Singular Bott-Chern Classes and the Arithmetic Grothendieck Riemann Roch Theorem for Closed Immersions

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Abstract. We study the singular Bott-Chern classes introduced by Bismut, Gillet and Soulé. Singular Bott-Chern classes are the main ingredient to define direct images for closed immersions in arithmetic K-theory. In this paper we give an axiomatic definition of a theory of singular Bott-Chern classes, study their properties, and classify all possible theories of this kind. We identify the theory defined by Bismut, Gillet and Soulé as the only one that satisfies the additional condition of being homogeneous. We include a proof of the arithmetic Grothendieck-Riemann-Roch theorem for closed immersions that generalizes a result of Bismut, Gillet and Soulé and was already proved by Zha. This result can be combined with the arithmetic Grothendieck-Riemann-Roch theorem for submersions to extend this theorem to arbitrary projective morphisms. As a byproduct of this study we obtain two results of independent interest. First, we prove a Poincaré lemma for the complex of currents with fixed wave front set, and second we prove that certain direct images of Bott-Chern classes are closed.

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0 Introduction

Chern-Weil theory associates to each hermitian vector bundle a family of closed characteristic forms that represent the characteristic classes of the vector bundle. The characteristic classes are compatible with exact sequences. But this is not true for the characteristic forms. The Bott-Chern classes measure the lack of compatibility of the characteristic forms with exact sequences.

The Grothendieck-Riemann-Roch theorem gives a formula that relates direct images and characteristic classes. In general this formula is not valid for the characteristic forms. The singular Bott-Chern classes measure, in a functorial way, the failure of an exact Grothendieck-Riemann-Roch theorem for closed immersions at the level of characteristic forms. In the same spirit, the analytic torsion forms measure the failure of an exact Grothendieck-Riemann-Roch theorem for submersions at the level of characteristic forms. Hence singular Bott-Chern classes and analytic torsion forms are analogous objects, the first for closed immersions and the second for submersions.

Let us give a more precise description of Bott-Chern classes and singular Bott-Chern classes. Let X be a complex manifold and let φ be a symmetric power series in r variables with real coefficients. Let $\overline{E}=(E,h)$ be a rank r holomorphic vector bundle provided with a hermitian metric. Using Chern-Weil theory, we can associate to $\overline{E}$ a differential form $\varphi(\overline{E})=\varphi(-K)$, where K is

the curvature tensor of E viewed as a matrix of 2-forms. The differential form $\varphi(\overline{E})$ is closed and is a sum of components of bidegree (p,p) for $p \geq 0$.
If

$$\overline{\xi}\colon 0 \longrightarrow \overline{E}' \longrightarrow \overline{E} \longrightarrow \overline{E}'' \longrightarrow 0$$

is a short exact sequence of holomorphic vector bundles provided with hermitian metrics, then the differential forms $\varphi(\overline{E})$ and $\varphi(\overline{E}' \oplus \overline{E}')$ may be different, but they represent the same cohomology class.
The Bott-Chern form associated to $\overline{\xi}$ is a solution of the differential equation

$$-2\partial\bar{\partial}\varphi(\overline{\xi}) = \varphi(\overline{E}' \oplus \overline{E}') - \varphi(\overline{E}) \tag{0.1}$$

obtained in a functorial way. The class of a Bott-Chern form modulo the image of ∂ and $\overline{\partial}$ is called a Bott-Chern class and is denoted by $\widetilde{\varphi}(\overline{\xi})$.
There are three ways of defining the Bott-Chern classes. The first one is the original definition of Bott and Chern [7]. It is based on a deformation between the connection associated to $\overline{E}$ and the connection associated to $\overline{E}' \oplus \overline{E}''$. This deformation is parameterized by a real variable.
In [17] Gillet and Soulé introduced a second definition of Bott-Chern classes that is based on a deformation between $\overline{E}$ and $\overline{E}' \oplus \overline{E}''$ parameterized by a projective line. This second definition is used in [4] to prove that the Bott-Chern classes are characterized by three properties

(i) The differential equation (0.1).

(ii) Functoriality (i.e. compatibility with pull-backs via holomorphic maps).

(iii) The vanishing of the Bott-Chern class of a orthogonally split exact sequence.

In [4] Bismut, Gillet and Soulé have a third definition of Bott-Chern classes based on the theory of superconnections. This definition is useful to link Bott-Chern classes with analytic torsion forms.
The definition of Bott-Chern classes can be generalized to any bounded exact sequence of hermitian vector bundles (see section 2 for details). Let

$$\overline{\xi}\colon 0 \longrightarrow (E_n, h_n) \longrightarrow \dots \longrightarrow (E_1, h_1) \longrightarrow (E_0, h_0) \longrightarrow 0$$

be a bounded acyclic complex of hermitian vector bundles; by this we mean a bounded acyclic complex of vector bundles, where each vector bundle is equipped with an arbitrarily chosen hermitian metric. Let

$$r = \sum_{i \text{ even}} \operatorname{rk}(E_i) = \sum_{i \text{ odd}} \operatorname{rk}(E_i).$$

As before, let φ be a symmetric power series in r variables. A Bott-Chern class associated to $\overline{\xi}$ satisfies the differential equation

$$-2\partial\bar{\partial}\widetilde{\varphi}(\overline{\xi}) = \varphi(\bigoplus_k \overline{E}_{2k}) - \varphi(\bigoplus_k \overline{E}_{2k+1}).$$

In particular, let "ch" denote the power series associated to the Chern character class. The Chern character class has the advantage of being additive for direct sums. Then, the Bott-Chern class associated to the long exact sequence $\overline{\xi}$ and to the Chern character class satisfies the differential equation

$$-2\partial\bar{\partial}\widetilde{\mathrm{ch}}(\overline{\xi}) = -\sum_{k=0}^{n}(-1)^i \,\mathrm{ch}(\overline{E}_k).$$

Let now $i\colon Y \longrightarrow X$ be a closed immersion of complex manifolds. Let $\overline{F}$ be a holomorphic vector bundle on Y provided with a hermitian metric. Let $\overline{N}$ be the normal bundle to Y in X provided also with a hermitian metric. Let

$$0 \longrightarrow \overline{E}_n \longrightarrow \overline{E}_{n-1} \longrightarrow \dots \longrightarrow \overline{E}_0 \longrightarrow i_*F \longrightarrow 0$$

be a resolution of the coherent sheaf i_*F by locally free sheaves, provided with hermitian metrics (following Zha [32] we shall call such a sequence a metric on the coherent sheaf i_*F). Let Td denote the Todd characteristic class. Then the Grothendieck-Riemann-Roch theorem for the closed immersion i implies that the current $i_*(\mathrm{Td}(\overline{N})^{-1}\,\mathrm{ch}(\overline{F}))$ and the differential form $\sum_k(-1)^k\,\mathrm{ch}(\overline{E}_k)$ represent the same class in cohomology. We denote $\overline{\xi}$ the data consisting in the closed embedding i, the hermitian bundle $\overline{N}$, the hermitian bundle $\overline{F}$ and the resolution $\overline{E}_* \longrightarrow i_*F$.

In the paper [5], Bismut, Gillet and Soulé introduced a current associated to the above situation. These currents are called singular Bott-Chern currents and denoted in [5] by $T(\overline{\xi})$. When the hermitian metrics satisfy a certain technical condition (condition A of Bismut) then the singular Bott-Chern current $T(\overline{\xi})$ satisfies the differential equation

$$-2\partial\bar{\partial}T(\overline{\xi}) = i_*(\mathrm{Td}(\overline{N})^{-1}\,\mathrm{ch}(\overline{F})) - \sum_{i=0}^{n}(-1)^i\,\mathrm{ch}(\overline{E}_i).$$

These singular Bott-Chern currents are among the main ingredients of the proof of Gillet and Soulé's arithmetic Riemann-Roch theorem. In fact it is the main ingredient of the arithmetic Riemann-Roch theorem for closed immersions [6]. This definition of singular Bott-Chern classes is based on the formalism of superconnections, like the third definition of ordinary Bott-Chern classes.

In his thesis [32], Zha gave another definition of singular Bott-Chern currents and used it to give a proof of a different version of the arithmetic Riemann-Roch theorem. This second definition is analogous to Bott and Chern's original definition. Nevertheless there is no explicit comparison between the two definitions of singular Bott-Chern currents.

One of the purposes of this note is to give a third construction of singular Bott-Chern currents, in fact of their classes modulo the image of ∂ and $\overline{\partial}$, which could be seen as analogous to the second definition of Bott-Chern classes. Moreover we will use this third construction to give an axiomatic definition of a theory of singular Bott-Chern classes. A theory of singular Bott-Chern classes is an assignment that, to each data $\overline{\xi}$ as above, associates a class of currents $T(\overline{\xi})$, that satisfies the analogue of conditions (i), (ii) and (iii). The main technical point of this axiomatic definition is that the conditions analogous to (i), (ii) and (iii) above are not enough to characterize the singular Bott-Chern classes. Thus we are led to the problem of classifying the possible theories of Bott-Chern classes, which is the other purpose of this paper.

We fix a theory T of singular Bott-Chern classes. Let Y be a complex manifold and let $\overline{N}$ and $\overline{F}$ be two hermitian holomorphic vector bundles on Y. We write $P = \mathbb{P}(N \oplus 1)$ for the projective completion of N. Let $s\colon Y \longrightarrow P$ be the inclusion as the zero section and let $\pi_P\colon P \longrightarrow Y$ be the projection. Let $\overline{K}_*$ be the Koszul resolution of $s_*\mathcal{O}_Y$ endowed with the metric induced by $\overline{N}$. Then we have a resolution by hermitian vector bundles

$$K(\overline{F},\overline{N})\colon \overline{K}_* \otimes \pi_P^*\overline{F} \longrightarrow s_*F.$$

To these data we associate a singular Bott-Chern class $T(K(\overline{F},\overline{N}))$. It turns out that the current

$$\frac{1}{(2\pi i)^{\operatorname{rk} N}}\int_{\pi_P} T(K(\overline{F},\overline{N})) = (\pi_P)_*T(K(\overline{F},\overline{N}))$$

is closed (see section 3 for general properties of the Bott-Chern classes that imply this property) and determines a characteristic class $C_T(F,N)$ on Y for the vector bundles N and F. Conversely, any arbitrary characteristic class for pairs of vector bundles can be obtained in this way. This allows us to classify the possible theories of singular Bott-Chern classes:

Claim (theorem 7.1). The assignment that sends a singular Bott-Chern class T to the characteristic class C_T is a bijection between the set of theories of singular Bott-Chern classes and the set of characteristic classes.

The next objective of this note is to study the properties of the different theories of singular Bott-Chern classes and of the corresponding characteristic classes. We mention, in the first place, that for the functoriality condition to make sense, we have to study the wave front sets of the currents representing the singular Bott-Chern classes. In particular we use a Poincaré Lemma for currents with fixed wave front set. This result implies that, in each singular Bott-Chern class, we can find a representative with controlled wave front set that can be pulled back with respect certain morphisms.

We also investigate how different properties of the singular Bott-Chern classes T are reflected in properties of the characteristic classes C_T. We thus characterize the compatibility of the singular Bott-Chern classes with the projection formula, by the property of C_T of being compatible with the projection formula. We also relate the compatibility of the singular Bott-Chern classes with the composition of successive closed immersions to an additivity property of the associated characteristic class.

Furthermore, we show that we can add a natural fourth axiom to the conditions analogue to (i), (ii) and (iii), namely the condition of being homogeneous (see section 9 for the precise definition).

Claim (theorem 9.11). There exists a unique homogeneous theory of singular Bott-Chern classes.

Thanks to this axiomatic characterization, we prove that this theory agrees with the theories of singular Bott-Chern classes introduced by Bismut, Gillet and Soulé [6], and by Zha [32]. In particular this provides us a comparison between the two definitions. We will also characterize the characteristic class C_{T^h} for the theory of homogeneous singular Bott-Chern classes.
The last objective of this paper is to give a proof of the arithmetic Riemann-Roch theorem for closed immersions. A version of this theorem was proved by Bismut, Gillet and Soulé and by Zha.
Next we will discuss the contents of the different sections of this paper. In section §1 we recall the properties of characteristic classes in analytic Deligne cohomology. A characteristic class is just a functorial assignment that associates a cohomology class to each vector bundle. The main result of this section is that any characteristic class is given by a power series on the Chern classes, with appropriate coefficients.
In section §2 we recall the theory of Bott-Chern forms and its main properties. The contents of this section are standard although the presentation is slightly different to the ones published in the literature.
In section §3 we study certain direct images of Bott-Chern forms. The main result of this section is that, even if the Bott-Chern classes are not closed, certain direct images of Bott-Chern classes are closed. This result generalizes previous results of Bismut, Gillet and Soulé and of Mourougane. This result is used to prove that the class C_T mentioned previously is indeed a cohomology class, but it can be of independent interest because it implies that several identities in characteristic classes are valid at the level of differential forms.
In section §4 we study the cohomology of the complex of currents with a fixed wave front set. The main result of this section is a Poincaré lemma for currents of this kind. This implies in particular a $\partial\bar{\partial}$-lemma. The results of this section are necessary to state the functorial properties of singular Bott-Chern classes.
In section §5 we recall the deformation of resolutions, that is a generalization of the deformation to the normal cone, and we also recall the construction of the Koszul resolution. These are the main geometric tools used to study singular Bott-Chern classes.
Sections §6 to §9 are devoted to the definition and study of the theories of singular Bott-Chern classes. Section §6 contains the definition and first properties. Section §7 is devoted to the classification theorem of such theories. In section §8 we study how properties of the theory of singular Bott-Chern classes and of the associated characteristic class are related. And in section §9 we define the theory of homogeneous singular Bott-Chern classes and we prove that it agrees with the theories defined by Bismut, Gillet and Soulé and by Zha.

Finally in section §10 we define arithmetic K-groups associated to a $\mathcal{D}_{\log}$-arithmetic variety $(\mathcal{X}, \mathcal{C})$ (in the sense of [13]) and push-forward maps for closed immersions of metrized arithmetic varieties, at the level of the arithmetic K-groups. After studying the compatibility of these maps with the projection formula and with the push-forward map at the level of currents, we prove a general Riemann-Roch theorem for closed immersions (theorem 10.28) that compares the direct images in the arithmetic K-groups with the direct images in the arithmetic Chow groups. This theorem is compatible, if we choose the theory of homogeneous singular Bott-Chern classes, with the arithmetic Riemann-Roch theorem for closed immersions proved by Bismut, Gillet and Soulé [6] and it agrees with the theorem proved by Zha [32]. Theorem 10.28, together with the arithmetic Grothendieck-Riemann-Roch theorem for submersions proved in [16], can be used to obtain an arithmetic Grothendieck-Riemann-Roch theorem for projective morphisms of regular arithmetic varieties.

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1 Characteristic classes in analytic Deligne cohomology

A characteristic class for complex vector bundles is a functorial assignment which, to each complex continuous vector bundle on a paracompact topological space X, assigns a class in a suitable cohomology theory of X. For example, if the cohomology theory is singular cohomology, it is well known that each characteristic class can be expressed as a power series in the Chern classes. This can be seen for instance, showing that continuous complex vector bundles on a paracompact space X can be classified by homotopy classes of maps from X to the classifying space $BGL_{\infty}(\mathbb{C})$ and that the cohomology of $BGL_{\infty}(\mathbb{C})$ is generated by the Chern classes (see for instance [28]).

The aim of this section is to show that a similar result is true if we restrict the class of spaces to the class of quasi-projective smooth complex manifolds, the class of maps to the class of algebraic maps and the class of vector bundles to the class of algebraic vector bundles and we choose analytic Deligne cohomology as our cohomology theory.

This result and the techniques used to prove it are standard. We will use the splitting principle to reduce to the case of line bundles and will then use the projective spaces as a model of the classifying space $BGL_1(\mathbb{C})$. In this section we also recall the definition of Chern classes in analytic Deligne cohomology and we fix some notations that will be used through the paper.

Definition 1.1. Let X be a complex manifold. For each integer p, *the analytic real Deligne complex* of X is

$$\begin{aligned}\mathbb{R}_{X,\mathcal{D}}(p) = (\underline{\mathbb{R}}(p) \longrightarrow \mathcal{O}_X \longrightarrow \Omega^1_X \longrightarrow \dots \longrightarrow \Omega^{p-1}_X)&\\ \cong s(\underline{\mathbb{R}}(p) \oplus F^p\Omega^*_X \longrightarrow \Omega^*_X),&\end{aligned}$$

where $\underline{\mathbb{R}}(p)$ is the constant sheaf $(2\pi i)^p\mathbb{R} \subseteq \underline{\mathbb{C}}$. The *analytic real Deligne cohomology of* X, denoted $H^*_{\mathcal{D}^{\mathrm{an}}}(X, \mathbb{R}(p))$, is the hyper-cohomology of the above complex.

Analytic Deligne cohomology satisfies the following result.

Theorem 1.2. *The assignment* $X \longmapsto H^*_{\mathcal{D}^{\mathrm{an}}}(X, \mathbb{R}(*)) = \bigoplus_p H^*_{\mathcal{D}^{\mathrm{an}}}(X, \mathbb{R}(p))$ *is a contravariant functor between the category of complex manifolds and holomorphic maps and the category of unitary bigraded rings that are graded commutative (with respect to the first degree) and associative. Moreover there exists a functorial map*

$$c\colon \operatorname{Pic}(X) = H^1(X, \mathcal{O}^*_X) \longrightarrow H^2_{\mathcal{D}^{\mathrm{an}}}(X, \mathbb{R}(1))$$

and, for each closed immersion of complex manifolds $i\colon Y \longrightarrow X$ *of codimension* p*, there exists a morphism*

$$i_*\colon H^*_{\mathcal{D}^{\mathrm{an}}}(Y, \mathbb{R}(*)) \longrightarrow H^{*+2p}_{\mathcal{D}^{\mathrm{an}}}(X, \mathbb{R}(*+p))$$

satisfying the properties

A1 Let X *be a complex manifold and let* E *be a holomorphic vector bundle of rank* r*. Let* $\mathbb{P}(E)$ *be the associated projective bundle and let* $\mathcal{O}(-1)$ *the tautological line bundle. The map*

$$\pi^*\colon H^*_{\mathcal{D}^{\mathrm{an}}}(X, \mathbb{R}(*)) \longrightarrow H^*_{\mathcal{D}^{\mathrm{an}}}(\mathbb{P}(E), \mathbb{R}(*))$$

induced by the projection $\pi\colon \mathbb{P}(E) \longrightarrow X$ *gives to the second ring a structure of left module over the first. Then the elements* $c(\mathrm{cl}(\mathcal{O}(-1)))^i$*,* $i = 0, \dots, r-1$ *form a basis of this module.*

A2 If X *is a complex manifold,* L *a line bundle,* s *a holomorphic section of* L *that is transverse to the zero section,* Y *is the zero locus of* s *and* $i\colon Y \longrightarrow X$ *the inclusion, then*

$$c(\mathrm{cl}(L)) = i_*(1_Y).$$

A3 If $j\colon Z \longrightarrow Y$ *and* $i\colon Y \longrightarrow X$ *are closed immersions of complex manifolds then* $(ij)_* = i_*j_*$*.*

A4 If $i\colon Y \longrightarrow X$ is a closed immersion of complex manifolds then, for every $a \in H^*_{\mathcal{D}^{\mathrm{an}}}(X,\mathbb{R}(*))$ and $b \in H^*_{\mathcal{D}^{\mathrm{an}}}(Y,\mathbb{R}(*))$

$$i_*(bi^*a) = (i_*b)a.$$

Proof. The functoriality is clear. The product structure is described, for instance, in [15]. The morphism c is defined by the morphism in the derived category

$$\mathcal{O}_X^*[1] \xleftarrow{\cong} s(\underline{\mathbb{Z}}(1) \to \mathcal{O}_X) \longrightarrow s(\underline{\mathbb{R}}(1) \to \mathcal{O}_X) = \mathbb{R}_{\mathcal{D}}(1).$$

The morphism i_* can be constructed by resolving the sheaves $\mathbb{R}_{\mathcal{D}}(p)$ by means of currents (see [26] for a related construction). Properties A3 and A4 follow easily from this construction.

By abuse of notation, we will denote by $c_1(\mathcal{O}(-1))$ the first Chern class of $\mathcal{O}(-1)$ with the algebro-geometric twist, in any of the groups $H^2(\mathbb{P}(E),\underline{\mathbb{R}}(1))$, $H^2(\mathbb{P}(E),\underline{\mathbb{C}})$, $H^1(\mathbb{P}(E),\Omega^1_{\mathbb{P}(E)})$. Then, we have sheaf isomorphisms (see for instance [22] for a related result),

$$\bigoplus_{i=0}^{r-1} \underline{\mathbb{R}}_X(p-i)[-2i] \longrightarrow R\pi_*\underline{\mathbb{R}}_{\mathbb{P}(E)}(p)$$

$$\bigoplus_{i=0}^{r-1} \Omega^*_X[-2i] \longrightarrow R\pi_*\Omega^*_{\mathbb{P}(E)}$$

$$\bigoplus_{i=0}^{r-1} F^{p-i}\Omega^*_X[-2i] \longrightarrow R\pi_*F^p\Omega^*_{\mathbb{P}(E)}$$

given, all of them, by $(a_0,\dots,a_{r-1}) \longmapsto \sum a_i c_1(\mathcal{O}(-1))^i$. Hence we obtain a sheaf isomorphism

$$\bigoplus_{i=0}^{r-1} \mathbb{R}_{X,\mathcal{D}}(p-i)[-2i] \longrightarrow R\pi_*\mathbb{R}_{\mathbb{P}(E),\mathcal{D}}(p)$$

from which property A1 follows. Finally property A2 in this context is given by the Poincare-Lelong formula (see [13] proposition 5.64). $\square$

Notation 1.3. For the convenience of the reader, we gather here together several notations and conventions regarding the differential forms, currents and Deligne cohomology that will be used through the paper.

Throughout this paper we will use consistently the algebro-geometric twist. In particular the Chern classes c_i, $i = 0,\dots$ in Betti cohomology will live in $c_i \in H^{2i}(X,\mathbb{R}(i))$; hence our normalizations differ from the ones in [18] where real forms and currents are used.

Moreover we will use the following notations. We will denote by $\mathscr{E}^*_X$ the sheaf of Dolbeault algebras of differential forms on X and by $\mathscr{D}^*_X$ the sheaf of Dolbeault complexes of currents on X (see [13] §5.4 for the structure of Dolbeault complex of $\mathscr{D}^*_X$). We will denote by $E^*(X)$ and by $D^*(X)$ the complexes of global sections of $\mathscr{E}^*_X$ and $\mathscr{D}^*_X$ respectively. Following [9] and [13] definition 5.10, we denote by $(\mathcal{D}^*(\_,*), \mathrm{d}_{\mathcal{D}})$ the functor that associates to a Dolbeault complex its corresponding Deligne complex. For shorthand, we will denote

$$\begin{aligned}\mathcal{D}^*(X,p) &= \mathcal{D}^*(E^*(X),p),\\ \mathcal{D}^*_D(X,p) &= \mathcal{D}^*(D^*(X),p).\end{aligned}$$

To keep track of the algebro-geometric twist we will use the conventions of [13] §5.4 regarding the current associated to a locally integrable differential form

$$[\omega](\eta) = \frac{1}{(2\pi i)^{\dim X}} \int_X \eta \wedge \omega$$

and the current associated with a subvariety Y

$$\delta_Y(\eta) = \frac{1}{(2\pi i)^{\dim Y}} \int_Y \eta.$$

With these conventions, we have a bigraded morphism $\mathcal{D}^*(X,*) \to \mathcal{D}^*_D(X,*)$ and, if Y has codimension p, the current δ_Y belongs to $\mathcal{D}^{2p}_D(X,p)$. Then $\mathcal{D}^*(X,p)$ and $\mathcal{D}^*_D(X,p)$ are the complex of global sections of an acyclic resolution of $\mathbb{R}_{X,\mathcal{D}}(p)$. Therefore

$$H^*_{\mathcal{D}^{\mathrm{an}}}(X,\mathbb{R}(p)) = H^*(\mathcal{D}(X,p)) = H^*(\mathcal{D}_D(X,p)).$$

If $f : X \to Y$ is a proper smooth morphism of complex manifolds of relative dimension e, then the integral along the fibre morphism

$$f_* : \mathcal{D}^k(X,p) \longrightarrow \mathcal{D}^{k-2e}(X,p-e)$$

is given by

$$f_*\omega = \frac{1}{(2\pi i)^e} \int_f \omega. \tag{1.4}$$

If $(\mathcal{D}^*(*), \mathrm{d}_{\mathcal{D}})$ is a Deligne complex associated to a Dolbeault complex, we will write

$$\widetilde{\mathcal{D}}^k(X,p) := \mathcal{D}^k(X,p)/\,\mathrm{d}_{\mathcal{D}}\,\mathcal{D}^{k-1}(X,p).$$

Finally, following [13] 5.14 we denote by $\bullet$ the product in the Deligne complex that induces the usual product in Deligne cohomology. Note that, if $\omega \in \bigoplus_p \mathcal{D}^{2p}(X,p)$, then for any $\eta \in \mathcal{D}^*(X,*)$ we have $\omega \bullet \eta = \eta \bullet \omega = \eta \wedge \omega$. Sometimes, in this case we will just write $\eta\omega := \eta \bullet \omega$.

We denote by $*$ the complex manifold consisting on one single point. Then

$$H^n_{\mathcal{D}^{\mathrm{an}}}(*,p) = \begin{cases} \mathbb{R}(p) := (2\pi i)^p\mathbb{R}, & \text{if } n=0,\ p\le 0,\\ \mathbb{R}(p-1) := (2\pi i)^{p-1}\mathbb{R}, & \text{if } n=1,\ p>0.\\ \{0\}, & \text{otherwise.}\end{cases}$$

The product structure in this case is the bigraded product that is given by complex number multiplication when the degrees allow the product to be non zero. We will denote by $\mathbb{D}$ this ring. This is the base ring for analytic Deligne cohomology. Note that, in particular, $H^1_{\mathcal{D}^{\mathrm{an}}}(*,1) = \mathbb{R} = \mathbb{C}/\mathbb{R}(1)$. We will denote by $\mathbf{1}_1$ the image of 1 in $H^1_{\mathcal{D}^{\mathrm{an}}}(*,1)$.
Following [23], theorem 1.2 implies the existence of a theory of Chern classes for holomorphic vector bundles in analytic Deligne cohomology. That is, to every vector bundle E, we can associate a collection of Chern classes $c_i(E) \in H^{2i}_{\mathcal{D}^{\mathrm{an}}}(X,\mathbb{R}(i))$, $i \geq 1$ in a functorial way.
We want to see that all possible characteristic classes in analytic Deligne cohomology can be derived from the Chern classes.

DEFINITION 1.5. Let $n \geq 1$ be an integer and let $r_1 \geq 1, \dots, r_n \geq 1$ be a collection of integers. A *theory of characteristic classes for n-tuples of vector bundles of rank $r_1,\dots,r_n$* is an assignment that, to each n-tuple of isomorphism classes of vector bundles $(E_1,\dots,E_n)$ over a complex manifold X, with $\mathrm{rk}(E_i) = r_i$, assigns a class

$$\mathrm{cl}(E_1,\dots,E_n) \in \bigoplus_{k,p} H^k_{\mathcal{D}^{\mathrm{an}}}(X,\mathbb{R}(p))$$

in a functorial way. That is, for every morphism $f\colon X \longrightarrow Y$ of complex manifolds, the equality

$$f^*(\mathrm{cl}(E_1,\dots,E_n)) = \mathrm{cl}(f^*E_1,\dots,f^*E_n)$$

holds

The first consequence of the functoriality and certain homotopy property of analytic Deligne cohomology classes is the following.

PROPOSITION 1.6. *Let* cl *be a theory of characteristic classes for n-tuples of vector bundles of rank $r_1,\dots,r_n$. Let X be a complex manifold and let $(E_1,\dots,E_n)$ be a n-tuple of vector bundles over X with* $\mathrm{rk}(E_i) = r_i$ *for all i. Let $1 \leq j \leq n$ and let*

$$0 \longrightarrow E'_j \longrightarrow E_j \longrightarrow E''_j \longrightarrow 0,$$

be a short exact sequence. Then the equality

$$\mathrm{cl}(E_1,\dots,E_j,\dots,E_n) = \mathrm{cl}(E_1,\dots,E'_j \oplus E''_j,\dots,E_n)$$

holds.

Proof. Let $\iota_0,\iota_\infty\colon X \longrightarrow X \times \mathbb{P}^1$ be the inclusion as the fiber over 0 and the fiber over ∞ respectively. Then there exists a vector bundle $\widetilde{E}_j$ on $X \times \mathbb{P}^1$ (see for instance [19] (1.2.3.1) or definition 2.5 below) such that $\iota_0^*\widetilde{E}_j \cong E_j$ and $\iota_\infty^*\widetilde{E}_j \cong E'_j \oplus E''_j$. Let $p_1\colon X \times \mathbb{P}^1 \longrightarrow X$ be the first projection. Let $\omega \in \bigoplus_{k,p} \mathcal{D}^k(X,p)$ be any $\mathrm{d}_{\mathcal{D}}$-closed form that represents

$\mathrm{cl}(p_1^*E_1,\dots,\widetilde{E}_j,\dots,p_1^*E_n)$. Then, by functoriality we know that $\iota_0^*\omega$ represents $\mathrm{cl}(E_1,\dots,E_j,\dots,E_n)$ and $\iota_\infty^*\omega$ represents $\mathrm{cl}(E_1,\dots,E_j'\oplus E_j'',\dots,E_n)$. We write

$$\beta=\frac{1}{2\pi i}\int_{\mathbb{P}^1}\frac{-1}{2}\log t\bar{t}\bullet\omega,$$

where t is the absolute coordinate of $\mathbb{P}^1$. Then

$$\mathrm{d}_{\mathcal{D}}\,\beta=\iota_\infty^*\omega-\iota_0^*\omega$$

which implies the result. □

A standard method to produce characteristic classes for vector bundles is to choose hermitian metrics on the vector bundles and to construct closed differential forms out of them. The following result shows that functoriality implies that the cohomology classes represented by these forms are independent from the hermitian metrics and therefore are characteristic classes. When working with hermitian vector bundles we will use the convention that, if E denotes the vector bundle, then $\overline{E}=(E,h)$ will denote the vector bundle together with the hermitian metric.

Proposition 1.7. *Let $n\geq 1$ be an integer and let $r_1\geq 1,\dots,r_n\geq 1$ be a collection of integers. Let* cl *be an assignment that, to each n-tuple $(\overline{E}_1,\dots,\overline{E}_n)=((E_1,h_1),\dots,(E_n,h_n))$ of isometry classes of hermitian vector bundles of rank $r_1,\dots,r_n$ over a complex manifold X, associates a cohomology class*

$$\mathrm{cl}(\overline{E}_1,\dots,\overline{E}_n)\in\bigoplus_{k,p}H^k_{\mathcal{D}}(X,\mathbb{R}(p))$$

such that, for each morphism $f:Y\to X$,

$$\mathrm{cl}(f^*\overline{E}_1,\dots,f^*\overline{E}_n)=f^*\,\mathrm{cl}(\overline{E}_1,\dots,\overline{E}_n).$$

Then the cohomology class $\mathrm{cl}(\overline{E}_1,\dots,\overline{E}_n)$ is independent from the hermitian metrics. Therefore it is a well defined characteristic class.

Proof. Let $1\leq j\leq n$ be an integer and let $\overline{E}_j'=(E_j,h_j')$ be the vector bundle underlying $\overline{E}_j$ with a different choice of metric. Let ι_0, ι_∞ and p_1 be as in the proof of proposition 1.6. Then we can choose a hermitian metric h on $p_1^*E_j$, such that $\iota_0^*(p_1^*E_j,h)=\overline{E}_j$ and $\iota_\infty^*(p_1^*E_j,h)=\overline{E}_j'$. Let ω be any smooth closed differential form on $X\times\mathbb{P}^1$ that represents $\mathrm{cl}(p_1^*\overline{E}_1,\dots,(p_1^*E_1,h),\dots,p_1^*\overline{E}_n)$. Then,

$$\beta=\frac{1}{2\pi i}\int_{\mathbb{P}^1}\frac{-1}{2}\log t\bar{t}\bullet\omega$$

satisfies

$$\mathrm{d}_{\mathcal{D}}\,\beta=\iota_\infty^*\omega-\iota_0^*\omega$$

which implies the result. □

We are interested in vector bundles that can be extended to a projective variety. Therefore we will restrict ourselves to the algebraic category. So, by a complex algebraic manifold we will mean the complex manifold associated to a smooth quasi-projective variety over $\mathbb{C}$. When working with an algebraic manifold, by a vector bundle we will mean the holomorphic vector bundle associated to an algebraic vector bundle.
We will denote by $\mathbb{D}[[x_1,\dots,x_r]]$ the ring of commutative formal power series. That is, the unknowns $x_1,\dots,x_r$ commute with each other and with $\mathbb{D}$. We turn it into a commutative bigraded ring by declaring that the unknowns x_i have bidegree $(2,1)$. The symmetric group in r elements, $\mathfrak{S}_r$ acts on $\mathbb{D}[[x_1,\dots,x_r]]$. The subalgebra of invariant elements is generated over $\mathbb{D}$ by the elementary symmetric functions. The main result of this section is the following

Theorem 1.8. *Let* cl *be a theory of characteristic classes for n-tuples of vector bundles of rank $r_1,\dots,r_n$. Then, there is a power series $\varphi\in\mathbb{D}[[x_1,\dots,x_r]]$ in $r=r_1+\dots+r_n$ variables with coefficients in the ring $\mathbb{D}$, such that, for each complex algebraic manifold X and each n-tuple of algebraic vector bundles $(E_1,\dots,E_n)$ over X with $\operatorname{rk}(E_i)=r_i$ this equality holds:*

$$\operatorname{cl}(E_1,\dots,E_n)=\varphi(c_1(E_1),\dots,c_{r_1}(E_1),\dots,c_1(E_n),\dots,c_{r_n}(E_n)). \tag{1.9}$$

Conversely, any power series φ as before determines a theory of characteristic classes for n-tuples of vector bundles of rank $r_1,\dots,r_n$, by equation (1.9).

Proof. The second statement is obvious from the properties of Chern classes. Since we are assuming X quasi-projective, given n algebraic vector bundles $E_1,\dots,E_n$ on X, there is a smooth projective compactification $\widetilde{X}$ and vector bundles $\widetilde{E}_1,\dots,\widetilde{E}_n$ on $\widetilde{X}$, such that $E_i=\widetilde{E}_i|_X$ (see for instance [14] proposition 2.2), we are reduced to the case when X is projective. In this case, analytic Deligne cohomology agrees with ordinary Deligne cohomology.
Let us assume first that $r_1=\dots=r_n=1$ and that we have a characteristic class cl for n line bundles. Then, for each n-tuple of positive integers $m_1,\dots,m_n$ we consider the space $\mathbb{P}^{m_1,\dots,m_n}=\mathbb{P}^{m_1}_{\mathbb{C}}\times\dots\times\mathbb{P}^{m_n}_{\mathbb{C}}$ and we denote by p_i the projection over the i-th factor. Then

$$\bigoplus_{k,p} H^k_{\mathcal{D}}(\mathbb{P}^{m_1,\dots,m_n},\mathbb{R}(p))=\mathbb{D}[x_1,\dots,x_n]\Big/(x_1^{m_1},\dots,x_n^{m_n})$$

is a quotient of the polynomial ring generated by the classes $x_i=c_1(p_i^*\mathcal{O}(1))$ with coefficients in the ring $\mathbb{D}$. Therefore, there is a polynomial $\varphi_{m_1,\dots,m_n}$ in n variables such that

$$\operatorname{cl}(p_1^*\mathcal{O}(1),\dots,p_1^*\mathcal{O}(1))=\varphi_{m_1,\dots,m_n}(x_1,\dots,x_n).$$

If $m_1\le m_1',\ \dots,\ m_n\le m_n'$ then, by functoriality, the polynomial $\varphi_{m_1,\dots,m_n}$ is the truncation of the polynomial $\varphi_{m_1',\dots,m_n'}$. Therefore there is a power series in n variables, φ such that $\varphi_{m_1,\dots,m_n}$ is the truncation of φ in the appropriate quotient of the polynomial ring.

Let $L_1,\dots,L_n$ be line bundles on a projective algebraic manifold that are generated by global sections. Then they determine a morphism $f\colon X \longrightarrow \mathbb{P}^{m_1,\dots,m_n}$ such that $L_i = f^* p_i^* \mathcal{O}(1)$. Therefore, again by functoriality, we obtain

$$\operatorname{cl}(L_1,\dots,L_n) = \varphi(c_1(L_1),\dots,c_1(L_n)).$$

From the class cl we can define a new characteristic class for $n+1$ line bundles by the formula

$$\operatorname{cl}'(L_1,\dots,L_n,M) = \operatorname{cl}(L_1 \otimes M^\vee,\dots,L_n \otimes M^\vee).$$

When $L_1,\dots,L_n$ and M are generated by global sections we have that there is a power series ψ such that

$$\operatorname{cl}'(L_1,\dots,L_n,M) = \psi(c_1(L_1),\dots,c_1(L_n),c_1(M)).$$

Moreover, when the line bundles $L_i \otimes M^\vee$ are also generated by global sections the following holds

$$\begin{aligned}\psi(c_1(L_1),\dots,c_1(L_n),c_1(M)) &= \varphi(c_1(L_1 \otimes M^\vee),\dots,c_1(L_n \otimes M^\vee))\\ &= \varphi(c_1(L_1) - c_1(M),\dots,c_1(L_n) - c_1(M)).\end{aligned}$$

Considering the system of spaces $\mathbb{P}^{m_1,\dots,m_n,m_{n+1}}$ with line bundles

$$L_i = p_i^*\mathcal{O}(1) \otimes p_{n+1}^*\mathcal{O}(1),\ i=1,\dots,n, \quad M = p_{n+1}^*\mathcal{O}(1),$$

we see that there is an identity of power series

$$\varphi(x_1 - y,\dots,x_n - y) = \psi(x_1,\dots,x_n,y).$$

Now let X be a projective complex manifold and let $L_1,\dots,L_n$ be arbitrary line bundles. Then there is a line bundle M such that M and $L_i' = L_i \otimes M$, $i=1,\dots,n$ are generated by global sections. Then we have

$$\begin{aligned}\operatorname{cl}(L_1,\dots,L_n) &= \operatorname{cl}(L_1' \otimes M^\vee,\dots,L_n' \otimes M^\vee)\\ &= \operatorname{cl}'(L_1',\dots,L_n',M)\\ &= \psi(c_1(L_1'),\dots,c_1(L_n'),c_1(M))\\ &= \varphi((c_1(L_1') - c_1(M),\dots,c_1(L_n') - c_1(M)))\\ &= \varphi(c_1(L_1),\dots,c_1(L_n)).\end{aligned}$$

The case of arbitrary rank vector bundles follows from the case of rank one vector bundles by proposition 1.6 and the splitting principle. We next recall the argument. Given a projective complex manifold X and vector bundles $E_1,\dots,E_n$ of rank $r_1,\dots,r_n$, we can find a proper morphism $\pi\colon \widetilde{X} \longrightarrow X$, with $\widetilde{X}$ a complex projective manifold, and such that the induced morphism

$$\pi^*\colon H^*_{\mathcal{D}}(X,\mathbb{R}(*)) \longrightarrow H^*_{\mathcal{D}}(\widetilde{X},\mathbb{R}(*))$$

is injective and every bundle $\pi^*(E_i)$ admits a holomorphic filtration

$$0 = K_{i,0} \subset K_{i,1} \subset \cdots \subset K_{i,r_i-1} \subset K_{i,r_i} = \pi^*(E_i),$$

with $L_{i,j} = K_{i,j}/K_{i,j-1}$ a line bundle. If cl is a characteristic class for n-tuples of vector bundles of rank $r_1,\dots,r_n$, we define a characteristic class for $r_1+\cdots+r_n$-tuples of line bundles by the formula

$$\begin{multline*}\mathrm{cl}'(L_{1,1},\dots,L_{1,r_1},\dots,L_{n,1},\dots,L_{n,r_n}) = \\ \mathrm{cl}(L_{1,1}\oplus\cdots\oplus L_{1,r_1},\dots,L_{n,1}\oplus\cdots\oplus, L_{n,r_n}).\end{multline*}$$

By the case of line bundles we know that there is a power series in $r_1+\cdots+r_n$ variables ψ such that

$$\mathrm{cl}'(L_{1,1},\dots,L_{1,r_1},\dots,L_{n,1},\dots,L_{n,r_n}) = \psi(c_1(L_{1,1}),\dots,c_1(L_{n,r_n})).$$

Since the class cl$'$ is symmetric under the group $\mathfrak{S}_{r_1}\times\cdots\times\mathfrak{S}_{r_n}$, the same is true for the power series ψ. Therefore ψ can be written in terms of symmetric elementary functions. That is, there is another power series in $r_1+\cdots+r_n$ variables φ, such that

$$\begin{multline*}\psi(x_{1,1},\dots,x_{n,r_n}) = \varphi(s_1(x_{1,1},\dots,x_{1,r_1}),\dots,s_{r_1}(x_{1,1},\dots,x_{1,r_1}),\dots \\ \dots,s_1(x_{n,1},\dots,x_{n,r_n}),\dots,s_{r_n}(x_{n,1},\dots,x_{n,r_n})),\end{multline*}$$

where s_i is the i-th elementary symmetric function of the appropriate number of variables. Then

$$\begin{aligned}\pi^*(\mathrm{cl}(E_1,\dots,E_n)) &= \mathrm{cl}(\pi^*E_1,\dots,\pi^*E_n))\\ &= \mathrm{cl}'(L_{1,1},\dots,L_{n,r_n})\\ &= \psi(c_1(L_{1,1}),\dots,c_1(L_{n,r_n}))\\ &= \varphi(c_1(\pi^*E_1),\dots,c_{r_1}(\pi^*E_1),\dots,c_1(\pi^*E_n),\dots,c_{r_n}(\pi^*E_n))\\ &= \pi^*\varphi(c_1(E_1),\dots,c_{r_1}(E_1),\dots,c_1(E_n),\dots,c_{r_n}(E_n)).\end{aligned}$$

Therefore, the result follows from the injectivity of π^*. □

Remark 1.10. It would be interesting to know if the functoriality of a characteristic class in enough to imply that it is a power series in the Chern classes for arbitrary complex manifolds and holomorphic vector bundles.

2 Bott-Chern classes

The aim of this section is to recall the theory of Bott-Chern classes. For more details we refer the reader to [7], [4], [19], [31], [14], [10] and [12]. Note however that the theory we present here is equivalent, although not identical, to the different versions that appear in the literature.

