Geometric Description of the Connecting Homomorphism for Witt Groups

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Abstract. We give a geometric setup in which the connecting homomorphism in the localization long exact sequence for Witt groups decomposes as the pull-back to the exceptional fiber of a suitable blow-up followed by a push-forward.

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1 Introduction

Witt groups form a very interesting cohomology theory in algebraic geometry. (For a survey, see [5].) Unlike the better known K-theory and Chow theory, Witt theory is not *oriented* in the sense of Levine-Morel [17] or Panin [22], as already visible on the non-standard projective bundle theorem, see Arason [2] and Walter [26]. Another way of expressing this is that push-forwards do not exist in sufficient generality for Witt groups. This "non-orientability" can make computations unexpectedly tricky. Indeed, the Witt groups of such elementary schemes as Grassmann varieties will appear for the first time in the companion article [6], whereas the corresponding computations for oriented cohomologies have been achieved more than 35 years ago in [16], using the well-known cellular decomposition of Grassmann varieties. See also [21] for general statements on cellular varieties.
In oriented theories, there is a very useful computational technique, recalled in Theorem 1.3 below, which allows inductive computations for families of cellular varieties. Our paper originates in an attempt to extend this result to the non-oriented setting of Witt theory. Roughly speaking, such an extension is possible "half of the time". In the remaining "half", some specific ideas must come in and reflect the truly non-oriented behavior of Witt groups. To explain this rough statement, let us fix the setup, which will remain valid for the entire paper.

Setup 1.1. We denote by $\mathcal{S}ch$ the category of separated connected noetherian $\mathbb{Z}[\frac{1}{2}]$-scheme. Let $X, Z \in \mathcal{S}ch$ be schemes and let $\iota : Z \hookrightarrow X$ be a regular closed immersion of codimension $c \geq 2$. Let $Bl = Bl_Z X$ be the blow-up of X along Z and E the exceptional fiber. Let $U = X - Z \cong Bl - E$ be the unaltered open complement. We have a commutative diagram

$$\begin{tikzcd} Z \arrow[r, hook, "\iota"] & X & U \arrow[l, hook', "\upsilon"'] \arrow[dl, hook', "\tilde{\upsilon}"] \\ E \arrow[u, "\tilde{\pi}"] \arrow[r, hook, "\tilde{\iota}"'] & Bl \arrow[u, "\pi"'] & \end{tikzcd} \qquad (1)$$

with the usual morphisms.

Consider now a cohomology theory with supports, say H^*

$$\cdots \xrightarrow{\partial} H^*_Z(X) \longrightarrow H^*(X) \xrightarrow{\upsilon^*} H^*(U) \xrightarrow{\partial} H^{*+1}_Z(X) \longrightarrow \cdots \qquad (2)$$

In this paper we shall focus on the case of Witt groups $H^* = \mathrm{W}^*$ but we take inspiration from H^* being an oriented cohomology theory. Ideally, we would like conditions for the vanishing of the connecting homomorphism $\partial = 0$ in the above localization long exact sequence. Even better would be conditions for the restriction υ^* to be split surjective. When H^* is an oriented theory, there is a well-known hypothesis under which such a splitting actually exists, namely:

Hypothesis 1.2. Assume that there exists an auxiliary morphism $\tilde{\alpha} : Bl \to Y$

$$\begin{tikzcd} Z \arrow[r, hook, "\iota"] & X & U \arrow[l, hook', "\upsilon"'] \arrow[dl, "\tilde{\upsilon}"'] \arrow[d, dotted, "\alpha"] \\ E \arrow[u, "\tilde{\pi}"] \arrow[r, hook, "\tilde{\iota}"'] & Bl \arrow[u, "\pi"'] \arrow[r, dotted, "\tilde{\alpha}"'] & Y \end{tikzcd} \tag{3}$$

such that $\alpha := \tilde{\alpha} \circ \tilde{\upsilon} : U \to Y$ is an $\mathbb{A}^*$-*bundle*, i.e. every point of Y has a Zariski neighborhood over which α is isomorphic to a trivial $\mathbb{A}^r$-bundle, for some $r \geq 0$. See Ex. 1.5 for an explicit example with X, Y and Z being Grassmann varieties.

Theorem 1.3 (The oriented technique). *Under Setup 1.1 and Hypothesis 1.2, assume X, Y and Z regular. Assume the cohomology theory H^* is homotopy invariant for regular schemes and* oriented*, in that it admits push-forwards along proper morphisms satisfying flat base-change. Then, the restriction $\upsilon^* : H^*(X) \to H^*(U)$ is split surjective with explicit section $\pi_* \circ \tilde{\alpha}^* \circ (\alpha^*)^{-1}$, where $\pi_* : H^*(Bl) \to H^*(X)$ is the push-forward. Hence the connecting homomorphism $\partial : H^*(U) \to H_Z^{*+1}(X)$ vanishes and the above localization long exact sequence* (2) *reduces to split short exact sequences $0 \to H_Z^*(X) \to H^*(X) \to H^*(U) \to 0$.*

Proof. By homotopy invariance, we have $\alpha^* : H^*(Y) \xrightarrow{\sim} H^*(U)$. By base-change, $\upsilon^* \circ \pi_* = \tilde{\upsilon}^*$ and since $\tilde{\upsilon}^* \circ \tilde{\alpha}^* = \alpha^*$, we have $\upsilon^* \circ \pi_* \circ \tilde{\alpha}^* \circ (\alpha^*)^{-1} = \mathrm{id}$. $\square$

The dichotomy between the cases where the above technique extends to Witt groups and the cases where is does not, comes from the duality. To understand this, recall that one can consider Witt groups $\mathrm{W}^*(X, L)$ with duality twisted by a line bundle L on the scheme X. Actually only the class of the twist L in $\mathrm{Pic}(X)/2$ really matters since we have square-periodicity isomorphisms for all $M \in \mathrm{Pic}(X)$

$$\mathrm{W}^*(X, L) \cong \mathrm{W}^*(X, L \otimes M^{\otimes 2}) \,. \tag{4}$$

Here is a condensed form of our Theorem 2.3 and Main Theorem 2.6 below:

Theorem 1.4. *Under Hypothesis 1.2, assume X, Y and Z regular. Let $L \in \mathrm{Pic}(X)$. Then there exists an integer $\lambda(L) \in \mathbb{Z}$ (defined by* (8) *below) such that:*

(A) If $\lambda(L) \equiv c - 1 \mod 2$ then the restriction $\upsilon^* : \mathrm{W}^*(X, L) \to \mathrm{W}^*(U, L_{|_U})$ is split surjective with a section given by the composition $\pi_* \circ \tilde{\alpha}^* \circ (\alpha^*)^{-1}$. Hence the connecting homomorphism $\mathrm{W}^*(U, L_{|_U}) \xrightarrow{\partial} \mathrm{W}_Z^{*+1}(X, L)$ vanishes and the localization long exact sequence reduces to split short exact sequences

$$0 \longrightarrow \mathrm{W}_Z^*(X, L) \longrightarrow \mathrm{W}^*(X, L) \longrightarrow \mathrm{W}^*(U, L_{|_U}) \longrightarrow 0 \,.$$

(B) If $\lambda(L) \equiv c \mod 2$ then the connecting homomorphism ∂ is equal to a composition of pull-backs and push-forwards: $\partial = \iota_* \circ \tilde{\pi}_* \circ \tilde{\iota}^* \circ \tilde{\alpha}^* \circ (\alpha^*)^{-1}$.

This statement requires some explanations. First of all, note that we have used push-forwards for Witt groups, along $\pi : Bl \to X$ in (A) and along $\tilde{\pi} : E \to Z$ and $\iota : Z \to X$ in (B). To explain this, recall that the push-forward in Witt theory is only conditionally defined. Indeed, given a proper morphism $f : X' \to X$ between (connected) regular schemes and given a line bundle $L \in \mathrm{Pic}(X)$, the push-forward homomorphism does not map $\mathrm{W}^*(X', f^*L)$ into $\mathrm{W}^*(X, L)$, as one could naively expect, but the second author and Hornbostel [8] showed that Grothendieck-Verdier duality yields a twist by the relative canonical line bundle $\omega_f \in \mathrm{Pic}(X')$:

$$\mathrm{W}^{i+\dim(f)}\big(\, X' \,,\, \omega_f \otimes f^*L \,\big) \xrightarrow{f_*} \mathrm{W}^i(\, X \,,\, L \,)\,. \tag{5}$$

Also note the shift by the relative dimension, $\dim(f) := \dim X' - \dim X$, which is not problematic, since we can always replace $i \in \mathbb{Z}$ by $i - \dim(f)$.

