Blow-up of Solutions to a Periodic Nonlinear Dispersive Rod Equation

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Abstract. In this paper, firstly we find an optimal constant for a convolution problem on the unit circle via the variational method. Then by using the optimal constant, we give a new and improved sufficient condition on the initial data to guarantee the corresponding strong solution blows up in finite time. We also analyze the corresponding ordinary difference equation associate to the convolution problem and give numerical simulation for the optimal constant.

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1 Introduction

Although a rod is always three-dimensional, if its diameter is much less than the axial length scale, one-dimensional equations can give a good description of the motion of the rod. Recently Dai [16] derived a new (one-dimensional) nonlinear dispersive equation including extra nonlinear terms involving second-order and third-order derivatives for a compressible hyperelastic material. The equation reads

$$v_\tau + \sigma_1 v v_\xi + \sigma_2 v_{\xi\xi\tau} + \sigma_3(2v_\xi v_{\xi\xi} + v v_{\xi\xi\xi}) = 0,$$

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where $v(\xi, \tau)$ represents the radial stretch relative to a pre-stressed state, $\sigma_1 \neq 0$, $\sigma_2 < 0$ and $\sigma_3 \leq 0$ are constants determined by the pre-stress and the material parameters. If one introduces the following transformations

$$\tau = \frac{3\sqrt{-\sigma_2}}{\sigma_1} t, \ \ \xi = \sqrt{-\sigma_2} x,$$

then the above equation turns into

$$u_t - u_{txx} + 3uu_x = \gamma(2u_x u_{xx} + uu_{xxx}), \tag{1.1}$$

where $\gamma = 3\sigma_3/(\sigma_1\sigma_2)$. In [17], the authors derived that value range of γ is from -29.4760 to 3.4174 for some special compressible materials. From the mathematical view point, we regard γ as a real number.

When $\gamma = 1$ in (1.1), we recover the shallow water (Camassa-Holm) equation derived physically by Camassa and Holm in [4] (found earlier by Fuchssteiner and Fokas [18] as a bi-Hamiltonian generalization of the KdV equation) by approximating directly the Hamiltonian for Euler's equations in the shallow water region, where $u(x,t)$ represents the free surface above a flat bottom. Recently, the alternative derivations of the Camassa-Holm equation as a model for water waves, respectively as the equation for geodesic flow on the diffeomorphism group of the circle were presented in [27] and respectively in [9, 29]. For the physical derivation, we refer to works in [10, 26]. Some satisfactory results have been obtained for this shallow water equation. Local well-posedness for the initial datum $u_0(x) \in H^s$ with $s > 3/2$ was proved by several authors, see [30, 32, 35]. For the initial data with lower regularity, we refer to [33] and [2]. While the regularized generalized Camassa–Holm equation was analyzed in [15]. Moreover, wave breaking for a large class of initial data has been established in [5, 7, 8, 30, 38, 39]. However, in [37], global existence of weak solutions is proved but uniqueness is obtained only under an a priori assumption that is known to hold only for initial data $u_0(x) \in H^1$ such that $u_0 - u_{0xx}$ is a sign-definite Radon measure (under this condition, global existence and uniqueness was shown in [12] also). Also it is worth to note that the global conservative solutions and global dissipative solutions (with energy being lost when wave breaking occurs) are constructed in [2, 22, 24] and [3, 25]. Recently, in [21], Himonas, Misiołek, Ponce and the third author showed the infinite propagation speed for the Camassa-Holm equation in the sense that a strong solution of the Cauchy problem with compact initial profile can not be compactly supported at any later time unless it is the zero solution, which is an improvement of previous results in this direction obtained in [6].

If $\gamma = 0$, (1.1) is the BBM equation, a well-known model for surface waves in a canal [1], and its solutions are global.