Let X be a complex manifold and let $\overline{E}=(E,h)$ be a rank r holomorphic vector bundle provided with a hermitian metric. Let $\phi\in\mathbb{D}[[x_1,\dots,x_r]]$ be a formal power series in r variables that is symmetric under the action of $\mathfrak{S}_r$. Let s_i, $i=1,\dots,r$ be the elementary symmetric functions in r variables. Then $\phi(x_1,\dots,x_r)=\varphi(s_1,\dots,s_r)$ for certain power series φ. By Chern-Weil theory we can obtain a representative of the class

$$\phi(E):=\varphi(c_1(E),\dots,c_r(E))\in\bigoplus_{k,p}H^k_{\mathcal{D}^{\mathrm{an}}}(X,\mathbb{R}(p))$$

as follows.
We denote also by ϕ the invariant power series in $r\times r$ matrices defined by ϕ. Let K be the curvature matrix of the hermitian holomorphic connection of (E,h). The entries of K in a particular trivialization of E are local sections of $\mathcal{D}^2(X,1)$. Then we write

$$\phi(E,h)=\phi(-K)\in\bigoplus_{k,p}\mathcal{D}^k(X,p).$$

The form $\phi(E,h)$ is well defined, closed, and it represents the class $\phi(E)$.
Now let

$$\overline{E}_*=(\dots\xrightarrow{f_{n+1}}\overline{E}_n\xrightarrow{f_n}\overline{E}_{n-1}\xrightarrow{f_{n-1}}\dots)$$

be a bounded acyclic complex of hermitian vector bundles; by this we mean a bounded acyclic complex of vector bundles, where each vector bundle is equipped with an arbitrarily chosen hermitian metric.
Write

$$r=\sum_{i\text{ even}}\mathrm{rk}(E_i)=\sum_{i\text{ odd}}\mathrm{rk}(E_i).$$

and let ϕ be a symmetric power series in r variables.
As before, we can define the Chern forms

$$\phi(\bigoplus_{i\text{ even}}(E_i,h_i))\text{ and }\phi(\bigoplus_{i\text{ odd}}(E_i,h_i)),$$

that represent the Chern classes $\phi(\bigoplus_{i\text{ even}}E_i)$ and $\phi(\bigoplus_{i\text{ odd}}E_i)$. The Chern classes are compatible with respect to exact sequences, that is,

$$\phi(\bigoplus_{i\text{ even}}E_i)=\phi(\bigoplus_{i\text{ odd}}E_i).$$

But, in general, this is not true for the Chern forms. This lack of compatibility with exact sequences on the level of Chern forms is measured by the Bott-Chern classes.

Definition 2.1. Let

$$\overline{E}_*=(\dots\xrightarrow{f_{n+1}}\overline{E}_n\xrightarrow{f_n}\overline{E}_{n-1}\xrightarrow{f_{n-1}}\dots)$$

be an acyclic complex of hermitian vector bundles, we will say that $\overline{E}_*$ is an *orthogonally split complex* of vector bundles if, for any integer n, the exact sequence

$$0 \longrightarrow \operatorname{Ker} f_n \longrightarrow \overline{E}_n \longrightarrow \operatorname{Ker} f_{n-1} \longrightarrow 0$$

is split, there is a splitting section $s_n \colon \operatorname{Ker} f_{n-1} \to E_n$ such that $\overline{E}_n$ is the orthogonal direct sum of $\operatorname{Ker} f_n$ and $\operatorname{Im} s_n$ and the metrics induced in the subbundle $\operatorname{Ker} f_{n-1}$ by the inclusion $\operatorname{Ker} f_{n-1} \subset \overline{E}_{n-1}$ and by the section s_n agree.

NOTATION 2.2. Let $(x:y)$ be homogeneous coordinates of $\mathbb{P}^1$ and let $t = x/y$ be the absolute coordinate. In order to make certain choices of metrics in a functorial way, we fix once and for all a partition of unity $\{\sigma_0, \sigma_\infty\}$, over $\mathbb{P}^1$ subordinated to the open cover of $\mathbb{P}^1$ given by the open subsets $\{\{|y| > 1/2|x|\}, \{|x| > 1/2|y|\}\}$. As usual we will write $\infty = (1:0)$, $0 = (0:1)$.

The fundamental result of the theory of Bott-Chern classes is the following theorem (see [7], [4], [19]).

THEOREM 2.3. *There is a unique way to attach to each bounded exact complex $\overline{E}_*$ as above, a class $\widetilde{\phi}(\overline{E}_*)$ in*

$$\bigoplus_k \widetilde{\mathcal{D}}^{2k-1}(X,k) = \bigoplus_k \mathcal{D}^{2k-1}(X,k)/\operatorname{Im}(\mathrm{d}_{\mathcal{D}})$$

satisfying the following properties

(i) (Differential equation)

$$\mathrm{d}_{\mathcal{D}}\, \widetilde{\phi}(\overline{E}_*) = \phi(\bigoplus_{i\ even} (E_i, h_i)) - \phi(\bigoplus_{i\ odd} (E_i, h_i)). \tag{2.4}$$

(ii) (Functoriality) $f^*\widetilde{\phi}(\overline{E}_*) = \widetilde{\phi}(f^*\overline{E}_*)$, for every holomorphic map $f\colon X' \longrightarrow X$.

(iii) (Normalization) If $\overline{E}_*$ is orthogonally split, then $\widetilde{\phi}(\overline{E}_*) = 0$.

Proof. We first recall how to prove the uniqueness.
Let $\overline{K}_i = (K_i, g_i)$, where $K_i = \operatorname{Ker} f_i$ and g_i is the metric induced by the inclusion $K_i \subset E_i$. Consider the complex manifold $X \times \mathbb{P}^1$ with projections p_1 and p_2. For every vector bundle F on X we will denote $F(i) = p_1^*F \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(i)$. Let $\widetilde{C}_* = \widetilde{C}(E_*)_*$ be the complex of vector bundles on $X \times \mathbb{P}^1$ given by $\widetilde{C}_i = E_i(i) \oplus E_{i-1}(i-1)$ with differential $d(s,t) = (t,0)$. Let $\widetilde{D}_* = \widetilde{D}(E_*)_*$ be the complex of vector bundles with $\widetilde{D}_i = E_{i-1}(i) \oplus E_{i-2}(i-1)$ and differential $d(s,t) = (t,0)$. Using notation 2.2 we define the map $\psi\colon \widetilde{C}(E_*)_i \longrightarrow \widetilde{D}(E_*)_i$ given by $\psi(s,t) = (f_i(s) - t \otimes y, f_{i-1}(t))$. It is a morphism of complexes.

DEFINITION 2.5. The *first transgression exact sequence* of E_* is given by

$$\operatorname{tr}_1(E_*)_* = \operatorname{Ker} \psi.$$

On $X \times \mathbb{A}^1$, the map $p_1^* E_i \longrightarrow \widetilde{C}(E_*)_i$ given by $s \longmapsto (s \otimes y^i, f_i(s) \otimes y^{i-1})$ induces an isomorphism of complexes

$$p_1^* E_* \longrightarrow \operatorname{tr}_1(E_*)_*|_{X \times \mathbb{A}^1}, \tag{2.6}$$

and in particular isomorphisms

$$\operatorname{tr}_1(E_*)_i|_{X \times \{0\}} \cong E_i. \tag{2.7}$$

Moreover, we have isomorphisms

$$\operatorname{tr}_1(E_*)_i|_{X \times \{\infty\}} \cong K_i \oplus K_{i-1}. \tag{2.8}$$

Definition 2.9. We will denote by $\operatorname{tr}_1(\overline{E}_*)_*$ the complex $\operatorname{tr}_1(E_*)_*$ provided with any hermitian metric such that the isomorphisms (2.7) and (2.8) are isometries. If we need a functorial choice of metric, we proceed as follows. On $X \times (\mathbb{P}^1 \setminus \{0\})$ we consider the metric induced by $\widetilde{C}$ on $\operatorname{tr}_1(E_*)_*$. On $X \times (\mathbb{P}^1 \setminus \{\infty\})$ we consider the metric induced by the isomorphism (2.6). We glue both metrics by means of the partition of unity of notation 2.2.

In particular, we have that $\operatorname{tr}_1(\overline{E}_*)|_{X \times \{\infty\}}$ is orthogonally split. We assume that there exists a theory of Bott-Chern classes satisfying the above properties. Thus, there exists a class of differential forms $\widetilde{\phi}(\operatorname{tr}_1(\overline{E}_*)_*)$ with the following properties. By (i) this class satisfies

$$\mathrm{d}_{\mathcal{D}} \widetilde{\phi}(\operatorname{tr}_1(\overline{E}_*)_*) = \phi(\bigoplus_{i \text{ even}} \operatorname{tr}_1(\overline{E}_*)_i)) - \phi(\bigoplus_{i \text{ odd}} \operatorname{tr}_1(\overline{E}_*)_i).$$

By (ii), it satisfies

$$\widetilde{\phi}(\operatorname{tr}_1(\overline{E}_*)_*) \mid_{X \times \{0\}} = \widetilde{\phi}(\operatorname{tr}_1(\overline{E}_*)_* \mid_{X \times \{0\}}) = \widetilde{\phi}(\overline{E}_*).$$

Finally, by (ii) and (iii) it satisfies

$$\widetilde{\phi}(\operatorname{tr}_1(\overline{E}_*)_*) \mid_{X \times \{\infty\}} = \widetilde{\phi}(\operatorname{tr}_1(\overline{E}_*)_* \mid_{X \times \{\infty\}}) = 0.$$

Let $\phi(\operatorname{tr}_1(\overline{E}_*)_*)$ be any representative of the class $\widetilde{\phi}(\operatorname{tr}_1(\overline{E}_*)_*)$.
Then, in the group $\bigoplus_k \widetilde{\mathcal{D}}^{2k-1}(X,k)$, we have

$$\begin{aligned}
0 &= \mathrm{d}_{\mathcal{D}} \frac{1}{2\pi i} \int_{\mathbb{P}^1} \frac{-1}{2} \log(t\bar{t}) \bullet \phi(\operatorname{tr}_1(\overline{E}_*)_*) \\
&= \frac{1}{2\pi i} \int_{\mathbb{P}^1} \left(\mathrm{d}_{\mathcal{D}} \frac{-1}{2} \log(t\bar{t}) \bullet \phi(\operatorname{tr}_1(\overline{E}_*)_*) - \frac{-1}{2} \log(t\bar{t}) \bullet \mathrm{d}_{\mathcal{D}} \phi(\operatorname{tr}_1(\overline{E}_*)_*) \right) \\
&= \widetilde{\phi}(\operatorname{tr}_1(\overline{E}_*)_*)|_{X \times \{\infty\}} - \widetilde{\phi}(\operatorname{tr}_1(\overline{E}_*)_*)|_{X \times \{0\}} \\
&\quad - \frac{1}{2\pi i} \int_{\mathbb{P}^1} \frac{-1}{2} \log(t\bar{t}) \bullet (\phi(\bigoplus_{i \text{ even}} \operatorname{tr}_1(\overline{E}_*)_i) - \phi(\bigoplus_{i \text{ odd}} \operatorname{tr}_1(\overline{E}_*)_i)) \\
&= -\widetilde{\phi}(\overline{E}_*) - \frac{1}{2\pi i} \int_{\mathbb{P}^1} \frac{-1}{2} \log(t\bar{t}) \bullet (\phi(\bigoplus_{i \text{ even}} \operatorname{tr}_1(\overline{E}_*)_i) - \phi(\bigoplus_{i \text{ odd}} \operatorname{tr}_1(\overline{E}_*)_i)).
\end{aligned}$$

Hence, if such a theory exists, it should satisfy the formula

$$\widetilde{\phi}(\overline{E}_*) = \frac{1}{2\pi i}\int_{\mathbb{P}^1} \frac{-1}{2}\log(t\bar{t}) \bullet (\phi(\bigoplus_{i \text{ odd}} \operatorname{tr}_1(\overline{E}_*)_i) - \phi(\bigoplus_{i \text{ even}} \operatorname{tr}_1(\overline{E}_*)_i)). \quad (2.10)$$

Therefore $\widetilde{\phi}(\overline{E}_*)$ is determined by properties (i), (ii) and (iii).
In order to prove the existence of a theory of functorial Bott-Chern forms, we have to see that the right hand side of equation (2.10) is independent from the choice of the metric on $\operatorname{tr}_1(\overline{E}_*)_*$ and that it satisfies the properties (i), (ii) and (iii). For this the reader can follow the proof of [4] theorem 1.29.

□

In view of the proof of theorem 2.3, we can define the Bott-Chern classes as follows.

DEFINITION 2.11. Let

$$\overline{E}_*\colon 0 \longrightarrow (E_n, h_n) \longrightarrow \dots \longrightarrow (E_1, h_1) \longrightarrow (E_0, h_0) \longrightarrow 0$$

be a bounded acyclic complex of hermitian vector bundles. Let

$$r = \sum_{i \text{ even}} \operatorname{rk}(E_i) = \sum_{i \text{ odd}} \operatorname{rk}(E_i).$$

Let $\phi \in \mathbb{D}[[x_1, \dots, x_r]]^{\mathfrak{S}_r}$ be a symmetric power series in r variables. Then the *Bott-Chern class* associated to ϕ and $\overline{E}_*$ is the element of $\bigoplus_{k,p} \widetilde{\mathcal{D}}^k(E_X, p)$ given by

$$\widetilde{\phi}(\overline{E_*}) = \frac{1}{2\pi i}\int_{\mathbb{P}^1} \frac{-1}{2}\log(t\bar{t}) \bullet (\phi(\bigoplus_{i \text{ odd}} \operatorname{tr}_1(\overline{E}_*)_i) - \phi(\bigoplus_{i \text{ even}} \operatorname{tr}_1(\overline{E}_*)_i)).$$

The following property is obvious from the definition.

LEMMA 2.12. *Let $\overline{E}_*$ be an acyclic complex of hermitian vector bundles. Then, for any integer k,*

$$\widetilde{\phi}(\overline{E}_*[k]) = (-1)^k \widetilde{\phi}(\overline{E}_*).$$

□

Particular cases of Bott-Chern classes are obtained when we consider a single vector bundle with two different hermitian metrics or a short exact sequence of vector bundles. Note however that, in order to fix the sign of the Bott-Chern classes on these cases, one has to choose the degree of the vector bundles involved, for instance as in the next definition.

DEFINITION 2.13. Let E be a holomorphic vector bundle of rank r, let h_0 and h_1 be two hermitian metrics and let ϕ be an invariant power series of r variables. We will denote by $\widetilde{\phi}(E, h_0, h_1)$ the Bott-Chern class associated to the complex

$$\overline{\xi}\colon 0 \longrightarrow (E, h_1) \longrightarrow (E, h_0) \longrightarrow 0,$$

where (E, h_0) sits in degree zero.

Therefore, this class satisfies

$$\mathrm{d}_{\mathcal{D}}\,\widetilde{\phi}(E,h_0,h_1)=\phi(E,h_0)-\phi(E,h_1).$$

In fact we can characterize $\widetilde{\phi}(E,h_0,h_1)$ axiomatically as follows.

Proposition 2.14. *Given ϕ, a symmetric power series in r variables, there is a unique way to attach, to each rank r vector bundle E on a complex manifold X and metrics h_0 and h_1, a class $\widetilde{\phi}(E,h_0,h_1)$ satisfying*

(i) $\mathrm{d}_{\mathcal{D}}\,\widetilde{\phi}(E,h_0,h_1)=\phi(E,h_0)-\phi(E,h_1)$.

(ii) $f^*\widetilde{\phi}(E,h_0,h_1)=\widetilde{\phi}(f^*(E,h_0,h_1))$ *for every holomorphic map* $f\colon Y\longrightarrow X$.

(iii) $\widetilde{\phi}(E,h,h)=0$.

Moreover, if we denote $\widetilde{E}:=\mathrm{tr}_1(\overline{\xi})_1$, *then it satisfies*

$$\widetilde{E}|_{X\times\{\infty\}}\cong(E,h_0),\quad \widetilde{E}|_{X\times\{0\}}\cong(E,h_1)$$

and

$$\widetilde{\phi}(E,h_0,h_1)=\frac{1}{2\pi i}\int_{\mathbb{P}^1}\frac{-1}{2}\log(t\bar{t})\bullet\phi(\widetilde{E}). \tag{2.15}$$

Proof. The axiomatic characterization is proved as in theorem 2.3. In order to prove equation (2.15), if we follow the notations of the proof of theorem 2.3 we have $K_0=(E,h_0)$ and $K_1=0$. Therefore $\mathrm{tr}_1(\overline{\xi})_0=p_1^*(E,h_0)$, while $\widetilde{E}:=\mathrm{tr}_1(\overline{\xi})_1$ satisfies $\widetilde{E}|_{X\times\{0\}}=(E,h_1)$ and $\widetilde{E}|_{X\times\{\infty\}}=(E,h_0)$. Using the antisymmetry of $\log t\bar{t}$ under the involution $t\mapsto 1/t$ we obtain

$$\widetilde{\phi}(E,h_0,h_1)=\widetilde{\phi}(\overline{\xi})=\frac{1}{2\pi i}\int_{\mathbb{P}^1}\frac{-1}{2}\log(t\bar{t})\bullet\phi(\widetilde{E}).$$

□

We can also treat the case of short exact sequences. If

$$\overline{\varepsilon}\colon 0\longrightarrow\overline{E}_2\longrightarrow\overline{E}_1\longrightarrow\overline{E}_0\longrightarrow 0$$

is a short exact sequence of hermitian vector bundles, by convention, we will assume that $\overline{E}_0$ sits in degree zero. This fixs the sign of $\widetilde{\phi}(\overline{\varepsilon})$.

Proposition 2.16. *Given ϕ, a symmetric power series in r variables, there is a unique way to attach, to each short exact sequence of hermitian vector bundles on a complex manifold X*

$$\overline{\varepsilon}\colon 0\longrightarrow\overline{E}_2\longrightarrow\overline{E}_1\longrightarrow\overline{E}_0\longrightarrow 0,$$

where $\overline{E}_1$ has rank r, a class $\widetilde{\phi}(\overline{\varepsilon})$ satisfying *(i)* $\mathrm{d}_{\mathcal{D}}\,\widetilde{\phi}(\overline{\varepsilon})=\phi(\overline{E}_0\oplus\overline{E}_2)-\phi(\overline{E}_1)$.

(ii) $f^*\widetilde{\phi}(\overline{\varepsilon})=\widetilde{\phi}(f^*(\overline{\epsilon}))$ *for every holomorphic map* $f\colon Y\longrightarrow X$.

(iii) $\widetilde{\phi}(\overline{\varepsilon})=0$ *whenever* $\overline{\varepsilon}$ *is orthogonally split.*

□

The following additivity result of Bott-Chern classes will be useful later.

Lemma 2.17. *Let* $\overline{A}_{*,*}$ *be a bounded exact sequence of bounded exact sequences of hermitian vector bundles. Let*

$$r=\sum_{i,j\ even}\mathrm{rk}(A_{i,j})=\sum_{i,j\ odd}\mathrm{rk}(A_{i,j})=\sum_{\substack{i\ odd\\ j\ even}}\mathrm{rk}(A_{i,j})=\sum_{\substack{i\ even\\ j\ odd}}\mathrm{rk}(A_{i,j}).$$

Let ϕ *be a symmetric power series in* r *variables. Then*

$$\widetilde{\phi}(\bigoplus_{k\ even}\overline{A}_{k,*})-\widetilde{\phi}(\bigoplus_{k\ odd}\overline{A}_{k,*})=\widetilde{\phi}(\bigoplus_{k\ even}\overline{A}_{*,k})-\widetilde{\phi}(\bigoplus_{k\ odd}\overline{A}_{*,k}).$$

Proof. The proof is analogous to the proof of proposition 6.13 and is left to the reader. □

Corollary 2.18. *Let* $\overline{A}_{*,*}$ *be a bounded double complex of hermitian vector bundles with exact rows, let*

$$r=\sum_{i+j\ even}\mathrm{rk}(A_{i,j})=\sum_{i+j\ odd}\mathrm{rk}(A_{i,j})$$

and let ϕ *be a symmetric power series in* r *variables. Then*

$$\widetilde{\phi}(\mathrm{Tot}\,\overline{A}_{*,*})=\widetilde{\phi}(\bigoplus_k\overline{A}_{*,k}[-k]).$$

Proof. Let k_0 be an integer such that $\overline{A}_{k,l}=0$ for $k<k_0$. For any integer n we denote by $\mathrm{Tot}_n=\mathrm{Tot}((\overline{A}_{k,l})_{k\geq n})$ the total complex of the exact complex formed by the rows with index greater or equal than n. Then $\mathrm{Tot}_{k_0}=\mathrm{Tot}(\overline{A}_{*,*})$. For each k there is an exact sequence of complexes

$$0\longrightarrow\mathrm{Tot}_{k+1}\longrightarrow\mathrm{Tot}_k\oplus\bigoplus_{l<k}\overline{A}_{l,*}[-l]\longrightarrow\bigoplus_{l\leq k}\overline{A}_{l,*}[-l]\longrightarrow 0,$$

which is orthogonally split in each degree. Therefore by lemma 2.17 we obtain

$$\widetilde{\phi}(\mathrm{Tot}_k\oplus\bigoplus_{l<k}\overline{A}_{l,*}[-l])=\widetilde{\phi}(\mathrm{Tot}_{k-1}\oplus\bigoplus_{l\leq k}\overline{A}_{l,*}[-l]).$$

Hence the result follows by induction. □

A particularly important characteristic class is the Chern character. This class is additive for exact sequences. Specializing lemma 2.17 and corollary 2.18 to the Chern character we obtain

Corollary 2.19. *With the hypothesis of lemma 2.17, the following equality holds:*

$$\sum_k (-1)^k \widetilde{\mathrm{ch}}(\overline{A}_{k,*}) = \sum_k (-1)^k \widetilde{\mathrm{ch}}(\overline{A}_{*,k}) = \widetilde{\mathrm{ch}}(\mathrm{Tot}\,\overline{A}_{*,*}).$$

□

Our next aim is to extend the Bott-Chern classes associated to the Chern character to metrized coherent sheaves. This extension is due to Zha [32], although it is still unpublished.

Definition 2.20. A *metrized coherent sheaf* $\overline{\mathcal{F}}$ on X is a pair $(\mathcal{F}, \overline{E}_* \to \mathcal{F})$ where $\mathcal{F}$ is a coherent sheaf on X and

$$0 \to \overline{E}_n \to \overline{E}_{n-1} \to \cdots \to \overline{E}_0 \to \mathcal{F} \to 0$$

is a finite resolution by hermitian vector bundles of the coherent sheaf $\mathcal{F}$. This resolution is also called the metric of $\overline{\mathcal{F}}$.
If $\overline{E}$ is a hermitian vector bundle, we will also denote by $\overline{E}$ the metrized coherent sheaf $(E, \overline{E} \xrightarrow{\mathrm{id}} E)$.

Note that the coherent sheaf 0 may have non trivial metrics. In fact, any exact sequence of hermitian vector bundles

$$0 \to \overline{A}_n \to \cdots \to \overline{A}_0 \to 0 \to 0$$

can be seen as a metric on 0. It will be denoted $\overline{0}_{A_*}$. A metric on 0 is said to be *orthogonally split* if the exact sequence is orthogonally split.
A morphism of metrized coherent sheaves $\overline{\mathcal{F}}_1 \to \overline{\mathcal{F}}_2$ is just a morphism of sheaves $\mathcal{F}_1 \to \mathcal{F}_2$. A sequence of metrized coherent sheaves

$$\overline{\varepsilon}: \qquad \dots \longrightarrow \overline{\mathcal{F}}_{n+1} \longrightarrow \overline{\mathcal{F}}_n \longrightarrow \overline{\mathcal{F}}_{n-1} \longrightarrow \dots$$

is said to be exact if it is exact as a sequence of coherent sheaves.

Definition 2.21. Let $\overline{\mathcal{F}} = (\mathcal{F}, \overline{E}_* \to \mathcal{F})$ be a metrized coherent sheaf. Then the *Chern character form* associated to $\overline{\mathcal{F}}$ is given by

$$\mathrm{ch}(\overline{\mathcal{F}}) = \sum_i (-1)^i \,\mathrm{ch}(\overline{E}_i).$$

Definition 2.22. *An exact sequence of metrized coherent sheaves with com-*

patible metrics is a commutative diagram

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \overline{E}_{n,1} & \to & \dots & \to & \overline{E}_{0,1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \overline{E}_{n,0} & \to & \dots & \to & \overline{E}_{0,0} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{F}_n & \to & \dots & \to & \mathcal{F}_0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\tag{2.23}
$$

where all the rows and columns are exact. The columns of this diagram are the individual metrics of each coherent sheaf. We will say that an exact sequence with compatible metrics is *orthogonally split* if each row of vector bundles is an orthogonally split exact sequence of hermitian vector bundles.

As in the case of exact sequences of hermitian vector bundles, the Chern character form is not compatible with exact sequences of metrized coherent sheaves and we can define a secondary Bott-Chern character which measures the lack of compatibility between the metrics.

Theorem 2.24. *1) There is a unique way to attach to every finite exact sequence of metrized coherent sheaves with compatible metrics*

$$\overline{\varepsilon}\colon \qquad 0 \to \overline{\mathcal{F}}_n \to \dots \to \overline{\mathcal{F}}_0 \to 0$$

on a complex manifold X a Bott-Chern secondary character

$$\widetilde{\mathrm{ch}}(\overline{\varepsilon}) \in \bigoplus_p \widetilde{\mathcal{D}}^{2p-1}(X,p)$$

such that the following axioms are satisfied:

(i) (Differential equation)

$$\mathrm{d}_{\mathcal{D}}\, \widetilde{\mathrm{ch}}(\overline{\varepsilon}) = \sum_k (-1)^k \,\mathrm{ch}(\overline{\mathcal{F}_k}).$$

(ii) (Functoriality) If $f\colon X' \longrightarrow X$ is a morphism of complex manifolds, that is tor-independent from the coherent sheaves $\mathcal{F}_k$, then

$$f^*(\widetilde{\mathrm{ch}})(\overline{\varepsilon}) = \widetilde{\mathrm{ch}}(f^*\overline{\varepsilon}),$$

where the exact sequence $f^*\overline{\varepsilon}$ exists thanks to the tor-independence.

(iii) (Horizontal normalization) If $\overline{\varepsilon}$ is orthogonally split then

$$\widetilde{\mathrm{ch}}(\overline{\varepsilon}) = 0.$$

2) There is a unique way to attach to every finite exact sequence of metrized coherent sheaves

$$\overline{\varepsilon}\colon \qquad 0 \to \overline{\mathcal{F}}_n \to \dots \to \overline{\mathcal{F}}_0 \to 0$$

on a complex manifold X a Bott-Chern secondary character

$$\widetilde{\mathrm{ch}}(\overline{\varepsilon}) \in \bigoplus_p \widetilde{\mathcal{D}}^{2p-1}(X,p)$$

such that the axioms (i), (ii) and (iii) above and the axiom (iv) below are satisfied:

(iv) (Vertical normalization) For every bounded complex of hermitian vector bundles

$$\dots \to \overline{A}_k \to \dots \to \overline{A}_0 \to 0$$

that is orthogonally split, and every bounded complex of metrized coherent sheaves

$$\overline{\varepsilon}\colon \qquad 0 \to \overline{\mathcal{F}}_n \to \dots \to \overline{\mathcal{F}}_0 \to 0$$

where the metrics are given by $\overline{E}_{i,*} \to \mathcal{F}_i$, if, for some i_0 we denote

$$\overline{\mathcal{F}}'_{i_0} = (\mathcal{F}_{i_0}, \overline{E}_{i_0,*} \oplus \overline{A}_* \to \mathcal{F}_{i_0})$$

and

$$\overline{\varepsilon}'\colon \qquad 0 \to \overline{\mathcal{F}}_n \to \dots \to \overline{\mathcal{F}}'_{i_0} \to \dots \to \overline{\mathcal{F}}_0 \to 0,$$

then $\widetilde{\mathrm{ch}}(\overline{\varepsilon}') = \widetilde{\mathrm{ch}}(\overline{\varepsilon})$.

Proof. 1) The uniqueness is proved using the standard deformation argument. By definition, the metrics of the coherent sheaves form a diagram like (2.23). On $X \times \mathbb{P}^1$, for each $j \geq 0$ we consider the exact sequences $\widetilde{E}_{*,j} = \mathrm{tr}_1(E_{*,j})$ associated to the rows of the diagram with the hermitian metrics of definition 2.9. Then, for each i,j there are maps $\mathrm{d}\colon \widetilde{E}_{i,j} \to \widetilde{E}_{i-1,j}$, and $\delta\colon \widetilde{E}_{i,j} \to \widetilde{E}_{i,j-1}$. We denote

$$\widetilde{\mathcal{F}}_i = \mathrm{Coker}(\delta\colon \widetilde{E}_{i,1} \to \widetilde{E}_{i,0}).$$

Using the definition of tr_1 and diagram chasing one can prove that there is a commutative diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \widetilde{E}_{n,1} & \to & \dots & \to & \widetilde{E}_{0,1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \widetilde{E}_{n,0} & \to & \dots & \to & \widetilde{E}_{0,0} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \widetilde{\mathcal{F}}_n & \to & \dots & \to & \widetilde{\mathcal{F}}_0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{2.25}$$

where all the rows and columns are exact. In particular this implies that the inclusions $i_0\colon X\to X\times\{0\}\to X\times\mathbb{P}^1$ and $i_\infty\colon X\to X\times\{\infty\}\to X\times\mathbb{P}^1$ are tor-independent from the sheaves $\widetilde{\mathcal{F}}_i$. But $i_0^*\widetilde{\mathcal{F}}_*$ is isometric with $\overline{\mathcal{F}}_*$ and $i_\infty^*\widetilde{\mathcal{F}}_*$ is orthogonally split. Hence, by the standard argument, axioms (i), (ii) and (iii) imply that

$$\widetilde{\mathrm{ch}}(\overline{\varepsilon})=\sum_j(-1)^j\widetilde{\mathrm{ch}}(\overline{E}_{*,j}). \tag{2.26}$$

To prove the existence we use equation (2.26) as definition. Then the properties of the Bott-Chern classes of exact sequences of hermitian vector bundles imply that axioms (i), (ii) and (iii) are satisfied.
Proof of 2). We first assume that such theory exists. Let

$$\dots\to\overline{A}_k\to\dots\to\overline{A}_0\to 0$$

be a bounded complex of hermitian vector bundles, non necessarily orthogonally split, and

$$\overline{\varepsilon}\colon\qquad 0\to\overline{\mathcal{F}}_n\to\dots\to\overline{\mathcal{F}}_0\to 0$$

a bounded complex of metrized coherent sheaves where the metrics are given by $\overline{E}_{i,*}\to\mathcal{F}_i$. As in axiom (iv), for some i_0 we denote

$$\overline{\mathcal{F}}'_{i_0}=(\mathcal{F}_{i_0},\overline{E}_{i,*}\oplus\overline{A}_*\to\mathcal{F}_{i_0})$$

and

$$\overline{\varepsilon}'\colon\qquad 0\to\overline{\mathcal{F}}_n\to\dots\to\overline{\mathcal{F}}'_{i_0}\to\dots\to\overline{\mathcal{F}}_0\to 0.$$

By axioms (i), (ii) and (iv), the class $(-1)^{i_0}(\widetilde{\mathrm{ch}}(\overline{\varepsilon}')-\widetilde{\mathrm{ch}}(\overline{\varepsilon}))$ satisfies the properties that characterize $\widetilde{\mathrm{ch}}(A_*)$. Therefore $\widetilde{\mathrm{ch}}(\overline{\varepsilon}')=\widetilde{\mathrm{ch}}(\overline{\varepsilon})+(-1)^{i_0}\widetilde{\mathrm{ch}}(A_*)$.
Fix again a number i_0 and assume that there is an exact sequence of resolutions

$$\begin{tikzcd} 0 \arrow[r] & \overline{A}_\bullet \arrow[r] \arrow[d] & \overline{E}'_{i_0,*} \arrow[r] \arrow[d] & \overline{E}_{i_0,*} \arrow[r] \arrow[d] & 0 \\ & 0 \arrow[r] & \mathcal{F}_{i_0} \arrow[r, equal] & \mathcal{F}_{i_0} & \end{tikzcd} \tag{2.27}$$

Let now $\overline{\varepsilon}'$ denote the exact sequence $\overline{\varepsilon}$ but with the metric $\overline{E}'_{i_0,*}$ in the position i_0. Let $\overline{\eta}_j$ denote the j-th row of the diagram (2.27). Again using a deformation argument one sees that

$$\widetilde{\mathrm{ch}}(\overline{\varepsilon}')-\widetilde{\mathrm{ch}}(\overline{\varepsilon})=(-1)^{i_0}\left(\widetilde{\mathrm{ch}}(\overline{A}_*)-\sum_j(-1)^j\,\widetilde{\mathrm{ch}}(\eta_j)\right). \tag{2.28}$$

Choose now a compatible system of metrics

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \overline{D}_{n,1} & \rightarrow & \dots & \rightarrow & \overline{D}_{0,1} & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \overline{D}_{n,0} & \rightarrow & \dots & \rightarrow & \overline{D}_{0,0} & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{F}_{n} & \rightarrow & \dots & \rightarrow & \mathcal{F}_{0} & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{2.29}$$

we denote by $\overline{\lambda}_j$ each row of the above diagram. For each i, choose a resolution $\overline{E}'_{i,*} \longrightarrow \mathcal{F}_i$ such that there exist exact sequences of resolutions

$$\begin{tikzcd}
0 \arrow[r] & \overline{A}_{i,*} \arrow[r] \arrow[d] & \overline{E}'_{i,*} \arrow[r] \arrow[d] & \overline{E}_{i,*} \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \mathcal{F}_i \arrow[r, equal] & \mathcal{F}_i &
\end{tikzcd} \tag{2.30}$$

and

$$\begin{tikzcd}
0 \arrow[r] & \overline{B}_{i,*} \arrow[r] \arrow[d] & \overline{E}'_{i,*} \arrow[r] \arrow[d] & \overline{D}_{i,*} \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \mathcal{F}_i \arrow[r, equal] & \mathcal{F}_i &
\end{tikzcd} \tag{2.31}$$

We denote by $\overline{\eta}_{i,j}$ each row of the diagram (2.30) and by $\overline{\mu}_{i,j}$ each row of the diagram (2.31). Then, by (2.28) and (2.26), we have

$$\begin{aligned}
\widetilde{\mathrm{ch}}(\overline{\varepsilon}) = \sum_j (-1)^j \, \widetilde{\mathrm{ch}}(\overline{\lambda}_j) + \sum_i (-1)^i (\widetilde{\mathrm{ch}}(\overline{B}_{i,*}) - \widetilde{\mathrm{ch}}(\overline{A}_{i,*})) \\
+ \sum_{i,j} (-1)^{i+j} (\widetilde{\mathrm{ch}}(\overline{\eta}_{i,j}) - \widetilde{\mathrm{ch}}(\overline{\mu}_{i,j}))
\end{aligned} \tag{2.32}$$

Thus, $\widetilde{\mathrm{ch}}(\overline{\varepsilon})$ is uniquely determined by axioms (i) to (iv). To prove the existence we use equation (2.32) as definition. We have to show that this definition is independent of the choices of the new resolutions. This independence follows from corollary 2.19. Once we know that the Bott-Chern classes are well defined, it is clear that they satisfy axioms (i), (ii), (iii) and (iv). $\square$

Proposition 2.33. *(Compatibility with exact squares) If*

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \to & \overline{\mathcal{F}}_{n+1,m+1} & \to & \overline{\mathcal{F}}_{n+1,m} & \to & \overline{\mathcal{F}}_{n+1,m-1} & \to & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \to & \overline{\mathcal{F}}_{n,m+1} & \to & \overline{\mathcal{F}}_{n,m} & \to & \overline{\mathcal{F}}_{n,m-1} & \to & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \to & \overline{\mathcal{F}}_{n-1,m+1} & \to & \overline{\mathcal{F}}_{n-1,m} & \to & \overline{\mathcal{F}}_{n-1,m-1} & \to & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

is a bounded commutative diagram of metrized coherent sheaves, where all the rows $\dots(\overline{\varepsilon}_{n-1})$, $(\overline{\varepsilon}_n)$, $(\overline{\varepsilon}_{n+1})$, $\dots$ *and all the columns* $(\overline{\eta}_{m-1})$, $(\overline{\eta}_m)$, $(\overline{\eta}_{m+1})$ *are exact, then*

$$\sum_n (-1)^n \widetilde{\mathrm{ch}}(\overline{\varepsilon}_n) = \sum_m (-1)^m \widetilde{\mathrm{ch}}(\overline{\eta}_m).$$

Proof. This follows from equation (2.32) and corollary 2.19. □

We will use the notation of definition 2.13 also in the case of metrized coherent sheaves.

It is easy to verify the following result.

Proposition 2.34. *Let*

$$(\overline{\varepsilon}) \qquad \dots \longrightarrow \overline{E}_{n+1} \longrightarrow \overline{E}_n \longrightarrow \overline{E}_{n-1} \longrightarrow \dots$$

be a finite exact sequence of hermitian vector bundles. Then the Bott-Chern classes obtained by theorem 2.24 and by theorem 2.3 agree. □

Proposition 2.35. *Let* $\overline{\mathcal{F}} = (\mathcal{F}, \overline{E}_* \to \mathcal{F})$ *be a metrized coherent sheaf. We consider the exact sequence of metrized coherent sheaves*

$$\overline{\varepsilon}\colon \qquad 0 \longrightarrow \overline{E}_n \to \dots \to \overline{E}_0 \to \overline{\mathcal{F}} \to 0,$$

where, by abuse of notation, $\overline{E}_i = (E_i, \overline{E}_i \xrightarrow{=} E_i)$. *Then* $\widetilde{\mathrm{ch}}(\overline{\varepsilon}) = 0$.

Proof. Define $\mathcal{K}_i = \mathrm{Ker}(E_i \to E_{i-1})$, $i = 1, \dots, n$ and $\mathcal{K}_0 = \mathrm{Ker}(E_0 \to \mathcal{F})$. Write

$$\overline{\mathcal{K}_i} = (\mathcal{K}_i, 0 \to \overline{E}_n \to \dots \to \overline{E}_{i+1} \to \mathcal{K}_i),\ i = 0, \dots, n,$$

and $\overline{\mathcal{K}}_{-1} = \overline{\mathcal{F}}$. If we prove that

$$\widetilde{\mathrm{ch}}(0 \to \overline{\mathcal{K}_i} \to \overline{E}_i \to \overline{\mathcal{K}}_{i-1} \to 0) = 0, \tag{2.36}$$

then we obtain the result by induction using proposition 2.33. In order to prove equation (2.36) we apply equation (2.32). To this end consider resolutions

$$\begin{aligned}
\overline{D}_{0,*} &\longrightarrow \mathcal{K}_{i-1}, & \overline{D}_{0,k} &= \overline{E}_{k+i}\\
\overline{D}_{1,*} &\longrightarrow E_{i}, & \overline{D}_{1,k} &= \overline{E}_{k+i+1} \oplus \overline{E}_{k+i}\\
\overline{D}_{2,*} &\longrightarrow \mathcal{K}_{i}, & \overline{D}_{2,k} &= \overline{E}_{k+i+1}
\end{aligned}$$

with the map $D_{2,k} \overset{\Delta}{\to} D_{1,k}$ given by $s \mapsto (s, \mathrm{d}\, s)$ and the map $D_{1,k} \overset{\nabla}{\to} D_{0,k}$ given by $(s,t) \mapsto t - \mathrm{d}\, s$. The differential of the complex $D_{1,k}$ is given by $(s,t) \mapsto (t,0)$. Using equations (2.32) and (2.26) we write the left hand side of equation (2.36) in terms of Bott-Chern classes of vector bundles. All the exact sequences involved are orthogonally split except maybe the sequences

$$\overline{\lambda}_k\colon \qquad 0 \to \overline{D}_{2,k} \to \overline{D}_{1,k} \to \overline{D}_{0,k} \to 0.$$

But now we consider the diagrams

$$\begin{tikzcd}
\overline{E}_{k+i+1} \arrow[r, "i_1"] \arrow[d, "\mathrm{id}"] & \overline{E}_{k+i+1} \oplus \overline{E}_{k+i} \arrow[r, "p_2"] \arrow[d, "f"] & \overline{E}_{k+i} \arrow[d, "\mathrm{id}"]\\
\overline{E}_{k+i+1} \arrow[r, "\Delta"] & \overline{E}_{k+i+1} \oplus \overline{E}_{k+i} \arrow[r, "\nabla"] & \overline{E}_{k+i}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\overline{E}_{k+i} \arrow[r, "i_2"] \arrow[d, "\mathrm{id}"] & \overline{E}_{k+i+1} \oplus \overline{E}_{k+i} \arrow[r, "p_1"] \arrow[d, "f"] & \overline{E}_{k+i+1} \arrow[d, "\mathrm{id}"]\\
\overline{E}_{k+i} \arrow[r, "i_2"] & \overline{E}_{k+i+1} \oplus \overline{E}_{k+i} \arrow[r, "p_1"] & \overline{E}_{k+i+1}
\end{tikzcd},$$

where i_i, i_2 are the natural inclusions, p_1 and p_2 are the projections and $f(s,t) = (s, t + f(s))$. These diagrams and corollary 2.19 imply that $\widetilde{\mathrm{ch}}(\overline{\lambda}_k) = 0$. □

Remark 2.37. In [32], Zha shows that the Bott-Chern classes associated to exact sequences of metrized coherent sheaves are characterized by proposition 2.34, proposition 2.35 and proposition 2.33. We prefer the characterization in terms of the differential equation, the functoriality and the normalization, because it relies on natural extensions of the corresponding axioms that define the Bott-Chern classes for exact sequences of hermitian vector bundles. Moreover, this approach will be used in a subsequent paper where we will study singular Bott-Chern classes associated to arbitrary proper morphisms.

The following generalization of proposition 2.35 will be useful later. Let

$$\varepsilon\colon 0 \to \mathcal{G}_n \to \mathcal{G}_{n-1} \to \cdots \to \mathcal{G}_0 \to \mathcal{F} \to 0$$

be a finite resolution of a coherent sheaf by coherent sheaves. Assume that we have a commutative diagram

$$
\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & \overline{E}_{1,n} & \to & \dots & \to & \overline{E}_{1,0} & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & \overline{E}_{0,n} & \to & \dots & \to & \overline{E}_{0,0} & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \to & \overline{\mathcal{G}}_n & \to & \dots & \to & \overline{\mathcal{G}}_0 & \to & \mathcal{F} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & 0 & & & &
\end{array}
$$

where the columns are exact, the rows are complexes and the $\overline{E}_{i,j}$ are hermitian vector bundles. The columns of this diagram define metrized coherent sheaves $\overline{\mathcal{G}}_i$. Let $\overline{\mathcal{F}}$ be the metrized coherent sheaf defined by the resolution $\mathrm{Tot}(\overline{E}_{*,*}) \longrightarrow \mathcal{F}$.

Proposition 2.38. *With the notations above, let $\overline{\varepsilon}$ be the exact sequence of metrized coherent sheaves*

$$\overline{\varepsilon}\colon 0 \to \overline{\mathcal{G}}_n \to \overline{\mathcal{G}}_{n-1} \to \cdots \to \overline{\mathcal{G}}_0 \to \overline{\mathcal{F}} \to 0$$

Then $\widetilde{\mathrm{ch}}(\overline{\varepsilon}) = 0$.

Proof. For each k, let $\mathrm{Tot}_k = \mathrm{Tot}((E_{*,j})_{j\geq k})$. There are inclusions $\mathrm{Tot}_k \longrightarrow \mathrm{Tot}_{k-1}$. Let $\overline{D}_{*,j} = s(\mathrm{Tot}_{j+1} \to \mathrm{Tot}_j)$ with the hermitian metric induced by $\overline{E}_{*,*}$. There are exact sequences of complexes

$$0 \longrightarrow \overline{E}_{*,j} \longrightarrow \overline{D}_{*,j} \longrightarrow s(\mathrm{Tot}_{j+1} \to \mathrm{Tot}_{j+1}) \longrightarrow 0 \tag{2.39}$$

that are orthogonally split at each degree. The third complex is orthogonally split. Therefore, if we denote by h_E and h_D the metric structures of $\mathcal{G}_j$ induced respectively by the first and second column of diagram (2.39), then

$$\widetilde{\mathrm{ch}}(\mathcal{G}_j, h_E, h_D) = 0. \tag{2.40}$$

There is a commutative diagram of resolutions

$$
\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \overline{D}_{1,n} & \to & \dots & \to & \overline{D}_{1,0} & \to & (\mathrm{Tot}_0)_1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \overline{D}_{0,n} & \to & \dots & \to & \overline{D}_{0,0} & \to & (\mathrm{Tot}_0)_0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{G}_n & \to & \dots & \to & \mathcal{G}_0 & \to & \mathcal{F} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
$$

where the rows of degree greater or equal than zero are orthogonally split. Hence the result follows from equation (2.26), equation (2.40) and proposition 2.33. □

Remark 2.41. We have only defined the Bott-Chern classes associated to the Chern character. Everything applies without change to any additive characteristic class. The reader will find no difficulty to adapt the previous results to any multiplicative characteristic class like the Todd genus or the total Chern class.