More trickily, if you are given a line bundle $M \in \mathrm{Pic}(X')$ and if you need a push-forward $\mathrm{W}^*(X', M) \to \mathrm{W}^{*-\dim(f)}(X, ?)$ along $f : X' \to X$, you first need to check that M is of the form $\omega_f \otimes f^*L$ for some $L \in \mathrm{Pic}(X)$, at least module squares. *Otherwise, you simply do not know how to push-forward.* This is precisely the source of the dichotomy of Theorem 1.4, as explained in Proposition 2.1 below.

At the end of the day, it is only possible to transpose to Witt groups the oriented technique of Theorem 1.3 when the push-forward π_* exists for Witt groups. But actually, the remarkable part of Theorem 1.4 is Case (B), that is our Main Theorem 2.6 below, which gives a description of the connecting homomorphism ∂ when we cannot prove it zero by the oriented method. This is the part where the non-oriented behavior really appears. See more in Remark 2.7. Main Theorem 2.6 is especially striking since the original definition of the connecting homomorphism given in [3, § 4] does not have such a geometric flavor of pull-backs and push-forwards but rather involves abstract techniques of triangulated categories, like symmetric cones, and the like. Our new geometric description is also remarkably simple to use in applications, see [6]. Here is the example in question.

EXAMPLE 1.5. Let k be a field of characteristic not 2. (We describe flag varieties over k by giving their k-points, as is customary.) Let $1 \leq d \leq n$. Fix a codimension one subspace k^{n-1} of k^n. Let $X = \mathrm{Gr}_d(n)$ be the Grassmann variety of d-dimensional subspaces $V_d \subset k^n$ and let $Z \subset X$ be the closed subvariety of those subspaces V_d contained in k^{n-1}. The open complement $U = X - Z$ consists of those $V_d \not\subset k^{n-1}$. For such $V_d \in U$, the subspace $V_d \cap k^{n-1} \subset k^{n-1}$ has dimension $d-1$. This construction defines an $\mathbb{A}^{n-d}$-bundle $\alpha : U \to Y := \mathrm{Gr}_{d-1}(n-1)$, mapping V_d to $V_d \cap k^{n-1}$. This situation relates the Grassmann variety $X = \mathrm{Gr}_d(n)$ to the smaller ones $Z = \mathrm{Gr}_d(n-1)$

and $Y = \mathrm{Gr}_{d-1}(n-1)$. Diagram (1) here becomes

$$\begin{tikzcd}
\mathrm{Gr}_d(n-1) \arrow[r, hook, "\iota"] & \mathrm{Gr}_d(n) & & U \arrow[ll, hook', "\upsilon"'] \arrow[d, "\alpha"] \arrow[dll, hook', "\tilde{\upsilon}"'] \\
E \arrow[u, "\tilde{\pi}"] \arrow[r, hook, "\tilde{\iota}"'] & Bl \arrow[u, "\pi"] \arrow[rr, "\tilde{\alpha}"'] & & \mathrm{Gr}_{d-1}(n-1)\,.
\end{tikzcd}$$

The blow-up Bl is the variety of pairs of subspaces $V_{d-1} \subset V_d$ in k^n, such that $V_{d-1} \subset k^{n-1}$. The morphisms $\pi : Bl \to X$ and $\tilde{\alpha} : Bl \to Y$ forget V_{d-1} and V_d respectively. The morphism $\tilde{\upsilon}$ maps $V_d \not\subset k^{n-1}$ to the pair $(V_d \cap k^{n-1}) \subset V_d$. Applying Theorem 1.3 to this situation, Laksov [16] computes the Chow groups of Grassmann varieties by induction. For Witt groups though, there are cases where the restriction $\mathrm{W}^*(X, L) \to \mathrm{W}^*(U, L_{|U})$ is not surjective (see [6, Cor. 6.7]). Nevertheless, thank to our geometric description of the connecting homomorphism, we have obtained a complete description of the Witt groups of Grassmann varieties, for all shifts and all twists, to appear in [6]. In addition to the present techniques, our computations involve other ideas, specific to Grassmann varieties, like Schubert cells and desingularisations thereof, plus some combinatorial bookkeeping by means of special Young diagrams. Including all this here would misleadingly hide the simplicity and generality of the present paper. We therefore chose to publish the computation of the Witt groups of Grassmann varieties separately in [6].

The paper is organized as follows. Section 2 is dedicated to the detailed explanation of the above dichotomy and the proof of the above Case (A), see Theorem 2.3. We also explain Case (B) in our Main Theorem 2.6 but its proof is deferred to Section 5. The whole Section 2 is written, as above, under the assumption that all schemes are regular. This assumption simplifies the statements but can be removed at the price of introducing dualizing complexes and coherent Witt groups, which provide the natural framework over non-regular schemes. This generalization is the purpose of Section 3. There, we even drop the auxiliary Hypothesis 1.2, i.e. the dotted part of Diagram (3). Indeed, our Main Lemma 3.5 gives a very general description of the connecting homomorphism applied to a Witt class over U, if that class comes from the blow-up Bl via restriction $\tilde{\upsilon}^*$. The proof of Main Lemma 3.5 occupies Section 4. Finally, Hypothesis 1.2 re-enters the game in Section 5, where we prove our Main Theorem 2.6 as a corollary of a non-regular generalization given in Theorem 5.1. For the convenience of the reader, we gathered in Appendix A the needed results about Picard groups, canonical bundles and dualizing complexes, which are sometimes difficult to find in the literature. The conscientious reader might want to start with that appendix.

2 The regular case

We keep notation as in Setup 1.1 and we assume all schemes to be regular. This section can also be considered as an expanded introduction.

As explained after Theorem 1.4 above, we have to decide when the push-forward along $\pi : Bl \to X$ and along $\tilde{\pi} : E \to Z$ exist. By (5), we need to determine the canonical line bundles $\omega_\pi \in \mathrm{Pic}(Bl)$ and $\omega_{\tilde{\pi}} \in \mathrm{Pic}(E)$. This is classical and is recalled in Appendix A. First of all, Proposition A.6 gives

$$\mathrm{Pic}\left(\begin{tikzcd} Z \arrow[r, hook, "\iota"] & X & U \arrow[l, hook', "\upsilon"'] \arrow[dl, "\tilde{\upsilon}"] \\ E \arrow[u, "\tilde{\pi}"] \arrow[r, hook, "\tilde{\iota}"'] & Bl \arrow[u, "\pi"'] & \end{tikzcd}\right) \cong \begin{tikzcd} \mathrm{Pic}(Z) \arrow[d, "\binom{1}{0}"'] & \mathrm{Pic}(X) \arrow[l, "\iota^*"'] \arrow[r, equal] \arrow[d, "\binom{1}{0}"'] & \mathrm{Pic}(X)\,. \\ \mathrm{Pic}(Z)\oplus\mathbb{Z} & \mathrm{Pic}(X)\oplus\mathbb{Z} \arrow[l, "\left(\begin{smallmatrix}\iota^* & 0\\ 0 & 1\end{smallmatrix}\right)"'] \arrow[ur, "(1\ 0)"'] & \end{tikzcd}$$

The $\mathbb{Z}$ summands in $\mathrm{Pic}(Bl)$ and $\mathrm{Pic}(E)$ are generated by $\mathcal{O}(E) = \mathcal{O}_{Bl}(-1)$ and $\mathcal{O}(E)_{|E} = \mathcal{O}_E(-1)$ respectively. Then Proposition A.11 gives the wanted

$$\begin{aligned} \omega_\pi &= (0, c-1) \quad \text{in} \quad \mathrm{Pic}(X)\oplus\mathbb{Z} \cong \mathrm{Pic}(Bl) \quad \text{and} \\ \omega_{\tilde{\pi}} &= (-\omega_\iota, c) \quad \text{in} \quad \mathrm{Pic}(Z)\oplus\mathbb{Z} \cong \mathrm{Pic}(E)\,. \end{aligned} \tag{6}$$

So, statistically, picking a line bundle $M \in \mathrm{Pic}(Bl)$ at random, there is a 50% chance of being able to push-forward $\mathrm{W}^*(Bl, M) \to \mathrm{W}^*(X, L)$ along π for some suitable line bundle $L \in \mathrm{Pic}(X)$. To justify this, observe that

$$\mathrm{coker}\big(\,\mathrm{Pic}(X) \xrightarrow{\pi^*} \mathrm{Pic}(Bl)\,\big)/2 \;\cong\; \mathbb{Z}/2$$

and tensoring by ω_π is a bijection, so half of the elements of $\mathrm{Pic}(Bl)/2$ are of the form $\omega_\pi \otimes \pi^*(L)$. The same probability of 50% applies to the push forward along $\tilde{\pi} : E \to Z$ but interestingly in complementary cases, as we summarize now.

