star(Q)} = \underline{a}\text{-}\deg(\mathscr{P}_Q)$ and $\deg(\mathscr{P}_Q)(\widetilde{\alpha}_i) > 0$, $i = 1, ..., m$, because $\mathscr{P}_Q$ satisfies $(2')$. We infer $y_Q = \sum_{i=1}^m n_i \det_{P_i}$ with $n_i > 0$, for $i = 1, ..., m$.
Claim 4: y_Q is the point of least distance to 0 *in* $\operatorname{conv}(d(B))$.
We have seen in Lemma 4.3.3 that

$$\operatorname{conv}\bigl(\{\, d(B) \,|\, T \subset B \,\}\bigr) = \bigcap_{\substack{P\supset T\\ P \text{ max. parabolic}}} \bigl\{\, x \in X^\star(T)^\vee \,|\, x(\det_P) \geq \underline{a}\text{-}\deg(\mathscr{P}_P) \,\bigr\}.$$

Thus, for any $x \in \operatorname{conv}(\{\, d(B) \,|\, T \subset B \,\})$ and any $i \in \{\, 1, ..., m \,\}$, we have $x(\det_{P_i}) > \underline{a}\text{-}\deg(\mathscr{P}_Q)(\det_{P_i}) = y_Q(\det_{P_i})$. Since $y_Q = \sum_{i=1}^m n_i \det_{P_i}$ with $n_i \geq 0$, $i = 1, ..., m$, we see that

$$(x - y_Q, y_Q) = \sum_{i=1}^m n_i\bigl(x(\det_{P_i}) - y_Q(\det_{P_i})\bigr) \geq 0,$$

so that $\|x\| \geq \|y_Q\|$. □

4.4 Semistable Reduction for Flagged Principal Bundles

Following our strategy from [22],[23], we want to prove a semistable reduction theorem for flagged principal bundles.

Theorem 4.4.1. *Let C be a smooth projective curve over the discrete valuation ring R with residue field k. Let $\{\, x_i\colon \operatorname{Spec}(R) \to C \,|\, i = 1, \dots, b \,\}$ be a finite set of disjoint sections, G a semisimple Chevalley group scheme over R, $\underline{P}$ a tuple of parabolic subgroups of G, and $\underline{a}$ an admissible stability parameter.*
Then, for any $\underline{a}$-semistable flagged principal G-bundle $(\mathscr{P}_K, \underline{s}_K)$ over C_K, there is a finite extension $R' \supset R$, such that $(\mathscr{P}_K, \underline{s}_K)$ extends to an $\underline{a}$-semistable flagged principal G-bundle over $C_{R'}$.

Proof. In order to ease notation, we will assume that $P_i = B$, $i = 1, ..., b$, for a fixed Borel subgroup B of G. For our main application, this case is sufficient. The other cases are proved in the same way. Write $S = \{\, x_1, \dots, x_b \,\}$, and consider S as a closed subscheme of C.

First Step: Find an arbitrary extension of $\mathscr{G}_K$ to $C_{R'}$.
We know ([22], First Step) that, after replacing R by a finite extension, we can always extend the principal G-bundle $\mathscr{P}_K$ to a principal bundle $\mathscr{P}_R$ over C_R. The reductions of $\mathscr{P}_{R|S}$ are parameterized by a scheme which is locally (over R) isomorphic to $G/B \times_R S$. Since this scheme is projective over R, the flaggings of $\mathscr{P}_{K|K\times_R S}$ extend uniquely to flaggings s_i of $\mathscr{P}_{R|S}$, $i = 1, ..., b$.
Second Step: Find a modification of $(\mathscr{P}_R, \underline{s})$.
Fix a local parameter $\pi \in R$. Assume that $(\mathscr{P}_k, \underline{s})$ is not semistable. Then, by Theorem 4.3.2, there is a canonical reduction of $\mathscr{P}_k$ to a parabolic subgroup $P \subset G$. Let $T \subset B \cap P$ be a maximal torus of G. The relative position of the reduction to P and to B at x_i is given by an element of $P\backslash G/B \cong W_P\backslash W$, $i = 1, ..., b$. Here, $W = N(T)/T$ is the Weyl group of G, and W_P is the Weyl group of the Levi quotient of P. For $i = 1, \dots, b$, we choose an element $w_i \in N(T)$ which defines the relative position at x_i.
We want to describe $(\mathscr{P}_R, \underline{s})$ by a glueing cocycle. Recall that any $g \in \prod_S G(\!(t)\!)(R)$ defines a principal G-bundle $\mathscr{P}_g$ on C together with a trivialization of the restrictions $\mathscr{P}_{g|C\setminus S}$ and $\mathscr{P}_{g|\widehat{\mathscr{O}}_{C,S}}$. In particular, the latter trivialization also defines flaggings at S.
As in [22], we choose a maximal parabolic subgroup $Q \supset P$. Then, there is a finite, disjoint set of sections U, such that we can find a cocycle $g \in \prod_S G(\!(t)\!)(R) \times \prod_U G(\!(t)\!)(R)$ and $g_0 \in \prod_S G(R)$, satisfying the following:

(1) gg_0 defines $(\mathscr{P}_R, \underline{s})$

(2) $g \mod \pi \in \prod_{S\cup U} P(\!(t)\!)(k)$ defines the canonical reduction of $\mathscr{P}_k$ to P.

(3) $(g_0)_{x_i\in S} \mod \pi = (w_i)_{x_i\in S} \in N(T)(k)$.

(4) Either g satisfies the conditions of [22], Proposition 7, or $g \in \prod_{S\cup U} P(\!(t)\!)(R)$.

(5) If $g \in \prod_{S\cup U} P(\!(t)\!)(R)$, then the maximal N, such that $(g_0)_{x_i\in S} \equiv (w_i)_{x_i\in S} \mod \pi^N$ is finite. Furthermore, $(g_0)_{x_i\in S} \mod \pi^{N+1} \notin \prod_{x_i\in S} Pw_iB$.

For the above cocycle gg_0, choose $z = \pi^{\ell/N}$ with ℓ maximal, such that the cocycle $zgg_0(w^{-1}z^{-1}w) = zgz^{-1}zg_0(w^{-1}z^{-1}w)$ is an $R[\pi^{1/N}]$-valued cocycle. This defines a flagged principal G-bundle $(\mathscr{P}', \underline{s}')$ which is another extension of $(\mathscr{P}_K, \underline{s}_K)$.
Third Step: Show that $(\mathscr{P}'_k, \underline{s}')$ is less unstable.
The Harder–Narasimhan strata (HN-strata) that we shall consider in the following are understood as Harder–Narasimhan strata in the stack $\mathrm{Bun}_{G,\underline{x},\underline{P},\underline{a}}$ of flagged principal G-bundles of type $(\underline{x}, \underline{P})$ with respect to the stability parameter $\underline{a}$.

Lemma 4.4.2. *Let $(\mathscr{P}_\eta, \underline{s})$ be a flagged principal G-bundle which specializes to the flagged principal G-bundle $(\mathscr{P}_0, \underline{s})$, i.e., assume that there is a family of* *flagged principal G-bundles parameterized by the complete discrete valuation ring R with special fiber $(\mathscr{P}_0, \underline{s})$ and generic fiber $(\mathscr{P}_\eta, \underline{s})$. Assume further that $(\mathscr{P}_\eta, \underline{s})$ has a canonical reduction defined over the generic point of R. Then,* $\mathrm{ideg}_{\underline{a}}(\mathscr{P}_\eta, \underline{s}) \leq \mathrm{ideg}_{\underline{a}}(\mathscr{P}_0, \underline{s})$. *If the flagged principal G-bundles $(\mathscr{P}_\eta, \underline{s})$ and $(\mathscr{P}_0, \underline{s})$ do not lie in the same HN-stratum, then* $\mathrm{ideg}_{\underline{a}}(\mathscr{P}_\eta, \underline{s}) < \mathrm{ideg}_{\underline{a}}(\mathscr{P}_0, \underline{s})$.

Proof. Let $\mathscr{P}_{P,\eta}$ denote the canonical reduction of $(\mathscr{P}_\eta, \underline{s})$. This induces a reduction $\mathscr{P}_{0,P}$ of the generic fiber by first extending the reduction to an open subset of the special fiber and then extending this to a reduction over the special fiber. Let us compare the contributions of the flaggings at the point x_i, $i = 1, ..., b$. First assume that the reduction $\mathscr{P}_{P,\eta}$ extends to the special fiber, locally at the point x_i. In this case, this extension coincides with $\mathscr{P}_{0,P}$ and we can apply the semicontinuity argument of Example 4.1.4 to see that the contribution of $\langle \det_P, a_i \rangle$ can at most increase in the special fiber.
In the other case, the reduction $\mathscr{P}_{P,\eta|\{x_i\}}$ can also be extended to a reduction of $\mathscr{P}_{x_i}$. We denote the corresponding reduction by $\mathscr{P}_{P,x_i}$. To this reduction, we can apply the same argument as before to see that the corresponding value of $\langle \det_P, a_i \rangle$ can at most increase in the special fiber.
Finally, let $\mathscr{P}_{\mathfrak{p}}^{\max}$ be the maximal subsheaf of $\mathscr{P} \times^G \mathrm{Lie}(G)$ that extends $\mathscr{P}_{\eta,P} \times^P \mathrm{Lie}(P)$. Then, in the special fiber over x_i, we have

$$\mathscr{P}^{\max}_{\mathfrak{p}|\{\, x_i, 0\,\}} \subset \mathscr{P}_{x_i,P} \times^P \mathrm{Lie}(P) \cap \mathscr{P}_{0,P} \times^P \mathrm{Lie}(P)_{|\{\, x_i, 0\,\}}, \quad i = 1, ..., b.$$

Since $\underline{a}$ is admissible, this implies

$$\begin{aligned}\mathrm{ideg}(\mathscr{P}_\eta, \underline{s}) \leq\ & \underline{a}\text{-}\deg(\mathscr{P}_{0,P}, \underline{s})(\det(P)) - \\ & - \deg\bigl(\mathrm{coker}(\mathscr{P}^{\max}_{\mathfrak{p},0} \to \mathscr{P}_{0,P} \times^P \mathrm{Lie}(P)\bigr) \cdot \\ & \cdot \bigl(1 - 2 \cdot \max\bigl\{\, |\langle \alpha, a_i\rangle| \,|\, \alpha \text{ a root of } G, i = 1, \dots, b \,\bigr\}\bigr) \\ \leq\ & \mathrm{ideg}(\mathscr{P}_0, \underline{s}).\end{aligned}$$

Therefore, we see that either $\mathrm{ideg}(\mathscr{P}_\eta) < \mathrm{ideg}(\mathscr{P}_0)$, or the canonical reduction $\mathscr{P}_P$ defines a reduction of $\mathscr{P}_0$ of the same parabolic degree, which must then be the canonical reduction by Theorem 4.3.2. □

Lemma 4.4.3. *Let $\mathscr{P}$ be a principal G-bundle and $(\mathscr{P}, \underline{s})$ and $(\mathscr{P}, \underline{s}')$ two flaggings of $\mathscr{P}$ of the same type. Let $\mathscr{P}_P$ be the canonical reduction of $(\mathscr{P}, \underline{s})$, and denote by w_i and $w_i' \in P\backslash G/B$ the elements defined by the relative position of the two reductions of $\mathscr{P}_{|\{x_i\}}$ to P and B given by s_i and s_i', respectively, $i = 1, ..., b$. Assume that w_i' specializes to w_i, $i = 1, ..., b$. Then, $(\mathscr{P}, \underline{s}')$ is less unstable than $(\mathscr{P}, \underline{s})$.*

Proof. Since $\underline{s}'$ specializes to $\underline{s}$, we can apply Lemma 4.4.2 to see that $\mathrm{ideg}(\mathscr{P}, \underline{s}) \geq \mathrm{ideg}(\mathscr{P}, \underline{s}')$. Assume that both flagged principal G-bundles lie in the same HN-stratum. Then, the canonical reduction of $(\mathscr{P}, \underline{s}')$ defines another reduction $\mathscr{P}_P'$ of $\mathscr{P}$ to P. Now, we may use Example 4.1.4 to see that the parabolic degree of $(\mathscr{P}_P', \underline{s})$ is bigger than the parabolic degree of $(\mathscr{P}_P, \underline{s}')$, because $w \neq w'$. □

Finally, as in [22], third step, choose a Levi subgroup L of Q, set $\mathscr{P}_Q := \mathscr{P}_P \times^P Q$, and consider the family $\mathscr{Q}_\lambda$ of principal Q-bundles over $C_k \times \mathbb{A}^1$ that is isomorphic to $\mathscr{P}_Q \times \mathbb{G}_m$ over $C \times \mathbb{G}_m$ and such that the fiber over 0 is $\mathscr{P}_Q/R_u(Q) \times^L Q$. Set $\mathscr{P}_\lambda := \mathscr{Q}_\lambda \times^Q G$. Note that the flagging of $\mathscr{P}_k$ induces a flagging for the whole family $(\mathscr{P}_\lambda, \underline{s}_\lambda)$; denote by $(\mathscr{P}_0, \underline{s}_0)$ the fiber over 0 of this family.

Lemma 4.4.4. *The flagged principal G-bundles $(\mathscr{P}_0, \underline{s}_0)$ and $(\mathscr{P}_k, \underline{s})$ lie in the same HN-stratum of* $\mathrm{Bun}_{G,\underline{x},\underline{P},\underline{a}}$.

Proof. The principal P-bundle $\mathscr{P}_P$ also defines a reduction $\mathscr{P}_{0,P}$ of $\mathscr{P}_0$ to P. For this reduction, $\underline{a}\text{-}\underline{\deg}(\mathscr{P}_{0,P}) = \underline{a}\text{-}\underline{\deg}(\mathscr{P}_P)$, because all terms in the definition of the degree depend only on the quotient of $\mathscr{P}_P/R_u(P)$. By Behrend's characterization of the canonical reduction, this implies that $\mathscr{P}_{0,P}$ is the canonical reduction of $\mathscr{P}_0$. □

Corollary 4.4.5. *The flagged principal G-bundle $(\mathscr{P}'_k, \underline{s}')$ is less unstable than $(\mathscr{P}_k, \underline{s})$.*

Proof. As in the case of principal bundles, we only need to compare the HN-strata of $(\mathscr{P}'_k, \underline{s}')$ and $(\mathscr{P}_k, \underline{s})$. If $\mathscr{P}'$ and $\mathscr{P}$ are isomorphic as principal G-bundles (i.e., without flagging), then the cocycle used to define $\mathscr{P}'$ satisfies (5). Then, we know that the element g'_0 specializes to w, in which case Lemma 4.4.3 proves our claim.
Otherwise, we can argue as in the case of principal bundles [23] to see that the reduction of $\mathscr{P}_0$ to Q does not lift to $\mathscr{P}'$. So, again we know that $\mathscr{P}'$ is less unstable. □

As in the case of principal bundles without flaggings, we can now argue as follows: start with an arbitrary unstable extension $(\mathscr{P}, \underline{s})$ of the flagged principal bundle $(\mathscr{P}_K, \underline{s}_K)$. Either the special fiber of $(\mathscr{P}, \underline{s})$ is semistable, or we can find another extension $(\mathscr{P}', \underline{s}')$ which is less unstable. Since the instability degree of $(\mathscr{P}, \underline{s})$ is finite, this process will stop after finitely many iterations. □

5 Construction of the Moduli Spaces

We will now carry out the GIT construction of the moduli spaces of flagged principal G-bundles. The strategy is roughly the same as in the case of principal G-bundles ([36], [38], [15]), i.e., we first introduce flagged pseudo G-bundles whose moduli spaces can be constructed with the help of decorated flagged vector bundles and then explain how we obtain the moduli spaces of flagged principal G-bundles from there. At the end, we will give the full construction of the moduli space of decorated flagged vector bundles, following and generalizing [37].

5.1 Reduction to a Problem for Decorated Vector Bundles

Fix the type $(\underline{x}, \underline{P})$ of the flagging and the semistability parameter $\underline{a}$. We want to adapt the construction of moduli spaces for principal bundles given in [15] to flagged principal G-bundles. Thus, we will fix a faithful representation $\varrho\colon G \longrightarrow \mathrm{SL}(V) \subset \mathrm{GL}(V)$ on a finite dimensional k-vector space V. Given a principal G-bundle $\mathscr{P}$ over C, we write $\mathscr{P}(V)$ or $\mathscr{P}(\varrho)$ for the vector bundle with fiber V that is associated with G via the representation ϱ, $\mathscr{P}_{\mathrm{SL}(V)} := \mathscr{P} \times^G \mathrm{SL}(V)$ for the corresponding principal $\mathrm{SL}(V)$-bundle, and $\mathscr{P}_{\mathrm{GL}(V)} := \mathscr{P} \times^G \mathrm{GL}(V)$ for the associated principal $\mathrm{GL}(V)$-bundle.

ϱ-Flagged Principal G-Bundles. — Let $\underline{P} = (P_1, ..., P_b)$ be a tuple of parabolic subgroups of $\mathrm{GL}(V)$. As before, we fix a tuple $\underline{x} = (x_1, ..., x_b)$ of distinct k-rational points. Then, a *ϱ-flagged principal G-bundle* (*of type* $(\underline{x}, \underline{P})$) is a tuple $(\mathscr{P}, \underline{s})$ that is composed of a principal G-bundle $\mathscr{P}$ and reductions $s_i\colon \{x_i\} \longrightarrow (\mathscr{P}_{\mathrm{GL}(V)} \times_C \{x_i\})/P_i$ of the associated principal $\mathrm{GL}(V)$-bundle at the points x_i, $i = 1, ..., b$. This time, the stability parameter will be a tuple $\underline{a} = (a_1, ..., a_b)$ with $a_i \in X^\star(P_i)^\vee_{\mathbb{Q},+}$, $i = 1, ..., b$.
Before we introduce the correct notion of semistability, we point out that, given a parabolic subgroup Q of G, a dominant character χ on Q, and a_i as above, there is no intrinsic way to define $\langle \chi_{s_i}, a_{s_i} \rangle$ (compare Section 4). Thus, we have to explain how we extend a parabolic subgroup of G and a dominant character on it to a parabolic subgroup of $\mathrm{GL}(V)$ and a dominant character on it. For this, we use the construction introduced in [36] and [15].
Fix a basis for V and let $\widetilde{T} \subset \mathrm{GL}(V)$ be the corresponding maximal torus of diagonal matrices. The basis yields a basis for $X^\star(\widetilde{T})$, i.e., an isomorphism $X^\star(\widetilde{T}) \cong \mathbb{Z}^n$. The symmetric bilinear map $\mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{Z}$, $((b_1, ..., b_n), (b'_1, ..., b'_m)) \longmapsto \sum_{i=1}^n b_i \cdot b'_i$ induces the symmetric bilinear map $(.,.)\colon X_\star(\widetilde{T}) \times X_\star(\widetilde{T}) \longrightarrow \mathbb{Z}$. Let $(.,.)_{\mathbb{K}}\colon X_{\star,\mathbb{K}}(\widetilde{T}) \times X_{\star,\mathbb{K}}(\widetilde{T}) \longrightarrow \mathbb{K}$ be its $\mathbb{K}$-bilinear extension to the vector space $X_{\star,\mathbb{K}}(\widetilde{T}) := X_\star(\widetilde{T}) \otimes_{\mathbb{Z}} \mathbb{K}$, $\mathbb{K} = \mathbb{Q}, \mathbb{R}$. Since the pairing $(.,.)$ is invariant under the Weyl group, it induces similar pairings on the character and cocharacter groups of any other maximal torus $\widetilde{T}' \subset \mathrm{GL}(V)$.
On the other hand, given a one-parameter subgroup $\lambda \in X_\star(\widetilde{T})$ and a character $\widetilde{\chi} \in X^\star(\widetilde{T})$, the composition $\widetilde{\chi} \circ \lambda\colon \mathbb{G}_m(k) \longrightarrow \mathbb{G}_m(k)$ is of the form $z \longmapsto z^{\langle \widetilde{\chi}, \lambda \rangle}$ and gives the duality pairing $\langle .,. \rangle\colon X^\star(\widetilde{T}) \times X_\star(\widetilde{T}) \longrightarrow \mathbb{Z}$. We let $\langle .,. \rangle_{\mathbb{K}}\colon X^\star_{\mathbb{K}}(\widetilde{T}) \times X_{\star,\mathbb{K}}(\widetilde{T}) \longrightarrow \mathbb{K}$, $\mathbb{K} = \mathbb{Q}, \mathbb{R}$, $X^\star_{\mathbb{K}}(\widetilde{T}) := X^\star(\widetilde{T}) \otimes_{\mathbb{Z}} \mathbb{K}$, be its extensions. Thus, any rational one-parameter subgroup $\lambda \in X_{\star,\mathbb{Q}}(\widetilde{T})$ gives rise to a character $\widetilde{\chi}_\lambda \in X^\star_{\mathbb{Q}}(\widetilde{T})$ defined by

$$(\lambda, \lambda')_{\mathbb{Q}} = \langle \widetilde{\chi}_\lambda, \lambda' \rangle_{\mathbb{Q}}, \quad \forall \lambda' \in X_{\star,\mathbb{Q}}(\widetilde{T}).$$

One checks that $\widetilde{\chi}_\lambda$ comes from a character of $Q := Q_{\mathrm{GL}(V)}(\lambda)$ that depends only on the conjugacy class of λ within Q. If the weighted flag of λ is, for example, $(\{0\} \subsetneq U \subsetneq V, (1))$, then

$$\begin{array}{rcl} \widetilde{\chi}_\lambda\colon Q_{\mathrm{GL}(V)}(\lambda) & \longrightarrow & \mathbb{G}_m(k) \\ \left(\begin{array}{c|c} g & \star \\ \hline 0 & h \end{array}\right) & \longmapsto & \det(g)^{\dim(U)-\dim(V)} \cdot \det(h)^{\dim(U)}. \end{array} \tag{1}$$

If $T \subset G$ is a maximal torus, then we may extend it to a maximal torus $\widetilde{T}$ of $\mathrm{GL}(V)$. The scalar product on $X^\star_{\mathbb{K}}(\widetilde{T})$ that we have obtained before restricts to a scalar product on $X^\star_{\mathbb{K}}(T)$. Lemma 2.8 in Chapter II of [32] implies that the scalar product on $X^\star_{\mathbb{K}}(T)$ thus obtained does not depend on the choice of the extending torus $\widetilde{T}$. Furthermore, it is invariant under the Weyl group $\mathscr{N}(T)/T$.
If $\lambda\colon \mathbb{G}_m(k) \longrightarrow G$ is a one-parameter subgroup, then we associate with it the parabolic subgroup $Q_G(\lambda)$, the anti-dominant character χ_λ, and the dominant character $\chi_{-\lambda} = -\chi_\lambda$. Likewise, we have $Q_{\mathrm{GL}(V)}(\lambda)$, the anti-dominant character $\widetilde{\chi}_\lambda$, and the dominant character $\widetilde{\chi}_{-\lambda} = -\widetilde{\chi}_\lambda$. Note that $Q_G(\lambda) = Q_{\mathrm{GL}(V)}(\lambda) \cap G$ and $\widetilde{\chi}_{\pm\lambda|Q_G(\lambda)} = \chi_{\pm\lambda}$.