3 Direct images of Bott-Chern classes

The aim of this section is to show that certain direct images of Bott-Chern classes are closed. This result is a generalization of results of Bismut, Gillet and Soulé [6] page 325 and of Mourougane [29] proposition 6. The fact that these direct images of Bott-Chern classes are closed implies that certain relations between characteristic classes are true at the level of differential forms (see corollary 3.7 and corollary 3.8).

In the first part of this section we deal with differential geometry. Thus all the varieties will be differentiable manifolds.

Let G_1 be a Lie group and let $\pi\colon N_2 \longrightarrow M_2$ be a principal bundle with structure group G_2 and connection ω_2. Assume that there is a left action of G_1 over N_2 that commutes with the right action of G_2 and such that the connection ω_2 is G_1-invariant.

Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be the Lie algebras of G_1 and G_2. Every element $\gamma \in \mathfrak{g}_1$ defines a tangent vector field γ^* over N_2 given by

$$\gamma^*_p = \left.\frac{d}{dt}\right|_{t=0} \exp(t\gamma)p.$$

Let $(\gamma^*)^V$ be the vertical component of γ^* with respect to the connection ω_2. For every point $p \in N_2$, we denote by $\varphi(\gamma, p) \in \mathfrak{g}_2$ the element characterized by $(\gamma^*)^V_p = \varphi(\gamma, p)^*_p$, where $\varphi(\gamma, p)^*$ is the fundamental vector field associated to $\varphi(\gamma, p)$.

The commutativity of the actions of G_1 and G_2 and the invariance of the connection ω_2 implies that, for $g \in G_1$ and $\gamma \in \mathfrak{g}_1$, the following equalities hold

$$L_{g*}(\gamma^*) = (\mathrm{ad}(g)\gamma^*), \tag{3.1}$$

$$L_{g*}(\gamma^*)^V = (\mathrm{ad}(g)\gamma^*)^V, \tag{3.2}$$

$$\varphi(\mathrm{ad}(g)\gamma, p) = \varphi(\gamma, g^{-1}p). \tag{3.3}$$

Let $\mathcal{G}_2$ be the vector bundle over M_2 associated to N_2 and the adjoint representation of G_2. That is,

$$\mathcal{G}_2 = N_2 \times \mathfrak{g}_2 \big/ \big\langle (pg, v) \sim (p, \mathrm{ad}(g)v) \big\rangle \ .$$

Thus, we can identify smooth sections of $\mathcal{G}_2$ with $\mathfrak{g}_2$-valued functions on N_2 that are invariant under the action of G_2. In this way, $\varphi(\gamma, p)$ determines a section

$$\varphi(\gamma) \in C^\infty(N_2, \mathfrak{g}_2)^{G_2} = C^\infty(M_2, \mathcal{G}_2).$$

Equation (3.3) implies that, for $g \in G_1$ and $\gamma \in \mathfrak{g}_1$,

$$\varphi(\operatorname{ad}(g)\gamma) = L^*_{g^{-1}}\varphi(\gamma).$$

We denote by Ω^{ω_2} the curvature of the connection ω_2. Let P be an invariant function on $\mathfrak{g}_2$, then $P(\Omega^{\omega_2} + \varphi(\gamma))$ is a well defined differential form on M_2.

Proposition 3.4. *Let P be an invariant function on $\mathfrak{g}_2$ and let μ be a current on M_2 invariant under the action of G_1. Then $\mu(P(\Omega^{\omega_2}+\varphi(\gamma)))$ is an invariant function on $\mathfrak{g}_1$.*

Proof. Let $g \in G_1$. Then,

$$\begin{aligned}\mu(P(\Omega^{\omega_2} + \varphi(\operatorname{ad}(g)\gamma))) &= \mu(P(\Omega^{\omega_2} + L^*_{g^{-1}}\varphi(\gamma)))\\ &= \mu(P(L^*_{g^{-1}}\Omega^{\omega_2} + L^*_{g^{-1}}\varphi(\gamma)))\\ &= L_{g^{-1}*}(\mu)(P(\Omega^{\omega_2} + \varphi(\gamma)))\\ &= \mu(P(\Omega^{\omega_2} + \varphi(\gamma)))\end{aligned}$$

□

Let now $N_1 \longrightarrow M_1$ be a principal bundle with structure group G_1 and provided with a connection ω_1. Then we can form the diagram

$$\begin{array}{ccc} N_1 \times N_2 & \xrightarrow{\pi_1} & N_1 \underset{G_1}{\times} N_2 \\ \Big\downarrow \pi' & & \Big\downarrow \pi \\ N_1 \times M_2 & \xrightarrow{\pi_2} & N_1 \underset{G_1}{\times} M_2 \\ & & \Big\downarrow q \\ & & M_1 \end{array}$$

Then π is a principal bundle with structure group G_2. The connections ω_1 and ω_2 induce a connection on the principal bundle π. The subbundle of horizontal vectors with respect to this connection is given by $\pi_{1*}(T^H N_1 \oplus T^H N_2)$. We will denote this connection by $\omega_{1,2}$. We are interested in computing the curvature $\omega_{1,2}$.

In fact, all the maps in the above diagram are fiber bundles provided with a connection. When applicable, given a vector field U in any of these spaces, we will denote by $U^{H,1}$ the horizontal lifting to $N_1 \times N_2$, by $U^{H,2}$ the horizontal lifting to $N_1 \underset{G_1}{\times} N_2$ and by $U^{H,3}$ the horizontal lifting to $N_1 \underset{G_1}{\times} M_2$.

The tangent space $T(N_1 \times N_2)$ can be decomposed as direct sum in the following ways

$$\begin{aligned} T(N_1 \times N_2) &= T^H N_1 \oplus T^V N_1 \oplus T^H N_2 \oplus T^V N_2 \\ &= T^H N_1 \oplus T^V N_1 \oplus T^H N_2 \oplus \operatorname{Ker} \pi_{1*}, \end{aligned} \tag{3.5}$$

For every point $(x,y) \in N_1 \times N_2$ we have that $(\operatorname{Ker} \pi_{1*})_{(x,y)} \subset T^V_x N_1 \oplus T_y N_2$. Moreover, there is an isomorphism $\mathfrak{g}_1 \longrightarrow (\operatorname{Ker} \pi_{1*})_{(x,y)}$ that sends an element $\gamma \in \mathfrak{g}_1$ to the element $(\gamma^*_x, -\gamma^*_y) \in T^V_x N_1 \oplus T_y N_2$.
The tangent space to $N_1 \underset{G_1}{\times} M_2$ can be decomposed as the sum of the subbundle of vertical vectors with respect to q and the subbundle of horizontal vectors defined by the connection ω_1. The horizontal lifting to $N_1 \times N_2$ of a vertical vector lies in $T^H N_2$ and the horizontal lifting of a horizontal vector lies in $T^H N_1$.
Let U, V be two vector fields on M_1 and let $U^{H,3}$, $V^{H,3}$ be the horizontal liftings to $N_1 \underset{G_1}{\times} M_2$. Then

$$\begin{aligned} \Omega^{\omega_{1,2}}(U^{H,3}, V^{H,3}) &= [U^{H,3}, V^{H,3}]^{H,2} - [U^{H,2}, V^{H,2}] \\ &= \pi_{1*}([U^{H,3}, V^{H,3}]^{H,1} - [U^{H,1}, V^{H,1}]) \\ &= \pi_{1*}([U^{H,3}, V^{H,3}]^{H,1} - [U,V]^{H,1} + [U,V]^{H,1} - [U^{H,1}, V^{H,1}]) \\ &= \pi_{1,*}([U^{H,3}, V^{H,3}]^{H,1} - [U,V]^{H,1} + \Omega^{\omega_1}(U,V)). \end{aligned}$$

But, we have

$$\begin{aligned} \Omega^{\omega_{1,2}}(U^{H,3}, V^{H,3}) &\in T^V N_2, \\ \Omega^{\omega_1}(U,V) &\in T^V N_1, \\ [U^{H,3}, V^{H,3}]^{H,1} - [U,V]^{H,1} &\in T^H N_2. \end{aligned}$$

Therefore, by the direct sum decomposition (3.5) we obtain that

$$\Omega^{\omega_{1,2}}(U^{H,3}, V^{H,3}) = ((\pi_{1*}\Omega^{\omega_1}(U,V)))^V,$$

where the vertical part is taken with respect to the fib re bundle π.
If U is a horizontal vector field over $N_1 \underset{G_1}{\times} M_2$ and V is a vertical vector field, a similar argument shows that $\Omega^{\omega_{1,2}}(U,V) = 0$. Finally, if U and V are vector fields on M_2, they determine vertical vector fields on $N_1 \underset{G_1}{\times} M_2$. Then the horizontal liftings $U^{H,1}$ and $V^{H,1}$ are induced by horizontal liftings of U and V to N_2. Therefore, reasoning as before we see that

$$\Omega^{\omega_{1,2}}(U,V) = \Omega^{\omega_2}(U,V).$$

Proposition 3.6. *Let G_1 and G_2 be Lie groups, with Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$. For $i = 1, 2$, let $N_i \longrightarrow M_i$ be a principal bundle with structure group*

G_i, provided with a connection ω_i. Assume that there is a left action of G_1 over N_2 that commutes with the right action of G_2 and that the connection ω_2 is invariant under the G_1-action. We form the G_2-principal bundle $\pi\colon N_1 \underset{G_1}{\times} N_2 \longrightarrow N_1 \underset{G_1}{\times} M_2$ with the induced connection $\omega_{1,2}$ and curvature $\Omega^{\omega_{1,2}}$. Let P be any invariant function on $\mathfrak{g}_2$. Thus $P(\Omega^{\omega_{1,2}})$ is a well defined closed differential form on $N_1 \underset{G_1}{\times} M_2$. Let μ be a current on M_2 invariant under the G_1-action. Being G_1 invariant, the current μ induces a current on $N_1 \underset{G_1}{\times} M_2$, that we denote also by μ. Let $q\colon N_1 \underset{G_1}{\times} M_2 \longrightarrow M_1$ be the projection. Then $q_*(P(\Omega^{\omega_{1,2}}) \wedge \mu)$ is a closed differential form on M_1.

Proof. Let $U \subset M_1$ be a trivializing open subset for N_1 and choose a trivialization of $N_1 \mid_U \cong U \times G_1$. With this trivialization, we can identify $\Omega^{\omega_1} \mid_U$ with a 2-form on U with values in $\mathfrak{g}_1$.
For $\gamma \in \mathfrak{g}_1$, we denote by

$$\psi_\mu(\gamma) = \mu(P(\Omega^{\omega_2} + \varphi(\gamma)))$$

the invariant function provided by proposition 3.4.
Then

$$q_*(P(\Omega^{\omega_{1,2}}) \wedge \mu) = \psi_\mu(\Omega^{\omega_1}).$$

Therefore, the result follows from the usual Chern-Weil theory. □

We go back now to complex geometry and analytic real Deligne cohomology and to the notations 1.3, in particular (1.4).

COROLLARY 3.7. *Let X be a complex manifold and let $\overline{E} = (E, h^E)$ be a rank r hermitian holomorphic vector bundle on X. Let $\pi\colon \mathbb{P}(E) \longrightarrow X$ be the associated projective bundle. On $\mathbb{P}(E)$ we consider the tautological exact sequence*

$$\overline{\xi}\colon 0 \longrightarrow \overline{\mathcal{O}(-1)} \longrightarrow \pi^*\overline{E} \longrightarrow \overline{Q} \longrightarrow 0$$

where all the vector bundles have the induced metric. Let P_1, P_2 and P_3 be invariant power series in 1, $r-1$ and r variables respectively with coefficients in $\mathbb{D}$. Let $P_1(\overline{\mathcal{O}(-1)})$ and $P_2(\overline{Q})$ be the associated Chern forms and let $\widetilde{P}_3(\overline{\xi})$ the associated Bott-Chern class. Then

$$\pi_*(P_1(\overline{\mathcal{O}(-1)}) \bullet P_2(\overline{Q}) \bullet \widetilde{P}_3(\overline{\xi})) \in \bigoplus_k \widetilde{\mathcal{D}}^{2k-1}(X,k)$$

is closed. Hence it defines a class in analytic real Deligne cohomology. This class does not depend on the hermitian metric of E.

Proof. We consider $\mathbb{C}^r$ with the standard hermitian metric. On the space $\mathbb{P}(\mathbb{C}^r)$ we have the tautological exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(\mathbb{C}^r)}(-1) \xrightarrow{f} \mathbb{C}^r \longrightarrow Q \longrightarrow 0.$$

Let $(x : y)$ be homogeneous coordinates on $\mathbb{P}^1$ and let $t = x/y$ be the absolute coordinate. Let p_1 and p_2 be the two projections of $M_2 = \mathbb{P}(\mathbb{C}^r) \times \mathbb{P}^1$. Let $\widetilde{E}$ be the cokernel of the map

$$\begin{array}{ccc} p_1^*\mathcal{O}_{\mathbb{P}(\mathbb{C}^r)}(-1) & \longrightarrow & p_1^*\mathcal{O}_{\mathbb{P}(\mathbb{C}^r)}(-1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(1) \oplus p_1^*\mathbb{C}^r \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(1) \\ s & \longmapsto & s \otimes y + f(s) \otimes x \end{array}$$

with the metric induced by the standard metric of $\mathbb{C}^r$ and the Fubini-Study metric of $\mathcal{O}_{\mathbb{P}(1)}(1)$.

Let N_2 be the principal bundle over M_2 formed by the triples (e_1, e_2, e_3), where e_1, e_2 and e_3 are unitary frames of $p_1^*\mathcal{O}_{\mathbb{P}(\mathbb{C}^r)}(-1)$, p_1^*Q and $\widetilde{E}$ respectively. The structure group of this principal bundle is $G_2 = U(1) \times U(r-1) \times U(r)$. Let ω_2 be the connection induced by the hermitian holomorphic connections on the vector bundles $p_1^*\mathcal{O}_{\mathbb{P}(\mathbb{C}^r)}(-1)$, p_1^*Q and $\widetilde{E}$.

Now we denote $M_1 = X$, and let N_1 be the bundle of unitary frames of $\overline{E}$. This is a principal bundle over M_1 with structure group $G_1 = U(r)$.

The group G_1 acts on the left on N_2. This action commutes with the right action of G_2 and the connection ω_2 is invariant under this action.

Let $\mu = [-\log(|t|)]$ be the current on M_2 associated to the locally integrable function $-\log(|t|)$. This current is invariant under the action of G_1 because this group acts trivially on the factor $\mathbb{P}^1$.

The invariant power series P_1, P_2 and P_3 determine an invariant function P on $\mathfrak{g}_2$, the Lie algebra of G_2.

Let ω_1 be the connection induced in N_1 by the holomorphic hermitian connection on $\overline{E}$. As before let $\omega_{1,2}$ be the connection on $N_1 \underset{G_1}{\times} N_2$ induced by ω_1 and ω_2 and let $q\colon N_1 \underset{G_1}{\times} M_2 \longrightarrow M_1$ be the projection. Observe that $N_1 \underset{G_1}{\times} M_2 = \mathbb{P}(E) \times \mathbb{P}^1$ and $q = \pi \circ p_1$.

By the projection formula and the definition of Bott-Chern classes we have

$$\pi_*(P_1(\overline{\mathcal{O}(-1)}) \wedge P_2(\overline{Q}) \wedge \widetilde{P}_3(\overline{\xi})) = q_*(\mu \bullet P(\Omega^{\omega_{1,2}})),$$

Therefore the fact that it is closed follows from 3.6. Since, for fixed P_1, P_2 and P_3, the construction is functorial on $(X.\overline{E})$, the fact that the class in analytic real Deligne cohomology does not depend on the choice of the hermitian metric follows from proposition 1.7. □

Corollary 3.8. *Let $\overline{E} = (E, h^E)$ be a hermitian holomorphic vector bundle on a complex manifold X. We consider the projective bundle $\pi\colon \mathbb{P}(E \oplus \mathbb{C}) \longrightarrow X$. Let $\overline{Q}$ be the universal quotient bundle on the space $\mathbb{P}(E \oplus \mathbb{C})$ with the induced metric. Then the following equality of differential forms holds*

$$\pi_* \sum_i (-1)^i \operatorname{ch}(\bigwedge^i \overline{Q}^\vee) = \pi_*(c_r(\overline{Q}) \operatorname{Td}^{-1}(\overline{Q})) = \operatorname{Td}^{-1}(\overline{E}).$$

Proof. Let $\overline{\xi}$ be the tautological exact sequence with induced metrics. We first prove that

$$\pi_*(c_r(\overline{Q})\,\mathrm{Td}(\overline{\mathcal{O}(-1)})) = 1.$$

We can write $\mathrm{Td}(\overline{\mathcal{O}(-1)}) = 1 + c_1(\overline{\mathcal{O}(-1)})\phi(\overline{\mathcal{O}(-1)})$ for certain power series ϕ. Since $c_{r+1}(\overline{E}\oplus\mathbb{C}) = 0$ we have

$$c_r(\overline{Q})c_1(\overline{\mathcal{O}(-1)}) = \mathrm{d}_{\mathcal{D}}\,\widetilde{c}_{r+1}(\overline{\xi}).$$

Therefore, by corollary 3.7, we have

$$\begin{aligned}\pi_*(c_r(\overline{Q})\,\mathrm{Td}(\overline{\mathcal{O}(-1)})) &= \pi_*(c_r(\overline{Q})) + \pi_*(c_r(\overline{Q})c_1(\overline{\mathcal{O}(-1)})\phi(\overline{\mathcal{O}(-1)}))\\ &= 1 + \mathrm{d}_{\mathcal{D}}\,\pi_*(\widetilde{c}_{r+1}(\overline{\xi})\phi(\overline{\mathcal{O}(-1)}))\\ &= 1.\end{aligned}$$

Then the corollary follows from corollary 3.7 by using the identity

$$\begin{aligned}\pi_*(c_r(\overline{Q})\,\mathrm{Td}^{-1}(\overline{Q})) = \pi_*(c_r(\overline{Q})\,\mathrm{Td}(\overline{\mathcal{O}(-1)})\pi^*\,\mathrm{Td}^{-1}(\overline{E}))\\ + \mathrm{d}_{\mathcal{D}}\,\pi_*(c_r(\overline{Q})\,\mathrm{Td}(\overline{\mathcal{O}(-1)})\widetilde{\mathrm{Td}^{-1}}(\overline{\xi})).\end{aligned}$$

□

The following generalization of corollary 3.7 provides many relations between integrals of Bott-Chern classes and is left to the reader.

Corollary 3.9. *Let X be a complex manifold and let $\overline{E} = (E, h^E)$ be a rank r hermitian holomorphic vector bundle on X. Let $\pi\colon \mathbb{P}(E) \longrightarrow X$ be the associated projective bundle. On $\mathbb{P}(E)$ we consider the tautological exact sequence*

$$\overline{\xi}\colon 0 \longrightarrow \overline{\mathcal{O}(-1)} \longrightarrow \pi^*\overline{E} \longrightarrow \overline{Q} \longrightarrow 0$$

where all the vector bundles have the induced metric. Let P_1 and P_2 be invariant power series in 1 and $r-1$ variables respectively with coefficients in $\mathbb{D}$ and let $P_3, \dots, P_k$ be invariant power series in r variables with coefficients in $\mathbb{D}$. Let $P_1(\overline{\mathcal{O}(-1)})$ and $P_2(\overline{Q})$ be the associated Chern forms and let $\widetilde{P}_3(\overline{\xi}), \dots, \widetilde{P}_k(\overline{\xi})$ be the associated Bott-Chern classes. Then

$$\pi_*(P_1(\overline{\mathcal{O}(-1)}) \bullet P_2(\overline{Q}) \bullet \widetilde{P}_3(\overline{\xi}) \bullet \dots \bullet \widetilde{P}_k(\overline{\xi}))$$

is a closed differential form on X for any choice of the ordering in computing the non associative product under the integral.

4 Cohomology of currents and wave front sets

The aim of this section is to prove the Poincaré lemma for the complex of currents with fixed wave front set. This implies in particular a certain $\partial\bar{\partial}$-lemma (corollary 4.7) that will allow us to control the singularities of singular Bott-Chern classes.

Let X be a complex manifold of dimension n. Following notation 1.3 recall that there is a canonical isomorphism

$$H^*_{\mathcal{D}^{\mathrm{an}}}(X, \mathbb{R}(p)) \cong H^*(\mathcal{D}^*_D(X, p)).$$

A current η can be viewed as a generalized section of a vector bundle and, as such, has a wave front set that is denoted by $\mathrm{WF}(\eta)$. The theory of wave front sets of distributions is developed in [25] chap. VIII. For the theory of wave front sets of generalized sections, the reader can consult [24] chap. VI. Although we will work with currents and hence with generalized sections of vector bundles, we will follow [25].

The wave front set of η is a closed conical subset of the cotangent bundle of X minus the zero section $T^*X_0 = T^*X \setminus \{0\}$. This set describes the points and directions of the singularities of η and it allows us to define certain products and inverse images of currents.

Let $S \subset T^*X_0$ be a closed conical subset, we will denote by $\mathscr{D}^*_{X,S}$ the subsheaf of currents whose wave front set is contained in S. We will denote by $D^*(X, S)$ its complex of global sections.

For every open set $U \subset X$ there is an appropriate notion of convergence in $\mathscr{D}^*_{X,S}(U)$ (see [25] VIII Definition 8.2.2). All references to continuity below are with respect to this notion of convergence.

We next summarize the basic properties of wave front sets.

Proposition 4.1. *Let u be a generalized section of a vector bundle and let P be a differential operator with smooth coefficients. Then*

$$\mathrm{WF}(Pu) \subseteq \mathrm{WF}(u).$$

Proof. This is [25] VIII (8.1.11). □

Corollary 4.2. *The sheaf $\mathscr{D}^*_{X,S}$ is closed under ∂ and $\bar{\partial}$. Therefore it is a sheaf of Dolbeault complexes.*

Let $f\colon X \longrightarrow Y$ be a morphism of complex manifolds. The *set of normal directions* of f is

$$N_f = \{(f(x), v) \in T^*Y \mid df(x)^t v = 0\}.$$

This set measures the singularities of f. For instance, if f is a smooth map then $N_f = 0$ whereas, if f is a closed immersion, N_f is the conormal bundle of $f(X)$. Let $S \subset T^*Y_0$ be a closed conical subset. We will say that f is transverse to S if $N_f \cap S = \emptyset$. We will denote

$$f^*S = \{(x, df(x)^t v) \in T^*X_0 \mid (f(x), v) \in S\}.$$

Theorem 4.3. *Let $f\colon X \longrightarrow Y$ be a morphism of complex manifolds that is transverse to S. Then there exists one and only one extension of the pull-back morphism $f^*\colon \mathscr{E}^*_Y \longrightarrow \mathscr{E}^*_X$ to a continuous morphism*

$$f^*\colon \mathscr{D}^*_{Y,S} \longrightarrow \mathscr{D}^*_{X,f^*S}.$$

In particular there is a continuous morphism of complexes

$$D^*(Y,S) \longrightarrow D^*(X,f^*S).$$

Proof. This follows from [25] theorem 8.2.4. □

We now recall the effect of correspondences on the wave front sets.
Let $K \in D^*(X \times Y)$, and let S be a conical subset of T^*Y_0. We will write

$$\begin{aligned} \mathrm{WF}(K)_X &= \{(x,\xi) \in T^*X_0 \mid \exists y \in Y, (x,y,\xi,0) \in \mathrm{WF}(K)\} \\ \mathrm{WF}'(K)_Y &= \{(y,\eta) \in T^*Y_0 \mid \exists x \in X, (x,y,0,-\eta) \in \mathrm{WF}(K)\} \\ \mathrm{WF}'(K) \circ S &= \{(x,\xi) \in T^*X_0 \mid \exists (y,\eta) \in S, (x,y,\xi,-\eta) \in \mathrm{WF}(K)\}. \end{aligned}$$

Theorem 4.4. *The image of the correspondence map*

$$\begin{array}{ccc} E_c^*(Y) & \longrightarrow & D^*(X) \\ \eta & \longmapsto & p_{1*}(K \wedge p_2^*(\eta)) \end{array}$$

is contained in $D^*(X, WF(K)_X)$. Moreover, if $S \cap \mathrm{WF}'(K)_Y = \emptyset$, then there exists one and only one extension to a continuous map

$$D_c^*(Y,S) \longrightarrow D^*(X,S'),$$

where $S' = \mathrm{WF}(K)_X \cup \mathrm{WF}'(K) \circ S$.

Proof. This is [25] theorem 8.2.13. □

We are now in a position to state and prove the Poincaré lemma for currents with fixed wave front set. As usual, we will denote by F the Hodge filtration of any Dolbeault complex.

Theorem 4.5 (Poincaré lemma). *Let S be any conical subset of T^*X_0. Then the natural morphism*

$$\iota\colon (E^*(X),F) \longrightarrow (D^*(X,S),F)$$

is a filtered quasi-isomorphism.

Proof. Let K be the Bochner-Martinelli integral operator on $\mathbb{C}^n \times \mathbb{C}^n$. It is the operator

$$\begin{array}{ccc} E_c^{p,q}(\mathbb{C}^n) & \longrightarrow & E^{p,q-1}(\mathbb{C}^n) \\ \varphi & \longmapsto & \int_{w \in \mathbb{C}^n} k(z,w) \wedge \varphi(w), \end{array}$$

where k is the Bochner-Martinelli kernel ([21] pag. 383). Thus k is a differential form on $\mathbb{C}^n \times \mathbb{C}^n$ with singularities only along the diagonal.
Using the explicit description of k in [21], it can be seen that $WF(k) = N^*\Delta_0$, the conormal bundle of the diagonal. By theorem 4.4, the operator K defines a continuous linear map from $\Gamma_c(\mathbb{C}^n, \mathscr{D}^*_{\mathbb{C}^n,S})$ to $\Gamma(\mathbb{C}^n, \mathscr{D}^*_{\mathbb{C}^n,S})$. This is the key

fact that allows us to adapt the proof of the Poincaré Lemma for arbitrary currents to the case of currents with fixed wave front set.
We will prove that the sheaf inclusion

$$(\mathscr{E}_X, F) \longrightarrow (\mathscr{D}_{X,S}, F)$$

is a filtered quasi-isomorphism. Then the theorem will follow from the fact that both are fine sheaves.
The previous statement is equivalent to the fact that, for any integer $p \geq 0$, the inclusion

$$\iota\colon \mathscr{E}_X^{p,*} \longrightarrow \mathscr{D}_{X,S}^{p,*}$$

is a quasi-isomorphism.
Let $x \in X$, since exactness can be checked at the level of stalks, we need to show that

$$\iota_x\colon \mathscr{E}_{X,x}^{p,*} \longrightarrow \mathscr{D}_{X,S,x}^{p,*}$$

is a quasi-isomorphism. let U be a coordinate neighborhood around x and let $x \in V \subset U$ be a relatively compact open subset.
Let $\rho \in C_c^\infty(U)$ be a function with compact support such that $\rho\mid_V = 1$. We define an operator

$$K\rho\colon \mathscr{D}_{X,S}^{p,q}(U) \longrightarrow \mathscr{D}_{X,S}^{p,q-1}(V).$$

If $T \in \mathscr{D}_{X,S}^{p,q}(U)$ and $\varphi \in E_c^*(V)$ is a test form, then

$$K\rho(T)(\varphi) = (-1)^{p+q} T(\rho K(\varphi)).$$

Hence, using that $\bar{\partial} K(\varphi) + K(\bar{\partial}\varphi) = \varphi$, and that $\varphi = \rho\varphi$, we have

$$(\bar{\partial} K\rho T + K\rho\bar{\partial} T + T)(\varphi) = -T(\bar{\partial}(\rho) \wedge K(\varphi)).$$

Observe that, even if the support of φ is contained in V, the support of $K(\varphi)$ can be $\mathbb{C}^n$; therefore the right hand side of the above equation may be non zero.
We compute

$$\begin{aligned} T(\bar{\partial}(\rho) \wedge K(\varphi)) &= T\left(\bar{\partial}(\rho) \wedge \int_{w\in\mathbb{C}^n} k(w,z) \wedge \varphi(w)\right) \\ &= T\left(\int_{w\in\mathbb{C}^n} \bar{\partial}(\rho) \wedge k(w,z) \wedge \varphi(w)\right). \end{aligned}$$

Since $\operatorname{supp}(\varphi) \subset V$ and $\bar{\partial}(\rho)|_V \equiv 0$, we can find a number $\epsilon > 0$ such that, if $\|z - w\| < \epsilon$, then $\bar{\partial}(\rho) \wedge k(w,z) \wedge \varphi(w) = 0$. Since the singularities of $k(w,z)$ are concentrated on the diagonal, it follows that the differential form $\bar{\partial}(\rho) \wedge k(w,z) \wedge \varphi(w)$ is smooth. Therefore, the current in V given by

$$\varphi \longmapsto T\left(\int_{w\in\mathbb{C}^n} \bar{\partial}(\rho) \wedge k(w,z) \wedge \varphi(w)\right),$$

is the current associated to the smooth differential form $T_z\left(\bar{\partial}(\rho)\wedge k(w,z)\right)$, where the subindex z means that T only acts on the z variable, being $w\in V$ a parameter. This smooth form will be denoted by $\Psi(T)$.
Summing up, we have shown that, for any current $T\in \mathscr{D}^{p,q}_{X,S}(U)$ there exists a smooth differential form $\Psi(T)\in \mathscr{E}^{p,q}_X(V)$ such that

$$T\mid_V = -\bar{\partial}K\rho T - K\rho\bar{\partial}T - \Psi(T).$$

Observe that we can not say that Ψ is a quasi-inverse of ι_x because it depends on the choice of ρ and it is not possible to choose a single ρ that can be applied to all T. Hence it is not a well defined operator at the level of stalks. Let now $T\in \mathscr{D}^{p,*}_{X,S,x}$ be closed. It is defined in some neighborhood of x, say U'. Applying the above procedure we find a smooth differential form $\Psi(T)$ defined on a relatively compact subset of U', say V', that is cohomologous to T. Hence the map induced by ι_x in cohomology is surjective. Let $\omega\in \mathscr{E}^{p,*}_{X,x}$ be closed and such that $\iota_x\omega=\bar{\partial}T$ for some $T\in \mathscr{D}^{p,*-1}_{X,S,x}$. We may assume that ω and T are defined is some neighborhood U'' of x. Then, on some relatively compact subset $V''\subset U''$, we have

$$\omega\mid_{V''} = \bar{\partial}T\mid_{V''} = -\bar{\partial}K\rho\omega - \bar{\partial}\Psi(T).$$

Since $K\rho\omega$ and $\Psi(T)$ are smooth differential forms we conclude that the map induced by ι_x in cohomology is injective. □

We will denote by $\mathcal{D}^*_D(X,S,p)$ the Deligne complex associated to $D^*(X,S)$. The following two results are direct consequences of theorem 4.5.

COROLLARY 4.6. *The inclusion $\mathcal{D}^*_D(X,S,p)\longrightarrow \mathcal{D}^*_D(X,p)$ induces an isomorphism*

$$H^*(\mathcal{D}^*_D(X,S,p))\cong H^*_{\mathcal{D}^{\mathrm{an}}}(X,\mathbb{R}(p)).$$

COROLLARY 4.7. *(i) Let $\eta\in\mathcal{D}^n_D(X,p)$ be a current such that*

$$\mathrm{d}_{\mathcal{D}}\,\eta\in\mathcal{D}^{n+1}_D(X,S,p),$$

then there is a current $a\in\mathcal{D}^{n-1}_D(X,p)$ such that $\eta+\mathrm{d}_{\mathcal{D}}\,a\in\mathcal{D}^n_D(X,S,p)$.

(ii) Let $\eta\in\mathcal{D}^n_D(X,S,p)$ be a current such that there is a current $a\in\mathcal{D}^{n-1}_D(X,p)$ with $\eta=\mathrm{d}_{\mathcal{D}}\,a$, then there is a current $b\in\mathcal{D}^{n-1}_D(X,S,p)$ such that $\eta=\mathrm{d}_{\mathcal{D}}\,b$.

□

5 DEFORMATION OF RESOLUTIONS

In this section we will recall the deformation of resolutions based on the Grassmannian graph construction of [1]. We will also recall the Koszul resolution associated to a section of a vector bundle.

The main theme is that given a bounded complex E_* of locally free sheaves (with some properties) on a complex manifold X, one can construct a bounded complex $\operatorname{tr}_1(E_*)_*$ over a certain manifold W. This new manifold has a birational map $\pi\colon W \longrightarrow X\times\mathbb{P}^1$, that is an isomorphism over $X\times\mathbb{P}^1\setminus\{\infty\}$. The complex $\operatorname{tr}_1(E_*)_*$ agrees with the original complex over $X\times\{0\}$ and is particularly simple over $\pi^{-1}(X\times\{\infty\})$. Thus $\operatorname{tr}_1(E_*)_*$ is a deformation of the original complex to a simpler one. The two examples we are interested in are: first, when the original complex is exact, then W agrees with $X\times\mathbb{P}^1$ and $\operatorname{tr}_1(E_*)_*$ was defined in 2.5. Its restriction to $\pi^{-1}(X\times\{\infty\})$ is split; second, when $i\colon Y\longrightarrow X$ is a closed immersion of complex manifolds, and E_* is a bounded resolution of $i_*\mathcal{O}_Y$, then W agrees with the deformation to the normal cone of Y and the restriction of $\operatorname{tr}_1(E_*)_*$ to $\pi^{-1}(X\times\{\infty\})$ is an extension of a Koszul resolution by a split complex. Note that, if we allow singularities, then the Grassmannian graph construction is much more general.

The deformation of resolutions is based on the Grassmannian graph construction of [1], and, in the form that we present here, has been developed in [6] and [20].

In order to fix notations we first recall the deformation to the normal cone and the Koszul resolution associated to the zero section of a vector bundle.

Let $Y\hookrightarrow X$ be a closed immersion of complex manifolds, with Y of pure codimension n. In the sequel we will use notation 2.2. Let $W=W_{Y/X}$ be the blow-up of $X\times\mathbb{P}^1$ along $Y\times\{\infty\}$. Since Y and $X\times\mathbb{P}^1$ are manifolds, W is also a manifold. The map $\pi\colon W\longrightarrow X\times\mathbb{P}^1$ is an isomorphism away from $Y\times\{\infty\}$; we will write P for the exceptional divisor of the blow-up. Then

$$P=\mathbb{P}(N_{Y/X}\otimes N^{-1}_{\infty/\mathbb{P}^1}\oplus\mathbb{C}).$$

Thus P can be seen as the projective completion of the vector bundle $N_{Y/X}\otimes N^{-1}_{\infty/\mathbb{P}^1}$. Note that $N_{\infty/\mathbb{P}^1}$ is trivial although not canonically trivial. Nevertheless we can choose to trivialize it by means of the section $y\in\mathcal{O}_{\mathbb{P}^1}(1)$. Sometimes we will tacitly assume this trivialization and omit $N_{\infty/\mathbb{P}^1}$ from the formulae.

The map $q_W\colon W\longrightarrow\mathbb{P}^1$, obtained by composing π with the projection $q\colon X\times\mathbb{P}^1\longrightarrow\mathbb{P}^1$, is flat and, for $t\in\mathbb{P}^1$, we have

$$q_W^{-1}(t)\cong\begin{cases}X\times\{t\}, & \text{if } t\neq\infty,\\ P\cup\widetilde{X}, & \text{if } t=\infty,\end{cases}$$

where $\widetilde{X}$ is the blow-up of X along Y, and $P\cap\widetilde{X}$ is, at the same time, the divisor at ∞ of P and the exceptional divisor of $\widetilde{X}$.

Following [6] we will use the following notations

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\pi_P"'] & W \arrow[d, "\pi"] \\ Y\times\{\infty\} \arrow[r, "i_\infty"] & X\times\mathbb{P}^1 \end{tikzcd}$$

$$\begin{aligned}
i\colon Y &\longrightarrow X, && \\
W_\infty &= \pi^{-1}(\infty) = P \cup \widetilde{X}, && \\
q\colon X \times \mathbb{P}^1 &\longrightarrow \mathbb{P}^1, && \text{the projection,}\\
p\colon X \times \mathbb{P}^1 &\longrightarrow X, && \text{the projection,}\\
q_W &= q \circ \pi && \\
p_W &= p \circ \pi && \\
q_Y\colon Y \times \mathbb{P}^1 &\longrightarrow \mathbb{P}^1, && \text{the projection,}\\
p_Y\colon Y \times \mathbb{P}^1 &\longrightarrow Y, && \text{the projection,}\\
j\colon Y \times \mathbb{P}^1 &\longrightarrow W && \text{the induced map,}\\
j_\infty\colon Y \times \{\infty\} &\longrightarrow P. &&
\end{aligned}$$

Given any map $g\colon Z \longrightarrow X \times \mathbb{P}^1$, we will denote $p_Z = p \circ g$ and $q_Z = q \circ g$. For instance $p_P = p \circ \pi \circ f = p_W \circ f = i \circ \pi_P$, where, in the last equality, we are identifying Y with $Y \times \{\infty\}$.
We next recall the construction of the Koszul resolution. Let Y be a complex manifold and let N be a rank n vector bundle. Let $P = \mathbb{P}(N \oplus \mathbb{C})$ be the projective bundle of lines in $N \oplus \mathbb{C}$. It is obtained by completing N with the divisor at infinity. Let $\pi_P\colon P \longrightarrow Y$ be the projection and let $s\colon Y \longrightarrow P$ be the zero section. On P there is a tautological short exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \pi_P^*(N \oplus \mathbb{C}) \longrightarrow Q \longrightarrow 0. \tag{5.1}$$

The above exact sequence and the inclusion $\mathbb{C} \longrightarrow \pi_P^*(N \oplus \mathbb{C})$ induce a section $\sigma\colon \mathcal{O}_P \longrightarrow Q$ that vanishes along the zero section $s(Y)$. By duality we obtain a morphism $Q^\vee \longrightarrow \mathcal{O}_P$ that induces a long exact sequence

$$0 \longrightarrow \bigwedge^n Q^\vee \longrightarrow \dots \longrightarrow \bigwedge^1 Q^\vee \longrightarrow \mathcal{O}_P \longrightarrow s_*\mathcal{O}_Y \longrightarrow 0.$$

If F is another vector bundle over Y, we obtain an exact sequence,

$$0 \longrightarrow \bigwedge^n Q^\vee \otimes \pi_P^* F \longrightarrow \dots \longrightarrow \bigwedge^1 Q^\vee \otimes \pi_P^* F \longrightarrow \pi_P^* F \longrightarrow s_* F \longrightarrow 0. \tag{5.2}$$

Definition 5.3. The *Koszul resolution* of $s_*(F)$ is the resolution (5.2). The complex

$$0 \longrightarrow \bigwedge^n Q^\vee \otimes \pi_P^* F \longrightarrow \dots \longrightarrow \bigwedge^1 Q^\vee \otimes \pi_P^* F \longrightarrow \pi_P^* F \longrightarrow 0$$

will be denoted by $K(F, N)$. When $\overline{N}$ is a hermitian vector bundle, the exact sequence (5.1) induces a hermitian metric on Q. If, moreover, $\overline{F}$ is also a hermitian vector bundle, all the vector bundles that appear in the Koszul resolution have an induced hermitian metric. We will denote by $K(\overline{F}, \overline{N})$ the corresponding complex of hermitian vector bundles.

In particular, we shall write $K(\overline{\mathcal{O}_Y}, \overline{N})$ if $F = \mathcal{O}_Y$ is endowed with the trivial metric $\|1\| = 1$, unless expressly stated otherwise.

We finish this section by recalling the results about deformation of resolutions that will be used in the sequel. For more details see [1] II.1, [6] Section 4 (c) and [20] Section 1.

Theorem 5.4. *Let $i : Y \hookrightarrow X$ be a closed immersion of complex manifolds, where Y may be empty. Let $U = X \setminus Y$. Let F be a vector bundle over Y and $E_* \longrightarrow i_*F \longrightarrow 0$ be a resolution of i_*F. Then there exists a complex manifold $W = W(E_*)$, called the Grassmannian graph construction, with a birational map $\pi\colon W \longrightarrow X \times \mathbb{P}^1$ and a complex of vector bundles, $\operatorname{tr}_1(E_*)_*$, over W such that*

(i) The map π is an isomorphism away from $Y \times \{\infty\}$. The restriction of $\operatorname{tr}_1(E_*)_*$ to $X \times (\mathbb{P}^1 \setminus \{\infty\})$ is isomorphic to $p_W^* E_*$ restricted to $X \times (\mathbb{P}^1 \setminus \{\infty\})$. Moreover, If $\widetilde{X}$ is the Zariski closure of $U \times \{\infty\}$ inside W, the restriction of $\operatorname{tr}_1(E_*)_*$ to $\widetilde{X}$ is split acyclic. In particular, if Y is empty or F is the zero vector bundle, hence E_* is acyclic in the whole X, then $W = X \times \mathbb{P}^1$ and $\operatorname{tr}_1(E_*)_*$ is the first transgression exact sequence introduced in 2.5.

(ii) When Y is non-empty and F is a non-zero vector bundle over Y, then $W(E_*)$ agrees with $W_{Y/X}$, the deformation to the normal cone of Y. Moreover, there is an exact sequence of resolutions on P

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_* & \longrightarrow & \operatorname{tr}_1(E_*)_* \mid_P & \longrightarrow & K(F, N_{Y/X} \otimes N^{-1}_{\infty/\mathbb{P}^1}) & \longrightarrow & 0\,, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & (j_\infty)_*F & \xrightarrow{=} & (j_\infty)_*F & &
\end{array}$$

where A_* is split acyclic and $K(F, N_{Y/X} \otimes N^{-1}_{\infty/\mathbb{P}^1})$ is the Koszul resolution.

(iii) Let $f\colon X' \longrightarrow X$ be a morphism of complex manifolds and assume that we are in one of the following cases:

(a) The map f is smooth.

(b) The map f is arbitrary and E_* is acyclic.

(c) f is transverse to Y.

Then $E'_* := f^*(E_*)$ is exact over $f^{-1}(U)$,

$$W' := W(E'_*) = W \underset{X}{\times} X',$$

with $f_W\colon W' \longrightarrow W$ the induced map, and we have $f_W^*(\operatorname{tr}_1(E_*)_*) = \operatorname{tr}_1(f^*(E_*))_*$.

(iv) *If the vector bundles E_i are provided with hermitian metrics, then one can choose a hermitian metric on* $\mathrm{tr}_1(E_*)_*$ *such that its restriction to* $X \times \{0\}$ *is isometric to* E_* *and the restriction to* $U \times \{\infty\}$ *is orthogonally split. We will denote by* $\mathrm{tr}_1(\overline{E}_*)_*$ *the complex* $\mathrm{tr}_1(E_*)_*$ *with such a choice of hermitian metrics. Moreover, this choice of metrics can be made functorial. That is, if f is a map as in item (iii), then*

$$f_W^*(\mathrm{tr}_1(\overline{E}_*)_*) = \mathrm{tr}_1(f^*(\overline{E}_*))_*$$

Proof. The case when E_* is acyclic has already been treated. For the case when Y is non-empty and F is non zero, we first recall the construction of the Grassmannian graph of an arbitrary complex from [20], which is more general than what we need here. If E is a vector bundle over X we will denote by $E(i)$ the vector bundle over $X \times \mathbb{P}^1$ given by $E(i) = p^*E \otimes q^*\mathcal{O}(i)$.