For general $\gamma \in \mathbb{R}$, the rod equation (1.1) was studied sketchily by the Constantin and Strauss in [13] first. Local well-posedness of strong solutions to (1.1) was established by applying Kato's theory [28] and some sufficient conditions on the initial data were found to guarantee the finite blow-up of the corresponding solutions for spatially nonperiodic case. Weak solutions was constructed in [14, 23]. Later, in [41], the third author proved the well-posedness result in detail, and various refined sufficient conditions on the initial data were found to guarantee the finite blow-up of the corresponding solutions for both spatially periodic and nonperiodic cases. Recently, blow-up criteria for a special class of initial data for the periodic rod equation was presented in [31, 42], where $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ is the unit circle. Furthermore, in [20], Guo and the third author have investigated the persistence properties for this rod equation. It should be mentioned that for $\gamma < 1$, (1.1) admits smooth solitary waves observed by Dai and Huo [17]. Let $u(x,t) = \phi(\xi), \xi = x - ct$ be the solitary wave to (1.1). It was shown that $\phi(\xi)$ satisfies

$$\pm\xi = -\sqrt{-\gamma}\left(\frac{1}{2}\pi + \arcsin\frac{2\gamma\phi - (\gamma+1)c}{(1-\gamma)c}\right) - \ln\frac{(\sqrt{c(c-\phi)} + \sqrt{c(c-\gamma\phi)})^2}{(1-\gamma)c\phi}$$

for $\gamma < 0$ and

$$\pm\xi = \sqrt{\gamma}\ln\frac{(\sqrt{c-\gamma\phi)} - \sqrt{\gamma(c-\phi)})^2}{(1-\gamma)c} - \ln\frac{(\sqrt{c-\gamma\phi} + \sqrt{c-\phi})^2}{(1-\gamma)\phi}$$

for $0 < \gamma < 1$. In [13] (see [40] also), Constantin and Strauss proved the stability of these solitary waves by applying a general theorem established by Grillakis, Shatah and Strauss [19].

We conclude this introduction by outlining the rest of the paper. In section 2, we recall the local well-posedness for (1.1) with initial datum $u_0 \in H^s$, $s > 3/2$, and the lifespan of the corresponding solution is finite if and only if its first-order derivative blows up. In section 3, formulation of the optimal constant for a convolution problem is settled by a variational method described in Struwe's book [36]. Then we solve the nonlinear ordinary differential equation in section 4. In section 5, a new blow-up criterion is established by applying the best constant for the convolution problem. Finally, in section 6, another representation will be showed, and a numerical simulation will be given.

2 Preliminaries

In this section, we concentrate on the periodic case. In [13, 41], it is proved that

Theorem 2.1 [13, 41] *Let the initial datum $u_0(x) \in H^s(\mathbb{S})$, $s > 3/2$. Then there exists $T = T(\|u_0\|_{H^s}) > 0$ and a unique solution u, which depends continuously on the initial datum u_0, to (1.1) such that*

$$u \in C([0,T); H^s(\mathbb{S})) \cap C^1\left([0,T); H^{s-1}(\mathbb{S})\right).$$

Moreover, the following two quantities E and F are invariants with respect to *time t for (1.1).*

$$\begin{cases} E(u)(t) = \int_{\mathbb{S}} \left(u^2(x,t) + u_x^2(x,t) \right) dx, \\ \\ F(u)(t) = \int_{\mathbb{S}} \left(u^3(x,t) + \gamma u(x,t) u_x^2(x,t) \right) dx. \end{cases}$$

Actually, the local well-posedness was proved for both periodic and nonperiodic case in the above paper.

The maximum value of T in Theorem 2.1 is called the lifespan of the solution, in general. If $T < \infty$, that is $\limsup_{t \uparrow T} \|u(.,t)\|_{H^s} = \infty$, we say that the solution blows up in finite time. The following theorem tells us that the solution blows up if and only if the first-order derivative blows up.

THEOREM 2.2 [13, 41] *Let $u_0(x) \in H^s(\mathbb{S})$, $s > 3/2$, and u be the corresponding solution to problem (1.1) with lifespan T. Then*

$$\sup_{x \in \mathbb{S}, 0 \le t < T} |u(x,t)| \le C(\|u_0\|_{H^1}). \tag{2.1}$$

T is bounded if and only if

$$\liminf_{t \uparrow T} \inf_{x \in \mathbb{S}} \{\gamma u_x(x,t)\} = -\infty. \tag{2.2}$$

For $\gamma \neq 0$, we set

$$m(t) := \inf_{x \in \mathbb{S}} \left(u_x(x,t) sign\{\gamma\} \right), \quad t \ge 0, \tag{2.3}$$

where $sign\{a\}$ is the sign function of $a \in \mathbb{R}$ and we set $m_0 := m(t = 0)$. Then for every $t \in [0,T)$ there exists at least one point $\xi(t) \in \mathbb{S}$ with $m(t) = u_x(\xi(t), t)$.

