The Index of Centralizers of Elements of Reductive Lie Algebras

Jean-Yves Charbonnel and Anne Moreau

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Abstract. For a finite dimensional complex Lie algebra, its index is the minimal dimension of stabilizers for the coadjoint action. A famous conjecture due to A.G. Elashvili says that the index of the centralizer of an element of a reductive Lie algebra is equal to the rank. That conjecture caught attention of several Lie theorists for years. It reduces to the case of nilpotent elements. In [Pa03a] and [Pa03b], D.I. Panyushev proved the conjecture for some classes of nilpotent elements (e.g. regular, subregular and spherical nilpotent elements). Then the conjecture has been proven for the classical Lie algebras in [Y06a] and checked with a computer programme for the exceptional ones [deG08]. In this paper we give an almost general proof of that conjecture.

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1. Introduction

In this note $\Bbbk$ is an algebraically closed field of characteristic 0.

1.1. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\Bbbk$ and consider the coadjoint representation of $\mathfrak{g}$. By definition, the *index* of $\mathfrak{g}$ is the minimal dimension of stabilizers $\mathfrak{g}^x$, $x \in \mathfrak{g}^*$, for the coadjoint representation:

$$\operatorname{ind}\mathfrak{g} \ := \ \min\{\dim \mathfrak{g}^x ;\ x \in \mathfrak{g}^*\}.$$

The definition of the index goes back to Dixmier [Di74]. It is a very important notion in representation theory and in invariant theory. By Rosenlicht's theorem [Ro63], generic orbits of an arbitrary algebraic action of a linear algebraic group on an irreducible algebraic variety are separated by rational invariants; in particular, if $\mathfrak{g}$ is an algebraic Lie algebra,

$$\operatorname{ind}\mathfrak{g} = \deg\operatorname{tr}\Bbbk(\mathfrak{g}^*)^{\mathfrak{g}},$$

where $\Bbbk(\mathfrak{g}^*)^{\mathfrak{g}}$ is the field of $\mathfrak{g}$-invariant rational functions on $\mathfrak{g}^*$. The index of a reductive algebra equals its rank. For an arbitrary Lie algebra, computing its index seems to be a wild problem. However, there is a large number of interesting results for several classes of nonreductive subalgebras of reductive Lie algebras. For instance, parabolic subalgebras and their relatives as nilpotent radicals, seaweeds, are considered in [Pa03a], [TY04], [J07]. The centralizers, or normalizers of centralizers, of elements form another interesting class of such subalgebras, [E85a], [Pa03a], [Mo06b]. The last topic is closely related to the theory of integrable Hamiltonian systems [Bol91]. Let us precise this link.

From now on, $\mathfrak{g}$ is supposed to be reductive. Denote by G the adjoint group of $\mathfrak{g}$. The symmetric algebra $\mathrm{S}(\mathfrak{g})$ carries a natural Poisson structure. By the so-called *argument shift method*, for x in $\mathfrak{g}^*$, we can construct a Poisson-commutative family $\mathcal{F}_x$ in $\mathrm{S}(\mathfrak{g}) = \Bbbk[\mathfrak{g}^*]$; see [MF78] or Remark 1.4. It is generated by the derivatives of all orders in the direction $x \in \mathfrak{g}^*$ of all elements of the algebra $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ of $\mathfrak{g}$-invariants of $\mathrm{S}(\mathfrak{g})$. Moreover, if $G.x$ denotes the coadjoint orbit of $x \in \mathfrak{g}^*$:

Theorem 1.1 ([Bol91], Theorems 2.1 and 3.2). *There is a Poisson-commutative family of polynomial functions on $\mathfrak{g}^*$, constructed by the argument shift method, such that its restriction to $G.x$ contains $\frac{1}{2}\dim(G.x)$ algebraically independent functions if and only if* $\operatorname{ind}\mathfrak{g}^x = \operatorname{ind}\mathfrak{g}$.

Denote by $\operatorname{rk}\mathfrak{g}$ the rank of $\mathfrak{g}$. Motivated by the preceding result of Bolsinov, A.G. Elashvili formulated a conjecture:

Conjecture 1.2 (Elashvili). *Let $\mathfrak{g}$ be a reductive Lie algebra. Then* $\operatorname{ind}\mathfrak{g}^x = \operatorname{rk}\mathfrak{g}$ *for all $x \in \mathfrak{g}^*$.*

Elashvili's conjecture also appears in the following problem: Is the algebra $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ of invariants in $\mathrm{S}(\mathfrak{g}^x)$ under the adjoint action a polynomial algebra? This question was formulated by A. Premet in [PPY07, Conjecture 0.1]. After that, O. Yakimova discovered a counterexample [Y07], but the question remains very interesting. As an example, under certain hypothesis and under the condition that Elashvili's conjecture holds, the algebra of invariants $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is polynomial in $\operatorname{rk}\mathfrak{g}$ variables, [PPY07, Theorem 0.3].

During the last decade, Elashvili's conjecture caught attention of many invariant theorists [Pa03a], [Ch04], [Y06a], [deG08]. To begin with, describe some easy but useful reductions. Since the $\mathfrak{g}$-modules $\mathfrak{g}$ and $\mathfrak{g}^*$ are isomorphic, it is equivalent to prove Conjecture 1.2 for centralizers of elements of $\mathfrak{g}$. On the other hand, by a result due to E.B. Vinberg [Pa03a], the inequality $\operatorname{ind}\mathfrak{g}^x \geq \operatorname{rk}\mathfrak{g}$ holds for all $x \in \mathfrak{g}$. So it only remains to prove the opposite one. Given $x \in \mathfrak{g}$, let $x = x_{\rm s} + x_{\rm n}$ be its Jordan decomposition. Then $\mathfrak{g}^x = (\mathfrak{g}^{x_{\rm s}})^{x_{\rm n}}$. The subalgebra $\mathfrak{g}^{x_{\rm s}}$ is reductive of rank $\operatorname{rk}\mathfrak{g}$. Thus, the verification of Conjecture 1.2 reduces to the case of nilpotent elements. At last, one can clearly restrict oneself to the case of simple $\mathfrak{g}$.

Review now the main results obtained so far on Elashvili's conjecture. If x is regular, then $\mathfrak{g}^x$ is a commutative Lie algebra of dimension $\operatorname{rk}\mathfrak{g}$. So, Conjecture 1.2 is obviously true in that case. Further, the conjecture is known for subregular nilpotent elements and nilpotent elements of height 2 and 3, [Pa03a], [Pa03b]. Remind that the *height* of a nilpotent element e is the maximal integer m such that $(\mathrm{ad} e)^m \neq 0$. More recently, O. Yakimova proved the conjecture in the classical case [Y06a]. To valid the conjecture in the exceptional types, W. de Graaf used the computer programme GAP, see [deG08]. Since there are many nilpotent orbits in the Lie algebras of exceptional type, it is difficult to present the results of such computations in a concise way. In 2004, the first author published a case-free proof of Conjecture 1.2 applicable to all simple Lie algebras; see [Ch04]. Unfortunately, the argument in [Ch04] has a gap in the final part of the proof which was pointed out by L. Rybnikov.

To summarize, so far, there is no conceptual proof of Conjecture 1.2. Nevertheless, according to Yakimova's works and de Graaf's works, we can claim:

THEOREM 1.3 ([Y06a], [deG08]). *Let $\mathfrak{g}$ be a reductive Lie algebra. Then $\operatorname{ind}\mathfrak{g}^x = \operatorname{rk}\mathfrak{g}$ for all $x \in \mathfrak{g}^*$.*

Because of the importance of Elashvili's conjecture in invariant theory, it would be very appreciated to find a general proof of Theorem 1.3 applicable to all finite-dimensional simple Lie algebras. The proof we propose in this paper is fresh and almost general. More precisely, it remains 7 isolated cases; one nilpotent orbit in type E_7 and six nilpotent orbits in type E_8 have to be considered separately. For these 7 orbits, the use of GAP is unfortunately necessary. In order to provide a complete proof of Theorem 1.3, we include in this paper the computations using GAP we made to deal with these remaining seven cases.

1.2. DESCRIPTION OF THE PAPER. Let us briefly explain our approach. Denote by $\mathcal{N}(\mathfrak{g})$ the nilpotent cone of $\mathfrak{g}$. As noticed previously, it suffices to prove $\operatorname{ind}\mathfrak{g}^e = \operatorname{rk}\mathfrak{g}$ for all e in $\mathcal{N}(\mathfrak{g})$. If the equality holds for e, it

does for all elements of $G.e$; we shortly say that $G.e$ satisfies Elashvili's conjecture.
From a nilpotent orbit $\mathcal{O}_{\mathfrak{l}}$ of a reductive factor $\mathfrak{l}$ of a parabolic subalgebra of $\mathfrak{g}$, we can construct a nilpotent orbit of $\mathfrak{g}$ having the same codimension in $\mathfrak{g}$ as $\mathcal{O}_{\mathfrak{l}}$ in $\mathfrak{l}$ and having other remarkable properties. The nilpotent orbits obtained in such a way are called *induced*; the other ones are called *rigid*. We refer the reader to Subsection 2.3 for more precisions about this topic. Using Bolsinov's criterion of Theorem 1.1, we first prove Theorem 1.3 for all induced nilpotent orbits and so the conjecture reduces to the case of rigid nilpotent orbits. To deal with rigid nilpotent orbits, we use methods developed in [Ch04] by the first author, and resumed in [Mo06a] by the second author, based on nice properties of Slodowy slices of nilpotent orbits.

In more details, the paper is organized as follows:

We state in Section 2 the necessary preliminary results. In particular, we investigate in Subsection 2.2 extensions of Bolsinov's criterion and we establish an important result (Theorem 2.7) which will be used repeatedly in the sequel. We prove in Section 3 the conjecture for all induced nilpotent orbits (Theorem 3.3) so that Elashvili's conjecture reduces to the case of rigid nilpotent orbits (Theorem 3.3). From Section 4, we handle the rigid nilpotent orbits: we introduce and study in Section 4 a property (P) given by Definition 4.2. Then, in Section 5, we are able to deal with almost all rigid nilpotent orbits. Still in Section 5, the remaining cases are dealt with set-apart by using a different approach.

1.3. Notations. • If E is a subset of a vector space V, we denote by $\mathrm{span}(E)$ the vector subspace of V generated by E. The grassmanian of all d-dimensional subspaces of V is denoted by $\mathrm{Gr}_d(V)$. By a *cone* of V, we mean a subset of V invariant under the natural action of $\Bbbk^* := \Bbbk\setminus\{0\}$ and by a *bicone* of $V \times V$ we mean a subset of $V \times V$ invariant under the natural action of $\Bbbk^* \times \Bbbk^*$ on $V \times V$.
• From now on, we assume that $\mathfrak{g}$ is semisimple of rank ℓ and we denote by $\langle .,. \rangle$ the Killing form of $\mathfrak{g}$. We identify $\mathfrak{g}$ to $\mathfrak{g}^*$ through $\langle .,. \rangle$. Unless otherwise specified, the notion of orthogonality refers to the bilinear form $\langle .,. \rangle$.
• Denote by $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ the algebra of $\mathfrak{g}$-invariant elements of $\mathrm{S}(\mathfrak{g})$. Let $f_1, \ldots, f_\ell$ be homogeneous generators of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ of degrees $d_1, \ldots, d_\ell$ respectively. We choose the polynomials $f_1, \ldots, f_\ell$ so that $d_1 \leq \cdots \leq d_\ell$. For $i = 1, \ldots, \ell$ and $(x, y) \in \mathfrak{g} \times \mathfrak{g}$, we may consider a shift of f_i in direction y: $f_i(x + ty)$ where $t \in \Bbbk$. Expanding $f_i(x + ty)$ as a polynomial

in t, we obtain

$$(1)\qquad f_i(x+ty) = \sum_{m=0}^{d_i} f_i^{(m)}(x,y)t^m; \qquad \forall (t,x,y) \in \Bbbk \times \mathfrak{g} \times \mathfrak{g}$$

where $y \mapsto (m!)f_i^{(m)}(x,y)$ is the differential at x of f_i of the order m in the direction y. The elements $f_i^{(m)}$ as defined by (1) are invariant elements of $\mathrm{S}(\mathfrak{g}) \otimes_{\Bbbk} \mathrm{S}(\mathfrak{g})$ under the diagonal action of G on $\mathfrak{g} \times \mathfrak{g}$. Note that $f_i^{(0)}(x,y) = f_i(x)$ while $f_i^{(d_i)}(x,y) = f_i(y)$ for all $(x,y) \in \mathfrak{g} \times \mathfrak{g}$.

Remark 1.4. The family $\mathcal{F}_x := \{f_i^{(m)}(x,.);\ 1 \leq i \leq \ell, 1 \leq m \leq d_i\}$ for $x \in \mathfrak{g}$, is a Poisson-commutative family of $\mathrm{S}(\mathfrak{g})$ by Mishchenko-Fomenko [MF78]. One says that the family $\mathcal{F}_x$ is constructed by the *argument shift method.*

• Let $i \in \{1, \ldots, \ell\}$. For x in $\mathfrak{g}$, we denote by $\varphi_i(x)$ the element of $\mathfrak{g}$ satisfying $(\mathrm{d}f_i)_x(y) = f_i^{(1)}(x,y) = \langle \varphi_i(x), y\rangle$, for all y in $\mathfrak{g}$. Thereby, φ_i is an invariant element of $\mathrm{S}(\mathfrak{g}) \otimes_{\Bbbk} \mathfrak{g}$ under the canonical action of G. We denote by $\varphi_i^{(m)}$, for $0 \leq m \leq d_i - 1$, the elements of $\mathrm{S}(\mathfrak{g}) \otimes_{\Bbbk} \mathrm{S}(\mathfrak{g}) \otimes_{\Bbbk} \mathfrak{g}$ defined by the equality:

$$(2)\qquad \varphi_i(x+ty) = \sum_{m=0}^{d_i-1} \varphi_i^{(m)}(x,y)t^m, \qquad \forall (t,x,y) \in \Bbbk \times \mathfrak{g} \times \mathfrak{g}.$$

• For $x \in \mathfrak{g}$, we denote by $\mathfrak{g}^x = \{y \in \mathfrak{g} \mid [y,x] = 0\}$ the centralizer of x in $\mathfrak{g}$ and by $\mathfrak{z}(\mathfrak{g}^x)$ the center of $\mathfrak{g}^x$. The set of regular elements of $\mathfrak{g}$ is

$$\mathfrak{g}_{\mathrm{reg}} := \{x \in \mathfrak{g} \mid \dim \mathfrak{g}^x = \ell\}$$

and we denote by $\mathfrak{g}_{\mathrm{reg,ss}}$ the set of regular semisimple elements of $\mathfrak{g}$. Both $\mathfrak{g}_{\mathrm{reg}}$ and $\mathfrak{g}_{\mathrm{reg,ss}}$ are G-invariant dense open subsets of $\mathfrak{g}$.
We denote by $C(x)$ the G-invariant cone generated by x and we denote by x_{s} and x_{n} the semisimple and nilpotent components of x respectively.
• The nilpotent cone of $\mathfrak{g}$ is $\mathcal{N}(\mathfrak{g})$. As a rule, for $e \in \mathcal{N}(\mathfrak{g})$, we choose an $\mathfrak{sl}_2$-triple (e,h,f) in $\mathfrak{g}$ given by the Jacobson-Morozov theorem [CMa93, Theorem 3.3.1]. In particular, it satisfies the equalities:

$$[h,e] = 2e, \qquad [e,f] = h, \qquad [h,f] = -2f$$

The action of $\mathrm{ad}h$ on $\mathfrak{g}$ induces a $\mathbb{Z}$-grading:

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)\ ,\ \mathfrak{g}(i) = \{x \in \mathfrak{g} \mid [h,x] = ix\}.$$

Recall that e, or $G.e$, is said to be *even* if $\mathfrak{g}(i) = 0$ for odd i. Note that $e \in \mathfrak{g}(2)$, $f \in \mathfrak{g}(-2)$ and that $\mathfrak{g}^e$, $\mathfrak{z}(\mathfrak{g}^e)$ and $\mathfrak{g}^f$ are all $\mathrm{ad}h$-stable.