Proposition 2.1. *With the notation of 1.1, assume X and Z regular. Recall that $c = \mathrm{codim}_X(Z)$. Let $M \in \mathrm{Pic}(Bl)$. Let $L \in \mathrm{Pic}(X)$ and $\ell \in \mathbb{Z}$ be such that $M = (L, \ell)$ in $\mathrm{Pic}(Bl) = \mathrm{Pic}(X) \oplus \mathbb{Z}$, that is, $M = \pi^* L \otimes \mathcal{O}(E)^{\otimes \ell}$.*

(A) If $\ell \equiv c-1 \mod 2$, we can push-forward along $\pi : Bl \to X$, as follows:

$$\mathrm{W}^*(Bl, M) \cong \mathrm{W}^*(Bl, \omega_\pi \otimes \pi^* L) \xrightarrow{\pi_*} \mathrm{W}^*(X, L)\,.$$

(B) If $\ell \equiv c \mod 2$, we can push-forward along $\tilde{\pi} : E \to Z$, as follows:

$$\mathrm{W}^*(E, M_{|E}) \cong \mathrm{W}^*\big(E, \omega_{\tilde{\pi}} \otimes \tilde{\pi}^*(\omega_\iota \otimes L_{|Z})\big) \xrightarrow{\tilde{\pi}_*} \mathrm{W}^{*-c+1}(Z, \omega_\iota \otimes L_{|Z}).$$

In each case, the isomorphism $\cong$ comes from square-periodicity in the twist (4) *and the subsequent homomorphism is the push-forward* (5).

Proof. We only have to check the congruences in $\mathrm{Pic}/2$. By (6), when $\ell \equiv c-1 \mod 2$, we have $[\omega_\pi \otimes \pi^* L] = [(L, \ell)] = [M]$ in $\mathrm{Pic}(Bl)/2$. When $\ell \equiv c \mod 2$, we have $[\omega_{\tilde{\pi}} \otimes \tilde{\pi}^*(\omega_\iota \otimes L_{|Z})] = [(L_{|Z}, \ell)] = [M_{|E}]$ in $\mathrm{Pic}(E)/2$. To apply (5), note that $\dim(\pi) = 0$ since π is birational and $\dim(\tilde{\pi}) = c-1$ since $E = \mathbb{P}_Z(C_{Z/X})$ is the projective bundle of the rank-c conormal bundle $C_{Z/X}$ over Z. ∎

So far, we have only used Setup 1.1. Now add Hypothesis 1.2 with Y regular.

Remark 2.2. Since Picard groups of regular schemes are homotopy invariant, the $\mathbb{A}^*$-bundle $\alpha : U \to Y$ yields an isomorphism $\alpha^* : \mathrm{Pic}(Y) \xrightarrow{\sim} \mathrm{Pic}(U)$. Let us identify $\mathrm{Pic}(Y)$ with $\mathrm{Pic}(U)$, and hence with $\mathrm{Pic}(X)$ as we did above since $c = \mathrm{codim}_X(Z) \geq 2$. We also have $\mathcal{O}(E)_{|_U} \simeq \mathcal{O}_U$. Putting all this together, the right-hand part of Diagram (3) yields the following on Picard groups:

$$\mathrm{Pic}\left(\begin{tikzcd} X & U \arrow[l, hook', "\upsilon"'] \arrow[d, "\alpha"] \arrow[dl, "\tilde{\upsilon}"'] \\ Bl \arrow[u, "\pi"] \arrow[r, "\tilde{\alpha}"'] & Y \end{tikzcd}\right) \cong \begin{tikzcd} \mathrm{Pic}(X) \arrow[r, equal] \arrow[d, "\binom{1}{0}"'] & \mathrm{Pic}(X) \arrow[d, equal] \\ \mathrm{Pic}(X) \oplus \mathbb{Z} \arrow[ur, "(1\ 0)"] & \mathrm{Pic}(X)\,. \arrow[l, "\binom{1}{\lambda}"'] \end{tikzcd}$$

Note that the lower right map $\mathrm{Pic}(X) \cong \mathrm{Pic}(Y) \xrightarrow{\tilde{\alpha}^*} \mathrm{Pic}(Bl) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}$ must be of the form $\binom{1}{\lambda}$ by commutativity (i.e. since $(1\ 0) \cdot \binom{1}{\lambda} = 1$) but there is no reason for its second component $\lambda : \mathrm{Pic}(X) \to \mathbb{Z}$ to vanish. This is indeed a key observation. In other words, we have two homomorphisms from $\mathrm{Pic}(X)$ to $\mathrm{Pic}(Bl)$, the direct one π^* and the circumvolant one $\tilde{\alpha}^* \circ (\alpha^*)^{-1} \circ \upsilon^*$ going via U and Y

$$\begin{tikzcd} \mathrm{Pic}(X) \arrow[r, "\upsilon^*", "\simeq"'] \arrow[d, "\pi^*"'] \arrow[dr, phantom, "\neq"] & \mathrm{Pic}(U) \arrow[d, "\simeq"', "(\alpha^*)^{-1}"] \\ \mathrm{Pic}(Bl) & \mathrm{Pic}(Y) \arrow[l, "\tilde{\alpha}^*"] \end{tikzcd} \tag{7}$$

and they do *not* coincide in general. The difference is measured by λ, which depends on the choice of Y and on the choice of $\tilde{\alpha} : Bl \to Y$, in Hypothesis 1.2. So, for every $L \in \mathrm{Pic}(X)$, the integer $\lambda(L) \in \mathbb{Z}$ is defined by the equation

$$\tilde{\alpha}^*\,(\alpha^*)^{-1}\,\upsilon^*(L) = \pi^*(L) \otimes \mathcal{O}(E)^{\otimes \lambda(L)} \tag{8}$$

in $\mathrm{Pic}(Bl)$. Under the isomorphism $\mathrm{Pic}(Bl) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}$, the above equation can be reformulated as $\tilde{\alpha}^*\,(\alpha^*)^{-1}\,\upsilon^*(L) = \big(L, \lambda(L)\big)$.

Theorem 2.3 (Partial analogue of Theorem 1.3). *With the notation of 1.1, assume Hypothesis 1.2 and assume X, Y, Z regular. Recall that $c = \mathrm{codim}_X(Z)$. Let $L \in \mathrm{Pic}(X)$ and consider the integer $\lambda(L) \in \mathbb{Z}$ defined in* (8) *above.*
If $\lambda(L) \equiv c-1 \mod 2$ then the restriction $\upsilon^* : \mathrm{W}^*(X, L) \to \mathrm{W}^*(U, L_{|_U})$ is split surjective, with an explicit section given by the composition $\pi_* \circ \tilde{\alpha}^* \circ (\alpha^*)^{-1}$

$$\begin{tikzcd} \mathrm{W}^*(X, L) & & \mathrm{W}^*(U, L_{|_U}) \arrow[d, "\simeq"', "(\alpha^*)^{-1}"] \\ \mathrm{W}^*(Bl\,,\, \omega_\pi \otimes \pi^* L) \cong \mathrm{W}^*\big(Bl\,,\, \tilde{\alpha}^*\,(\alpha^*)^{-1} L_{|_U}\big) \arrow[u, "\pi_*"] & & \mathrm{W}^*\big(Y\,,\, (\alpha^*)^{-1}\, L_{|_U}\big) \arrow[ll, "\tilde{\alpha}^*"] \end{tikzcd}$$

Proof. The whole point is that π_* can be applied after $\tilde{\alpha}^* \circ (\alpha^*)^{-1}$, that is, on $\mathrm{W}^*\big(Bl\,,\,\tilde{\alpha}^*\,(\alpha^*)^{-1}\,\upsilon^*(L)\big)$. This holds by Proposition 2.1 (A) applied to

$$M := \tilde{\alpha}^*\,(\alpha^*)^{-1}\,\upsilon^*(L) \overset{(8)}{=} (L, \lambda(L)) \in \mathrm{Pic}(X) \oplus \mathbb{Z} = \mathrm{Pic}(Bl)\,. \tag{9}$$

The assumption $\lambda(L) \equiv c-1 \bmod 2$ expresses the hypothesis of Proposition 2.1 (A). Checking that we indeed have a section goes as in the oriented case, see Thm. 1.3:

$$\upsilon^* \circ \pi_* \circ \tilde{\alpha}^* \circ (\alpha^*)^{-1} = \tilde{\upsilon}^* \circ \tilde{\alpha}^* \circ (\alpha^*)^{-1} = \alpha^* \circ (\alpha^*)^{-1} = \mathrm{id}\,.$$

The first equality uses base-change [8, Thm. 6.9] on the left-hand cartesian square:

$$\begin{tikzcd} X & U \arrow[l, hook', "\upsilon"'] & & L & L_{|U} \\ Bl \arrow[u, "\pi"] & U \arrow[l, hook', "\tilde{\upsilon}"'] \arrow[u, "\mathrm{id}"'] & & \omega_\pi \otimes \pi^* L & L_{|U} \end{tikzcd}$$

with respect to the right-hand line bundles. Note that $(\omega_\pi)_{|U} = \mathcal{O}_U$ by (6). $\square$

Remark 2.4. In the above proof, see (9), we do not apply Proposition 2.1 to M being π^*L, as one could first expect; see Remark 2.2. Consequently, our condition on L, namely $\lambda(L) \equiv c-1 \bmod 2$, does not only depend on the codimension c of Z in X but also involves (hidden in the definition of λ) the particular choice of the auxiliary scheme Y and of the morphism $\tilde{\alpha} : Bl \to Y$ of Hypothesis 1.2.