Let $\widetilde{C}_*$ be the complex of locally free sheaves given by $\widetilde{C}_i = E_i(i) \oplus E_{i-1}(i-1)$ with differential given by $\mathrm{d}(a,b) = (b,0)$. On $X \times (\mathbb{P}^1 \setminus \{\infty\})$ we consider, for each i, the inclusion of vector bundles $\gamma_i \colon E_i \hookrightarrow \widetilde{C}_i$ given by $s \longmapsto (s \otimes y^i, \mathrm{d}\, s \otimes y^{i-1})$. Let G be the product of the Grassmann bundles $Gr(n_i, \widetilde{C}_i)$ that parametrize rank $n_i = \mathrm{rk}\, E_i$ subbundles of $\widetilde{C}_i$ over $X \times \mathbb{P}^1$. The inclusion $\gamma_* \colon \bigoplus E_i \longrightarrow \bigoplus \widetilde{C}_i$ induces a section s of G over $X \times \mathbb{A}^1$.

Then $W(E_*)$ is defined to be the closure of $s(X \times \mathbb{A}^1)$ in G. Since the projection from G to $X \times \mathbb{P}^1$ is proper, the same is true for the induced map $\pi \colon W \longrightarrow X \times \mathbb{P}^1$. For each i, the induced map $W \longrightarrow Gr(n_i, \widetilde{C}_i)$ defines a subbundle $\mathrm{tr}_1(E_*)_i$ of $\pi^*\widetilde{C}_i$. This subbundle agrees with E_i over $X \times \mathbb{A}^1$. The differential of $\widetilde{C}_*$ induces a differential on $\mathrm{tr}_1(E_*)_*$.

Assume now that the bundles E_i are provided with hermitian metrics. Using the Fubini-Study metric of $\mathcal{O}(1)$ we obtain induced metrics on $\widetilde{C}_i$. Over $\pi^{-1}(X \times (\mathbb{P}^1 \setminus \{\infty\}))$ we induce a metric on $\mathrm{tr}_1(E_*)_i$ by means of the identification with E_i. Over $\pi^{-1}(X \times (\mathbb{P}^1 \setminus \{0\}))$ we consider on $\mathrm{tr}_1(E_*)_i$ the metric induced by $\widetilde{C}_i$. We glue together both metrics with the partition of unity $\{\sigma_0, \sigma_\infty\}$ of notation 2.2.

In the case we are interested there is a more explicit description of $\mathrm{tr}_1(E_*)_*$ given in [6] Section 4 (c). Namely, $\mathrm{tr}_1(E_*)_i$ is the kernel of the morphism

$$\phi \colon p_W^*\widetilde{C}_i = p_W^*E_i(i) \oplus p_W^*E_{i-1}(i-1) \longrightarrow p_W^*E_{i-1}(i) \oplus p_W^*E_{i-2}(i-1) \quad (5.5)$$

given by $\phi(s,t) = (\mathrm{d}\, s - t \otimes y, \mathrm{d}\, t)$.

The only statements that are not explicitly proved in [6] Section 4 (c) or [20] Section 1 are the functoriality when f is not smooth and the properties of the explicit choice of metrics.

If the complex E_* is acyclic, then the same is true for $E'_* = f^*E_*$. In this case $W = X \times \mathbb{P}^1$ and $W' = X' \times \mathbb{P}^1$. Then the functoriality follows from the definition of $\mathrm{tr}_1(E_*)_*$.

Assume now that we are in case (iii)c. We can form the Cartesian square

$$\begin{tikzcd} Y' \arrow[r, "i'"] \arrow[d, "g"'] & X' \arrow[d, "f"] \\ Y \arrow[r, "i"] & X \end{tikzcd}$$

where i' is also a closed immersion of complex manifolds. Then we have that E'_* is a resolution of $i'_* g^* F$. Hence $W' = W(E'_*)$ is the deformation to the normal cone of Y' and therefore $W' = W \underset{X}{\times} X'$. Again the functoriality of $\operatorname{tr}_1(E_*)_*$ can be checked using the explicit construction of [20] Section 1 that we have recalled above. $\square$

Remark 5.6. (i) The definition of $\operatorname{tr}_1(E_*)$ can be extended to any bounded chain complex over a integral scheme (see [20]).

(ii) There is a sign difference in the definition of the inclusion γ used in [20] and the one used in [6]. We have followed the signs of the first reference.

6 Singular Bott-Chern classes

Throughout this section we will use notation 1.3. In particular we will write

$$\begin{aligned} \widetilde{\mathcal{D}}^n_D(X,p) &= \mathcal{D}^n_D(X,p)/\operatorname{d}_{\mathcal{D}} \mathcal{D}^{n-1}_D(X,p), \\ \widetilde{\mathcal{D}}^n_D(X,S,p) &= \mathcal{D}^n_D(X,S,p)/\operatorname{d}_{\mathcal{D}} \mathcal{D}^{n-1}_D(X,S,p). \end{aligned}$$

A particularly important current is $W_1 \in \mathcal{D}^1_D(\mathbb{P}^1,1)$ given by

$$W_1 = [\frac{-1}{2}\log\|t\|^2]. \tag{6.1}$$

With the above convention, this means that

$$W_1(\eta) = \frac{1}{2\pi i}\int_{\mathbb{P}^1}\frac{-1}{2}\log\|t\|^2 \bullet \eta. \tag{6.2}$$

By the Poincaré-Lelong equation

$$\operatorname{d}_{\mathcal{D}} W_1 = \delta_\infty - \delta_0. \tag{6.3}$$

Note that the current W_1 was used in the construction of Bott-Chern classes (definition 2.11) and will also have a role in the definition of singular Bott-Chern classes.
Before defining singular Bott-Chern classes we need to define the objects that give rise to them.

DEFINITION 6.4. Let $i\colon Y \longrightarrow X$ be a closed immersion of complex manifolds. Let N be the normal bundle of Y and let h_N be a hermitian metric on N. We denote $\overline{N} = (N, h_N)$. Let r_N be the rank of N, that agrees with the codimension of Y in X. Let $\overline{F} = (F, h_F)$ be a hermitian vector bundle on Y of rank r_F. Let $\overline{E}_* \to i_*F$ be a metric on the coherent sheaf i_*F. The four-tuple

$$\overline{\xi} = (i, \overline{N}, \overline{F}, \overline{E}_*). \tag{6.5}$$

is called a *hermitian embedded vector bundle*. The number r_F will be called the *rank* of $\overline{\xi}$ and the number r_N will be called the *codimension* of $\overline{\xi}$.
By convention, any exact complex of hermitian vector bundles on X will be considered a hermitian embedded vector bundle of any rank and codimension.

Obviously, to any hermitian embedded vector bundle we can associate the metrized coherent sheaf $(i_*F, \overline{E}_* \to i_*F)$.

DEFINITION 6.6. A *singular Bott-Chern class* for a hermitian embedded vector bundle $\overline{\xi}$ is a class $\widetilde{\eta} \in \bigoplus_p \widetilde{\mathcal{D}}_D^{2p-1}(X, p)$ such that

$$\mathrm{d}_{\mathcal{D}}\, \eta = \sum_{i=0}^{n} (-1)^i [\mathrm{ch}(\overline{E}_i)] - i_*([\mathrm{Td}^{-1}(\overline{N})\, \mathrm{ch}(\overline{F})]) \tag{6.7}$$

for any current $\eta \in \tilde{\eta}$.

The existence of this class is guaranteed by the Grothendieck-Riemann-Roch theorem, which implies that the two currents in the right hand side of equation (6.7) are cohomologous.
Even if we have defined singular Bott-Chern classes as classes of currents with arbitrary singularities, it is an important observation that in each singular Bott-Chern class we can find representatives with controlled singularities. Let $N^*_{Y,0}$ be the conormal bundle of Y with the zero section deleted. It is a closed conical subset of $T^*_0(X)$. Since the current

$$\sum_{i=0}^{n} (-1)^i [\mathrm{ch}(\overline{E}_i)] - i_*([\mathrm{Td}^{-1}(\overline{N})\, \mathrm{ch}(\overline{F})]) = \sum_{i=0}^{n} (-1)^i [\mathrm{ch}(\overline{E}_i)] - \mathrm{Td}^{-1}(\overline{N})\, \mathrm{ch}(\overline{F})\delta_Y$$

belongs to $\mathcal{D}^*_D(X, N^*_{Y,0}, p)$, by corollary 4.7, we obtain

PROPOSITION 6.8. *Let $\overline{\xi} = (i, \overline{N}, \overline{F}, \overline{E}_*)$ be a hermitian embedded vector bundle as before. Then any singular Bott-Chern class for $\overline{\xi}$ belongs to the subset*

$$\bigoplus_p \widetilde{\mathcal{D}}_D^{2p-1}(X, N^*_{Y,0}, p) \subset \bigoplus_p \widetilde{\mathcal{D}}_D^{2p-1}(X, p).$$

□

This result will allow us to define inverse images of singular Bott-Chern classes for certain maps.
Let $f\colon X' \longrightarrow X$ be a morphism of complex manifolds that is transverse to Y. We form the Cartesian square

$$\begin{tikzcd} Y' \arrow[r, "i'"] \arrow[d, "g"] & X' \arrow[d, "f"] \\ Y \arrow[r, "i"] & X \end{tikzcd} .$$

Observe that, by the transversality hypothesis, the normal bundle to Y' on X' is the inverse image of the normal bundle to Y on X and $f^{*}E_{*}$ is a resolution of $i'_{*}g^{*}F$. Thus we write $f^{*}\overline{\xi} = (i', f^{*}\overline{N}, g^{*}\overline{F}, f^{*}\overline{E}_{*})$, which is a hermitian embedded vector bundle.
By proposition 6.8, given any singular Bott-Chern class $\widetilde{\eta}$ for ξ, we can find a representative $\eta \in \bigoplus_{p} \mathcal{D}_{D}^{2p-1}(X, N^{*}_{Y,0}, p)$. By theorem 4.3, there is a well defined current $f^{*}\eta$ and it is a singular Bott-Chern current for $f^{*}\xi$. Therefore we can define $f^{*}(\widetilde{\eta}) = \widetilde{f^{*}(\eta)}$. Again by theorem 4.3, this class does not depend on the choice of the representative η.
Our next objective is to study the possible definitions of functorial singular Bott-Chern classes.

Definition 6.9. Let r_F and r_N be two integers. A *theory of singular Bott-Chern classes of rank r_F and codimension r_N* is an assignment which, to each hermitian embedded vector bundle $\overline{\xi} = (i\colon Y \longrightarrow X, \overline{N}, \overline{F}, \overline{E}_{*})$ of rank r_F and codimension r_N, assigns a class of currents

$$T(\overline{\xi}) \in \bigoplus_{p} \widetilde{\mathcal{D}}_{D}^{2p-1}(X, p)$$

satisfying the following properties

(i) (Differential equation) The following equality holds

$$\mathrm{d}_{\mathcal{D}} T(\overline{\xi}) = \sum_{i} (-1)^{i} [\mathrm{ch}(\overline{E}_{i})] - i_{*}([\mathrm{Td}^{-1}(\overline{N}) \,\mathrm{ch}(\overline{F})]). \tag{6.10}$$

(ii) (Functoriality) For every morphism $f\colon X' \longrightarrow X$ of complex manifolds that is transverse to Y, then

$$f^{*}T(\overline{\xi}) = T(f^{*}\overline{\xi}).$$

(iii) (Normalization) Let $\overline{A} = (A_{*}, g_{*})$ be a non-negatively graded orthogonally split complex of vector bundles. Write $\overline{\xi} \oplus \overline{A} = (i\colon Y \longrightarrow X, \overline{N}, \overline{F}, \overline{E}_{*} \oplus \overline{A}_{*})$. Then $T(\overline{\xi}) = T(\overline{\xi} \oplus \overline{A})$. Moreover, if $X = \operatorname{Spec}\mathbb{C}$ is one point, $Y = \emptyset$ and $\overline{E}_{*} = 0$, then $T(\overline{\xi}) = 0$.

A *theory of singular Bott-Chern classes* is an assignment as before, for all positive integers r_F and r_M. When the inclusion i and the bundles F and N are clear from the context, we will denote $T(\overline{\xi})$ by $T(\overline{E}_*)$. Sometimes we will have to restrict ourselves to complex algebraic manifolds and algebraic vector bundles. In this case we will talk of *theory of singular Bott-Chern classes for algebraic vector bundles*.

Remark 6.11. (i) Recall that the case when $Y = \emptyset$ and $\overline{E}_*$ is any bounded exact sequence of hermitian vector bundles is considered a hermitian embedded vector bundle of arbitrary rank. In this case, the properties above imply that

$$T(\overline{\xi}) = [\widetilde{\mathrm{ch}}(\overline{E}_*)],$$

where $\widetilde{\mathrm{ch}}$ is the Bott-Chern class associated to the Chern character. That is, for acyclic complexes, any theory of singular Bott-Chern classes agrees with the Bott-Chern classes associated to the Chern character.

(ii) If the map f is transverse to Y, then either $f^{-1}(Y)$ is empty or it has the same codimension as Y. Moreover, it is clear that f^*F has the same rank as F. Therefore, the properties of singular Bott-Chern classes do not mix rank or codimension. This is why we have defined singular Bott-Chern classes for a particular rank and codimension.

(iii) By contrast with the case of Bott-Chern classes, the properties above are not enough to characterize singular Bott-Chern classes.

For the rest of this section we will assume the existence of a theory of singular Bott-Chern classes and we will obtain some consequences of the definition. We start with the compatibility of singular Bott-Chern classes with exact sequences and Bott-Chern classes.

Let

$$\overline{\chi}\colon 0 \longrightarrow \overline{F}_n \longrightarrow \dots \longrightarrow \overline{F}_1 \longrightarrow \overline{F}_0 \longrightarrow 0 \tag{6.12}$$

be a bounded exact sequence of hermitian vector bundles on Y. For $j = 0, \dots, n$, let $\overline{E}_{j,*} \longrightarrow i_*F_j$ be a resolution, and assume that they fit in a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \overline{E}_{n,*} \arrow[r] \arrow[d] & \cdots \arrow[r] & \overline{E}_{1,*} \arrow[r] \arrow[d] & \overline{E}_{0,*} \arrow[r] \arrow[d] & 0\ , \\ 0 \arrow[r] & i_*F_n \arrow[r] & \cdots \arrow[r] & i_*F_1 \arrow[r] & i_*F_0 \arrow[r] & 0 \end{tikzcd}$$

with exact rows. We write $\overline{\xi_j} = (i\colon Y \longrightarrow X, \overline{N}, \overline{F}_j, \overline{E}_{j,*})$. For each k, we denote by $\overline{\eta}_k$ the exact sequence

$$0 \longrightarrow \overline{E}_{n,k} \longrightarrow \dots \longrightarrow \overline{E}_{1,k} \longrightarrow \overline{E}_{0,k} \longrightarrow 0.$$

Proposition 6.13. *With the above notations, the following equation holds:*

$$T(\bigoplus_{j\ even}\overline{\xi}_j)-T(\bigoplus_{j\ odd}\overline{\xi}_j)=\sum_k(-1)^k[\widetilde{\mathrm{ch}}(\overline{\eta}_k)]-i_*([\mathrm{Td}^{-1}(\overline{N})\widetilde{\mathrm{ch}}(\overline{\chi})]).$$

Here the direct sum of hermitian embedded vector bundles, involving the same embedding and the same hermitian normal bundle, is defined in the obvious manner.

Proof. We consider the construction of theorem 5.4 for each of the exact sequences $\overline{\eta}_k$ and the exact sequence $\overline{\chi}$. For each k, we have $W_X:=W(\overline{\eta}_k)=X\times\mathbb{P}^1$ and we denote $W_Y:=W(\overline{\chi})=Y\times\mathbb{P}^1$. On W_Y we consider the transgression exact sequence $\mathrm{tr}_1(\overline{\chi})_*$ and on W_X we consider the transgression exact sequences $\mathrm{tr}_1(\overline{\eta_k})_*$. We denote by $j\colon W_Y\longrightarrow W_X$ the induced morphism. Then there is an exact sequence (of exact sequences)

$$\dots\longrightarrow\mathrm{tr}_1(\overline{\eta}_1)_*\longrightarrow\mathrm{tr}_1(\overline{\eta}_0)_*\longrightarrow j_*\,\mathrm{tr}_1(\overline{\chi})_*\longrightarrow 0.$$

We denote

$$\begin{aligned}\mathrm{tr}_1(\overline{\chi})_+&=\bigoplus_{j\text{ even}}\mathrm{tr}_1(\overline{\chi})_j, &\mathrm{tr}_1(\overline{\chi})_-&=\bigoplus_{j\text{ odd}}\mathrm{tr}_1(\overline{\chi})_j,\\ \mathrm{tr}_1(\overline{\eta}_k)_+&=\bigoplus_{j\text{ even}}\mathrm{tr}_1(\overline{\eta}_k)_j, &\mathrm{tr}_1(\overline{\eta}_k)_-&=\bigoplus_{j\text{ odd}}\mathrm{tr}_1(\overline{\eta}_k)_j,\end{aligned}$$

and

$$\begin{aligned}\mathrm{tr}_1(\overline{\xi})_+&=(j\colon W_Y\longrightarrow W_X,p_Y^*\overline{N},\mathrm{tr}_1(\overline{\chi})_+,\mathrm{tr}_1(\overline{\eta}_*)_+),\\ \mathrm{tr}_1(\overline{\xi})_-&=(j\colon W_Y\longrightarrow W_X,p_Y^*\overline{N},\mathrm{tr}_1(\overline{\chi})_-,\mathrm{tr}_1(\overline{\eta}_*)_-),\end{aligned}$$

where here $p_Y\colon W_Y\longrightarrow Y$ denotes the projection.
We consider the current on $X\times\mathbb{P}^1$ given by $W_1\bullet\big(T(\mathrm{tr}_1(\overline{\xi})_+)-T(\mathrm{tr}_1(\overline{\xi})_-)\big)$. This current is well defined because the wave front set of W_1 is the conormal bundle of $(X\times\{0\})\cup(X\times\{\infty\})$, whereas the wave front set of $T(\mathrm{tr}_1(\overline{\xi})_\pm)$ is the conormal bundle of $Y\times\mathbb{P}^1$.
By the functoriality of the transgression exact sequences, we obtain that

$$\mathrm{tr}_1(\overline{\xi})_+\mid_{X\times\{0\}}=\bigoplus_{j\text{ even}}\overline{\xi}_j,\quad \mathrm{tr}_1(\overline{\xi})_-\mid_{X\times\{0\}}=\bigoplus_{j\text{ odd}}\overline{\xi}_j.$$

Moreover, using the fact that, for any bounded acyclic complex of hermitian vector bundles $\overline{E}_*$, the exact sequence $\mathrm{tr}_1(\overline{E}_*)\mid_{X\times\{\infty\}}$ is orthogonally split, we have an isometry

$$\mathrm{tr}_1(\overline{\xi})_+\mid_{X\times\{\infty\}}\cong\mathrm{tr}_1(\overline{\xi})_-\mid_{X\times\{\infty\}}.$$

We now denote by $p_X\colon W_X\longrightarrow X$ the projection. Using the properties that define a theory of singular Bott-Chern classes, in the group $\bigoplus_p\widetilde{\mathcal{D}}^{2p-1}_D(X,N^*_{Y,0},p)$, the following holds

$$
\begin{aligned}
0 &= \mathrm{d}_{\mathcal{D}}(p_X)_*\left(W_1 \bullet T(\mathrm{tr}_1(\overline{\xi})_+) - W_1 \bullet T(\mathrm{tr}_1(\overline{\xi})_-)\right)\\
&= \left(T(\mathrm{tr}_1(\overline{\xi})_+) - T(\mathrm{tr}_1(\overline{\xi})_-)\right)|_{X\times\{\infty\}} - \left(T(\mathrm{tr}_1(\overline{\xi})_+) - T(\mathrm{tr}_1(\overline{\xi})_-)\right)|_{X\times\{0\}}\\
&- (p_X)_*\sum_k(-1)^k W_1 \bullet (\mathrm{ch}(\mathrm{tr}_1(\overline{\eta}_k)_+) - \mathrm{ch}(\mathrm{tr}_1(\overline{\eta}_k)_-))\\
&+ (p_X)_*\left(W_1 \bullet j_*\left[\mathrm{Td}^{-1}(p_Y^*\overline{N})\,\mathrm{ch}(\mathrm{tr}_1(\overline{\chi})_+) - \mathrm{Td}^{-1}(p_Y^*\overline{N})\,\mathrm{ch}(\mathrm{tr}_1(\overline{\chi})_-)\right]\right)\\
&= -T(\bigoplus_{j \text{ even}} \overline{\xi}_j) + T(\bigoplus_{j \text{ odd}} \overline{\xi}_j) + \sum(-1)^k[\widetilde{\mathrm{ch}}(\overline{\eta}_k)] - i_*[\mathrm{Td}^{-1}(\overline{N}) \bullet \widetilde{\mathrm{ch}}(\overline{\chi})],
\end{aligned}
$$

which implies the proposition. □

The following result is a consequence of proposition 6.13 and theorem 2.24.

Corollary 6.14. *Let $Y \longrightarrow X$ be a closed immersion of complex manifolds. Let $\overline{\chi}$ be an exact sequence of hermitian vector bundles on Y as* (6.12). *For each j, let $\xi_j = (i\colon Y \longrightarrow X, \overline{N}, \overline{F}_j, \overline{E}_{j,*})$ be a hermitian embedded vector bundle. We denote by $\overline{\varepsilon}$ the induced exact sequence of metrized coherent sheaves. Then*

$$
T(\bigoplus_{j \text{ even}} \overline{\xi}_j) - T(\bigoplus_{j \text{ odd}} \overline{\xi}_j) = [\widetilde{\mathrm{ch}}(\overline{\varepsilon})] - i_*([\mathrm{Td}^{-1}(\overline{N})\widetilde{\mathrm{ch}}(\overline{\chi})]).
$$

□

We now study the effect of changing the metric of the normal bundle N.

Proposition 6.15. *Let $\overline{\xi}_0 = (i, \overline{N}_0, \overline{F}, \overline{E}_*)$ be a hermitian embedded vector bundle, where $\overline{N}_0 = (N, h_0)$. Let h_1 be another metric in the vector bundle N and write $\overline{N}_1 = (N, h_1)$, $\overline{\xi}_1 = (i, \overline{N}_1, \overline{F}, \overline{E}_*)$. Then*

$$
T(\overline{\xi}_0) - T(\overline{\xi}_1) = -i_*[\widetilde{\mathrm{Td}^{-1}}(N, h_0, h_1)\,\mathrm{ch}(\overline{F})].
$$

Proof. The proof is completely analogous to the proof of proposition 6.13. □

We now study the case when Y is the zero section of a completed vector bundle. Let $\overline{F}$ and $\overline{N}$ be hermitian vector bundles over Y. We denote $P = \mathbb{P}(N \oplus \mathbb{C})$, the projective bundle of lines in $N \oplus \mathcal{O}_Y$. Let $s\colon Y \longrightarrow P$ denote the zero section and let $\pi_P\colon P \longrightarrow Y$ denote the projection. Let $K(\overline{F}, \overline{N})$ be the Koszul resolution of definition 5.3. We will use the notations before this definition. The following result is due to Bismut, Gillet and Soulé for the particular choice of singular Bott-Chern classes defined in [6].

Theorem 6.16. *Let T be a theory of singular Bott-Chern classes of rank r_F and codimension r_N. Let Y be a complex manifold and let $\overline{F}$ and $\overline{N}$ be hermitian vector bundles of rank r_F and r_N respectively. Then the current $(\pi_P)_*(T(K(\overline{F}, \overline{N})))$ is closed. Moreover the cohomology class that it represents does not depend on the metric of N and F and determines a characteristic class for pairs of vector bundles of rank r_F and r_N. We denote this class by $C_T(F, N)$.*

Proof. We have that

$$\begin{aligned}
\mathrm{d}_{\mathcal{D}}&(\pi_P)_*(T(K(\overline{F},\overline{N})))\\
&= (\pi_P)_*(\mathrm{d}_{\mathcal{D}}\, T(K(\overline{F},\overline{N})))\\
&= (\pi_P)_*\left(\sum_{k=0}^{r}(-1)^k[\mathrm{ch}(\bigwedge^k \overline{Q}^{\vee})\pi_P^*\,\mathrm{ch}(\overline{F})] - s_*[\mathrm{Td}^{-1}(\overline{N})\,\mathrm{ch}(\overline{F})]\right)\\
&= \big((\pi_P)_*[c_r(\overline{Q})\,\mathrm{Td}^{-1}(\overline{Q})] - [\mathrm{Td}^{-1}(\overline{N})]\big)\,\mathrm{ch}(\overline{F})).
\end{aligned}$$

Therefore, the fact that the current $(\pi_P)_*(T(K(\overline{F},\overline{N})))$ is closed follows from corollary 3.8. The fact that this class is functorial on $(Y,\overline{N},\overline{F})$ is clear from the construction Thus, the fact that it does not depend on the hermitian metrics of N and F follows from proposition 1.7. □

Remark 6.17. By theorem 1.8 we know that, if we restrict ourselves to the algebraic category, $C_T(F,N)$ is given by a power series on the Chern classes with coefficients in $\mathbb{D}$. By degree reasons

$$C_T(F,N) \in \bigoplus_p H^{2p-1}_{\mathcal{D}^{\mathrm{an}}}(Y,\mathbb{R}(p)).$$

Let $\mathbf{1}_1 \in H^1_{\mathcal{D}}(*,\mathbb{R}(1))$ be the element determined by the constant function with value 1 in $\mathcal{D}^1(*,1)$. Then $C_T(F,N)/\mathbf{1}_1$ is a power series in the Chern classes of N and F with real coefficients.

7 Classification of theories of singular Bott-Chern classes

The aim of this section is to give a complete classification of the possible theories of singular Bott-Chern classes. This classification is given in terms of the characteristic class C_T introduced in the previous section.

Theorem 7.1. *Let r_F and r_N be two positive integers. Let C be a characteristic class for pairs of vector bundles of rank r_F and r_N. Then there exists a unique theory T_C of singular Bott-Chern classes of rank r_F and codimension r_N such that $C_{T_C} = C$.*

Proof. We first prove the uniqueness. Assume that T is a theory of singular Bott-Chern classes such that $C_T = C$. Let $\overline{\xi} = (i\colon Y \longrightarrow X,\overline{N},\overline{F},\overline{E}_*)$ be a hermitian embedded vector bundle as in section 6. Let W be the deformation to the normal cone of Y. We will use all the notations of section 5. In particular, we will denote by $p_{\widetilde{X}}\colon \widetilde{X} \longrightarrow X$ and $p_P\colon P \longrightarrow X$ the morphisms induced by restricting p_W. Recall that p_P can be factored as

$$P \xrightarrow{\pi_P} Y \xrightarrow{i} X.$$

The normal vector bundle to the inclusion $j\colon Y\times\mathbb{P}^1 \longrightarrow W$ is isomorphic to $p_Y^*N\otimes q_Y^*\mathcal{O}(-1)$. We provide it with the hermitian metric induced by the metric of N and the Fubini-Study metric of $\mathcal{O}(-1)$ and we denote it by $\overline{N}'$.

By theorem 5.4 we have a complex of hermitian vector bundles, $\operatorname{tr}_1(E_*)_*$ such that the restriction $\operatorname{tr}_1(E_*)_*|_{X\times\{0\}}$ is isometric to E_*, the restriction $\operatorname{tr}_1(E_*)_*|_{\widetilde{X}}$ is orthogonally split and there is an exact sequence on P

$$0 \longrightarrow A_* \longrightarrow \operatorname{tr}_1(E_*)_*|_P \longrightarrow K(F,N) \longrightarrow 0,$$

where A_* is split acyclic and $K(F,N)$ is the Koszul resolution. Recall that we have trivialized $N^{-1}_{\infty/\mathbb{P}^1}$ by means of the section y of $\mathcal{O}_{\mathbb{P}^1}(1)$. We choose a hermitian metric in every bundle of A_* such that it becomes orthogonally split. For each k we will denote by $\overline{\eta}_k$ the exact sequence of hermitian vector bundles

$$0 \longrightarrow \overline{A}_k \longrightarrow \operatorname{tr}_1(\overline{E}_*)_k|_P \longrightarrow K(\overline{F},\overline{N})_k \longrightarrow 0. \tag{7.2}$$

Observe that the current W_1 is defined as the current associated to a locally integrable differential form. The pull-back of this form to W is also locally integrable. Therefore it defines a current on W that we also denote by W_1. Moreover, since the wave front sets of W_1 and of $T(\operatorname{tr}_1(\overline{E}_*)_*)$ are disjoint, there is a well defined current $W_1 \bullet T(\operatorname{tr}_1(\overline{E}_*)_*)$. Then, using the properties of singular Bott-Chern classes in definition 6.9, the equality

$$\begin{aligned}
0 &= \mathrm{d}_{\mathcal{D}}(p_W)_*\left(W_1 \bullet T(\operatorname{tr}_1(\overline{E}_*)_*)\right)\\
&= (p_{\widetilde{X}})_*(T(\operatorname{tr}_1(\overline{E}_*)_*)|_{\widetilde{X}}) + (p_P)_*(T(\operatorname{tr}_1(\overline{E}_*)_*)|_P) - T(\overline{\xi})\\
&\quad - (p_W)_*\left(W_1 \bullet \left(\sum_k (-1)^k \operatorname{ch}(\operatorname{tr}_1(\overline{E}_*)_*) - (j_*(\operatorname{ch}(p_Y^*\overline{F})\operatorname{Td}^{-1}(\overline{N}'))\right)\right)
\end{aligned}$$

holds in the group $\bigoplus_k \widetilde{\mathcal{D}}^{2k-1}(X,k)$. By properties 6.9(ii) and 6.9(iii), $T(\operatorname{tr}_1(\overline{E}_*)_*)|_{\widetilde{X}} = T(\operatorname{tr}_1(\overline{E}_*)_*|_{\widetilde{X}}) = 0$.
By proposition 6.13 we have

$$T(\operatorname{tr}_1(\overline{E}_*)_*|_P) = T(K(\overline{F},\overline{N})) - \sum_k (-1)^k[\widetilde{\operatorname{ch}}(\overline{\eta}_k)].$$

Moreover, we have

$$(p_P)_*(T(K(\overline{F},\overline{N}))) = i_*(\pi_P)_*(T(K(\overline{F},\overline{N}))) = i_*C_T(F,N).$$

By the definition of N' and the choice of its metric, there are two differential forms a, b on Y, such that

$$\operatorname{ch}(p_Y^*\overline{F})\operatorname{Td}^{-1}(\overline{N}') = p_Y^*(a) + p_Y^*(b) \wedge q_Y^*(c_1(\mathcal{O}(-1))).$$

We denote $\omega = -c_1(\mathcal{O}(-1))$. By the properties of the Fubini-Study metric, ω is invariant under the involution of $\mathbb{P}^1$ that sends t to $1/t$. Then

$$(p_W)_*\left(W_1 \bullet (j_*(\operatorname{ch}(p_Y^*\overline{F})\operatorname{Td}^{-1}(\overline{N}'))\right) = i_*(p_Y)_*(W_1 \bullet (p_Y^*a + p_Y^*b\omega)) = 0$$

because the current W_1 changes sign under the involution $t \longmapsto 1/t$.
Summing up, we have obtained the equation

$$T(\overline{\xi}) = -(p_W)_*\left(\sum_k (-1)^k W_1 \bullet \operatorname{ch}(\operatorname{tr}_1(\overline{E}_*)_k)\right) \\ - \sum_k (-1)^k (p_P)_*[\widetilde{\operatorname{ch}}(\overline{\eta}_k)] + i_* C_T(F,N). \quad (7.3)$$

Hence the singular Bott-Chern class is characterized by the properties of definition 6.9 and the characteristic class C_T.
In order to prove the existence of a theory of singular Bott-Chern classes, we use equation (7.3) to define a class $T_C(\xi)$ as follows.

Definition 7.4. Let C be a characteristic class for pairs of vector bundles of rank r_F and r_N as in theorem 7.1. Let $\overline{\xi} = (i\colon Y \longrightarrow X, \overline{N}, \overline{F}, \overline{E}_*)$ be as in definition 6.9. Let $\overline{A}_*$, $\operatorname{tr}_1(\overline{E}_*)_*$ and $\overline{\eta}_*$ be as in (7.2). Then we define

$$T_C(\overline{\xi}) = -(p_W)_*\left(\sum_k (-1)^k W_1 \bullet \operatorname{ch}(\operatorname{tr}_1(\overline{E}_*)_k)\right) \\ - \sum_k (-1)^k (p_P)_*[\widetilde{\operatorname{ch}}(\overline{\eta}_k)] + i_* C(F,N). \quad (7.5)$$

We have to prove that this definition does not depend on the choice of the metric of $\operatorname{tr}_1(\overline{E}_*)_*$ or the metric of $\overline{A}_*$, that T_C satisfies the properties of definition 6.9 and that the characteristic class C_{T_C} agrees with C.
First we prove the independence from the metrics. We denote by h_k the hermitian metric on $\operatorname{tr}_1(\overline{E}_*)_k$ and by g_k the hermitian metric on A_k. Let h'_k and g'_k be another choice of metrics satisfying also that (A_*, g'_*) is orthogonally split, that $(\operatorname{tr}_1(E_*)_k, h'_k)|_{X\times\{0\}}$ is isometric to $\overline{E}_k$ and that $(\operatorname{tr}_1(E_*)_k, h'_k)|_{\widetilde{X}}$ is orthogonally split. We denote by $\overline{\eta}'_k$ the exact sequence η_k provided with the metrics g' and h'. Then, in the group $\bigoplus_p \widetilde{\mathcal{D}}^{2p-1}(X,p)$, we have

$$\sum_k (-1)^k (p_P)_*[\widetilde{\operatorname{ch}}(\overline{\eta}_k)] - \sum_k (-1)^k (p_P)_*[\widetilde{\operatorname{ch}}(\overline{\eta}'_k)] = \\ \sum_k (-1)^k (p_P)_*\left[\widetilde{\operatorname{ch}}(A_k, g_k, g'_k)\right] - \sum_k (-1)^k (p_P)_*\left[\widetilde{\operatorname{ch}}(\operatorname{tr}_1(E_*)_k|_P, h_k, h'_k)\right]. \quad (7.6)$$

Observe that the first term of the right hand side vanishes due to the hypothesis of A_* being orthogonally split for both metrics.

Moreover, we also have,

$$(p_W)_*\left(\sum_k (-1)^k W_1 \bullet \mathrm{ch}(\mathrm{tr}_1(E_*)_k, h_k)\right) - \\ (p_W)_*\left(\sum_k (-1)^k W_1 \bullet \mathrm{ch}(\mathrm{tr}_1(E_*)_k, h'_k)\right) = \\ (p_W)_*\left(\sum_k (-1)^k W_1 \bullet \mathrm{d}_{\mathcal{D}}\, \widetilde{\mathrm{ch}}(\mathrm{tr}_1(E_*)_k, h_k, h'_k)\right). \quad (7.7)$$

But, in the group $\bigoplus_p \widetilde{\mathcal{D}}^{2p-1}(X,p)$,

$$(p_W)_*\left(\sum_k (-1)^k W_1 \bullet \mathrm{d}_{\mathcal{D}}\, \widetilde{\mathrm{ch}}(\mathrm{tr}_1(E_*)_k, h_k, h'_k)\right) = \\ \sum_k (-1)^k (p_{\widetilde{X}})_*[\widetilde{\mathrm{ch}}(\mathrm{tr}_1(E_*)_k, h_k, h'_k)]|_{\widetilde{X}} \\ + \sum_k (-1)^k (p_P)_*[\widetilde{\mathrm{ch}}(\mathrm{tr}_1(E_*)_k, h_k, h'_k)]|_P) \\ - \sum_k (-1)^k [\widetilde{\mathrm{ch}}(\mathrm{tr}_1(E_*)_k, h_k, h'_k)]|_{X\times\{0\}}. \quad (7.8)$$

The last term of the right hand side vanishes because the metrics h_k and h'_k agree when restricted to $X \times \{0\}$ and the first term vanishes by the hypothesis that $\mathrm{tr}_1(E_*)_*|_{\widetilde{X}}$ is orthogonally split with both metrics. Combining equations (7.6), (7.7) and (7.8) we obtain that the right hand side of equation (7.5) does not depend on the choice of metrics.

We next prove the property (i) of definition 6.9. We compute

$$\mathrm{d}_{\mathcal{D}}\, T_C(\overline{\xi}) = -\sum_k (-1)^k \left((p_{\widetilde{X}})_* \mathrm{ch}(\mathrm{tr}_1(\overline{E}_*)_k|_{\widetilde{X}}) + (p_P)_* \mathrm{ch}(\mathrm{tr}_1(\overline{E}_*)_k|_P)\right) \\ + \sum_k (-1)^k \mathrm{ch}(\mathrm{tr}_1(\overline{E}_*)_k|_{X\times\{0\}}) \\ - \sum_k (-1)^k (p_P)_* \left(\mathrm{ch}(\overline{A}_k) + \mathrm{ch}(K(\overline{F}, \overline{N})_k) - \mathrm{ch}(\mathrm{tr}_1(\overline{E}_*)_k|_P)\right).$$

Using that $\overline{A}_*$ and that $\mathrm{tr}_1(\overline{E}_*)_*|_{\widetilde{X}}$ are orthogonally split and corollary 3.8 we obtain

$$\begin{aligned} \mathrm{d}_{\mathcal{D}}\, T_C(\overline{\xi}) &= \sum_k (-1)^k \mathrm{ch}(\overline{E}_k) - \sum_k (-1)^k (p_P)_* \mathrm{ch}(K(\overline{F}, \overline{N})_k) \\ &= \sum_k (-1)^k [\mathrm{ch}(\overline{E}_k)] - (p_P)_*[c_r(\overline{Q})\, \mathrm{Td}^{-1}(\overline{Q})] \\ &= \sum_k (-1)^k [\mathrm{ch}(\overline{E}_k)] - i_*[\mathrm{ch}(\overline{F})\, \mathrm{Td}^{-1}(\overline{N})]. \end{aligned}$$

We now prove the normalization property. We consider first the case when $Y = \emptyset$ and $\overline{E}_*$ is a non-negatively graded orthogonally split complex. We denote by

$$\overline{K}_i = \operatorname{Ker}(\mathrm{d}_i \colon E_i \longrightarrow E_{i-1})$$

with the induced metric. By hypothesis there are isometries

$$\overline{E}_i = \overline{K}_i \oplus \overline{K}_{i-1}.$$

Under these isometries, the differential is $\mathrm{d}(s,t) = (t,0)$. Following the explicit construction of $\operatorname{tr}_1(E_*)$ given in [20], recalled in definition 2.5, we see that

$$\operatorname{tr}_1(E_*)_i = p^*K_i \otimes q^*\mathcal{O}(i) \oplus p^*K_{i-1} \otimes q^*\mathcal{O}(i-1) = K_i(i) \oplus K_{i-1}(i-1).$$

Moreover, we can induce a metric on $\operatorname{tr}_1(E_*)_*$ satisfying the hypothesis of definition 2.9 by means of the metric of the bundles K_i and the Fubini-Study metric on the bundles $\mathcal{O}(i)$. It is clear that the second and third terms of the right hand side of equation (7.3) are zero. For the first term we have

$$\begin{aligned} &\sum_k (-1)^k (p_W)_* W_1 \bullet \left(\operatorname{ch}(\operatorname{tr}_1(\overline{E}_*)_k)\right) \\ &\qquad = (p_W)_* \left(\sum_k (-1)^k W_1 \bullet \operatorname{ch}(\overline{K}_k(k) \overset{\perp}{\oplus} \overline{K}_{k-1}(k-1)) \right) \\ &\qquad = (p_W)_* \left(W_1 \bullet (a + b \wedge \omega) \right), \end{aligned}$$

where ω is the Fubini-Study $(1,1)$-form on $\mathbb{P}^1$ and a, b are inverse images of differential forms on X. Therefore we obtain that $T_C(\overline{E}_*) = 0$.
Now let $\overline{\xi} = (i \colon Y \longrightarrow X, \overline{N}, \overline{F}, \overline{E}_*)$ and let $\overline{B}_*$ be a non-negatively graded orthogonally split complex of vector bundles. By [20] section 1.1, we have that $W(E_* \oplus B_*) = W(E_*)$ and that

$$\operatorname{tr}_1(E_* \oplus B_*) = \operatorname{tr}_1(E_*) \oplus \pi^* \operatorname{tr}_1(B_*).$$

In order to compute $T_C(\overline{\xi})$, we have to consider the exact sequences of hermitian vector bundles over P

$$\overline{\eta}_k \colon 0 \longrightarrow \overline{A}_k \longrightarrow \operatorname{tr}_1(\overline{E}_*)_k|_P \longrightarrow K(\overline{F}, \overline{N})_k \longrightarrow 0,$$

whereas, in order to compute $T_C(\overline{\xi} \oplus \overline{B}_*)$, we consider the sequences

$$\begin{aligned} &\overline{\eta}'_k \colon \\ &0 \longrightarrow \overline{A}_k \oplus \pi^*(\operatorname{tr}_1(\overline{B})_k)|_P \longrightarrow \operatorname{tr}_1(\overline{E}_*)_k \oplus \pi^*(\operatorname{tr}_1(\overline{B})_k)|_P \longrightarrow K(\overline{F}, \overline{N})_k \longrightarrow 0. \end{aligned}$$

By the additivity of Bott-Chern classes, we have that $\widetilde{\mathrm{ch}}(\overline{\eta}_k) = \widetilde{\mathrm{ch}}(\overline{\eta}'_k)$. Therefore

$$\begin{aligned} T_C(\overline{\xi} \oplus \bar{B}_*) - T_C(\overline{\xi}) &= -(p_W)_* \left(\sum_k (-1)^k W_1 \bullet \mathrm{ch}(\mathrm{tr}_1(\overline{E}_* \oplus \overline{B}_*)_k) \right) \\ &\quad + (p_W)_* \left(\sum_k (-1)^k W_1 \bullet \mathrm{ch}(\mathrm{tr}_1(\overline{E}_*)_k) \right) \\ &= -(p_W)_* \left(\sum_k (-1)^k W_1 \bullet \mathrm{ch}(\mathrm{tr}_1(\overline{B}_*)_k) \right) \\ &= 0. \end{aligned}$$

The proof of the functoriality is left to the reader.
Finally we prove that $C_{T_C} = C$. Let Y be a complex manifold and let $\overline{F}$ and $\overline{N}$ be two hermitian vector bundles. We write $X = \mathbb{P}(N \oplus \mathbb{C})$. Let $i\colon Y \longrightarrow X$ be the inclusion given by the zero section and let $\pi_X : X \longrightarrow Y$ be the projection. On X we have the tautological exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \pi_X^*(N \oplus \mathbb{C}) \longrightarrow Q \longrightarrow 0$$

and the Koszul resolution, denoted $K(\overline{F}, \overline{N})$. We denote

$$\overline{\xi} = (i\colon Y \longrightarrow X, \overline{N}, \overline{F}, K(\overline{F}, \overline{N})).$$

Using the definition of T_C, that is, equation (7.5), and the fact that T_C satisfies the properties of definition 6.9, hence equation (7.3) is satisfied, we obtain that

$$i_* C(F, N) = i_* C_{T_C}(F, N)$$

Applying $(\pi_X)_*$ we obtain that $C(F, N) = C_{T_C}(F, N)$ which finishes the proof of theorem 7.1. □

8 Transitivity and projection formula

We now investigate how different properties of the characteristic class C_T are reflected in the corresponding theory of singular Bott-Chern classes.