• All topological terms refer to the Zariski topology. If Y is a subset of a topological space X, we denote by $\overline{Y}$ the closure of Y in X.

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Contents

1. Introduction 387
2. Preliminary results 392
3. Proof of Theorem 1.3 for induced nilpotent orbits 398
4. The Slodowy slice and the property (P) 400
5. Proof of Theorem 1.3 for rigid nilpotent orbits 408
Appendix A. Proof of Proposition 5.10: explicit computations. 413
References 418

2. Preliminary results

We start in this section by reviewing some facts about the differentials of generators of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$. Then, the goal of Subsection 2.2 is Theorem 2.7. We collect in Subsection 2.3 basic facts about induced nilpotent orbits.

2.1. Differentials of generators of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$. According to subsection 1.3, the elements $\varphi_1,\ldots,\varphi_\ell$ of $\mathrm{S}(\mathfrak{g}) \otimes_{\Bbbk} \mathfrak{g}$ are the differentials of $f_1,\ldots,f_\ell$ respectively. Since $f_i(g(x)) = f_i(x)$ for all $(x,g) \in \mathfrak{g} \times G$, the element $\varphi_i(x)$ centralizes x for all $x \in \mathfrak{g}$. Moreover:

Lemma 2.1. (i)[Ri87, Lemma 2.1] *The elements $\varphi_1(x),\ldots,\varphi_\ell(x)$ belong to $\mathfrak{z}(\mathfrak{g}^e)$.*
(ii)[Ko63, Theorem 9] *The elements $\varphi_1(x),\ldots,\varphi_\ell(x)$ are linearly independent elements of $\mathfrak{g}$ if and only if x is regular. Moreover, if so, $\varphi_1(x),\ldots,\varphi_\ell(x)$ is a basis of $\mathfrak{g}^x$.*

We turn now to the elements $\varphi_i^{(m)}$, for $i = 1,\ldots,\ell$ and $0 \leq m \leq d_i - 1$, defined in Subsection 1.3 by (2). Recall that d_i is the degree of the homogeneous polynomial f_i, for $i = 1,\ldots,\ell$. The integers $d_1-1,\ldots,d_\ell-1$ are thus the exponents of $\mathfrak{g}$. By a classical result [Bou02, Ch. V, §5, Proposition 3], we have $\sum d_i = \mathrm{b}_{\mathfrak{g}}$ where $\mathrm{b}_{\mathfrak{g}}$ is the dimension of Borel subalgebras of $\mathfrak{g}$. For (x,y) in $\mathfrak{g} \times \mathfrak{g}$, we set:

$$(3) \qquad V_{x,y} := \mathrm{span}\{\varphi_i^{(m)}(x,y) \ ; \quad 1 \leq i \leq \ell, 0 \leq m \leq d_i - 1\}.$$

The subspaces $V_{x,y}$ will play a central role throughout the note.

Remark 2.2. (1) For $(x,y) \in \mathfrak{g}\times\mathfrak{g}$, the dimension of $V_{x,y}$ is at most $\mathrm{b}_{\mathfrak{g}}$ since $\sum d_i = \mathrm{b}_{\mathfrak{g}}$. Moreover, for all (x,y) in a nonempty open subset of $\mathfrak{g}\times\mathfrak{g}$, the equality holds [Bol91]. Actually, in this note, we do not need this observation.
(2) By Lemma 2.1(ii), if x is regular, then $\mathfrak{g}^x$ is contained in $V_{x,y}$ for all $y \in \mathfrak{g}$. In particular, if so, $\dim [x, V_{x,y}] = \dim V_{x,y} - \ell$.

The subspaces $V_{x,y}$ were introduced and studied by Bolsinov in [Bol91], motivated by the maximality of Poisson-commutative families in $\mathrm{S}(\mathfrak{g})$. These subspaces have been recently exploited in [PY08] and [CMo08]. The following results are mostly due to Bosinov, [Bol91]. We refer to [PY08] for a more recent account about this topic. We present them in a slightly different way:

Lemma 2.3. *Let (x,y) be in $\mathfrak{g}_{\mathrm{reg}}\times\mathfrak{g}$.*
(i) *The subspace $V_{x,y}$ of $\mathfrak{g}$ is the sum of the subspaces $\mathfrak{g}^{x+ty}$ where t runs through any nonempty open subset of $\Bbbk$ such that $x+ty$ is regular for all t in this subset.*
(ii) *The subspace $\mathfrak{g}^y + V_{x,y}$ is a totally isotropic subspace of $\mathfrak{g}$ with respect to the Kirillov form K_y on $\mathfrak{g}\times\mathfrak{g}$, $(v,w)\mapsto \langle y,[v,w]\rangle$. Furthermore, $\dim(\mathfrak{g}^y+V_{x,y})^{\perp} \geq \frac{1}{2}\dim G.y$.*
(iii) *The subspaces $[x,V_{x,y}]$ and $[y,V_{x,y}]$ are equal.*

Proof. (i) Let O be a nonempty open subset of $\Bbbk$ such that $x+ty$ is regular for all t in O. Such an open subset does exist since x is regular. Denote by V_O the sum of all the subspaces $\mathfrak{g}^{x+ty}$ where t runs through O. For all t in O, $\mathfrak{g}^{x+ty}$ is generated by $\varphi_1(x+ty),\ldots,\varphi_\ell(x+ty)$, cf. Lemma 2.1(ii). As a consequence, V_O is contained in $V_{x,y}$. Conversely, for $i=1,\ldots,\ell$ and for $t_1,\ldots,t_{d_i}$ pairwise different elements of O, $\varphi_i^{(m)}(x,y)$ is a linear combination of $\varphi_i(x+t_1y),\ldots,\varphi_i(x+t_{d_i}y)$; hence $\varphi_i^{(m)}(x,y)$ belongs to V_O. Thus $V_{x,y}$ is equal to V_O, whence the assertion.
(ii) results from [PY08, Proposition A4]. Notice that in (ii) the inequality is an easy consequence of the first statement.
At last, [PY08, Lemma A2] gives us (iii). $\square$

Let σ and σ_i, for $i=1,\ldots,\ell$, be the maps

$$\begin{array}{ccc} \mathfrak{g}\times\mathfrak{g} & \xrightarrow{\sigma} & \Bbbk^{\mathrm{b}_{\mathfrak{g}}+\ell} \\ (x,y) & \longmapsto & (f_i^{(m)}(x,y))_{\substack{1\leq i\leq \ell,\\ 0\leq m\leq d_i}} \end{array}, \qquad \begin{array}{ccc} \mathfrak{g}\times\mathfrak{g} & \xrightarrow{\sigma_i} & \Bbbk^{d_i+1} \\ (x,y) & \longmapsto & (f_i^{(m)}(x,y))_{0\leq m\leq d_i} \end{array}$$

respectively, and denote by $\sigma'(x,y)$ and $\sigma'_i(x,y)$ the tangent map at (x,y) of σ and σ_i respectively. Then $\sigma'_i(x,y)$ is given by the differentials of the $f_i^{(m)}$'s at (x,y) and $\sigma'(x,y)$ is given by the elements $\sigma'_i(x,y)$.

Lemma 2.4. *Let (x,y) and (v,w) be in $\mathfrak{g}\times\mathfrak{g}$.*
(i) *For $i=1,\ldots,\ell$, $\sigma'_i(x,y)$ maps (v,w) to*

$$\begin{aligned}(\langle\varphi_i(x),v\rangle, \langle\varphi_i^{(1)}(x,y),v\rangle + \langle\varphi_i^{(0)}(x,y),w\rangle,\\ \ldots,\langle\varphi_i^{(d_i-1)}(x,y),v\rangle + \langle\varphi_i^{(d_i-2)}(x,y),w\rangle,\langle\varphi_i(y),w\rangle).\end{aligned}$$

(ii) *Suppose that $\sigma'(x,y)(v,w)=0$. Then, for w' in $\mathfrak{g}$, $\sigma'(x,y)(v,w')=0$ if and only if $w-w'$ is orthogonal to $V_{x,y}$.*
(iii) *For $x\in\mathfrak{g}_{\rm reg}$, $\sigma'(x,y)(v,w')=0$ for some $w'\in\mathfrak{g}$ if and only if $v\in[x,\mathfrak{g}]$.*

Proof. (i) The verifications are easy and left to the reader.
(ii) Since $\sigma'(x,y)(v,w)=0$, $\sigma'(x,y)(v,w')=0$ if and only if $\sigma'(x,y)(v,w-w')=0$ whence the statement by (i).
(iii) Suppose that x is regular and suppose that $\sigma'(x,y)(v,w')=0$ for some $w'\in\mathfrak{g}$. Then by (i), v is orthogonal to the elements $\varphi_1(x),\ldots,\varphi_\ell(x)$. So by Lemma 2.1(ii), v is orthogonal to $\mathfrak{g}^x$. Since $\mathfrak{g}^x$ is the orthogonal complement of $[x,\mathfrak{g}]$ in $\mathfrak{g}$, we deduce that v lies in $[x,\mathfrak{g}]$. Conversely, since $\sigma(x,y)=\sigma(g(x),g(y))$ for all g in G, the element $([u,x],[u,y])$ belongs to the kernel of $\sigma'(x,y)$ for all $u\in\mathfrak{g}$. So, the converse implication follows. □

2.2. On Bolsinov's criterion. Let a be in $\mathfrak{g}$ and denote by π the map

$$\begin{array}{rcl}\mathfrak{g}\times G.a & \xrightarrow{\pi} & \mathfrak{g}\times\Bbbk^{\mathrm{b}_\mathfrak{g}+\ell}\\ (x,y) & \longmapsto & (x,\sigma(x,y)).\end{array}$$

Remark 2.5. Recall that the family $(\mathcal{F}_x)_{x\in\mathfrak{g}}$ constructed by the argument shift method consists of all elements $f_i^{(m)}(x,.)$ for $i=1,\ldots,\ell$ and $1\leq m\leq d_i$, see Remark 1.4. By definition of the morphism π, there is a family constructed by the argument shift method whose restriction to $G.a$ contains $\frac{1}{2}\dim G.a$ algebraically independent functions if and only if π has a fiber of dimension $\frac{1}{2}\dim G.a$.

In view of Theorem 1.1 and the above remark, we now concentrate on the fibers of π. For $(x,y)\in\mathfrak{g}\times G.a$, denote by $F_{x,y}$ the fiber of π at $\pi(x,y)$:

$$F_{x,y} := \{x\}\times\{y'\in G.a \mid \sigma(x,y')=\sigma(x,y)\}.$$

Lemma 2.6. *Let (x,y) be in $\mathfrak{g}\times G.a$.*
(i) *The irreducible components of $F_{x,y}$ have dimension at least $\frac{1}{2}\dim G.a$.*

(ii) *The fiber $F_{x,y}$ has dimension $\frac{1}{2}\dim G.a$ if and only if any irreducible component of $F_{x,y}$ contains an element (x,y') such that $(\mathfrak{g}^{y'}+V_{x,y'})^{\perp}$ has dimension $\frac{1}{2}\dim G.a$.*

Proof. We prove (i) and (ii) all together. The tangent space $\mathrm{T}_{x,y'}(F_{x,y})$ of $F_{x,y}$ at (x,y') in $F_{x,y}$ identifies to the subspace of elements w of $[y',\mathfrak{g}]$ such that $\sigma'(x,y')(0,w)=0$. Hence, by Lemma 2.4(ii),

$$\mathrm{T}_{x,y'}(F_{x,y}) = [y',\mathfrak{g}]\cap V_{x,y'}^{\perp} = (\mathfrak{g}^{y'}+V_{x,y'})^{\perp},$$

since $[y',\mathfrak{g}]=(\mathfrak{g}^{y'})^{\perp}$. But by Lemma 2.3(ii), $(\mathfrak{g}^{y'}+V_{x,y'})^{\perp}$ has dimension at least $\frac{1}{2}\dim G.a$; so does $\mathrm{T}_{x,y'}(F_{x,y})$. This proves (i). Moreover, the equality holds if and only if $(\mathfrak{g}^{y'}+V_{x,y'})^{\perp}$ has dimension $\frac{1}{2}\dim G.a$, whence the statement (ii). □

Theorem 2.7. *The following conditions are equivalent:*

1. $\mathrm{ind}\,\mathfrak{g}^a=\ell$;
2. π *has a fiber of dimension* $\frac{1}{2}\dim G.a$;
3. *there exists* $(x,y)\in\mathfrak{g}\times G.a$ *such that* $(\mathfrak{g}^y+V_{x,y})^{\perp}$ *has dimension* $\frac{1}{2}\dim G.a$;
4. *there exists x in $\mathfrak{g}_{\mathrm{reg}}$ such that* $\dim(\mathfrak{g}^a+V_{x,a})=\frac{1}{2}(\dim\mathfrak{g}+\dim\mathfrak{g}^a)$;
5. *there exists x in $\mathfrak{g}_{\mathrm{reg}}$ such that* $\dim V_{x,a}=\frac{1}{2}\dim G.a+\ell$;
6. $\sigma(\mathfrak{g}\times\{a\})$ *has dimension* $\frac{1}{2}\dim G.a+\ell$.

Proof. By Theorem 1.1 and Remark 2.5, we have (1)⇔(2). Moreover, by Lemma 2.6(ii), we have (2)⇔(3).
(3)⇔(4): If (4) holds, so does (3). Indeed, if so,

$$\dim\mathfrak{g}-\frac{1}{2}\dim G.a=\frac{1}{2}(\dim\mathfrak{g}+\dim\mathfrak{g}^a)=\dim(\mathfrak{g}^a+V_{x,a}).$$

Conversely, suppose that (3) holds. By Lemma 2.3(ii), $\mathfrak{g}^y+V_{x,y}$ has maximal dimension $\frac{1}{2}(\dim\mathfrak{g}+\dim\mathfrak{g}^y)$. So the same goes for all (x,y) in a G-invariant nonempty open subset of $\mathfrak{g}\times G.a$. Hence, since the map $(x,y)\mapsto V_{x,y}$ is G-equivariant, there exists x in $\mathfrak{g}_{\mathrm{reg}}$ such that

$$\dim(V_{x,a}+\mathfrak{g}^a)=\frac{1}{2}(\dim\mathfrak{g}+\dim\mathfrak{g}^a).$$

(4)⇔(5): Let x be in $\mathfrak{g}_{\mathrm{reg}}$. By Lemma 2.3(iii), $[x,V_{x,a}]=[a,V_{x,a}]$. Hence $\mathfrak{g}^a\cap V_{x,a}$ has dimension ℓ by Remark 2.2(2). As a consequence,

$$\dim(\mathfrak{g}^a+V_{x,a})=\dim\mathfrak{g}^a+\dim V_{x,a}-\ell,$$

whence the equivalence.