Proposition 5.1.1. *The assignment $\lambda \longmapsto (Q_G(\lambda), \chi_{-\lambda})$ ($\lambda \longmapsto (Q_G(\lambda), \chi_\lambda)$) is a surjection from the set of one-parameter subgroups of G onto the set of pairs consisting of a parabolic subgroup of G and a dominant (anti-dominant) character on that parabolic subgroup.*

Proof. See Section 3.2 of [15]. □

We say that a ϱ-flagged principal G-bundle $(\mathscr{P}, \underline{s})$ is *$\underline{a}$-(semi)stable*, if, for every one-parameter subgroup $\lambda\colon \mathbb{G}_m(k) \longrightarrow G$ and every reduction of $\mathscr{P}$ to the parabolic subgroup $Q := Q_G(\lambda)$, the inequality

$$\deg\bigl(\mathscr{P}_Q(\chi_{-\lambda})\bigr) + \sum_{i=1}^{b} \bigl\langle (\widetilde{\chi}_{-\lambda})_{s_i}, a_{s_i} \bigr\rangle (\le) 0$$

holds true.

Associated Flagged Principal Bundles and Semistability. — Now, we return to the situation where we are given a type $(\underline{x}, \underline{P})$ with $\underline{x}$ as usual and $\underline{P} = (P_1, ..., P_b)$ a tuple of parabolic subgroups of G and a stability parameter $\underline{a} = (a_1, ..., a_b)$ with $a_i \in X^\star(P_i)^\vee_{\mathbb{Q},+}$, $i = 1, ..., b$. As we have explained in Section 4, we may view a_i as a rational one-parameter subgroup of G with $P_G(a_i) = P_i$, $i = 1, ..., b$. We set $\varrho_\star(\underline{P}) := (\widetilde{P}_1, ..., \widetilde{P}_b) := (P_{\mathrm{GL}(V)}(a_1), ..., P_{\mathrm{GL}(V)}(a_b))$ and $\varrho_\star(\underline{a}) := (\widetilde{a}_1, ..., \widetilde{a}_b) := (\varrho \circ a_1, ..., \varrho \circ a_s)$. Next note that any flagged principal G-bundle $(\mathscr{P}, \underline{s})$ of type $(\underline{x}, \underline{P})$ defines the ϱ-flagged principal $\mathrm{GL}(V)$-bundle $(\mathscr{P}_{\mathrm{GL}(V)}, \varrho_\star(\underline{s}))$, $\varrho_\star(\underline{s}) = (\widetilde{s}_1, ..., \widetilde{s}_b)$, with

$$\widetilde{s}_i\colon \{x_i\} \xrightarrow{s_i} \mathscr{P}_{|\{x_i\}}/P_i \hookrightarrow \mathscr{P}_{\mathrm{GL}(V)|\{x_i\}}/\widetilde{P}_i, \quad i = 1, ..., b.$$

Lemma 5.1.2. *A flagged principal G-bundle $(\mathscr{P},\underline{s})$ of type $(\underline{x},\underline{P})$ is $\underline{a}$-(semi)stable, if and only if the associated ϱ-flagged principal* $\mathrm{GL}(V)$*-bundle* $(\mathscr{P}_{\mathrm{GL}(V)},\varrho_\star(\underline{s}))$ *of type* $(\underline{x},\varrho_\star(\underline{P}))$ *is* $\varrho_\star(\underline{a})$*-(semi)stable.*

Proof. By Proposition 5.1.1, $(\mathscr{P},\underline{s})$ is $\underline{a}$-(semi)stable, if and only if, for every one-parameter subgroup $\lambda\colon \mathbb{G}_m(k)\longrightarrow G$ and every reduction $\mathscr{P}_Q$ of $\mathscr{P}$ to $Q:=Q_G(\lambda)$, one has

$$\deg\bigl(\mathscr{P}_Q(\chi_{-\lambda})\bigr)+\sum_{i=1}^{b}\bigl\langle(\chi_{-\lambda})_{s_i},a_{s_i}\bigr\rangle(\leq)0.$$

Our contention therefore reduces to the trivial fact $\langle(\chi_{-\lambda})_{s_i},a_{s_i}\rangle = \langle(\widetilde{\chi}_{-\lambda})_{s_i},\widetilde{a}_{s_i}\rangle$, $i=1,...,b$. $\square$

Another Formulation of Semistability for ϱ-Flagged Principal Bundles. — Before we may introduce even more general objects, we have to reformulate the notion of $\underline{a}$-(semi)stability. The first trivial reformulation is that we may say that $(\mathscr{P},\underline{s})$ is $\underline{a}$-(semi)stable, if, for every one-parameter subgroup $\lambda\colon \mathbb{G}_m(k)\longrightarrow G$ and every reduction of $\mathscr{P}$ to the parabolic subgroup $Q:=Q_G(\lambda)$, the inequality

$$\deg\bigl(\mathscr{P}_Q(\chi_{\lambda})\bigr)+\sum_{i=1}^{b}\bigl\langle(\widetilde{\chi}_{\lambda})_{s_i},a_{s_i}\bigr\rangle(\geq)0$$

holds true.
Next, assume we are given a principal G-bundle $\mathscr{P}$, a one-parameter subgroup $\lambda\colon \mathbb{G}_m(k)\longrightarrow G$ with weighted flag

$$\bigl(V_\bullet(\lambda),\beta_\bullet(\lambda)\bigr)=\bigl(\{0\}\subsetneq V_1\subsetneq\cdots\subsetneq V_t\subsetneq V,(\beta_1,...,\beta_t)\bigr),$$

and a reduction $\mathscr{P}_Q$ of $\mathscr{P}$ to $Q:=Q_G(\lambda)$. Then, we obtain an induced reduction $\mathscr{P}_{Q_{\mathrm{GL}(V)}(\lambda)}$ of the principal $\mathrm{GL}(V)$-bundle $\mathscr{P}_{\mathrm{GL}(V)}$ to $Q_{\mathrm{GL}(V)}(\lambda)$. The datum of that reduction is equivalent to the datum of a filtration

$$E_\bullet(\mathscr{P}_Q):\{0\}\subsetneq E_1\subsetneq\cdots\subsetneq E_t\subsetneq E\quad\text{with }\mathrm{rk}(E_i)=\dim(V_i),\ i=1,...,t.$$

Using (1), one easily computes

$$\deg\bigl(\mathscr{P}_Q(\chi_\lambda)\bigr)=\sum_{i=1}^{t}\beta_i\cdot\bigl(\deg(E)\cdot\mathrm{rk}(E_i)-\deg(E_i)\cdot\mathrm{rk}(E)\bigr).\qquad(2)$$

Note that a parabolic subgroup of $\mathrm{GL}(V)$ is the stabilizer of a flag in V. Thus, the tuple $\underline{P}$ of parabolic subgroups of $\mathrm{GL}(V)$ gives quotients $V\longrightarrow W_{ij}$, and subspaces $V_{ij}:=\ker(V\longrightarrow W_{ij})$, $j=1,...,t_i$, $i=1,...,b$, such that $V_{ij}\subsetneq V_{ij+1}$, $j=1,...,t_i-1$, $i=1,...,b$.

Next, let $\lambda\colon \mathbb{G}_m(k) \longrightarrow G$ be a one-parameter subgroup with weighted flag

$$\big(V_\bullet(\lambda),\beta_\bullet(\lambda)\big)=\big(\{0\}\subsetneq V_1'\subsetneq\cdots\subsetneq V_t'\subsetneq V,(\beta_1,...,\beta_t)\big)$$

and a a rational one-parameter subgroup of G with weighted flag

$$\big(V_\bullet(a),\beta_\bullet(a)\big)=\left(\{0\}\subsetneq V_1\subsetneq\cdots\subsetneq V_\tau\subsetneq V,\frac{1}{\dim(V)}\cdot(a_1,...,a_\tau)\right).$$

In addition, define

$$Q_h:=V/V_h,\quad R_{ih}:=V_i'/(V_i'\cap V_h),\quad r_{ih}:=\dim(R_{ih}),\quad i=1,...,t,h=1,...,\tau,$$

$r:=\dim(V)$. We claim that

$$\langle\widetilde{\chi}_\lambda,a\rangle=\sum_{i=1}^{t}\Big(\beta_i\cdot\sum_{h=1}^{\tau}a_h\cdot\big(r\cdot r_{ih}-r_j\cdot\dim(Q_h)\big)\Big).\tag{3}$$

By bilinearity, this has to be checked only for $\tau=t=1$, $\beta_1=1$, and $a_1=r$. In this case, it follows easily from the definitions and (1).

Finally, suppose we are given a stability parameter $\underline{a}=(a_1,...,a_b)$ with $a_i\in X^\star(P_i)^\vee_{\mathbb{Q},+}$, $i=1,...,b$. Then, we write $\beta_\bullet(a_i)=:(1/r)\cdot(a_{i1},...,a_{it_i})$, $i=1,...,b$. The parabolic subgroups $P_1,...,P_b$ define quotients $V\longrightarrow W_{ij}$, and we set $r_{ij}:=\dim(W_{ij})$, $j=1,...,t_i$, $i=1,...,b$. Suppose that $(\mathscr{P},\underline{s})$ is a ϱ-flagged principal G-bundle of type $(\underline{x},\underline{P})$. Then, we have the associated vector bundle E, and the reductions s_i define quotients $q_{ij}\colon E_i\longrightarrow Q_{ij}$ with $\dim(Q_{ij})=r_{ij}$, $j=1,...,t_i$, $i=1,...,b$. For any subbundle $\{0\}\subsetneq F\subsetneq E$, we set

$$\underline{a}\text{-deg}(F):=\deg(F)-\sum_{i=1}^{b}\sum_{j=1}^{t_i}a_{ij}\cdot\dim\big(q_{ij}(F)\big).$$

Putting (2) and (3) together, we infer the following characterization of semistability.

Proposition 5.1.3. *The ϱ-flagged principal G-bundle is $\underline{a}$-(semi)stable, if and only if, for every one-parameter subgroup λ of G and every reduction $\mathscr{P}_Q$ of $\mathscr{P}$ to $Q:=Q_G(\lambda)$, the inequality*

$$\sum_{i=1}^{t}\beta_i\big(\underline{a}\text{-deg}(E)\cdot\mathrm{rk}(E_i)-\underline{a}\text{-deg}(E_i)\cdot\mathrm{rk}(E)\big)(\geq)0$$

is verified. Here,

$$E_\bullet\big(\mathscr{P}_Q\big)=\{0\}\subsetneq E_1\subsetneq\cdots\subsetneq E_t\subsetneq E\quad\textit{and}\quad\beta_\bullet(\lambda)=(\beta_1,...,\beta_t).$$

Reminder on Pseudo G-Bundles. — Following the general strategy from [36] and [15], we will first embed principal G-bundles into pseudo G-bundles which in turn can be embedded into decorated vector bundles for which we finally can do the GIT-calculations. We have already chosen to view principal G-bundles as principal $\mathrm{GL}(V)$-bundles together with a reduction to G, i.e., as pairs $(\mathscr{P}, \sigma)$ that consist of a principal $\mathrm{GL}(V)$-bundle $\mathscr{P}$ and a section $\sigma\colon C \longrightarrow \mathscr{P}/G$. Given such a pair $(\mathscr{P}, \sigma)$, let E be the corresponding vector bundle. Then,

$$\mathscr{P} = \mathscr{I}som(V \otimes \mathscr{O}_C, E) \subset \mathscr{H}om(V \otimes \mathscr{O}_C, E) = \mathrm{Spec}\big(\mathscr{S}ym^\star(V \otimes E^\vee)\big).$$

Moreover, the good quotient $\mathscr{H}om(V \otimes \mathscr{O}_C, E)/\!\!/G = \mathrm{Spec}(\mathscr{S}ym^\star(V \otimes E)^G)$ exists and there is the open embedding

$$\mathscr{I}som(V \otimes \mathscr{O}_C, E)/G \subset \mathscr{H}om(V \otimes \mathscr{O}_C, E)/\!\!/G.$$

Thus, σ is given by a non-trivial homomorphism $\tau\colon \mathscr{S}ym^\star(V \otimes E^\vee)^G \longrightarrow \mathscr{O}_C$. This suggests the following definition: a *pseudo G-bundle* (E, τ) consists of a vector bundle E with trivial determinant $\det(E) \cong \mathscr{O}_C$ and a non-trivial homomorphism $\tau\colon \mathscr{S}ym^\star(V \otimes E^\vee)^G \longrightarrow \mathscr{O}_C$ of $\mathscr{O}_C$-algebras. Not any homomorphism τ gives rise to a principal G-bundle, but the following result ([36], Corollary 3.4) gives an important characterization when it does.

Lemma 5.1.4. *Let (E, τ) be a pseudo G-bundle with associated section $\sigma\colon C \longrightarrow \mathscr{H}om(V \otimes \mathscr{O}_C, E)/\!\!/G$. Then, (E, τ) is a principal G-bundle, if and only if there exists a point $x \in C$, such that*

$$\sigma(x) \in \mathrm{Isom}(V, E_{|\{x\}})/G.$$

For our purposes, we therefore look at the following objects: a *ϱ-flagged pseudo G-bundle* $(E, \tau, \underline{q})$ is a pseudo G-bundle (E, τ) together with quotients

$$q_{ij}\colon E_{|\{x_i\}} \longrightarrow Q_{ij}$$

onto k-vector spaces Q_{ij}, $j = 1, ..., t_i$, $i = 1, ..., b$, such that

$$\ker(q_{ij}) \subseteq \ker(q_{ij+1}), \quad j = 1, ..., t_i - 1, i = 1, ..., b. \tag{4}$$

The tuple $(\underline{x}, \underline{r})$ with $\underline{r} = (r_{ij} := \dim(Q_{ij}), j = 1, ..., t_i, i = 1, ..., s)$ will be referred to as the *type of the flagging.* There is an obvious notion of *isomorphism of ϱ-flagged pseudo G-bundles.*

The algebra $\mathscr{S}ym^\star(V \otimes E^\vee)^G$ is finitely generated, so that the morphism τ is determined, for $s \gg 0$, by its restriction $\tau_{\le s}\colon \bigoplus_{i=1}^s \mathscr{S}ym^i(V \otimes E^\vee)^G \longrightarrow \mathscr{O}_C$. In particular, ϱ-flagged pseudo-G-bundles form an algebraic stack, locally of finite type. Lemma 5.1.4 implies that the stack of ϱ-flagged G-bundles is an open substack of the stack of ϱ-flagged pseudo G-bundles. Following [15], we choose $s \gg 0$, such that

a) $\mathrm{Sym}^\star(V\otimes k^{r\vee})^G$ is generated by elements in degree $\leq s$.

b) $\mathrm{Sym}^{(s!)}(V\otimes k^{r\vee})^G$ is generated by elements in degree 1, i.e., by the elements in the vector space $\mathrm{Sym}^{s!}(V\otimes k^{r\vee})^G$.

Set

$$\mathbb{V}_s(E):=\bigoplus_{\substack{(d_1,\dots,d_s):\\ d_i\geq 0,\sum id_i=s!}}\Big(\mathrm{Sym}^{d_1}\big((V\otimes E^\vee)^G\big)\otimes\cdots\otimes\mathrm{Sym}^{d_s}\big(\mathrm{Sym}^s(V\otimes E^\vee)^G\big)\Big).$$

Then, τ induces morphisms

$$\tau^{s!}\colon \mathscr{S}ym^{s!}(V\otimes E^\vee)^G\longrightarrow \mathscr{O}_C$$

and

$$\varphi\colon \mathbb{V}_s(E)\twoheadrightarrow \mathscr{S}ym^{s!}(V\otimes E^\vee)^G\longrightarrow \mathscr{O}_C.$$

Homogeneous Representations. — Instead of the representation $\mathbb{V}_s$, we can also allow a more general class of representations without complicating the arguments. This might be useful for other applications, too. A representation $\kappa\colon \mathrm{GL}_r(k)\longrightarrow \mathrm{GL}(U)$ is called a *polynomial representation*, if it extends to a (multiplicative) map $\overline{\kappa}\colon M_r(k)\longrightarrow \mathrm{End}(U)$. We say that κ is *homogeneous of degree* $u\in\mathbb{Z}$, if

$$\kappa\big(z\cdot\mathbb{E}_r\big)=z^u\cdot\mathrm{id}_U,\quad \forall z\in\mathbb{G}_m(k).$$

Let $P(r,u)$ be the abelian category of homogeneous polynomial representations of $\mathrm{GL}_r(k)$ of degree u. It comes with the duality functor

$$\begin{array}{rccc} {}^\star\colon & P(r,u) & \longrightarrow & P(r,u)\\ & \kappa & \longmapsto & \big(\kappa\circ\mathrm{id}^\vee_{\mathrm{GL}_r(k)}\big)^\vee. \end{array}$$

Here, $.^\vee$ stands for the corresponding dual representation. An example for a representation in $P(r,u)$ is the *u-th divided power* $(\mathrm{Sym}^u(\mathrm{id}_{\mathrm{GL}_r(k)}))^\star$, i.e., the representation of $\mathrm{GL}_r(k)$ on

$$D^u(W):=\big(\mathrm{Sym}^u(W^\vee)\big)^\vee,\quad W:=k^r.$$

More generally, we look, for $u,v>0$, at the $\mathrm{GL}_r(k)$-module

$$\mathbb{D}^{u,v}(W):=\bigoplus_{\substack{(u_1,\dots,u_v):\\ u_i\geq 0,\sum_{i=1}^v u_i=u}}\big(D^{u_1}(W)\otimes\cdots\otimes D^{u_v}(W)\big). \tag{5}$$

Lemma 5.1.5. *Let $\kappa\colon \mathrm{GL}_r(k)\longrightarrow\mathrm{GL}(U)$ be a homogeneous polynomial representation of degree u. Then, there exists an integer $v>0$, such that U is a quotient of the $\mathrm{GL}(U)$-module $\mathbb{D}^{u,v}(W)$. If κ is homogeneous, but not polynomial, then it is a quotient of $\mathbb{D}^{u,v}(W)\otimes(\bigwedge^r W)^{\otimes -w}$ for some $w>0$.*

Proof. The proof of Proposition 5.3 in [27] shows that any representation $\kappa'\colon \mathrm{GL}_r(k) \longrightarrow \mathrm{GL}(U')$ in $P(r,u)$ is, for suitable $v > 0$, a sub-representation of the representation of $\mathrm{GL}_r(k)$ on the vector space

$$\bigoplus_{\substack{(u_1,\dots,u_v):\\ u_i\ge 0, \sum_{i=1}^v u_i = u}} \big(\mathrm{Sym}^{u_1}(W)\otimes\cdots\otimes \mathrm{Sym}^{u_v}(W)\big).$$

Applying this result to the dual $\kappa^\star\colon \mathrm{GL}_r(k) \longrightarrow \mathrm{GL}(U^\star)$ of κ proves the first assertion.
The second assertion follows from the obvious fact that $U\otimes(\bigwedge^r W)^{\otimes w}$ will be a polynomial representation for large w. □

Remark 5.1.6. As is apparent from the construction in [27], the above result also holds over the ring of integers.