Remark 2.5. The legitimate question is now to decide what to do in the remaining case, that is, when $\lambda(L) \equiv c \bmod 2$. As announced, this is the central goal of our paper (Thm. 2.6 below). So, let $L \in \mathrm{Pic}(X)$ be a twist such that push-forward along $\pi : Bl \to X$ cannot be applied to define a section to the restriction $\mathrm{W}^*(X,L) \to \mathrm{W}^*(U, L_{|U})$ as above. Actually, we can find examples of such line bundles for which this restriction is simply not surjective (see Ex. 1.5). The natural problem then becomes to compute the possibly non-zero connecting homomorphism $\partial : \mathrm{W}^*(U, L_{|U}) \to \mathrm{W}^{*+1}_Z(X,L)$. Although not absolutely necessary, it actually simplifies the formulation of Theorem 2.6 to use *dévissage* from [9, §6], i.e. the fact that push-forward along a regular closed immersion is an isomorphism

$$\iota_* : \mathrm{W}^{*-c}(Z, \omega_\iota \otimes L_{|Z}) \overset{\sim}{\to} \mathrm{W}^*_Z(X,L)\,. \tag{10}$$

Using this isomorphism, we can replace the Witt groups with supports by Witt groups of Z in the localization long exact sequence, and obtain a long exact sequence

$$\begin{tikzcd} \cdots \; \mathrm{W}^*(X,L) \arrow[r, "\upsilon^*"] & \mathrm{W}^*(U, L_{|U}) \arrow[r, "\partial'"] \arrow[dr, "\partial"'] & \mathrm{W}^{*+1-c}(Z, \omega_\iota \otimes L_{|Z}) \arrow[r] \arrow[d, "\iota_*", "\simeq"'] & \mathrm{W}^{*+1}(X,L) \; \cdots \\ & & \mathrm{W}^{*+1}_Z(X,L) \arrow[ur] & \end{tikzcd} \tag{11}$$

We now want to describe ∂' when $\lambda(L) \equiv c \mod 2$ (otherwise $\partial' = 0$ by Thm. 2.3).
By the complete dichotomy of Proposition 2.1, we know that when the push-forward $\pi_* : \mathrm{W}^*(Bl, M) \to \mathrm{W}^*(X, ?)$ does not exist, here for $M = \tilde{\alpha}^* \, (\alpha^*)^{-1} \, \upsilon^*(L)$ by (9), then the following composition $\tilde{\pi}_* \circ \tilde{\iota}^*$ exists and starts from the very group where π_* cannot be defined and arrives in the very group where ∂' itself arrives:

$$\mathrm{W}^*(Bl, M) \xrightarrow{\tilde{\iota}^*} \mathrm{W}^*(E, M_{|E}) \xrightarrow{\tilde{\pi}_*} \mathrm{W}^{*-c+1}(Z, \omega_\iota \otimes L_{|Z}) \,.$$

Hence, in a moment of exaltation, if we blindly apply this observation at the precise point where the oriented technique fails for Witt groups, we see that when we cannot define a section to restriction by the formula $\pi_* \circ \big(\tilde{\alpha}^* \circ (\alpha^*)^{-1}\big)$ we can instead define a mysterious homomorphism $(\tilde{\pi}_* \circ \tilde{\iota}^*) \circ \big(\tilde{\alpha}^* \circ (\alpha^*)^{-1}\big)$.

Theorem 2.6 (Main Theorem in regular case). *With the notation of 1.1, assume Hypothesis 1.2 and assume X, Y, Z regular. Let $L \in \mathrm{Pic}(X)$ and recall the integer $\lambda(L) \in \mathbb{Z}$ defined by* (8).
If $\lambda(L) \equiv c \mod 2$ then the composition $\tilde{\pi}_* \circ \tilde{\iota}^* \circ \tilde{\alpha}^* \circ (\alpha^*)^{-1}$ is equal to the connecting homomorphism ∂' of (11)*, that is, the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{W}^{*+1-c}(Z, \omega_\iota \otimes L_{|Z}) & & \mathrm{W}^*(U, L_{|U}) \arrow[ll, "\partial'"'] \arrow[d, "(\alpha^*)^{-1}", "\simeq"'] \\
\mathrm{W}^*\big(E \,, \omega_{\tilde{\pi}} \otimes \tilde{\pi}^*(\omega_\iota \otimes L_{|Z})\big) \cong \mathrm{W}^*\big(E \,, \tilde{\iota}^* \, \tilde{\alpha}^* \, (\alpha^*)^{-1} L_{|U}\big) \arrow[u, "\tilde{\pi}_*"] & & \mathrm{W}^*\big(Y, (\alpha^*)^{-1} \, L_{|U}\big) \arrow[ll, "\tilde{\iota}^* \, \tilde{\alpha}^*"]
\end{tikzcd}$$

This statement implies Thm. 1.4 (B) since $\partial = \iota_* \, \partial'$ by (11). Its proof will be given after generalization to the non-regular setting, at the end of Section 5.

Remark 2.7. Let us stress the peculiar combination of Theorem 2.3 and Theorem 2.6. Start with a Witt class w_U over the open $U \subset X$, for the duality twisted by some $L \in \mathrm{Pic}(U) = \mathrm{Pic}(X)$, and try to extend w_U to a Witt class w_X over X:

$$\begin{tikzcd}
\partial'(w_U) & & w_X \arrow[r, dotted, mapsto, "\upsilon^*"] & w_U \arrow[d, mapsto, "(\alpha^*)^{-1}"] \\
w_E \arrow[u, dotted, mapsto, "\tilde{\pi}_*"] \arrow[urr, phantom, "\boxed{\leftarrow 50\% \,\big|\, 50\% \rightarrow}"] & & w_{Bl} \arrow[ll, mapsto, "\tilde{\iota}^*"'] \arrow[u, dotted, mapsto, "\pi_*"'] & w_Y \arrow[l, mapsto, "\tilde{\alpha}^*"']
\end{tikzcd}$$

Then, *either* we can apply the same construction as for oriented theories, i.e. push-forward the class $w_{Bl} := \tilde{\alpha}^* \circ (\alpha^*)^{-1}(w_U)$ from Bl to X along π, constructing in this way an extension $w_X := \pi_*(w_{Bl})$ as wanted, *or* this last push-forward π_* is forbidden on w_{Bl} because of the twist, in which case the Witt class w_U might simply not belong to the image of restriction υ^*. The latter means that w_U might have a non-zero boundary $\partial'(w_U)$ over Z, which then deserves to be computed. The little miracle precisely is that in order to compute this $\partial'(w_U)$, it suffices to resume the above process where it failed, i.e. with w_{Bl}, and, since we cannot push it forward along π, we can consider the bifurcation of Proposition 2.1 and restrict this class w_{Bl} to the exceptional fiber E, say $w_E := \tilde{\iota}^* w_{Bl}$, and then push it forward along $\tilde{\pi}$. Of course, this does not construct an extension of w_U anymore, since this new class $\tilde{\pi}_*(w_E)$ lives over Z, not over X. Indeed, there is no reason a priori for this new class to give anything sensible at all. Our Main Theorem is that this construction in fact gives a formula for the boundary $\partial'(w_U)$.

BOTTOM LINE: *Essentially the same geometric recipe of pull-back and push-forward either splits the restriction or constructs the connecting homomorphism. In particular, the connecting homomorphism is explicitly described in both cases.*

3 THE NON-REGULAR CASE

In Section 2, we restricted our attention to the regular case in order to grasp the main ideas. However, most results can be stated in the greater generality of separated and noetherian $\mathbb{Z}[\frac{1}{2}]$-schemes admitting a dualizing complex. The goal of this section is to provide the relevant background and to extend Theorem 1.4 to this non-regular setting, see Main Lemma 3.5.