(2)⇔(6): Suppose that (2) holds. By Lemma 2.6, $\frac{1}{2}\dim G.a$ is the minimal dimension of the fibers of π. So, $\pi(\mathfrak{g}\times G.a)$ has dimension

$$\dim \mathfrak{g} + \dim G.a - \frac{1}{2}\dim G.a = \dim \mathfrak{g} + \frac{1}{2}\dim G.a.$$

Denote by τ the restriction to $\pi(\mathfrak{g}\times G.a)$ of the projection map $\mathfrak{g}\times \Bbbk^{\mathrm{b}_{\mathfrak{g}}+\ell} \to \Bbbk^{\mathrm{b}_{\mathfrak{g}}+\ell}$. Then $\tau\circ\pi$ is the restriction of σ to $\mathfrak{g}\times G.a$. Since σ is a G-invariant map, $\sigma(\mathfrak{g}\times\{a\}) = \sigma(\mathfrak{g}\times G.a)$. Let $(x,y)\in \mathfrak{g}_{\mathrm{reg,ss}}\times G.a$. The fiber of τ at $z=\sigma(x,y)$ is $G.x$ since x is a regular semisimple element of $\mathfrak{g}$. Hence,

$$\dim \sigma(\mathfrak{g}\times\{a\}) \quad = \quad \dim \pi(\mathfrak{g}\times G.a) - (\dim \mathfrak{g} - \ell) = \frac{1}{2}\dim G.a + \ell$$

and we obtain (6).
Conversely, suppose that (6) holds. Then $\pi(\mathfrak{g}\times G.a)$ has dimension $\dim \mathfrak{g} + \frac{1}{2}\dim G.a$ by the above equality. So the minimal dimension of the fibers of π is equal to

$$\dim \mathfrak{g} + \dim G.a - (\dim \mathfrak{g} + \frac{1}{2}\dim G.a) = \frac{1}{2}\dim G.a$$

and (2) holds. □

2.3. Induced and rigid nilpotent orbits. The definitions and results of this subsection are mostly extracted from [Di74], [Di75], [LS79] and [BoK79]. We refer to [CMa93] and [TY05] for recent surveys.
Let $\mathfrak{p}$ be a proper parabolic subalgebra of $\mathfrak{g}$ and let $\mathfrak{l}$ be a reductive factor of $\mathfrak{p}$. We denote by $\mathfrak{p}_{\mathrm{u}}$ the nilpotent radical of $\mathfrak{p}$. Denote by L the connected closed subgroup of G whose Lie algebra is $ad\,\mathfrak{l}$ and denote by P the normalizer of $\mathfrak{p}$ in G.

Theorem 2.8 ([CMa93],Theorem 7.1.1). *Let $\mathcal{O}_{\mathfrak{l}}$ be a nilpotent orbit of $\mathfrak{l}$. There exists a unique nilpotent orbit $\mathcal{O}_{\mathfrak{g}}$ in $\mathfrak{g}$ whose intersection with $\mathcal{O}_{\mathfrak{l}}+\mathfrak{p}_{\mathrm{u}}$ is a dense open subset of $\mathcal{O}_{\mathfrak{l}}+\mathfrak{p}_{\mathrm{u}}$. Moreover, the intersection of $\mathcal{O}_{\mathfrak{g}}$ and $\mathcal{O}_{\mathfrak{l}}+\mathfrak{p}_{\mathrm{u}}$ consists of a single P-orbit and* $\mathrm{codim}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}}) = \mathrm{codim}_{\mathfrak{l}}(\mathcal{O}_{\mathfrak{l}})$.

The orbit $\mathcal{O}_{\mathfrak{g}}$ only depends on $\mathfrak{l}$ and not on the choice of a parabolic subalgebra $\mathfrak{p}$ containing it [CMa93, Theorem 7.1.3]. By definition, the orbit $\mathcal{O}_{\mathfrak{g}}$ is called the *induced orbit from* $\mathcal{O}_{\mathfrak{l}}$; it is denoted by $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}})$. If $\mathcal{O}_{\mathfrak{l}}=0$, then we call $\mathcal{O}_{\mathfrak{g}}$ a *Richardson orbit*. For example all even nilpotent orbits are Richardson [CMa93, Corollary 7.1.7]. In turn, not all nilpotent orbits are induced from another one. A nilpotent orbit which is not induced in a proper way from another one is called *rigid*.
We shall say that $e\in\mathcal{N}(\mathfrak{g})$ is an induced (respectively rigid) nilpotent element of $\mathfrak{g}$ if the G-orbit of e is an induced (respectively rigid) nilpotent orbit of $\mathfrak{g}$. The following results are deeply linked to the properties of

the sheets of $\mathfrak{g}$ and the deformations of its G-orbits. We refer to [BoK79] about these notions.

THEOREM 2.9. (i) *Let x be a non nilpotent element of $\mathfrak{g}$ and let $\mathcal{O}_{\mathfrak{g}}$ be the induced nilpotent orbit from the adjoint orbit of x_n in $\mathfrak{g}^{x_s}$. Then $\mathcal{O}_{\mathfrak{g}}$ is the unique nilpotent orbit contained in $\overline{C(x)}$ whose dimension is $\dim G.x$. Furthermore, $\overline{C(x)} \cap \mathcal{N}(\mathfrak{g}) = \overline{\mathcal{O}_{\mathfrak{g}}}$ and $\overline{C(x)} \cap \mathcal{N}(\mathfrak{g})$ is the nullvariety in $\overline{C(x)}$ of f_i where i is an element of $\{1, \ldots, \ell\}$ such that $f_i(x) \neq 0$.*
(ii) *Conversely, if $\mathcal{O}_{\mathfrak{g}}$ is an induced nilpotent orbit, there exists a non nilpotent element x of $\mathfrak{g}$ such that $\overline{C(x)} \cap \mathcal{N}(\mathfrak{g}) = \overline{\mathcal{O}_{\mathfrak{g}}}$.*

Proof. (i) Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ having $\mathfrak{g}^{x_s}$ as a Levi factor. Denote by $\mathfrak{p}_u$ its nilpotent radical and by P the normalizer of $\mathfrak{p}$ in G. Let $\mathcal{O}'$ be the adjoint orbit of x_n in $\mathfrak{g}^{x_s}$.

Claim 2.10. Let C be the P-invariant closed cone generated by x and let C_0 be the subset of nilpotent elements of C. Then $C = \Bbbk x_s + \overline{\mathcal{O}'} + \mathfrak{p}_u$, $C_0 = \overline{\mathcal{O}'} + \mathfrak{p}_u$ and C_0 is an irreducible subset of dimension $\dim P(x)$.

Proof. The subset $x_s + \overline{\mathcal{O}'} + \mathfrak{p}_u$ is an irreducible closed subset of $\mathfrak{p}$ containing $P(x)$. Moreover, its dimension is equal to

$$\dim \mathcal{O}' + \dim \mathfrak{p}_u = \dim \mathfrak{g}^{x_s} - \dim \mathfrak{g}^{x} + \dim \mathfrak{p}_u = \dim \mathfrak{p} - \dim \mathfrak{g}^{x}.$$

Since the closure of $P(x)$ and $x_s + \overline{\mathcal{O}'} + \mathfrak{p}_u$ are both irreducible subsets of $\mathfrak{g}$, they coincide. As a consequence, the set $\Bbbk x_s + \overline{\mathcal{O}'} + \mathfrak{p}_u$ is contained in C. Since the former set is clearly a closed conical subset of $\mathfrak{g}$ containing x, $C = \Bbbk x_s + \overline{\mathcal{O}'} + \mathfrak{p}_u$. Then we deduce that $C_0 = \overline{\mathcal{O}'} + \mathfrak{p}_u$. □

Denote by $G \times_P \mathfrak{g}$ the quotient of $G \times \mathfrak{g}$ under the right action of P given by $(g, z).p := (gp, p^{-1}(z))$. The map $(g, z) \mapsto g(z)$ from $G \times \mathfrak{g}$ to $\mathfrak{g}$ factorizes through the quotient map from $G \times \mathfrak{g}$ to $G \times_P \mathfrak{g}$. Since G/P is a projective variety, the so obtained map from $G \times_P \mathfrak{g}$ to $\mathfrak{g}$ is closed. Since C and C_0 are closed P-invariant subsets of $\mathfrak{g}$, $G \times_P C$ and $G \times_P C_0$ are closed subsets of $G \times_P \mathfrak{g}$. Hence $\overline{C(x)} = G(C)$ and $G(C_0)$ is a closed subset of $\mathfrak{g}$. So, by the claim, the subset of nilpotent elements of $\overline{C(x)}$ is irreducible since C_0 is irreducible. Since there are finitely many nilpotent orbits, the subset of nilpotent elements of $\overline{C(x)}$ is the closure of one nilpotent orbit. Denote it by $\tilde{\mathcal{O}}$ and prove $\tilde{\mathcal{O}} = \mathcal{O}_{\mathfrak{g}}$.
For all k, l in $\{1, \ldots, \ell\}$, denote by $p_{k,l}$ the polynomial function

$$p_{k,l} := f_k(x)^{d_l} f_l^{d_k} - f_l(x)^{d_k} f_k^{d_l}$$

Then $p_{k,l}$ is G-invariant and homogeneous of degree $d_k d_l$. Moreover $p_{k,l}(x) = 0$. As a consequence, $\overline{C(x)}$ is contained in the nullvariety of

the functions $p_{k,l}$, $1 \leq k,l \leq \ell$. Hence the nullvariety of f_i in $\overline{C(x)}$ is contained in the nilpotent cone of $\mathfrak{g}$ since it is the nullvariety in $\mathfrak{g}$ of the functions $f_1,\ldots,f_\ell$. Then $\dim \tilde{\mathcal{O}} = \dim \overline{C(x)} - 1 = \dim G.x$. Since $\overline{\mathcal{O}' + \mathfrak{p}_{\mathrm{u}}}$ is contained in $\overline{C(x)}$, Theorem 2.8 tells us that $\mathcal{O}_{\mathfrak{g}}$ is contained in $\overline{C(x)}$. Moreover by Theorem 2.8, $\mathcal{O}_{\mathfrak{g}}$ has dimension $\dim G.x$, whence $\tilde{\mathcal{O}} = \mathcal{O}_{\mathfrak{g}}$. All statements of (i) are now clear.
(ii) By hypothesis, $\mathcal{O}_{\mathfrak{g}} = \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}})$, where $\mathfrak{l}$ is a proper Levi subalgebra of $\mathfrak{g}$ and $\mathcal{O}_{\mathfrak{l}}$ a nilpotent orbit in $\mathfrak{l}$. Let x_{s} be an element of the center of $\mathfrak{l}$ such that $\mathfrak{g}^{x_{\mathrm{s}}} = \mathfrak{l}$, let x_{n} be an element of $\mathcal{O}_{\mathfrak{l}}$ and set $x = x_{\mathrm{s}} + x_{\mathrm{n}}$. Since $\mathfrak{l}$ is a proper subalgebra, the element x is not nilpotent. So by (i), the subset of nilpotent elements of $\overline{C(x)}$ is the closure of $\mathcal{O}_{\mathfrak{g}}$. □

3. Proof of Theorem 1.3 for induced nilpotent orbits

Let e be an induced nilpotent element. Let x be a non nilpotent element of $\mathfrak{g}$ such that $\overline{C(x)} \cap \mathcal{N}(\mathfrak{g}) = \overline{G.e}$. Such an element does exist by Theorem 2.9(ii).
As an abbreviation, we set:

$$\begin{array}{lcl} \Bbbk^{\underline{d}} & := & \Bbbk^{d_1+1} \times \cdots \times \Bbbk^{d_\ell+1} \simeq \Bbbk^{\mathrm{b}_{\mathfrak{g}}+\ell}, \\ \Bbbk^{\underline{d}\times} & := & (\Bbbk^{d_1+1} \setminus \{0\}) \times \cdots \times (\Bbbk^{d_\ell+1} \setminus \{0\}), \\ \mathbb{P}^{\underline{d}} & := & \mathbb{P}(\Bbbk^{d_1+1}) \times \cdots \times \mathbb{P}(\Bbbk^{d_\ell+1}) = \mathbb{P}^{d_1} \times \cdots \times \mathbb{P}^{d_\ell}. \end{array}$$

For $j = 1,\ldots,\ell$, recall that σ_j is the map:

$$\mathfrak{g} \times \mathfrak{g} \xrightarrow{\sigma_j} \Bbbk^{d_j+1},\ (x,y) \mapsto (f_j^{(m)}(x,y))_{0\leq m\leq d_j}.$$

Let $\mathcal{B}_j$ be the nullvariety of σ_j in $\mathfrak{g} \times \mathfrak{g}$ and let $\mathcal{B}$ be the union of $\mathcal{B}_1,\ldots,\mathcal{B}_\ell$; it is a bicone of $\mathfrak{g} \times \mathfrak{g}$. Denote by ρ and τ the canonical maps:

$$\begin{array}{ccc} (\mathfrak{g} \times \mathfrak{g}) \setminus \{0\} & \xrightarrow{\rho} & \mathbb{P}(\mathfrak{g} \times \mathfrak{g}) \\ \Bbbk^{\mathrm{d}\times} & \xrightarrow{\tau} & \mathbb{P}^{\underline{d}}. \end{array}$$

Let σ^* be the restriction to $(\mathfrak{g} \times \mathfrak{g}) \setminus \mathcal{B}$ of σ; it has values in $\Bbbk^{\underline{d}\times}$. Since $\sigma_j(sx,sy) = s^{d_j}\sigma_j(x,y)$ for all $(x,y) \in \mathfrak{g} \times \mathfrak{g}$ and $j = 1,\ldots,\ell$, the map $\tau\circ\sigma^*$ factors through ρ. Denote by $\overline{\sigma^*}$ the map from $\rho(\mathfrak{g} \times \mathfrak{g} \setminus \mathcal{B})$ to $\mathbb{P}^{\underline{d}}$ making the following diagram commutative:

$$\begin{tikzcd} \mathfrak{g} \times \mathfrak{g} \setminus \mathcal{B} \arrow[r, "\sigma^*=\sigma|_{\mathfrak{g}\times\mathfrak{g}\setminus\mathcal{B}}"] \arrow[d, "\rho"'] & \Bbbk^{\underline{d}\times} \arrow[d, "\tau"] \\ \rho(\mathfrak{g} \times \mathfrak{g} \setminus \mathcal{B}) \arrow[r, dashed, "\overline{\sigma^*}"] & \mathbb{P}^{\underline{d}} \end{tikzcd}$$

and let Γ be the graph of the restriction to $\rho(\mathfrak{g} \times \overline{C(x)} \setminus \mathcal{B})$ of $\overline{\sigma^*}$.

Lemma 3.1. *The set* Γ *is a closed subset of* $\mathbb{P}(\mathfrak{g}\times\mathfrak{g})\times\mathbb{P}^{\underline{d}}$.