LEMMA 2.3 [13] *Let $u(t)$ be the solution to (1.1) on $[0,T)$ with initial data $u_0 \in H^s(\mathbb{S})$, $s > 3/2$, as given by Theorem 2.1. Then the function $m(t)$ is almost everywhere differentiable on $[0,T)$, with*

$$\frac{dm(t)}{dt} = u_{tx}(\xi(t), t), \quad a.e. \ on \ (0,T).$$

Consideration of the quantity $m(t)$ for wave breaking comes from an idea of Seliger [34] originally. The rigorous regularity proof is given in [8] for the Camassa-Holm equation.

Set $Q^s = (1 - \partial_x^2)^{s/2}$, then the operator Q^{-2} can be expressed by

$$Q^{-2} f = G * f = \int_{\mathbb{T}} G(x-y) f(y) dy$$

for any $f \in L^2(\mathbb{S})$ with

$$G(x) = \frac{\cosh(x - [x] - 1/2)}{2\sinh(1/2)}, \tag{2.4}$$

where $[x]$ denotes the integer part of x. Then equation (1.1) can be rewritten as

$$u_t + \gamma u u_x + \partial_x Q^{-2}\left(\frac{3-\gamma}{2}u^2 + \frac{\gamma}{2}u_x^2\right) = 0. \tag{2.5}$$

Just as in [13, 41], it is easy to derive a equation for $m(t)$ from (2.5) as

$$\frac{dm}{dt} = -\frac{\gamma}{2}m^2 + \frac{3-\gamma}{2}u^2(\xi(t),t) - \left[G * \left(\frac{3-\gamma}{2}u^2 + \frac{\gamma}{2}u_x^2\right)\right](\xi(t),t) \tag{2.6}$$

a.e. on $(0,T)$, where $m(t)$ and $\xi(t)$ are defined in (2.3) and Lemma 2.3.
If $\gamma = 3$, it turns out that (2.6) is a Riccati type equation with negative initial data for any nonconstant u_0. So the solutions to (1.1) in periodic case definitely blow up in finite time with arbitrary nonconstant initial data u_0.
In what follows, we assume that $0 < \gamma < 3$.

3 The best constant for a convolution problem-formulation

To prove the blow-up result, one of the basic ingredients is to analyze equation (2.6). It is clear that the difficult part is the convolution term.
In this section, we consider the following convolution problem

$$G * \left(f^2 + \frac{\alpha}{2}f_x^2\right)(x),$$

where G, defined by (2.4), is the Green function for Q^{-2} in the unit circle, $\alpha > 0$ is a constant, and function f belongs to $H^1(\mathbb{S})$.
Direct computation which already done in [43] yields

$$G * (f^2 + \frac{1}{2}f_x^2)(x) \geq \frac{1}{2}f^2(x),$$

for any $x \in \mathbb{S}$.
Therefore,

$$G * \left(f^2 + \frac{\alpha}{2}f_x^2\right)(x) \geq \min\{\alpha, 1\} G * (f^2 + \frac{1}{2}f_x^2)(x) \geq \min\{\alpha, 1\}\frac{1}{2}f^2(x).$$

Our goal is to find an optimal constant $C(\alpha)$ for the following inequality:

$$G * \left(f^2 + \frac{\alpha}{2}f_x^2\right)(x) \geq C(\alpha) f^2(x), \tag{3.1}$$

for all $f \in H^1(\mathbb{S})$.

For this purpose, let

$$\mathcal{A} = \{f \in H^1(\mathbb{S}) \mid \|f\|_{L^\infty} = 1\}$$

and

$$I[f](x) = G * \left(f^2 + \frac{\alpha}{2} f_x^2\right)(x) = \int_{\mathbb{S}} G(x-y)\left(f^2(y) + \frac{\alpha}{2} f_x^2(y)\right) dy.$$

Since $I[f]$ is a translation invariant on the unit circle $\mathbb{S}$, we can assume that $\mathcal{A}$ is defined on the interval $[0,1]$ with $f \geq 0$ and $f(0) = f(1) = 1$ without loss of generality. Hence finding the best constant for the problem (3.1) is equivalent to finding the minimum value for

$$I[f](0) = \frac{1}{2\sinh(1/2)} \int_0^1 \cosh(x - 1/2)\left(f^2(x) + \frac{\alpha}{2} f_x^2(x)\right) dx.$$