REMARK 3.1. The *coherent Witt groups* $\tilde{\mathrm{W}}^*(X, K_X)$ of a scheme $X \in \mathcal{S}ch$ (see 1.1) are defined using the derived category $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ of complexes of $\mathcal{O}_X$-modules whose cohomology is coherent and bounded. Since X is noetherian and separated, $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ is equivalent to its subcategory $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}(X))$ of bounded complexes of coherent $\mathcal{O}_X$-modules; see for instance [8, Prop. A.4]. The duality is defined using the derived functor $\mathrm{RHom}(-, K_X)$ where $K_X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ is a *dualizing complex* (see [19, § 3] or [8, § 2]), meaning that the functor $\mathrm{RHom}(-, K_X)$ defines a duality on $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$. For example, a scheme is Gorenstein if and only if $\mathcal{O}_X$ itself is an injectively bounded dualizing complex and, in that case, all other dualizing complexes are shifted line bundles (see Lemma A.7). Regular schemes are Gorenstein, and for them, coherent Witt groups coincide with the usual "locally free" Witt groups $\mathrm{W}^*(X, L)$ (i.e. the ones defined using bounded complexes of locally free sheaves instead of coherent ones). For any line bundle L, we still have a square-periodicity isomorphism

$$\tilde{\mathrm{W}}(X, K_X) \cong \tilde{\mathrm{W}}(X, K_X \otimes L^{\otimes 2}) \tag{12}$$

given by the multiplication by the class in $\mathrm{W}^0(X, L^{\otimes 2})$ of the canonical form $L \to L^\vee \otimes L^{\otimes 2}$, using the pairing between locally free and coherent Witt groups. For any closed embedding $Z \hookrightarrow X$ with open complement $\upsilon : U \hookrightarrow X$, the restriction $K_U := \upsilon^* K_X$ is a dualizing complex [19, Thm. 3.12] and the general triangulated framework of [3] gives a localization long exact sequence

$$\cdots \xrightarrow{\partial} \tilde{\mathrm{W}}^*_Z(X, K_X) \longrightarrow \tilde{\mathrm{W}}^*(X, K_X) \longrightarrow \tilde{\mathrm{W}}^*(U, K_U) \xrightarrow{\partial} \tilde{\mathrm{W}}^{*+1}_Z(X, K_X) \longrightarrow \cdots \tag{13}$$

As for K-theory, no such sequence holds in general for singular schemes and locally free Witt groups.

Remark 3.2. For coherent Witt groups, the push-forward along a proper morphism $f : X' \to X$ takes the following very round form: If K_X is a dualizing complex on X then $f^! K_X$ is a dualizing complex on X' ([8, Prop. 3.9]) and the functor $\mathrm{R}\, f_* : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X') \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ induces a *push-forward* ([8, Thm. 4.4])

$$f_* : \tilde{\mathrm{W}}^i(X', f^! K_X) \to \tilde{\mathrm{W}}^i(X, K_X)\,. \tag{14}$$

Recall that $f^! : \mathrm{D}_{\mathrm{Qcoh}}(X) \to \mathrm{D}_{\mathrm{Qcoh}}(X')$ is the right adjoint of $\mathrm{R}\, f_*$. If we twist the chosen dualizing complex K_X by a line bundle $L \in \mathrm{Pic}(X)$, this is transported to X' via the following formula (see [8, Thm. 3.7])

$$f^!(K_X \otimes L) \simeq f^!(K_X) \otimes f^* L\,. \tag{15}$$

In the regular case, push-forward maps are also described in Nenashev [20].

Remark 3.3. Let us also recall from [8, Thm. 4.1] that the pull-back

$$f^* : \tilde{\mathrm{W}}^i(X, K_X) \to \tilde{\mathrm{W}}^i(X', \mathrm{L}f^* K_X)$$

along a finite Tor-dimension morphism $f : X' \to X$ is defined if $\mathrm{L}f^*(K_X)$ is a dualizing complex (this is not automatically true). Together with the push-forward, this pull-back satisfies the usual flat base-change formula (see [8, Thm. 5.5]).
A regular immersion $f : X' \hookrightarrow X$ has finite Tor-dimension since it is even perfect (see [1, p. 250]). Moreover, in that case, $\mathrm{L}f^*$ is the same as $f^!$ up to a twist and a shift (see Proposition A.8), hence it preserves dualizing complexes.

Proposition 3.4. *In Setup 1.1, let K_X be a dualizing complex on X. Let $L \in \mathrm{Pic}(X)$ and $\ell \in \mathbb{Z}$. Then $K = \pi^!(K_X) \otimes \pi^* L \otimes \mathcal{O}(E)^{\otimes \ell}$ is a dualizing complex on Bl and any dualizing complex has this form, for some $L \in \mathrm{Pic}(X)$ and $\ell \in \mathbb{Z}$. Moreover, the dichotomy of Proposition 2.1 here becomes:*

(A) If $\ell \equiv 0 \mod 2$, we can push-forward along $\pi : Bl \to X$, as follows:

$$\tilde{\mathrm{W}}^*(Bl, K) \cong \tilde{\mathrm{W}}^*\big(Bl, \pi^!(K_X \otimes L)\big) \xrightarrow{\pi_*} \tilde{\mathrm{W}}^*\big(X, K_X \otimes L\big)\,.$$

(B) If $\ell \equiv 1 \mod 2$, we can push-forward along $\tilde{\pi} : E \to Z$, as follows:

$$\tilde{\mathrm{W}}^*(E, \mathrm{L}\tilde{\iota}^* K) \cong \tilde{\mathrm{W}}^{*+1}\big(E, \tilde{\pi}^! \iota^!(K_X \otimes L)\big) \xrightarrow{\tilde{\pi}_*} \tilde{\mathrm{W}}^{*+1}\big(Z, \iota^!(K_X \otimes L)\big).$$

As before, in both cases, the first isomorphism $\cong$ comes from square-periodicity (12) *and the second morphism is push-forward* (14).

Proof. The complex $K_{Bl} := \pi^! K_X$ is a dualizing complex on Bl by Remark 3.2. By Lemma A.7 and Proposition A.6 (i), all dualizing complexes on Bl are of the form $K = \pi^!(K_X) \otimes \pi^* L \otimes \mathcal{O}(E)^{\otimes \ell}$, for unique $L \in \operatorname{Pic}(X)$ and $\ell \in \mathbb{Z}$. We only need to check the relevant parity for applying (12). Case (A) follows easily from (15) by definition of K and parity of ℓ. In (B), we need to compare $\mathrm{L}\tilde{\iota}^* K$ and $\tilde{\pi}^! \iota^! (K_X \otimes L)[1]$. By Proposition A.11 (iv), we know that $\tilde{\iota}^!(-) \cong \tilde{\iota}^* \mathcal{O}(E)[-1] \otimes \mathrm{L}\tilde{\iota}^*(-)$. We apply this and (15) in the second equality below, the first one using simply that $\iota \tilde{\pi} = \pi \tilde{\iota}$:

$$\begin{aligned}\tilde{\pi}^! \iota^! (K_X \otimes L)[1] &\cong \tilde{\iota}^! \pi^! (K_X \otimes L)[1] \cong \tilde{\iota}^* \mathcal{O}(E)[-1] \otimes \mathrm{L}\tilde{\iota}^* \big(\pi^!(K_X) \otimes \pi^* L\big)[1] \cong \\ &\cong \tilde{\iota}^* \mathcal{O}(E) \otimes \mathrm{L}\tilde{\iota}^* (K \otimes \mathcal{O}(E)^{\otimes -\ell}) \cong \tilde{\iota}^* \mathcal{O}(E)^{\otimes (1-\ell)} \otimes \mathrm{L}\tilde{\iota}^* K.\end{aligned}$$

Since $1-\ell$ is even, $\tilde{\iota}^* \mathcal{O}(E)^{\otimes(1-\ell)}$ is a square, as desired. ☐

We now want to give the key technical result of the paper, which is an analogue of Theorem 1.4 in the non-regular setting. The idea is to describe the connecting homomorphism on Witt classes over U which admit an extension to the blow-up Bl. The key fact is the existence of an additional twist on Bl, namely the twist by $\mathcal{O}(E)$, which disappears on U (see A.1) and hence allows Case (B) below.