Proof. Let $\widetilde{\Gamma}$ be the inverse image of Γ by the map $\rho\times\tau$. Then $\widetilde{\Gamma}$ is the intersection of the graph of σ and $(\mathfrak{g}\times\overline{C(x)}\setminus\mathcal{B})\times\Bbbk^{\underline{d}\times}$. Since $\sigma^{-1}(\Bbbk^{\underline{d}}\setminus\Bbbk^{\underline{d}\times})$ is contained in $\mathcal{B}$, $\widetilde{\Gamma}$ is the intersection of the graph of σ and $(\mathfrak{g}\times\overline{C(x)}\setminus\{0\})\times\Bbbk^{\underline{d}\times}$; so $\widetilde{\Gamma}$ is closed in $(\mathfrak{g}\times\overline{C(x)})\setminus\{0\}\times\Bbbk^{\underline{d}\times}$. As a consequence, Γ is closed in $\mathbb{P}(\mathfrak{g}\times\mathfrak{g})\times\mathbb{P}^{\underline{d}}$, since $\mathbb{P}(\mathfrak{g}\times\mathfrak{g})\times\mathbb{P}^{\underline{d}}$ is endowed with the quotient topology. □

Denote by Z the closure of $\sigma(\mathfrak{g}\times\overline{C(x)})$ in $\Bbbk^{\underline{d}}$.

Lemma 3.2. *There exists an open subset* U *of* Z*, contained in* $\sigma(\mathfrak{g}\times\overline{C(x)})$*, such that* $U\cap\sigma(\mathfrak{g}\times G.e)$ *is not empty.*

Proof. Let Γ_2 be the projection of $\Gamma\subset\mathbb{P}(\mathfrak{g}\times\mathfrak{g})\times\mathbb{P}^{\underline{d}}$ to $\mathbb{P}^{\underline{d}}$. By Lemma 3.1, Γ_2 is a closed subset of $\mathbb{P}^{\underline{d}}$ since $\mathbb{P}(\mathfrak{g}\times\mathfrak{g})$ is complete. So $\tau^{-1}(\Gamma_2)$ is a closed subset of $\Bbbk^{\underline{d}\times}$. Moreover,

$$\tau^{-1}(\Gamma_2)=\sigma(\mathfrak{g}\times\overline{C(x)}\setminus\mathcal{B})$$

since $\sigma(\mathfrak{g}\times\overline{C(x)})$ is stable under the action of $\Bbbk^*\times\cdots\times\Bbbk^*$ on $\Bbbk^{\underline{d}}$. Hence the open subset $Z\cap\Bbbk^{\underline{d}\times}$ of Z is contained in $\sigma(\mathfrak{g}\times\overline{C(x)})$. But for all y in $\mathfrak{g}$ such that $f_j(y)\neq 0$ for any j, $\sigma(y,e)$ belongs to $\Bbbk^{\underline{d}\times}$. Thus, the open subset $U=Z\cap\Bbbk^{\underline{d}\times}$ of Z is convenient and the lemma follows. □

We are now ready to prove the main result of this section:

Theorem 3.3. *Assume that* $\mathrm{ind}\,\mathfrak{a}^x=\mathrm{rk}\,\mathfrak{a}$ *for all reductive subalgebras* $\mathfrak{a}$ *strictly contained in* $\mathfrak{g}$ *and for all* x *in* $\mathfrak{a}$*. Then for all induced nilpotent orbits* $\mathcal{O}_{\mathfrak{g}}$ *in* $\mathfrak{g}$ *and for all* e *in* $\mathcal{O}_{\mathfrak{g}}$*,* $\mathrm{ind}\,\mathfrak{g}^e=\ell$.

Proof. Let $\mathcal{O}_{\mathfrak{g}}$ be an induced nilpotent orbit and let e be in $\mathcal{O}_{\mathfrak{g}}$. Using Theorem 2.9(ii), we let x be a non nilpotent element of $\mathfrak{g}$ such that $\overline{C(x)}\cap\mathcal{N}(\mathfrak{g})=\overline{\mathcal{O}_{\mathfrak{g}}}$. Since x is not nilpotent, $\mathfrak{g}^x$ is the centralizer in the reductive Lie algebra $\mathfrak{g}^{x_{\rm s}}$ of the nilpotent element $x_{\rm n}$ of $\mathfrak{g}^{x_{\rm s}}$. Since $\mathfrak{g}^{x_{\rm s}}$ is strictly contained in $\mathfrak{g}$ and has rank ℓ, the index of $\mathfrak{g}^x$ is equal to ℓ by hypothesis. Besides, by Theorem 2.7, (1)⇒(6), applied to x,

$$\dim\sigma(\mathfrak{g}\times\{x\})=\frac{1}{2}\dim G.x+\ell.$$

Since σ is G-invariant, $\sigma(\mathfrak{g}\times\{x\})=\sigma(\mathfrak{g}\times G.x)$. Hence for all z in a dense subset of $\sigma(\mathfrak{g}\times G.x)$, the fiber of the restriction of σ to $\mathfrak{g}\times G.x$ at z has minimal dimension

$$\dim\mathfrak{g}+\dim G.x-(\frac{1}{2}\dim G.x+\ell)=\dim\mathfrak{g}+\frac{1}{2}\dim G.x-\ell.$$

Denote by Z the closure of $\sigma(\mathfrak{g}\times\overline{C(x)})$ in $\Bbbk^{\underline{d}}$. We deduce from the above equality that Z has dimension

$$\begin{aligned}\dim\mathfrak{g}+\dim C(x)-(\dim\mathfrak{g}+\frac{1}{2}\dim G.x-\ell)\\ &= \dim C(x)-\frac{1}{2}\dim G.x+\ell\\ &= \frac{1}{2}\dim G.e+\ell+1,\end{aligned}$$

since $\dim C(x)=\dim G.x+1=\dim G.e+1$.
By Lemma 3.2, there exists an open subset U of Z contained in $\sigma(\mathfrak{g}\times\overline{C(x)})$ having a nonempty intersection with $\sigma(\mathfrak{g}\times G.e)$. Let i be in $\{1,\ldots,\ell\}$ such that $f_i(x)\neq 0$. For $z\in\Bbbk^{\underline{d}}$, we write $z=(z_{i,j})_{\substack{1\leq i\leq\ell\\ 0\leq j\leq d_i}}$ its coordinates. Let $\mathcal{V}_i$ be the nullvariety in U of the coordinate z_{i,d_i}. Then $\mathcal{V}_i$ is not empty by the choice of U. Since U is irreducible and since z_{i,d_i} is not identically zero on U, $\mathcal{V}_i$ is equidimensional of dimension $\frac{1}{2}\dim G.e+\ell$. By Theorem 2.9(i), the nullvariety of f_i in $\overline{C(x)}$ is equal to $\overline{G.e}$. Hence $\sigma^{-1}(\mathcal{V}_i)\cap(\mathfrak{g}\times\overline{C(x)})=\sigma^{-1}(U)\cap(\mathfrak{g}\times\overline{G.e})$ is an open subset of $\mathfrak{g}\times\overline{G.e}$. So $\sigma(\mathfrak{g}\times G.e)$ has dimension $\frac{1}{2}\dim G.e+\ell$. Then by Theorem 2.7, (6)$\Rightarrow$(1), the index of $\mathfrak{g}^e$ is equal to ℓ. □

From that point, our goal is to prove Theorem 1.3 for rigid nilpotent elements; Theorem 3.3 tells us that this is enough to complete the proof.

4. The Slodowy slice and the property (P)

In this section, we introduce a property (P) in Definition 4.2 and we prove that $e\in\mathcal{N}(\mathfrak{g})$ has Property (P) if and only if $\operatorname{ind}\mathfrak{g}^e=\ell$ (Theorem 4.13). Then, we will show in the next section that all rigid nilpotent orbits of $\mathfrak{g}$ but seven orbits (one in the type E_7 and six in the type E_8) do have Property (P).

4.1. Blowing up of $\mathcal{S}$. Let e be a nilpotent element of $\mathfrak{g}$ and consider an $\mathfrak{sl}_2$-triple (e,h,f) containing e as in Subsection 1.3. The *Slodowy slice* is the affine subspace $\mathcal{S}:=e+\mathfrak{g}^f$ of $\mathfrak{g}$ which is a transverse variety to the adjoint orbit $G.e$. Denote by $B_e(\mathcal{S})$ the blowing up of $\mathcal{S}$ centered at e and let $p:B_e(\mathcal{S})\to\mathcal{S}$ be the canonical morphism. The variety $\mathcal{S}$ is smooth and $p^{-1}(e)$ is a smooth irreducible hypersurface of $B_e(\mathcal{S})$. The use of the blowing-up $B_e(\mathcal{S})$ for the computation of the index was initiated by the first author in [Ch04] and resumed by the second author in [Mo06a]. Here, we use again this technique to study the index of $\mathfrak{g}^e$. Describe first the main tools extracted from [Ch04] we need.

For Y an open subset of $B_e(\mathcal{S})$, we denote by $\Bbbk[Y]$ the algebra of regular functions on Y. By [Ch04, Théorème 3.3], we have:

Theorem 4.1. *The following two assertions are equivalent:*
(A) *the equality* $\operatorname{ind}\mathfrak{g}^e = \ell$ *holds,*
(B) *there exists an affine open subset* $Y \subset B_e(\mathcal{S})$ *such that* $Y \cap p^{-1}(e) \neq \emptyset$ *and satisfying the following property:*

> *for any regular map* $\varphi \in \Bbbk[Y] \otimes_\Bbbk \mathfrak{g}$ *such that* $\varphi(x) \in [\mathfrak{g}, p(x)]$ *for all* $x \in Y$, *there exists* $\psi \in \Bbbk[Y] \otimes_\Bbbk \mathfrak{g}$ *such that* $\varphi(x) = [\psi(x), p(x)]$ *for all* $x \in Y$.

An open subset $\Omega \subset B_e(\mathcal{S})$ is called *a big open subset* if $B_e(\mathcal{S}) \setminus \Omega$ has codimension at least 2 in $B_e(\mathcal{S})$. As explained in [Ch04, Section 2], there exists a big open subset Ω of $B_e(\mathcal{S})$ and a regular map

$$\alpha : \Omega \to \mathrm{Gr}_\ell(\mathfrak{g})$$

such that $\alpha(x) = \mathfrak{g}^{p(x)}$ if $p(x)$ is regular. Furthermore, the map α is uniquely defined by this condition. In fact, this result is a consequence of [Sh94, Ch. VI, Theorem 1]. From now on, α stands for the so-defined map. Since $p^{-1}(e)$ is an hypersurface and since Ω is a big open subset of $B_e(\mathcal{S})$, note that $\Omega \cap p^{-1}(e)$ is a nonempty set. In addition, $\alpha(x) \subset \mathfrak{g}^{p(x)}$ for all $x \in \Omega$.

Definition 4.2. We say that e *has Property* (P) if $\mathfrak{z}(\mathfrak{g}^e) \subset \alpha(x)$ for all x in $\Omega \cap p^{-1}(e)$.

Remark 4.3. Suppose that e is regular. Then $\mathfrak{g}^e$ is a commutative algebra, i.e. $\mathfrak{z}(\mathfrak{g}^e) = \mathfrak{g}^e$. If $x \in \Omega \cap p^{-1}(e)$, then $\alpha(x) = \mathfrak{g}^e$ since $p(x) = e$ is regular in this case. On the other hand, $\operatorname{ind}\mathfrak{g}^e = \dim\mathfrak{g}^e = \ell$ since e is regular. So e has Property (P) and $\operatorname{ind}\mathfrak{g}^e = \ell$.

4.2. On the property (P). This subsection aims to show: Property (P) holds for e if and only if $\operatorname{ind}\mathfrak{g}^e = \ell$. As a consequence of Remark 4.3, we can (and will) assume that e is a nonregular nilpotent element of $\mathfrak{g}$. As a first step, we will state in Corollary 4.12 that, if (P) holds, then so does the assertion (B) of Theorem 4.1.

Let $L_\mathfrak{g}$ be the $\mathrm{S}(\mathfrak{g})$-submodule of $\varphi \in \mathrm{S}(\mathfrak{g}) \otimes_\Bbbk \mathfrak{g}$ satisfying $[\varphi(x), x] = 0$ for all x in $\mathfrak{g}$. It is known that $L_\mathfrak{g}$ is a free module of basis $\varphi_1, \ldots, \varphi_\ell$, cf. [Di79]. We investigate an analogous property for the Slodowy slice $\mathcal{S} = e + \mathfrak{g}^f$. We denote by $\mathcal{S}_{\mathrm{reg}}$ the intersection of $\mathcal{S}$ and $\mathfrak{g}_{\mathrm{reg}}$. As e is nonregular, the set $(\mathcal{S} \setminus \mathcal{S}_{\mathrm{reg}})$ contains e.

Lemma 4.4. *The set* $\mathcal{S} \setminus \mathcal{S}_{\mathrm{reg}}$ *has codimension* 3 *in* $\mathcal{S}$ *and each irreducible component of* $\mathcal{S} \setminus \mathcal{S}_{\mathrm{reg}}$ *contains* e.

Proof. Let us consider the morphism

$$\begin{array}{ccc} G\times \mathcal{S} & \longrightarrow & \mathfrak{g} \\ (g,x) & \longmapsto & g(x) \end{array}$$

By a Slodowy's result [Sl80], this morphism is a smooth morphism. So its fibers are equidimensional of dimension $\dim \mathfrak{g}^f$. In addition, by [V72], $\mathfrak{g}\setminus\mathfrak{g}_{\rm reg}$ is a G-invariant equidimensional closed subset of $\mathfrak{g}$ of codimension 3. Hence $\mathcal{S}\setminus\mathcal{S}_{\rm reg}$ is an equidimensional closed subset of $\mathcal{S}$ of codimension 3.

Denoting by $t\mapsto g(t)$ the one parameter subgroup of G generated by ${\rm ad}h$, $\mathcal{S}$ and $\mathcal{S}\setminus\mathcal{S}_{\rm reg}$ are stable under the action of $t^{-2}g(t)$ for all t in $\Bbbk^*$. Furthermore, for all x in $\mathcal{S}$, $t^{-2}g(t)(x)$ goes to e when t goes to ∞, whence the lemma. □

Denote by $\Bbbk[\mathcal{S}]$ the algebra of regular functions on $\mathcal{S}$ and denote by $L_{\mathcal{S}}$ the $\Bbbk[\mathcal{S}]$-submodule of $\varphi\in\Bbbk[\mathcal{S}]\otimes_{\Bbbk}\mathfrak{g}$ satisfying $[\varphi(x),x]=0$ for all x in $\mathcal{S}$.