Fix natural numbers u, v and let A be any vector bundle on the curve C, that is, we do not assume A to have rank r. Then, we set for $w \ge 0$

$$\mathbb{D}^{u,v}(A) := \bigoplus_{\substack{(u_1,\dots,u_v):\\ u_i\ge 0, \sum_{i=1}^v u_i = u}} \big(D^{u_1}(A)\otimes\cdots\otimes D^{u_v}(A)\big), \quad D^w(A) := \big(\mathscr{S}\!ym^w(A^\vee)\big)^\vee.$$

Remark 5.1.7. Any surjective homomorphism $\psi\colon A \longrightarrow B$ between vector bundles induces a surjective homomorphism

$$\mathbb{D}^{u,v}(\psi)\colon \mathbb{D}^{u,v}(A) \longrightarrow \mathbb{D}^{u,v}(B).$$

Decorated Flagged Vector Bundles. — Now, fix a line bundle L on C. A *decorated flagged vector bundle of type* $(r,d,\underline{x},\underline{r},u,v,L)$ is a tuple $(E,\underline{q},\varphi)$ which consists of a vector bundle E on C of rank r and degree d, a non-trivial homomorphism

$$\varphi\colon \mathbb{D}^{u,v}(E) \longrightarrow L,$$

and a flagging $\underline{q} = (q_{ij}\colon E_{|\{x_i\}} \longrightarrow Q_{ij}, j = 1,\dots,t_i, i = 1,\dots,s)$ of type $\underline{r} = (r_{ij}, j = 1,\dots,t_i, i = 1,\dots,s)$. The moduli functors for the objects we have considered, so far, are straightforward to define (just form the isomorphism classes in the corresponding stack). For decorated flagged vector bundles, this is slightly more delicate. Thus, we give the definition. A *family of decorated flagged vector bundles of type* $(r,d,\underline{x},\underline{r},u,v,L)$ (*parameterized by the scheme* S) is a tuple $(E_S,\underline{q}_S,\mathscr{N}_S,\varphi_S)$ which consists of a vector bundle E_S of rank r on $S\times C$ and fiberwise of degree d, a tuple $\underline{q}_S = (q_{S,ij}\colon E_{S|S\times\{x_i\}} \longrightarrow Q_{S,ij})$ of surjections onto vector bundles $Q_{S,ij}$ of rank r_{ij}, $j = 1,\dots,t_i$, $i = 1,\dots,b$, subject to the conditions in (4), a line bundle $\mathscr{N}_S$ on S, and a homomorphism $\varphi_S\colon \mathbb{D}^{u,v}(E_S) \longrightarrow \pi_C^\star(L)\otimes\pi_S^\star(\mathscr{N}_S)$ which is non-trivial on every fiber $\{s\}\times C$. Two such families $(E_S,\underline{q}_S,\mathscr{N}_S,\varphi_S)$ and $(E'_S,\underline{q}'_S,\mathscr{N}'_S,\varphi'_S)$ are said to be *isomorphic*, if there exist isomorphisms $\psi_S\colon E_S \longrightarrow E'_S$ and $\chi_S\colon \mathscr{N}_S \longrightarrow \mathscr{N}'_S$ fulfilling

$$q_{S,ij} = q'_{S,ij}\circ\psi_{S|S\times\{x_i\}}, \quad j = 1,\dots,t_i,\ i = 1,\dots,b,$$

and
$$\varphi_S = \big(\mathrm{id}_{\pi_C^\star(L)} \otimes \pi_S^\star(\chi_S)\big)^{-1} \circ \varphi'_S \circ \mathbb{D}^{u,v}(\psi_S).$$
Thus, we may form the functor that assigns to every scheme the set of isomorphism classes of families of decorated vector bundles of type $(r, d, \underline{x}, \underline{r}, u, v, L)$ parameterized by S.

By Lemma 5.1.5, the representation $\mathbb{V}_s$ can be written as the quotient of $\mathbb{D}^{u,v}(W) \otimes (\bigwedge^r W)^{\otimes -s!}$. Now, suppose we are given a vector bundle E_S on $S \times C$, a homomorphism $\tau_S \colon \mathscr{S}ym^\star(V \otimes E_S^\vee)^G \longrightarrow \mathscr{O}_{S\times C}$ and a flagging $\underline{q}_S$ of type $(\underline{x}, \underline{r})$ of E_S. Then, the determinant of E_S is isomorphic to the pullback of a line bundle $\mathscr{D}_S$ on S. Choose an isomorphism $\det(E_S) \cong \pi_S^\star(\mathscr{D}_S)$, and set $\mathscr{N}_S := \mathscr{D}^{\otimes s!}$, so that τ_S gives rise to
$$\mathbb{D}^{u,v}(E_S) \otimes \mathscr{N}_S^\vee \twoheadrightarrow \mathbb{V}_s(E_S) \twoheadrightarrow \mathscr{S}ym^{s!}(V \otimes E_S^\vee)^G \longrightarrow \mathscr{O}_{S\times C}.$$
Thus, we obtain the family $(E_S, \underline{q}_S, \mathscr{N}_S, \varphi_S)$ of decorated flagged vector bundles. Its isomorphism class does not depend on the choice of the isomorphism $\det(E_S) \cong \pi_S^\star(\mathscr{D}_S)$, so that this construction gives rise to a natural transformation of functors.

Lemma 5.1.8. *The above natural transformation applied to $S = \mathrm{Spec}(K)$, K an algebraically closed extension of k, is injective.*

Proof. The proof is the same as the one of Lemma 5.1.1 in [15]. □

We now come to the definition of semistability. Fix the stability parameter $\underline{a}$ for the flagging. Here, we view $\underline{a} = (a_{ij}, j = 1, ..., t_i, i = 1, ..., s)$ as a tuple of rational numbers, and we assume that

- $a_{ij} > 0$, $j = 1, ..., t_i$, $i = 1, ..., s$;
- $\sum_{j=1}^{t_i} a_{ij} < 1$, $i = 1, ..., s$.

Then, given a decorated flagged vector bundle $(E, \underline{q}, \varphi)$ and a weighted filtration $(E_\bullet, \beta_\bullet)$ of E, we define
$$M_{\underline{a}}(E_\bullet, \beta_\bullet) := \sum_{j=1}^{t} \beta_j \cdot \big(\underline{a}\text{-}\deg(E) \cdot \mathrm{rk}(E_j) - \underline{a}\text{-}\deg(E_j) \cdot \mathrm{rk}(E)\big).$$
The quantity $\mu(E_\bullet, \beta_\bullet; \varphi)$ is obtained as follows. Let η be the generic point of the curve C and let $\mathbb{E}$ stand for the restriction of E to $\{\eta\}$. Then, the restricted homomorphism $\varphi_{|\{\eta\}}$ gives a point
$$\sigma_\eta \in \mathbb{P}\big(\mathbb{D}^{u,v}(\mathbb{E})\big).$$
We may choose a one-parameter subgroup $\lambda_K \colon \mathbb{G}_m(K) \longrightarrow \mathrm{SL}(\mathbb{E})$, $K := k(C)$, whose weighted flag agrees with the restriction of $(E_\bullet, \beta_\bullet)$ to $\{\eta\}$ and define
$$\mu\big(E_\bullet, \beta_\bullet; \varphi\big) := \mu(\lambda_K, \sigma_\eta).$$
This does not depend on the choice of λ_K.

Remark 5.1.9. By construction, the vector bundle $\mathbb{D}^{u,v}(E)$ is a subbundle of $(E^{\otimes u})^{\oplus N}$ for some $N > 0$. Set $E_{t+1} := E$ and, for $(i_1, ..., i_u) \in \{ 1, ..., t+1 \}^{\times u}$,

$$E_{i_1} \star \cdots \star E_{i_u} := \big(E_{i_1} \otimes \cdots \otimes E_{i_u}\big)^{\oplus N} \cap \mathbb{D}^{u,v}(\mathbb{E}).$$

For a weighted filtration $(E_\bullet, \beta_\bullet)$ of the vector bundle E, we define the *associated (integral) weight vector*

$$\big(\underbrace{\gamma_1, \dots, \gamma_1}_{(\operatorname{rk} E_1)\times}, \underbrace{\gamma_2, \dots, \gamma_2}_{(\operatorname{rk} E_2 - \operatorname{rk} E_1)\times}, \dots, \underbrace{\gamma_{t+1}, \dots, \gamma_{t+1}}_{(\operatorname{rk} E - \operatorname{rk} E_t)\times}\big) := \sum_{l=1}^{t} \beta_l \cdot \gamma_r^{(\operatorname{rk} E_l)}. \tag{6}$$

(Note that we recover $\beta_l = (\gamma_{l+1} - \gamma_l)/r$, $l = 1, ..., t$.)
With these concepts, one readily verifies

$$\mu\big(E_\bullet, \beta_\bullet; \varphi\big) = \\ -\min\Big\{ \gamma_{i_1} + \cdots + \gamma_{i_u} \,\big|\, (i_1, ..., i_u) \in \{ 1, ..., t+1 \}^{\times u} : \varphi_{|(E_{i_1} \star \cdots \star E_{i_u})} \not\equiv 0 \Big\}.$$

To define semistability, we also fix a positive rational number δ. Then, we say that a decorated flagged vector bundle is $(\underline{a}, \delta)$-*(semi)stable*, if the inequality

$$M_{\underline{a}}(E_\bullet, \beta_\bullet) + \delta \cdot \mu\big(E_\bullet, \beta_\bullet; \varphi\big)(\geq)0$$

holds for any weighted filtration $(E_\bullet, \beta_\bullet)$ of E.

Boundedness. — The starting point for the GIT construction is the boundedness of the family of $(\underline{a}, \delta)$-semistable decorated flagged vector bundles of type $(r, d, \underline{x}, \underline{r}, u, v, L)$. This property is a consequence of the following statement.

Proposition 5.1.10. *Fix the type $(r, d, \underline{x}, \underline{r}, u, v, L)$ and the stability parameter δ. Then, there is a positive constant D_0, such that, given a tuple $\underline{a} = (a_{ij}, j = 1, ..., t_i, 1 = 1, ..., s)$ of positive rational numbers with $\sum_{j=1}^{t_i} a_{ij} < 1$ for $i = 1, ..., s$ and an $(\underline{a}, \delta)$-semistable decorated flagged vector bundle $(E, \underline{q}, \varphi)$ of type $(r, d, \underline{x}, \underline{r}, u, v, L)$, one finds*

$$\mu_{\max}(E) \leq \frac{d}{r} + D_0.$$

Proof. Let $(F, \underline{\widetilde{q}})$ be any vector bundle with a flagging of type $\underline{r}$. Setting $R := \max\{ r_{ij} \,|\, j = 1, ..., t_i, i = 1, ..., s \}$, we derive, for $\underline{a}$ as in the proposition, the obvious estimate

$$\deg(F) \geq \underline{a}\text{-}\deg(F) \geq \deg(F) - s \cdot R.$$

Now, let $(E, \underline{q}, \varphi)$ be as above and $0 \subsetneq F \subsetneq E$ a subbundle. For the weighted filtration $(E_\bullet : 0 \subsetneq F \subsetneq E, \beta_\bullet = (1))$, one checks

$$\mu(E_\bullet, \beta_\bullet; \varphi) \leq u \cdot (r-1).$$

Together with these two estimates, the condition of $(\underline{a},\delta)$-semistability implies

$$
\begin{aligned}
& d\cdot \mathrm{rk}(F) - \deg(F)\cdot r + s\cdot r\cdot R + \delta\cdot u\cdot (r-1)\\
&\quad \geq \underline{a}\text{-}\deg(E)\cdot \mathrm{rk}(F) - \underline{a}\text{-}\deg(F)\cdot r + \delta\cdot \mu(E_\bullet,\beta_\bullet;\varphi) \geq 0.
\end{aligned}
$$

We transform this into the inequality

$$
\mu(F) \leq \frac{d}{r} + \frac{s\cdot r\cdot R + \delta\cdot u\cdot (r-1)}{\mathrm{rk}(F)\cdot r} \leq \frac{d}{r} + \underbrace{s\cdot R + \frac{\delta\cdot u\cdot (r-1)}{r}}_{=:D_0}.
$$

This is the assertion we made. □

5.2 The Moduli Space of Decorated Flagged Vector Bundles

Suppose we are given a constant D. Then, we let $\mathfrak{S}$ be the bounded family of isomorphism classes of vector bundles of rank r and degree d with $\mu_{\max}(E) \leq D$. We also fix an ample line bundle $\mathscr{O}_C(1)$ of degree one on C, a natural number n, such that $E(n)$ is globally generated and $H^1(E(n)) = \{0\}$ for any vector bundle E, such that $[E]\in \mathfrak{S}$, as well as a vector space Y of dimension $d + r(n+1-g)$. Now, a *quotient family of decorated flagged vector bundles of type* $(r,d,\underline{x},\underline{r},u,v,L)$ (*parameterized by the scheme* S) is a tuple $(k_S,\underline{q}_S,\mathscr{N}_S,\varphi_S)$ which consists of a quotient $k_S\colon Y\otimes \pi_C^\star(\mathscr{O}_C(n)) \longrightarrow E_S$, a tuple $\underline{q}_S = (q_{S,ij}\colon E_{S|S\times\{x_i\}} \longrightarrow Q_{S,ij}, j=1,...,t_i, i=1,...,b)$, a line bundle $\mathscr{N}_S$ on S, and a homomorphism $\varphi_S\colon \mathbb{D}^{u,v}(E_S)\longrightarrow \pi_C^\star(L)\otimes\pi_S^\star(\mathscr{N}_S)$ with the following properties:

- E_S is a vector bundle on $S\times C$, such that $[E_{S|\{s\}\times C}]\in\mathfrak{S}$, for every $s\in S(k)$,
- $\pi_{S\star}(k_S\otimes \mathrm{id}_{\pi_C^\star(\mathscr{O}_C(n))})\colon Y\otimes\mathscr{O}_S \longrightarrow \pi_{S\star}(E_S\otimes\pi_C^\star(\mathscr{O}_C(n)))$ is an isomorphism,
- $\underline{q}_S$ consists of surjections onto vector bundles $Q_{S,ij}$ of rank r_{ij}, $j=1,...,t_i$, $i=1,...,b$, subject to the conditions in (4), and
- φ_S is non-trivial on every fiber $\{s\}\times C$.

Two such families $(k_S,\underline{q}_S,\mathscr{N}_S,\varphi_S)$ and $(k'_S,\underline{q}'_S,\mathscr{N}'_S,\varphi'_S)$ are said to be *isomorphic*, if there exist isomorphisms $\psi_S\colon E_S\longrightarrow E'_S$ and $\chi_S\colon\mathscr{N}_S\longrightarrow\mathscr{N}'_S$, fulfilling

$$
k_S = k'_S\circ\psi_S,\ q_{S,ij} = q'_{S,ij}\circ\psi_{S|S\times\{x_i\}},\ j=1,...,t_i, i=1,...,b,
$$

$$
\varphi_S = \big(\mathrm{id}_{\pi_C^\star(L)}\otimes\pi_S^\star(\chi_S)\big)^{-1}\circ\varphi'_S\circ\mathbb{D}^{u,v}(\psi_S).
$$

Suppose we are also given stability parameters $\underline{a}$ and δ as above. Then, we take $D = D_0$ from Proposition 5.1.10. The first step toward the construction of the moduli spaces is the construction of a suitable parameter space:

Proposition 5.2.1. *Fix the input data* $(r, d, \underline{x}, \underline{r}, u, v, L)$, *and let* D_0 *be a before. Then, the functor that assigns to a scheme* S *the set of isomorphism classes of quotient families of type* $(r, d, \underline{x}, \underline{r}, u, v, L)$ *is representable by a quasi-projective scheme* $\mathfrak{P}$.

By its universal property, the parameter scheme $\mathfrak{P}$ comes with a natural action of $\mathrm{GL}(Y)$. The next theorem is the main GIT-result that we will prove.

Theorem 5.2.2. i) *There are open subschemes* $\mathfrak{P}^{(\underline{a},\delta)\text{-(s)s}}$ *whose* k*-rational points are the classes of tuples* $(q\colon Y \otimes \mathscr{O}_C(-n) \longrightarrow E, \underline{q}, \varphi)$, *such that* $(E, \underline{q}, \varphi)$ *is an* $(\underline{a}, \delta)$*-(semi)stable decorated flagged vector bundle of type* $(r, d, \underline{x}, \underline{r}, u, v, L)$.
ii) *The good quotient*

$$\mathscr{M}(r, d, \underline{x}, \underline{r}, u, v, L)^{(\underline{a},\delta)\text{-ss}} := \mathfrak{P}^{(\underline{a},\delta)\text{-ss}} /\!\!/ \mathrm{GL}(Y)$$

exists as a projective scheme over $\mathrm{Spec}(k)$, *and the geometric quotient*

$$\mathscr{M}(r, d, \underline{x}, \underline{r}, u, v, L)^{(\underline{a},\delta)\text{-s}} := \mathfrak{P}^{(\underline{a},\delta)\text{-s}} / \mathrm{GL}(Y)$$

as an open subscheme of $\mathscr{M}(r, d, \underline{x}, \underline{r}, u, v, L)^{(\underline{a},\delta)\text{-ss}}$.

Let $\mathrm{M}(r, d, \underline{x}, \underline{r}, u, v, L)^{(\underline{a},\delta)\text{-(s)s}}$ stand for the functor that associates with a scheme S the set of isomorphism classes of families of $(\underline{a}, \delta)$-(semi)stable decorated flagged vector bundles of type $(r, d, \underline{x}, \underline{r}, u, v, L)$ parameterized by S. We infer from the above theorem:

Corollary 5.2.3. *The scheme* $\mathscr{M}(r, d, \underline{x}, \underline{r}, u, v, L)^{(\underline{a},\delta)\text{-(s)s}}$ *is the coarse moduli scheme for the functor* $\mathrm{M}(r, d, \underline{x}, \underline{r}, u, v, L)^{(\underline{a},\delta)\text{-(s)s}}$.

Remark 5.2.4. The divided powers are clearly defined over the integers. Therefore, the above theorem also works in the relative setting, i.e., for a curve $\mathscr{C} \longrightarrow \mathrm{Spec}(R)$, possessing a section. The justification has already been given in Remark 3.2.4.

Now that we have stated our main result on the moduli spaces of decorated flagged vector bundles and have explained how we get from flagged principal G-bundles to decorated flagged vector bundles, we must next show how to work our way back from the above theorem to get moduli spaces of flagged principal G-bundles. This will be the content of the next sections.