Proposition 8.1. *Let $i\colon Y \hookrightarrow X$ be a closed immersion of complex manifolds. Let $\overline{F}$ be a hermitian vector bundle on Y and $\overline{G}$ a hermitian vector bundle on X. Let $\overline{N}$ denote the normal bundle to Y provided with a hermitian metric. Let $\overline{E}_*$ be a finite resolution of i_*F by hermitian vector bundles. We denote $\overline{\xi} = (i\colon Y \longrightarrow X, \overline{N}, \overline{F}, \overline{E}_*)$ and $\overline{\xi} \otimes \overline{G} = (i\colon Y \longrightarrow X, \overline{N}, \overline{F} \otimes i^*\overline{G}, \overline{E}_* \otimes \overline{G})$. Then*

$$T(\overline{\xi} \otimes \overline{G}) - T(\overline{\xi}) \bullet \mathrm{ch}(\overline{G}) = i_*(C_T(F \otimes i^*G, N)) - i_*(C_T(F, N)) \bullet \mathrm{ch}(\overline{G}).$$

Proof. Since the construction of $\operatorname{tr}_1(E_*)_*$ is local on X and Y and compatible with finite sums, we have that

$$W(E_*) = W(E_* \otimes G), \qquad \operatorname{tr}_1(\overline{E}_* \otimes \overline{G})_* = \operatorname{tr}_1(\overline{E}_*)_* \otimes p_W^* \overline{G}.$$

We first compute

$$\begin{aligned}(p_W)_* &\left(\sum_k (-1)^k W_1 \bullet \operatorname{ch}(\operatorname{tr}_1(\overline{E}_* \otimes \overline{G})_*) \right) \\ &= (p_W)_* \left(\sum_k (-1)^k W_1 \bullet \operatorname{ch}(\operatorname{tr}_1(\overline{E}_*)_*) p_W^* \operatorname{ch}(\overline{G}) \right) \\ &= (p_W)_* \left(\sum_k (-1)^k W_1 \bullet \operatorname{ch}(\operatorname{tr}_1(\overline{E}_*)_*) \right) \operatorname{ch}(\overline{G}). \end{aligned} \tag{8.2}$$

The Koszul resolution of $i_*(F \otimes i^* G)$ is given by

$$K(F \otimes i^* G, N) = K(F, N) \otimes p_P^* G.$$

For each $k \geq 0$, we will denote by $\overline{\eta}_k \otimes p_P^* \overline{G}$ the exact sequence

$$0 \longrightarrow \overline{A}_k \otimes p_P^* \overline{G} \longrightarrow \operatorname{tr}_1(\overline{E}_* \otimes \overline{G})_k|_P \longrightarrow K(\overline{F}, \overline{N})_k \otimes p_P^* \overline{G} \longrightarrow 0.$$

Then, we have

$$(p_P)_*[\widetilde{\operatorname{ch}}(\overline{\eta}_k \otimes p_P^* \overline{G})] = (p_P)_*[\widetilde{\operatorname{ch}}(\overline{\eta}_k) \bullet p_P^* \operatorname{ch}(\overline{G})] = (p_P)_*[\widetilde{\operatorname{ch}}(\overline{\eta}_k)] \bullet \operatorname{ch}(\overline{G}) \tag{8.3}$$

Thus the proposition follows from equation (8.2), equation (8.3) and formula (7.3). $\square$

Definition 8.4. We will say that a theory of singular Bott-Chern classes is *compatible with the projection formula* if, whenever we are in the situation of proposition 8.1, the following equality holds:

$$T(\overline{\xi} \otimes \overline{G}) = T(\overline{\xi}) \bullet \operatorname{ch}(\overline{G}).$$

We will say that a characteristic class C (of pairs of vector bundles) is *compatible with the projection formula* if it satisfies

$$C(F, N) = C(\mathcal{O}_Y, N) \bullet \operatorname{ch}(F).$$

Corollary 8.5. *A theory of singular Bott-Chern classes T is compatible with the projection formula if and only if it is the case for the associated characteristic class C_T.*

Proof. Assume that C_T is compatible with the projection formula and that we are in the situation of proposition 8.1. Then

$$\begin{aligned} i_*C_T(F\otimes i^*G,N)) &= i_*(C_T(\mathcal{O}_Y,N)\bullet \operatorname{ch}(F\otimes i^*G))\\ &= i_*(C_T(\mathcal{O}_Y,N)\bullet \operatorname{ch}(F)i^*\operatorname{ch}(G))\\ &= i_*(C_T(\mathcal{O}_Y,N)\bullet \operatorname{ch}(F))\operatorname{ch}(G)\\ &= i_*(C_T(F,N))\bullet \operatorname{ch}(G). \end{aligned}$$

Thus, by proposition 8.1, T is compatible with the projection formula.
Assume that T is compatible with the projection formula. Let $\overline{F}$ and $\overline{N}$ be hermitian vector bundles over a complex manifold Y. Let $s\colon Y\hookrightarrow P:=\mathbb{P}(N\oplus\mathbb{C})$ be the zero section and let $\pi\colon P\longrightarrow Y$ be the projection. Then

$$\begin{aligned} C_T(F,N) &= \pi_*(T(K(\overline{F},\overline{N})))\\ &= \pi_*(T(K(\overline{\mathcal{O}}_Y,\overline{N})\otimes\pi^*\overline{F}))\\ &= \pi_*(T(K(\overline{\mathcal{O}}_Y,\overline{N}))\bullet\pi^*\operatorname{ch}(F))\\ &= \pi_*(T(K(\overline{\mathcal{O}}_Y,\overline{N})))\bullet\operatorname{ch}(F)\\ &= C_T(\mathcal{O}_Y,N)\bullet\operatorname{ch}(F). \end{aligned}$$

□

We will next investigate the relationship between singular Bott-Chern classes and compositions of closed immersions. Thus, let

$$\begin{tikzcd} Y \arrow[r, hook, "i_{Y/X}"] \arrow[rr, hook, bend right, "i_{Y/M}"'] & X \arrow[r, hook, "i_{X/M}"] & M \end{tikzcd}$$

be a composition of closed immersions. Assume that the normal bundles $N_{Y/X}$, $N_{X/M}$ and $N_{Y/M}$ are provided with hermitian metrics. We will denote by $\overline{\varepsilon}$ the exact sequence

$$\overline{\varepsilon}\colon 0\to\overline{N}_{Y/X}\to\overline{N}_{Y/M}\to i^*_{Y/X}\overline{N}_{X/M}\to 0. \tag{8.6}$$

Let $P_{X/M}=\mathbb{P}(N_{X/M}\oplus\mathbb{C})$ be the projective completion of the normal cone to X in M. Then there is an isomorphism

$$N_{Y/P_{X/M}}\cong N_{Y/X}\oplus i^*_{Y/X}N_{X/M}. \tag{8.7}$$

We denote by $\overline{N}_{Y/P_{X/M}}$ the vector bundle on the left hand side with the hermitian metric induced by the isomorphism (8.7).
Let $\overline{F}$ be a hermitian vector bundle over Y, let $\overline{E}_*\longrightarrow(i_{Y/X})_*F$ be a resolution by hermitian vector bundles. Let $\overline{E}'_{*,*}$ be a complex of complexes of vector

bundles over M, such that, for each $k \geq 0$, $\overline{E}'_{k,*} \longrightarrow (i_{X/M})_* E_k$ is a resolution, and there is a commutative diagram of resolutions

$$\begin{tikzcd} \cdots \arrow[r] & E'_{k+1,*} \arrow[r] \arrow[d] & E'_{k,*} \arrow[r] \arrow[d] & E'_{k-1,*} \arrow[r] \arrow[d] & \cdots . \\ \cdots \arrow[r] & (i_{X/M})_* E_{k+1} \arrow[r] & (i_{X/M})_* E_k \arrow[r] & (i_{X/M})_* E_{k-1} \arrow[r] & \cdots \end{tikzcd}$$

It follows that we have a resolution $\operatorname{Tot}(\overline{E}'_{*,*}) \longrightarrow (i_{Y/M})_* F$ of $(i_{Y/M})_* F$ by hermitian vector bundles.

Notation 8.8. We will denote

$$\begin{aligned} \overline{\xi}_{Y \hookrightarrow X} &= (i_{Y/X}, \overline{N}_{Y/X}, \overline{F}, \overline{E}_*), \\ \overline{\xi}_{Y \hookrightarrow M} &= (i_{Y/M}, \overline{N}_{Y/M}, \overline{F}, \operatorname{Tot}(\overline{E}'_{*,*})), \\ \overline{\xi}_{X \hookrightarrow M,k} &= (i_{X/M}, \overline{N}_{X/M}, \overline{E}_k, \overline{E}'_{k,*}). \end{aligned}$$

We will also denote by $\overline{\xi}_{Y \hookrightarrow P_{X/M}}$ the hermitian embedded vector bundle

$$\left(Y \hookrightarrow P_{X/M}, \overline{N}_{Y/P_{X/M}}, \overline{F}, \operatorname{Tot}(\pi^*_{P_{X/M}} \overline{E}_* \otimes K(\mathcal{O}_X, \overline{N}_{X/M})) \right).$$

Let T be a theory of singular Bott-Chern classes, and let C_T be its associated characteristic class. Our aim now is to relate $T(\overline{\xi}_{Y \hookrightarrow X})$, $T(\overline{\xi}_{Y \hookrightarrow M})$ and $T(\overline{\xi}_{X \hookrightarrow M,k})$.
Let W_X be the deformation to the normal cone of X in M. As before we denote by $j_X \colon X \times \mathbb{P}^1 \longrightarrow W_X$ the inclusion.
We denote by W the deformation to the normal cone of $j_X(Y \times \mathbb{P}^1)$ in W_X.
This double deformation is represented in figure 1. There is a proper map $q_W \colon W \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. The fibers of q_W over the corners of $\mathbb{P}^1 \times \mathbb{P}^1$ are as follows:

$$\begin{aligned} q_W^{-1}(0,0) &= M, \\ q_W^{-1}(\infty,0) &= \widetilde{M}_X \times \{0\} \cup P_{X/M}, \\ q_W^{-1}(0,\infty) &= \widetilde{M}_Y \cup P_{Y/M}, \\ q_W^{-1}(\infty,\infty) &= \widetilde{M}_X \times \{\infty\} \cup \widetilde{P}_{X/M} \cup P_{Y/P_{X/M}}, \end{aligned}$$

where $\widetilde{M}_X$ and $\widetilde{M}_Y$ are the blow-up of M along X and Y respectively, $P_{Y/M} = \mathbb{P}(N_{Y/M} \oplus \mathbb{C})$ is the projective completion of the normal cone to Y in M, $P_{Y/P_{X/M}}$ of the normal cone to Y in $P_{X/M}$ and $\widetilde{P}_{X/M}$ is the blow-up of $P_{X/M}$

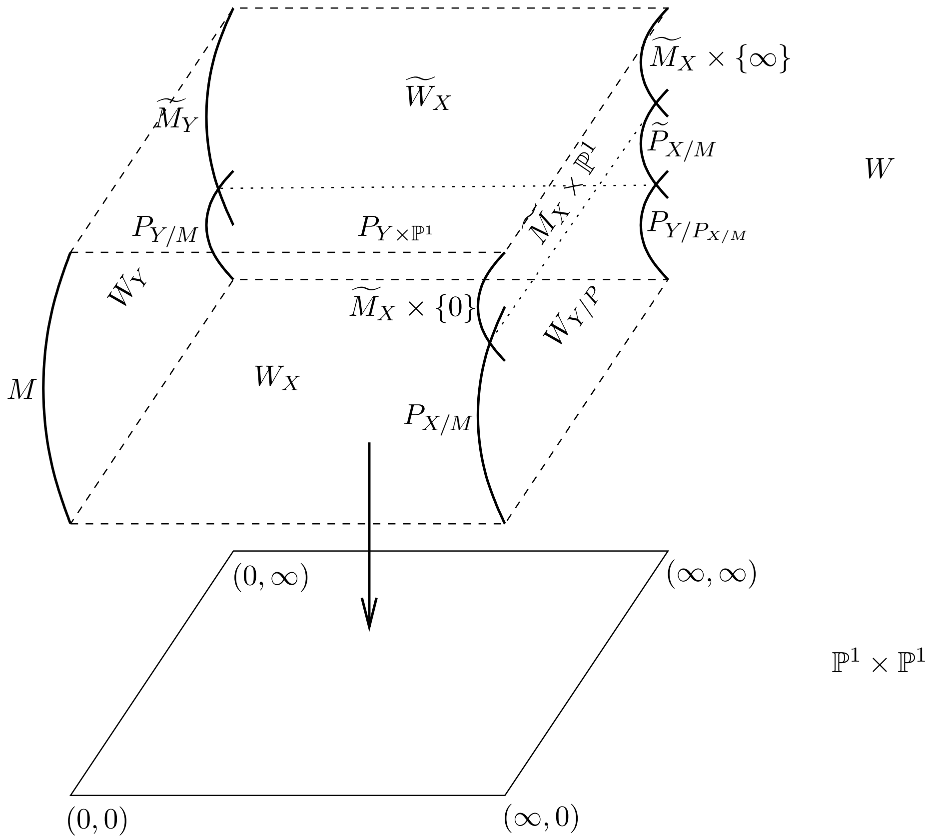

Figure 1: Double deformation

along Y. The preimages by π of the different faces of $\mathbb{P}^1 \times \mathbb{P}^1$ are as follows:

$$\begin{aligned}
q_W^{-1}(\mathbb{P}^1 \times \{0\}) &= W_X,\\
q_W^{-1}(\{0\} \times \mathbb{P}^1) &= W_Y,\\
q_W^{-1}(\mathbb{P}^1 \times \{\infty\}) &= \widetilde{W}_X \cup P_{Y\times\mathbb{P}^1},\\
q_W^{-1}(\{\infty\} \times \mathbb{P}^1) &= \widetilde{M}_X \times \mathbb{P}^1 \cup W_{Y/P},
\end{aligned}$$

where W_Y is the deformation to the normal cone of Y in M, the component $\widetilde{W}_X$ is the blow-up of W_X along $j_X(Y \times \mathbb{P}^1)$, while $P_{Y\times\mathbb{P}^1} = \mathbb{P}(N_{Y\times\mathbb{P}^1/W_X} \oplus \mathbb{C})$ is the projective completion of the normal cone to $j_X(Y \times \mathbb{P}^1)$ in W_X and $W_{Y/P}$ is the deformation to the normal cone of Y inside $P_{X/M}$. All the above subvarieties will be called boundary components of W.

We will use the following notations for the different maps.

$$\begin{array}{ll}
p_X\colon X\times\mathbb{P}^1\longrightarrow X & p_Y\colon Y\times\mathbb{P}^1\longrightarrow Y\\
p_{Y\times\mathbb{P}^1}\colon Y\times\mathbb{P}^1\times\mathbb{P}^1\longrightarrow Y\times\mathbb{P}^1 & p_{\widetilde{M}_X\times\mathbb{P}^1}\colon \widetilde{M}_X\times\mathbb{P}^1\longrightarrow M\\
p_{W_{Y/P}}\colon W_{Y/P}\longrightarrow M & p_{W_Y}\colon W_Y\longrightarrow M\\
p_{W_X}\colon W_X\longrightarrow M & p_{P_{Y\times\mathbb{P}^1}}\colon P_{Y\times\mathbb{P}^1}\longrightarrow M\\
p_{\widetilde{W}_X}\colon \widetilde{W}_X\longrightarrow M & p_{P_{Y/P_{X/M}}}\colon P_{Y/P_{X/M}}\longrightarrow M\\
p_{P_{X/M}}\colon P_{X/M}\longrightarrow M & p_{\widetilde{P}_{X/M}}\colon \widetilde{P}_{X/M}\longrightarrow M\\
p_{P_{Y/M}}\colon P_{Y/M}\longrightarrow M & p_W\colon W\longrightarrow M\\
j_Y\colon Y\times\mathbb{P}^1\longrightarrow W_Y & j'_Y\colon Y\times\mathbb{P}^1\longrightarrow W_X\\
j_{Y\times\mathbb{P}^1}\colon Y\times\mathbb{P}^1\times\mathbb{P}^1\longrightarrow W & i_{Y/P_{X/M}}\colon Y\longrightarrow P_{X/M}\\
\pi_{P_{X/M}}\colon P_{X/M}\longrightarrow X & \pi_{P_{Y/M}}\colon P_{Y/M}\longrightarrow Y\\
\pi_{P_{Y/P}}\colon P_{Y/P_{X/M}}\longrightarrow Y & \pi_{P_{Y\times\mathbb{P}^1}}\colon P_{Y\times\mathbb{P}^1}\longrightarrow Y\times\mathbb{P}^1\\
\pi_{\widetilde{M}_X}\colon \widetilde{M}_X\longrightarrow M & \pi_{\widetilde{M}_Y}\colon \widetilde{M}_Y\longrightarrow M
\end{array}$$

Note that the map $p_{\widetilde{M}_X\times\mathbb{P}^1}$ factors through the blow-up $\widetilde{M}_X\longrightarrow M$ and the map $p_{\widetilde{W}_X}$ factors through the blow-up $\widetilde{M}_Y\longrightarrow M$, whereas the maps $p_{W_{Y/P}}$, $p_{P_{X/M}}$ and $p_{\widetilde{P}_{X/M}}$ factor through the inclusion $X\hookrightarrow M$ and the maps $p_{P_{Y\times\mathbb{P}^1}}$, $p_{P_{Y/M}}$ and $p_{P_{Y/P_{X/M}}}$ factor through the inclusion $Y\hookrightarrow M$.
The normal bundle to $X\times\mathbb{P}^1$ in W_X is isomorphic to $p_X^*N_{X/M}\otimes q_X^*\mathcal{O}(-1)$ and we consider on it the metric induced by the metric on $\overline{N}_{X/M}$ and the Fubini-Study metric on $\mathcal{O}(-1)$. We denote it by $\overline{N}_{X\times\mathbb{P}^1/W_X}$. The normal bundle to $Y\times\mathbb{P}^1$ in W_X satisfies

$$\begin{aligned}
N_{Y\times\mathbb{P}^1/W_X}|_{Y\times\{0\}}&\cong N_{Y/M}\\
N_{Y\times\mathbb{P}^1/W_X}|_{Y\times\{\infty\}}&\cong N_{Y/X}\oplus i^*_{Y/X}N_{X/M}.
\end{aligned}$$

On $N_{Y\times\mathbb{P}^1/W_X}$ we choose a hermitian metric such that the above isomorphisms are isometries. Finally, on the normal bundle to $Y\times\mathbb{P}^1\times\mathbb{P}^1$ in W, we define a metric using the same procedure as the definition of the metric of $\overline{N}_{X\times\mathbb{P}^1/W_X}$.
On W_X we obtain a sequence of resolutions $\operatorname{tr}_1(\overline{E}')_{n,*}\longrightarrow(j_X)_*p_X^*E_n$. They form a complex of complexes $\operatorname{tr}_1(\overline{E}')_{*,*}$ and the associated total complex $\operatorname{Tot}(\operatorname{tr}_1(\overline{E}')_{*,*})$ provides us with a resolution

$$\operatorname{Tot}(\operatorname{tr}_1(\overline{E}')_{*,*})_*\longrightarrow(j'_Y)_*p_Y^*F. \tag{8.9}$$

The restriction of $\operatorname{Tot}(\operatorname{tr}_1(\overline{E}')_{*,*})$ to M is $\operatorname{Tot}(\overline{E}'_{*,*})$. The restriction of each complex $\operatorname{tr}_1(\overline{E}')_{n,*}$ to $\widetilde{M}_X\times\{0\}$ is orthogonally split. Therefore the restriction of $\operatorname{Tot}(\operatorname{tr}_1(\overline{E}'))$ to $\widetilde{M}_X\times\{0\}$ is the total complex of a complex of orthogonally split complexes. So it is acyclic although not necessarily orthogonally split. The restriction of each complex $\operatorname{tr}_1(\overline{E}')_{n,*}$ to $P_{X/M}$ fits in an exact sequence

$$0 \longrightarrow \overline{A}_{n,*} \longrightarrow \operatorname{tr}_1(\overline{E}')_{n,*}|_{P_{X/M}} \longrightarrow \pi^*_{P_{X/M}}\overline{E}_n \otimes K(\overline{\mathcal{O}}_X, \overline{N}_{X/M})_* \longrightarrow 0.$$

These exact sequences glue together giving a commutative diagram

$$\begin{tikzcd} \operatorname{Tot}(\overline{A}_{*,*}) \arrow[r, hook] \arrow[d] & \operatorname{Tot}(\operatorname{tr}_1(\overline{E}')_{*,*}|_{P_{X/M}}) \arrow[r, two heads] \arrow[d] & \operatorname{Tot}(\pi^*_{P_{X/M}}\overline{E}_* \otimes K(\overline{\mathcal{O}}_X, \overline{N}_{X/M})_*) \arrow[d] \\ 0 \arrow[r, hook] & (i_{Y/P_{X/M}})_*F \arrow[r, two heads] & (i_{Y/P_{X/M}})_*F \end{tikzcd}$$

where the rows are short exact sequences. Even if the complexes $(\overline{A}_n)_*$ are orthogonally split, this is not necessarily the case for $\operatorname{Tot}(\overline{A}_{*,*})$. To ease the notation we will denote $\overline{A}_* = \operatorname{Tot}(\overline{A}_{*,*})$.

Applying theorem 5.4 to the resolution (8.9), we obtain a complex of hermitian vector bundles $\widetilde{E}'_* = \operatorname{tr}_1(\operatorname{Tot}(\operatorname{tr}_1(\overline{E}')_{*,*}))$ which is a resolution of the coherent sheaf $(j_{Y\times\mathbb{P}^1})_* p^*_{Y\times\mathbb{P}^1} p^*_Y F$.

We now study the restriction of $\widetilde{E}'_*$ to each of the boundary components of W.

- The restriction of $\widetilde{E}'_*$ to W_X is just $\operatorname{Tot}(\operatorname{tr}_1(\overline{E}'))$ which has already been described. For each $k \geq 0$, we will denote by η^1_k the short exact sequence of hermitian vector bundles on $P_{X/M}$

  $$\overline{A}_k \hookrightarrow \operatorname{Tot}(\operatorname{tr}_1(\overline{E}')_{*,*}|_{P_{X/M}})_k \twoheadrightarrow \operatorname{Tot}(\pi^*_{P_{X/M}}\overline{E} \otimes K(\mathcal{O}_X, \overline{N}_{X/M}))_k\ ,$$

  whereas, for each $n, k \geq 0$ we will denote by $\eta^1_{n,k}$ the short exact sequence

  $$\overline{A}_{n,k} \hookrightarrow \operatorname{tr}_1(\overline{E}')_{n,k}|_{P_{X/M}} \twoheadrightarrow \pi^*_{P_{X/M}}\overline{E}_n \otimes K(\mathcal{O}_X, \overline{N}_{X/M})_k\ .$$

- Its restriction to W_Y is $\operatorname{tr}_1(\operatorname{Tot}(\overline{E}'))$. It is a resolution of $(j_Y)_* p^*_Y F$. Its restriction to $\widetilde{M}_Y$ is orthogonally split, whereas its restriction to $P_{Y/M}$ fits in an exact sequence

  $$0 \longrightarrow \overline{B}_* \longrightarrow \operatorname{tr}_1(\operatorname{Tot}(\overline{E}'))_*|_{P_{Y/M}} \longrightarrow \pi^*_{P_{Y/M}}\overline{F} \otimes K(\overline{\mathcal{O}}_Y, \overline{N}_{Y/M}) \longrightarrow 0.$$

  For each $k \geq 0$ we will denote by η^2_k the degree k piece of the above exact sequence.

- Its restriction to $\widetilde{M}_X \times \mathbb{P}^1$ is an acyclic complex, such that its further restriction to $\widetilde{M}_X \times \{0\}$ is acyclic and its restriction to $\widetilde{M}_X \times \{\infty\}$ is orthogonally split.

- Its restriction to $W_{Y/P}$ fits in a short exact sequence

$$0 \to \operatorname{tr}_1(\overline{A}_*) \to \widetilde{E}'_*|_{W_{Y/P}} \to \operatorname{tr}_1(\operatorname{Tot}(\pi^*_{P_{X/M}}\overline{E} \otimes K(\overline{\mathcal{O}}_X, \overline{N}_{X/M}))) \to 0.$$

For each $k \geq 0$, we will denote by μ^1_k the exact sequence of hermitian vector bundles over $W_{Y/P}$ given by the piece of degree k of this exact sequence. The three terms of the above exact sequence become orthogonally split when restricted to $\widetilde{P}_{X/M}$. By contrast, when restricted to $P_{Y/P_{X/M}}$ they fit in a commutative diagram

$$\begin{tikzcd}
\overline{C}^1_* \arrow[r, hook] \arrow[d] & \overline{C}^2_* \arrow[r, two heads] \arrow[d] & \overline{C}^3 \arrow[d] \\
\operatorname{tr}_1(\overline{A})_*|_{P_{Y/P_{X/M}}} \arrow[r, hook] \arrow[d] & \widetilde{E}'_*|_{P_{Y/P_{X/M}}} \arrow[r, two heads] \arrow[d] & \overline{D}^2_* \arrow[d] \\
0 \arrow[r, hook] & \overline{D}^1_* \arrow[r, two heads] & \overline{D}^1_*
\end{tikzcd}$$

where the complexes $\overline{C}^i_*$ are orthogonally split, and

$$\begin{aligned}
\overline{D}^1_* &= \pi^*_{P_{Y/P}}\overline{F} \otimes K(\overline{\mathcal{O}}_Y, \overline{N}_{Y/P_{X/M}}),\\
\overline{D}^2_* &= \operatorname{tr}_1(\operatorname{Tot}(\pi^*_{P_{X/M}}\overline{E} \otimes K(\overline{\mathcal{O}}_X, \overline{N}_{X/M})))|_{P_{Y/P_{X/M}}}.
\end{aligned}$$

For each $k \geq 0$, we will denote by η^3_k the exact sequence corresponding to the piece of degree k of the second row of the above diagram, by η^4_k that of the second column and by η^5_k that of the third column. Notice that the map in the third row is an isometry. We assume that the metric on C^1_* is chosen in such a way that the first column is an isometry. Since the complexes $\overline{C}^i_*$ are orthogonally split, by lemma 2.17 we obtain

$$\sum_k (-1)^k \left(\widetilde{\operatorname{ch}}(\eta^3_k) - \widetilde{\operatorname{ch}}(\eta^4_k) + \widetilde{\operatorname{ch}}(\eta^5_k)\right) = 0. \tag{8.10}$$

Note that the restriction of μ^1_k to $P_{X/M}$ agrees with η^1_k, whereas its restriction to $P_{Y/P_{X/M}}$ agrees with η^3_k.

- Its restriction to $\widetilde{W}_X$ is orthogonally split.

- Finally its restriction to $P_{Y\times\mathbb{P}^1}$ fits in an exact sequence

$$\overline{D}_* \hookrightarrow \widetilde{E}'_*|_{P_{Y\times\mathbb{P}^1}} \twoheadrightarrow \pi^*_{P_{Y\times\mathbb{P}^1}} p^*_{Y\times\mathbb{P}^1}\overline{F} \otimes K(\mathcal{O}_{Y\times\mathbb{P}^1}, \overline{N}_{Y\times\mathbb{P}^1/W_X}),$$

where $\overline{D}_*$ is orthogonally split. For each $k \geq 0$ we will denote by μ_k^2 the piece of degree k of this exact sequence. Note that the restriction of μ_k^2 to $P_{Y/M}$ agrees with η_k^2 and the restriction of μ_k^2 to $P_{Y/P_{X/M}}$ agrees with η_k^4.

On $\mathbb{P}^1 \times \mathbb{P}^1$ we denote the two projections by p_1 and p_2. Since the currents $p_1^* W_1$ and $p_2^* W_1$ have disjoint wave front sets we can define the current $W_2 = p_1^* W_1 \bullet p_2^* W_1 \in \mathcal{D}_D^2(\mathbb{P}^1 \times \mathbb{P}^1, 2)$ which satisfies

$$\mathrm{d}_{\mathcal{D}} W_2 = (\delta_{\{\infty\}\times\mathbb{P}^1} - \delta_{\{0\}\times\mathbb{P}^1}) \bullet p_2^* W_1 - p_1^* W_1 \bullet (\delta_{\mathbb{P}^1\times\{\infty\}} - \delta_{\mathbb{P}^1\times\{0\}}). \quad (8.11)$$

The key point in order to study the compatibility of singular Bott-Chern classes and composition of closed immersions is that, in the group $\bigoplus_p \widetilde{\mathcal{D}}^{2p-1}(M,p)$, we have

$$\mathrm{d}_{\mathcal{D}}(p_W)_* \left(\sum_k (-1)^k W_2 \bullet \mathrm{ch}(\widetilde{E}'_k)\right) = 0.$$

We compute this class using the equation (8.11). It can be decomposed as follows.

$$\begin{aligned}
\mathrm{d}_{\mathcal{D}}(p_W)_* \left(\sum_k (-1)^k W_2 \bullet \mathrm{ch}(\widetilde{E}'_k)\right) = & \\
(p_{\widetilde{M}_X\times\mathbb{P}^1})_* \left(\sum_k (-1)^k W_1 \bullet \mathrm{ch}(\widetilde{E}'_k|_{\widetilde{M}_X\times\mathbb{P}^1})\right) & \qquad \text{(a)}\\
+ (p_{W_{Y/P}})_* \left(\sum_k (-1)^k W_1 \bullet \mathrm{ch}(\widetilde{E}'_k|_{W_{Y/P}})\right) & \qquad \text{(b)}\\
- (p_{W_Y})_* \left(\sum_k (-1)^k W_1 \bullet \mathrm{ch}(\widetilde{E}'_k|_{W_Y})\right) & \qquad \text{(c)}\\
- (p_{\widetilde{W}_X})_* \left(\sum_k (-1)^k W_1 \bullet \mathrm{ch}(\widetilde{E}'_k|_{\widetilde{W}_X})\right) & \qquad \text{(d)}\\
- (p_{P_{Y\times\mathbb{P}^1}})_* \left(\sum_k (-1)^k W_1 \bullet \mathrm{ch}(\widetilde{E}'_k|_{P_{Y\times\mathbb{P}^1}})\right) & \qquad \text{(e)}\\
+ (p_{W_X})_* \left(\sum_k (-1)^k W_1 \bullet \mathrm{ch}(\widetilde{E}'_k|_{W_X})\right) & \qquad \text{(f)}\\
=: I_a + I_b - I_c - I_d - I_e + I_f &
\end{aligned}$$

We compute each of the above terms.
(a) Since the restriction $\widetilde{E}'|_{\widetilde{M}_X\times\{\infty\}}$ is orthogonally split, we have

$$I_a = -(\pi_{\widetilde{M}_X})_* \widetilde{\mathrm{ch}}(\widetilde{E}'|_{\widetilde{M}_X\times\{0\}}).$$

But, using lemma 2.17 and the fact, for each k, the complexes $\operatorname{tr}_1(\overline{E}')_{k,*}|_{\widetilde{M}_X}$ are orthogonally split, we obtain that $I_a = 0$.
(b) We compute

$$\begin{aligned}
I_b =&(p_{W_{Y/P}})_*\left(\sum_k (-1)^k W_1 \bullet \operatorname{ch}(\widetilde{E}'_k|_{W_{Y/P}})\right)\\
=&(p_{W_{Y/P}})_*\Bigg(W_1 \bullet \sum_k (-1)^k (-\operatorname{d}_{\mathcal{D}}\widetilde{\operatorname{ch}}(\mu^1_k) + \operatorname{ch}(\operatorname{tr}_1(\overline{A}_*)_k)\\
&\qquad + \operatorname{ch}(\operatorname{tr}_1(\operatorname{Tot}(\pi^*_{P_{X/M}}\overline{E}\otimes K(\mathcal{O}_X, \overline{N}_{X/M})))_k))\Bigg)\\
=&\sum_k (-1)^k(-(p_{P_{Y/P_{X/M}}})_*\widetilde{\operatorname{ch}}(\eta^3_k) - (p_{\widetilde{P}_{X/M}})_*\widetilde{\operatorname{ch}}(\mu^1_k|_{\widetilde{P}_{X/M}}) + (p_{P_{X/M}})_*\widetilde{\operatorname{ch}}(\eta^1_k))\\
&- \widetilde{\operatorname{ch}}(\overline{A})\\
&- (i_{X/M})_*(\pi_{P_{X/M}})_* T(\overline{\xi}_{Y\hookrightarrow P_{X/M}}) + (i_{Y/M})_* C_T(F, N_{Y/P_{X/M}})\\
&- \sum_k (-1)^k (p_{P_{Y/P_{X/M}}})\widetilde{\operatorname{ch}}(\eta^5_k),
\end{aligned}$$

where $\xi_{Y\hookrightarrow P_{X/M}}$ is as in notation 8.8.
By corollary 2.19 and the fact that the exact sequences $\overline{A}_{k,*}$ are orthogonally split, the term $\widetilde{\operatorname{ch}}(\overline{A})$ vanishes.
Also by corollary 2.19 we can see that

$$\sum_k (-1)^k (p_{\widetilde{P}_{X/M}})_*\widetilde{\operatorname{ch}}(\mu^1_k|_{\widetilde{P}_{X/M}})$$

vanishes.
Therefore we conclude

$$\begin{aligned}
I_b =& \sum_k (-1)^k(-(p_{P_{Y/P_{X/M}}})_*\widetilde{\operatorname{ch}}(\eta^3_k) + (p_{P_{X/M}})_*\widetilde{\operatorname{ch}}(\eta^1_k)) - (p_{P_{Y/P_{X/M}}})\widetilde{\operatorname{ch}}(\eta^5_k)\\
&- (i_{X/M})_*(\pi_{P_{X/M}})_* T(\overline{\xi}_{Y\hookrightarrow P_{X/M}}) + (i_{Y/M})_* C_T(F, N_{Y/P_{X/M}}).
\end{aligned}$$

(c) By the definition of singular Bott-Chern forms we have

$$I_c = -T(\overline{\xi}_{Y\hookrightarrow M}) + (i_{Y/M})_* C_T(F, N_{Y/M}) - \sum_k (-1)^k (p_{P_{Y/M}})_*\widetilde{\operatorname{ch}}(\eta^2_k),$$

(d) Since the restriction of $\widetilde{E}'_*$ to $\widetilde{W}_X$ is orthogonally split, we have $I_d = 0$.

(e) We compute

$$\begin{aligned}I_e =&(p_{P_{Y\times\mathbb{P}^1}})_*\left(\sum_k(-1)^kW_1\bullet\operatorname{ch}(\widetilde{E}'_k|_{P_{Y\times\mathbb{P}^1}})\right)\\ =&(p_{P_{Y\times\mathbb{P}^1}})_*\Bigg(W_1\bullet\sum_k(-1)^k\big(-\mathrm{d}_{\mathcal{D}}\,\widetilde{ch}(\mu_k^2)+\operatorname{ch}(\overline{D}_k)\\ &\qquad+\operatorname{ch}(\pi^*_{P_{Y\times\mathbb{P}^1}}p_Y^*\overline{F}\otimes K(\mathcal{O}_{Y\times\mathbb{P}^1},\overline{N}_{Y\times\mathbb{P}^1/W_X})_k)\big)\Bigg).\end{aligned}$$

The term $\sum(-1)^k\operatorname{ch}(\overline{D}_k)$ vanishes because the complex D_* is orthogonally split. We have

$$\begin{aligned}&\sum_k(-1)^k(p_{P_{Y\times\mathbb{P}^1}})_*(W_1\bullet\operatorname{ch}(\pi^*_{P_{Y\times\mathbb{P}^1}}p_Y^*\overline{F}\otimes K(\overline{\mathcal{O}}_{Y\times\mathbb{P}^1},\overline{N}_{Y\times\mathbb{P}^1/W_X})_k))\\ &=(i_{Y/M})_*\operatorname{ch}(\overline{F})\bullet(p_Y)_*\left(W_1\bullet\pi^*_{P_{Y\times\mathbb{P}^1}}\sum_k(-1)^k\operatorname{ch}(K(\overline{\mathcal{O}}_{Y\times\mathbb{P}^1},\overline{N}_{Y\times\mathbb{P}^1/W_X})_k)\right)\\ &\qquad=(i_{Y/M})_*\operatorname{ch}(\overline{F})\bullet(p_Y)_*\left(W_1\bullet\operatorname{Td}^{-1}(\overline{N}_{Y\times\mathbb{P}^1/W_X})\right)\\ &\qquad\qquad=(i_{Y/M})_*\operatorname{ch}(\overline{F})\bullet\widetilde{\operatorname{Td}^{-1}}(\overline{\varepsilon}_N),\end{aligned}\tag{8.12}$$

where $\overline{\varepsilon}_N$ is the exact sequence (8.6).
Therefore we obtain

$$\begin{aligned}I_e=&-\sum_k(-1)^k(p_{P_{Y/P_{X/M}}})_*\widetilde{\operatorname{ch}}(\eta_k^4)+\sum_k(-1)^k(p_{P_{Y/M}})_*\widetilde{\operatorname{ch}}(\eta_k^2)\\ &+(i_{Y/M})_*\operatorname{ch}(\overline{F})\bullet\widetilde{\operatorname{Td}^{-1}}(\overline{\varepsilon}_N).\end{aligned}$$

(f) Finally we have

$$\begin{aligned}I_f=&-\sum_k(-1)^kT(\overline{\xi}_{X\hookrightarrow M,k})+\sum_k(-1)^k(i_{X/M})_*C_T(E_k,N_{X/M})\\ &-\sum_{k,l}(-1)^{k+l}(p_{P_{X/M}})_*\widetilde{\operatorname{ch}}(\eta^1_{k,l}).\end{aligned}$$

By corollary 2.19 we have that

$$\sum_{m,l}(-1)^{m+l}(p_{P_{X/M}})_*\widetilde{\operatorname{ch}}(\eta^1_{m,l})=\sum_k(-1)^k(p_{P_{X/M}})_*\widetilde{\operatorname{ch}}(\eta^1_k).$$

Thus

$$\begin{aligned}I_f=&-\sum_k(-1)^kT(\overline{\xi}_{X\hookrightarrow M,k})+\sum_k(-1)^k(i_{X/M})_*C_T(E_k,N_{X/M})\\ &-\sum_k(-1)^k(p_{P_{X/M}})_*\widetilde{\operatorname{ch}}(\eta^1_k).\end{aligned}$$

Summing up all the terms we have computed, and taking into account equation (8.10) and the fact that

$$C_T(F, N_{Y/M}) = C_T(F, N_{Y/P_{X/M}})$$

we have obtained the following partial result.

Lemma 8.13. *Let $i_{Y/M} = i_{X/M} \circ i_{Y/X}$ be a composition of closed immersions of complex manifolds. Let T be a theory of singular Bott-Chern classes with C_T its associated characteristic class. Let $\overline{\xi}_{Y\hookrightarrow M}$, $\overline{\xi}_{X\hookrightarrow M,k}$ and $\overline{\xi}_{Y\hookrightarrow P_{X/M}}$ be as in notation 8.8, and let $\overline{\varepsilon}$ be as in* (8.6)*. Then, in the group $\bigoplus_p \widetilde{\mathcal{D}}^{2p-1}(M,p)$, the equation*

$$\begin{aligned} T(\overline{\xi}_{Y\hookrightarrow M}) = \sum_k (-1)^k T(\overline{\xi}_{X\hookrightarrow M,k}) - \sum_k (-1)^k (i_{X/M})_* C_T(E_k, N_{X/M}) \\ + (i_{X/M})_*(\pi_{P_{X/M}})_* T(\overline{\xi}_{Y\hookrightarrow P_{X/M}}) + (i_{Y/M})_* \operatorname{ch}(\overline{F}) \bullet \widetilde{\operatorname{Td}^{-1}}(\overline{\varepsilon}_N) \quad (8.14)\end{aligned}$$

holds.

In order to compute the third term of the right hand side of equation (8.14) we consider the following situation

$$\begin{tikzcd} Y \times_X P_{X/M} \arrow[r, "j"] \arrow[d, bend right, "\pi"'] & P_{X/M} \arrow[d, bend right, "\pi"'] \\ Y \arrow[r, "i"] \arrow[u, bend right, "s"'] & X \arrow[u, bend right, "s"'] \end{tikzcd} .$$

To ease the notation, we denote $P_{X/M}$ by P, $Y \underset{X}{\times} P_{X/M}$ by X' and we denote by P' the projective completion of the normal cone to X' in P and by $\pi_{P'} \colon P' \longrightarrow X'$, $\pi_{X'/Y} \colon X' \longrightarrow Y$ and $\pi_{P'/Y} \colon P' \longrightarrow Y$ the projections. Observe that X and X' intersect transversely along Y. Moreover, $N_{Y/X'} = i^*_{Y/X} N_{X/M}$, $N_{X'/P} = \pi^*_{X'/Y} N_{Y/X}$ and $N_{Y/P} = N_{Y/X} \oplus N_{Y/X'}$. We use these identifications to define metrics on $N_{Y/X'}$, $N_{X'/P}$ and $N_{Y/P}$. Therefore the exact sequence

$$0 \longrightarrow \overline{N}_{Y/X'} \longrightarrow \overline{N}_{Y/P} \longrightarrow i^*_{Y/X'} \overline{N}_{X'/P} \longrightarrow 0$$

is orthogonally split.