LEMMA 4.5. *The module $L_{\mathcal{S}}$ is a free module of basis $\varphi_1|_{\mathcal{S}},\dots,\varphi_\ell|_{\mathcal{S}}$ where $\varphi_i|_{\mathcal{S}}$ is the restriction to $\mathcal{S}$ of φ_i for $i=1,\dots,\ell$.*

Proof. Let φ be in $L_{\mathcal{S}}$. There are regular functions $a_1,\dots,a_\ell$ on $\mathcal{S}_{\rm reg}$ satisfying

$$\varphi(x)=a_1(x)\varphi_1|_{\mathcal{S}}(x)+\cdots+a_\ell(x)\varphi_\ell|_{\mathcal{S}}(x)$$

for all $x\in\mathcal{S}_{\rm reg}$, by Lemma 2.1(ii). By Lemma 4.4, $\mathcal{S}\setminus\mathcal{S}_{\rm reg}$ has codimension 3 in $\mathcal{S}$. Hence $a_1,\dots,a_\ell$ have polynomial extensions to $\mathcal{S}$ since $\mathcal{S}$ is normal. So the maps $\varphi_1|_{\mathcal{S}},\dots,\varphi_\ell|_{\mathcal{S}}$ generate $L_{\mathcal{S}}$. Moreover, by Lemma 2.1(ii) for all $x\in\mathcal{S}_{\rm reg}$, $\varphi_1(x),\dots,\varphi_\ell(x)$ are linearly independent, whence the statement. □

The following proposition accounts for an important step to interpret Assertion (B) of Theorem 4.1:

PROPOSITION 4.6. *Let φ be in $\Bbbk[\mathcal{S}]\otimes_{\Bbbk}\mathfrak{g}$ such that $\varphi(x)\in[\mathfrak{g},x]$ for all x in a nonempty open subset of $\mathfrak{g}$. Then there exists a polynomial map $\psi\in\Bbbk[\mathcal{S}]\otimes_{\Bbbk}\mathfrak{g}$ such that $\varphi(x)=[\psi(x),x]$ for all $x\in\mathcal{S}$.*

Proof. Since $\mathfrak{g}^x$ is the orthogonal complement of $[x,\mathfrak{g}]$ in $\mathfrak{g}$, our hypothesis says that $\varphi(x)$ is orthogonal to $\mathfrak{g}^x$ for all x in a nonempty open subset $\mathcal{S}'$ of $\mathcal{S}$. The intersection $\mathcal{S}'\cap\mathcal{S}_{\rm reg}$ is not empty; so by Lemma 2.1(ii), $\langle\varphi(x),\varphi_i|_{\mathcal{S}}(x)\rangle=0$ for all $i=1,\dots,\ell$ and for all $x\in\mathcal{S}'\cap\mathcal{S}_{\rm reg}$. Therefore, by continuity, $\langle\varphi(x),\varphi_i|_{\mathcal{S}}(x)\rangle=0$ for all $i=1,\dots,\ell$ and all $x\in\mathcal{S}$. Hence $\varphi(x)\in[x,\mathfrak{g}]$ for all $x\in\mathcal{S}_{\rm reg}$ by Lemma 2.1(ii) again. Consequently by Lemma 4.4, Lemma 4.5 and the proof of the main theorem of [Di79], there exists an element $\psi \in \Bbbk[\mathcal{S}] \otimes_{\Bbbk} \mathfrak{g}$ which satisfies the condition of the proposition. □

Let $u_1, \ldots, u_m$ be a basis of $\mathfrak{g}^f$ and let $u_1^*, \ldots, u_m^*$ be the corresponding coordinate system of $\mathcal{S} = e + \mathfrak{g}^f$. There is an affine open subset $Y \subset B_e(\mathcal{S})$ with $Y \cap p^{-1}(e) \neq \emptyset$ such that $\Bbbk[Y]$ is the set of linear combinations of monomials in $(u_1^*)^{-1}, u_1^*, \ldots, u_m^*$ whose total degree is nonnegative. In particular, we have a global coordinates system $u_1^*, v_2^*, \ldots, v_m^*$ on Y satisfying the relations:

$$u_2^* = u_1^* v_2^* \quad , \ldots , \quad u_m^* = u_1^* v_m^*. \tag{4}$$

Note that, for $x \in Y$, we so have: $p(x) = e + u_1^*(x)(u_1 + v_2^*(x)u_2 + \cdots + v_m^*(x)u_m)$. So, the image of Y by p is the union of $\{e\}$ and the complementary in $\mathcal{S}$ of the nullvariety of u_1^*. Let Y' be an affine open subset of Y contained in Ω and having a nonempty intersection with $p^{-1}(e)$. Denote by $L_{Y'}$ the set of regular maps φ from Y' to $\mathfrak{g}$ satisfying $[\varphi(x), p(x)] = 0$ for all $x \in Y'$.

Lemma 4.7. *Suppose that e has Property* (P). *For each $z \in \mathfrak{z}(\mathfrak{g}^e)$, there exists $\psi_z \in \Bbbk[Y'] \otimes_{\Bbbk} \mathfrak{g}$ such that $z - u_1^*\psi_z$ belongs to $L_{Y'}$.*

Proof. Let z be in $\mathfrak{z}(\mathfrak{g}^e)$. Since $Y' \subset \Omega$, for each $y \in Y'$, there exists an affine open subset U_y of Y' containing y and regular maps $\nu_1, \ldots, \nu_\ell$ from U_y to $\mathfrak{g}$ such that $\nu_1(x), \ldots, \nu_\ell(x)$ is a basis of $\alpha(x)$ for all $x \in U_y$. Let y be in Y'. We consider two cases:
(1) Suppose $p(y) = e$.
Since e has Property (P), there exist regular functions $a_1, \ldots, a_\ell$ on U_y satisfying

$$z = a_1(x)\nu_1(x) + \cdots + a_\ell(x)\nu_\ell(x),$$

for all $x \in U_y \cap p^{-1}(e)$. The intersection $U_y \cap p^{-1}(e)$ is the set of zeroes of u_1^* in U_y. So there exists a regular map ψ from U_y to $\mathfrak{g}$ which satisfies the equality:

$$z - u_1^*\psi = a_1\nu_1 + \cdots + a_\ell\nu_\ell.$$

Hence $[z - u_1^*(x)\psi(x), p(x)] = 0$ for all $x \in U_y$ since $\alpha(x)$ is contained in $\mathfrak{g}^{p(x)}$ for all $x \in \Omega$.
(2) Suppose $p(y) \neq e$.
Then we can assume that $U_y \cap p^{-1}(e) = \emptyset$ and the map $\psi = (u_1^*)^{-1}z$ satisfies the condition: $[z - u_1^*(x)\psi(x), p(x)] = 0$ for all $x \in U_y$.
In both cases (1) or (2), we have found a regular map ψ_y from U_y to $\mathfrak{g}$ satisfying: $[z - (u_1^*\psi_y)(x), p(x)] = 0$ for all $x \in U_y$.
Let $y_1, \ldots, y_k$ be in Y' such that the open subsets $U_{y_1}, \ldots, U_{y_k}$ cover Y'. For $i = 1, \ldots, k$, we denote by ψ_i a regular map from U_{y_i} to $\mathfrak{g}$ such that $z - u_1^*\psi_i$ is in $\Gamma(U_{y_i}, \mathcal{L})$ where $\mathcal{L}$ is the localization of $L_{Y'}$ on Y'.

Then for $i, j = 1, \ldots, k$, $\psi_i - \psi_j$ is in $\Gamma(U_{y_i} \cap U_{y_j}, \mathcal{L})$. Since Y' is affine, $\mathrm{H}^1(Y', \mathcal{L}) = 0$. So, for $i = 1, \ldots, l$, there exists $\widetilde{\psi}_i$ in $\Gamma(U_{y_i}, \mathcal{L})$ such that $\widetilde{\psi}_i - \widetilde{\psi}_j$ is equal to $\psi_i - \psi_j$ on $U_{y_i} \cap U_{y_j}$ for all i, j. Then there exists a well-defined map ψ_z from Y' to $\mathfrak{g}$ whose restriction to U_{y_i} is equal to $\psi_i - \widetilde{\psi}_i$ for all i, and such that $z - u_1^*\psi_z$ belongs to $L_{Y'}$. Finally, the map ψ_z verifies the required property. $\square$

Let z be in $\mathfrak{z}(\mathfrak{g}^e)$. We denote by φ_z the regular map from Y to $\mathfrak{g}$ defined by:

$$\varphi_z(x) = [z, u_1] + v_2^*(x)[z, u_2] + \cdots + v_m^*(x)[z, u_m], \quad \text{for all } x \in Y. \tag{5}$$

Corollary 4.8. *Suppose that e has Property* (P) *and let z be in $\mathfrak{z}(\mathfrak{g}^e)$. There exists ψ_z in $\Bbbk[Y'] \otimes_{\Bbbk} \mathfrak{g}$ such that $\varphi_z(x) = [\psi_z(x), p(x)]$ for all $x \in Y'$.*

Proof. By Lemma 4.7, there exists ψ_z in $\Bbbk[Y'] \otimes_{\Bbbk} \mathfrak{g}$ such that $z - u_1^*\psi_z$ is in $L_{Y'}$. Then

$$u_1^*\varphi_z(x) = [z, p(x)] = [z - u_1^*\psi_z(x), p(x)] + u_1^*[\psi_z(x), p(x)],$$

for all $x \in Y'$. So the map ψ_z is convenient, since u_1^* is not identically zero on Y'. $\square$

The following lemma is easy but helpful for Proposition 4.10:

Lemma 4.9. *Let v be in $\mathfrak{g}^e$. Then, v belongs to $\mathfrak{z}(\mathfrak{g}^e)$ if and only if $[v, \mathfrak{g}^f] \subset [e, \mathfrak{g}]$.*

Proof. Since $[x, \mathfrak{g}]$ is the orthogonal complement of $\mathfrak{g}^x$ in $\mathfrak{g}$ for all $x \in \mathfrak{g}$, we have:

$$\begin{gathered}[v, \mathfrak{g}^f] \subset [e, \mathfrak{g}] \iff \langle [v, \mathfrak{g}^f], \mathfrak{g}^e \rangle = 0 \\ \iff \langle [v, \mathfrak{g}^e], \mathfrak{g}^f \rangle = 0 \iff [v, \mathfrak{g}^e] \subset [f, \mathfrak{g}].\end{gathered}$$

But $\mathfrak{g}$ is the direct sum of $\mathfrak{g}^e$ and $[f, \mathfrak{g}]$ and $[v, \mathfrak{g}^e]$ is contained in $\mathfrak{g}^e$ since $v \in \mathfrak{g}^e$. Hence $[v, \mathfrak{g}^f]$ is contained in $[e, \mathfrak{g}]$ if and only if v is in $\mathfrak{z}(\mathfrak{g}^e)$. $\square$

Proposition 4.10. *Suppose that e has Property* (P) *and let φ be in $\Bbbk[Y] \otimes_{\Bbbk} \mathfrak{g}$ such that $\varphi(x) \in [\mathfrak{g}, p(x)]$ for all $x \in Y$. Then there exists ψ in $\Bbbk[Y'] \otimes_{\Bbbk} \mathfrak{g}$ such that $\varphi(x) = [\psi(x), p(x)]$ for all $x \in Y'$.*

Proof. Since φ is a regular map from Y to $\mathfrak{g}$, there is a nonnegative integer d and $\widetilde{\varphi} \in \Bbbk[\mathcal{S}] \otimes_{\Bbbk} \mathfrak{g}$ such that

$$(u_1^*)^d(x)\varphi(x) = (\widetilde{\varphi} \circ p)(x), \ \forall x \in Y \tag{6}$$

and $\widetilde{\varphi}$ is a linear combination of monomials in $u_1^*, \ldots, u_m^*$ whose total degree is at least d. By hypothesis on φ, we deduce that for all $x \in \mathcal{S}$

such that $u_1^*(x) \neq 0$, $\widetilde{\varphi}(x)$ is in $[\mathfrak{g}, x]$. Hence by Proposition 4.6, there exists $\widetilde{\psi}$ in $\Bbbk[\mathcal{S}] \otimes_k \mathfrak{g}$ such that $\widetilde{\varphi}(x) = [\widetilde{\psi}(x), x]$ for all $x \in \mathcal{S}$.
Denote by $\widetilde{\psi}'$ the sum of monomials of degree at least d in $\widetilde{\psi}$ and denote by ψ' the element of $\Bbbk[Y] \otimes_\Bbbk \mathfrak{g}$ satisfying

$$(u_1^*)^d(x)\psi'(x) = (\widetilde{\psi}'\circ p)(x), \ \forall x \in Y. \tag{7}$$

Then we set, for $x \in Y$, $\varphi'(x) := \varphi(x) - [\psi'(x), p(x)]$. We have to prove the existence of an element ψ'' in $\Bbbk[Y'] \otimes_\Bbbk \mathfrak{g}$ such that $\varphi'(x) = [\psi''(x), p(x)]$ for all $x \in Y'$.
• If $d = 0$, then $\varphi = \widetilde{\varphi}\circ p$, $\psi' = \psi$ and $\varphi' = 0$; so ψ' is convenient in that case.
• If $d = 1$, we can write

$$\begin{aligned} u_1^*(x)\varphi(x) &= \widetilde{\varphi}(p(x)) \\ &= [\widetilde{\psi}(p(x)), e + u_1^*(x)(u_1 + v_2^*(x)u_2 + \cdots + v_m^*(x)u_m)], \end{aligned}$$

for all $x \in Y$, whence we deduce

$$\begin{aligned} &u_1^*(x)(\varphi(x) - [\psi'(x), p(x)]) \\ &\qquad = [\tilde{\psi}(e), e + u_1^*(x)(u_1 + v_2^*(x)u_2 + \cdots + v_m^*(x)u_m)] \end{aligned}$$

for all $x \in Y$. Hence $\tilde{\psi}(e)$ belongs to $\mathfrak{g}^e$ and $[\widetilde{\psi}(e), u_i] \in [e, \mathfrak{g}]$ for all $i = 1, \ldots, m$, since $\varphi(x) \in [e, \mathfrak{g}]$ for all $x \in Y \cap p^{-1}(e)$. Then $\widetilde{\psi}(e)$ is in $\mathfrak{z}(\mathfrak{g}^e)$ by Lemma 4.9. So by Corollary 4.8, φ' has the desired property.
• Suppose $d > 1$. For $\underline{i} = (i_1, \ldots, i_m) \in \mathbb{N}^m$, we set $|\underline{i}| := i_1 + \cdots + i_m$ and we denote by $\psi_{\underline{i}}$ the coefficient of $(u_1^*)^{i_1} \cdots (u_m^*)^{i_m}$ in $\widetilde{\psi}$. By Corollary 4.8, it suffices to prove:

$$\begin{cases} \psi_{\underline{i}} = 0 & \text{if } |\underline{i}| < d-1; \\ \psi_{\underline{i}} \in \mathfrak{z}(\mathfrak{g}^e) & \text{if } |\underline{i}| = d-1 \end{cases}.$$

For $\underline{i} \in \mathbb{N}^m$ and $j \in \{1, \ldots, m\}$, we define the element $\underline{i}(j)$ of $\mathbb{N}^m$ by:

$$i(j) := (i_1, \ldots, i_{j-1}, i_j + 1, i_{j+1}, \ldots, i_m).$$

It suffices to prove:

Claim 4.11. For $|i| \leq d-1$, $\psi_{\underline{i}}$ is an element of $\mathfrak{g}^e$ such that $[\psi_{\underline{i}}, u_j] + [\psi_{\underline{i}(j)}, e] = 0$ for $j = 1, \ldots, m$.