MAIN LEMMA 3.5. *In Setup 1.1, assume that X has a dualizing complex K_X and let $K_U = \upsilon^*(K_X)$ and $K_{Bl} = \pi^!(K_X)$; see Remarks 3.1 and 3.2. Let $i \in \mathbb{Z}$.*

(A) The following composition vanishes:

$$\tilde{\mathrm{W}}^i(Bl, K_{Bl}) \xrightarrow{\tilde{\upsilon}^*} \tilde{\mathrm{W}}^i(U, K_U) \xrightarrow{\partial} \tilde{\mathrm{W}}^{i+1}_Z(X, K_X)\,.$$

(B) The following composition (well-defined since $\tilde{\upsilon}^* \mathcal{O}(E) \simeq \mathcal{O}_U$)

$$\tilde{\mathrm{W}}^i\big(Bl, K_{Bl} \otimes \mathcal{O}(E)\big) \xrightarrow{\tilde{\upsilon}^*} \tilde{\mathrm{W}}^i\big(U, K_U \otimes \tilde{\upsilon}^* \mathcal{O}(E)\big) \cong \tilde{\mathrm{W}}^i(U, K_U) \xrightarrow{\partial} \tilde{\mathrm{W}}^{i+1}_Z(X, K_X)$$

coincides with the composition

$$\begin{tikzcd}
\tilde{\mathrm{W}}^i(Bl, \mathcal{O}(E) \otimes K_{Bl}) \arrow[d, "\tilde{\iota}^*"] & & \tilde{\mathrm{W}}^{i+1}_Z(X, K_X) \\
\tilde{\mathrm{W}}^i\big(E, \mathrm{L}\tilde{\iota}^*(\mathcal{O}(E) \otimes K_{Bl})\big) \arrow[r, phantom, "\cong"] & \tilde{\mathrm{W}}^{i+1}\big(E, \tilde{\pi}^! \iota^! K_X\big) \arrow[r, "\tilde{\pi}_*"] & \tilde{\mathrm{W}}^{i+1}\big(Z, \iota^! K_X\big) \arrow[u, "\iota_*"']
\end{tikzcd}$$

where the latter isomorphism $\cong$ is induced by the composition

$$\mathrm{L}\tilde{\iota}^*(\mathcal{O}(E) \otimes K_{Bl}) \cong \tilde{\iota}^*(\mathcal{O}(E)) \otimes \mathrm{L}\tilde{\iota}^*(K_{Bl}) \cong \tilde{\iota}^! K_{Bl}[1] \cong \tilde{\pi}^! \iota^! K_X[1]\,. \quad (16)$$

The proof of this result occupies Section 4. Here are just a couple of comments on the statement. Let us first of all explain the announced sequence of isomorphisms (16). The first one holds since $\mathrm{L}\tilde{\iota}^*$ is a tensor functor and since $\mathcal{O}(E)$ is a line bundle (hence is flat). The second one holds by Proposition A.9 (v). The last one follows by definition of K_{Bl} and the fact that $\iota\,\tilde{\pi} = \pi\,\tilde{\iota}$. Finally, note that we use the pull-back $\tilde{\iota}^*$ on coherent Witt groups as recalled in Remark 3.3.

4 THE MAIN ARGUMENT

Surprisingly enough for a problem involving the blow-up $Bl = Bl_Z(X)$ of X along Z, see (1), the case where $\operatorname{codim}_X(Z) = 1$ is also interesting, even though, of course, in that case $Bl = X$ and $E = Z$. In fact, this case is crucial for the proof of Main Lemma 3.5 and this is why we deal with it first. In the "general" proof where $\operatorname{codim}_X(Z)$ is arbitrary, we will apply the case of codimension one to $\tilde{\iota} : E \hookrightarrow Bl$. Therefore, we use the following notation to discuss codimension one.

NOTATION 4.1. Let $B \in \mathcal{S}ch$ be a scheme with a dualizing complex K_B and $\tilde{\iota} : E \hookrightarrow B$ be a prime divisor, that is, a regular closed immersion of codimension one, of a subscheme $E \in \mathcal{S}ch$. Let $\mathcal{O}(E)$ be the line bundle on B associated to E (see Definition A.1). Let $\tilde{v} : U \hookrightarrow B$ be the open immersion of the open complement

$$E \xhookrightarrow{\tilde{\iota}} B \xhookleftarrow{\tilde{v}} U$$

$U = B - E$ and let K_U be the dualizing complex $\tilde{v}^*(K_B)$ on U.

LEMMA 4.2 (Main Lemma in codimension one). *With Notation 4.1, let $i \in \mathbb{Z}$. Then:*

(A) The composition

$$\tilde{\mathrm{W}}^i(B, K_B) \xrightarrow{\tilde{v}^*} \tilde{\mathrm{W}}^i(U, K_U) \xrightarrow{\partial} \tilde{\mathrm{W}}^{i+1}_E(B, K_B)$$

is zero.

(B) The composition

$$\tilde{\mathrm{W}}^i\big(B, K_B \otimes \mathcal{O}(E)\big) \xrightarrow{\tilde{v}^*} \tilde{\mathrm{W}}^i\big(U, K_U \otimes \tilde{v}^*\mathcal{O}(E)\big) \cong \tilde{\mathrm{W}}^i(U, K_U) \xrightarrow{\partial} \tilde{\mathrm{W}}^{i+1}_E(B, K_B)$$

coincides with the composition

$$\begin{tikzcd}
\tilde{\mathrm{W}}^i(B, \mathcal{O}(E) \otimes K_B) \arrow[d, "\tilde{\iota}^*"'] & & \tilde{\mathrm{W}}^{i+1}_E(B, K_B) \\
\tilde{\mathrm{W}}^i\big(E, \mathrm{L}\tilde{\iota}^*(\mathcal{O}(E) \otimes K_B)\big) \arrow[r, phantom, "\cong"] & \tilde{\mathrm{W}}^i\big(E, \tilde{\iota}^! K_B[1]\big) \arrow[r, phantom, "\cong"] & \tilde{\mathrm{W}}^{i+1}\big(E, \tilde{\iota}^! K_B\big) \arrow[u, "\tilde{\iota}_*"']
\end{tikzcd}$$

where the first isomorphism $\cong$ is induced by the following isomorphism

$$\mathrm{L}\tilde{\iota}^*(\mathcal{O}(E) \otimes K_B) \cong \tilde{\iota}^*(\mathcal{O}(E)) \otimes \mathrm{L}\tilde{\iota}^*(K_B) \cong \tilde{\iota}^!(K_B)[1]. \tag{17}$$

Proof. Case (A) is simple: The composition of two consecutive morphisms in the localization long exact sequence (13) is zero. Case (B) is the nontrivial one. The isomorphisms (17) are the same as in (16).

At this stage, we upload the definition of the connecting homomorphism for Witt groups $\partial : \tilde{\mathrm{W}}^i(U, K_U) \to \tilde{\mathrm{W}}^{i+1}_E(B, K_B)$, which goes as follows: Take a non-degenerate symmetric space (P, ϕ) over U for the i^{th}-shifted duality with values in K_U; there exists a possibly degenerate symmetric pair (Q, ψ) over B for the same duality (with values in K_B) which restricts to (P, ϕ) over U; compute its symmetric cone $d(Q, \psi)$, which is essentially the cone of ψ equipped with a natural metabolic form; see [3, § 4] or [4, Def. 2.3] for instance; for any choice of such a pair (Q, ψ), the boundary $\partial(P, \phi) \in \tilde{\mathrm{W}}^{i+1}_E(B, K_B)$ is the Witt class of $d(Q, \psi)$.

There is nothing really specific to dualizing complexes here. The above construction is a purely triangular one, as long as one uses the *same* duality for the ambient scheme B, for the open $U \subset B$ and for the Witt group of B with supports in the closed complement E. The subtlety of statement (B) is that we start with a twisted duality on the scheme B which is not the duality used for ∂, but which agrees with it on U by the first isomorphism $\cong$ in statement (B).

Now, take an element in $\tilde{\mathrm{W}}^i(B, \mathcal{O}(E) \otimes K_B)$. It is the Witt-equivalence class of a symmetric space (P, ϕ) over B with respect to the i^{th}-shifted duality with values in $\mathcal{O}(E) \otimes K_B$. The claim of the statement is that, modulo the above identifications of dualizing complexes, we should have

$$\partial(\tilde{v}^*(P, \phi)) = \tilde{\imath}_*(\tilde{\imath}^*(P, \phi)) \tag{18}$$

in $\tilde{\mathrm{W}}^{i+1}_E(B, K_B)$. By the above discussion, in order to compute $\partial(\tilde{v}^*(P, \phi))$, we need to find a symmetric pair (Q, ψ) over B, for the duality given by K_B, and such that $\tilde{v}^*(Q, \psi) = \tilde{v}^*(P, \phi)$. Note that we cannot take for (Q, ψ) the pair (P, ϕ) itself because (P, ϕ) is symmetric for the twisted duality $\mathcal{O}(E) \otimes K_B$ on B. Nevertheless, it is easy to "correct" (P, ϕ) as follows.