From now on, we follow the variational method discussed in a comprehensive book written by Struwe [36].
It is clear that

$$\begin{aligned} \min\{\alpha, 1\} \frac{1}{2\sinh(1/2)} \int_0^1 \left(f^2(x) + f_x^2(x)\right) dx &\leq I[f](0) \\ \leq \quad \max\{\alpha, 1\} \frac{\cosh(1/2)}{2\sinh(1/2)} \int_0^1 \left(f^2(x) + f_x^2(x)\right) dx, \end{aligned}$$

for any $f \in \mathcal{A}$. The above inequality means that $I[f](0)$ is equivalent to the H^1-norm of f.
Suppose that $\{f_k\}_{k=1}^\infty$ is a minimizing sequence, i.e., $I[f_k](0) \to \inf_{f \in \mathcal{A}} I[f](0)$, as $k \to \infty$. Hence it is easy to show that there exists a subsequence $\{f_{kj}\}_{j=1}^\infty \subset \{f_k\}_{k=1}^\infty$, denoted it by $\{f_k\}_{k=1}^\infty$ also, and a function $g \in \mathcal{A}$ with $f_k \to g$ as $k \to \infty$. For the details we refer to [38].
Due to the identities $\cosh(3x) = \cosh^3(x) + 3\cosh(x)\sinh^2(x)$ and $\sinh(3x) = 4\sinh^3(x) + 3\sinh(x)$, we have

$$\begin{aligned} & I\left[\cosh(x - 1/2)/\cosh(1/2)\right](0) \\ = & \frac{1}{2\sinh(1/2)\cosh^2(1/2)} \\ & \times \int_0^1 \left(\cosh^3(x - 1/2) + \frac{\alpha}{2}\cosh(x-1/2)\sinh^2(x - 1/2)\right) dx \\ = & \frac{1}{2\sinh(1/2)\cosh^2(1/2)} \int_{-1/2}^{1/2} \left(\cosh(3x) + \left(\frac{\alpha}{2} - 3\right)\cosh(x)\sinh^2(x)\right) dx \\ = & \frac{2\sinh(3/2) + (\alpha - 6)\sinh^3(1/2)}{6\sinh(1/2)\cosh^2(1/2)} = \frac{6 + (\alpha + 2)\sinh^2(1/2)}{6\cosh^2(1/2)} \\ = & 1 - \frac{(4-\alpha)\sinh^2(1/2)}{6\cosh^2(1/2)} < 1 = I[1](0), \end{aligned}$$

provided that $\alpha < 4$.
For the case $\alpha \geq 4$, we consider the family

$$\frac{\beta + \cosh(x - 1/2)}{\beta + \cosh(1/2)},$$

where $\beta > 0$ is a constant to be determined later. By the same steps, one can get

$$\begin{aligned} & I\left[\frac{\beta + \cosh(x - 1/2)}{\beta + \cosh(1/2)}\right](0) \\ = & \quad \frac{6(\beta+1)\sinh(1/2) + (\alpha+2)\sinh^3(1/2) + 6\beta\cosh(1/2)\sinh(1/2) + 3\beta}{6\sinh(1/2)(\beta + \cosh(1/2))^2}. \end{aligned}$$

Direct computation yields

$$I\left[\frac{\beta + \cosh(x - 1/2)}{\beta + \cosh(1/2)}\right](0) < 1,$$

provided that

$$\frac{6\beta(\beta - 1)}{\sinh^2(1/2)} > \alpha - 4.$$

The above inequality implies that 1 is not the minimizer for $I[f](0)$, in other words, there exists region U where the value of g is strictly less than 1.
Let ϕ be a smooth function with compact support in U. One can choose ϵ is sufficient small such that $g + \epsilon\phi \in \mathcal{A}$. Now we set

$$i(t) = I[g + t\epsilon\phi](0) = \int_0^1 G(x)\left((g + t\epsilon\phi)^2 + \frac{\alpha}{2}(g_x + t\epsilon\phi_x)^2\right)dx,$$

where $t \in \mathbb{R}$ such that $g + t\epsilon\phi \in \mathcal{A}$. Since g is the minimizer, we have

$$0 = i'(0) = \epsilon\int_0^1 (2Gg\phi + \alpha G g_x \phi_x)dx = \epsilon\int_0^1 (2Gg - \alpha(Gg_x)_x)\phi dx.$$