Indeed, by Lemma 4.9, if

$$[\psi_{\underline{i}}, u_j] + [\psi_{\underline{i}(j)}, e] = 0 \text{ and } \psi_{\underline{i}} \in \mathfrak{g}^e$$

for all $j = 1, \ldots, m$, then $\psi_{\underline{i}} \in \mathfrak{z}(\mathfrak{g}^e)$. Furthermore, if

$$[\psi_{\underline{i}}, u_j] + [\psi_{\underline{i}(j)}, e] = 0 \text{ and } \psi_{\underline{i}} \in \mathfrak{g}^e \text{ and } \psi_{\underline{i}(j)} \in \mathfrak{g}^e$$

for all $j = 1, \dots, m$, then $\psi_{\underline{i}} = 0$ since $\mathfrak{z}(\mathfrak{g}^e) \cap \mathfrak{g}^f = 0$. So only remains to prove Claim 4.11.
We prove the claim by induction on $|\underline{i}|$. Arguing as in the case $d = 1$, we prove the claim for $|\underline{i}| = 0$. We suppose the claim true for all $|\underline{i}| \leq l-1$ for some $0 < l \leq d-2$. We have to prove the statement for all $|i| \leq l$. By what foregoes and by induction hypothesis, $\psi_{\underline{i}} = 0$ for $|\underline{i}| \leq l-2$. For $k = l+1, l+2$, we consider the ring $\Bbbk[\tau_k]$ where $\tau_k^k = 0$. Since $(u_1^*)^d$ vanishes on the set of $\Bbbk[\tau_{l+1}]$-points $x = x_0 + x_1\tau_{l+1} + \cdots + x_l\tau_{l+1}^l$ of Y whose source x_0 is a zero of u_1^*,

$$0 = [\widetilde{\psi}(e + \tau_{l+1}v), e + \tau_{l+1}v] = \sum_{|i|=l} \tau_{l+1}^l [\psi_{\underline{i}}, e](u_1^*)^{i_1} \cdots (u_m^*)^{i_m}(v),$$

for all $v \in \mathfrak{g}^f$. So $\psi_{\underline{i}} \in \mathfrak{g}^e$ for $|\underline{i}| = l$.
For $|\underline{i}|$ equal to l, the term in

$$\tau_{l+2}^{l+1}(u_1^*)^{i_1} \cdots (u_{i_{j-1}}^*)^{i_{j-1}}(u_{i_j+1}^*)^{i_j+1}(u_{i_{j+1}}^*)^{i_{j+1}} \cdots (u_m^*)^{i_m}(v)$$

of $[\widetilde{\psi}(e + \tau_{l+2}v), e + \tau_{l+2}v]$ is equal to $[\psi_{\underline{i}(j)}, e] + [\psi_{\underline{i}}, u_j]$. Since $(u_1^*)^d$ vanishes on the set of $\Bbbk[\tau_{l+2}]$-points of Y whose source is a zero of u_1^*, this term is equal to 0, whence the claim. □

Recall that Y' is an affine open subset of Y contained in Ω and having a nonempty intersection with $p^{-1}(e)$.

Corollary 4.12. *Suppose that e has Property* (P). *Let φ be in $\Bbbk[Y'] \otimes_{\Bbbk} \mathfrak{g}$ such that $\varphi(x) \in [\mathfrak{g}, p(x)]$ for all $x \in Y'$. Then there exists ψ in $\Bbbk[Y'] \otimes_{\Bbbk} \mathfrak{g}$ such that $\varphi(x) = [\psi(x), p(x)]$ for all $x \in Y'$.*

Proof. For $a \in \Bbbk[Y]$, denote by $D(a)$ the principal open subset defined by a. Let $D(a_1), \dots, D(a_m)$ be an open covering of Y' by principal open subsets of Y, with $a_1, \dots, a_k$ in $\Bbbk[Y]$. Since φ is a regular map from Y' to $\mathfrak{g}$, there is $m_i \geq 0$ such that $a_i^{m_i}\varphi$ is the restriction to Y' of some regular map φ_i from Y to $\mathfrak{g}$. For m_i big enough, φ_i vanishes on $Y \setminus D(a_i)$; hence $\varphi_i(x) \in [\mathfrak{g}, p(x)]$ for all $x \in Y$. So, by Proposition 4.6, there is a regular map ψ_i from Y' to $\mathfrak{g}$ such that $\varphi_i(x) = [\psi_i(x), p(x)]$ for all $x \in Y'$. Then for all $x \in D(a_i)$, we have $\varphi(x) = [a_i(x)^{-m_i}\psi_i(x), p(x)]$. Since Y' is an affine open subset of Y, there exists a regular map ψ from Y' to $\mathfrak{g}$ which satisfies the condition of the corollary. □

We are now in position to prove the main result of this section:

Theorem 4.13. *The equality* $\operatorname{ind} \mathfrak{g}^e = \ell$ *holds if and only if e has Property* (P).

Proof. By Corollary 4.12, if e has Property (P), then Assertion (B) of Theorem 4.1 is satisfied. Conversely, suppose that $\mathrm{ind}\,\mathfrak{g}^e = \ell$ and show that e has Property (P). By Theorem 4.1, (A)⇒(B), Assertion (B) is satisfied. We choose an affine open subset Y' of Y, contained in Ω, such that $Y' \cap p^{-1}(e) \neq$ and verifying the condition of the assertion (B). Let $z \in \mathfrak{z}(\mathfrak{g}^e)$. Recall that the map φ_z is defined by (5). Let x be in Y'. If $u_1^*(x) \neq 0$, then $\varphi_z(x)$ belongs to $[\mathfrak{g}, p(x)]$ by (5). If $u_1^*(x) = 0$, then by Lemma 4.9, $\varphi_z(x)$ belongs to $[e, \mathfrak{g}]$. So there exists a regular map ψ from Y' to $\mathfrak{g}$ such that $\varphi_z(x) = [\psi(x), p(x)]$ for all $x \in Y'$ by Assertion (B). Hence we have

$$[z - u_1^*\psi(x), p(x)] = 0,$$

for all $x \in Y'$ since $(u_1^*\varphi_z)(x) = [z, p(x)]$ for all $x \in Y$. So $\alpha(x)$ contains z for all x in $\Omega \cap Y' \cap p^{-1}(e)$. Since $p^{-1}(e)$ is irreducible, we deduce that e has Property (P). $\square$

4.3. A NEW FORMULATION OF THE PROPERTY (P). Recall that Property (P) is introduced in Definition 4.2. As has been noticed in the proof of Lemma 4.4, the morphism $G \times \mathcal{S} \to \mathfrak{g}, (g, x) \mapsto g(x)$ is smooth. As a consequence, the set $\mathcal{S}_{\mathrm{reg}}$ of $v \in \mathcal{S}$ such that v is regular is a nonempty open subset of $\mathcal{S}$. For x in $\mathcal{S}_{\mathrm{reg}}$, $\mathfrak{g}^{e+t(x-e)}$ has dimension ℓ for all t in a nonempty open subset of $\Bbbk$ since $x = e + (x - e)$ is regular. Furthermore, since $\Bbbk$ has dimension 1, [Sh94, Ch. VI, Theorem 1] asserts that there is a unique regular map

$$\beta_x : \Bbbk \to \mathrm{Gr}_\ell(\mathfrak{g})$$

satisfying $\beta_x(t) = \mathfrak{g}^{e+t(x-e)}$ for all t in a nonempty open subset of $\Bbbk$. Recall that Y is an affine open subset of $B_e(\mathcal{S})$ with $Y \cap p^{-1}(e) \neq \emptyset$ and that $u_1^*, v_2^*, \ldots, v_m^*$ is a global coordinates system of Y, cf. (4). Let $\mathcal{S}'_{\mathrm{reg}}$ be the subset of x in $\mathcal{S}_{\mathrm{reg}}$ such that $u_1^*(x) \neq 0$. For x in $\mathcal{S}'_{\mathrm{reg}}$, we denote by $\widetilde{x}$ the element of Y whose coordinates are $0, v_2^*(x), \ldots, v_m^*(x)$.

LEMMA 4.14. *Let x be in $\mathcal{S}'_{\mathrm{reg}}$.*
(i) *The subspace $\beta_x(0)$ is contained in $\mathfrak{g}^e$.*
(ii) *If $\widetilde{x} \in \Omega$, then $\alpha(\widetilde{x}) = \beta_x(0)$.*

Proof. (i) The map β_x is a regular map and $[\beta_x(t), e + t(x - e)] = 0$ for all t in a nonempty open subset of $\Bbbk$. So, $\beta_x(0)$ is contained in $\mathfrak{g}^e$.
(ii) Since $\mathcal{S}'_{\mathrm{reg}}$ has an empty intersection with the nullvariety of u_1^* in $\mathcal{S}$, the restriction of p to $p^{-1}(\mathcal{S}'_{\mathrm{reg}})$ is an isomorphism from $p^{-1}(\mathcal{S}'_{\mathrm{reg}})$ to $\mathcal{S}'_{\mathrm{reg}}$. Furthermore, $\beta_x(t) = \alpha(p^{-1}(e + tx - te))$ for any t in $\Bbbk$ such that $e + t(x - e)$ belongs to $\mathcal{S}'_{\mathrm{reg}}$ and $p^{-1}(e + tx - te)$ goes to $\widetilde{x}$ when t goes to 0. Hence $\beta_x(0)$ is equal to $\alpha(\widetilde{x})$ since α and β are regular maps. $\square$

COROLLARY 4.15. *The element e has Property* (P) *if and only if* $\mathfrak{z}(\mathfrak{g}^e) \subset \beta_x(0)$ *for all x in a nonempty open subset of* $\mathcal{S}_{\rm reg}$.

Proof. The map $x \mapsto \widetilde{x}$ from $\mathcal{S}'_{\rm reg}$ to Y is well-defined and its image is an open subset of $Y \cap p^{-1}(e)$. Let $\mathcal{S}''_{\rm reg}$ be the set of $x \in \mathcal{S}'_{\rm reg}$ such that $\widetilde{x} \in \Omega$ and let Y'' be the image of $\mathcal{S}''_{\rm reg}$ by the map $x \mapsto \widetilde{x}$. Then $\mathcal{S}''_{\rm reg}$ is open in $\mathcal{S}_{\rm reg}$ and Y'' is dense in $\Omega \cap p^{-1}(e)$ since $p^{-1}(e)$ is irreducible. Furthermore, the image of a dense open subset of $\mathcal{S}''_{\rm reg}$ by the map $x \mapsto \widetilde{x}$ is dense in Y''. Since α is regular, e has property (P) if and only if $\alpha(x)$ contains $\mathfrak{z}(\mathfrak{g}^e)$ for all x in a dense subset of Y''. By Lemma 4.14(ii), the latter property is equivalent to the fact that $\beta_x(0)$ contains $\mathfrak{z}(\mathfrak{g}^e)$ for all x in a dense open subset of $\mathcal{S}''_{\rm reg}$. □

COROLLARY 4.16. (i) *If $\mathfrak{z}(\mathfrak{g}^e)$ is generated by $\varphi_1(e), \dots, \varphi_\ell(e)$, then e has Property* (P).
(ii) *If $\mathfrak{z}(\mathfrak{g}^e)$ has dimension 1, then e has Property* (P).

Proof. Recall that $\varphi_i(e)$ belongs to $\mathfrak{z}(\mathfrak{g}^e)$, for all $i = 1, \dots, \ell$, by Lemma 2.1(i). Moreover, for all x in $\mathcal{S}_{\rm reg}$ and all $i = 1, \dots, \ell$, $\varphi_i(e + t(x-e))$ belongs to $\mathfrak{g}^{e+t(x-e)}$ for any t in $\Bbbk$. So by continuity, $\varphi_i(e)$ belongs to $\beta_x(0)$. As a consequence, whenever $\mathfrak{z}(\mathfrak{g}^e)$ is generated by $\varphi_1(e), \dots, \varphi_\ell(e)$, e has Property (P) by Corollary 4.15.
(ii) is an immediate consequence of (i) since $\varphi_1(e) = e$ by our choice of d_1. □

5. PROOF OF THEOREM 1.3 FOR RIGID NILPOTENT ORBITS

We intend to prove in this section the following theorem:

THEOREM 5.1. *Suppose that $\mathfrak{g}$ is reductive and let e be a rigid nilpotent element of $\mathfrak{g}$. Then the index of $\mathfrak{g}^e$ is equal to ℓ.*

Theorem 5.1 will complete the proof of Theorem 1.3 by Theorem 3.3. As explained in introduction, we can assume that $\mathfrak{g}$ is simple. We consider two cases, according to $\mathfrak{g}$ has classical type or exceptional type.

5.1. THE CLASSICAL CASE. Assume that $\mathfrak{g}$ is simple of classical type. More precisely, assume that $\mathfrak{g}$ is one of the Lie algebras $\mathfrak{sl}_{\ell+1}(\Bbbk)$, $\mathfrak{so}_{2\ell+1}(\Bbbk)$, $\mathfrak{sp}_{2\ell}(\Bbbk)$, $\mathfrak{so}_{2\ell}(\Bbbk)$.

LEMMA 5.2. *Let m be a positive integer such that $x^m - {\rm tr}\, x^m$ belongs to $\mathfrak{g}$ for all x in $\mathfrak{g}$. Then e^m belongs to the subspace generated by $\varphi_1(e), \dots, \varphi_\ell(e)$.*

Proof. Recall that $L_\mathfrak{g}$ is the submodule of elements φ of $\mathrm{S}(\mathfrak{g}) \otimes_\Bbbk \mathfrak{g}$ such that $[x, \varphi(x)] = 0$ for all x in $\mathfrak{g}$. According to [Di79], $L_\mathfrak{g}$ is a free module generated by the $\varphi_i' s$. For all x in $\mathfrak{g}$, $[x, x^m] = 0$. Hence there exist polynomial functions $a_1, \dots, a_\ell$ on $\mathfrak{g}$ such that

$$x^m - \operatorname{tr} x^m = a_1(x)\varphi_1(x) + \cdots + a_\ell(x)\varphi_\ell(x)$$

for all x in $\mathfrak{g}$, whence the lemma. □

Theorem 5.3. *Let e be a rigid nilpotent element. Then $\mathfrak{z}(\mathfrak{g}^e)$ is generated by powers of e. In particular, the index of $\mathfrak{g}^e$ is equal to ℓ.*

Proof. Let us prove the first assertion. If $\mathfrak{g}$ has type A or C, then $\mathfrak{z}(\mathfrak{g}^e)$ is generated by powers of e by [Mo06c, Théorème 1.1.8] or [Y06b]. So we can assume that $\mathfrak{g}$ has type B or D.
Set $n := 2\ell + 1$ if $\mathfrak{g}$ has type B_ℓ and $n := 2\ell$ if $\mathfrak{g}$ has type D_ℓ. Denote by $(n_1, \dots, n_k)$, with $n_1 \geq \cdots \geq n_k$, the partition of n corresponding to the nilpotent element e. By [Mo06c, Théorème 1.1.8] or [Y06b], $\mathfrak{z}(\mathfrak{g}^e)$ is *not* generated by powers of e if and only if n_1 and n_2 are both odd integers and $n_3 < n_2$. On the other hand, since e is rigid, n_k is equal to 1, $n_i \leq n_{i+1} \leq n_i + 1$ and all odd integers of the partition $(n_1, \dots, n_k)$ have a multiplicity different from 2 [Ke83, Sp82, ch. II] or [CMa93, Corollary 7.3.5]. Hence, the preceding criterion is not satisfied for e. Then, the second assertion results from Lemma 5.2, Corollary 4.16(i) and Theorem 4.13. □

Remark 5.4. Yakimova's proof of Elashvili's conjecture in the classical case is shorter and more elementary [Y06a]. The results of Section 4 will serve the exceptional case in a more relevant way.