5.3 The Moduli Space for ϱ-Flagged Pseudo G-Bundles

Let D, $\mathfrak{S}$, n, and Y be as above. A *quotient family of* ϱ*-flagged pseudo* G*-bundles of type* $(\underline{x}, \underline{r})$ (*parameterized by the scheme* S) is a tuple $(k_S, \tau_S, \underline{q}_S)$ which is composed of a quotient $k_S\colon Y \otimes \pi_C^\star(\mathscr{O}_C(n)) \longrightarrow E_S$, a homomorphism $\tau_S\colon \mathscr{S}ym^\star(V \otimes E_S^\vee) \longrightarrow \mathscr{O}_{S\times C}$, and a tuple $\underline{q}_S = (q_{S,ij}\colon E_{S|S\times\{x_i\}} \longrightarrow Q_{S,ij}, j = 1, ..., t_i, i = 1, ..., b)$, such that

- E_S is a vector bundle on $S \times C$, such that $[E_{S|\{s\}\times C}] \in \mathfrak{S}$, for every $s \in S(k)$,
- $\pi_{S\star}(k_S \otimes \mathrm{id}_{\pi_C^\star(\mathscr{O}_C(n))})\colon Y \otimes \mathscr{O}_S \longrightarrow \pi_{S\star}(E_S \otimes \pi_C^\star(\mathscr{O}_C(n)))$ is an isomorphism, and
- τ_S is non-trivial on every fiber $\{s\} \times C$.

For these quotient families, we have an obvious notion of *isomorphism*.

Proposition 5.3.1. *Fix the input data D and $(\underline{x}, \underline{r})$. The functor that assigns to a scheme S the set of isomorphism classes of quotient families of ϱ-flagged pseudo G-bundles of type $(\underline{x}, \underline{r})$ is representable by a quasi-projective scheme $\mathfrak{F}_{\varrho\text{-FLPsBun}}$.*
Let $\mathfrak{Q}$ be the quasi-projective scheme that parameterizes quotients $q\colon Y \otimes \mathscr{O}_C(-n) \longrightarrow E$, such that $[E] \in \mathfrak{S}$ and $H^0(q(n))$ is an isomorphism. The natural morphism $\mathfrak{F}_{\varrho\text{-FLPsBun}} \longrightarrow \mathfrak{Q}$ induces a projective morphism $\mathfrak{F}_{\varrho\text{-FLPsBun}}/\!\!/\mathbb{G}_m(k) \longrightarrow \mathfrak{Q}$. (Here, the $\mathbb{G}_m(k)$-action comes from the embedding of $\mathbb{G}_m(k)$ into $\mathrm{GL}(Y)$ as the group of homotheties and the natural $\mathrm{GL}(Y)$-action on $\mathfrak{F}_{\varrho\text{-FLPsBun}}$.)

Fix stability parameters $\underline{a}$ and δ as before. We say that a ϱ-flagged pseudo G-bundle $(E, \tau, \underline{q})$ is $(\underline{a}, \delta)$-*(semi)stable*, if the associated decorated flagged vector bundle $(E, \underline{q}, \varphi)$ is so. Given the type $(\underline{x}, \underline{r})$, we define the moduli functor $\mathrm{M}(\varrho, \underline{x}, \underline{r})^{(\underline{a},\delta)\text{-(s)s}}$ as the functor that assigns to a scheme S the isomorphism classes of $(\underline{a}, \delta)$-(semi)stable ϱ-flagged pseudo G-bundles parameterized by S. In order to obtain the moduli spaces, we proceed as follows.
The natural transformation from the functor of isomorphism classes of families of ϱ-flagged pseudo G-bundles into the functor of decorated flagged vector bundles gives rise to the $\mathrm{GL}(Y)$-equivariant morphism

$$\begin{tikzcd} \mathrm{AD}\colon \mathfrak{F}_{\varrho\text{-FLPsBun}} \arrow[rr] \arrow[dr] & & \mathfrak{P} . \arrow[dl] \\ & \mathfrak{Q} & \end{tikzcd}$$

The subgroup $\mathbb{G}_m(k) = \mathbb{G}_m(k) \cdot \mathrm{id}_V$ acts trivially on $\mathfrak{P}$ and $\mathfrak{Q}$, so that AD induces the $\mathrm{SL}(Y)$-equivariant morphism

$$\begin{tikzcd} \overline{\mathrm{AD}}\colon \mathfrak{F}_{\varrho\text{-FLPsBun}}/\!\!/\mathbb{G}_m(k) \arrow[rr] \arrow[dr] & & \mathfrak{P} . \arrow[dl] \\ & \mathfrak{Q} & \end{tikzcd}$$

By Proposition 5.3.1, the scheme $\mathfrak{F}_{\varrho\text{-FLPsBun}}/\!\!/\mathbb{G}_m(k)$ is proper over $\mathfrak{Q}$, so that $\overline{\mathrm{AD}}$ is a proper morphism. According to Lemma 5.1.8, it is also an injective map. Altogether, we realize that $\overline{\mathrm{AD}}$ is a finite map.

Theorem 5.2.2 claims that there are the $\mathrm{SL}(Y)$-invariant open subsets $\mathfrak{P}^{(\underline{a},\delta)\text{-(s)s}}$ that correspond to the $(\underline{a},\delta)$-(semi)stable decorated flagged vector bundles. By definition,

$$\mathfrak{F}^{(\underline{a},\delta)\text{-(s)s}}_{\varrho\text{-FLPsBun}} := \mathrm{AD}^{-1}\big(\mathfrak{P}^{(\underline{a},\delta)\text{-(s)s}}\big)$$

is set the of $(\underline{a},\delta)$-(semi)stable ϱ-flagged pseudo G-bundles, and we find

$$\mathfrak{F}^{(\underline{a},\delta)\text{-(s)s}}_{\varrho\text{-FLPsBun}}/\!\!/\mathbb{G}_m(k) = \overline{\mathrm{AD}}^{-1}\big(\mathfrak{P}^{(\underline{a},\delta)\text{-(s)s}}\big).$$

We have seen that the good quotient $\mathfrak{P}^{(\underline{a},\delta)\text{-ss}}/\!\!/\,\mathrm{SL}(Y)$ exists as a projective scheme that contains the geometric quotient $\mathfrak{P}^{(\underline{a},\delta)\text{-s}}/\,\mathrm{SL}(Y)$ as an open subscheme. Since $\overline{\mathrm{AD}}$ is finite, the quotients

$$\mathscr{M}(\varrho,\underline{x},\underline{r})^{(\underline{a},\delta)\text{-(s)s}} := \big(\mathfrak{F}^{(\underline{a},\delta)\text{-(s)s}}_{\varrho\text{-FLPsBun}}/\!\!/\mathbb{G}_m(k)\big)/\!\!/\,\mathrm{SL}(Y)$$

also exist. The scheme $\mathscr{M}(\varrho,\underline{x},\underline{r})^{(\underline{a},\delta)\text{-ss}}$ is a projective good quotient and $\mathscr{M}(\varrho,\underline{x},\underline{r})^{(\underline{a},\delta)\text{-s}}$, an open subscheme of $\mathscr{M}^{(\underline{a},\delta)\text{-ss}}(\varrho,\underline{r})$, is a geometric quotient. Since

$$\begin{aligned}\big(\mathfrak{F}^{(\underline{a},\delta)\text{-(s)s}}_{\varrho\text{-FLPsBun}}/\!\!/\mathbb{G}_m(k)\big)/\!\!/\,\mathrm{SL}(Y) &= \mathfrak{F}^{(\underline{a},\delta)\text{-(s)s}}_{\varrho\text{-FLPsBun}}/\!\!/\big(\mathbb{G}_m(k)\times\mathrm{SL}(Y)\big)\\ &= \mathfrak{F}^{(\underline{a},\delta)\text{-(s)s}}_{\varrho\text{-FLPsBun}}/\!\!/\,\mathrm{GL}(Y),\end{aligned}$$

the scheme $\mathscr{M}(\varrho,\underline{x},\underline{r})^{(\underline{a},\delta)\text{-ss}}$ is the moduli space we were striving at. (More details on the above arguments may be found in the paper [15].) This construction implies the following result.

Theorem 5.3.2. *The coarse moduli spaces* $\mathscr{M}(\varrho,\underline{x},\underline{r})^{(\underline{a},\delta)\text{-(s)s}}$ *for the functors* $\mathrm{M}(\varrho,\underline{x},\underline{r})^{(\underline{a},\delta)\text{-(s)s}}$ *exist, the scheme* $\mathscr{M}(\varrho,\underline{x},\underline{r})^{(\underline{a},\delta)\text{-ss}}$ *being projective.*

Remark 5.3.3. The construction of this moduli space does not immediately generalize to curves over a base ring. Let us explain the remedy.
We assume that G and the representation $\varrho\colon G \longrightarrow \mathrm{GL}(V_{\mathbb{Z}})$ are defined over the integers. By Seshadri's generalization of GIT relative to base varieties which are defined over Nagata rings [39], the algebra

$$\mathrm{Sym}^\star(V_{\mathbb{Z}}\otimes\mathbb{Z}^r)^G$$

is a finitely generated $\mathbb{Z}$-algebra, and we have the good quotients

$$\pi\colon \mathrm{Hom}(V_{\mathbb{Z}},\mathbb{Z}^r) \longrightarrow \mathrm{Hom}(V_{\mathbb{Z}},\mathbb{Z}^r)/\!\!/G := \mathrm{Spec}\big(\mathrm{Sym}^\star(V_{\mathbb{Z}}\otimes\mathbb{Z}^r)^G\big) \longrightarrow \mathrm{Spec}(\mathbb{Z})$$

and

$$\begin{array}{c}\overline{\pi}\colon \mathbb{P}\big(\mathrm{Hom}(V_{\mathbb{Z}},\mathbb{Z}^r)^\vee\big) \dashrightarrow \mathbb{P}\big(\mathrm{Hom}(V_{\mathbb{Z}},\mathbb{Z}^r)^\vee\big)/\!\!/G := \mathrm{Proj}\big(\mathrm{Sym}^\star(V_{\mathbb{Z}}\otimes\mathbb{Z}^r)^G\big)\\ \hfill\big\downarrow\\ \hfill\mathrm{Spec}(\mathbb{Z}).\end{array}$$

The quotient

$$\pi^0\colon \mathrm{Isom}(V_{\mathbb{Z}}, \mathbb{Z}^r) \longrightarrow \mathrm{Isom}(V_{\mathbb{Z}}, \mathbb{Z}^r)/G$$

is a principal G-bundle and thus a universal categorical quotient. However, the quotients π and $\overline{\pi}$ are not necessarily universal categorical quotients. This fact accounts for the slight modifications which we do have to make. The good quotient parameterizes orbits of geometric points with respect to the equivalence relation that two points map to the same point in the quotient, if and only if the closures of their orbits intersect. This implies the following.

Lemma 5.3.4. *Let $Z \hookrightarrow \mathrm{Spec}(\mathbb{Z})$ be a closed subscheme. Then, the canonical morphisms*

$$\big(\mathrm{Hom}(V_{\mathbb{Z}}, \mathbb{Z}^r) \times_{\mathrm{Spec}(\mathbb{Z})} Z\big)/\!\!/G \longrightarrow \big(\mathrm{Hom}(V_{\mathbb{Z}}, \mathbb{Z}^r)/\!\!/G\big) \times_{\mathrm{Spec}(\mathbb{Z})} Z$$

and

$$\Big(\mathbb{P}\big(\mathrm{Hom}(V_{\mathbb{Z}}, \mathbb{Z}^r)^\vee\big) \times_{\mathrm{Spec}(\mathbb{Z})} Z\Big)/\!\!/G \longrightarrow \Big(\mathbb{P}\big(\mathrm{Hom}(V_{\mathbb{Z}}, \mathbb{Z}^r)^\vee\big)/\!\!/G\Big) \times_{\mathrm{Spec}(\mathbb{Z})} Z$$

are bijective on geometric points.

Let us write

$$\big(\mathrm{Hom}(V_{\mathbb{Z}}, \mathbb{Z}^r) \times_{\mathrm{Spec}(\mathbb{Z})} Z\big)\widetilde{/\!\!/}G := \big(\mathrm{Hom}(V_{\mathbb{Z}}, \mathbb{Z}^r)/\!\!/G\big) \times_{\mathrm{Spec}(\mathbb{Z})} Z$$

and

$$\Big(\mathbb{P}\big(\mathrm{Hom}(V_{\mathbb{Z}}, \mathbb{Z}^r)^\vee\big) \times_{\mathrm{Spec}(\mathbb{Z})} Z\Big)\widetilde{/\!\!/}G := \Big(\mathbb{P}\big(\mathrm{Hom}(V_{\mathbb{Z}}, \mathbb{Z}^r)^\vee\big))/\!\!/G\Big) \times_{\mathrm{Spec}(\mathbb{Z})} Z.$$

Next, assume that E is a locally free sheaf on the scheme Y which is of finite type over $\mathrm{Spec}(R)$, R a Nagata ring. Then, we may easily construct the geometric quotient

$$\begin{aligned}&\widetilde{\mathscr{H}} := \mathscr{H}\!om(V \otimes \mathscr{O}_Y, E)\widetilde{/\!\!/}G := \\ &\Big(\mathscr{I}\!som(R^r \otimes \mathscr{O}_Y, E) \times_{\mathrm{Spec}(R)} \big(\mathrm{Hom}(V_{\mathbb{Z}}, \mathbb{Z}^r) \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(R)\big)\widetilde{/\!\!/}G\Big)/\mathrm{GL}_r(R),\end{aligned}$$

using local trivializations. The construction of $\mathscr{H}\!om(V \otimes \mathscr{O}_Y, E)\widetilde{/\!\!/}G$ clearly commutes with base changes $Y' \longrightarrow Y$. Moreover, we have the natural morphism

$$\mathscr{H}\!om(V \otimes \mathscr{O}_Y, E)/\!\!/G \longrightarrow \mathscr{H}\!om(V \otimes \mathscr{O}_Y, E)\widetilde{/\!\!/}G$$

which is bijective, by Lemma 5.3.4. This construction has an algebraic counterpart. Define $\widetilde{\pi}\colon \widetilde{\mathscr{H}} \longrightarrow Y$ as the projection map, and let

$$\widetilde{\mathscr{S}\!ym}^\star(E^\vee \otimes V)^G$$

be the sheaf $\widetilde{\pi}_\star(\mathscr{O}_{\widetilde{\mathscr{H}}})$. Then, we obtain the homomorphism

$$\mathrm{ps}(E)\colon \widetilde{\mathscr{S}\!ym}^\star(E^\vee \otimes V)^G \longrightarrow \mathscr{S}\!ym^\star(E^\vee \otimes V)^G$$

that induces the bijective map $\mathscr{H}om(V\otimes\mathscr{O}_Y,E)/\!\!/G\longrightarrow\widetilde{\mathscr{H}}$.
Now, assume that $\mathscr{C}\longrightarrow \operatorname{Spec}(R)$ is a curve over the Nagata ring R and that $S\longrightarrow\operatorname{Spec}(R)$ is a scheme of finite type over R. Then, a *family of weak pseudo G-bundles on $\mathscr{C}$ parameterized by S*, is a pair $(E_S,\widetilde{\tau_S})$ that consists of a locally free sheaf E_S of rank $\dim(V)$ on $S\times_{\operatorname{Spec}(R)}\mathscr{C}$, such that $\det(E_S)$ is a pullback from S, and a homomorphism

$$\widetilde{\tau}\colon \widetilde{\mathscr{S}ym}^{\star}(E_S^{\vee}\otimes V)^G\longrightarrow\mathscr{O}_{S\times_{\operatorname{Spec}(R)}\mathscr{C}}$$

whose fibers over S are non-trivial. Unlike the pseudo G-bundles that we had considered before, there is a pull-back for weak pseudo G-bundles, so that there are reasonable stacks and moduli functors for them. In the same manner, we can define *ϱ-flagged weak pseudo G-bundles* and families of such.
Next, suppose that the algebra $\operatorname{Sym}^\star(V_{\mathbb{Z}}\otimes\mathbb{Z}^r)^G$ is generated in degrees $\le s$. By Remark 5.1.6, we may write

$$\bigoplus_{\substack{(d_1,\dots,d_s):\\ d_i\ge 0,\ \sum id_i=s!}}\operatorname{Sym}^{d_1}\big((V_{\mathbb{Z}}\otimes\mathbb{Z}^r)^G\big)\otimes\cdots\otimes\operatorname{Sym}^{d_s}\big(\operatorname{Sym}^i(V_{\mathbb{Z}}\otimes\mathbb{Z}^r)^G\big)$$

as the quotient of $\mathbb{D}^{s!,v}(\mathbb{Z}^r)$, for an appropriate integer $v>0$. As before, we may therefore associate with a family of ϱ-flagged weak pseudo G-bundles a family of ϱ-flagged decorated vector bundles.
We also point out the following result:

Lemma 5.3.5. *Let G be a reductive algebraic group, X and Y projective schemes equipped with a G-action, and $\pi\colon X\longrightarrow Y$ a finite and G-equivariant morphism. Suppose $\mathscr{L}$ is a G-linearized ample line bundle on Y. Then, for any point $x\in X$ and any one-parameter subgroup $\lambda\colon\mathbb{G}_m\longrightarrow G$, one has*

$$\mu_{\pi^\star(\mathscr{L})}(\lambda,x)=\mu_{\mathscr{L}}\big(\lambda,\pi(x)\big).$$

Proof. This is Lemma 2.1 in [38] and also holds in positive characteristic: simply replace the G-module splitting by a splitting of the induced $\mathbb{G}_m$-module. □

In particular, we may apply this lemma to the finite morphism

$$\pi\colon\mathbb{P}\big(\operatorname{Hom}(V,k^r)^\vee\big)/\!\!/G\longrightarrow\mathbb{P}\big(\operatorname{Hom}(V,k^r)^\vee\big)\widetilde{/\!\!/}G.$$

(Note that the ample line bundle $\mathscr{N}$ on the left hand space with which we compute the μ-function is indeed the pullback of the ample line bundle $\mathscr{L}$ on the right hand space with respect to which we compute the μ-function. Indeed, for $r\gg 0$, $\mathscr{N}$ is constructed from the invariant global sections in $\mathscr{O}_{\mathbb{P}(\operatorname{Hom}(V,k^r)^\vee)}(r)$ whereas $\mathscr{L}$ is constructed from those invariant sections that extend to $\mathbb{P}(\operatorname{Hom}(V_{\mathbb{Z}},\mathbb{Z}^r)^\vee)$.) The lemma therefore shows that, if we use the above new construction to associate with a principal G-bundle $\mathscr{P}=(E,\tau)$ a decorated vector bundle (E,φ), we may still characterize those weighted

filtrations $(E_\bullet, \beta_\bullet)$ that arise from reductions of $\mathscr{P}$ to one-parameter subgroups of G by the condition "$\mu(E_\bullet, \beta_\bullet; \varphi) = 0$", as in [15], Lemma 5.4.2.
These considerations clearly show that the moduli spaces of ϱ-flagged weak pseudo G-bundles on $\mathscr{C}$ may be constructed from the moduli spaces of ϱ-flagged decorated vector bundles in the same way as before.
Note that, for ϱ-flagged principal G-bundles, nothing changes, because $\mathscr{I}som(V \otimes \mathscr{O}_Y, E)/G$ is still an open subscheme of $\mathscr{H}om(V \otimes \mathscr{O}_Y, E)\widetilde{/\!\!/}G$.

S-equivalence. — As usual, the points in the moduli space will be in one to one correspondence to the S-equivalence classes of $(\underline{a}, \delta)$-semistable pseudo G-bundles. So, in order to identify the closed points of $\mathscr{M}(\varrho, \underline{x}, \underline{r})^{(\underline{a},\delta)\text{-ss}}$, we have to explain this equivalence relation.
Suppose that $(E, \tau, \underline{q})$ is an $(\underline{a}, \delta)$-semistable ϱ-flagged pseudo G-bundle with associated decorated flagged vector bundle $(E, \underline{q}, \varphi)$ and that $(E_\bullet, \beta_\bullet)$ is a weighted filtration of E with

$$M_{\underline{a}}(E_\bullet, \beta_\bullet) + \delta \cdot \mu\bigl(E_\bullet, \beta_\bullet; \varphi\bigr) = 0.$$

We first define the *associated admissible deformation* $\mathrm{df}_{(E_\bullet,\beta_\bullet)}(E, \tau, \underline{q}) = (E_{\mathrm{df}}, \tau_{\mathrm{df}}, \underline{q}_{\mathrm{df}})$. We set $E_{\mathrm{df}} = \bigoplus_{i=0}^t E_{i+1}/E_i$. Let $\lambda\colon \mathbb{G}_m(k) \longrightarrow \mathrm{SL}_r(k)$ be a one-parameter subgroup whose weighted flag $(W_\bullet(\lambda), \beta_\bullet(\lambda))$ in k^r satisfies:

- $\dim(W_i) = \mathrm{rk}(E_i)$, $i = 1, ..., t$, in $W_\bullet(\lambda) : 0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_t \subsetneq k^r$;
- $\beta_\bullet(\lambda) = \beta_\bullet$.