Therefore the equation for g in the region $g < 1$ reads

$$\alpha(Gg_x)_x = 2Gg, \text{ with } G(x) = \frac{\cosh(x - 1/2)}{2\sinh(1/2)}.$$

Just as what was done in [38], we have the following claim that $g < 1$ at all points except 0 and 1. So the equation for g is

$$\alpha(Gg_x)_x = 2Gg, \text{ in } (0,1), \text{ with } g(0) = g(1) = 1. \tag{3.2}$$

After changing variable we can rewrite (3.2) as

$$\cosh(x)g''(x) + \sinh(x)g'(x) - \frac{2}{\alpha}\cosh(x)g(x) = 0, \tag{3.3}$$

$x \in (-1/2, 1/2)$, where prime means taking derivative with respect to x.
If $\alpha = 1$, equation (3.3) has been solved in [43] as

$$g = \frac{1 + \arctan(\sinh(x - 1/2))\sinh(x - 1/2)}{1 + \arctan(\sinh(1/2))\sinh(1/2)}, \tag{3.4}$$

for $x \in [0, 1]$.
For general case $\alpha \neq 1$, we will solve the equation in the next section.
However, here we can find the optimal constant for the functional I achieved by g satisfying the equation (3.3). Actually, from the equation (3.2), one has

$$Gg^2 + \frac{\alpha}{2} G g_x^2 = \frac{\alpha}{2} \left(G g g_x \right)_x .$$

Therefore,

$$\begin{aligned} I[g](0) &= \int_0^1 \frac{\alpha}{2} \left(G g g_x \right)_x \\ &= \frac{\alpha}{2\tanh(1/2)} \left(g(1/2) g'(1/2 - 0) - g(-1/2) g'(-1/2 + 0) \right) \\ &= \frac{\alpha}{\tanh(1/2)} g'(1/2 - 0), \end{aligned}$$

since $g(x)$ is an even function on $[-1/2, 1/2]$, and $g(-1/2) = g(1/2) = 1$.
Hence, we have the following theorem

THEOREM 3.1 *For all $f \in H^1(\mathbb{S})$, and $\alpha > 0$, the following inequality holds*

$$G * \left(f^2 + \frac{\alpha}{2} f_x^2 \right)(x) \geq C(\alpha) f^2(x), \tag{3.5}$$

with

$$C(\alpha) = \frac{\alpha}{\tanh(1/2)} g'(1/2 - 0), \tag{3.6}$$

where $g(x)$ is an even function on $[-1/2, 1/2]$ satisfying (3.3).

$$C(1) = \frac{1}{2} + \frac{\arctan(\sinh(1/2))}{2\sinh(1/2) + 2\arctan(\sinh(1/2))\sinh^2(1/2)} \approx 0.869,$$

which has been founded in [43].

REMARK 3.1 *From the above variational approach, it implies that $C(\alpha) < 1$ for any $\alpha > 0$.*

4 Solve the ordinary differential equation

For our convenience, we rewrite the equation (3.3) as

$$\cosh(x)u_\lambda''(x) + \sinh(x)u_\lambda'(x) - \lambda(\lambda+1)\cosh(x)u_\lambda(x) = 0, \tag{4.1}$$

$x \in (-1/2, 1/2)$, with $\lambda = \frac{\sqrt{\alpha+8}-\sqrt{\alpha}}{2\sqrt{\alpha}} > 0$.
Now, letting $s = \sinh(x)$ and $v_\lambda(s) = u_\lambda(x)$, then (4.1) changes to

$$(1+s^2)v_\lambda''(s) + 2sv_\lambda'(s) - \lambda(\lambda+1)v_\lambda(s) = 0. \tag{4.2}$$