5.2. The exceptional case. We let in this subsection $\mathfrak{g}$ be simple of exceptional type and we assume that e is a nonzero rigid nilpotent element of $\mathfrak{g}$. The dimension of the center of centralizers of nilpotent elements has been recently described in [LT08, Theorem 4]. On the other hand, we have explicit computations for the rigid nilpotent orbits in the exceptional types due to A.G. Elashvili. These computations are collected in [Sp82, Appendix of Chap. II] and a complete version was published later in [E85b]. From all this, we observe that the center of $\mathfrak{g}^e$ has dimension 1 in most cases. In more details, we have:

Proposition 5.5. *Let e be nonzero rigid nilpotent element of $\mathfrak{g}$.*
(i) *Suppose that $\mathfrak{g}$ has type G_2, F_4 or E_6. Then $\dim \mathfrak{z}(\mathfrak{g}^e) = 1$.*
(ii) *Suppose that $\mathfrak{g}$ has type E_7. If $\mathfrak{g}^e$ has dimension 41, then $\dim \mathfrak{z}(\mathfrak{g}^e) = 2$; otherwise $\dim \mathfrak{z}(\mathfrak{g}^e) = 1$.*
(iii) *Suppose that $\mathfrak{g}$ has type E_8. If $\mathfrak{g}^e$ has dimension 112, 84, 76, or 46, then $\dim \mathfrak{z}(\mathfrak{g}^e) = 2$, if $\mathfrak{g}^e$ has dimension 72, then $\dim \mathfrak{z}(\mathfrak{g}^e) = 3$; otherwise $\dim \mathfrak{z}(\mathfrak{g}^e) = 1$.*

By Corollary 4.16(ii), $\mathrm{ind}\,\mathfrak{g}^e = \ell$ whenever $\dim \mathfrak{z}(\mathfrak{g}^e) = 1$. So, as an immediate consequence of Proposition 5.5, we obtain:

Corollary 5.6. *Suppose that either* $\mathfrak{g}$ *has type* G_2, F_4, E_6, *or* $\mathfrak{g}$ *has type* E_7 *and* $\dim \mathfrak{g}^e \neq 41$, *or* $\mathfrak{g}$ *has type* E_8 *and* $\dim \mathfrak{g}^e \not\in \{112, 84, 76, 72, 46\}$. *Then* $\dim \mathfrak{z}(\mathfrak{g}^e) = 1$ *and the index of* $\mathfrak{g}^e$ *is equal to* ℓ.

According to Corollary 5.6, it remains 7 cases; there are indeed two rigid nilpotent orbits of codimension 46 in E_8. We handle now these remaining cases. We process here in a different way; we study technical conditions on $\mathfrak{g}^e$ under which $\mathrm{ind}\,\mathfrak{g}^e = \ell$. For the moment, we state general results about the index.
Let $\mathfrak{a}$ be an algebraic Lie algebra. Recall that the stabilizer of $\xi \in \mathfrak{a}^*$ for the coadjoint representation is denoted by $\mathfrak{a}^\xi$ and that ξ is regular if $\dim \mathfrak{a}^\xi = \mathrm{ind}\,\mathfrak{a}$. Choose a commutative subalgebra $\mathfrak{t}$ of $\mathfrak{a}$ consisted of semisimple elements of $\mathfrak{a}$ and denote by $\mathfrak{z}_{\mathfrak{a}}(\mathfrak{t})$ the centralizer of $\mathfrak{t}$ in $\mathfrak{a}$. Then $\mathfrak{a} = \mathfrak{z}_{\mathfrak{a}}(\mathfrak{t}) \oplus [\mathfrak{t}, \mathfrak{a}]$. The dual $\mathfrak{z}_{\mathfrak{a}}(\mathfrak{t})^*$ of $\mathfrak{z}_{\mathfrak{a}}(\mathfrak{t})$ identifies to the orthogonal complement of $[\mathfrak{t}, \mathfrak{a}]$ in $\mathfrak{a}^*$. Thus, $\xi \in \mathfrak{z}_{\mathfrak{a}}(\mathfrak{t})^*$ if and only if $\mathfrak{t}$ is contained in $\mathfrak{a}^\xi$.

Lemma 5.7. *Suppose that there exists* ξ *in* $\mathfrak{z}_{\mathfrak{a}}(\mathfrak{t})^*$ *such that* $\dim(\mathfrak{a}^\xi \cap [\mathfrak{t}, \mathfrak{a}]) \leq 2$. *Then*

$$\mathrm{ind}\,\mathfrak{a} \leq \mathrm{ind}\,\mathfrak{z}_{\mathfrak{a}}(\mathfrak{t}) + 1.$$

Proof. Let T be the closure in $\mathfrak{z}_{\mathfrak{a}}(\mathfrak{t})^* \times \mathrm{Gr}_3([\mathfrak{t}, \mathfrak{a}])$ of the subset of elements (η, E) such that η is a regular element of $\mathfrak{z}_{\mathfrak{a}}(\mathfrak{t})^*$ and E is contained in $\mathfrak{a}^\eta$. The image T_1 of T by the projection from $\mathfrak{z}_{\mathfrak{a}}(\mathfrak{t})^* \times \mathrm{Gr}_3([\mathfrak{t}, \mathfrak{a}])$ to $\mathfrak{z}_{\mathfrak{a}}(\mathfrak{t})^*$ is closed in $\mathfrak{z}_{\mathfrak{a}}(\mathfrak{t})^*$. By hypothesis, T_1 is not equal to $\mathfrak{z}_{\mathfrak{a}}(\mathfrak{t})^*$ since for all η in T_1, $\dim(\mathfrak{a}^\eta \cap [\mathfrak{t}, \mathfrak{a}]) \geq 3$. Hence there exists a regular element ξ_0 in $\mathfrak{z}_{\mathfrak{a}}(\mathfrak{t})^*$ such that $\dim(\mathfrak{a}^{\xi_0} \cap [\mathfrak{t}, \mathfrak{a}]) \leq 2$. Since $\mathfrak{t}$ is contained in $\mathfrak{a}^{\xi_0}$,

$$\mathfrak{a}^{\xi_0} = \mathfrak{z}_{\mathfrak{a}}(\mathfrak{t})^{\xi_0} \oplus \mathfrak{a}^{\xi_0} \cap [\mathfrak{t}, \mathfrak{a}].$$

If $[\mathfrak{t}, \mathfrak{a}] \cap \mathfrak{a}^{\xi_0} = \{0\}$ then $\mathrm{ind}\,\mathfrak{a}$ is at most $\mathrm{ind}\,\mathfrak{z}_{\mathfrak{a}}(\mathfrak{t})$. Otherwise, $\mathfrak{a}^{\xi_0}$ is not a commutative subalgebra since $\mathfrak{t}$ is contained in $\mathfrak{a}^{\xi_0}$. Hence ξ_0 is not a regular element of $\mathfrak{a}^*$, so $\mathrm{ind}\,\mathfrak{a} < \dim \mathfrak{a}^{\xi_0}$. Since $\dim \mathfrak{a}^{\xi_0} \leq \mathrm{ind}\,\mathfrak{z}_{\mathfrak{a}}(\mathfrak{t}) + 2$, the lemma follows. □

From now on, we assume that $\mathfrak{a} = \mathfrak{g}^e$. As a rigid nilpotent element of $\mathfrak{g}$, e is a nondistinguished nilpotent element. So we can choose a nonzero commutative subalgebra $\mathfrak{t}$ of $\mathfrak{g}^e$ consisted of semisimple elements. Denote by $\mathfrak{l}$ the centralizer of $\mathfrak{t}$ in $\mathfrak{g}$. As a Levi subalgebra of $\mathfrak{g}$, $\mathfrak{l}$ is a reductive Lie algebra whose rank is ℓ. Moreover its dimension is strictly smaller than $\dim \mathfrak{g}$. In the preceding notations, we have $\mathfrak{z}_{\mathfrak{g}^e}(\mathfrak{t}) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{t})^e = \mathfrak{l}^e$. Let $\mathfrak{t}_1$ be a commutative subalgebra of $\mathfrak{l}^e$ containing $\mathfrak{t}$ and consisting of semisimple elements of $\mathfrak{l}$. Then $[\mathfrak{t},\mathfrak{g}^e]$ is stable under the adjoint action of $\mathfrak{t}_1$. For λ in $\mathfrak{t}_1^*$, denote by $\mathfrak{g}^e_\lambda$ the λ-weight space of the adjoint action of $\mathfrak{t}_1$ in $\mathfrak{g}^e$.

Lemma 5.8. *Let $\lambda \in \mathfrak{t}_1^*$ be a nonzero weight of the adjoint action of $\mathfrak{t}_1$ in $\mathfrak{g}^e$. Then $-\lambda$ is also a weight for this action and λ and $-\lambda$ have the same multiplicity. Moreover, $\mathfrak{g}^e_\lambda$ is contained in $[\mathfrak{t},\mathfrak{g}^e]$ if and only if the restriction of λ to $\mathfrak{t}$ is not identically zero.*

Proof. By definition, $\mathfrak{g}^e_\lambda \cap \mathfrak{l}^e = \{0\}$ if and only if the restriction of λ to $\mathfrak{t}$ is not identically zero. So $\mathfrak{g}^e_\lambda$ is contained in $[\mathfrak{t},\mathfrak{g}^e]$ if and only if the restriction of λ to $\mathfrak{t}$ is not equal to 0 since

$$\mathfrak{g}^e_\lambda = (\mathfrak{g}^e_\lambda \cap \mathfrak{l}^e) \oplus (\mathfrak{g}^e_\lambda \cap [\mathfrak{t},\mathfrak{g}^e]).$$

The subalgebra $\mathfrak{t}_1$ is contained in a reductive factor of $\mathfrak{g}^e$. So we can choose h and f such that $\mathfrak{t}_1$ is contained in $\mathfrak{g}^e\cap\mathfrak{g}^f$. As a consequence, any weight of the adjoint action of $\mathfrak{t}_1$ in $\mathfrak{g}^f$ is a weight of the adjoint action of $\mathfrak{t}_1$ in $\mathfrak{g}^e$ with the same multiplicity. Furthermore, the $\mathfrak{t}_1$-module $\mathfrak{g}^f$ for the ajoint action is isomorphic to the $\mathfrak{t}_1$-module $(\mathfrak{g}^e)^*$ for the coadjoint action. So $-\lambda$ is a weight of the adjoint action of $\mathfrak{t}_1$ in $\mathfrak{g}^f$ with the same multiplicity as λ. Hence $-\lambda$ is a weight of the adjoint action of $\mathfrak{t}_1$ in $\mathfrak{g}^e$ with the same multiplicity as λ, whence the lemma. □

Choose pairwise different elements $\lambda_1,\dots,\lambda_r$ of $\mathfrak{t}_1^*$ so that the weights of the adjoint action of $\mathfrak{t}_1$ in $\mathfrak{g}^e$ which are not identically zero on $\mathfrak{t}$ are precisely the elements $\pm\lambda_i$. For $i=1,\dots,r$, let $v_{i,1},\dots,v_{i,m_i}$ and $w_{i,1},\dots,w_{i,m_i}$ be basis of $\mathfrak{g}^e_{\lambda_i}$ and $\mathfrak{g}^e_{-\lambda_i}$ respectively. Then we set:

$$q_i := \det\left([v_{i,k},w_{i,l}]\right)_{1\leq k,l\leq m_i} \in \mathrm{S}(\mathfrak{l}^e).$$

Proposition 5.9. *Suppose that* $\operatorname{ind}\mathfrak{l}^e = \ell$ *and suppose that one of the following two conditions is satisfied:*

1. *for $i=1,\dots,r$, $q_i \neq 0$,*
2. *there exists j in $\{1,\dots,r\}$ such that $q_i\neq 0$ for all $i\neq j$ and such that the basis $v_{j,1},\dots,v_{j,m_j}$ and $w_{j,1},\dots,w_{j,m_j}$ can be chosen so that*

$$\det\left([v_{j,k},w_{j,l}]\right)_{1\leq k,l\leq m_j-1} \neq 0.$$

Then, $\operatorname{ind}\mathfrak{g}^e = \ell$.

Proof. First, observe that $\operatorname{ind}\mathfrak{g}^e - \operatorname{ind}\mathfrak{g}$ is an even integer. Indeed, we have:

$$\operatorname{ind}\mathfrak{g}^e - \operatorname{ind}\mathfrak{g} = (\operatorname{ind}\mathfrak{g}^e - \dim\mathfrak{g}^e) + (\dim\mathfrak{g}^e - \dim\mathfrak{g}) + (\dim\mathfrak{g} - \operatorname{ind}\mathfrak{g}).$$

But the integers $\operatorname{ind}\mathfrak{g}^e - \dim\mathfrak{g}^e$, $\dim\mathfrak{g}^e - \dim\mathfrak{g}$ and $\dim\mathfrak{g} - \operatorname{ind}\mathfrak{g}$ are all even integers. Thereby, if $\operatorname{ind}\mathfrak{g}^e \leq \operatorname{ind}\mathfrak{g}+1$, then $\operatorname{ind}\mathfrak{g}^e \leq \operatorname{ind}\mathfrak{g}$. In

turn, by Vinberg's inequality (cf. Introduction), we have $\operatorname{ind}\mathfrak{g}^e \geq \operatorname{ind}\mathfrak{g}$. Hence, it suffices to prove $\operatorname{ind}\mathfrak{g}^e \leq \operatorname{ind}\mathfrak{l}^e + 1$ since our hypothesis says that $\operatorname{ind}\mathfrak{l}^e = \ell = \operatorname{ind}\mathfrak{g}$. Now, by Lemma 5.7, if there exists ξ in $(\mathfrak{l}^e)^*$ such that $(\mathfrak{g}^e)^\xi \cap [\mathfrak{t}, \mathfrak{g}^e]$ has dimension at most 2, then we are done.
Denote by $\mathfrak{l}_1$ the centralizer of $\mathfrak{t}_1$ in $\mathfrak{g}$. Then $\mathfrak{l}_1$ is contained in $\mathfrak{l}$ and $\mathfrak{l}^e = \mathfrak{l}_1^e \oplus [\mathfrak{t}_1, \mathfrak{l}^e]$ and $(\mathfrak{l}_1^e)^*$ identifies to the orthogonal of $[\mathfrak{t}_1, \mathfrak{l}^e]$ in the dual of $\mathfrak{l}^e$. Moreover, for $i = 1, \ldots, r$, q_i belongs to $\mathrm{S}(\mathfrak{l}_1^e)$. For ξ in $(\mathfrak{l}_1^e)^*$, denote by B_ξ the bilinear form

$$\begin{array}{ccc} [\mathfrak{t}, \mathfrak{g}^e] \times [\mathfrak{t}, \mathfrak{g}^e] & \longrightarrow & \Bbbk \\ (v, w) & \longmapsto & \xi([v, w]) \end{array}$$

and denote by $\ker B_\xi$ its kernel. For $i = 1, \ldots, r$, $-q_i(\xi)^2$ is the determinant of the restriction of B_ξ to the subspace

$$(\mathfrak{g}^e_{\lambda_i} \oplus \mathfrak{g}^e_{-\lambda_i}) \times (\mathfrak{g}^e_{\lambda_i} \oplus \mathfrak{g}^e_{-\lambda_i})$$

in the basis $v_{i,1}, \ldots, v_{i,m_i}, w_{i,1}, \ldots, w_{i,m_i}$.
If (1) holds, we can find ξ in $(\mathfrak{l}_1^e)^*$ such that $\ker B_\xi$ is zero. If (2) holds, we can find ξ in $(\mathfrak{l}_1^e)^*$ such that $\ker B_\xi$ has dimension 2 since B_ξ is invariant under the adjoint action of $\mathfrak{t}_1$. But $\ker B_\xi$ is equal to $(\mathfrak{g}^e)^\xi \cap [\mathfrak{t}, \mathfrak{g}^e]$. Hence such a ξ satisfies the required inequality and the proposition follows. □