Then, the given filtration $E_\bullet$ corresponds to a reduction of the structure group of $\mathscr{I}som(\mathscr{O}_C^{\oplus r}, E)$ to $Q(\lambda)$. On the other hand, λ defines a decomposition

$$\mathrm{Sym}^\star\bigl(V \otimes (k^r)^\vee\bigr)^G = \bigoplus_{i \in \mathbb{Z}} U^i,$$

U^i being the eigenspace to the character $z \longmapsto z^i$, $i \in \mathbb{Z}$. With $U_i := \bigoplus_{j \le i} U^i$, we define the filtration

$$\cdots \subset U_{i-1} \subset U_i \subset U_{i+1} \subset \cdots \subset \mathrm{Sym}^\star\bigl(V \otimes (k^r)^\vee\bigr)^G. \tag{7}$$

Observe that $Q(\lambda)$ fixes this filtration. Thus, we obtain a $Q(\lambda)$-module structure on

$$\bigoplus_{i \in \mathbb{Z}} U_i/U_{i-1} \cong \mathrm{Sym}^\star\bigl(V \otimes (k^r)^\vee\bigr)^G. \tag{8}$$

Next, we write $Q(\lambda) = \mathscr{R}_u(Q(\lambda)) \rtimes L(\lambda)$ where $L(\lambda) \cong \mathrm{GL}(W_1/W_0) \times \cdots \times \mathrm{GL}(k^r/W_t)$ is the centralizer of λ. Note that (8) is an isomorphism of $L(\lambda)$-modules. The process of passing from E to E_{df} corresponds to first reducing the structure group to $Q(\lambda)$, then extending it to $L(\lambda)$ via

$Q(\lambda) \longrightarrow Q(\lambda)/\mathscr{R}_u(Q(\lambda)) \cong L(\lambda)$, and then extending it to $\mathrm{GL}_r(k)$ via the inclusion $L(\lambda) \subset \mathrm{GL}_r(k)$. By (7), there is a filtration

$$\cdots \subset \mathscr{U}_{i-1} \subset \mathscr{U}_i \subset \mathscr{U}_{i+1} \subset \cdots \subset \mathscr{S}ym^\star(V \otimes E^\vee)^G,$$

and, by (8), we have a canonical isomorphism

$$\mathscr{S}ym^\star(V \otimes E_{\mathrm{df}}^\vee)^G \cong \bigoplus_{i\in\mathbb{Z}} \mathscr{U}_i/\mathscr{U}_{i-1}. \tag{9}$$

Observe that the modules $\mathscr{U}_i$ and $\mathscr{U}_i/\mathscr{U}_{i-1}$, $i \in \mathbb{Z}$, are graded by the degree in the algebra $\mathscr{S}ym^\star(V \otimes E^\vee)^G$, so that the algebra in (9) is in fact bigraded. We now look at the subalgebra $\mathscr{S}_\mu$ consisting of the components of bidegree (d, i) where either $d = 0$ or $d > 0$ and

$$\frac{i}{d} = \frac{1}{s!} \cdot \mu(E_\bullet, \beta_\bullet; \varphi).$$

Then, τ clearly induces a non-trivial homomorphism τ_μ on $\mathscr{S}_\mu$, and we define τ_{df} as τ_μ on $\mathscr{S}_\mu$ and as zero on the other components. The flagging $\underline{q}_{\mathrm{df}}$ of E_{df} is obtained by a similar procedure.

Remark 5.3.6. If $(E, \tau, \underline{q})$ is a ϱ-flagged *principal* G-bundle and $\delta \gg 0$, the arguments of [15], proof of Theorem 5.4.1, show that admissible deformations are associated with weighted filtrations $(E_\bullet, \beta_\bullet)$, such that $M_{\underline{a}}(E_\bullet, \beta_\bullet) = 0$ and $\mu(E_\bullet, \beta_\bullet; \varphi) = 0$. In that case, $\mathscr{S}_0 = \mathscr{U}_0$. Recall that $\mu(E_\bullet, \beta_\bullet; \varphi) = 0$ means that $(E_\bullet, \beta_\bullet)$ comes from a reduction of $\mathscr{P} = (E, \tau)$ to a parabolic subgroup ([15], Lemma 5.4.2).

A ϱ-flagged pseudo G-bundle $(E, \tau, \underline{q})$ is said to be $(\underline{a}, \delta)$-*polystable*, if it is $(\underline{a}, \delta)$-semistable and equivalent to every admissible deformation $\mathrm{df}_{(E_\bullet,\beta_\bullet)}(E, \tau, \underline{q}) = (E_{\mathrm{df}}, \tau_{\mathrm{df}}, \underline{q}_{\mathrm{df}})$ associated with a filtration $(E_\bullet, \beta_\bullet)$ of E with $M_{\underline{a}}(E_\bullet, \beta_\bullet) + \delta \cdot \mu\big(E_\bullet, \beta_\bullet; \varphi\big) = 0$.

Lemma 5.3.7. *Let $(E, \tau, \underline{q})$ be an $(\underline{a}, \delta)$-semistable ϱ-flagged pseudo G-bundle. Then, there exists an $(\underline{a}, \delta)$-polystable admissible deformation $\mathrm{gr}(E, \tau, \underline{q})$ of $(E, \tau, \underline{q})$. The ϱ-flagged pseudo G-bundle $\mathrm{gr}(E, \tau, \underline{q})$ is unique up to equivalence.*

In general, not every admissible deformation will immediately lead to a polystable ϱ-flagged pseudo G-bundle, but any iteration of admissible deformations (leading to non-equivalent ϱ-flagged pseudo G-bundles) will do so after finitely many steps. We call two $(\underline{a}, \delta)$-semistable ϱ-flagged pseudo G-bundles $(E, \tau, \underline{q})$ and $(E', \tau', \underline{q}')$ *S-equivalent*, if $\mathrm{gr}(E, \tau, \underline{q})$ and $\mathrm{gr}(E', \tau', \underline{q}')$ are equivalent.

Sketch of proof of Lemma 5.3.7. The lemma follows from our GIT construction of the moduli space. As is well-known, two points $y, y' \in \mathfrak{F}^{(\underline{a},\delta)\text{-ss}}$, $\mathfrak{F} := \mathfrak{F}_{\varrho\text{-FLPsBun}}$, will be mapped to the same point in the quotient, if and only

if the closures of their orbits intersect. Let us call the resulting equivalence relation *orbit equivalence*. Let $y \in \mathfrak{F}^{(\underline{a},\delta)\text{-ss}}$ be a point and $\lambda\colon \mathbb{G}_m(k) \longrightarrow \mathrm{SL}(Y)$ a one parameter subgroup with $\mu(\lambda, y) = 0$. Define $y_\infty(\lambda) := \lim_{z\to\infty} \lambda(z)\cdot y$. By the Hilbert–Mumford criterion (see [32], p. 53, i), and Lemma 0.3), orbit equivalence is the equivalence relation that is generated by $y \sim y_\infty(\lambda)$, $y \in \mathfrak{F}^{(\underline{a},\delta)\text{-ss}}$, λ a one-parameter subgroup of $\mathrm{SL}(Y)$ with $\mu(\lambda, y) = 0$.
On the other hand, if y represents the ϱ-flagged pseudo G-bundle $(E, \tau, \underline{q})$, then λ induces a weighted filtration $(E_\bullet, \beta_\bullet)$ with $M_{\underline{a}}(E_\bullet, \beta_\bullet) + \delta \cdot \mu(E_\bullet, \beta_\bullet; \varphi) = 0$ and $y_\infty(\lambda)$ represents the admissible deformation $\mathrm{df}_{(E_\bullet,\beta_\bullet)}(E, \tau, \underline{q})$. Conversely, any admissible deformation of $(E, \tau, \underline{q})$ comes from a one-parameter subgroup λ of $\mathrm{SL}(Y)$ with $\mu(\lambda, y) = 0$. The assertion of the lemma now results from the fact that the closure of any orbit contains a unique closed orbit.
The details of the above proof consist of a very careful but routine analysis of the computations with the Hilbert–Mumford criterion (which will be performed in Section 5.6). $\square$

Corollary 5.3.8. *The closed points of the moduli space $\mathscr{M}(\varrho, \underline{x}, \underline{r})^{(\underline{a},\delta)\text{-ss}}$ are in one to one correspondence to the S-equivalence classes of $(\underline{a}, \delta)$-semistable ϱ-flagged pseudo G-bundles of type $\underline{r}$, or, equivalently, to the isomorphism classes of $(\underline{a}, \delta)$-polystable ϱ-flagged pseudo G-bundles of type $\underline{r}$.*

5.4 The Moduli Spaces for ϱ-Flagged Principal G-Bundles

Let us remind the reader of the set-up for ϱ-flagged principal G-bundles. First, we fix an element $\vartheta \in \pi_1(G)$, a tuple $\underline{x} = (x_1, ..., x_b)$ of distinct k-rational points on C, and a tuple $\underline{P} = (P_1, ..., P_b)$ of parabolic subgroups of $\mathrm{GL}(V)$. The tuple $\underline{P}$ gives rise to a tuple $\underline{r} = (r_{ij}, j = 1, ..., t_i, i = 1, ..., b)$ of positive integers.
Let $\underline{a} = (a_1, ..., a_b)$ be a stability parameter where $a_i \in X^\star(P_i)^\vee_{\mathbb{Q},+}$, $i = 1, ..., b$. Then, representing a_i by a rational one-parameter subgroup, we obtain a weighted flag $(V_\bullet(a_i), \beta_\bullet(a_i))$ in V, $i = 1, ..., b$. The tuple $\beta_\bullet(a_i)$ does not depend on the choice of the representative for a_i. Hence, we get the well-defined tuple $\underline{a}^\varrho = (a^\varrho_{ij}, j = 1, ..., t_i, i = 1, ..., b)$ via

$$\left(a^\varrho_{i1}, ..., a^\varrho_{it_i}\right) := r \cdot \beta_\bullet(a_i), \quad i = 1, ..., b.$$

Proposition 5.4.1. *There is a positive rational number δ_0, such that for every rational number $\delta > \delta_0$ and every ϱ-flagged principal G-bundle $(\mathscr{P}, \underline{s})$ of type $(\underline{x}, \underline{P})$ with associated ϱ-flagged pseudo G-bundle $(E, \tau, \underline{q})$ of type $(\underline{x}, \underline{r})$ the following properties are equivalent:*

i) *$(\mathscr{P}, \underline{s})$ is an $\underline{a}$-(semi)stable ϱ-flagged principal G-bundle.*

ii) *$(E, \tau, \underline{q})$ is an $(\underline{a}^\varrho, \delta)$-(semi)stable ϱ-flagged pseudo G-bundle.*

Proof. First note that the set of isomorphism classes of $\underline{a}$-semistable ϱ-flagged principal G-bundles of type $(\underline{x}, \underline{P})$ is bounded. Indeed, given a parabolic subgroup Q of G, we write the pair $(Q, \det_Q)$ as $(Q_G(\lambda), \chi_{-\lambda})$ for some one-parameter subgroup λ of G. Since there only finitely many conjugacy classes of parabolic subgroups of G, it is clear that we may find a constant D_1 with

$$\big\langle (\widetilde{\chi}_\lambda)_{s_i}, a_{s_i} \big\rangle = -\big\langle (\widetilde{\chi}_{-\lambda})_{s_i}, a_{s_i} \big\rangle \geq D_1,$$

for any reduction $\mathscr{P}_Q$ of $\mathscr{P}$ to Q and $i = 1, ..., b$. The condition of $\underline{a}$-semistability thus gives the estimate

$$\deg\big(\mathscr{P}_Q(\det_Q)\big) \geq \sum_{i=1}^{b} \big\langle (\widetilde{\chi}_\lambda)_{s_i}, a_{s_i} \big\rangle \geq b \cdot D_1.$$

Therefore, the degree of instability of $\mathscr{P}$ as a principal G-bundle is bounded from below by a constant that depends only on the input data. As is well known (see, e.g., [4]) this implies that $\mathscr{P}$ belongs to a bounded family of isomorphism classes of principal G-bundles.
Using Proposition 5.1.3, the rest of the arguments are now identical to those given in the proof of Theorem 5.4.1 in [15]. □

As is obvious from Lemma 5.1.4, there is an open and $\mathrm{GL}(Y)$-invariant subscheme

$$\mathfrak{F}_{\varrho\text{-FlBun}} \subset \mathfrak{F}_{\varrho\text{-FlPsBun}}$$

that parameterizes the ϱ-flagged principal G-bundles. We claim that

$$\mathfrak{F}^{\underline{a}\text{-ss}}_{\varrho\text{-FlBun}} := \mathfrak{F}^{(\underline{a}^\varrho, \delta)\text{-ss}}_{\varrho\text{-FlPsBun}} \cap \mathfrak{F}_{\varrho\text{-FlBun}}$$

is a saturated open subset, i.e., for every point $f \in \mathfrak{F}^{\underline{a}\text{-ss}}_{\varrho\text{-FlBun}}$, the closure of the orbit $\mathrm{GL}(Y) \cdot f$ inside $\mathfrak{F}^{(\underline{a}^\varrho, \delta)\text{-ss}}_{\varrho\text{-FlPsBun}}$ is contained in $\mathfrak{F}^{\underline{a}\text{-ss}}_{\varrho\text{-FlBun}}$. The discussion of S-equivalence of ϱ-flagged pseudo G-bundles shows that this statement is equivalent to the fact that the set of isomorphism classes of $\underline{a}$-semistable ϱ-flagged principal bundles is closed under S-equivalence inside the set of isomorphism classes of $(\underline{a}^\varrho, \delta)$-semistable ϱ-flagged pseudo G-bundles. To see this, note that, by Remark 5.3.6, an admissible deformation of the ϱ-flagged principal bundle $(E, \tau, \underline{q})$ is associated with a weighted filtration $(E_\bullet(\mathscr{P}_Q), \beta_\bullet(\mathscr{P}_Q))$, coming from a reduction $\mathscr{P}_Q$ of $\mathscr{P}$ to a parabolic subgroup Q of G, such that

$$M_{\underline{a}}\big(E_\bullet(\mathscr{P}_Q), \beta_\bullet(\mathscr{P}_Q)\big) = 0.$$

It is easy to verify that $(E_{\mathrm{df}}, \tau_{\mathrm{df}})$ in $\mathrm{df}_{(E_\bullet(\delta), \beta_\bullet(\delta))}(E, \tau, \underline{q}) = (E_{\mathrm{df}}, \tau_{\mathrm{df}}, \underline{q}_{\mathrm{df}})$ defines again a principal G-bundle. (In fact, $\mathscr{P}$ is obtained from $\mathscr{P}_Q$ by means of extending the structure group via $Q \subset G$. Extending the structure group of $\mathscr{P}_Q$ via $Q \longrightarrow L \subset G$, L a Levi subgroup of Q, yields the principal bundle $\mathscr{P}_{\mathrm{df}}$ corresponding to $(E_{\mathrm{df}}, \tau_{\mathrm{df}})$.)

Since $\mathfrak{F}^{\underline{a}\text{-ss}}_{\varrho\text{-FlBun}}$ is a saturated subset of $\mathfrak{F}^{(\underline{a}^{\varrho},\delta)\text{-ss}}_{\varrho\text{-FlPsBun}}$, there is an open subset $U \subset \mathfrak{F}^{(\underline{a}^{\varrho},\delta)\text{-ss}}_{\varrho\text{-FlPsBun}} /\!\!/ \operatorname{GL}(Y)$, such that $\mathfrak{F}^{\underline{a}\text{-ss}}_{\varrho\text{-FlBun}}$ is the preimage of U under the quotient map $\mathfrak{F}^{(\underline{a}^{\varrho},\delta)\text{-ss}}_{\varrho\text{-FlPsBun}} \longrightarrow \mathfrak{F}^{(\underline{a}^{\varrho},\delta)\text{-ss}}_{\varrho\text{-FlPsBun}} /\!\!/ \operatorname{GL}(Y)$, and

$$U = \mathfrak{F}^{\underline{a}\text{-ss}}_{\varrho\text{-FlBun}} /\!\!/ \operatorname{GL}(Y)$$

is the good quotient. Likewise, we see that the geometric quotient $\mathfrak{F}^{\underline{a}\text{-s}}_{\varrho\text{-FlBun}} / \operatorname{GL}(Y)$ does exist. We define

$$\mathscr{M}(\vartheta, \varrho, \underline{x}, \underline{P})^{\underline{a}\text{-(s)s}} := \mathfrak{F}^{\underline{a}\text{-(s)s}}_{\varrho\text{-FlBun}} /\!\!/ \operatorname{GL}(Y).$$

Theorem 5.4.2. *Assume that the stability parameter $\underline{a}$ is such that $\sum_{j=1}^{t_i} a^{\varrho}_{ij} < 1$ for $i = 1, ..., b$. Then, the moduli spaces $\mathscr{M}(\vartheta, \varrho, \underline{x}, \underline{P})^{\underline{a}\text{-(s)s}}$ for the functors that assign to a scheme S the set of isomorphism classes of families of $\underline{a}$-(semi)stable ϱ-flagged principal G-bundles of topological type ϑ and type $(\underline{x}, \underline{P})$ exist as quasi-projective schemes.*

Finally, we note that the same argument as in Theorem 5.4.4 in [15] gives the following result:

Theorem 5.4.3 (Semistable reduction). *Assume that the representation $\varrho\colon G \longrightarrow \operatorname{GL}(V)$ is of low separable index or that G is an adjoint group, $\varrho\colon G \longrightarrow \operatorname{GL}(\operatorname{Lie}(G))$ is the adjoint representation, and that the characteristic of k is larger than the height of ϱ. Then, $\mathscr{M}(\vartheta, \varrho, \underline{x}, \underline{P})^{\underline{a}\text{-(s)s}}$ is projective.*

5.5 The Moduli Spaces for Flagged Principal G-Bundles

We fix $\vartheta \in \pi_1(G)$, $\underline{x} = (x_1, ..., x_b)$, and the tuple $\underline{P} = (P_1, ..., P_s)$ of parabolic subgroups of G. Let $\underline{a} = (a_1, ..., a_b)$ be a stability parameter with $a_i \in X^\star(P_i)^\vee_{\mathbb{Q},+}$, $i = 1, ..., b$.
For the moment, let $\varrho\colon G \longrightarrow \operatorname{GL}(V)$ be any (not necessarily faithful) representation. We assume that we may represent the a_i by rational one-parameter subgroups that do not lie in the kernel of ϱ. Then, the same construction as in the last section provides us with a tuple $\underline{a}^{\varrho} = (a^{\varrho}_{ij}, j = 1, ..., t_i, i = 1, ..., b)$ of positive rational numbers. We say that the stability parameter $\underline{a}$ is *ϱ-admissible*, if the condition

$$\sum_{j=1}^{t_i} a^{\varrho}_{ij} < 1, \quad i = 1, ..., b,$$

is verified.

Lemma 5.5.1. *The stability parameter $\underline{a}$ is* Ad-*admissible, if and only it is admissible in the sense of the definition following Remark* 4.1.5.

Proof. Let a be a rational one-parameter subgroup of the maximal torus $T \subset G$. The eigenspaces of a are direct sums of root spaces, and a acts on the space for the root α with the weight $\langle\alpha, a\rangle$. The Lie algebra of T is contained in the eigenspace to the weight zero. Since, for every root α, $-\alpha$ is also a root, the weights of the eigenspaces of a are (in increasing order) $-\gamma_s, ..., -\gamma_1, 0, \gamma_1, ..., \gamma_s$. If $(a_1, ..., a_t) = \dim(G) \cdot \beta_\bullet(a)$, we infer

$$\sum_{j=1}^{t} a_j = 2\gamma_s.$$

The condition $\sum_{j=1}^{t} a_j < 1$ thus amounts to the condition $\gamma_s < 1/2$. Since $|\langle\alpha, a\rangle| \leq \gamma_s$ for all roots and equality holds for at least one root, these considerations establish our claim. □

Note that there is a $\mathrm{GL}(Y)$-invariant closed subscheme

$$\mathfrak{F}_{\mathrm{FlBun}} \hookrightarrow \mathfrak{F}_{\varrho\text{-FlBun}}$$

that parameterizes the flagged principal G-bundles. Recall that we have verified in Lemma 5.1.2 the compatibility of the notions of (semi)stability. Theorem 5.4.2 thus immediatly implies:

Theorem 5.5.2. *Let $\underline{a}$ be a stability parameter, such that there exists a faithful representation $\varrho\colon G \longrightarrow \mathrm{GL}(V)$ for which $\underline{a}$ is ϱ-admissible. Then, the moduli spaces $\mathscr{M}(\vartheta, \underline{x}, \underline{P})^{\underline{a}\text{-(s)s}}$ for the functors of isomorphism classes of families of $\underline{a}$-(semi)stable flagged principal G-bundles of topological type ϑ and type $(\underline{x}, \underline{P})$ exist as quasi-projective schemes. They are projective by Theorem* 4.4.1.