In general, the solution to (4.2) can be represented as the following power series:

$$v_\lambda(s) = 1 + \sum_{n=1}^{\infty} \frac{\prod_{k=0}^{n-1}(\lambda-2k)(\lambda+1+2k)}{(2n)!} s^{2n}, \tag{4.3}$$

with convergence radius of 1. Hence it is convergent at $s = \sinh(1/2)$.
It is easy to find that, (4.3) is a polynomial with finite terms for λ being a positive even number, i.e., $\lambda = 2m$, $k \in \mathbb{N}$. For $\lambda = 2m+1$, the solution to (4.2) can be obtained by

$$v_\lambda(s) = -v_1(s) \int_0^s \frac{d\tau}{v_1^2(\tau)(1+\tau^2)}, \tag{4.4}$$

where

$$v_1(s) = s\left(1 + \sum_{n=1}^{\infty} \frac{\prod_{k=1}^{n}(\lambda+2k)(\lambda+1-2k)}{(2n+1)!} s^{2n}\right)$$

is another solution to (4.2), which is independent of (4.3).
Due to the strategic steps established here, we can write down the solutions to (4.2) for $\lambda \in \mathbb{N}$. For example, we have

$$u_1(x) = -s\int_0^s \frac{d\tau}{\tau^2(1+\tau^2)} = 1 + s\arctan s = 1 + \sinh(x)\arctan(\sinh(x)),$$

(3.4) is recovered again. We also can write down the following solutions.

$$u_2(x) = 1 + 3s^2 = 1 + 3\sinh^2(x),$$

$$\begin{aligned} u_3(x) &= -s\left(1+\frac{5}{3}s^2\right)\int_0^s \frac{d\tau}{\tau^2\left(1+\frac{5}{3}\tau^2\right)^2(1+\tau^2)} \\ &= 1 + \frac{15}{4}s^2 + s\left(\frac{9}{4}+\frac{15}{4}s^2\right)\arctan s \\ &= 1 + \frac{15}{4}\sinh(x)^2 + s\left(\frac{9}{4}+\frac{15}{4}\sinh(x)^2\right)\arctan\left(\sinh(x)\right), \end{aligned}$$

$$u_4(x) = 1 + 10s^2 + \frac{35}{3}s^4 = 1 + 10\sinh^2(x) + \frac{35}{3}\sinh^4(x),$$

$$\begin{aligned} u_5(x) &= 1 + \frac{105}{64}s^2(7+9s^2) + \frac{15}{64}s(15+70s^2+63s^4)\arctan s \\ &= 1 + \frac{105}{64}\sinh^2(x)\left(7+9\sinh^2(x)\right) \\ &\quad + \frac{15}{64}\sinh(x)\left(15+70\sinh^2(x)+63\sinh^4(x)\right)\arctan\left(\sinh(x)\right), \end{aligned}$$

$$\begin{aligned} u_6(x) &= 1 + 21s^2 + 63s^4 + \frac{231}{5}s^6 \\ &= 1 + 21\sinh^2(x) + 63\sinh^4(x) + \frac{231}{5}\sinh^6(x). \end{aligned}$$

For general $\lambda > 0$, we only have the form of (4.2) at present. We will do some computation in section 6.

5 Blow-up criteria

After local well-posedness of strong solutions (see Theorem 2.1) is established, the next question is whether this local solution can exist globally. As far as we know, the only available global existence result is for the case $\gamma = 1$: see the paper by Constantin [5] for a PDE approach, and the paper by Constantin and McKean [11] for an approach based on the integrable structure of the equation. If the solution exists only for finite time, how about the behavior of the solution when it blows up? What induces the blow-up? On the other hand, to find sufficient conditions to guarantee the finite time blow-up or global existence is of great interest, especially for sufficient conditions added on the initial data.
The main theorem of this section is as following:

THEOREM 5.1 *Let $0 < \gamma < 3$. Assume that $u_0 \in H^2(\mathbb{S})$ satisfies $m_0 < 0$ and*

$$m_0^2 > \frac{3-\gamma}{2\gamma}\left(1 - C\left(\frac{2\gamma}{3-\gamma}\right)\right)\frac{\cosh(1/2)}{\sinh(1/2)}\|u_0\|^2_{H^1(\mathbb{S})}, \tag{5.1}$$

where $C\left(\frac{2\gamma}{3-\gamma}\right)$ is defined by (3.6). Then the life span $T > 0$ of the corresponding solution to (1.1) is finite.

REMARK 5.1 *When $\gamma = 1$, we recover the theorem established in [43]. The cases for $\gamma < 0$ and $\gamma > 3$ were discussed in [41, 42].*

First, we have the following blow-up result for a Riccati type ordinary differential equation.