As in Definition A.1, we have a canonical homomorphism of line bundles:

$$\sigma_E : \mathcal{O}(E)^\vee \to \mathcal{O}_B\,.$$

The pair $(\mathcal{O}(E)^\vee, \sigma_E)$ is symmetric in the derived category $\mathrm{D^b}(\mathrm{VB}(B))$ of vector bundles over B, with respect to the 0^{th}-shifted duality twisted by $\mathcal{O}(E)^\vee$, because the target of σ_E is the dual of its source: $(\mathcal{O}(E)^\vee)^\vee[0] \otimes \mathcal{O}(E)^\vee \cong \mathcal{O}_B$. Let us define the wanted symmetric pair (Q, ψ) in $\mathrm{D^b_{coh}}(B)$ for the i^{th}-shifted duality with values in K_B as the following product:

$$(Q, \psi) := (\mathcal{O}(E)^\vee, \sigma_E) \otimes (P, \phi)\,.$$

Note that we tensor a complex of vector bundles with a coherent one to get a coherent one, following the formalism of [4, § 4] where such external products are denoted by $\star$. We claim that the restriction of (Q, ψ) to U is nothing but $\tilde{v}^*(P, \phi)$. This is easy to check since $\mathcal{O}(E)_{|U} = \mathcal{O}_U$ via σ_E (see A.1), which means $(\mathcal{O}(E)^\vee, \sigma_E)_{|U} = 1_U$. So, by the construction of the connecting homomorphism ∂ recalled at the beginning of the proof, we know that $\partial(\tilde{v}^*(P, \phi))$ can be computed as $d(Q, \psi)$. This reads:

$$\partial(\tilde{v}^*(P, \phi)) = d\big((\mathcal{O}(E)^\vee, \sigma_E) \otimes (P, \phi)\big).$$

Now, we use that (P,ϕ) is non-degenerate and that therefore (see [4, Rem. 5.4] if necessary) we can take (P,ϕ) out of the above symmetric cone $d(...)$, i.e.

$$\partial(\tilde{v}^*(P,\phi)) = d(\mathcal{O}(E)^\vee, \sigma_E) \otimes (P,\phi)\,. \tag{19}$$

Let us compute the symmetric cone $d(\mathcal{O}(E)^\vee,\sigma_E) =: (C,\chi)$. Note that this only involves vector bundles. We define C to be the cone of σ_E and we equip it with a symmetric form $\chi : C \overset{\sim}{\to} C^\vee[1]\otimes\mathcal{O}(E)^\vee$ for the duality used for $(\mathcal{O}(E),\sigma_E)$ but shifted by one, that is, for the 1^{st} shifted duality with values in $\mathcal{O}(E)^\vee$. One checks that (C,χ) is given by the following explicit formula:

$$\begin{tikzcd}
C = \arrow[d, "\chi"'] & \big(\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathcal{O}(E)^\vee \arrow[r, "\sigma_E"] \arrow[d, "-1"'] & \mathcal{O}_B \arrow[r] \arrow[d, "1"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots\big) \\
C^\vee[1]\otimes\mathcal{O}(E)^\vee = & \big(\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathcal{O}(E)^\vee \arrow[r, "-(\sigma_E)^\vee"'] & \mathcal{O}_B \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots\big)
\end{tikzcd} \tag{20}$$

where the complexes have $\mathcal{O}_B$ in degree zero. Now, observe that the complex C is a resolution of $\tilde{\iota}_*(\mathcal{O}_E)$ over B, by Definition A.1, that is, $C \simeq \tilde{\iota}_*(\mathcal{O}_E)$ in the derived category of B. Moreover, by Propositions A.8 and A.9 (ii), we have $\tilde{\iota}^!(\mathcal{O}(E)^\vee[1]) = \omega_{\tilde{\iota}}[-1]\otimes\tilde{\iota}^*(\mathcal{O}(E)^\vee[1]) \cong \mathcal{O}_E$. Using this, one checks the conceptually obvious fact that χ is also the push-forward along the perfect morphism $\tilde{\iota}$ of the unit form on $\mathcal{O}_E$. See Remark 4.3 below for more details. This means that we have an isometry in $\mathrm{D}^{\mathrm{b}}_E(\mathrm{VB}(B))$

$$d\big(\mathcal{O}(E)^\vee,\sigma_E\big) = \tilde{\iota}_*(1_E)$$

of symmetric spaces with respect to the 1^{st} shifted duality with values in $\mathcal{O}(E)^\vee$. Plugging this last equality in (19), and using the projection formula (see Remark 4.3) we obtain

$$\partial(\tilde{v}^*(P,\phi)) = \tilde{\iota}_*(1_E)\otimes(P,\phi) = \tilde{\iota}_*\big(1_E\otimes\tilde{\iota}^*(P,\phi)\big) = \tilde{\iota}_*\big(\tilde{\iota}^*(P,\phi)\big)\,.$$

This is the claimed equality (18). $\square$

Remark 4.3. In the above proof, we use the "conceptually obvious fact" that the push-forward of the unit form on $\mathcal{O}_E$ is indeed the χ of (20). This fact is obvious to the expert but we cannot provide a direct reference for this exact statement. However, if the reader does not want to do this lengthy verification directly, the computation of [8, § 7.2] can be applied essentially verbatim. The main difference is that here, we are considering a push-forward of locally free instead of coherent Witt groups along a regular embedding. Such a push-forward is constructed using the same tensor formalism as the proper push-forwards for coherent Witt groups considered in loc. cit. along morphisms that are proper, perfect and Gorenstein, which is true of a regular embedding. In loc. cit. there is an assumption that the schemes are Gorenstein, ensuring that

the line bundles are dualizing complexes. But here, the dualizing objects for our category of complexes of locally free sheaves are line bundles anyway and the extra Gorenstein assumption is irrelevant.
Moreover, the projection formula used in the above proof is established in complete generality for non necessarily regular schemes by the same method as in [8, § 5.7] using the pairing between the locally free derived category and the coherent one to the coherent one. More precisely, this pairing is just a restriction of the quasi-coherent pairing $\mathrm{D}_{\mathrm{Qcoh}} \times \mathrm{D}_{\mathrm{Qcoh}} \xrightarrow{\otimes} \mathrm{D}_{\mathrm{Qcoh}}$ of loc. cit. to these subcategories. By the general tensor formalism of [10], for any morphism $f : X \to Y$ as above, for any object A (resp. B) in the quasi-coherent derived category of X (resp. Y), we obtain a projection morphism in $\mathrm{D}_{\mathrm{Qcoh}}(Y)$

$$\mathrm{R}\, f_*(A) \otimes B \longrightarrow \mathrm{R}\, f_*(A \otimes \mathrm{L}f^*(B)),$$

see [10, Prop. 4.2.5]. It is an isomorphism by [8, Thm. 3.7]. We actually only use it for A a complex of locally free sheaves and B a complex with coherent and bounded cohomology. The projection formula is implied by [10, Thm. 5.5.1].

Proof of Main Lemma 3.5. Case (A) follows from the codimension one case and the compatibility of push-forwards with connecting homomorphisms (here along the identity of U). Case (B) follows from the outer commutativity of the following diagram:

$$\begin{tikzcd}
\tilde{\mathrm{W}}^i(U, K_U) \arrow[r, "\partial"] & \tilde{\mathrm{W}}^{i+1}_Z(X, K_X) & \tilde{\mathrm{W}}^{i+1}(Z, \iota^! K_X) \arrow[l, "\iota_*"'] \\
\tilde{\mathrm{W}}^i(U, K_U) \arrow[u, equal] \arrow[r, "\partial"] & \tilde{\mathrm{W}}^{i+1}_E(Bl, K_{Bl}) \arrow[u, "\pi_*"'] & \tilde{\mathrm{W}}^{i+1}(E, \tilde{\pi}^! \iota^! K_X) \arrow[l, "\tilde{\iota}_*"'] \arrow[u, "\tilde{\pi}_*"'] \\
\tilde{\mathrm{W}}^i(Bl, \mathcal{O}(E) \otimes K_{Bl}) \arrow[u, "\tilde{v}^*"] \arrow[r, "\tilde{\iota}^*"] & \tilde{\mathrm{W}}^i\big(Bl, \mathrm{L}\tilde{\iota}^*(\mathcal{O}(E) \otimes K_{Bl})\big) \arrow[r, phantom, "\cong"] & \tilde{\mathrm{W}}^{i+1}(E, \tilde{\iota}^! K_{Bl}) \arrow[u, "\cong"']
\end{tikzcd} \tag{21}$$

We shall now verify the inner commutativity of this diagram. The upper left square of (21) commutes by compatibility of push-forward with connecting homomorphisms. The upper right square of (21) simply commutes by functoriality of push-forward applied to $\iota \circ \tilde{\pi} = \pi \circ \tilde{\iota}$. Most interestingly, the lower part of (21) commutes by Lemma 4.2 applied to the codimension one inclusion $\tilde{\iota} : E \hookrightarrow Bl$. ∎

5 The Main Theorem in the non-regular case

Without regularity assumptions, we have shown in Main Lemma 3.5 how to compute the connecting homomorphism $\partial : \tilde{\mathrm{W}}^*(U, K_U) \to \tilde{\mathrm{W}}^{*+1}_Z(X, K_X)$ on those Witt classes over U which come from $Bl = Bl_Z(X)$ by restriction $\tilde{v}^*$. The whole point of adding Hypothesis 1.2 is precisely to split $\tilde{v}^*$, that is, to construct for each Witt class on U an extension on Bl. In the regular case, this follows from homotopy invariance of Picard groups and Witt groups. In the non-regular setting, things are a little more complicated. Let us give the statement and comment on the hypotheses afterwards (see Remark 5.2).