The proof of the following proposition is given in Appendix A since it relies on explicit computations:

Proposition 5.10. (i) *Suppose that either* $\mathfrak{g}$ *has type* E_7 *and* $\dim \mathfrak{g}^e = 41$ *or,* $\mathfrak{g}$ *has type* E_8 *and* $\dim \mathfrak{g}^e \in \{112, 72\}$*. Then, for suitable choices of* $\mathfrak{t}$ *and* $\mathfrak{t}_1$*, Condition* (1) *of Proposition* 5.9 *is satisfied.*
(ii) *Suppose that* $\mathfrak{g}$ *has type* E_8 *and that* $\mathfrak{g}^e$ *has dimension* 84, 76*, or* 46*. Then, for suitable choices of* $\mathfrak{t}$ *and* $\mathfrak{t}_1$*, Condition* (2) *of Proposition* 5.5 *is satisfied.*

5.3. Proof of Theorem 1.3. We are now in position to complete the proof of Theorem 1.3:

Proof of Theorem 1.3. We argue by induction on the dimension of $\mathfrak{g}$. If $\mathfrak{g}$ has dimension 3, the statement is known. Assume now that $\operatorname{ind}\mathfrak{l}^{e'} = \operatorname{rk}\mathfrak{l}$ for any reductive Lie algebras $\mathfrak{l}$ of dimension at most $\dim\mathfrak{g} - 1$ and any $e' \in \mathcal{N}(\mathfrak{l})$. Let $e \in \mathcal{N}(\mathfrak{g})$ be a nilpotent element of $\mathfrak{g}$. By Theorem 3.3 and Theorem 5.3, we can assume that e is rigid and that $\mathfrak{g}$ is simple of exceptional type. Furthermore by Corollary 5.6, we can assume that $\dim\mathfrak{z}(\mathfrak{g}^e) > 1$. Then we consider the different cases given by Proposition 5.10.
If, either $\mathfrak{g}$ has type E_7 and $\dim\mathfrak{g}^e = 41$, or $\mathfrak{g}$ has type E_8 and $\dim\mathfrak{g}^e$ equals 112, 72, or 46, then Condition (1) of Proposition 5.9 applies for

suitable choices of $\mathfrak{t}$ and $\mathfrak{t}_1$ by Proposition 5.10. Moreover, if $\mathfrak{l} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{t})$, then $\mathfrak{l}$ is a reductive Lie algebra of rank ℓ and strictly contained in $\mathfrak{g}$. So, from our induction hypothesis, we deduce that $\operatorname{ind}\mathfrak{g}^e = \ell$ by Proposition 5.9.
If $\mathfrak{g}$ has type E_8 and $\dim\mathfrak{g}^e$ equals 84, 76, or 46, then Condition (2) of Proposition 5.9 applies for suitable choices of $\mathfrak{t}$ and $\mathfrak{t}_1$ by Proposition 5.10. Arguing as above, we deduce that $\operatorname{ind}\mathfrak{g}^e = \ell$. $\square$

Appendix A. Proof of Proposition 5.10: explicit computations.

This appendix aims to prove Proposition 5.10. We prove Proposition 5.10 for each case by using explicit computations made with the help of `GAP`; our programmes are presented below (two cases are detailed; the other ones are similar). Explain the general approach. In our programmes, $x[1], \dots$ are root vectors generating the nilradical of the Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and the representative e (denoted by `e` in our programmes) of the rigid orbit is chosen so that e and h belong to $\mathfrak{b}$ and $\mathfrak{h}$ respectively. The element e is given by the tables of [GQT80]. In fact, in [GQT80], they use the programme Lie which induces minor changes in the numbering. Then, we exhibit suitable tori $\mathfrak{t}$ and $\mathfrak{t}_1$ of $\mathfrak{g}$ contained in $\mathfrak{g}^e$ which satisfies conditions (1) or (2) of Proposition 5.9. In each case, our torus $\mathfrak{t}$ is one dimensional; we define it by a generator, called `t` in our programmes. Its centralizer in $\mathfrak{g}^e$ is denoted by `le`. The torus $\mathfrak{t}_1$ has dimension at most 4. It is defined by a basis denoted by `Bt1`. The weights of $\mathfrak{t}_1$ for the adjoint action of $\mathfrak{t}_1$ on $\mathfrak{g}^e$ are given by their values on the basis `Bt1` of $\mathfrak{t}_1$. We list in a matrix `W` almost all weights which have a positive value at `Bt1`. The other weights have multiplicity 1. In our programmes, by the term `S` we check that no weight is forgotten; this term has to be zero. Then, the matrices corresponding to the weights given by `W` are given by a function `A`. Their determinants correspond to the q_i's in the notations of Proposition 5.9. If there is only one other weight, the corresponding matrix is denoted by `a`. At last, we verify that these matrices have the desired property depending on the situations (i) or (ii) of Proposition 5.10.
As examples, we detail below two cases:
(1) the case of E_7 with $\dim\mathfrak{g}^e = 41$ where we intend to check that Condition (1) of Proposition 5.9 is satisfied;
(2) the case of E_8 with $\dim\mathfrak{g}^e = 84$ where we intend to check that Condition (2) of Proposition 5.9 is satisfied.

(1) E_7, $\dim \mathfrak{g}^e = 41$: In this case, with our choices, $\dim \mathfrak{t} = 1$, $\dim \mathfrak{l}^e = 23$ and $\dim \mathfrak{t}_1 = 3$. The order of matrices to be considered is at most 2.

```
L := SimpleLieAlgebra("E",7,Rationals);;
R := RootSystem(L);;
x := PositiveRootVectors(R);; y := NegativeRootVectors(R);;
e := x[14]+x[26]+x[28]+x[49];;
c := LieCentralizer(L,Subspace(L,[e]));
Bc := BasisVectors(Basis(c));;
> <Lie algebra of dimension 41 over Rationals>
z := LieCentre(c);; Bz := BasisVectors(Basis(z));;
t := Bc[Dimension(c)];;
le := LieCentralizer(L,Subspace(L,[t,e]));
> <Lie algebra of dimension 23 over Rationals>
n := function(k)
     if k=2 then return 1;;
     elif k=-2 then return 1;;
     elif k=1 then return 8;;
     elif k=-1 then return 8;; fi;; end;;
#The function n assigns to each weight of t
#the dimension of the corresponding
#weight subspace.
M := function(k)  local m;;
   m := function(j,k)
      if j=1 then
          return Position(List([1..Dimension(c)],
              i->t*Bc[i]-k*Bc[i]),0*x[1]);;
          else
             return m(j-1,k)
                 +Position(List([m(j-1,k)+1..Dimension(c)],
          i->t*Bc[i]-k*Bc[i]),0*x[1]);;
          fi;;
     end;;
     return List([1..n(k)],i->m(i,k));;
end;;
Bt1 := [Bc[41],Bc[40],Bc[39]];;
N := function(k,p) local n;;
     n := function(j,k,p)
          if j=1 then
              return Position(List([1..8],
          i->Bt1[2]*Bc[M(k)[i]]-p*Bc[M(k)[i]]),0*x[1]);;
          else
```

```
          return n(j-1,k,p)
               +Position(List([n(j-1,k,p)+1..8],
        i->Bt1[2]*Bc[M(k)[i]]-p*Bc[M(k)[i]]),0*x[1]);;
        fi;;
     end;;
     return List([1..4],i->M(k)[n(i,k,p)]);;
end;;
r := function(t)
     if t=1 then return 1;
     elif t=-1 then return 1;;
     elif t=0 then return 2;;
     fi;;
end;;
Q := function(k,s,t) local q;;
  q := function(j,k,s,t)
     if j=1 then
          return Position(List([1..4],
     i->Bt1[3]*Bc[N(k,s)[i]]-t*Bc[N(k,s)[i]]),0*x[1]);;
     else
          return q(j-1,k,s,t)
                 +Position(List([q(j-1,k,s,t)+1..4],
     i->Bt1[3]*Bc[N(k,s)[i]]-t*Bc[N(k,s)[i]]),0*x[1]);;
          fi;;
     end;;
     return List([1..r(t)],i->N(k,s)[q(i,k,s,t)]);;
end;;
W := [[1,1,1],[1,-1,1],[1,1,-1],[1,-1,-1],
                               [1,1,0],[1,-1,0]];;
S := 2*(1+Sum(List([1..Length(W)],
               i->Length(Q(W[i][1],W[i][2],W[i][3])))))
                   +Dimension(le)-Dimension(c);
> 0
A := function(i) return
          List([1..r(W[i][3])],t->List([1..r(W[i][3])],
     s->Bc[Q(W[i][1],W[i][2],W[i][3])[s]]*
                Bc[Q(-W[i][1],-W[i][2],-W[i][3])[t]]));;
end;;
A(1);A(2);A(3);A(4);A(5);A(6);
> [ [ (-1)*v.63 ] ]
> [ [ v.63 ] ]
> [ [ v.63 ] ]
```

```
> [ [ (-1)*v.63 ] ]
> [ [ (-1)*v.57+(-1)*v.60, (-1)*v.63 ],
                           [ (-1)*v.63, 0*v.1 ] ]
> [ [ (-1)*v.57+(-1)*v.60, (-1)*v.63 ],
                              [ (-1)*v.63, 0*v.1 ] ]
a := Bc[M(2)[1]]*Bc[M(-2)[1]];
> v.133
```

In conclusion, Condition (1) of Proposition 5.9 is satisfied for $\mathfrak{t} := \Bbbk$`t` and $\mathfrak{t}_1$:=span(`Bt1`).

(2) E_8, $\dim \mathfrak{g}^e = 84$: In this case, with our choices, $\dim \mathfrak{t} = 1$, $\dim \mathfrak{l}^e = 48$ and $\dim \mathfrak{t}_1 = 3$. The matrix `A(7)` has order 5 and it is singular of rank 4. The order of the other matrices is at most 2.

```
L := SimpleLieAlgebra("E",8,Rationals);;
R := RootSystem(L);;
x := PositiveRootVectors(R);; y := NegativeRootVectors(R);;
e := x[54]+x[61]+x[77]+x[97];;
c := LieCentralizer(L,Subspace(L,[e]));
Bc := BasisVectors(Basis(c));;
> <Lie algebra of dimension 84 over Rationals>
z := LieCentre(c);; Bz := BasisVectors(Basis(z));;
t := Bc[Dimension(c)];;
le := LieCentralizer(L,Subspace(L,[t,e]));
> <Lie algebra of dimension 48 over Rationals>
n := function(k)
     if k=2 then return 1;;
     elif k=-2 then return 1;;
     elif k=1 then return 17;;
     elif k=-1 then return 17;;
     fi;;
end;;
M := function(k)  local m;;
   m := function(j,k)
      if j=1 then
          return Position(List([1..Dimension(c)],
        i->Bc[84]*Bc[i]-k*Bc[i]),0*x[1]);;
      else
          return m(j-1,k)
            +Position(List([m(j-1,k)+1..Dimension(c)],
          i->Bc[84]*Bc[i]-k*Bc[i]), 0*x[1]);;
          fi;;
```

```
     end;;
     return List([1..n(k)],i->m(i,k));;
end;;
r := function(k,t)
     if k=1 and t=1  then return 4;;
     elif k=-1 and t=-1  then return 4;;
     elif k=1 and t=-1  then return 4;;
     elif k=-1 and t=1  then return 4;;
     elif k=1 and t=0  then return 9;;
     elif k=-1 and t=0  then return 9;;
     fi;;
end;;
Bt1 := [Bc[84],Bc[83],Bc[82]];;
N := function(k,t) local p;;
   p := function(j,k,t)
      if j=1 then
         return Position(List([1..n(k)],
       i->Bt1[2]*Bc[M(k)[i]]-t*Bc[M(k)[i]]),0*x[1]);;
      else
         return p(j-1,k,t)
              +Position(List([p(j-1,k,t)+1..n(k)],
       i->Bt1[2]*Bc[M(k)[i]]-t*Bc[M(k)[i]]),0*x[1]);;
          fi;;
     end;;
     return List([1..r(k,t)],i->M(k)[p(i,k,t)]);;
end;;
m := function(k,s,t)
     if k=1 and s=1 and t=-1 then return 2;;
     elif k=-1 and s=-1 and t=1 then  return 2;;
     elif k=1 and s=1 and t=0 then return 2;;
     elif k=-1 and s=-1 and t=0 then return 2;;
     elif k=1 and s=-1 and t=1 then  return 2;;
     elif k=-1 and s=1 and t=-1 then return 2;;
     elif k=1 and s=-1 and t=0 then return 2;;
     elif k=-1 and s=1 and t=0 then  return 2;;
     elif k=1 and s=0 and t=1 then return 2;;
     elif k=-1 and s=0 and t=-1 then return 2;;
     elif k=1 and s=0 and t=-1 then return 2;;
     elif k=-1 and s=0 and t=1 then return 2;;
     elif k=1 and s=0 and t=0 then return 5;;
     elif k=-1 and s=0 and t=0 then return 5;;
```

```
     fi;;
end;;
Q := function(k,s,t) local q;;
  q := function(j,k,s,t)
    if j=1 then
        return Position(List([1..r(k,s)],
     i->Bt1[3]*Bc[N(k,s)[i]]-t*Bc[N(k,s)[i]]),0*x[1]);;
    else
        return q(j-1,k,s,t)
              +Position(List([q(j-1,k,s,t)+1..r(k,s)],
     i->Bt1[3]*Bc[N(k,s)[i]]-t*Bc[N(k,s)[i]]),0*x[1]);;
         fi;;
     end;;
     return List([1..m(k,s,t)],i->N(k,s)[q(i,k,s,t)]);;
end;;
W := [[1,1,-1],[1,1,0],[1,-1,1],[1,-1,0],
                        [1,0,1],[1,0,-1],[1,0,0]];;
S := 2 + 2*Sum(List([1..Length(W)],
               i->Length(Q(W[i][1],W[i][2],W[i][3]))))
                  + Dimension(le)-Dimension(c);;
A := function(i) return
     List([1..m(W[i][1],W[i][2],W[i][3])],
        t->List([1..m(W[i][1],W[i][2],W[i][3])],
        s->Bc[Q(W[i][1],W[i][2],W[i][3])[s]]*
               Bc[Q(-W[i][1],-W[i][2],-W[i][3])[t]]));;
end;;
# A(1), A(2), A(3), A(5), A(6) are nonsingular.
# A(7) is singular of order 5 of rank 4; its minor
List([1..4],s->List([1..4],
t->Bc[Q(W[7][1],W[7][2],W[7][3])[s]]*
              Bc[Q(-W[7][1],-W[7][2],-W[7][3])[t]]));;
# is different from 0.
a := Bc[M(2)[1]]*Bc[M(-2)[1]];;
```

In conclusion, Condition (2) of Proposition 5.9 is satisfied for $\mathfrak{t} := \Bbbk\mathtt{t}$ and $\mathfrak{t}_1$:=span(Bt1).

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Jean-Yves Charbonnel
Université Paris 7 - CNRS
Institut de Mathématiques de Jussieu
Théorie des groupes
Case 7012
Bâtiment Chevaleret
75205 Paris Cedex 13
France
jyc@math.jussieu.fr

Anne Moreau
LMA
Téléport 2 - BP 30179
Boulevard Marie et Pierre Curie
86962 Futuroscope Chasseneuil Cedex
France
anne.moreau@math.univ-poitiers.fr