Lemma 5.2 [43] *Assume that a differentiable function $y(t)$ satisfies*

$$y'(t) \leq -Cy^2(t) + K, \tag{5.2}$$

with constants $C, K > 0$. If the initial datum $y(0) = y_0 < -\sqrt{\frac{K}{C}}$, then the solution to (5.2) goes to $-\infty$ in finite time.

Secondly, let us recall the best constant for a Sobolev inequality proved in [38].

$$\|f\|^2_{L^\infty(\mathbb{S})} \leq \frac{\cosh(1/2)}{2\sinh(1/2)} \|f\|^2_{H^1(\mathbb{S})}, \tag{5.3}$$

for $f \in H^1(\mathbb{S})$. Moreover, it is an optimal constant for the Sobolev imbedding $H^1 \subset L^\infty$ in the sense that (5.3) holds if and only if $f(x) = \lambda G(x-y)$ for some $\lambda, y \in \mathbb{R}$.
We start the proof for the main theorem from (2.6).

$$\begin{aligned}
\frac{dm}{dt} &= -\frac{\gamma}{2}m^2 + \frac{3-\gamma}{2}u^2(\xi(t),t) - \left[G * \left(\frac{3-\gamma}{2}u^2 + \frac{\gamma}{2}u_x^2\right)\right](\xi(t),t) \\
&= -\frac{\gamma}{2}m^2 + \frac{3-\gamma}{2}u^2(\xi(t),t) - \frac{3-\gamma}{2}\left[G * \left(u^2 + \frac{1}{2}\frac{2\gamma}{3-\gamma}u_x^2\right)\right](\xi(t),t) \\
&\leq -\frac{\gamma}{2}m^2 + \frac{3-\gamma}{2}u^2(\xi(t),t) - \frac{3-\gamma}{2}C\left(\frac{2\gamma}{3-\gamma}\right)u^2(\xi(t),t) \\
&\leq -\frac{\gamma}{2}m^2 + \frac{3-\gamma}{2}\left(1 - C\left(\frac{2\gamma}{3-\gamma}\right)\right)u^2(\xi(t),t) \\
&\leq -\frac{\gamma}{2}m^2 + \frac{3-\gamma}{2}\left(1 - C\left(\frac{2\gamma}{3-\gamma}\right)\right)\frac{\cosh(1/2)}{2\sinh(1/2)}\|u_0\|^2_{H^1(\mathbb{S})},
\end{aligned}$$

where we used (3.5) and (5.3).
So, the proof can be completed by using the condition in Theorem 5.1 and Lemma 5.2.

6 Another presentation and numerical simulation

We have the equation (4.2), and the local solution to (4.2) can be represented as the power series (4.4), which is convergent at $s = \sinh(1/2) = 0.521\cdots$.
By the transformation of variables

$$z := -s^2, y_\lambda(s) := v_\lambda(s)$$

and let

$$a := -\lambda/2, b := (\lambda+1)/2, c := 1/2,$$

then

$$v_\lambda(s) = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \prod_{k=0}^{n-1} \frac{(a+k)(b+k)}{c+k} =: F(a,b,c;z), \tag{6.1}$$

where $F(a, b, c; z)$ is called the hypergeometric function, a regular solution of the hypergeometric differential equation

$$z(1-z)y''_\lambda(z) + [c-(a+b+1)z]y'_\lambda(z) - aby_\lambda(z) = 0.$$

Since

$$F'(a, b, c; z) = \frac{ab}{c}F(a+1, b+1, c+1; z),$$

we obtain the analytic expression

$$C(\alpha) = \frac{\cosh^2(1/2)F(a+1, b+1, c+1; -\sinh^2(1/2))}{\sinh(1/2)F(a, b, c; -\sinh^2(1/2))},$$

where $\alpha = 2/(\lambda(\lambda+1))$.

Although the value of $C(\alpha)$ for each λ can be obtained by calling the hypergeometric functions in softwares such as Mathematica, Maple or MATLAB, generally the calculation based on (6.1), thus the calculations are not efficient. In the following, we give an efficient method for calculating $C(\alpha)$.

Define

$$q_\lambda(s) := \frac{v'_\lambda(s)}{\lambda(\lambda+1)v_\lambda(s)}, \quad \lambda > 0,$$

then $q_\lambda(s)$ is the solution of the following initial value problem of the first order ordinary differential equation

$$q'_\lambda(s) + \frac{2s}{1+s^2}q_\lambda(s) + \mu q_\lambda^2(s) = \frac{1}{1+s^2}, \quad q_\lambda(0) = 0, \tag{6.2}$$

where $\mu := \lambda(\lambda+1)$.