Corollary 5.5.3. *Assume that the stability parameter $\underline{a}$ is admissible. Then, the moduli spaces $\mathscr{M}(\vartheta, \underline{x}, \underline{P})^{\underline{a}\text{-(s)s}}$ exist as projective schemes.*

Proof. If G is an adjoint group, the quasi-projectivity of the moduli space is a restatement of Theorem 5.4.2, taking into account Lemma 5.5.1. Properness follows from Theorem 4.4.1.
In general, one can use Ramanathan's method to construct the moduli space for an arbitrary semisimple group from the one of the adjoint group. (Observe that every flagged principal G-bundle $(\mathscr{P}, \underline{s})$ defines in a natural way an adjoint flagged principal G-bundle $\mathrm{Ad}(\mathscr{P}, \underline{s})$, such that $(\mathscr{P}, \underline{s})$ is $\underline{a}$-semistable, if and only if $\mathrm{Ad}(\mathscr{P}, \underline{s})$ is so.) The necessary techniques are described in Section 5 of [16]. □

Remark 5.5.4. i) The corollary gives a complete construction of the moduli spaces of flagged principal G-bundles in all characteristics. Note that we do not need it for our applications, because we are allowed to make the stability parameter $\underline{a}$ as small as we wish to (cf. the proof of Proposition 4.2.2). Thus, having prescribed any faithful representation ϱ, we may for our purposes assume that $\underline{a}$ is ϱ-admissible.
ii) Note that, in our application, we need only the moduli spaces for stability parameters of coprime type. For these stability parameters, the properness of the moduli space implies the semistable reduction theorem, by Lemma 3.3.1.

iii) Suppose that R is, as in Corollary 3.3.4, a ring of finite type over $\mathbb{Z}$, regular and of dimension at most 1. Assume that $\mathscr{C} \longrightarrow \mathrm{Spec}(R)$ is a smooth projective curve. We claim that in this setting, we can construct our moduli space $\mathscr{M}_{\mathscr{C}}(\vartheta, \underline{x}, \underline{P})^{\underline{a}\text{-ss}}$ as a projective scheme over $\mathrm{Spec}(R)$. The only case in which this is not completely obvious is the case when $\mathrm{Spec}(R)$ dominates $\mathrm{Spec}(\mathbb{Z})$. By Remark 5.3.3, we know that we can construct $\mathscr{M}_{\mathscr{C}}(\vartheta, \underline{x}, \underline{P})^{\underline{a}\text{-ss}}$ as a quasi-projective scheme; let $\overline{\mathscr{M}} \longrightarrow \mathrm{Spec}(R)$ be the closure that is obtained as the quotient of the closure of the locus $\underline{a}$-semistable flagged principal G-bundles in $\mathfrak{F}^{(\underline{a},\delta)\text{-ss}}_{\varrho\text{-FLPsBun}}$. By Proposition 2.1.2 and Remark 2.1.3, the moduli space $\mathscr{M}_{\mathscr{C}}(\vartheta, \underline{x}, \underline{P})^{\underline{a}\text{-ss}}$ is irreducible, so that the same holds for $\overline{\mathscr{M}}$. Let C_η be the generic fiber of $\mathscr{C}$ over $\mathrm{Spec}(R)$. We know that the generic fiber of $\overline{\mathscr{M}}$ is the projective moduli space $\mathscr{M}_{C_\eta}(\vartheta, \underline{x}, \underline{P})^{\underline{a}\text{-ss}}$. By the same argument as before, this moduli space is irreducible and, hence, connected. If $r \in \mathrm{Spec}(R)$ is a closed point, and C_r is the fiber of $\mathscr{C}$ over r, then the semistable reduction theorem (Theorem 4.4.1 and 5.4.3) implies that $\mathscr{M}_{C_r}(\vartheta, \underline{x}, \underline{P})^{\underline{a}\text{-ss}}$ is a connected component of the fiber of $\overline{\mathscr{M}}$ over r. Thus, we have to show that $\overline{\mathscr{M}} \longrightarrow \mathrm{Spec}(R)$ has connected fibers. This follows from Stein factorization: indeed, if we factorize $\overline{\mathscr{M}} \longrightarrow \mathrm{Spec}(R') \longrightarrow \mathrm{Spec}(R)$, such that the morphism $\overline{\mathscr{M}} \longrightarrow \mathrm{Spec}(R')$ has connected fibers, then $\mathrm{Spec}(R') \longrightarrow \mathrm{Spec}(R)$ must be an isomorphism. This follows, because it is an isomorphism at the generic point (the generic fiber of $\overline{\mathscr{M}}$ was already connected) and R is assumed to be normal.

5.6 Construction of the Moduli Spaces for Decorated Flagged Vector Bundles

In this section, we will first give the proof of Proposition 5.2.1 by an explicit construction and then carry out the most difficult parts in the proof of Theorem 5.2.2.

Construction of the Parameter Space. — We fix the type $(r, d, \underline{x}, \underline{r}, u, v, L)$. Again, we pick a point $x_0 \in C$ and write $\mathscr{O}_C(1)$ for $\mathscr{O}_C(x_0)$. By Proposition 5.1.10, we can choose an integer n_0, such that, for every $n \geq n_0$ and every $(\underline{a}, \delta)$-semistable decorated flagged vector bundle $(E, \underline{q}, \varphi)$ of type $(r, d, \underline{x}, \underline{r}, u, v, L)$, the following conclusions are true:

- $H^1(E(n)) = \{0\}$ and $E(n)$ is globally generated;
- $H^1(\det(E)(rn)) = \{0\}$ and $\det(E)(rn)$ is globally generated.

Furthermore, we suppose:

- $H^1(L(un)) = \{0\}$ and $L(un)$ is globally generated.

Choose some $n \geq n_0$ and set $l := d + rn + r(1-g)$. Let Y be a k-vector space of dimension l. We define $\mathfrak{Q}^0$ as the quasi-projective scheme parameterizing equivalence classes of quotients $k\colon Y \otimes \mathscr{O}_C(-n) \longrightarrow E$ where E is a vector bundle of rank r and degree d on C and $H^0(k(n))$ is an isomorphism. Then, there is the universal quotient

$$k_{\mathfrak{Q}^0}\colon Y\otimes\pi_C^\star\big(\mathscr{O}_C(-n)\big)\longrightarrow E_{\mathfrak{Q}^0}$$

on $\mathfrak{Q}^0\times C$. Set

$$\mathscr{H}:=\operatorname{Hom}\big(\mathbb{D}^{u,v}(Y),L(un)\big)\quad\text{and}\quad\mathfrak{H}:=\mathbb{P}(\mathscr{H}^\vee)\times\mathfrak{Q}^0.$$

We let

$$k_{\mathfrak{H}}\colon Y\otimes\pi_C^\star\big(\mathscr{O}_C(-n)\big)\longrightarrow E_{\mathfrak{H}}$$

be the pullback of $k_{\mathfrak{Q}^0}$ to $\mathfrak{H}\times C$. Now, on $\mathfrak{H}\times C$, there is the tautological homomorphism

$$s_{\mathfrak{H}}\colon\mathbb{D}^{u,v}(Y)\otimes\mathscr{O}_{\mathfrak{H}}\longrightarrow\pi_C^\star\big(L(un)\big)\otimes\pi_{\mathfrak{H}}^\star\big(\mathscr{O}_{\mathfrak{H}}(1)\big).$$

Let $\mathfrak{T}$ be the closed subscheme defined by the condition that $s_{\mathfrak{H}}\otimes\pi_C^\star(\mathrm{id}_{\mathscr{O}_C(-un)})$ vanishes on

$$\ker\Big(\mathbb{D}^{u,v}(Y)\otimes\pi_C^\star\big(\mathscr{O}_C(-un)\big)\longrightarrow\mathbb{D}^{u,v}(E_{\mathfrak{H}})\Big)\quad\text{(cf. Remark 5.1.7).}$$

Let

$$k_{\mathfrak{T}}\colon Y\otimes\pi_C^\star\big(\mathscr{O}_C(-n)\big)\longrightarrow E_{\mathfrak{T}}$$

be the restriction of $k_{\mathfrak{H}}$ to $\mathfrak{T}\times C$. By definition, there is the universal homomorphism

$$\varphi_{\mathfrak{T}}\colon\mathbb{D}^{u,v}\big(E_{\mathfrak{T}}\big)\longrightarrow\pi_C^\star(L)\otimes\pi_{\mathfrak{T}}^\star(\mathfrak{N}_{\mathfrak{T}}).$$

Here, $\mathfrak{N}_{\mathfrak{T}}$ is the restriction of $\mathscr{O}_{\mathfrak{H}}(1)$ to $\mathfrak{T}$.
Next, let $\mathfrak{G}_{ij}$ be the Graßmann variety that parameterizes the r_{ij}-dimensional quotients of the vector space Y, $j=1,...,t_i$, $i=1,...,s$, and set $\mathfrak{G}:=\mathsf{X}_{j=1,...,t_i,i=1,...,s}\,\mathfrak{G}_{ij}$. We construct the parameter space $\mathfrak{P}$ as a closed subscheme of $\mathfrak{T}\times\mathfrak{G}$: on the scheme $\widetilde{\mathfrak{P}}:=\mathfrak{T}\times\mathfrak{G}$, there are the tautological quotients

$$\widetilde{q}_{\widetilde{\mathfrak{P}},ij}\colon Y\otimes\mathscr{O}_{\widetilde{\mathfrak{P}}\times C}\longrightarrow\widetilde{R}_{\widetilde{\mathfrak{P}},ij},\quad j=1,...,t_i,\ i=1,....,s.$$

We define the closed subscheme $\mathfrak{P}$ by the condition that $\widetilde{q}_{\widetilde{\mathfrak{P}},ij}$ vanishes on the kernel of the restriction of $k_{\widetilde{\mathfrak{P}}}$ to $\widetilde{\mathfrak{P}}\times\{x_i\}$, for all $j=1,...,t_i$, $i=1,....,s$. Let $\mathfrak{N}_{\mathfrak{P}}$ be the pullback of $\mathfrak{N}_{\mathfrak{T}}$ to $\mathfrak{P}$. Similarly, we may pull back $k_{\mathfrak{T}}$ and $\varphi_{\mathfrak{T}}$ from $\mathfrak{T}\times C$ to $\mathfrak{P}\times C$ in order to obtain

$$k_{\mathfrak{P}}\colon Y\otimes\pi_C^\star\big(\mathscr{O}_C(-n)\big)\longrightarrow E_{\mathfrak{P}}$$

and

$$\varphi_{\mathfrak{P}}\colon\mathbb{D}^{u,v}\big(E_{\mathfrak{P}}\big)\longrightarrow\pi_C^\star(L)\otimes\pi_{\mathfrak{P}}^\star(\mathfrak{N}_{\mathfrak{P}}).$$

Finally, on $\mathfrak{P} \times \{x_i\}$, we have the quotients

$$q_{\mathfrak{P},ij} \colon E_{\mathfrak{P}|\mathfrak{P}\times\{x_i\}} \longrightarrow R_{\mathfrak{P},ij}, \quad j = 1, ..., t_i,\ i = 1, ..., s.$$

We call $(E_{\mathfrak{P}}; \underline{q}_{\mathfrak{P}}; \varphi_{\mathfrak{P}})$ the *universal family.* This name is justified, because any family of decorated flagged vector bundles parameterized by a scheme S is locally induced by a morphism to $\mathfrak{P}$ and this universal family.
Finally, we note that there is a canonical action of the group $\mathrm{GL}(Y)$ on the parameter space $\mathfrak{P}$, and it will be our task to construct the good and the geometric quotient of the open subsets that parameterize the semistable and the stable objects, respectively. Since the center $\mathbb{G}_m(k) \cdot \mathrm{id}_Y$ acts trivially on $\mathfrak{P}$, it suffices to construct the respective quotients for the action of $\mathrm{SL}(Y)$.

The Map to the Gieseker Space. — Let Jac^d be the Jacobian variety that classifies the line bundles of degree d on C, and choose a Poincaré sheaf $\mathscr{P}$ on $\mathrm{Jac}^d \times C$. By our assumptions on n, the sheaf

$$\mathscr{K}_1 := \mathscr{H}\!om\Big(\bigwedge^r(Y) \otimes \mathscr{O}_{\mathrm{Jac}^d}, \pi_{\mathrm{Jac}^d\,\star}\big(\mathscr{P} \otimes \pi_C^\star(\mathscr{O}_C(rn))\big)\Big)$$

is locally free. We set $\mathbb{K}_1 := \mathbb{P}(\mathscr{K}_1^\vee)$. By replacing $\mathscr{P}$ with $\mathscr{P} \otimes \pi^\star_{\mathrm{Jac}^d}$ (sufficiently ample)$^\vee$, we may assume that $\mathscr{O}_{\mathbb{K}_1}(1)$ is very ample. Let $\mathfrak{d} \colon \mathfrak{P} \longrightarrow \mathrm{Jac}^d$ be the morphism associated with $\bigwedge^r(E_{\mathfrak{P}})$, and let $\mathfrak{A}_{\mathfrak{P}}$ be a line bundle on $\mathfrak{P}$ with $\bigwedge^r(E_{\mathfrak{P}}) \cong (\mathfrak{d} \times \mathrm{id}_C)^\star(\mathscr{P}) \otimes \pi_{\mathfrak{P}}^\star(\mathfrak{A}_{\mathfrak{P}})$. Then,

$$\bigwedge^r\big(k_{\mathfrak{P}} \otimes \mathrm{id}_{\pi_C^\star(\mathscr{O}_C(n))}\big) \colon \bigwedge^r(Y) \otimes \mathscr{O}_{\mathfrak{P}} \longrightarrow (\mathfrak{d} \times \mathrm{id}_C)^\star(\mathscr{P}) \otimes \pi_C^\star\big(\mathscr{O}_C(rn)\big) \otimes \pi_{\mathfrak{P}}^\star(\mathfrak{A}_{\mathfrak{P}})$$

defines a morphism $\iota_1 \colon \mathfrak{P} \longrightarrow \mathbb{K}_1$ with $\iota_1^\star(\mathscr{O}_{\mathbb{K}_1}(1)) \cong \mathfrak{A}_{\mathfrak{P}}$.
Define $\mathbb{K}_2 := \mathbb{P}(\mathscr{H}^\vee)$ (see above) as well as the *Gieseker space* $\mathbb{G} := \mathbb{K}_1 \times \mathbb{K}_2 \times \mathfrak{G}$, and let

$$\iota := (\iota_1 \times \mathrm{id}_{\mathbb{K}_2} \times \mathrm{id}_{\mathfrak{G}}) \colon \mathfrak{P} \longrightarrow \mathbb{G}$$

be the natural, $\mathrm{SL}(Y)$-equivariant, and injective morphism. Using the ample line bundles on the $\mathfrak{G}_{ij}$ that are induced by the Plücker embedding, we find, for every tuple $\underline{e} := (e_1; e_2; \varepsilon_{ij}, j = 1, ..., t_i, i = 1, ..., s)$ of positive rational numbers, the $\mathrm{SL}(Y)$-linearized ample $\mathbb{Q}$-line bundle

$$\mathscr{L}_{\underline{e}} := \mathscr{O}\big(e_1; e_2; \varepsilon_{ij}, j = 1, ..., t_i, i = 1, ..., s\big)$$

on the Gieseker space $\mathbb{G}$.
Linearize the $\mathrm{SL}(Y)$-action on $\mathbb{G}$ in $\mathscr{L}_{\underline{e}}$ with

$$e_1 := l - u\cdot\delta - \sum_{i=1}^{s}\sum_{j=1}^{t_i} r_{ij}\cdot a_{ij}, \quad e_2 := r\cdot\delta, \quad \varepsilon_{ij} := r\cdot a_{ij},\ j = 1, ..., t_i,\ i = 1, ..., s, \tag{10}$$

and denote by $\mathbb{G}^{\underline{e}\text{-(s)s}}$ the sets of points in $\mathbb{G}$ that are $\mathrm{SL}(Y)$-(semi)stable with respect to the linearization in the line bundle $\mathscr{L}_{\underline{e}}$.

Theorem 5.6.1. *Given a point $p \in \mathfrak{P}$, denote by $(E_p; \underline{q}_p; \varphi_p)$ the restriction of the universal family to $\mathfrak{P} \times \{p\}$. Then, for n large enough, the following two properties hold true.*
i) *The preimages $\iota^{-1}(\mathbb{G}^{\underline{e}\text{-(s)s}})$ consist exactly of those points $p \in \mathfrak{P}$ for which $(E_p; \underline{q}_p; \varphi_p)$ is an $(\underline{a}, \delta)$-(semi)stable decorated flagged vector bundle of type $(r, d, \underline{x}, \underline{r}, u, v, L)$.*
ii) *The morphism*

$$\iota' \colon \mathfrak{P}^{\underline{e}\text{-ss}} \longrightarrow \mathbb{G}^{\underline{e}\text{-ss}},$$

induced by restricting the morphism ι to the preimage $\mathfrak{P}^{\underline{e}\text{-ss}}$ of $\mathbb{G}^{\underline{e}\text{-ss}}$, is proper.

The proof resembles the one of Theorem 2.11 in [37] and Theorem 4.4.1 in [15]. A part of it will be explained in the following section.