We apply the previous lemma to the composition of closed inclusions

$$Y \hookrightarrow X' \hookrightarrow P,$$

the vector bundle $\overline{F}$ over Y and the resolutions

$$\begin{gathered} \pi^* \overline{F} \otimes j^* K(\overline{\mathcal{O}}_X, \overline{N}_{X/M})_* \longrightarrow s_* F \\ \pi^* \overline{E}_* \otimes K(\overline{\mathcal{O}}_X, \overline{N}_{X/M})_k \longrightarrow j_*(\pi^* F \otimes j^* K(\mathcal{O}_X, N_{X/M})_k). \end{gathered}$$

We denote by $\overline{\xi}_{Y\hookrightarrow P}$ and $\overline{\xi}_{X'\hookrightarrow P,k}$ the hermitian embedded vector bundles corresponding to the above resolutions. If $i_{Y/P'}\colon Y\hookrightarrow P'$ is the induced inclusion, we denote by $\overline{\xi}_{Y\hookrightarrow P'}$ the hermitian embedded vector bundle

$$\left(i_{Y/P'},\overline{N}_{Y/P'},\overline{F},\operatorname{Tot}(\pi^{*}_{P'}j^{*}K(\overline{\mathcal{O}}_X,\overline{N}_{X/M})\otimes K(\overline{\mathcal{O}}_{X'},\overline{N}_{X'/P})\otimes(\pi_{P'/Y})^{*}\overline{F})\right).$$

Note that the hermitian embedded vector bundle $\overline{\xi}_{Y\hookrightarrow P}$ agrees with the hermitian embedded vector bundle denoted $\overline{\xi}_{Y\hookrightarrow P_{X/M}}$ in lemma 8.13. Moreover, we have that

$$\overline{\xi}_{X'\hookrightarrow P,k}=\pi^{*}\overline{\xi}_{Y\hookrightarrow X}\otimes K(\overline{\mathcal{O}}_X,\overline{N}_{X/M})_k.$$

Applying lemma 8.13, we obtain

$$\begin{aligned}T(\overline{\xi}_{Y\hookrightarrow P_{X/M}})=&\sum_k(-1)^kT(\overline{\xi}_{X'\hookrightarrow P_{X/M},k})\\&-\sum_k(-1)^kj_{*}C_T(\pi^{*}F\otimes j^{*}K(\mathcal{O}_X,N_{X/M})_k,N_{X'/P})\\&+j_{*}(\pi_{P'})_{*}T(\overline{\xi}_{Y\hookrightarrow P'})\end{aligned}\quad(8.15)$$

By proposition 8.1,

$$\begin{aligned}\sum_k(-1)^kT(\overline{\xi}_{X'\hookrightarrow P_{X/M},k})&=\sum_k(-1)^kT(\pi^{*}\overline{\xi}_{Y\hookrightarrow X}\otimes K(\overline{\mathcal{O}}_X,\overline{N}_{X/M})_k)\\&=T(\pi^{*}\overline{\xi}_{Y\hookrightarrow X})\bullet\sum_k(-1)^k\operatorname{ch}(K(\overline{\mathcal{O}}_X,\overline{N}_{X/M})_k)\\&+\sum_k(-1)^kj_{*}C_T(\pi^{*}F\otimes j^{*}K(\mathcal{O}_X,N_{X/M})_k,N_{X'/P})\\&-\sum_k(-1)^kj_{*}C_T(\pi^{*}F,N_{X'/P})\bullet\operatorname{ch}(K(\mathcal{O}_X,N_{X/M})_k)\end{aligned}\quad(8.16)$$

We now want to compute the term $(i_{X/M})_{*}(\pi_{P_{X/M}})_{*}j_{*}(\pi_{P'})_{*}T(\overline{\xi}_{Y\hookrightarrow P'})$.
Observe that we can identify

$$P'=\mathbb{P}(i^{*}_{Y/X}N_{X/M}\oplus\mathbb{C})\underset{Y}{\times}\mathbb{P}(s^{*}N_{X'/P}\oplus\mathbb{C}),$$

where $s^{*}N_{X'/P}$ is canonically isomorphic to $N_{Y/X}$.
Moreover

$$(i_{X/M})_{*}(\pi_{P_{X/M}})_{*}j_{*}(\pi_{P'})_{*}T(\overline{\xi}_{Y\hookrightarrow P'})=(i_{Y/M})_{*}(\pi_{P'/Y})_{*}T(\overline{\xi}_{Y\hookrightarrow P'}).$$

Definition 8.17. We denote

$$C^{\mathrm{ad}}_T(F,N_{Y/X},i^{*}_{Y/X}N_{X/M})=(\pi_{P'/Y})_{*}T(\overline{\xi}_{Y\hookrightarrow P'})$$

and we define

$$\rho(F,N_{Y/X},i^{*}_{Y/X}N_{X/M})=C_T(F,N_{Y/M})-C^{\mathrm{ad}}_T(F,N_{Y/X},i^{*}_{Y/X}N_{X/M}).\quad(8.18)$$

Lemma 8.19. *The current* $C_T^{\mathrm{ad}}(F, N_{Y/X}, i_{Y/X}^* N_{X/M})$ *is closed and defines a characteristic class of triples of vector bundles. Therefore* ρ *is also a characteristic class. Moreover the class* ρ *does not depend on the theory of singular Bott-Chern classes* T.

Proof. The fact that $C_T^{\mathrm{ad}}(F, N_{Y/X}, i_{Y/X}^* N_{X/M})$ is closed and determines a characteristic class is proved as in 6.16. The independence of ρ from to T is seen as follows. We denote by $\overline{K}'_*$ the complex

$$\operatorname{Tot}(\pi_{P'}^* j^* K(\overline{\mathcal{O}}_X, \overline{N}_{X/M}) \otimes K(\overline{\mathcal{O}}_{X'}, \overline{N}_{X'/P})) \otimes (\pi_{P'/Y})^* \overline{F}.$$

This complex is a resolution of $(i_{Y/P'})_* \overline{F}$
Let W be the blow-up of $P' \times \mathbb{P}^1$ along $Y \times \infty$, and let $\operatorname{tr}_1(\overline{K}')_*$ be the deformation of complexes on W given by theorem 5.4. Just by looking at the rank of the different vector bundles we see that the restriction of $\operatorname{tr}_1(\overline{K}')_*$ to $P_{Y/P'}$, the exceptional divisor of this blow-up, is isomorphic (although not necessarily isometric) to the Koszul complex $K(\overline{F}, \overline{N}_{X/M})_*$. Then, by equation (7.3)

$$\begin{aligned}
T(\overline{\xi}_{Y \hookrightarrow P'}) - (i_{Y/P'})_* C_T(F, N_{Y/M}) = \\
&- (p_W)_* \left(W_1 \bullet \sum_k (-1)^k \operatorname{ch}(\operatorname{tr}_1(\overline{K}')_k) \right) \\
&- \sum_k (-1)^k (p_P)_* \widetilde{\operatorname{ch}}(\operatorname{tr}_1(\overline{K}')_k|_{P_{Y/P'}}, K(\overline{F}, \overline{N}_{X/M})_k).
\end{aligned}$$

Since the right hand side of this equation does not depend on the theory T, the result is proved. $\square$

Using equations (8.15), (8.16), lemma 8.19 and the projection formula, we obtain

$$\begin{aligned}
(\pi_{P_{X/M}})_* T(\overline{\xi}_{Y \hookrightarrow P_{X/M}}) &= \big(T(\overline{\xi}_{Y \hookrightarrow X}) - (i_{Y/X})_* C_T(F, N_{Y/X})\big) \\
&\quad \bullet (\pi_{P_{X/M}})_* \sum_k (-1)^k \operatorname{ch}(K(\mathcal{O}_X, \overline{N}_{X/M})_k) \\
&\quad + (\pi_{P_{X/M}})_* j_* (\pi_{P'})_* T(\overline{\xi}_{Y \hookrightarrow P'}) \\
&= \big(T(\overline{\xi}_{Y \hookrightarrow X}) - (i_{Y/X})_* C_T(F, N_{Y/X})\big) \bullet \operatorname{Td}^{-1}(\overline{N}_{X/M}) \\
&\quad + (i_{Y/X})_* C_T^{\mathrm{ad}}(F, N_{Y/X}, i_{Y/X}^* N_{X/M}) \\
&= \big(T(\overline{\xi}_{Y \hookrightarrow X}) - (i_{Y/X})_* C_T(F, N_{Y/X})\big) \bullet \operatorname{Td}^{-1}(\overline{N}_{X/M}) \\
&\quad + (i_{Y/X})_* C_T(F, N_{Y/M}) - \rho(F, N_{Y/X}, i_{Y/X}^* N_{X/M}).
\end{aligned} \tag{8.20}$$

Joining this equation and lemma 8.13 we obtain the main relationship between singular Bott-Chern classes and composition of closed immersions.

Proposition 8.21. *Let $i_{Y/M} = i_{X/M} \circ i_{Y/X}$ be a composition of closed immersions of complex manifolds. Let T be a theory of singular Bott-Chern classes with C_T its associated characteristic class. Let $\overline{\xi}_{Y\hookrightarrow M}$, $\overline{\xi}_{X\hookrightarrow M,k}$ and $\overline{\xi}_{Y\hookrightarrow P_{X/M}}$ be as in notation 8.8 and let $\overline{\varepsilon}$ be as in* (8.6). *Then, in the group $\bigoplus_p \widetilde{\mathcal{D}}^{2p-1}(M,p)$, we have the equation*

$$\begin{aligned} T(\overline{\xi}_{Y\hookrightarrow M}) = \sum_k (-1)^k T(\overline{\xi}_{X\hookrightarrow M,k}) + (i_{X/M})_*(T(\overline{\xi}_{Y\hookrightarrow X}) \bullet \mathrm{Td}^{-1}(\overline{N}_{X/M})) \\ + (i_{Y/M})_* \operatorname{ch}(\overline{F}) \bullet \widetilde{\mathrm{Td}^{-1}}(\overline{\varepsilon}_N) \\ + (i_{Y/M})_* C_T^{\mathrm{ad}}(F, N_{Y/X}, i_{Y/X}^* N_{X/M}) \\ - (i_{X/M})_*((i_{Y/X})_* C_T(F, N_{Y/X}) \bullet \mathrm{Td}^{-1}(N_{X/M})) \\ - (i_{X/M})_* \sum_k (-1)^k C_T(E_k, N_{X/M}) \end{aligned}$$

We can simplify the formula of proposition 8.21 if we assume that our theory of singular Bott-Chern classes is compatible with the projection formula.

Corollary 8.22. *With the hypothesis of proposition 8.21, assume furthermore that T is compatible with the projection formula. Then*

$$\begin{aligned} T(\overline{\xi}_{Y\hookrightarrow M}) = \sum_k (-1)^k T(\overline{\xi}_{X\hookrightarrow M,k}) + (i_{X/M})_*(T(\overline{\xi}_{Y\hookrightarrow X}) \bullet \mathrm{Td}^{-1}(\overline{N}_{X/M})) \\ + (i_{Y/M})_* \operatorname{ch}(\overline{F}) \bullet \widetilde{\mathrm{Td}^{-1}}(\overline{\varepsilon}_N) \\ + (i_{Y/M})_* \Big[C_T^{\mathrm{ad}}(F, N_{Y/X}, i_{Y/X}^* N_{X/M}) - C_T(F, N_{Y/X}) \bullet \mathrm{Td}^{-1}(i_{Y/X}^* N_{X/M})) \\ - C_T(F, i_{Y/X}^* N_{X/M}) \bullet \mathrm{Td}^{-1}(N_{Y/X}) \Big] \end{aligned}$$

Proof. Since T is compatible with the projection formula, then C_T is also. Therefore, using the Grothendieck-Riemann-Roch theorem for closed immersions at the level of analytic Deligne cohomology classes, we have

$$\begin{aligned} \sum_k (-1)^k C_T(E_k, N_{X/M}) &= C_T(\mathcal{O}_X, N_{X/M}) \bullet \sum_k (-1)^k \operatorname{ch}(E_k) \\ &= C_T(\mathcal{O}_X, N_{X/M}) \bullet (i_{Y/X})_*(\operatorname{ch}(F) \bullet \mathrm{Td}^{-1}(N_{Y/X})) \\ &= (i_{Y/X})_*(i_{Y/X}^* C_T(\mathcal{O}_X, N_{X/M}) \bullet \operatorname{ch}(F) \bullet \mathrm{Td}^{-1}(N_{Y/X})) \\ &= (i_{Y/X})_*(C_T(F, i_{Y/X}^* N_{X/M}) \bullet \mathrm{Td}^{-1}(N_{Y/X})), \end{aligned}$$

which implies the result. □

Definition 8.23. Let T be a theory of singular Bott-Chern classes. We will say that T is *transitive* if the equation

$$T(\overline{\xi}_{Y\hookrightarrow M}) = \sum_k (-1)^k T(\overline{\xi}_{X\hookrightarrow M,k}) + (i_{X/M})_*(T(\overline{\xi}_{Y\hookrightarrow X}) \bullet \operatorname{Td}^{-1}(\overline{N}_{X/M})) \\ + (i_{Y/M})_* \operatorname{ch}(\overline{F}) \bullet \widetilde{\operatorname{Td}^{-1}}(\overline{\varepsilon}_N) \quad (8.24)$$

holds. When equation (8.24) is satisfied for a particular choice of complex immersions and resolutions, we say that the theory T is *transitive with respect to this particular choice.*

We now introduce an abstract version of definition 8.17.

Definition 8.25. Given any characteristic class C of pairs of vector bundles, we will denote

$$C^{\rho}(F, N_1, N_2) := C(F, N_1 \oplus N_2) - \rho(F, N_1, N_2),$$

where ρ is the characteristic class of definition 8.17.

Note that, when T is a theory of singular Bott-Chern classes we have

$$C^{\rho}_T(F, N_1, N_2) = C^{\mathrm{ad}}_T(F, N_1, N_2).$$

Definition 8.26. We will say that a characteristic class C (of pairs of vector bundles) is *ρ-Todd additive* (in the second variable) if it satisfies

$$C(F, N_1 \oplus N_2) = C(F, N_1) \bullet \operatorname{Td}^{-1}(N_2) + C(F, N_2) \bullet \operatorname{Td}^{-1}(N_1) + \rho(F, N_1, N_2)$$

or, equivalently,

$$C^{\rho}(F, N_1, N_2) = C(F, N_1) \bullet \operatorname{Td}^{-1}(N_2) + C(F, N_2) \bullet \operatorname{Td}^{-1}(N_1).$$

A direct consequence of corollary 8.22 is

Corollary 8.27. *Let T be a theory of singular Bott-Chern classes that is compatible with the projection formula. Then it is transitive if and only if the associated characteristic class C_T is ρ-Todd additive.*

Since we are mainly interested in singular Bott-Chern classes that are transitive and compatible with the projection formula, we will study characteristic classes that are compatible with the projection formula and ρ-Todd-additive in the second variable. Since we want to express any characteristic class in terms of a power series we will restrict ourselves to the algebraic category.

Proposition 8.28. *Let C be a class that is compatible with the projection formula and ρ-Todd additive in the second variable. Then C determines a power series $\phi_C(x)$ given by*

$$C(\mathcal{O}_Y, L) = \phi_C(c_1(L)), \quad (8.29)$$

for every complex algebraic manifold Y and algebraic line bundle Y. Conversely, given any power series in one variable $\phi(x)$, there exists a unique characteristic class for algebraic vector bundles that is compatible with the projection formula and ρ-Todd additive in the second variable such that equation (8.29) *holds.*

Proof. This result follows directly from the splitting principle and theorem 1.8. ☐

Remark 8.30. The utility of corollary 8.27 and proposition 8.28 is limited by the fact that we do not know an explicit formula for the class $\rho(\mathcal{O}_Y, N_1, N_2)$. This class is related with the arithmetic difference between $\mathbb{P}_Y(N_1 \oplus N_2 \oplus \mathbb{C})$ and $\mathbb{P}_Y(N_1 \oplus \mathbb{C}) \underset{Y}{\times} \mathbb{P}_Y(N_2 \oplus \mathbb{C})$, the second space being simpler than the first. The main ingredients needed to compute this class are the Bott-Chern classes of the tautological exact sequence. Therefore the work of Mourougane [29] might be useful for computing this class.

Recall that an additive genus is a characteristic class for algebraic vector bundles S such that

$$S(N_1 \oplus N_2) = S(N_1) + S(N_2).$$

Let $\phi(x) = \sum_{i=0}^{\infty} a_i x^i$ be a power series in one variable. There is a one to one correspondence between additive genus and power series characterized by the condition that $S(L) = \phi(c_1(L))$, for each line bundle L.
Since the class ρ does not depend on the theory T it cancels out when considering the difference between two different theories of singular Bott-Chern classes.

Proposition 8.31. *Let C_1 and C_2 be two characteristic classes for pairs of algebraic vector bundles that are compatible with the projection formula and ρ-Todd-additive in the second variable. Then there is a unique additive genus S_{12} such that*

$$C_1(F,N) - C_2(F,N) = \mathrm{ch}(F) \bullet \mathrm{Td}(N)^{-1} \bullet S_{12}(N). \tag{8.32}$$

We can summarize the results of this section in the following theorem.

Theorem 8.33. *There is a one to one correspondence between theories of singular Bott-Chern classes for complex algebraic manifolds that are transitive and compatible with the projection formula, and formal power series $\phi(x) \in \mathbb{R}[[x]]$. To each theory of singular Bott-Chern classes corresponds the power series ϕ such that*

$$C_T(\mathcal{O}_Y, L) = \mathbf{1}_1 \bullet \phi(c_1(L)), \tag{8.34}$$

for every complex algebraic manifold Y and every algebraic line bundle L. To each power series ϕ it corresponds a unique class C, compatible with the projection formula and ρ-Todd-additive in the second variable, characterized by equation (8.34) *and a theory of singular Bott-Chern given by definition 7.4.*

Even if we do not know the exact value of the class ρ another consequence of corollary 8.27 is that, in order to prove the transitivity of a theory of singular Bott-Chern classes it is enough to check it for a particular class of compositions.

Corollary 8.35. *Let T be a theory of singular Bott-Chern classes compatible with the projection formula. Then T is transitive if and only if for any compact complex manifold Y and vector bundles N_1, N_2, the theory T is transitive with respect to the composition of inclusions*

$$Y \hookrightarrow \mathbb{P}_Y(N_1 \oplus \mathbb{C}) \hookrightarrow \mathbb{P}_Y(N_1 \oplus \mathbb{C}) \times_Y \mathbb{P}_Y(N_2 \oplus \mathbb{C})$$

and the Koszul resolutions. □

We can make the previous corollary a little more explicit. Let π_1 and π_2 be the projections from $P := \mathbb{P}_Y(N_1 \oplus \mathbb{C}) \times_Y \mathbb{P}_Y(N_2 \oplus \mathbb{C})$ to $P_1 := \mathbb{P}_Y(N_1 \oplus \mathbb{C})$ and $P_2 := \mathbb{P}_Y(N_2 \oplus \mathbb{C})$ respectively. Let $\overline{K}_1 = K(\overline{\mathcal{O}}_Y, \overline{N}_1)$ and $\overline{K}_2 = K(\overline{\mathcal{O}}_Y, \overline{N}_2)$ be the Koszul resolutions in P_1 and P_2 respectively. Then,

$$\overline{K} = \pi_1^* K_1 \otimes \pi_2^* K_2$$

is a resolution of $\mathcal{O}_Y$ in P. Then the theory T is transitive in this case if

$$T(\overline{K}) = \pi_2^* T(\overline{K}_2) \bullet \pi_1^*(c_{r_1}(\overline{Q}_1) \bullet \operatorname{Td}^{-1}(\overline{Q}_1)) + (i_1)_*(T(\overline{K}_1) \bullet p_1^* \operatorname{Td}^{-1}(\overline{N}_2)),$$

where r_1 is the rank of N_1, $\overline{Q}_1$ is the tautological quotient bundle in P_1 with the induced metric, $i_1 \colon P_1 \longrightarrow P$ is the inclusion and $p_1 \colon P_1 \longrightarrow Y$ is the projection.

The singular Bott-Chern classes that we have defined depend on the choice of a hermitian metric on the normal bundle and behave well with respect inverse images. Nevertheless, when one is interested in covariant functorial properties and, in particular, in a composition of closed immersions, it might be interesting to consider a variant of singular Bott-Chern classes that depend on the choice of metrics on the tangent bundles to Y and X.

Notation 8.36. Let $\overline{\xi} = (i \colon Y \longrightarrow X, \overline{N}, \overline{F}, \overline{E}_* \to i_*F)$ be a hermitian embedded vector bundle. Let $\overline{T}_X$ and $\overline{T}_Y$ be the tangent bundles to X and Y provided with hermitian metrics. As usual we write $\operatorname{Td}(Y) = \operatorname{Td}(\overline{T}_Y)$ and $\operatorname{Td}(X) = \operatorname{Td}(\overline{T}_X)$. We put

$$\overline{\xi}_c = (i \colon Y \longrightarrow X, \overline{T}_X, \overline{T}_Y, \overline{F}, \overline{E}_* \to i_*F).$$

By abuse of notation we will also say that $\overline{\xi}_c$ is a hermitian embedded vector bundle. In this situation we denote by $\overline{\xi}_{N_{Y/X}}$ the exact sequence of hermitian vector bundles

$$\overline{\xi}_{N_{Y/X}} \colon 0 \longrightarrow \overline{T}_Y \longrightarrow i^* \overline{T}_X \longrightarrow \overline{N}_{Y/X} \longrightarrow 0.$$

If there is no danger of confusion we will denote $\overline{N} = \overline{N}_{Y/X}$ and therefore $\overline{\xi}_N = \overline{\xi}_{N_{Y/X}}$.

Definition 8.37. Let T be a theory of singular Bott-Chern classes. Then the *covariant singular Bott-Chern class* associated to T is given by

$$T_c(\overline{\xi}_c) = T(\overline{\xi}) + i_*(\operatorname{ch}(\overline{F}) \bullet \widetilde{\operatorname{Td}^{-1}}(\overline{\xi}_{N_{Y/X}}) \operatorname{Td}(Y)) \tag{8.38}$$

Proposition 8.39. *The covariant singular Bott-Chern classes satisfy the following properties*

(i) The class $T_c(\overline{\xi}_c)$ does not depend on the choice of the metric on $N_{Y/X}$.

(ii) The differential equation

$$\mathrm{d}_{\mathcal{D}} T_c(\overline{\xi}_c) = \sum_k (-1)^k \operatorname{ch}(\overline{E}_k) - i_*(\operatorname{ch}(\overline{F}) \bullet \operatorname{Td}(Y)) \bullet \operatorname{Td}^{-1}(X) \tag{8.40}$$

holds.

(iii) If the theory T is compatible with the projection formula, then

$$T_c(\overline{\xi}_c \otimes \overline{G}) = T_c(\overline{\xi}_c) \bullet \operatorname{ch}(\overline{G}).$$

(iv) If, moreover, the theory T is transitive, then, using notation 8.8 adapted to the current setting, we have

$$\begin{aligned} T_c(\overline{\xi}_{Y\hookrightarrow M,c}) = \sum_k &(-1)^k T_c(\overline{\xi}_{X\hookrightarrow M,k,c}) \\ &+ (i_{X/M})_*(T_c(\overline{\xi}_{Y\hookrightarrow X,c}) \bullet \operatorname{Td}(X)) \bullet \operatorname{Td}^{-1}(M). \end{aligned} \tag{8.41}$$

(v) With the hypothesis of corollary 6.14, we have

$$T_c(\bigoplus_{j\ even} \overline{\xi}_{j,c}) - T_c(\bigoplus_{j\ odd} \overline{\xi}_{j,c}) = [\widetilde{\operatorname{ch}}(\overline{\varepsilon})] - i_*([\widetilde{\operatorname{ch}}(\overline{\chi}) \bullet \operatorname{Td}(Y)]) \bullet \operatorname{Td}^{-1}(X). \tag{8.42}$$

Proof. All the statements follow from straightforward computations. □

9 Homogeneous singular Bott-Chern classes

In this section we will show that, by adding a natural fourth axiom to definition 6.9, we obtain a unique theory of singular Bott-Chern classes that we call homogeneous singular Bott-Chern classes, and we will compare it with the classes previously defined by Bismut, Gillet and Soulé and by Zha.

In the paper [6], Bismut, Gillet and Soulé introduced a theory of singular Bott-Chern classes that is the main ingredient in their construction of direct images for closed immersions.

Strictly speaking, the construction of [6] only produces a theory of singular Bott-Chern classes in the sense of this paper when the metrics involved satisfy a technical condition, called Condition (A) of Bismut. Nevertheless, there is a unique way to extend the definition of [6] from metrics satisfying Bismut's condition (A) to general metrics in such a way that one obtains a theory of singular Bott-Chern classes in the sense of this paper.

In his thesis [32], Zha gave another definition of singular Bott-Chern classes, and he also used them to define direct images for closed immersions in Arakelov theory.

We will recall the construction of both theories of singular Bott-Chern classes and we will show that they agree with the theory of homogeneous singular Bott-Chern classes.

We warn the reader that the normalizations we use differ from the normalizations in [6] and [32]. The main difference is that we insist on using the algebro-geometric twist in cohomology, whereas in the other two papers the authors use cohomology with real coefficients.

Let r_F and r_N be two positive integers. Let Y be a complex manifold and let $\overline{F}$ and $\overline{N}$ be two hermitian vector bundles of rank r_F and r_N respectively. Let $P = \mathbb{P}(N \oplus \mathbb{C})$ and let s be the zero section. We will follow the notations of definition 5.3. Then $T(K(\overline{F}, \overline{N}))$ satisfies the differential equation

$$\mathrm{d}_{\mathcal{D}}\, T(K(\overline{F}, \overline{N})) = c_{r_N}(\overline{Q})\, \mathrm{Td}^{-1}(\overline{Q})\, \mathrm{ch}(\pi_P^*\overline{F}) - s_*(\mathrm{ch}(\overline{F})\, \mathrm{Td}^{-1}(\overline{N})).$$

Therefore, the class

$$\widetilde{e}_T(\overline{F}, \overline{N}) := T(K(\overline{F}, \overline{N})) \bullet \mathrm{Td}(\overline{Q}) \bullet \mathrm{ch}^{-1}(\pi_P^*\overline{F})$$

satisfies the simpler equation

$$\mathrm{d}_{\mathcal{D}}\, \widetilde{e}_T(\overline{F}, \overline{N}) = [c_{r_N}(\overline{Q})] - \delta_Y. \tag{9.1}$$

Observe that the right hand side of this equation belongs to $\mathcal{D}_D^{2r_N}(P, r_N)$. Thus it seems natural to introduce the following definition.

Definition 9.2. Let T be a theory of singular Bott-Chern classes of rank $r_F > 0$ and codimension r_N. Then the class

$$\widetilde{e}_T(\overline{F}, \overline{N}) = T(K(\overline{F}, \overline{N})) \bullet \mathrm{Td}(\overline{Q}) \bullet \mathrm{ch}^{-1}(\pi_P^*\overline{F})$$

is called the *Euler-Green class associated to* T. The class $T(K(\overline{F}, \overline{N}))$ is said to be *homogeneous* if

$$\widetilde{e}_T(\overline{F}, \overline{N}) \in \widetilde{\mathcal{D}}_D^{2r_N-1}(P, r_N).$$

A theory of singular Bott-Chern classes of rank 0 is said to be *homogeneous* if it agrees with the theory of Bott-Chern classes associated to the Chern character. Finally, a theory of singular Bott-Chern classes is said to be *homogeneous* if its restrictions to all ranks and codimensions are homogeneous.

The main interest of the above definition is the following result.

Theorem 9.3. *Given two positive integers r_F and r_N there exists a unique theory of homogeneous singular Bott-Chern classes of rank r_F and codimension r_N.*

Proof. The proof of this result is based on the theory of Euler-Green classes. Let $P = \mathbb{P}(N \oplus \mathbb{C})$ be as before, and let s denote the zero section of P. Let D_∞ be the subvariety of P that parametrizes the lines contained in N. Then $D_\infty = \mathbb{P}(N)$.

Lemma 9.4. *There exists a unique class $\widetilde{e}(P,\overline{Q},s) \in \widetilde{\mathcal{D}}_D^{2r_N-1}(P,r_N)$ such that*

(i) It satisfies

$$\mathrm{d}_{\mathcal{D}}\, \widetilde{e}(P,\overline{Q},s) = [c_{r_N}(\overline{Q})] - \delta_Y. \tag{9.5}$$

(ii) The restriction $\widetilde{e}(P,\overline{Q},s)|_{D_\infty} = 0$.

Proof. We first show the uniqueness. Assume that $\widetilde{e}$ and $\widetilde{e}'$ are two classes that satisfy the hypothesis of the theorem. Then $\widetilde{e}' - \widetilde{e}$ is closed. Hence it determines a cohomology class in $H^{2r_N-1}_{\mathcal{D}^{\mathrm{an}}}(P,r_N)$. Since, by theorem 1.2, the restriction

$$H^{2r_N-1}_{\mathcal{D}^{\mathrm{an}}}(P,r_N) \longrightarrow H^{2r_N-1}_{\mathcal{D}^{\mathrm{an}}}(D_\infty,r_N) \tag{9.6}$$

is an isomorphism, condition (ii) implies that $\widetilde{e}' = \widetilde{e}$. Now we prove the existence. Since Y is the zero locus of the section s, that is transversal to the zero section of Q, we know that the currents $[c_{r_N}]$ and δ_Y are cohomologous. Therefore there exists an element $\widetilde{a} \in \widetilde{\mathcal{D}}_D^{2r_N-1}(P,r_N)$ such that $\mathrm{d}_{\mathcal{D}}\,\widetilde{a} = [c_{r_N}(\overline{Q})] - \delta_Y$. Since $\overline{Q}$ restricted to D_∞ splits as an orthogonal direct sum

$$\overline{Q}|_{D_\infty} = \overline{S} \oplus \overline{\mathbb{C}} \tag{9.7}$$

where the metric on the factor $\mathbb{C}$ is trivial, and the section s restricts to the constant section 1, we obtain that $([c_{r_N}(\overline{Q})] - \delta_Y)|_{D_\infty} = 0$. Therefore $\widetilde{a}$ determines a class in $H^{2r_N-1}_{\mathcal{D}^{\mathrm{an}}}(P,r_N)$. Using again that (9.6) is an isomorphism, we find an element $\widetilde{b} \in H^{2r_N-1}_{\mathcal{D}^{\mathrm{an}}}(P,r_N)$, such that $\widetilde{e} = \widetilde{a} - \widetilde{b}$ satisfies the conditions of the lemma. □

We continue with the proof of theorem 9.3. We first prove the uniqueness. Let T be a theory of homogeneous singular Bott-Chern classes. The splitting (9.7) implies easily that the restriction of the Koszul resolution $K(\overline{F},\overline{N})$ to D_∞ is orthogonally split. By the functoriality of singular Bott-Chern classes, $T(K(\overline{F},\overline{N}))|_{D_\infty} = 0$. Thus the class

$$\widetilde{e}_T(\overline{F},\overline{N}) := T(K(\overline{F},\overline{N})) \bullet \mathrm{Td}(\overline{Q}) \bullet \mathrm{ch}^{-1}(\pi_P^*\overline{F}) \in \widetilde{\mathcal{D}}_D^{2r_N-1}(P,r_N)$$

satisfies the two conditions of lemma 9.4. Therefore $\widetilde{e}_T(\overline{F},\overline{N}) = \widetilde{e}(P,\overline{Q},s)$ and

$$T(K(\overline{F},\overline{N})) = \widetilde{e}(P,\overline{Q},s) \bullet \mathrm{Td}^{-1}(\overline{Q}) \bullet \mathrm{ch}(\pi_P^*\overline{F}), \tag{9.8}$$

where the right hand side does not depend on the theory T. In consequence we have that

$$C_T(F,N) = (\pi_P)_* T(K(\overline{F},\overline{N})) \tag{9.9}$$

does not depend on the theory T. Thus by the uniqueness in theorem 7.1 we obtain the uniqueness here.
For the existence we observe

Lemma 9.10. *The current*

$$C(F,N) = (\pi_P)_*(\widetilde{e}(P,\overline{Q},s) \bullet \operatorname{Td}^{-1}(\overline{Q})) \bullet \operatorname{ch}(\overline{F})$$

is a characteristic class for pairs of vector bundles of rank r_F and r_N.

Proof. We first compute, using equation (9.5) and corollary 3.8,

$$\begin{aligned}
\mathrm{d}_{\mathcal{D}}\, C(F,N) &= (\pi_P)_* \left(\mathrm{d}_{\mathcal{D}}\, \widetilde{e}(P,\overline{Q},s) \bullet \operatorname{Td}^{-1}(\overline{Q})\right) \bullet \operatorname{ch}(\overline{F}) \\
&= (\pi_P)_* \left(([c_{r_N}(\overline{Q})] - \delta_Y) \bullet \operatorname{Td}^{-1}(\overline{Q})\right) \bullet \operatorname{ch}(\overline{F}) \\
&= (\pi_P)_* \left(c_{r_N}(\overline{Q}) \bullet \operatorname{Td}^{-1}(\overline{Q})\right) \bullet \operatorname{ch}(\overline{F}) - \operatorname{Td}^{-1}(\overline{N}) \bullet \operatorname{ch}(\overline{F}) \\
&= 0.
\end{aligned}$$

Thus $C(F,N)$ determines a cohomology class. This class is functorial by construction. By proposition 1.7 this class does not depend on the metric and defines a characteristic class. □

By the existence in theorem 7.1 we obtain a theory of singular Bott-Chern classes T_C that is easily seen to be homogeneous. □

A reformulation of theorem 9.3 is

Theorem 9.11. *There exists a unique way to associate to each hermitian embedded vector bundle $\overline{\xi} = (i\colon Y \longrightarrow X, \overline{N}, \overline{F}, \overline{E}_*)$ a class of currents*

$$T^h(\overline{\xi}) \in \bigoplus_p \widetilde{\mathcal{D}}_D^{2p-1}(X, N^*_{Y,0}, p)$$

that we call homogeneous singular Bott-Chern class, satisfying the following properties

(i) *(Differential equation) The equality*

$$\mathrm{d}_{\mathcal{D}}\, T^h(\overline{\xi}) = \sum_i (-1)^i [\operatorname{ch}(\overline{E}_i)] - i_*([\operatorname{Td}^{-1}(\overline{N}) \operatorname{ch}(\overline{F})]) \tag{9.12}$$

holds.

(ii) *(Functoriality) For every morphism $f\colon X' \longrightarrow X$ of complex manifolds that is transverse to Y,*

$$f^* T^h(\overline{\xi}) = T^h(f^* \overline{\xi}).$$

(iii) *(Normalization) Let* $\overline{A}=(A_*,g_*)$ *be a non-negatively graded orthogonally split complex of vector bundles. Write* $\overline{\xi}\oplus\overline{A}=(i\colon Y\longrightarrow X,\overline{N},\overline{F},\overline{E}_*\oplus\overline{A}_*)$. *Then* $T^h(\overline{\xi})=T^h(\overline{\xi}\oplus\overline{A})$. *Moreover, if* $X=\operatorname{Spec}\mathbb{C}$ *is one point,* $Y=\emptyset$ *and* $\overline{E}_*=0$, *then* $T^h(\overline{\xi})=0$.

(iv) *(Homogeneity) If* $r_F=\operatorname{rk}(F)>0$ *and* $r_N=\operatorname{rk}(N)>0$, *then, with the notations of definition 9.2,*

$$T^h(K(\overline{F},\overline{N}))\bullet\operatorname{Td}(\overline{Q})\bullet\operatorname{ch}^{-1}(\pi_P^*\overline{F})\in\widetilde{\mathcal{D}}_D^{2r_N-1}(P,r_N).$$

□

The class $\widetilde{e}(P,\overline{Q},s)$ of lemma 9.4 is a particular case of the Euler-Green classes introduced by Bismut, Gillet and Soulé in [6]. The basic properties of the Euler-Green classes are summarized in the following results.

Proposition 9.13. *Let* X *be a complex manifold, let* $\overline{E}$ *be a hermitian holomorphic vector bundle of rank* r *and let* s *be a holomorphic section of* E *that is transverse to the zero section. Denote by* Y *the zero locus of* s. *There is a unique way to assign to each* $(X,\overline{E},s)$ *as before a class of currents*

$$\widetilde{e}(X,\overline{E},s)\in\widetilde{\mathcal{D}}_D^{2r-1}(X,N_{Y,0}^*,r)$$

satisfying the following properties

(i) *(Differential equation)*

$$\mathrm{d}_{\mathcal{D}}\,\widetilde{e}(X,\overline{E},s)=c_r(\overline{E})-\delta_Y. \tag{9.14}$$

(ii) *(Functoriality) If* $f\colon X'\longrightarrow X$ *is a morphism transverse to* Y *then*

$$\widetilde{e}(X',f^*\overline{E},f^*s)=f^*\widetilde{e}(X,\overline{E},s). \tag{9.15}$$

(iii) *(Multiplicativity) Let* $\overline{E}_1$ *and* $\overline{E}_2$ *be hermitian holomorphic vector bundles, and let* s_1 *and* s_2 *be holomorphic sections of* $\overline{E}_1$ *and* $\overline{E}_2$ *respectively that are transverse to the zero section and with zero locus* Y_1 *and* Y_2. *We write* $\overline{E}=\overline{E}_1\oplus\overline{E}_2$ *and* $s=s_1\oplus s_2$. *Assume that* s *is transverse to the zero section; hence* Y_1 *and* Y_2 *meet transversely. With this hypothesis we have*

$$\begin{aligned}\widetilde{e}(X,\overline{E},s)&=\widetilde{e}(X,\overline{E}_1,s_1)\wedge c_{r_2}(\overline{E}_2)+\delta_{Y_1}\wedge\widetilde{e}(X,\overline{E}_2,s_2)\\&=\widetilde{e}(X,\overline{E}_1,s_1)\wedge\delta_{Y_2}+c_{r_1}(\overline{E}_1)\wedge\widetilde{e}(X,\overline{E}_2,s_2).\end{aligned}$$

(iv) *(Line bundles) If* $\overline{L}$ *is a hermitian line bundle and* s *is a section of* L, *then*

$$\widetilde{e}(X,\overline{L},s)=-\log\|s\|. \tag{9.16}$$

Proof. Bismut, Gillet and Soulé prove the existence by constructing explicitly an Euler-Green current in the total space of E and pulling it back to X by the section s. For the uniqueness, first we see that properties (i) and (ii) imply that, if h_0 and h_1 are two hermitian metrics in E, then

$$\widetilde{e}(X,(E,h_0),s)-\widetilde{e}(X,(E,h_1),s)=\widetilde{c}_r(E,h_0,h_1). \tag{9.17}$$

We now consider $\pi\colon P=\mathbb{P}(E\oplus\mathbb{C})\longrightarrow X$, with the tautological exact sequence

$$0\longrightarrow\mathcal{O}(-1)\longrightarrow\pi^*E\oplus\mathbb{C}\longrightarrow Q\longrightarrow 0$$

On Q we consider the metric induced by the metric of $\overline{E}$ and the trivial metric on the factor $\mathbb{C}$, and let s_Q the section of Q induced by the section 1 of $\mathbb{C}$. Let D_∞ be as in lemma 9.4. Then properties (ii) to (iv) imply that $\widetilde{e}(P,\overline{Q},s_Q)|_{D_\infty}=0$. Hence by lemma 9.4 $\widetilde{e}$ is uniquely determined. Finally, let $f\colon X\longrightarrow P$ be the map given by $x\longmapsto(s(x):-1)$. Then $f^*Q\cong E$, although they are not necessarily isometric, and $f^*s_Q=s$. Therefore, the functoriality and equation (9.17) determine $\widetilde{e}(X,\overline{E},s)$.

To prove the existence, we use lemma 9.4, functoriality and equation (9.17) to define the Euler-Green classes. It is easy to show that they are well defined and satisfy properties (i) to (iv). □

Equation (9.8) relating homogeneous singular Bott-Chern classes and Euler-Green classes in a particular case can be generalized to arbitrary vector bundles.

Proposition 9.18. *Let X be a complex manifold, $\overline{E}$ a hermitian vector bundle over X, s a section of E transversal to the zero section and $i\colon Y\longrightarrow X$ the zero locus of s. Let $K(\overline{E})$ be the Koszul resolution of $i_*\mathcal{O}_Y$ determined by $\overline{E}$ and s. We can identify $N_{Y/X}$ with i^*E. We denote by $\overline{N}_{Y/X}$ the vector bundle with the metric induced by the above identification. Then*

$$T^h(i,\overline{\mathcal{O}}_Y,\overline{N}_{Y/X},K(\overline{E}))=\widetilde{e}(X,\overline{E},s)\bullet\operatorname{Td}^{-1}(\overline{E}).$$

Proof. Let $P=\mathbb{P}(E\oplus\mathbb{C})$. We follow the notation of proposition 9.13. We denote by h_0 the original metric of $\overline{E}$ and by h_1 the metric induced by the isomorphism $E\cong f^*Q$. Observe that h_0 and h_1 agree when restricted to Y, because the preimage of $\overline{Q}$ by the zero section agrees with $\overline{E}$. Hence there is an isometry $\overline{N}_{Y/X}\cong i^*f^*\overline{Q}$. We denote $T^h(K(\overline{E}))=T^h(i,\overline{\mathcal{O}}_Y,\overline{N}_{Y/X},K(\overline{E}))$.

Then we have

$$\begin{aligned}
T^h(K(\overline{E})) &= f^*T^h(K(\overline{\mathcal{O}_X},\overline{E})) + \sum_i (-1)^i \widetilde{\mathrm{ch}}(\bigwedge^i E^\vee, h_0, h_1)\\
&= f^*(\widetilde{e}(P,\overline{Q},s_Q)\bullet \mathrm{Td}^{-1}(\overline{Q})) + \widetilde{c}_r(E,h_0,h_1)\bullet \mathrm{Td}^{-1}(E,h_1)\\
&\qquad + c_r(E,h_0)\bullet \widetilde{\mathrm{Td}^{-1}}(E,h_0,h_1)\\
&= \widetilde{e}(X,\overline{E},s)\bullet \mathrm{Td}^{-1}(E,h_1) - \widetilde{c}_r(E,h_0,h_1)\bullet \mathrm{Td}^{-1}(E,h_1)\\
&\qquad + \widetilde{c}_r(E,h_0,h_1)\bullet \mathrm{Td}^{-1}(E,h_1) + c_r(E,h_0)\bullet \widetilde{\mathrm{Td}^{-1}}(E,h_0,h_1)\\
&= \widetilde{e}(X,\overline{E},s)\bullet \mathrm{Td}^{-1}(E,h_0) - \widetilde{e}(X,\overline{E},s)\bullet \mathrm{d}_{\mathcal{D}}\widetilde{\mathrm{Td}^{-1}}(E,h_0,h_1)\\
&\qquad + c_r(E,h_0)\bullet \widetilde{\mathrm{Td}^{-1}}(E,h_0,h_1)\\
&= \widetilde{e}(X,\overline{E},s)\bullet \mathrm{Td}^{-1}(E,h_0) - \mathrm{d}_{\mathcal{D}}\widetilde{e}(X,\overline{E},s)\bullet \widetilde{\mathrm{Td}^{-1}}(E,h_0,h_1)\\
&\qquad + c_r(E,h_0)\bullet \widetilde{\mathrm{Td}^{-1}}(E,h_0,h_1)\\
&= \widetilde{e}(X,\overline{E},s)\bullet \mathrm{Td}^{-1}(E,h_0) + i_*\widetilde{\mathrm{Td}^{-1}}(E,h_0,h_1)|_Y\\
&= \widetilde{e}(X,\overline{E},s)\bullet \mathrm{Td}^{-1}(\overline{E}),
\end{aligned}$$

which concludes the proof. □

Theorem 9.19. *The theory of homogeneous singular Bott-Chern classes is compatible with the projection formula and transitive.*

Proof. We have

$$\begin{aligned}
C_{T^h}(F,N) &= (\pi_P)_*T^h(K(\overline{F},\overline{N}))\\
&= (\pi_P)_*(\widetilde{e}(P,\overline{Q},s)\bullet \mathrm{Td}^{-1}(\overline{Q})\bullet \mathrm{ch}(\pi_P^*\overline{F}))\\
&= (\pi_P)_*(\widetilde{e}(P,\overline{Q},s)\bullet \mathrm{Td}^{-1}(\overline{Q}))\bullet \mathrm{ch}(\overline{F})\\
&= C_{T^h}(\mathcal{O}_Y,N)\bullet \mathrm{ch}(F).
\end{aligned}$$

Thus C_{T^h} is compatible with the projection formula.
We now prove the transitivity. Let Y, N_1 and N_2 be as in corollary 8.35. We follow the notation after this corollary. Then applying proposition 9.18 we obtain

$$T^h(\overline{K}) = \widetilde{e}(P,\pi_1^*\overline{Q}_1\oplus\pi_2^*\overline{Q}_2, s_1+s_2)\bullet \mathrm{Td}^{-1}(\pi_1^*\overline{Q}_1\oplus\pi_2^*\overline{Q}_2), \tag{9.20}$$

where s_i denote the tautological section of $\overline{Q}_i$ or its preimage by π_i.
Then, by proposition 9.13 (iii), taking into account that $Y_1 = P_2$,

$$\begin{aligned}
T^h(\overline{K}) = \pi_1^*(c_{r_1}(\overline{Q}_1)\,\mathrm{Td}^{-1}(\overline{Q}_1))\bullet \pi_2^*(\widetilde{e}(P_2,\overline{Q}_2,s_2)\,\mathrm{Td}^{-1}(\overline{Q}_2))&\\
+ (i_1)_*(\widetilde{e}(P_1,\overline{Q}_1,s_1)\,\mathrm{Td}^{-1}(\overline{Q}_1)\bullet p_1^*\,\mathrm{Td}^{-1}(\overline{N}_2)).&
\end{aligned} \tag{9.21}$$

Applying again proposition 9.18 we obtain

$$T^h(\overline{K}) = \pi_1^*(c_{r_1}(\overline{Q}_1)\operatorname{Td}^{-1}(\overline{Q}_1)) \bullet \pi_2^*(T^h(\overline{K}_2)) + (i_1)_*(T^h(\overline{K}_1) \bullet p_1^* \operatorname{Td}^{-1}(\overline{N}_2)). \tag{9.22}$$

Thus, by corollary 8.35 the theory of homogeneous singular Bott-Chern classes is transitive. □

We next recall the construction of singular Bott-Chern classes of Bismut, Gillet and Soulé. Let $i\colon Y \longrightarrow X$ be a closed immersion of complex manifolds and let $\overline{\xi} = (i, \overline{N}, \overline{F}, \overline{E}_*)$ be a hermitian embedded vector bundle. We consider the associated complex of sheaves

$$0 \to E_n \xrightarrow{v} \dots \xrightarrow{v} E_0 \to 0,$$

where we denote by v the differential of this complex.
This complex is exact for all $x \in X \setminus Y$. The cohomology sheaves of this complex are holomorphic vector bundles on Y which we denote by

$$H_n = \mathcal{H}_n(E_*|_Y), \quad H = \bigoplus_n H_n.$$

For each $x \in Y$ and $U \in T_xX$ we denote by $\partial_U v(x)$ the derivative of the map v calculated in any holomorphic trivialization of E near x. Then $\partial_U v(x)$ acts on H_x. Moreover, this action only depends on the class y of U in N_x. We denote it by $\partial_y v(x)$. Moreover $(\partial_y v(x))^2 = 0$; therefore the pull-back of H to the total space of N together with $\partial_y v$ is a complex that we denote by $(H, \partial_y v)$.
On the total space of N, the interior multiplication by $y \in N$ turns $\bigwedge N^\vee$ into a Koszul complex. By abuse of notation we denote also by ι_y the operator $\iota_y \otimes 1$ acting on $\bigwedge N^\vee \otimes F$. There is a canonical isomorphism between the complexes $(H, \partial_y v)$ and $(\bigwedge N^\vee \otimes F, \iota_y)$. An explicit description of this isomorphism can be found in [3] §1.
Let v^* be the adjoint of the operator v with respect to the metrics of $\overline{E}_*$. Then we have an identification of vector bundles over Y

$$H_k = \{f \in E_k \mid vf = v^*f = 0\}.$$

This identification induces a hermitian metric on H_k, and hence on H. Note that the metrics on N and F also induce a hermitian metric on $\bigwedge N^\vee \otimes F$.