Thus

$$C(\alpha) = \frac{\cosh^2(1/2)q_\lambda(\sinh(1/2))}{\sinh(1/2)}. \tag{6.3}$$

By using MATLAB programme, we can plot the graph of $C(\alpha)$ as Fig. 1. Here α is taken from 0.01 to 10 with equal step length 0.01. The detailed MATLAB code is given in the appendix.

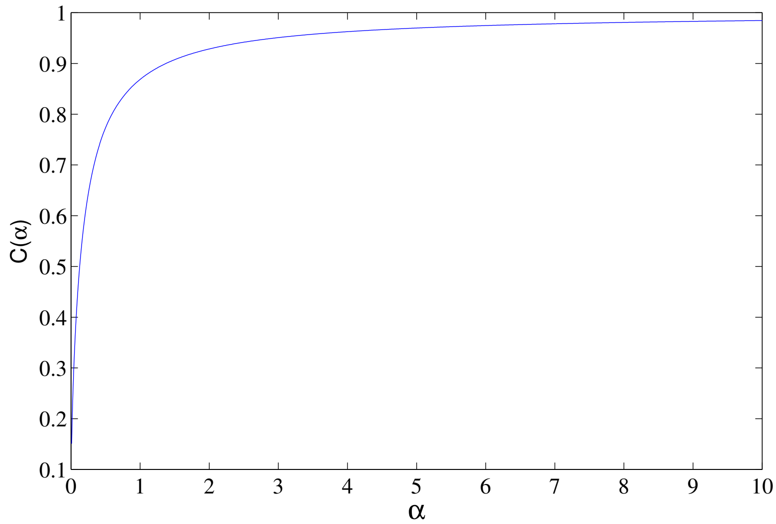

Fig. 1

From Fig. 1 we see that $C(\alpha)$ is a strictly increasing function of α which can be proved analytically as follows.
Differentiate both sides of (6.2) with respect to μ, we have a linear differential equation of $q_{\lambda\mu}(s) = \partial q_\lambda(s)/\partial\mu$:

$$q'_{\lambda\mu}(s) + 2\left(\mu q_\lambda(s) + \frac{s}{1+s^2}\right) q_{\lambda\mu}(s) + q_\lambda^2(s) = 0, \quad q_\lambda(0) = 0. \tag{6.4}$$

The solution to (6.4) is

$$q_{\lambda\mu}(s) = -\frac{1}{1+s^2}\int_0^s (1+\tau^2)\exp\left(-2\mu\int_\tau^s q_\lambda(t)\,\mathrm{d}t\right) q_\lambda^2(\tau)\,\mathrm{d}\tau < 0. \tag{6.5}$$

While

$$q_{\lambda\alpha}(s) = \frac{\partial q_\lambda(s)}{\partial\mu}\frac{\partial\mu}{\partial\alpha} = q_{\lambda\mu}\cdot\frac{-2}{\alpha^2} > 0.$$

This completes the proof due to (6.3).

7 Appendix

Fig. 1 is plotted by two MATLAB routines for calculating $C(\alpha)$. We use MATLAB because of its advantage of efficient vector operations. The argument alpha can be a vector.

1. The main function routine, MATLAB m-file named bestc.m:

   function C=bestc(alpha)

   % Input: alpha, may be a scalar or a vector;

   % Output: C, the best coefficient corresponding to alpha, When alpha is a vector, C is also a vector with the same size as alpha.

   T=.52109530549374738495; % T = sinh(1/2)

   G=2.4401300568286909964; % G = T+Tˆ(-1)

   options=odeset('RelTol',2.221e-14,'ABsTol',1e-15); % Set ODE solver's relative error tolerance and absolute error tolerance.

   [S,Y]=ode45(@rod,[0,T],0*alpha,options,alpha); % Call ODE solver ode45.

   C=Y(end,:)*G;

   % End of the main routine.

2. The ODE-file named rod.m is as follows:

   function dqds = rod(s,q,alpha)

   dqds = (1-2*s*q)/(1+sˆ2)-2*q.ˆ2./alpha;

   % End of the ODE-file.

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