Elements of the Proof of Theorem 5.6.1. — Let p be a point in the parameter space $\mathfrak{P}$, such that the decorated flagged vector bundle $(E_p; \underline{q}_p; \varphi_p)$ is $(\underline{a}, \delta)$-(semi)stable. In this section, we will demonstrate that the Gieseker point $\iota(p)$ is (semi)stable with respect to the chosen linearization of the $\mathrm{SL}(Y)$-action.
By the Hilbert–Mumford criterion, we have to show that, for every one-parameter subgroup $\lambda \colon \mathbb{G}_m(k) \longrightarrow \mathrm{SL}(Y)$, the inequality

$$\begin{aligned} \mu_{\mathscr{L}_{\underline{e}}}\big(\lambda, \iota(p)\big) \quad &= \quad e_1 \cdot \mu_{\mathscr{O}_{\mathbb{K}_1}(1)}\big(\lambda, \iota_1(t)\big) + e_2 \cdot \mu_{\mathscr{O}_{\mathbb{K}_2}(1)}\big(\lambda, \iota_2(t)\big) \\ &\quad + \sum_{i=1}^{s} \sum_{j=1}^{t_i} \varepsilon_{ij} \cdot \mu_{\mathscr{O}_{\mathfrak{G}_{ij}}(1)}(\lambda, q_{ij}) \quad (\geq) \quad 0 \end{aligned} \tag{11}$$

is satisfied. The one-parameter subgroup λ provides us with the weighted flag $(Y_\bullet(\lambda), \delta_\bullet(\lambda))$ in the vector space Y. We write

$$Y_\bullet(\lambda) : 0 =: Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_\tau \subsetneq Y_{\tau+1} := Y; \quad \delta_\bullet(\lambda) = (\delta_1, ..., \delta_\tau).$$

We remind the reader that there is an integer $N > 0$ (which is the number of summands in (5)), such that

$$\mathbb{D}^{u,v}(Y) \subset Y_{u,N} := \big(Y^{\otimes u}\big)^{\oplus N}.$$

Let $k_p \colon Y \otimes \mathscr{O}_C(-n) \longrightarrow E_p$ be the quotient corresponding to p. For $h \in \{1, ..., \tau\}$, define $l_h := \dim(Y_h)$ and $\mathscr{F}_h := k_p(Y_h \otimes \mathscr{O}_C(-n))$. Now, using (11), we compute

$$\begin{aligned} \mu_{\mathscr{L}_{\underline{e}}}\big(\lambda, \iota(p)\big) \quad &= \quad e_1 \cdot \sum_{h=1}^{\tau} \delta_h \cdot \big(l \cdot \mathrm{rk}(\mathscr{F}_h) - l_h \cdot r\big) + e_2 \cdot \mu_{\mathscr{O}_{\mathbb{K}_2}(1)}\big(\lambda, \iota_2(t)\big) + \\ &\quad + \sum_{i=1}^{s} \sum_{j=1}^{t_i} \varepsilon_{ij} \cdot \sum_{h=1}^{\tau} \delta_h \cdot \big(l \cdot \dim(q_{ij}(\mathscr{F}_h)) - l_h \cdot r_{ij}\big). \end{aligned}$$

We first inspect the quantity $\mu_{\mathscr{O}_{\mathbb{K}_2}(1)}(\lambda, \iota_2(t))$. To this end, let $\widetilde{E}_h$ be the subbundle of E_p that is generated by $\mathscr{F}_h$, $h = 0, ..., \tau + 1$. Note that improper inclusions may occur among the bundles $\widetilde{E}_h$, i.e., there might exist indices $h' < h$ with $\widetilde{E}_{h'} = \widetilde{E}_h$. We eliminate these improper inclusions in order to find the filtration

$$E_\bullet : 0 =: E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_t \subsetneq E_{t+1} := E_p.$$

With each index $j \in \{ 1, ..., t \}$, we associate the set

$$T(j) := \Big\{ h \in \{ 1, ..., \tau \} \,\Big|\, \widetilde{E}_h = E_j \Big\}$$

and the positive rational number

$$\beta_j := \sum_{h \in T(j)} \delta_h. \tag{12}$$

Setting $\beta_\bullet := (\beta_1, ..., \beta_t)$, we have defined the weighted filtration $(E_\bullet, \beta_\bullet)$ of E. In addition, we define the function $J\colon \{ 1, ..., \tau \} \longrightarrow \{ 1, ..., t \}$ by requiring that $\widetilde{E}_h = E_{J(h)}$, $h = 1, ..., \tau$. For an index $j \in \{ 0, ..., t+1 \}$, we set

$$\begin{array}{rcl} \underline{h}(j) & := & \min\big\{ h = 1, ..., \tau \,|\, \widetilde{E}_h = E_j \big\}, \quad \underline{Y}_j := Y_{\underline{h}(j)}, \\ \overline{h}(j) & := & \max\big\{ h = 1, ..., \tau \,|\, \widetilde{E}_h = E_j \big\}, \quad \overline{Y}_j := Y_{\overline{h}(j)}, \end{array}$$

and also, for $j = 1, ..., t$,

$$\widetilde{Y}_j := \underline{Y}_j / \overline{Y}_{j-1}.$$

Next, given an index tuple $(i_1, ..., i_u) \in I := \{ 1, ..., t+1 \}^{\times u}$, we introduce the vector space

$$\widetilde{Y}_{i_1,\dots,i_u} := \big(\widetilde{Y}_{i_1} \otimes \cdots \otimes \widetilde{Y}_{i_u}\big)^{\oplus N}.$$

We fix a basis $\underline{y}$ for Y that consists of eigenvectors for the one-parameter subgroup λ and has the property

$$\langle y_1, ..., y_{l_h} \rangle = Y_h, \quad h = 0, ..., \tau + 1.$$

Using this basis, we may view $(\widetilde{Y}_{i_1,\dots,i_u})^{\oplus N}$ as a subspace of $Y_{\nu,N}$, and declare

$$\widetilde{Y}^\star_{i_1,\dots,i_u} := \widetilde{Y}_{i_1,\dots,i_u} \cap \mathbb{D}^{u,v}(Y).$$

If we are also given a weight vector $\underline{\gamma} = (\gamma_1, ..., \gamma_l)$, we let $\lambda(\underline{y}, \underline{\gamma})$ be the one-parameter subgroup with $\lambda(\underline{y}, \underline{\gamma})(y_i) = z^{\gamma_i} \cdot y_i$, $z \in \mathbb{G}_m(k)$, $i = 1, ..., l$. Apparently,

$$\lambda = \lambda(\underline{y}, \underline{\gamma}) \quad \text{for } \underline{\gamma} = \sum_{h=1}^{\tau} \delta_h \cdot \gamma_l^{(l_h)}.$$

We also define the one-parameter subgroups $\lambda^h := \lambda(\underline{y}, \gamma_l^{(l_h)})$, $h = 1, ..., \tau$. Then, the subspaces $\widetilde{Y}^\star_{i_1,\dots,i_u}$, $(i_1, ..., i_u) \in I$, that we have just defined are eigenspaces for all the one-parameter subgroups $\lambda^1, ..., \lambda^\tau$. Indeed, define for $\underline{i} \in I$ and $j \in \{ 0, ..., t+1 \}$,

$$\nu_j(\underline{i}) = \#\big\{ i_k \le j \,|\, k = 1, ..., u \big\}.$$

Since $\underline{h}(j) \le h$ holds precisely when $j \le J(h)$, the one-parameter subgroup λ^h acts on $\widetilde{Y}^\star_{i_1,...,i_u}$ with weight $l_h \cdot u - l \cdot \nu_{J(h)}(i_1, ..., i_u)$, $\underline{i} = (i_1, ..., i_u) \in I$, $h = 1, ..., \tau$.
The homomorphism φ_p is determined by the homomorphism

$$F_p \colon \mathbb{D}^{u,v}(Y) \longrightarrow H^0\big(L(un)\big).$$

Therefore,

$$\begin{aligned} &\mu_{\mathscr{O}_{\mathbb{K}_2(1)}}(\lambda, F_p) \ge \qquad\qquad (13)\\ &-\min\Big\{ \sum_{h=1}^{\tau} \delta_h\big(l_h \cdot u - l \cdot \nu_{J(h)}(i_1, ..., i_u)\big) \,\big|\, \underline{i} = (i_1, ..., i_u) \in I : F_{p|\widetilde{Y}^\star_{i_1,...,i_u}} \not\equiv 0 \Big\}. \end{aligned}$$

Let $\underline{i}_0 = (i^0_1, ..., i^0_u) \in I$ be an index tuple, such that the minimum in the second formula in Remark 5.1.9 is achieved for this index tuple.

Lemma 5.6.2. *The restricted homomorphism $F_{p|\widetilde{Y}^\star_{i^0_1,...,i^0_u}}$ is non-trivial.*

Proof. Under the surjection $\mathbb{D}^{u,v}(Y \otimes \mathscr{O}_C(n)) \longrightarrow \mathbb{D}^{u,v}(E_p(n))$ that is induced by k_p, the vector space $F_{p|\widetilde{Y}^\star_{i^0_1,...,i^0_u}}$ maps to the global sections of the bundle $E_{i^0_1}(n) \star \cdots \star E_{i^0_u}(n)$, and

$$\Big(\mathbb{D}^{u,v}(Y) \cap \big(Y'_{i^0_1} \otimes \cdots \otimes Y'_{i^0_u}\big)^{\oplus N} \Big) \otimes \mathscr{O}_C(un) \quad \text{with } Y'_j := \bigoplus_{k=1}^{j} \widetilde{Y}_k,\ j = 1, ..., t,$$

generically generates that bundle. To see these assertions, observe that

$$D^{u_1}(Y) \otimes \cdots \otimes D^{u_v}(Y) \subset Y^{\otimes u}, \text{ for } u_1 + \cdots + u_v = u,$$

is, by definition, the submodule that is invariant under action of $\varSigma_{u_1} \times \cdots \times \varSigma_{u_v}$, $\varSigma_w$ being the symmetric group in w letters, $w > 0$. The intersection

$$D^{u_1}(Y) \otimes \cdots \otimes D^{u_v}(Y) \cap \big(Y'_{i^0_1} \otimes \cdots \otimes Y'_{i^0_u}\big)$$

is consequently of the form

$$D^{u_1}(Y'_{i^\star_1}) \otimes \cdots \otimes D^{u_v}(Y'_{i^\star_v})$$

where $i^\star_1$ is the smallest index among i^0_1,...,$i^0_{u_1}$, $i^\star_2$ is the smallest index among $i^0_{u_1+1}$,...,$i^0_{u_2}$, and so on. The map $(Y \otimes \mathscr{O}_C(-n))^{\otimes u} \longrightarrow E_p^{\otimes u}$ is certainly equivariant under the $(\varSigma_{u_1} \times \cdots \times \varSigma_{u_v})$-action and is easily seen to induce a surjection

$D^{u_1}(Y \otimes \mathscr{O}_C(-n)) \otimes \cdots \otimes D^{u_v}(Y \otimes \mathscr{O}_C(-n)) \longrightarrow D^{u_1}(E_p) \otimes \cdots \otimes D^{u_v}(E_p)$. Since the isomorphism $Y \longrightarrow H^0(E_p(n))$ maps Y'_j to the global sections of $E_j(n)$, $j = 1, ..., t$, and Y'_j generically generates the bundle E_j, we see that $D^{u_1}(Y'_{i_1^\star}) \otimes \cdots \otimes D^{u_v}(Y'_{i_v^\star})$ generically generates

$$D^{u_1}(E_{i_1^\star}) \otimes \cdots \otimes D^{u_v}(E_{i_v^\star}) = \bigl(D^{u_1}(E_p) \otimes \cdots \otimes D^{u_v}(E_p)\bigr) \cap \bigl(E_{i_1^0} \otimes \cdots \otimes E_{i_u^0}\bigr).$$

Therefore, if $F_{p|\widetilde{Y}^\star_{i_1^0,...,i_u^0}}$ were zero, we would find indices $i'_j \le i_j^0$, $j = 1, ..., u$, where at least one inequality is strict, such that $F_{p|\widetilde{Y}^\star_{i'_1,...,i'_u}} \not\equiv 0$. By the same argument as before, this would imply that the restriction of φ_p to $E_{i'_1} \star \cdots \star E_{i'_u}$ was non-trivial. But clearly

$$\gamma_{i'_1} + \cdots + \gamma_{i'_u} < \gamma_{i_1^0} + \cdots + \gamma_{i_u^0}.$$

This contradicts our choice of $\underline{i}_0$. $\square$

Using (13), we find

$$\begin{aligned} \mu_{\mathscr{O}_{\mathbb{K}_2(1)}}(\lambda, F_p) &\ge -\sum_{h=1}^{\tau} \delta_h \cdot \bigl(l_h \cdot u - l \cdot \nu_{J(h)}(i_1^0, ..., i_u^0)\bigr) \\ &\ge -\sum_{j=1}^{t} \beta_j \cdot \bigl(h^0(E_j(n)) \cdot u - l \cdot \nu_j(i_1^0, ..., i_u^0)\bigr). \qquad (14) \end{aligned}$$

We note our first estimate:

$$\begin{aligned} \mu_{\mathscr{L}_{\underline{e}}}\bigl(\lambda, \iota(p)\bigr) &\ge e_1 \cdot \sum_{h=1}^{\tau} \delta_h \cdot \bigl(l \cdot \mathrm{rk}(\mathscr{F}_h) - l_h \cdot r\bigr) + \\ &\quad + e_2 \cdot \sum_{j=1}^{t} \beta_j \cdot \bigl(l \cdot \nu_j(i_1^0, ..., i_u^0) - h^0(E_j(n)) \cdot u\bigr) + \qquad (15) \\ &\quad + \sum_{i=1}^{s}\sum_{j=1}^{t_i} \varepsilon_{ij} \cdot \sum_{h=1}^{\tau} \delta_h \cdot \bigl(l \cdot \dim(q_{ij}(\mathscr{F}_h)) - l_h \cdot r_{ij}\bigr). \end{aligned}$$

For $j \in \{1, ..., t\}$, choose $h^\star(j) \in T(j)$, such that

$$\begin{aligned} & e_1 \cdot \bigl(l \cdot \mathrm{rk}(\mathscr{F}_{h^\star(j)}) - l_{h^\star(j)} \cdot r\bigr) + \\ & \quad + \sum_{i=1}^{s}\sum_{j=1}^{t_i} \varepsilon_{ij} \cdot \bigl(l \cdot \dim(q_{ij}(\mathscr{F}_{h^\star(j)})) - l_{h^\star(j)} \cdot r_{ij}\bigr) \\ & = \min\Bigl\{ e_1 \cdot \bigl(l \cdot \mathrm{rk}(\mathscr{F}_h) - l_h \cdot r\bigr) + \sum_{i=1}^{s}\sum_{j=1}^{t_i} \varepsilon_{ij} \cdot \bigl(l \cdot \dim(q_{ij}(\mathscr{F}_h)) - l_h \cdot r_{ij}\bigr) \;\Big|\; \\ & \qquad\qquad h \in T(j) \Bigr\}. \end{aligned}$$

Together with (15), we arrive at our second estimate:

$$
\begin{aligned}
\mu_{\mathscr{L}_{\underline{e}}}\big(\lambda, \iota(p)\big) \quad \geq \quad & e_1 \cdot \sum_{k=1}^{t} \beta_k \cdot \big(l \cdot \mathrm{rk}(\mathscr{F}_{h^\star(k)}) - l_{h^\star(k)} \cdot r\big) + \\
& + e_2 \cdot \sum_{k=1}^{t} \beta_k \cdot \big(l \cdot v_k(\underline{i}_0) - h^0(E_k(n)) \cdot u\big) + \\
& + \sum_{i=1}^{s} \sum_{j=1}^{t_i} \varepsilon_{ij} \cdot \sum_{k=1}^{t} \beta_k \cdot \big(l \cdot \dim(q_{ij}(\mathscr{F}_{h^\star(k)})) - l_{h^\star(k)} \cdot r_{ij}\big).
\end{aligned} \tag{16}
$$

Plugging in the definition (10) of the linearization parameters, Formula (16) transforms into

$$
\begin{aligned}
& \mu_{\mathscr{L}_{\underline{e}}}\big(\lambda, \iota(p)\big) \\
\geq \quad & \sum_{k=1}^{t} \beta_k \cdot \Big(l^2 \cdot \mathrm{rk}(\mathscr{F}_{h^\star(k)}) - l \cdot u \cdot \delta \cdot \mathrm{rk}(\mathscr{F}_{h^\star(k)}) - \\
& - l \cdot \sum_{i=1}^{s} \sum_{j=1}^{t_i} r_{ij} \cdot a_{ij} \cdot \mathrm{rk}(\mathscr{F}_{h^\star(k)}) - l \cdot l_{h^\star(k)} \cdot r\Big) + \\
& + r \cdot \delta \cdot \sum_{k=1}^{t} \beta_k \cdot l \cdot \nu_k(\underline{i}_0) + r \cdot \sum_{i=1}^{s} \sum_{j=1}^{t_i} a_{ij} \cdot \sum_{k=1}^{t} \beta_k \cdot l \cdot \dim(q_{ij}(\mathscr{F}_{h^\star(k)})).
\end{aligned}
$$

Note that $l_{h^\star(k)} \leq h^0(\mathscr{F}_{h^\star(k)}(n))$, so that we find

$$
\begin{aligned}
& \mu_{\mathscr{L}_{\underline{e}}}\big(\lambda, \iota(p)\big) \\
\geq \quad & \sum_{k=1}^{t} \beta_k \Big(l^2 \cdot \mathrm{rk}(\mathscr{F}_{h^\star(k)}) - l \cdot u \cdot \delta \cdot \mathrm{rk}(\mathscr{F}_{h^\star(k)}) - \\
& - l \cdot \sum_{i=1}^{s} \sum_{j=1}^{t_i} r_{ij} \cdot a_{ij} \cdot \mathrm{rk}(\mathscr{F}_{h^\star(k)}) - l \cdot h^0(\mathscr{F}_{h^\star(k)}(n)) \cdot r\Big) + \\
& + r \cdot \delta \cdot \sum_{k=1}^{t} \beta_k \cdot l \cdot \nu_k(\underline{i}_0) + r \cdot \sum_{i=1}^{s} \sum_{j=1}^{t_i} a_{ij} \cdot \sum_{k=1}^{t} \beta_k \cdot l \cdot \dim(q_{ij}(\mathscr{F}_{h^\star(k)})).
\end{aligned}
$$

We divide the quantity on the right hand side by l and rearrange it, until we get

$$\begin{aligned} \mu_{\mathscr{L}_{\underline{e}}}\big(\lambda,\iota(p)\big) \quad \geq \quad & \sum_{k=1}^{t} \beta_k \cdot \big(l \cdot \mathrm{rk}(\mathscr{F}_{h^\star(k)}) - h^0(\mathscr{F}_{h^\star(k)}(n)) \cdot r\big) + \\ & +\delta \cdot \sum_{k=1}^{t} \beta_k \cdot \big(r \cdot \nu_k(\underline{i}_0) - u \cdot \mathrm{rk}(E_k)\big) + \\ & + \sum_{i=1}^{s}\sum_{j=1}^{t_i} a_{ij} \sum_{k=1}^{t} \beta_k\big(r \cdot \dim(q_{ij}(\mathscr{F}_{h^\star(k)})) - \mathrm{rk}(\mathscr{F}_{h^\star(k)}) \cdot r_{ij}\big). \end{aligned} \tag{17}$$

By our choice of $\underline{i}_0$, the number $\sum_{k=1}^{t} \beta_k \cdot (r \cdot \nu_k(\underline{i}_0) - u \cdot \mathrm{rk}(E_k))$ equals $\mu(E_\bullet, \beta_\bullet; \varphi_p)$. Our contention is therefore a consequence of the next result.