Definition 9.23. We say that $\overline{\xi} = (i, \overline{N}, \overline{F}, \overline{E}_*)$ satisfies Bismut assumption (A) if the canonical isomorphism between $(H, \partial_y v)$ and $(\bigwedge N^\vee \otimes F, \iota_y)$ is an isometry.

Proposition 9.24. *Let $\overline{\xi} = (i, \overline{N}, \overline{F}, \overline{E}_*)$ be as before, with $\overline{N} = (N, h_N)$ and $\overline{F} = (F, h_F)$. Then there exist metrics h'_{E_k} over E_k such that the hermitian embedded vector bundle $\overline{\xi}' = (i, \overline{N}, \overline{F}, (E_*, h'_{E_*}))$ satisfies Bismut assumption (A).*

Proof. This is [3] proposition 1.6. □

Let ∇^E be the canonical hermitian holomorphic connection on E and let $V = v + v^*$. Then

$$A_u = \nabla^E + \sqrt{u}V$$

is a superconnection on E.
Let ∇^H be the canonical hermitian connection on H. Then

$$B = \nabla^H + \partial_y v + (\partial_y v)^*$$

is a superconnection on H.
Let N_H be the number operator on the complex (E, v), that is, N_H acts on E_k by multiplication by k, and let Tr_s denote the supertrace. Recall that here we are using the symbol [ ] to denote the current associated to a locally integrable differential form and the symbol δ_Y to denote the current integration along a subvariety, both with the normalizations of notation 1.3.
For $0 < Re(s) \leq 1/2$ let $\zeta_E(s)$ be the current on X given by the formula

$$\zeta_E(s) = \frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} \Big\{ \left[\mathrm{Tr}_s\left(N_H \exp(-A_u^2)\right)\right] - i_* \left[\int_N \mathrm{Tr}_s\left(N_H \exp(-B^2)\right)\right] \Big\} \,\mathrm{d}\, u. \quad (9.25)$$

This current is well defined and extends to a current that depends holomorphically on s near 0.

Definition 9.26. Assume that $\overline{\xi} = (i, \overline{N}, \overline{F}, \overline{E}_*)$ satisfies Bismut assumption (A). Then we denote

$$T^{BGS}(\overline{\xi}) = -\frac{1}{2}\zeta'_E(0).$$

By abuse of notation we will denote also by $T^{BGS}(\overline{\xi})$ its class in $\widetilde{\bigoplus}_p \widetilde{\mathcal{D}}_D^{2p-1}(X, p)$.
Let now $\overline{\xi} = (i, \overline{N}, \overline{F}, (E_*, h_{E_*}))$ be general and let $\overline{\xi}' = (i, \overline{N}, \overline{F}, (E_*, h'_{E_*}))$ be any hermitian embedded vector bundle satisfying assumption (A) provided by proposition 9.24. Then we denote

$$T^{BGS}(\overline{\xi}) = T^{BGS}(\overline{\xi}') + \sum_i (-1)^i \widetilde{\mathrm{ch}}(E_i, h_{E_i}, h'_{E_i}),$$

where $\widetilde{\mathrm{ch}}(E_i, h_{E_i}, h'_{E_i})$ is as in definition 2.13.

Remark 9.27. This definition only agrees (up to a normalization factor) with the definition in [6] for hermitian embedded vector bundles that satisfy assumption (A).

Theorem 9.28. *The assignment that, to each hermitian embedded vector bundle $\overline{\xi}$, associates the current $T^{BGS}(\overline{\xi})$, is a theory of singular Bott-Chern classes that agrees with T^h.*

Proof. First we have to show that, when $\overline{\xi}$ does not satisfy assumption (A) then $T^{BGS}(\overline{\xi})$ is well defined. Assume that $\overline{\xi}'' = (i, \overline{N}, \overline{F}, (E_*, h'_{E_*}))$ is another choice of hermitian embedded vector bundle satisfying assumption (A). By lemma 2.17 we have that

$$\widetilde{\mathrm{ch}}(E_i, h_i, h'_i) + \widetilde{\mathrm{ch}}(E_i, h'_i, h''_i) + \widetilde{\mathrm{ch}}(E_i, h''_i, h_i) = 0.$$

By [6] theorem 2.5 we have that

$$T^{BGS}(\overline{\xi}') - T^{BGS}(\overline{\xi}'') = \sum_i (-1)^i \widetilde{\mathrm{ch}}(E_i, h'_{E_i}, h''_{E_i}).$$

Summing up we obtain that $T^{BGS}(\overline{\xi})$ is well defined.
If the hermitian embedded vector bundle $\overline{\xi}$ satisfies Bismut assumption (A) then, by [6] theorem 1.9, $T^{BGS}(\overline{\xi})$ satisfies equation (6.10). If $\overline{\xi}$ does not satisfy assumption (A) then, combining [6] theorem 1.9 and equation (2.4), we also obtain that $T^{BGS}(\overline{\xi})$ satisfies equation (6.10).
The functoriality property is [6] theorem 1.10.
In order to prove the normalization property, let $\overline{\xi} = (i\colon Y \longrightarrow X, \overline{N}, \overline{F}, \overline{E}_*)$ be a hermitian embedded vector bundle that satisfies assumption (A) and let $\overline{A}$ be a non-negatively graded orthogonally split complex of vector bundles on X. Observe that $\overline{A}$ is also a (trivial) hermitian embedded vector bundle. Then $\overline{A}$ and $\overline{\xi} \oplus \overline{A}$ also satisfy assumption (A). By [6] theorem 2.9

$$T^{BGS}(\overline{\xi} \oplus \overline{A}) = T^{BGS}(\overline{\xi}) + T^{BGS}(\overline{A}).$$

But by [5] remark 2.3, $T^{BGS}(\overline{A})$ agrees with the Bott-Chern class associated to the Chern character and the exact complex $\overline{A}$. Since A is orthogonally split we have $T^{BGS}(\overline{A}) = 0$. Now the case when ξ does not satisfy assumption (A) follows from the definition.
By [6] theorem 3.17, with the hypothesis of proposition 9.18, we have that

$$\begin{aligned} T^{BGS}(i, \overline{\mathcal{O}}_Y, \overline{N}_{Y/X}, K(\overline{E})) &= \widetilde{e}(X, \overline{E}, s) \bullet \mathrm{Td}^{-1}(\overline{E}) \\ &= T^h(i, \overline{\mathcal{O}}_Y, \overline{N}_{Y/X}, K(\overline{E})). \end{aligned}$$

From this it follows that $C_{T^{BGS}} = C_{T^h}$ and by theorem 7.1, $T^{BGS} = T^h$. □

We now recall Zha's construction. Note that, in order to obtain a theory of singular Bott-Chern classes, we have changed the normalization convention from the one used by Zha. Note also that Zha does not define explicitly a singular Bott-Chern class, but such a definition is implicit in his definition of direct images for closed immersions. Let Y be a complex manifold and let $\overline{N} = (N, h)$ be a hermitian vector bundle. We denote $P = \mathbb{P}(N \oplus \mathbb{C})$. Let

$\pi\colon P \longrightarrow Y$ denote the projection and let $\iota\colon Y \longrightarrow P$ denote the inclusion as the zero section. On P we consider the tautological exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \pi^* N \oplus \mathcal{O}_P \longrightarrow Q \longrightarrow 0.$$

Let h_1 denote the hermitian metric on $Q^\vee$ induced by the metric of N and the trivial metric on $\mathcal{O}_P$ and let h_0 denote the semi-definite hermitian form on $Q^\vee$ induced by the map $Q^\vee \longrightarrow \mathcal{O}_P$ obtained from the above exact sequence and the trivial metric on $\mathcal{O}_P$. Let $h_t = (1-t^2)h_0 + t^2 h_1$. It is a hermitian metric on $Q^\vee$. We will denote $\overline{Q}_t^\vee = (Q^\vee, h_t)$. Let ∇_t be the associated hermitian holomorphic connection and let N_t denote the endomorphism defined by

$$\frac{\mathrm{d}}{\mathrm{d}\, t} \langle v, w\rangle_t = \langle N_t v, w\rangle .$$

For each $n \geq 1$, let Det denote the alternate n-linear form on the space of n by n matrices such that

$$\det(A) = \operatorname{Det}(A, \dots, A).$$

We denote $\det(B; A) = \operatorname{Det}(B, A, \dots, A)$.
Zha introduced the differential form

$$\widetilde{e}_Z(\overline{Q}^\vee) = \frac{-1}{2} \lim_{s\to 0} \int_s^1 \det(N_t, \nabla_t^2)\,\mathrm{d}\, t \tag{9.29}$$

which is a smooth form on $P \setminus \iota(Y)$, locally integrable on P. Hence it defines a current, also denoted by $\widetilde{e}_Z(\overline{Q}^\vee)$ on P. The important property of this current is that it satisfies

$$\mathrm{d}_{\mathcal{D}}\, \widetilde{e}_Z(Q^\vee) = c_n(\overline{Q}_1) - \delta_Y. \tag{9.30}$$

In [32], Zha denotes by $C(\overline{Q}^\vee)$ a form that differs from $\widetilde{e}_Z$ by the normalization factor and the sign. We denote it by $\widetilde{e}_Z$ because it agrees with the Euler-Green current introduced in [6].

Proposition 9.31. *The equality*

$$\widetilde{e}_Z(Q^\vee) = \widetilde{e}(P, \overline{Q}_1, s_Q)$$

holds.

Proof. With the notations of lemma 9.4, both classes satisfy equation (9.30) and their restriction to D_∞ is zero. By lemma 9.4 they agree. □

Definition 9.32. Let $\overline{\xi} = (i\colon Y \longrightarrow X, \overline{N}, \overline{F}, \overline{E}_*)$ be as in definition 6.9. Let $\overline{A}_*$, $\operatorname{tr}_1(\overline{E})_*$ and $\overline{\eta}_*$ be as in (7.2). Then we define

$$\begin{aligned} T^Z(\overline{\xi}) = &-(p_W)_* \left(\sum_k (-1)^k W_1 \bullet \operatorname{ch}(\operatorname{tr}_1(\overline{E})_k) \right) \\ &- \sum_k (-1)^k (p_P)_* [\widetilde{\operatorname{ch}}(\overline{\eta}_k)] + (p_P)_* (\operatorname{ch}(\pi_p^* \overline{F}) \operatorname{Td}^{-1}(\overline{Q}_1) \widetilde{e}_Z(\overline{Q}_1^\vee)). \end{aligned} \tag{9.33}$$

It follows directly from the definition that T^Z is the theory of singular Bott-Chern classes associated to the class

$$C_Z(F,N) = (p_P)_*(\operatorname{ch}(\pi_p^*\overline{F})\operatorname{Td}^{-1}(\overline{Q}_1)\widetilde{e}_Z(\overline{Q}_1^{\vee})). \tag{9.34}$$

Theorem 9.35. *The theory of singular Bott-Chern classes T^Z agrees with the theory of homogeneous singular Bott-Chern classes T^h.*

Proof. The result follows directly from theorem 7.1, equation (9.34) and proposition 9.18. □

Next we want to use 8.33 to give another characterization of T^h. To this end we only need to compute the characteristic class $C_{T^h}(\mathcal{O}_Y, L)$ for a line bundle L as a power series in $c_1(L)$.

Theorem 9.36. *The theory of homogeneous singular Bott-Chern classes of algebraic vector bundles is the unique theory of singular Bott-Chern classes of algebraic vector bundles that is compatible with the projection formula and transitive and that satisfies*

$$C_{T^h}(\mathcal{O}_Y, L) = \mathbf{1}_1 \bullet \phi(c_1(L)),$$

where ϕ is the power series

$$\phi(x) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}H_{n+1}}{(n+2)!}x^n,$$

and where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, $n \geq 1$ are the harmonic numbers.

We already know that T^h is compatible with the projection formula and transitive. Thus it only remains to compute the power series ϕ.
Let $\overline{L} = (L, h_L)$ be a hermitian line bundle over a complex manifold Y. Let z be a system of holomorphic coordinates of Y. Let e be a local section of L and let $h(z) = h(e_z, e_z)$. Let $P = \mathbb{P}(L \oplus \mathbb{C})$, with $\pi\colon P \longrightarrow Y$ the projection and $\iota\colon Y \longrightarrow P$ the zero section. We choose homogeneous coordinates on P given by $(z, (x:y))$, here $(x:y)$ represents the line of $L_z \oplus \mathbb{C}$ generated by $xe(z) + y\mathbf{1}$, where $\mathbf{1}$ is a generator of $\mathbb{C}$ of norm 1. On the open set $y \neq 0$ we will use the absolute coordinate $t = x/y$. Let

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \pi^*(L \oplus \mathbb{C}) \longrightarrow Q \longrightarrow 0$$

be the tautological exact sequence. The section $s = \{\mathbf{1}\}$ is a global section of Q that vanishes along the zero section. Moreover we have

$$\|s\|^2_{(z,(x:y))} = \frac{x\bar{x}h(z)}{y\bar{y} + x\bar{x}h(z)} = \frac{t\bar{t}h}{1 + t\bar{t}h}.$$

Then (recall that we are using the algebro-geometric normalization)

$$\begin{aligned}
c_1(\overline{Q}) &= \partial\bar{\partial}\log\|s\|^2 && (9.37)\\
&= \partial\bar{\partial}\log\frac{t\bar{t}h}{1+t\bar{t}h} && (9.38)\\
&= \partial\left(\frac{1+t\bar{t}h}{t\bar{t}h}\frac{t\bar{\partial}(\bar{t}h)(1+t\bar{t}h)-t^2\bar{t}h\bar{\partial}(\bar{t}h)}{(1+t\bar{t}h)^2}\right) && (9.39)\\
&= \partial\left(\frac{t\bar{\partial}(\bar{t}h)}{t\bar{t}h(1+t\bar{t}h)}\right) && (9.40)\\
&= \partial\left(\frac{\bar{\partial}(\bar{t}h)}{\bar{t}h}\right)\frac{1}{1+t\bar{t}h}-\frac{\bar{t}\partial(ht)\wedge\bar{\partial}(\bar{t}h)}{\bar{t}h(1+t\bar{t}h)^2} && (9.41)\\
&= \frac{\pi^*c_1(\overline{L})}{1+t\bar{t}h}-\frac{\partial(th)\wedge\bar{\partial}(\bar{t}h)}{h(1+t\bar{t}h)^2}. && (9.42)
\end{aligned}$$

We now consider the Koszul resolution

$$\overline{K}\colon 0 \longrightarrow Q^\vee \xrightarrow{s} \mathcal{O}_p \longrightarrow \iota_*\mathcal{O}_X \longrightarrow 0.$$

We denote by $T^h(\overline{K})$ the singular Bott-Chern class associated to this Koszul complex. Then, by proposition 9.13 and proposition 9.18,

$$T^h(\overline{K}) = -\frac{1}{2}\operatorname{Td}^{-1}(\overline{Q})\log\|s\|^2.$$

In order to compute $\pi_*T^h(\overline{K})$ we have to compute first $\pi_*c_1(\overline{Q})^n\log\|s\|^2$. But

$$c_1(\overline{Q})^n = \frac{\pi^*c_1(\overline{L})^n}{(1+t\bar{t}h)^n} - n\left(\frac{\pi^*c_1(\overline{L})}{(1+t\bar{t}h)}\right)^{n-1}\frac{\partial(th)\wedge\bar{\partial}(\bar{t}h)}{h(1+t\bar{t}h)^2}.$$

Therefore

$$\begin{aligned}
\pi_*c_1(\overline{Q})^n\log\|s\|^2 &= -nc_1(\overline{L})^{n-1}\frac{1}{2\pi i}\int_{\mathbb{P}^1}\frac{\partial(th)\wedge\bar{\partial}(\bar{t}h)}{h(1+t\bar{t}h)^{n+1}}\log\frac{t\bar{t}h}{1+t\bar{t}h}\\
&= -nc_1(\overline{L})^{n-1}\frac{1}{2\pi i}\int_0^{2\pi}\int_0^\infty\log\frac{r^2}{1+r^2}\frac{-2ir\,\mathrm{d}\,\theta\,\mathrm{d}\,r}{(1+r^2)^{n+1}}\\
&= nc_1(\overline{L})^{n-1}\int_0^1\log(1-w)w^{n-1}\,\mathrm{d}\,w\\
&= -c_1(\overline{L})^{n-1}H_n,
\end{aligned}$$

where H_n, $n\geq 1$ are the harmonic numbers. Since

$$Td^{-1}(\overline{Q}) = \frac{1-\exp(-c_1(\overline{Q}))}{c_1(\overline{Q})} = \sum_{n=0}^\infty\frac{(-1)^n}{(n+1)!}c_1(\overline{Q})^n,$$

we obtain

$$C_{T^h}(\mathcal{O}_Y, L) = \pi_* T^h(\overline{K}) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}H_{n+1}}{(n+2)!}c_1(\overline{L})^n\mathbf{1}_1.$$

Then, a reformulation of proposition 8.31 is

Corollary 9.43. *Let T be a theory of singular Bott-Chern classes for algebraic vector bundles that is compatible with the projection formula and transitive. Then there is a unique additive genus S_T such that*

$$C_T(F, N) - C_{T^h}(F, N) = \operatorname{ch}(F) \bullet \operatorname{Td}(N)^{-1} \bullet S_T(N). \tag{9.44}$$

Conversely, any additive genus determines a theory of singular Bott-Chern classes by the formula (9.44).

10 The arithmetic Riemann-Roch theorem for regular closed immersions

In this section we recall the definition of arithmetic Chow groups and arithmetic K-groups. We see that each choice of an additive theory of singular Bott-Chern classes allows us to define direct images for closed immersions in arithmetic K-theory. Once the direct images for closed immersions are defined, we prove the arithmetic Grothendieck-Riemann-Roch theorem for closed immersions. A version of this theorem was proved earlier by Bismut, Gillet and Soulé [6] when there is a commutative diagram

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "i"] \arrow[dr, "f"'] & \mathcal{X} \arrow[d, "g"] \\ & \mathcal{Z} \end{tikzcd},$$

where i is a closed immersion and f and g are smooth over $\mathbb{C}$. The version of this theorem given in this paper is due to Zha [32], but still unpublished. The theorem of Bismut, Gillet and Soulé compares $g_* \widehat{\operatorname{ch}}(i_*\overline{E})$ with $f_* \widehat{\operatorname{ch}}(\overline{E})$, whereas the theorem of Zha compares directly $\widehat{\operatorname{ch}}(i_*\overline{E})$ with $i_* \widehat{\operatorname{ch}}(\overline{E})$. The main difference between the theorem of Bismut, Gillet and Soulé and that of Zha is the kind of arithmetic Chow groups they use. In the first case these groups are only covariant for proper morphisms that are smooth over $\mathbb{C}$; thus the Grothendieck-Riemann-Roch can only be stated for a diagram as above, while in the second case a version of these groups that are covariant for arbitrary proper morphisms is used.

Since each choice of a theory of singular Bott-Chern classes gives rise to a different definition of direct images for closed immersions, the arithmetic Grothendieck-Riemann-Roch theorem will have a correction term that depends on the theory of singular Bott-Chern classes used. In the particular case of the homogeneous singular Bott-Chern classes, which are the theories used by Bismut, Gillet and Soulé and by Zha, this correction term vanishes and we obtain the simplest formula. In this case the arithmetic Grothendieck-Riemann-Roch theorem is formally identical to the classical one.

Let (A, Σ, F_∞) be an arithmetic ring [18]. Since we will allow the arithmetic varieties to be non regular and we will use Chow groups indexed by dimension, following [20] we will assume that the ring A is equidimensional and Jacobson. Let F be the field of fractions of A. An *arithmetic variety* $\mathcal{X}$ is a scheme flat and quasi-projective over A such that $\mathcal{X}_F = \mathcal{X} \times \operatorname{Spec} F$ is smooth. Then $X := \mathcal{X}_\Sigma$ is a complex algebraic manifold, which is endowed with an anti-holomorphic automorphism F_∞. One also associates to $\mathcal{X}$ the real variety $X_\mathbb{R} = (X, F_\infty)$.

Following [13], to each regular arithmetic variety we can associate different kinds of arithmetic Chow groups. Concerning arithmetic Chow groups, we shall use the terminology and notation in op. cit. §4 and §6.

Let $\mathcal{D}_{\log}$ be the Deligne complex of sheaves defined in [13] section 5.3; we refer to op. cit. for the precise definition and properties. A $\mathcal{D}_{\log}$-*arithmetic variety* is a pair $(\mathcal{X}, \mathcal{C})$ consisting of an arithmetic variety $\mathcal{X}$ and a complex of sheaves $\mathcal{C}$ on $X_\mathbb{R}$ which is a $\mathcal{D}_{\log}$-complex (see op. cit. section 3.1).

We are interested in the following $\mathcal{D}_{\log}$-complexes of sheaves:

(i) The Deligne complex $\mathcal{D}_{\mathrm{l,a},X}$ of differential forms on X with logarithmic and arbitrary singularities. That is, for every Zariski open subset U of X, we write

$$E^*_{\mathrm{l,a},X}(U) = \varinjlim_{\overline{U}} \Gamma(\overline{U}, \mathscr{E}^*_{\overline{U}}(\log B)),$$

where the limit is taken over all diagrams

$$\begin{tikzcd} U \arrow[r, "\overline{\iota}"] \arrow[dr, "\iota"'] & \overline{U} \arrow[d, "\beta"] \\ & X \end{tikzcd}$$

such that $\overline{\iota}$ is an open immersion, β is a proper morphism, $B = \overline{U} \setminus U$, is a normal crossing divisor and $\mathscr{E}^*_{\overline{U}}(\log B)$ denotes the sheaf of smooth differential forms on U with logarithmic singularities along B introduced in [8] .

For any Zariski open subset $U \subseteq X$, we put

$$\mathcal{D}^*_{\mathrm{l,a},X}(U,p) = (\mathcal{D}^*_{\mathrm{l,a},X}(U,p), \mathrm{d}_\mathcal{D}) = (\mathcal{D}^*(E_{\mathrm{l,a},X}(U),p), \mathrm{d}_\mathcal{D}).$$

If U is now a Zariski open subset of $X_\mathbb{R}$, then we write

$$\mathcal{D}^*_{\mathrm{l,a},X}(U,p) = (\mathcal{D}^*_{\mathrm{l,a},X}(U,p), \mathrm{d}_\mathcal{D}) = (\mathcal{D}^*_{\mathrm{l,a},X}(U_\mathbb{C},p)^\sigma, \mathrm{d}_\mathcal{D}),$$

where σ is the involution $\sigma(\eta) = \overline{F^*_\infty \eta}$ as in [13] notation 5.65.

Note that the sections of $\mathcal{D}^*_{\mathrm{l,a},X}$ over an open set $U \subset X$ are differential forms on U with logarithmic singularities along $X \setminus U$ and arbitrary singularities along $\overline{X} \setminus X$, where $\overline{X}$ is an arbitrary compactification of X. Therefore the complex of global sections satisfy

$$\mathcal{D}^*_{\mathrm{l,a},X}(X,*) = \mathcal{D}^*(X,*),$$

where the right hand side complex has been introduced in section §1. The complex $\mathcal{D}^*_{\mathrm{l,a},X}$ is a particular case of the construction of [12] section 3.6.

(ii) The Deligne complex $\mathcal{D}_{\mathrm{cur},X}$ of currents on X. This is the complex introduced in [13] definition 6.30.

When $\mathcal{X}$ is regular, applying the theory of [13] we can define the arithmetic Chow groups $\widehat{\mathrm{CH}}^*(\mathcal{X}, \mathcal{D}_{\mathrm{l,a},X})$ and $\widehat{\mathrm{CH}}^*(\mathcal{X}, \mathcal{D}_{\mathrm{cur},X})$. These groups satisfy the following properties

(i) There are natural morphisms

$$\widehat{\mathrm{CH}}^*(\mathcal{X}, \mathcal{D}_{\mathrm{l,a},X}) \longrightarrow \widehat{\mathrm{CH}}^*(\mathcal{X}, \mathcal{D}_{\mathrm{cur},X})$$

and, when applicable, all properties below will be compatible with these morphisms.

(ii) There is a product structure that turns $\widehat{\mathrm{CH}}^*(\mathcal{X}, \mathcal{D}_{\mathrm{l,a},X})_{\mathbb{Q}}$ into an associative and commutative algebra. Moreover, it turns $\widehat{\mathrm{CH}}^*(\mathcal{X}, \mathcal{D}_{\mathrm{cur},X})_{\mathbb{Q}}$ into a $\widehat{\mathrm{CH}}^*(\mathcal{X}, \mathcal{D}_{\mathrm{l,a},X})_{\mathbb{Q}}$-module.

(iii) If $f\colon \mathcal{Y} \longrightarrow \mathcal{X}$ is a map of regular arithmetic varieties, there are pull-back morphisms

$$f^*\colon\ \widehat{\mathrm{CH}}^*(\mathcal{X}, \mathcal{D}_{\mathrm{l,a},X}) \longrightarrow \widehat{\mathrm{CH}}^*(\mathcal{Y}, \mathcal{D}_{\mathrm{l,a},Y}).$$

If moreover, f is smooth over F, there are pull-back morphisms

$$f^*\colon\ \widehat{\mathrm{CH}}^*(\mathcal{X}, \mathcal{D}_{\mathrm{cur},X}) \longrightarrow \widehat{\mathrm{CH}}^*(\mathcal{Y}, \mathcal{D}_{\mathrm{cur},Y}).$$

The inverse image is compatible with the product structure.

(iv) If $f\colon \mathcal{Y} \longrightarrow \mathcal{X}$ is a proper map of regular arithmetic varieties of relative dimension d, there are push-forward morphisms

$$f_*\colon\ \widehat{\mathrm{CH}}^*(\mathcal{Y}, \mathcal{D}_{\mathrm{cur},Y}) \longrightarrow \widehat{\mathrm{CH}}^{*-d}(\mathcal{X}, \mathcal{D}_{\mathrm{cur},X}).$$

If moreover, f is smooth over F, there are push-forward morphisms

$$f_*\colon\ \widehat{\mathrm{CH}}^*(\mathcal{Y}, \mathcal{D}_{\mathrm{l,a},Y}) \longrightarrow \widehat{\mathrm{CH}}^{*-d}(\mathcal{X}, \mathcal{D}_{\mathrm{l,a},X}).$$

The push-forward morphism satisfies the projection formula and is compatible with base change.

(v) The groups $\widehat{\mathrm{CH}}^{*}(\mathcal{X},\mathcal{D}_{\mathrm{l,a},X})$ are naturally isomorphic to the groups defined by Gillet and Soulé in [18] (see [12] theorem 3.33). When X is generically projective, the groups $\widehat{\mathrm{CH}}^{*}(\mathcal{X},\mathcal{D}_{\mathrm{cur},X})$ are isomorphic to analogous groups introduced by Kawaguchi and Moriwaki [27] and are very similar to the weak arithmetic Chow groups introduced by Zha (see [11]).

(vi) There are well-defined maps

$$\begin{aligned}\zeta\colon\ \widehat{\mathrm{CH}}^{p}(\mathcal{X},\mathcal{C}) &\longrightarrow \mathrm{CH}^{p}(\mathcal{X}),\\ \mathrm{a}\colon \widetilde{\mathcal{C}}^{2p-1}(X_{\mathbb{R}},p) &\longrightarrow \widehat{\mathrm{CH}}^{p}(\mathcal{X},\mathcal{C}),\\ \omega\colon\ \widehat{\mathrm{CH}}^{p}(\mathcal{X},\mathcal{C}) &\longrightarrow \mathrm{Z}\mathcal{C}^{2p}(X_{\mathbb{R}},p),\end{aligned}$$

where $\mathcal{C}$ is either $\mathcal{D}_{\mathrm{l,a},X}$ or $\mathcal{D}_{\mathrm{cur},X}$. For the precise definition of these maps see [13] notation 4.12.

When $\mathcal{X}$ is not necessarily regular, following [20] and combining with the definition of [13] we can define the arithmetic Chow groups indexed by dimension $\widehat{\mathrm{CH}}_{*}(\mathcal{X},\mathcal{D}_{\mathrm{l,a},X})$ and $\widehat{\mathrm{CH}}_{*}(\mathcal{X},\mathcal{D}_{\mathrm{cur},X})$ (see [12] section 5.3).
They have the following properties (see [20]).

(i) If $\mathcal{X}$ is regular and equidimensional of dimension n then there are isomorphisms

$$\begin{aligned}\widehat{\mathrm{CH}}_{*}(\mathcal{X},\mathcal{D}_{\mathrm{l,a},X}) &\cong \widehat{\mathrm{CH}}^{n-*}(\mathcal{X},\mathcal{D}_{\mathrm{l,a},X}),\\ \widehat{\mathrm{CH}}_{*}(\mathcal{X},\mathcal{D}_{\mathrm{cur},X}) &\cong \widehat{\mathrm{CH}}^{n-*}(\mathcal{X},\mathcal{D}_{\mathrm{cur},X}).\end{aligned}$$

(ii) If $f\colon \mathcal{Y}\longrightarrow\mathcal{X}$ is a proper map between arithmetic varieties then there is a push-forward map

$$f_{*}\colon\ \widehat{\mathrm{CH}}_{*}(\mathcal{Y},\mathcal{D}_{\mathrm{cur},Y}) \longrightarrow \widehat{\mathrm{CH}}_{*}(\mathcal{X},\mathcal{D}_{\mathrm{cur},X}).$$

If f is smooth over F then there is a push-forward map

$$f_{*}\colon\ \widehat{\mathrm{CH}}_{*}(\mathcal{Y},\mathcal{D}_{\mathrm{l,a},Y}) \longrightarrow \widehat{\mathrm{CH}}_{*}(\mathcal{X},\mathcal{D}_{\mathrm{l,a},X}).$$

(iii) If $f\colon \mathcal{Y}\longrightarrow\mathcal{X}$ is a flat map or, more generally, a local complete intersection (l.c.i) map of relative dimension d, there are pull-back morphisms

$$f^{*}\colon\ \widehat{\mathrm{CH}}_{*}(\mathcal{X},\mathcal{D}_{\mathrm{l,a},X}) \longrightarrow \widehat{\mathrm{CH}}_{*+d}(\mathcal{Y},\mathcal{D}_{\mathrm{l,a},Y}).$$

If moreover, f is smooth over F, there are pull-back morphisms

$$f^{*}\colon\ \widehat{\mathrm{CH}}_{*}(\mathcal{X},\mathcal{D}_{\mathrm{cur},X}) \longrightarrow \widehat{\mathrm{CH}}_{*+d}(\mathcal{Y},\mathcal{D}_{\mathrm{cur},Y}).$$

(iv) If $f\colon \mathcal{Y} \longrightarrow \mathcal{X}$ is a morphism of arithmetic varieties with $\mathcal{X}$ regular, then there is a cap product

$$\widehat{\mathrm{CH}}^{p}(\mathcal{X}, \mathcal{D}_{\mathrm{l,a},X}) \otimes \widehat{\mathrm{CH}}_{d}(\mathcal{Y}, \mathcal{D}_{\mathrm{l,a},Y}) \longrightarrow \widehat{\mathrm{CH}}_{d-p}(\mathcal{Y}, \mathcal{D}_{\mathrm{l,a},Y})_{\mathbb{Q}},$$

and a similar cap-product with the groups $\widehat{\mathrm{CH}}_{d}(\mathcal{Y}, \mathcal{D}_{\mathrm{cur},Y})$. This product is denoted by $y \otimes x \mapsto y._{f} x$,

For more properties of these groups see [20].
We will define now the arithmetic K-groups in this context. As a matter of convention, in the sequel we will use slanted letters to denote a object defined over A and the same letter in roman type for the corresponding object defined over $\mathbb{C}$. For instance we will denote a vector bundle over $\mathcal{X}$ by $\mathcal{E}$ and the corresponding vector bundle over X by E.

Definition 10.1. A *hermitian vector bundle* on an arithmetic variety $\mathcal{X}$, $\overline{\mathcal{E}}$, is a locally free sheaf $\mathcal{E}$ with a hermitian metric h_E on the vector bundle E induced on X, that is invariant under F_∞. A sequence of hermitian vector bundles on $\mathcal{X}$

$$(\overline{\varepsilon}) \qquad \dots \longrightarrow \overline{\mathcal{E}}_{n+1} \longrightarrow \overline{\mathcal{E}}_{n} \longrightarrow \overline{\mathcal{E}}_{n-1} \longrightarrow \dots$$

is said to be exact if it is exact as a sequence of vector bundles.
A *metrized coherent sheaf* is a pair $\overline{\mathcal{F}} = (\mathcal{F}, \overline{E}_* \to F)$, where $\mathcal{F}$ is a coherent sheaf on $\mathcal{X}$ and $\overline{E}_* \to F$ is a resolution of the coherent sheaf $F = \mathcal{F}_{\mathbb{C}}$ by hermitian vector bundles, that is defined over $\mathbb{R}$, hence is invariant under F_∞. We assume that the hermitian metrics are also invariant under F_∞.

Recall that to every hermitian vector bundle we can associate a collection of Chern forms, denoted by c_p. Moreover, the invariance of the hermitian metric under F_∞ implies that the Chern forms will be invariant under the involution σ. Thus

$$c_p(\overline{\mathcal{E}}) \in \mathcal{D}^{2p}_{\mathrm{l,a},X}(X_{\mathbb{R}}, p) = \mathcal{D}^{2p}(X, p)^{\sigma}.$$

We will denote also by $c_p(\overline{\mathcal{E}})$ its image in $\mathcal{D}^{2p}_{\mathrm{cur},X}(X_{\mathbb{R}}, p)$. In particular we have defined the Chern character $\mathrm{ch}(\overline{\mathcal{E}})$ in either of the groups $\bigoplus_p \mathcal{D}^{2p}_{\mathrm{l,a},X}(X_{\mathbb{R}}, p)$ or $\bigoplus_p \mathcal{D}^{2p}_{\mathrm{cur},X}(X_{\mathbb{R}}, p)$. Moreover, to each finite exact sequence $(\overline{\varepsilon})$ of hermitian vector bundles on $\mathcal{X}$ we can attach a secondary Bott-Chern class $\widetilde{\mathrm{ch}}(\overline{\varepsilon})$. Again, the fact that the sequence is defined over A and the invariance of the metrics with respect to F_∞ imply that

$$\widetilde{\mathrm{ch}}(\overline{\varepsilon}) \in \bigoplus_p \widetilde{\mathcal{D}}^{2p-1}_{\mathrm{l,a},X}(X_{\mathbb{R}}, p) = \bigoplus_p \widetilde{\mathcal{D}}^{2p-1}(X, p)^{\sigma}.$$

We will denote also by $\widetilde{\mathrm{ch}}(\overline{\varepsilon})$ its image in $\bigoplus_p \widetilde{\mathcal{D}}^{2p-1}_{\mathrm{cur},X}(X_{\mathbb{R}}, p)$. The Bott-Chern classes associated to exact sequences of metrized coherent sheaves enjoy the same properties.

Definition 10.2. Let $\mathcal{X}$ be an arithmetic variety and let $\mathcal{C}^*(*)$ be one of the two $\mathcal{D}_{\log}$-complexes $\mathcal{D}_{\mathrm{l,a},X}$ or $\mathcal{D}_{\mathrm{cur},X}$. The *arithmetic K-group* associated to the $\mathcal{D}_{\log}$-arithmetic variety $(\mathcal{X}, \mathcal{C})$ is the abelian group $\widehat{K}(\mathcal{X}, \mathcal{C})$ generated by pairs $(\overline{\mathcal{E}}, \eta)$, where $\overline{\mathcal{E}}$ is a hermitian vector bundle on $\mathcal{X}$ and $\eta \in \bigoplus_{p\geq 0} \widetilde{\mathcal{C}}^{2p-1}(X_{\mathbb{R}}, p)$, modulo relations

$$(\overline{\mathcal{E}}_1, \eta_1) + (\overline{\mathcal{E}}_2, \eta_2) = (\overline{\mathcal{E}}, \tilde{\mathrm{ch}}(\overline{\varepsilon}) + \eta_1 + \eta_2) \tag{10.3}$$

for each short exact sequence

$$(\overline{\varepsilon}) \qquad 0 \longrightarrow \overline{\mathcal{E}}_1 \longrightarrow \overline{\mathcal{E}} \longrightarrow \overline{\mathcal{E}}_2 \longrightarrow 0\ .$$

The *arithmetic K'-group* associated to the $\mathcal{D}_{\log}$-arithmetic variety $(\mathcal{X}, \mathcal{C})$ is the abelian group $\widehat{K}'(\mathcal{X}, \mathcal{C})$ generated by pairs $(\overline{\mathcal{F}}, \eta)$, where $\overline{\mathcal{F}}$ is a metrized coherent sheaf on $\mathcal{X}$ and $\eta \in \bigoplus_{p\geq 0} \widetilde{\mathcal{C}}^{2p-1}(X_{\mathbb{R}}, p)$, modulo relations

$$(\overline{\mathcal{F}}_1, \eta_1) + (\overline{\mathcal{F}}_2, \eta_2) = (\overline{\mathcal{F}}, \tilde{\mathrm{ch}}(\overline{\varepsilon}) + \eta_1 + \eta_2) \tag{10.4}$$

for each short exact sequence of metrized coherent sheaves

$$(\overline{\varepsilon}) \qquad 0 \longrightarrow \overline{\mathcal{F}}_1 \longrightarrow \overline{\mathcal{F}} \longrightarrow \overline{\mathcal{F}}_2 \longrightarrow 0\ .$$

We now give some properties of the arithmetic K-groups. As their proofs are similar, in the essential points, to those of analogous statements in, for example, [18] in the regular case and [20] in the singular case, we omit them.

(i) We have natural morphisms

$$\widehat{K}(\mathcal{X}, \mathcal{D}_{\mathrm{l,a},X}) \longrightarrow \widehat{K}(\mathcal{X}, \mathcal{D}_{\mathrm{cur},X}) \text{ and } \widehat{K}'(\mathcal{X}, \mathcal{D}_{\mathrm{l,a},X}) \longrightarrow \widehat{K}'(\mathcal{X}, \mathcal{D}_{\mathrm{cur},X}).$$

When applicable, all properties below will be compatible with these morphisms.

(ii) $\widehat{K}(\mathcal{X}, \mathcal{D}_{\mathrm{l,a},X})$ is a ring. The product structure is given by

$$(\overline{\mathcal{F}}_1, \eta_1)\cdot(\overline{\mathcal{F}}_2, \eta_2) = (\overline{\mathcal{F}}_1 \otimes \overline{\mathcal{F}}_2, \mathrm{ch}(\overline{\mathcal{F}}_1) \bullet \eta_2 + \eta_1 \bullet \mathrm{ch}(\overline{\mathcal{F}}_2) + \mathrm{d}_{\mathcal{D}}\, \eta_1 \bullet \eta_2) \tag{10.5}$$

(iii) $\widehat{K}(\mathcal{X}, \mathcal{D}_{\mathrm{cur},X})$ is a $\widehat{K}(\mathcal{X}, \mathcal{D}_{\mathrm{l,a},X})$-module.

(iv) There are natural maps

$$\widehat{K}(\mathcal{X}, \mathcal{C}) \longrightarrow \widehat{K}'(\mathcal{X}, \mathcal{C})$$

that, when $\mathcal{X}$ is regular, are isomorphisms.

(v) The groups $\widehat{K}'(\mathcal{X}, \mathcal{D}_{\mathrm{l,a},X})$ and $\widehat{K}'(\mathcal{X}, \mathcal{D}_{\mathrm{cur},X})$ are $\widehat{K}(\mathcal{X}, \mathcal{D}_{\mathrm{l,a},X})$-modules.

(vi) There are natural maps

$$\omega\colon \widehat{K}'(\mathcal{X},\mathcal{C}) \longrightarrow \bigoplus_p Z\mathcal{C}^{2p}(p)$$

that send the class of a pair $(\overline{\mathcal{F}},\eta)$ with $\overline{\mathcal{F}}=(\mathcal{F},\overline{E}_*\to\mathcal{F}_{\mathbb{C}})$ to the form (or current)

$$\omega(\overline{\mathcal{F}},\eta)=\sum_i(-1)^i\operatorname{ch}(\overline{E}_i)+\mathrm{d}_{\mathcal{D}}\,\eta.$$

(vii) When $\mathcal{X}$ is regular, there exists a Chern character,

$$\widehat{\operatorname{ch}}\colon \widehat{K}(\mathcal{X},\mathcal{C})_{\mathbb{Q}} \longrightarrow \bigoplus_p \widehat{\operatorname{CH}}^p(\mathcal{X},\mathcal{C})_{\mathbb{Q}},$$

that is an isomorphism. Moreover, if $\mathcal{C}=\mathcal{D}_{\mathrm{l,a},X}$ this isomorphism is compatible with the product structure. If $\mathcal{X}$ is not regular, there is a biadditive pairing

$$\widehat{K}(\mathcal{X},\mathcal{D}_{\mathrm{l,a},X})\otimes\widehat{\operatorname{CH}}_*(\mathcal{X},\mathcal{D}_{\mathrm{l,a},X})\longrightarrow\widehat{\operatorname{CH}}_*(\mathcal{X},\mathcal{D}_{\mathrm{l,a},X})_{\mathbb{Q}},$$

and a similar pairing with the groups $\widehat{\operatorname{CH}}_*(\mathcal{X},\mathcal{D}_{\mathrm{cur},X})$, which is denoted in both cases by $\alpha\otimes x\mapsto\widehat{\operatorname{ch}}(\alpha)\cap x$. For the properties of this product see [20] pg. 496.

(viii) If $\mathcal{Y}$ and $\mathcal{X}$ are arithmetic varieties and $f\colon\mathcal{Y}\to\mathcal{X}$ is a morphism of arithmetic varieties, f induces a morphism of rings:

$$f^*\colon\widehat{K}(\mathcal{X},\mathcal{D}_{\mathrm{l,a},X})\to\widehat{K}(\mathcal{Y},\mathcal{D}_{\mathrm{l,a},Y}).$$

When f is flat, the inverse image is also defined for the groups $\widehat{K}'(\mathcal{X},\mathcal{D}_{\mathrm{l,a},X})$. Moreover, if $f_{\mathbb{C}}$ is smooth, the inverse image can be defined for the groups $\widehat{K}(\mathcal{X},\mathcal{D}_{\mathrm{cur},X})$ and, when in addition f is flat, for the groups $\widehat{K}'(\mathcal{X},\mathcal{D}_{\mathrm{cur},X})$.

In what follows we will be interested in direct images for closed immersions. Since the direct images in arithmetic K-theory will depend on the choice of a metric, we have the following

Definition 10.6. A *metrized arithmetic variety* is a pair $(\mathcal{X},h_X)$ consisting of an arithmetic variety $\mathcal{X}$ and a hermitian metric on the complex tangent bundle T_X that is invariant under F_∞.

Let $(\mathcal{X},h_X)$ and $(\mathcal{Y},h_Y)$ be metrized arithmetic varieties and let $i\colon\mathcal{Y}\longrightarrow\mathcal{X}$ be a closed immersion. Over the complex numbers, we are in the situation of notation 8.36. In particular we have a canonical exact sequence of hermitian vector bundles

$$\overline{\xi}_N\colon 0\longrightarrow\overline{T}_Y\longrightarrow i^*\overline{T}_X\longrightarrow\overline{N}_{Y/X}\longrightarrow 0 \tag{10.7}$$

where the tangent bundles T_Y, T_X are endowed with the hermitian metrics h_Y, h_X respectively and the normal bundle $N_{Y/X}$ is endowed with an arbitrary hermitian metric h_N. We will follow the conventions of notation 8.36.
We next define push-forward maps, via a closed immersion, for the elements of the arithmetic K-group of a metrized arithmetic variety. We will define two kinds of push-forward maps. One will depend only on a metric on the complex normal bundle $N_{Y/X}$. By contrast, the second will depend on the choice of metrics on the complex tangent bundles T_X and T_Y. The second definition allows us to see $K'(\_, \mathcal{D}_{\mathrm{cur},Y})$ as a functor from the category whose objects are metrized arithmetic varieties and whose morphisms are closed immersions to the category of abelian groups.
As we deal with hermitian vector bundles and metrized coherent sheaves, both definitions will involve the choice of a theory of singular Bott-Chern classes. In order for the push forward to be well defined in K-theory we need a minimal additivity property for the singular Bott-Chern classes.

Definition 10.8. A theory of singular Bott-Chern classes T is called *additive* if for any closed embedding of complex manifolds $i\colon Y \hookrightarrow X$ and any hermitian embedded vector bundles $\overline{\xi}_1 = (i, \overline{N}, \overline{F}_1, \overline{E}_{1,*})$, $\overline{\xi}_2 = (i, \overline{N}, \overline{F}_2, \overline{E}_{2,*})$ the equation
$$T(\overline{\xi}_1 \oplus \overline{\xi}_2) = T(\overline{\xi}_1) + T(\overline{\xi}_2)$$
is satisfied.
Let C be a characteristic class for pairs of vector bundles. We say that it is *additive* (in the first variable) if
$$C(F_1 \oplus F_2, N) = C(F_1, N) + C(F_2, N)$$
for any vector bundles F_1, F_2, N on a complex manifold X.

The following statement follows directly from equation 7.5:

Proposition 10.9. *A theory of singular Bott-Chern classes T is additive if and only if the corresponding characteristic class C_T is additive in the first variable.*

Note that a theory of singular Bott-Chern classes consists in joining theories of singular Bott-Chern classes in arbitrary rank and codimension (definition 6.9). The property of being additive gives a compatibility condition for these theories, by respect to the hermitian vector bundles $\overline{F}$ (with the notation used in definition 6.9). Note also that if a theory of singular Bott-Chern classes is compatible with the projection formula then it is additive.

Definition 10.10. Let T be an additive theory of singular Bott-Chern classes, and let T_c be the associated covariant class as in definition 8.37. Let $i\colon (\mathcal{Y}, h_Y) \longrightarrow (\mathcal{X}, h_X)$ be a closed immersion of metrized arithmetic varieties and let $\overline{N} = \overline{N}_{Y/X} = (N_{Y/X}, h_N)$ be a choice of a hermitian metric on the complex normal bundle. The *push-forward maps*
$$i_*^{T_c}, i_*^{T}\colon \widehat{K}(\mathcal{Y}, \mathcal{D}_{\mathrm{cur},Y}) \longrightarrow \widehat{K}'(\mathcal{X}, \mathcal{D}_{\mathrm{cur},X})$$

are defined by

$$
\begin{aligned}
i_*^{T_c}(\overline{\mathcal{F}},\eta) &= [((i_*\mathcal{F},\overline{E}_* \to (i_*\mathcal{F})_{\mathbb{C}}),0)] - [(0,T_c(\overline{\xi}_c))] \\
&\qquad + [(0,i_*(\eta\operatorname{Td}(Y) i^*\operatorname{Td}^{-1}(X)))] \qquad (10.11)\\
i_*^{T}(\overline{\mathcal{F}},\eta) &= [((i_*\mathcal{F},\overline{E}_* \to (i_*\mathcal{F})_{\mathbb{C}}),0)] - [(0,T(\overline{\xi}))] \\
&\qquad + [(0,i_*(\eta\operatorname{Td}^{-1}(\overline{N}_{Y/X})))]. \qquad (10.12)
\end{aligned}
$$

Here

$$0 \to \overline{E}_n \to \dots \to \overline{E}_1 \to \overline{E}_0 \to (i_*\mathcal{F})_{\mathbb{C}} \to 0$$

is a finite resolution of the coherent sheaf $(i_*\mathcal{F})_{\mathbb{C}}$ by hermitian vector bundles, $\overline{\xi} = (i,\overline{N}_{X/Y},\overline{\mathcal{F}}_{\mathbb{C}},\overline{E}_*)$ is the induced hermitian embedded vector bundle on X, and $\overline{\xi}_c = (i,\overline{T}_X,\overline{T}_Y,\overline{\mathcal{F}}_{\mathbb{C}},\overline{E}_*)$ as in definition 8.37.
We can extend this definition to push-forward maps

$$i_*^{T_c}, i_*^{T} \colon \widehat{K}'(\mathcal{Y},\mathcal{D}_{\mathrm{cur},Y}) \longrightarrow \widehat{K}'(\mathcal{X},\mathcal{D}_{\mathrm{cur},X})$$

by the rule

$$
\begin{aligned}
i_*^{T_c}(\overline{\mathcal{F}},\eta) &= [((i_*\mathcal{F},\operatorname{Tot}(\overline{E}_{*,*}) \to (i_*\mathcal{F})_{\mathbb{C}}),0)] - \sum_i (-1)^i[(0,T_c(\overline{\xi}_{i,c}))] \\
&\qquad + [(0,i_*(\eta\operatorname{Td}(Y) i^*\operatorname{Td}^{-1}(X)))], \qquad (10.13)\\
i_*^{T}(\overline{\mathcal{F}},\eta) &= [((i_*\mathcal{F},\operatorname{Tot}(\overline{E}_{*,*}) \to (i_*\mathcal{F})_{\mathbb{C}}),0)] - \sum_i (-1)^i[(0,T(\overline{\xi}_i))] \\
&\qquad + [(0,i_*(\eta\operatorname{Td}^{-1}(\overline{N}_{Y/X})))], \qquad (10.14)
\end{aligned}
$$

where $0 \to \overline{E}_n \to \dots \to \overline{E}_0 \to \mathcal{F}_{\mathbb{C}} \to 0$ is a resolution of $\mathcal{F}_{\mathbb{C}}$ by hermitian vector bundles, $\overline{E}_{*,*}$ is a complex of complexes of vector bundles over X, such that, for each $i \geq 0$, $\overline{E}_{i,*} \to i_*E_i$ is also a resolution by hermitian vector bundles and $\overline{\xi}_i = (i,\overline{N}_{X/Y},\overline{E}_i,\overline{E}_{i,*})$ is the induced hermitian embedded vector bundle and $\overline{\xi}_{i,c}$ is as in definition 8.37. We suppose that there is a commutative diagram of resolutions

$$
\begin{tikzcd}
\cdots \arrow[r] & E_{k+1,*} \arrow[r] \arrow[d] & E_{k,*} \arrow[r] \arrow[d] & E_{k-1,*} \arrow[r] \arrow[d] & \cdots . \\
\cdots \arrow[r] & i_*E_{k+1} \arrow[r] & i_*E_k \arrow[r] & i_*E_{k-1} \arrow[r] & \cdots
\end{tikzcd}
$$

hence a resolution $\operatorname{Tot}(\overline{E}_{*,*}) \longrightarrow (i_*\mathcal{F})_{\mathbb{C}}$ by hermitian vector bundles.

Note that, whenever the push-forward i_*^T appears, we will assume that we have chosen a metric on $N_{Y/X}$.
The two push-forward maps are related by the equation

$$i_*^{T_c}(\overline{\mathcal{F}},\eta) = i_*^{T}(\overline{\mathcal{F}},\eta) - \left[\left(0, i_*\left(\omega(\overline{\mathcal{F}},\eta)\widetilde{\operatorname{Td}^{-1}}(\overline{\xi}_N)\operatorname{Td}(Y)\right)\right)\right], \qquad (10.15)$$

where $\overline{\xi}_N$ is the exact sequence (10.7).

PROPOSITION 10.16. *The push-forward maps i_*^T, $i_*^{T_c}$ are well defined. That is, they do not depend on the choice of a representative of a class in $\widehat{K}$, nor on the choice of metrics on the coherent sheaf $(i_*\mathcal{F})_{\mathbb{C}}$. The first one does not depend on the choice of metrics on T_X nor on T_Y, whereas the second one does not depend on the choice of a metric on the normal bundle $N_{Y/X}$. Moreover, if i is a regular closed immersion or $\mathcal{X}$ is a regular arithmetic variety, then $i_*^{T_c}$ and i_*^T can be lifted to maps*

$$i_*^{T_c}, i_*^T \colon \widehat{K}(\mathcal{Y}, \mathcal{D}_{\mathrm{cur},Y}) \longrightarrow \widehat{K}(\mathcal{X}, \mathcal{D}_{\mathrm{cur},Y}).$$

Proof. The fact that i_*^T only depends on the metric on $\overline{N}$ and not on the metrics on T_X and T_Y and that for $i_*^{T_c}$ is the opposite, follows directly from the definition in the first case and from proposition 8.39 in the second.
We will only prove the other statements for $i_*^{T_c}$, as the other case is analogous. We first prove the independence from the metric chosen on the coherent sheaf $(i_*\mathcal{F})_{\mathbb{C}}$. If $\overline{E}_* \to (i_*\mathcal{F})_{\mathbb{C}}$, $\overline{E}'_* \to (i_*\mathcal{F})_{\mathbb{C}}$ are two such metrics, inducing the hermitian embedded vector bundles $\overline{\xi}$ respectively $\overline{\xi}'$, then, using corollary 6.14

$$T_c(\overline{\xi}'_c) - T_c(\overline{\xi}_c) = T(\overline{\xi}') - T(\overline{\xi}) = \widetilde{\mathrm{ch}}(\overline{\varepsilon}),$$

where $\overline{\varepsilon}$ is the exact complex of hermitian embedded vector bundles

$$\overline{\varepsilon} \colon 0 \longrightarrow \overline{\xi} \longrightarrow \overline{\xi}' \longrightarrow 0,$$

where $\overline{\xi}'$ sits in degree zero.
Therefore, by equation 10.4,

$$\begin{aligned}[((i_*\mathcal{F}, \overline{E}_* \to (i_*\mathcal{F})_{\mathbb{C}}), 0)] - [(0, T_c(\overline{\xi_c}))] \\ = [((i_*\mathcal{F}, \overline{E}'_* \to (i_*\mathcal{F})_{\mathbb{C}}), 0)] - [(0, T_c(\overline{\xi}'_c))].\end{aligned}$$

Since the last term of equation 10.11 does not depend on the metric on $(i_*\mathcal{F})_{\mathbb{C}}$, we obtain that $i_*^{T_c}$ does not depend on this metric.
For proving that the push-forward map $i_*^{T_c}$ is well defined it remains to show the independence from the choice of a representative of a class in $\widehat{K}(\mathcal{Y}, \mathcal{D}_{\mathrm{cur},Y})$. We consider an exact sequence of hermitian vector bundles on $\mathcal{Y}$

$$\overline{\varepsilon} \colon 0 \longrightarrow \overline{\mathcal{F}}_1 \longrightarrow \overline{\mathcal{F}} \longrightarrow \overline{\mathcal{F}}_2 \longrightarrow 0$$

and two classes $\eta_1, \eta_2 \in \bigoplus_{p \geq 0} \widetilde{\mathcal{D}}_{\mathrm{cur}}^{2p-1}(Y, p)$. We also denote $\overline{\varepsilon}$ the induced exact sequence of hermitian vector bundles on Y. We have to prove

$$i_*^{T_c}([(\overline{\mathcal{F}}, \eta_1 + \eta_2 + \widetilde{\mathrm{ch}}(\overline{\varepsilon})]) = i_*^{T_c}([(\overline{\mathcal{F}_1}, \eta_1)]) + i_*^{T_c}([(\overline{\mathcal{F}_2}, \eta_2)]). \tag{10.17}$$

Since it is clear that $i_*^{T_c}(0, \eta_1 + \eta_2) = i_*^{T_c}(0, \eta_1) + i_*^{T_c}(0, \eta_2)$, we are led to prove

$$i_*^{T_c}([(\overline{\mathcal{F}}, \widetilde{\mathrm{ch}}(\overline{\varepsilon})]) = i_*^{T_c}([(\overline{\mathcal{F}_1}, 0)]) + i_*^{T_c}([(\overline{\mathcal{F}_2}, 0)]). \tag{10.18}$$

We choose metrics on the coherent sheaves $(i_*\mathcal{F}_1)_{\mathbb{C}}$, $(i_*\mathcal{F}_2)_{\mathbb{C}}$ and $(i_*\mathcal{F})_{\mathbb{C}}$ respectively:

$$\overline{E}_{1,*} \longrightarrow (i_*\mathcal{F}_1)_{\mathbb{C}}\ ,\ \overline{E}_{2,*} \longrightarrow (i_*\mathcal{F}_2)_{\mathbb{C}}\ ,\ \overline{E}_{*} \longrightarrow (i_*\mathcal{F})_{\mathbb{C}}.$$

We denote $\overline{\xi}_1, \overline{\xi}_2, \overline{\xi}$ the induced hermitian embedded vector bundles. We obtain an exact sequence of metrized coherent sheaves on $\mathcal{X}$:

$$\overline{\nu}\colon 0 \longrightarrow \overline{i_*\mathcal{F}_1} \longrightarrow \overline{i_*\mathcal{F}} \longrightarrow \overline{i_*\mathcal{F}_2} \longrightarrow 0.$$

Then, using the fact that the theory T is additive and equation (8.42) we have

$$T_c(\overline{\xi}_{1,c}) + T_c(\overline{\xi}_{2,c}) - T_c(\overline{\xi}_c) = [\widetilde{\mathrm{ch}}(\overline{\nu})] - i_*([\widetilde{\mathrm{ch}}(\overline{\varepsilon}) \bullet \mathrm{Td}(Y)]) \bullet \mathrm{Td}^{-1}(X). \quad (10.19)$$

Moreover, by the relation (10.4),

$$[(\overline{i_*\mathcal{F}_1}, 0)] + [(\overline{i_*\mathcal{F}_2}, 0)] = [(\overline{i_*\mathcal{F}}, \widetilde{\mathrm{ch}}(\overline{\nu}))]. \quad (10.20)$$

Hence, we compute,

$$\begin{aligned}
i_*^{T_c}([(\overline{\mathcal{F}}, \widetilde{\mathrm{ch}}(\overline{\varepsilon})]) &- i_*^{T_c}([(\overline{\mathcal{F}_1}, 0)]) - i_*^{T_c}([(\overline{\mathcal{F}_2}, 0)]) \\
&= [(i_*\overline{\mathcal{F}}, 0)] - [(i_*\overline{\mathcal{F}_1}, 0)] - [(i_*\overline{\mathcal{F}_2}, 0)] \\
&\quad - [(0, T_c(\overline{\xi}_c))] + [(0, T_c(\overline{\xi_{1,c}}))] + [(0, T_c(\overline{\xi_{2,c}}))] \\
&\quad + [(0, i_*([\widetilde{\mathrm{ch}}(\overline{\varepsilon})] \bullet \mathrm{Td}(Y) \bullet i^* \mathrm{Td}^{-1}(X)))] \\
&= -[(0, i_*([\widetilde{\mathrm{ch}}(\overline{\varepsilon})] \bullet \mathrm{Td}(Y) \bullet i^* \mathrm{Td}^{-1}(X))))] \\
&\quad + [(0, i_*([\widetilde{\mathrm{ch}}(\overline{\varepsilon})] \bullet \mathrm{Td}(Y) \bullet i^* \mathrm{Td}^{-1}(X))))] \\
&= 0.
\end{aligned}$$

The proof that $i_*^{T_c}$ for metrized coherent sheaves is well defined is similar. The proof of its independence from choice of a metric on $N_{Y/X}$ or from the choice of the resolutions and metrics in X is the same as before. Now let

$$0 \longrightarrow \overline{\mathcal{F}}' \longrightarrow \overline{\mathcal{F}} \longrightarrow \overline{\mathcal{F}}'' \longrightarrow 0$$

be a short exact sequence of metrized coherent sheaves on $\mathcal{Y}$. This means that we have resolutions $\overline{E}'_* \to \mathcal{F}'_{\mathbb{C}}$, $\overline{E}_* \to \mathcal{F}_{\mathbb{C}}$ and $\overline{E}''_* \to \mathcal{F}''_{\mathbb{C}}$. Using theorem 2.24 we can suppose that there is a commutative diagram of resolutions

$$\begin{array}{ccccccccc}
0 & \to & \overline{E}'_* & \to & \overline{E}_* & \to & \overline{E}''_* & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{F}'_{\mathbb{C}} & \to & \mathcal{F}_{\mathbb{C}} & \to & \mathcal{F}''_{\mathbb{C}} & \to & 0,
\end{array} \quad (10.21)$$

with exact rows. Moreover, we can assume that the complexes of complexes $\overline{E}'_{*,*}$, $\overline{E}_{*,*}$, $\overline{E}''_{*,*}$ used in definition 10.10 are chosen compatible with diagram (10.21). Thus we obtain a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{Tot}\,\overline{E}'_{*,*} & \to & \mathrm{Tot}\,\overline{E}_{*,*} & \to & \mathrm{Tot}\,\overline{E}''_{*,*} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & i_*\mathcal{F}'_{\mathbb{C}} & \to & i_*\mathcal{F}_{\mathbb{C}} & \to & i_*\mathcal{F}''_{\mathbb{C}} & \to & 0.
\end{array} \quad (10.22)$$

We denote by $\overline{\nu}$ the exact sequence of metrized coherent sheaves on X defined by diagram (10.22). We denote $\overline{\chi}_i$ the exact sequence of hermitian vector bundles on Y

$$\overline{\chi}_i \colon 0 \longrightarrow \overline{E}'_i \longrightarrow \overline{E}_i \longrightarrow \overline{E}''_i \longrightarrow 0,$$

and by $\overline{\varepsilon}$ the exact sequence of metrized coherent sheaves on X

$$\overline{\varepsilon}_i \colon 0 \longrightarrow \overline{i_*E}'_i \longrightarrow \overline{i_*E_i} \longrightarrow \overline{i_*E}''_i \longrightarrow 0.$$

Moreover, let $\overline{\xi}_i$, $\overline{\xi}'_i$ and $\overline{\xi}''_i$ denote the hermitian embedded vector bundles defined by the above resolutions and $\overline{E}_i$, $\overline{E}'_i$ and $\overline{E}''_i$ respectively and let $\overline{\xi}_{i,c}$, $\overline{\xi}'_{i,c}$ and $\overline{\xi}''_{i,c}$ be as in definition 8.37. Then, using proposition 2.38 and equation (8.42) we obtain

$$\begin{aligned}\widetilde{\mathrm{ch}}(\overline{\nu}) &= \sum_i (-1)^i \widetilde{\mathrm{ch}}(\overline{\varepsilon}) \\ &= \sum_i (-1)^i (T_c(\overline{\xi}'_{i,c}) + T_c(\overline{\xi}''_{i,c}) - T_c(\overline{\xi}_{i,c})) \\ &\quad + \sum_i (-1)^i i_*(\widetilde{\mathrm{ch}}(\overline{\chi}_i) \bullet \mathrm{Td}(Y)) \bullet \mathrm{Td}^{-1}(X)\end{aligned} \tag{10.23}$$

Now the proof follows as before, but using equation (10.23) instead of equation (10.19).

If $\mathcal{X}$ is a regular arithmetic variety, the lifting property follows from the isomorphism between the $\widehat{K}$-groups and the $\widehat{K}'$-groups.

Suppose now that $i \colon \mathcal{Y} \longrightarrow \mathcal{X}$ is a regular closed immersion and let $[\overline{\mathcal{F}}, \eta] \in \widehat{K}(\mathcal{Y}, \mathcal{D}_{\mathrm{cur},Y})$. Then it follows from [2] III that the coherent sheaf $i_*\mathcal{F}$ can be resolved

$$0 \longrightarrow \mathcal{E}_n \longrightarrow \dots \longrightarrow \mathcal{E}_0 \longrightarrow i_*\mathcal{F} \longrightarrow 0$$

with $\mathcal{E}_i$ locally free sheaves on $\mathcal{X}$. Moreover we endow the vector bundles E_i induced on X with hermitian metrics and so we obtain a metric on the coherent sheaf $i_*\mathcal{F}$ and the corresponding hermitian embedded vector bundle $\overline{\xi}$. Using the independence from the resolutions and on the metrics we see that the equation 10.11 defines an element in $\widehat{K}(\mathcal{X}, \mathcal{D}_{\mathrm{cur},X})$. ☐

Proposition 10.24. *For any element $\alpha \in \widehat{K}'(\mathcal{Y}, \mathcal{D}_{\mathrm{cur},Y})$ we have*

$$\omega(i_*^{T_c}(\alpha))\,\mathrm{Td}(X) = i_*(\omega(\alpha)\,\mathrm{Td}(Y)) \tag{10.25}$$

$$\omega(i_*^{T}(\alpha)) = i_*(\omega(\alpha)\,\mathrm{Td}^{-1}(N_{Y/X})) \tag{10.26}$$

Proof. We will prove the statement only for $i_*^{T_c}$. We consider first a class of the form $[\overline{\mathcal{F}}, 0]$. Using equation (8.38) we obtain, after choosing a metric $\overline{E}_i \longrightarrow (i_*\mathcal{F})_{\mathbb{C}}$, and considering the induced hermitian embedded vector bundle

$\overline{\xi_c}$:

$$\begin{aligned}\omega(i_*^{T_c}([\overline{\mathcal{F}},0]))\operatorname{Td}(X)&=\Big(\sum(-1)^i\operatorname{ch}(\overline{E_i})-\mathrm{d}_{\mathcal{D}}\,T_c(\overline{\xi_c})\Big)\operatorname{Td}(X)\\&=i_*(\operatorname{ch}(\overline{F})\bullet\operatorname{Td}(Y)\bullet i^*\operatorname{Td}^{-1}(X)i^*(\operatorname{Td}(X)))\\&=i_*(\operatorname{ch}(\overline{F})\bullet\operatorname{Td}(Y))\\&=i_*(\omega([\overline{\mathcal{F}},0])\operatorname{Td}(Y))\end{aligned}$$

Taking now a class of the form $[0,\eta]$ we obtain:

$$\begin{aligned}\omega(i_*^{T_c}([0,\eta]))\operatorname{Td}(X)&=\mathrm{d}_{\mathcal{D}}\left(i_*(\eta\operatorname{Td}(Y)i^*\operatorname{Td}^{-1}(X))\right)\operatorname{Td}(X)\\&=i_*\,\mathrm{d}_{\mathcal{D}}(\eta\operatorname{Td}(Y))\\&=i_*(\omega([0,\eta])\operatorname{Td}(Y))\end{aligned}$$

and hence the equality 10.25 is proved. □

The next proposition explains the terminology "compatible with the projection formula" and "transitive" that we used for theories of singular Bott-Chern classes. The second statement is the main reason to introduce the push-forward $i_*^{T_c}$.

Proposition 10.27. *If the theory of singular Bott-Chern classes is compatible with the projection formula, we have that, for $\alpha\in\widehat{K}'(\mathcal{Y},\mathcal{D}_{\mathrm{cur},Y})$ and $\beta\in\widehat{K}(\mathcal{X},\mathcal{D}_{\mathrm{l,a},X})$ the following equalities hold*

$$\begin{aligned}i_*^{T_c}(\alpha i^*\beta)&=i_*^{T_c}(\alpha)\beta,\\ i_*^{T}(\alpha i^*\beta)&=i_*^{T}(\alpha)\beta.\end{aligned}$$

If moreover the theory of singular Bott-Chern classes is transitive and $j\colon(\mathcal{Z},h_Z)\longrightarrow(\mathcal{Y},h_Y)$ is another closed immersion of metrized arithmetic varieties, then

$$(i\circ j)_*^{T_c}=i_*^{T_c}\circ j_*^{T_c}.$$

Proof. We prove first the projection formula. For simplicity we will treat the case when $\alpha\in\widehat{K}(\mathcal{Y},\mathcal{D}_{\mathrm{cur},Y})$. Let $\alpha=(\overline{\mathcal{F}},\eta)$, let $\overline{\xi_c}=(i,\overline{T}_X,\overline{T}_Y,\overline{\mathcal{F}}_{\mathbb{C}},\overline{E}_*)$ be a hermitian embedded vector bundle and let $\beta=(\overline{\mathcal{E}},\chi)$. Using equations (10.11) and (10.5), we obtain

$$\begin{aligned}i_*^{T_c}(\alpha i^*\beta)-i_*^{T_c}(\alpha)\beta=&-\sum_i(-1)^i\operatorname{ch}(\overline{E}_i)\bullet\chi+\mathrm{d}_{\mathcal{D}}(T_c(\overline{\xi}_c))\bullet\chi\\&+i_*(\operatorname{ch}((\overline{\mathcal{F}})_{\mathbb{C}})\bullet\operatorname{Td}(Y)))\bullet\operatorname{Td}^{-1}(X)\bullet\chi\\&+T_c(\overline{\xi}_c)\bullet\operatorname{ch}(\overline{\mathcal{E}}_{\mathbb{C}})-T_c(\overline{\xi}_c\otimes\overline{\mathcal{E}}_{\mathbb{C}})\\=&\,T_c(\overline{\xi}_c\otimes\overline{\mathcal{E}}_{\mathbb{C}})-T_c(\overline{\xi}_c)\bullet\operatorname{ch}(\overline{\mathcal{E}}_{\mathbb{C}}).\end{aligned}$$

Therefore, if T is compatible with the projection formula, then the projection formula holds.

The fact that, if moreover T is transitive then $(i\circ j)_*^{T_c}=i_*^{T_c}\circ j_*^{T_c}$ follows directly from the definition and equation (8.41). □

If $i\colon \mathcal{Y} \longrightarrow \mathcal{X}$ is a regular closed immersion between arithmetic varieties, then the normal cone $\mathcal{N}_{\mathcal{Y}/\mathcal{X}}$ is a locally free sheaf. The choice of a hermitian metric on $N_{Y/X}$ determines a hermitian vector bundle $\overline{\mathcal{N}}_{\mathcal{Y}/\mathcal{X}}$. If now $i\colon (\mathcal{Y}, h_Y) \longrightarrow (\mathcal{X}, h_X)$ is a closed immersion between regular metrized arithmetic varieties, then the tangent bundles $\mathcal{T}_{\mathcal{Y}}$ and $\mathcal{T}_{\mathcal{X}}$ are virtual vector bundles. Since over $\mathbb{C}$ they define vector bundles, we can provide them with hermitian metrics and denote the hermitian virtual vector bundles by $\overline{\mathcal{T}}_{\mathcal{X}}$ and $\overline{\mathcal{T}}_{\mathcal{Y}}$. There are well defined clases $\widehat{\operatorname{Td}}(\mathcal{Y}) = \widehat{\operatorname{Td}}(\overline{\mathcal{T}}_{\mathcal{Y}})$ and $\widehat{\operatorname{Td}}(\mathcal{X}) = \widehat{\operatorname{Td}}(\overline{\mathcal{T}}_{\mathcal{X}})$.
The arithmetic Grothendieck-Riemann-Roch theorem for closed immersions compares the direct images in the arithmetic K-groups with the direct images in the arithmetic Chow groups.

Theorem 10.28 ([6], [32]). *Let T be a theory of singular Bott-Chern classes and let S_T be the additive genus of corollary 9.43.*

(i) Let $i\colon \mathcal{Y} \longrightarrow \mathcal{X}$ be a regular closed immersion between arithmetic varieties. Assume that we have chosen a hermitian metric on the complex bundle $N_{Y/X}$. Then, for any $\alpha = (\overline{\mathcal{F}}, \eta) \in \widehat{K}(\mathcal{Y}, \mathcal{D}_{\mathrm{cur},Y})$ the equation

$$\widehat{\operatorname{ch}}(i_*^T(\alpha)) = i_*(\widehat{\operatorname{ch}}(\alpha)\widehat{\operatorname{Td}}^{-1}(\overline{\mathcal{N}}_{\mathcal{Y}/\mathcal{X}})) - \mathrm{a}(i_*(\operatorname{ch}(\mathcal{F}_{\mathbb{C}})\operatorname{Td}^{-1}(N_{Y/X})S_T(N)) \tag{10.29}$$

holds.

(ii) Let $i\colon (\mathcal{Y}, h_Y) \longrightarrow (\mathcal{X}, h_X)$ be a closed immersion between regular metrized arithmetic varieties. Then, for any $\alpha = (\overline{\mathcal{F}}, \eta) \in \widehat{K}(\mathcal{Y}, \mathcal{D}_{\mathrm{cur},Y})$ the equation

$$\widehat{\operatorname{ch}}(i_*^{T_c}(\alpha))\widehat{\operatorname{Td}}(\mathcal{X}) = i_*(\widehat{\operatorname{ch}}(\alpha)\widehat{\operatorname{Td}}(\mathcal{Y})) - \mathrm{a}(i_*(\operatorname{ch}(\mathcal{F}_{\mathbb{C}})\operatorname{Td}(Y)S_T(N))) \tag{10.30}$$

holds.

Proof. The proof follows the classical pattern of the deformation to the normal cone as in [6] and [32].
Let $\mathcal{W}$ be the deformation to the normal cone to $\mathcal{Y}$ in $\mathcal{X}$. We will follow the notation of section 5. Since i is a regular closed immersion, there is a finite resolution by locally free sheaves

$$0 \to \mathcal{E}_n \to \cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to i_*\mathcal{F} \to 0.$$

We choose hermitian metrics on the complex bundles $E_i = (\mathcal{E}_i)_{\mathbb{C}}$. The immersion $j\colon \mathcal{Y} \times \mathbb{P}^1 \longrightarrow \mathcal{W}$ is also a regular immersion. The construction of theorem 5.4 is valid over the arithmetic ring A. Therefore we have a resolution by hermitian vector bundles

$$0 \to \widetilde{\mathcal{G}}_n \to \cdots \to \widetilde{\mathcal{G}}_1 \to \widetilde{\mathcal{G}}_0 \to i_*\mathcal{F} \to 0.$$

such that its restriction to $\mathcal{X}\times\{0\}$ is isometric to $\mathcal{E}_*$. Its restriction to $\widetilde{\mathcal{X}}$ is orthogonally split, and its restriction to $\mathcal{P}=\mathbb{P}(\mathcal{N}_{\mathcal{Y}/\mathcal{X}}\oplus\mathcal{O}_{\mathcal{Y}})$ fits in a short exact sequence

$$0\longrightarrow \overline{\mathcal{A}}_*\longrightarrow \widetilde{\mathcal{E}}_*|_{\mathcal{P}}\longrightarrow K(\overline{\mathcal{F}},\overline{\mathcal{N}}_{\mathcal{Y}/\mathcal{X}})\longrightarrow 0,$$

where $\overline{\mathcal{A}}_*$ is orthogonally split and $K(\overline{\mathcal{F}},\overline{\mathcal{N}}_{\mathcal{Y}/\mathcal{X}})$ is the Koszul resolution. We denote by $\overline{\eta}_k$ the piece of degree k of this exact sequence. Let t be the absolute coordinate of $\mathbb{P}^1$. It defines a rational function in $\mathcal{W}$ and

$$\widehat{\operatorname{div}}(t)=(\mathcal{X}_0+\mathcal{P}+\widetilde{\mathcal{X}},(0,-\frac{1}{2}\log t\bar{t}))$$

The key point of the proof of the theorem is that, in the group $\widehat{\mathrm{CH}}^*(\mathcal{X},\mathcal{D}_{\mathrm{cur},X})$, we have

$$(p_{\mathcal{W}})_*(\widehat{\operatorname{ch}}(\widetilde{\mathcal{E}}_*)\widehat{\operatorname{div}}(t))=0.$$

Using the definition of the product in the arithmetic Chow rings we obtain

$$\begin{aligned}(p_{\mathcal{W}})_*(\widehat{\operatorname{ch}}(\widetilde{\mathcal{E}}_*)\widehat{\operatorname{div}}(t))=\widehat{\operatorname{ch}}(\overline{\mathcal{E}}_*)-(p_{\widetilde{\mathcal{X}}})_*\widehat{\operatorname{ch}}(\widetilde{\mathcal{E}}_*|_{\widetilde{\mathcal{X}}})-(p_{\widetilde{\mathcal{P}}})_*\widehat{\operatorname{ch}}(\widetilde{\mathcal{E}}_*|_{\mathcal{P}})\\ +\operatorname{a}((p_W)_*(\operatorname{ch}((\widetilde{\mathcal{E}}_*)_{\mathbb{C}})\bullet W_1)).\end{aligned}\tag{10.31}$$

But we have

$$\widehat{\operatorname{ch}}(\overline{\mathcal{E}}_*)=\widehat{\operatorname{ch}}(i_*^T(\overline{\mathcal{F}}))+\operatorname{a}(T(\overline{\xi})),\tag{10.32}$$

$$(p_{\widetilde{\mathcal{X}}})_*\widehat{\operatorname{ch}}(\widetilde{\mathcal{E}}_*|_{\widetilde{\mathcal{X}}})=0,\tag{10.33}$$

$$(p_{\widetilde{\mathcal{P}}})_*\widehat{\operatorname{ch}}(\widetilde{\mathcal{E}}_*|_{\mathcal{P}})=i_*(\pi_{\mathcal{P}})_*(\widehat{\operatorname{ch}}(K(\overline{\mathcal{F}},\overline{\mathcal{N}}_{\mathcal{Y}/\mathcal{X}}))-\sum_k(-1)^k\operatorname{a}(\widetilde{\operatorname{ch}}(\overline{\eta}_k))).\tag{10.34}$$

Moreover, by equation (7.3),

$$\begin{aligned}\operatorname{a}((p_W)_*(\operatorname{ch}((\widetilde{\mathcal{E}}_*)_{\mathbb{C}})\bullet W_1))=-\operatorname{a}(T(\overline{\xi}))-\sum_k(-1)^k\operatorname{a}(\widetilde{\operatorname{ch}}(\overline{\eta}_k)))\\ +\operatorname{a}(i_*C_T(\mathcal{F}_{\mathbb{C}},\mathcal{N}_{\mathbb{C}})).\end{aligned}\tag{10.35}$$

Thus we are led to compute $i_*(\pi_{\mathcal{P}})_*\widehat{\operatorname{ch}}(K(\overline{\mathcal{F}},\overline{\mathcal{N}}_{\mathcal{Y}/\mathcal{X}}))$. This is done in the following two lemmas.

Lemma 10.36. *Let $\mathcal{Y}$ be an arithmetic variety, $\overline{\mathcal{N}}$ a rank r hermitian vector bundle over $\mathcal{Y}$ and denote $\mathcal{P}=\mathbb{P}^1(\mathcal{N}\oplus\mathcal{O}_{\mathcal{Y}})$, and $\overline{\mathcal{Q}}$ the tautological quotient bundle. Let $\mathcal{Y}_0$ be the cycle defined by the zero section of $\mathcal{P}$. Then*

$$\widehat{c}_r(\overline{\mathcal{Q}})=(\mathcal{Y}_0,(c_r(\overline{\mathcal{Q}}_{\mathbb{C}}),\widetilde{e}(\mathcal{P}_{\mathbb{C}},\overline{\mathcal{Q}}_{\mathbb{C}},s))),\tag{10.37}$$

where $\widetilde{e}(\mathcal{P}_{\mathbb{C}},\overline{\mathcal{Q}}_{\mathbb{C}},s)$ is the Euler-Green current of lemma 9.4.

Proof. We know that $\widehat{c}_r(\overline{\mathcal{Q}}) = (\mathcal{Y}_0, (c_r(\overline{\mathcal{Q}}_{\mathbb{C}}), \widetilde{e}))$ for certain Green current $\widetilde{e}$. By definition this Green current satisfies

$$\mathrm{d}_{\mathcal{D}}\,\widetilde{e} = c_r(\overline{\mathcal{Q}}_{\mathbb{C}}) - \delta_{\mathcal{Y}_{\mathbb{C}}}.$$

Moreover, since the restriction of $\overline{\mathcal{Q}}_{\mathbb{C}}$ to D_∞ has a global section of constant norm we have that $\widetilde{e}|_{D_\infty} = 0$. Therefore, by lemma 9.4,

$$\widetilde{e} = \widetilde{e}(\mathcal{P}_{\mathbb{C}}, \overline{\mathcal{Q}}_{\mathbb{C}}, s).$$

$\square$

Lemma 10.38. *The following equality hold:*

$$(\pi_{\mathcal{P}})_*\widehat{\mathrm{ch}}(K(\overline{\mathcal{F}},\overline{\mathcal{N}})_*) = \\ \widehat{\mathrm{ch}}(\overline{\mathcal{F}})\widehat{\mathrm{Td}^{-1}}(\overline{\mathcal{N}}) + \mathrm{a}(C_T(\overline{\mathcal{F}},\overline{\mathcal{N}}) - \mathrm{ch}(\mathcal{F}_{\mathbb{C}})\,\mathrm{Td}^{-1}(N_{Y/X})S_T(N)). \quad (10.39)$$

Proof. We just compute, using lemma 10.36,

$$\begin{aligned}(\pi_{\mathcal{P}})_*\widehat{\mathrm{ch}}(K(\overline{\mathcal{F}},\overline{\mathcal{N}})_*) &= (\pi_{\mathcal{P}})_*\sum_k(-1)^k\widehat{\mathrm{ch}}(\bigwedge^k\overline{\mathcal{Q}}^\vee)\widehat{\mathrm{ch}}(\pi_{\mathcal{P}}^*\overline{\mathcal{F}})\\
&= (\pi_{\mathcal{P}})_*(\widehat{c}_r(\overline{\mathcal{Q}})\widehat{\mathrm{Td}^{-1}}(\overline{\mathcal{Q}}))\widehat{\mathrm{ch}}(\overline{\mathcal{F}})\\
&= \widehat{\mathrm{Td}^{-1}}(\overline{\mathcal{N}})\widehat{\mathrm{ch}}(\overline{\mathcal{F}}) + \mathrm{a}((\pi_P)_*(\widetilde{e}\,\mathrm{Td}^{-1}(\overline{Q}))\,\mathrm{ch}(\overline{F}))\\
&= \widehat{\mathrm{Td}^{-1}}(\overline{\mathcal{N}})\widehat{\mathrm{ch}}(\overline{\mathcal{F}}) + \mathrm{a}((\pi_P)_*(T^h(K(\overline{F},\overline{N})))\,\mathrm{ch}(\overline{F}))\\
&= \widehat{\mathrm{Td}^{-1}}(\overline{\mathcal{N}})\widehat{\mathrm{ch}}(\overline{\mathcal{F}}) + \mathrm{a}(C_{T^h}(F,N))\\
&= \widehat{\mathrm{Td}^{-1}}(\overline{\mathcal{N}})\widehat{\mathrm{ch}}(\overline{\mathcal{F}}) + C_T(F,N) - \mathrm{a}(\mathrm{Td}^{-1}(N)\,\mathrm{ch}(F)S_T(N)).\end{aligned}$$

$\square$

The equation (10.29) follows by combining equations (10.31), (10.32), (10.33), (10.34), (10.35) and (10.39).
The equation (10.30) follows from equation (10.29) by a straightforward computation. $\square$

Since T is homogeneous if and only if $S_T = 0$, in view of this result, the theory of homogeneous singular Bott-Chern classes is characterized for being the unique theory of singular Bott-Chern classes that provides an exact arithmetic Grothendieck-Riemann-Roch theorem for closed immersions. By contrast, if one uses a theory of singular Bott-Chern classes that is not homogeneous, there is an analogy between the genus S_T and the R-genus that appears in the arithmetic Grothendieck-Riemann-Roch theorem for submersions.
Since there is a unique theory of homogeneous singular Bott-Chern classes, the following definition is natural.

Definition 10.40. Let $i\colon (\mathcal{Y},h_Y) \longrightarrow (\mathcal{X},h_X)$ be a closed immersion of metrized arithmetic varieties, the *push-forward map*

$$i_*\colon \widehat{K}'(\mathcal{Y},\mathcal{D}_{\mathrm{cur},Y}) \longrightarrow \widehat{K}'(\mathcal{X},\mathcal{D}_{\mathrm{cur},Y})$$

is defined as $i_* = i_*^{T_c^h}$.

Corollary 10.41. *The push-forward map makes $\widehat{K}'(\_,\mathcal{D}_{\mathrm{cur},Y})$ and $\widehat{K}(\_,\mathcal{D}_{\mathrm{cur},Y})$ functors from the category of regular metrized arithmetic varieties and closed immersions to the category of abelian groups.*

Corollary 10.42. *Let $i\colon (\mathcal{Y},h_Y) \longrightarrow (\mathcal{X},h_X)$ be a closed immersion of regular metrized arithmetic varieties, then*

$$\widehat{\mathrm{ch}}(i_*^T(\alpha))\widehat{\mathrm{Td}}(\mathcal{X}) = i_*(\widehat{\mathrm{ch}}(\alpha)\widehat{\mathrm{Td}}(\mathcal{Y})). \tag{10.43}$$

Remark 10.44. Combining theorem 10.28 with [16] we can obtain an arithmetic Grothendieck-Riemann-Roch theorem for projective morphisms of regular arithmetic varieties.
In a forthcoming paper we will show that the higher torsion forms used to define the direct images for submersions can also be characterized axiomatically.

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