Proposition 5.6.3. *Having fixed the input data r, d, u, v, and L, as well as the stability parameters $\underline{a}$ and δ, there exists an n_1, such that any $(\underline{a}, \delta)$-(semi)stable decorated flagged vector bundle (E, L, φ) of type $(r, d, \underline{x}, \underline{r}, u, v, L)$ has the following property: Let*

$$0 \subsetneq \mathscr{F}_1 \subsetneq \cdots \subsetneq \mathscr{F}_t \subsetneq E$$

be a filtration of E by not necessarily saturated subsheaves, such that $0 < \mathrm{rk}(\mathscr{F}_1) < \cdots < \mathrm{rk}(\mathscr{F}_t) < r$, let

$$E_\bullet : 0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_t \subsetneq E$$

be the filtration of E by the subbundles $E_i := \ker(E \longrightarrow (E/\mathscr{F}_i)/\mathrm{Torsion}(E/\mathscr{F}_i))$, $i = 1, ..., t$, and let $\beta_\bullet = (\beta_1, ..., \beta_t)$ be a tuple of positive rational numbers. Then, for all $n \geq n_1$,

$$\begin{aligned} 0 \quad \leq \quad & \sum_{k=1}^{t} \beta_k \cdot \big(h^0(E(n)) \cdot \mathrm{rk}(\mathscr{F}_k) - h^0(\mathscr{F}_k(n)) \cdot r\big) + \delta \cdot \mu(E_\bullet, \beta_\bullet; \varphi) + \\ & + \sum_{i=1}^{s}\sum_{j=1}^{t_i} a_{ij} \cdot \sum_{k=1}^{t} \beta_k \cdot \big(r \cdot \dim(q_{ij}(\mathscr{F}_k)) - \mathrm{rk}(\mathscr{F}_k) \cdot r_{ij}\big). \end{aligned}$$

Proof. We choose $n_1 \geq n_0$, so that $h^1(E(n)) = 0$ and $l := h^0(E(n)) = d + r(n + 1 - g)$. First, we assume that the sheaves $\mathscr{F}_1(n), ..., \mathscr{F}_t(n)$ are all globally generated and have trivial first cohomology spaces. The same holds then for $E_1(n)$,...,$E_t(n)$. Let $\mathscr{T}_i$ be the torsion sheaf $E_i/\mathscr{F}_i$, $i = 1, ..., t$. Since $H^1(\mathscr{F}_i(n)) = \{0\}$, the map $H^0(E_i(n)) \longrightarrow \mathscr{T}_i$ is surjective, so that

$$h^0\big(E_i(n)\big) = h^0\big(\mathscr{F}_i(n)\big) + \dim(\mathscr{T}_i), \quad i = 1, ..., t. \tag{18}$$

Invoking $\sum_{j=1}^{t_i} a_{ij} < 1$, $i = 1, ..., s$, once more, we discover

$$\sum_{j=1}^{t_i} a_{ij} \cdot \dim(q_{ij}(E_k)) \leq \sum_{j=1}^{t_i} a_{ij} \cdot \dim(q_{ij}(\mathscr{F}_k)) + \dim(\mathscr{T}_{k|\{x_i\}}), \quad i = 1, ..., s.$$

In this case, we consequently find

$$
\begin{aligned}
& \sum_{k=1}^{t}\beta_k\cdot\bigl(h^0(E(n))\cdot\mathrm{rk}(E_k)-h^0(\mathscr{F}_k(n))\cdot r\bigr)+\delta\cdot\mu(E_\bullet,\beta_\bullet;\varphi)+\\
& +\sum_{i=1}^{s}\sum_{j=1}^{t_i}a_{ij}\cdot\sum_{k=1}^{t}\beta_k\cdot\bigl(r\cdot\dim(q_{ij}(\mathscr{F}_k))-\mathrm{rk}(\mathscr{F}_k)\cdot r_{ij}\bigr)\\
\geq\quad & \sum_{k=1}^{t}\beta_k\cdot\bigl(h^0(E(n))\cdot\mathrm{rk}(E_k)-h^0(E_k(n))\cdot r\bigr)+\delta\cdot\mu(E_\bullet,\beta_\bullet;\varphi)+\\
& +\sum_{i=1}^{s}\sum_{j=1}^{t_i}a_{ij}\cdot\sum_{k=1}^{t}\beta_k\cdot\bigl(r\cdot\dim(q_{ij}(E_k))-\mathrm{rk}(E_k)\cdot r_{ij}\bigr) \qquad (19)\\
=\quad & \sum_{k=1}^{t}\beta_k\cdot\bigl(\deg(E)\cdot\mathrm{rk}(E_k)-\deg(E_k)\cdot r\bigr)+\delta\cdot\mu(E_\bullet,\beta_\bullet;\varphi)+\\
& +\sum_{i=1}^{s}\sum_{j=1}^{t_i}a_{ij}\cdot\sum_{k=1}^{t}\beta_k\cdot\bigl(r\cdot\dim(q_{ij}(E_k))-\mathrm{rk}(E_k)\cdot r_{ij}\bigr)\\
=\quad & M_{\underline{a}}(E_\bullet,\beta_\bullet)+\delta\cdot\mu(E_\bullet,\beta_\bullet;\varphi)\quad(\geq)\quad 0.
\end{aligned}
$$

Let $\mathfrak{S}$ be the bounded family of isomorphism classes of locally free sheaves E of rank r and degree d on C for which there exists an $(\underline{a},\delta)$-semistable decorated flagged vector bundle $(E,\underline{q},\varphi)$ of type $(r,d,\underline{x},\underline{r},u,v,L)$. Suppose that we have fixed some positive constant K. Then, we divide the locally free sheaves $\mathscr{F}$ on C that may occur as subsheaves of sheaves in the family $\mathfrak{S}$ into two classes:

A. $\mu(\mathscr{F})\geq d/r-K$

B. $\mu(\mathscr{F})<d/r-K$.

By the Langer–LePotier–Simpson estimate [28], there are non-negative constants K_1 and K_2 which depend only on r, such that any locally free $\mathscr{O}_C$-module A on C of rank at most r satisfies

$$
\begin{aligned}
& h^0(A)\leq\\
& \mathrm{rk}(A)\cdot\left(\frac{\mathrm{rk}(A)-1}{\mathrm{rk}(A)}\cdot\bigl[\mu_{\max}(A)+K_1+1\bigr]_+ +\frac{1}{\mathrm{rk}(A)}\cdot\bigl[\mu(A)+K_2+1\bigr]_+\right).
\end{aligned}
$$

For a sheaf A in Class B, this leads to

$$
h^0\bigl(A(n)\bigr)\leq\mathrm{rk}(A)\cdot\left(\frac{d}{r}+n+1+(r-1)(K_0+K_1)+K_2-\frac{1}{r}\cdot K\right),
$$

if the right hand side is positive. There exists an integer $n'(K)=n'(r,d,K_1,K_2,K)$ such that this holds for $n\geq n'(K)$. Furthermore, we estimate

$$
\begin{aligned}
& h^0(E(n))\cdot\mathrm{rk}(A)-h^0(A(n))\cdot r\geq\\
&\qquad -(r-1)rg-(r-2)(r-1)r(K_0+K_1)-(r-2)rK_2+K=:L.
\end{aligned}
$$

We choose K so large that

$$L \geq \delta \cdot u \cdot (r-1) + \left(\sum_{i=1}^{s}\sum_{j=1}^{t_i} a_{ij}\right) \cdot (r-1)^2.$$

Suppose that all the sheaves $\mathscr{F}_1, ..., \mathscr{F}_t$ belong to Class B. Then,

$$\begin{aligned}
& \sum_{k=1}^{t} \beta_k \cdot \big(h^0(E(n)) \cdot \mathrm{rk}(\mathscr{F}_k) - h^0(\mathscr{F}_k(n)) \cdot r\big) - \delta \cdot u \cdot (r-1) \cdot \sum_{k=1}^{l} \beta_k + \\
& + \sum_{i=1}^{s}\sum_{j=1}^{t_i} a_{ij} \cdot \sum_{k=1}^{t} \beta_k \cdot \big(r \cdot \dim(q_{ij}(\mathscr{F}_k)) - \mathrm{rk}(\mathscr{F}_k) \cdot r_{ij}\big) \\
\geq \; & \sum_{k=1}^{l} \beta_k \cdot \big(L - \delta \cdot u \cdot (r-1)\big) - \sum_{i=1}^{s}\sum_{j=1}^{t_i} a_{ij} \cdot \sum_{k=1}^{t} \beta_k \cdot \mathrm{rk}(E_k) \cdot r_{ij} \qquad (20) \\
\geq \; & \sum_{k=1}^{l} \beta_k \cdot \left(L - \delta \cdot u \cdot (r-1) - (r-1)^2 \cdot \left(\sum_{i=1}^{s}\sum_{j=1}^{t_i} a_{ij}\right)\right) \quad > \quad 0.
\end{aligned}$$

Note that the sheaves in Class A form a bounded family: the ranks and degrees of those sheaves belong to finite sets and their $\mu_{\max}$ is bounded by $\mu_{\max}(E)$, $[E] \in \mathfrak{S}$. Hence, there is an $n''(K)$, such that, for any $n \geq n''(K)$ and any sheaf A of Class A, one finds that $A(n)$ is globally generated and that $h^1(A(n)) = 0$. Set $n_1 := \max\{\, n'(K), n''(K)\,\}$. We have to verify our assertion. To do so, we set $I := \{\,1, ..., t\,\}$, $I_{\mathrm{A}} := \{\, i \in I \,|\, \mathscr{F}_i$ is in Class A$\,\}$, and $I_{\mathrm{B}} := \{\, i \in I \,|\, \mathscr{F}_i$ is in Class B$\,\}$, so that $I = I_{\mathrm{A}} \sqcup I_{\mathrm{B}}$. Write $I_{\mathrm{A/B}} = \{\, i_1^{\mathrm{A/B}}, ..., i_{t_{\mathrm{A/B}}}^{\mathrm{A/B}}\,\}$ with $i_1^{\mathrm{A/B}} < \cdots < i_{t_{\mathrm{A/B}}}^{\mathrm{A/B}}$. This gives the weighted filtrations

$$\big(E_\bullet^{\mathrm{A/B}} : 0 \subsetneq E_{i_1^{\mathrm{A/B}}} \subsetneq \cdots \subsetneq E_{i_{t_{\mathrm{A/B}}}^{\mathrm{A/B}}} \subsetneq E, \beta_\bullet^{\mathrm{A/B}} = (\beta_{i_1^{\mathrm{A/B}}}, ..., \beta_{i_{t_{\mathrm{A/B}}}^{\mathrm{A/B}}})\big).$$

It is then easy to see that

$$\mu\big(E_\bullet, \beta_\bullet; \varphi\big) \geq \mu\big(E_\bullet^{\mathrm{A}}, \beta_\bullet^{\mathrm{A}}; \varphi\big) - u \cdot (r-1) \cdot \sum_{j=1}^{t_{\mathrm{B}}} \beta_{i_j^{\mathrm{B}}}. \qquad (21)$$

Equation (21) together with the formulae (19) and (20) finally establishes the contention of the Proposition. □

The Remaining Steps. — The converse assertion, namely the fact that $(E_p, \underline{q}_p, \varphi_p)$ is $(\underline{a}, \delta)$-(semi)stable, if the Gieseker point associated with p is (semi)stable with respect to the linearization in $\mathscr{L}_{\underline{e}}$, is proved along similar lines, but is easier. The same holds for the proof of properness of the Gieseker map. The reader should combine the above arguments with those from [37] and [15] to fill in the details.

5.7 Construction of the Parameter Spaces for ϱ-Flagged Pseudo G-Bundles

We next include the explicit construction of the parameter space $\mathfrak{F}_{\varrho\text{-FLPsBun}}$ that will make the asserted properties in Proposition 5.3.1 evident.
There is a quasi-projective quot scheme $\mathfrak{Q}$ which parameterizes quotients $k\colon Y\otimes \mathscr{O}_C(-n)\longrightarrow E$ where E is a vector bundle of rank r and degree zero, such that $\mu_{\max}(E)\le D$, and where $H^0(k(n))$ is an isomorphism. The scheme $\mathfrak{Q}\times C$ carries the *universal quotient*

$$k_{\mathfrak{Q}}\colon Y\otimes \pi_C^\star\big(\mathscr{O}_C(-n)\big)\longrightarrow E_{\mathfrak{Q}}.$$

For the vector bundle $E_{\mathfrak{Q}}$, as for any vector bundle of rank r, we have the canonical isomorphism

$$E_{\mathfrak{Q}}^\vee \cong \bigwedge^{r-1}(E_{\mathfrak{Q}})\otimes\Big(\bigwedge^{r}(E_{\mathfrak{Q}})\Big)^\vee.$$

Since the restriction of $(\bigwedge^r(E_{\mathfrak{Q}}))^\vee$ to any fiber $\{k\}\times C$, $k\in\mathfrak{Q}$, is trivial, there is a line bundle $\mathscr{A}$ on $\mathfrak{Q}$, such that

$$\Big(\bigwedge^{r}(E_{\mathfrak{Q}})\Big)^\vee\cong \pi_{\mathfrak{Q}}^\star(\mathscr{A}).$$

Gathering all this information, we find a surjection

$$\mathscr{S}ym^\star\Big(V\otimes\bigwedge^{r-1}\big(Y\otimes\pi_C^\star(\mathscr{O}_C(-n))\big)\otimes\pi_{\mathfrak{Q}}^\star(\mathscr{A})\Big)^G\longrightarrow \mathscr{S}ym^\star\big(V\otimes E_{\mathfrak{Q}}^\vee\big)^G.$$

For a point $[q\colon Y\otimes\mathscr{O}_C(-n)\longrightarrow E]\in\mathfrak{Q}$, any homomorphism $\tau\colon \mathscr{S}ym^\star(V\otimes E^\vee)^G\longrightarrow\mathscr{O}_C$ of $\mathscr{O}_C$-algebras is determined by the composite homomorphism

$$\bigoplus_{i=1}^{s}\mathscr{S}ym^i\Big(V\otimes\bigwedge^{r-1}\big(Y\otimes\mathscr{O}_C(-n)\big)\Big)^G\longrightarrow\mathscr{O}_C$$

of $\mathscr{O}_C$-modules. Noting that

$$\mathscr{S}ym^i\Big(V\otimes\bigwedge^{r-1}\big(Y\otimes\mathscr{O}_C(-n)\big)\Big)^G\cong \mathrm{Sym}^i\Big(V\otimes\bigwedge^{r-1}Y\Big)^G\otimes\mathscr{O}_C\big(-i(r-1)n\big),$$

τ is determined by a collection of homomorphisms

$$\varphi_i\colon \mathrm{Sym}^i\Big(V\otimes\bigwedge^{r-1}Y\Big)^G\otimes\mathscr{O}_C\longrightarrow\mathscr{O}_C\big(i(r-1)n\big),\quad i=1,...,s.$$

Since φ_i is determined by the induced linear map on global sections, we will construct the parameter space inside

$$\overline{\mathfrak{Y}}:=\bigoplus_{i=1}^{s}\mathscr{H}om\Big(\mathscr{S}ym^i\Big(V\otimes\bigwedge^{r-1}Y\otimes\pi_{\mathfrak{Q}}^\star(\mathscr{A})\Big)^G,H^0\Big(\mathscr{O}_C\big(i(r-1)n\big)\Big)\otimes\mathscr{O}_{\mathfrak{Q}}\Big).$$

Write $\pi\colon \overline{\mathfrak{Y}} \longrightarrow \mathfrak{Q}$ for the bundle projection and observe that, over $\overline{\mathfrak{Y}} \times C$, there are universal homomorphisms

$$\widetilde{\varphi}^i\colon \mathscr{S}ym^i\Big(V\otimes \bigwedge^{r-1} Y\otimes\big(\pi_{\mathfrak{Q}}\circ(\pi\times \mathrm{id}_C)\big)^\star(\mathscr{A})\Big)^G \to H^0\Big(\mathscr{O}_C\big(i(r-1)n\big)\Big)\otimes\mathscr{O}_{\overline{\mathfrak{Y}}\times C},\,.$$

$i = 1, ..., s$. Define $\varphi^i = \mathrm{ev}\circ\widetilde{\varphi}^i$ as the composition of $\widetilde{\varphi}^i$ with the evaluation map $\mathrm{ev}\colon H^0(\mathscr{O}_C(i(r-1)n))\otimes\mathscr{O}_{\overline{\mathfrak{Y}}\times C}\longrightarrow \pi_C^\star(\mathscr{O}_C(i(r-1)n))$, $i = 1, ..., s$. We twist φ^i by $\mathrm{id}_{\pi_C^\star(\mathscr{O}_C(-i(r-1)n))}$ and put the resulting maps together to obtain the homomorphism

$$\varphi\colon \mathscr{V}_{\overline{\mathfrak{Y}}} \longrightarrow \mathscr{O}_{\overline{\mathfrak{Y}}\times C}$$

with

$$\mathscr{V}_{\overline{\mathfrak{Y}}} := \bigoplus_{i=1}^{s}\mathscr{S}ym^i\Big(V\otimes\bigwedge^{r-1}\Big(Y\otimes\pi_C^\star\big(\mathscr{O}_C(-n)\big)\Big)\otimes\big(\pi_{\mathfrak{Q}}\circ(\pi\times\mathrm{id}_C)\big)^\star(\mathscr{A})\Big)^G.$$

Next, φ yields a homomorphism of $\mathscr{O}_{\overline{\mathfrak{Y}}\times C}$-algebras

$$\widetilde{\tau}_{\overline{\mathfrak{Y}}}\colon \mathscr{S}ym^\star\big(\mathscr{V}_{\overline{\mathfrak{Y}}}\big)\longrightarrow\mathscr{O}_{\overline{\mathfrak{Y}}\times C}.$$

On the other hand, there is a surjective homomorphism

$$\beta\colon \mathscr{S}ym^\star(\mathscr{V}_{\overline{\mathfrak{Y}}})\longrightarrow\mathscr{S}ym^\star\Big(V\otimes(\pi\times\mathrm{id}_C)^\star\big(E_{\mathfrak{Q}}^\vee\big)\Big)^G$$

of graded algebras where the left hand algebra is graded by assigning the weight i to the elements in $\mathscr{S}ym^i(\cdots)^G$. The parameter space $\mathfrak{Y}$ is defined by the condition that $\widetilde{\tau}_{\overline{\mathfrak{Y}}}$ factorizes over β, i.e., setting $E_{\mathfrak{Y}} := ((\pi\times\mathrm{id}_C)^\star(E_{\mathfrak{Q}}))_{|\mathfrak{Y}\times C}$, there is a homomorphism

$$\tau_{\mathfrak{Y}}\colon \mathscr{S}ym^\star\big(V\otimes E_{\mathfrak{Y}}^\vee\big)^G\longrightarrow\mathscr{O}_{\mathfrak{Y}\times C}$$

with $\widetilde{\tau}_{\overline{\mathfrak{Y}}|\mathfrak{Y}\times C} = \tau_{\mathfrak{Y}}\circ\beta$. Formally, $\mathfrak{Y}$ is defined as the scheme theoretic intersection of the closed subschemes

$$\mathfrak{Y}_d := \Big\{\, y\in\overline{\mathfrak{Y}}\,\big|\,\widetilde{\tau}^d_{\overline{\mathfrak{Y}}|\{y\}\times C}\colon \ker\big(\beta^d_{|\{y\}\times C}\big)\longrightarrow\mathscr{O}_C \text{ is trivial}\Big\},\quad d\geq 0.$$

The family $(E_{\mathfrak{Y}},\tau_{\mathfrak{Y}})$ is the *universal family of pseudo G-bundles parameterized by $\mathfrak{Y}$*.

Remark 5.7.1. i) The scheme $\mathfrak{Y}$ is equipped with a natural $\mathrm{GL}(Y)$-action, and the vector bundle $E_{\mathfrak{Y}}$ is linearized with respect to this group action.
ii) Note that elimination theory shows that there is an open subscheme $\mathfrak{Y}^0$ that parameterizes the principal G-bundles. Moreover, there exists a *universal principal G-bundle* $\mathscr{P}_{\mathfrak{Y}^0}$ on $\mathfrak{Y}^0\times C$.
iii) There is a locally closed and $\mathrm{GL}(Y)$-invariant subscheme $\mathfrak{Y}^{\vartheta,\geq h}\subset\mathfrak{Y}^0$ which parameterizes those principal G-bundles $\mathscr{P}$ that have topological type ϑ and instability degree at least h. By construction, every such principal bundle $\mathscr{P}$ is represented by at least one point in $\mathfrak{Y}^{\vartheta,\geq h}$, so that we have a surjective map $\mathfrak{Y}^{\vartheta,\geq h}\longrightarrow\mathrm{Bun}_G^{\vartheta,\geq h}$. In fact, this map identifies $\mathrm{Bun}_G^{\vartheta,\geq h}$ with the quotient $[\mathfrak{Y}^{\vartheta,\geq h}/\,\mathrm{GL}(Y)]$.

We proceed to parameterize ϱ-flagged pseudo G-bundles. For this, we fix the tuple $\underline{x} = (x_1, ..., x_b)$ of points on C and the type $\underline{r} = (r_{ij}, j = 1, ..., t_i, i = 1, ..., b)$ of the flaggings. The tuple $(r_{i1}, ..., r_{it_i})$ determines the conjugacy class of a parabolic subgroup of $\mathrm{GL}(V)$. Pick representatives $\widetilde{P}_i$ for these conjugacy classes, $i = 1, ..., b$, and define

$$\widetilde{\mathfrak{F}}_i := \Big(\mathscr{I}som\big(V \otimes \mathscr{O}_{\mathfrak{Y}}, E_{\mathfrak{Y}|(\mathfrak{Y}\times\{x_i\})}\big)\Big)/\widetilde{P}_i, \quad i = 1, ..., b,$$

and

$$\mathfrak{F}_{\varrho\text{-FlPsBun}} := \widetilde{\mathfrak{F}}_1 \times_{\mathfrak{Y}} \cdots \times_{\mathfrak{Y}} \widetilde{\mathfrak{F}}_b.$$

Remark 5.7.2. Since the vector bundle $E_{\mathfrak{Y}}$ is linearized, $\widetilde{\mathfrak{F}}_i$, $i = 1, ..., b$, and $\mathfrak{F}_{\varrho\text{-FlPsBun}}$ inherit $\mathrm{GL}(Y)$-actions. The equivalence relation on geometric points that is induced by the group action on $\mathfrak{F}_{\varrho\text{-FlPsBun}}$ is isomorphy of ϱ-flagged pseudo G-bundles.

Acknowledgments

Alexander Schmitt acknowledges support by the DFG via a Heisenberg fellowship and via the Schwerpunkt program “Globale Methoden in der Komplexen Geometrie—Global Methods in Complex Geometry”. The article was partly written during the visit of A.S. to the Institut des Hautes Études Scientifiques where he benefitted from support of the European Commission through its 6th Framework Program “Structuring the European Research Area” and the contract Nr. RITA-CT-2004-505493 for the provision of Transnational Access implemented as Specific Support Action.
Jochen Heinloth was partially supported by the Leibniz-Preis of H. Esnault and E. Viehweg. He would like to thank G. Harder for raising the question a long time ago and U. Stuhler for his interest.

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