Main Theorem 5.1. *In Setup 1.1, assume that X has a dualizing complex K_X and equip U with the restricted complex $K_U = \upsilon^*(K_X)$. Assume Hypothesis 1.2 and further make the following hypotheses:*

(a) There exists a dualizing complex K_Y on Y such that $\alpha^* K_Y = K_U$.

(b) The $\mathbb{A}^*$-bundle α induces an isomorphism $\tilde{\mathrm{W}}^*(Y, K_Y) \xrightarrow{\sim} \tilde{\mathrm{W}}^*(U, K_U)$.

(c) The morphism $\tilde{\alpha}$ is of finite Tor dimension and $\mathrm{L}\tilde{\alpha}^*(K_Y)$ is dualizing.

(d) Sequence (25) is exact: $\mathbb{Z} \to \mathrm{Pic}(Bl) \to \mathrm{Pic}(U)$. (See Proposition A.3.)

Then $\mathrm{L}\tilde{\alpha}^*(K_Y) \simeq \pi^! K_X \otimes \mathcal{O}(E)^{\otimes n}$ for some $n \in \mathbb{Z}$, and the following holds true:

(A) If n can be chosen even, the composition $\pi_* \tilde{\alpha}^* (\alpha^*)^{-1}$ is a section of υ^*.

(B) If n can be chosen odd, the composition $\iota_* \tilde{\pi}_* \tilde{\iota}^* \tilde{\alpha}^* (\alpha^*)^{-1}$ coincides with the connecting homomorphism $\partial : \tilde{\mathrm{W}}^*(U, K_U) \to \tilde{\mathrm{W}}^{*+1}_E(X, K_X)$.

Proof. By (c) and Remark 3.2 respectively, both $\mathrm{L}\tilde{\alpha}^*(K_Y)$ and $\pi^! K_X$ are dualizing complexes on Bl. By Lemma A.7 (i), they differ by a shifted line bundle: $\mathrm{L}\tilde{\alpha}^*(K_Y) \simeq \pi^! K_X \otimes L[m]$ with $L \in \mathrm{Pic}(Bl)$ and $m \in \mathbb{Z}$. Restricting to U, we get

$$\begin{aligned} K_U \otimes \tilde{\upsilon}^* L[m] &\simeq \tilde{\upsilon}^* \pi^! K_X \otimes \tilde{\upsilon}^* L[m] \\ &\simeq \tilde{\upsilon}^*(\pi^! K_X \otimes L[m]) \simeq \tilde{\upsilon}^* \,\mathrm{L}\tilde{\alpha}^*(K_Y) \simeq \alpha^* K_Y \simeq K_U \end{aligned}$$

where the first equality holds by flat base-change ([8, Thm. 5.5]). Thus, $\tilde{\upsilon}^* L[m]$ is the trivial line bundle on U by Lemma A.7 (ii). So $m = 0$ and, by (d), $L \simeq \mathcal{O}(E)^{\otimes n}$ for some $n \in \mathbb{Z}$. This gives $\mathrm{L}\tilde{\alpha}^*(K_Y) \simeq \pi^! K_X \otimes \mathcal{O}(E)^{\otimes n}$ as claimed.

We now consider coherent Witt groups. By (c) and Remark 3.3, $\tilde{\alpha}$ induces a morphism $\tilde{\alpha}^* : \tilde{\mathrm{W}}^*(Y, K_Y) \to \tilde{\mathrm{W}}(Bl, \mathrm{L}\tilde{\alpha}^* K_Y)$. By Lemma A.12, the flat morphism α induces a homomorphism $\alpha^* : \tilde{\mathrm{W}}^*(Y, K_Y) \to \tilde{\mathrm{W}}^*(U, K_U)$ which is assumed to be an isomorphism in (b). So, we can use $(\alpha^*)^{-1}$. When n is even, we have

$$\upsilon^* \pi_* \tilde{\alpha}^* (\alpha^*)^{-1} = \tilde{\upsilon}^* \tilde{\alpha}^* (\alpha^*)^{-1} = \alpha^* (\alpha^*)^{-1} = \mathrm{id}$$

where the first equality holds by flat base-change ([8, Thm. 5.5]). This proves (A). On the other hand, when n is odd, we have

$$\iota_* \tilde{\pi}_* \tilde{\iota}^* \tilde{\alpha}^* (\alpha^*)^{-1} = \partial\, \tilde{\upsilon}^* \tilde{\alpha}^* (\alpha^*)^{-1} = \partial\, \alpha^* (\alpha^*)^{-1} = \partial$$

where the first equality holds by Main Lemma 3.5 (B). □

Remark 5.2. Hypothesis (a) in Theorem 5.1 is always true when Y admits a dualizing complex and homotopy invariance holds over Y for the Picard group (e.g. Y regular). Homotopy invariance for coherent Witt groups should hold in general but only appears in the literature when Y is Gorenstein, see Gille [12]. This means that Hypothesis (b) is a mild one. Hypothesis (d) is discussed in Proposition A.3.

Remark 5.3. In Theorem 5.1, the equation $\mathrm{L}\tilde{\alpha}^*(K_Y) \simeq \pi^! K_X \otimes \mathcal{O}(E)^{\otimes n}$, for $n \in \mathbb{Z}$, should be considered as a non-regular analogue of Equation (8). In Remark 2.2, we discussed the compatibility of the various lines bundles on the schemes X, U, Y and Bl. Here, we need to control the relationship between dualizing complexes instead and we do so by restricting to U and by using the exact sequence (25). Alternatively, one can remove Hypothesis (d) and directly assume the relation $\mathrm{L}\tilde{\alpha}^*(K_Y) \simeq \pi^!(K_X) \otimes \mathcal{O}(E)^{\otimes n}$ for some $n \in \mathbb{Z}$. This might hold in some particular examples even if (25) is not exact.

For the convenience of the reader, we include the proofs of the following facts.

Lemma 5.4. *If X is Gorenstein, then Z and Bl are Gorenstein. If X is regular, Bl is regular.*

Proof. By Prop. A.8, $\pi^!(\mathcal{O}_X)$ is the line bundle ω_π. Since π is proper, $\pi^!$ preserves injectively bounded dualizing complexes and ω_π is dualizing and since it is a line bundle, Bl is Gorenstein. The same proof holds for Z, since $\iota^!(\mathcal{O}_X)$ is ω_ι (shifted) which is also a line bundle by Prop. A.11. For regularity, see [18, Thm. 1.19]. □

Proof of Theorem 2.6. Note that all the assumptions of Theorem 5.1 are fulfilled in the regular case, that is, in the setting of Section 2. Indeed, if X and Y are regular, Bl and U are regular, and the dualizing complexes on X, Y, Bl and U are simply shifted line bundles. The morphism $\alpha^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(U)$ is then an isomorphism (homotopy invariance) and $\tilde{\alpha}$ is automatically of finite Tor dimension, as any morphism to a regular scheme. Finally, the sequence on Picard groups is exact by Proposition A.3.
Let $K_X = L$ be the chosen line bundle on X. Then set $L_U := K_U = \upsilon^* L$ and choose $L_Y = K_Y$ to be the unique line bundle (up to isomorphism) such that $\alpha^* L_Y = L_U$. By (8), we have $\tilde{\alpha}^* L_Y = \pi^* L \otimes \mathcal{O}(E)^{\otimes \lambda(L)} = \pi^! L \otimes \mathcal{O}(E)^{\otimes(\lambda(L)-c+1)}$, where the last equality holds since $\pi^! L = \mathcal{O}(E)^{\otimes(c-1)} \otimes \pi^* L$ by Proposition A.11 (vi). In other words, we have proved that $\tilde{\alpha}^* K_Y = \pi^! K_X \otimes \mathcal{O}(E)^{\otimes(\lambda(L)-c+1)}$. In Theorem 5.1, we can then take $n = \lambda(L) - c + 1$ and the parity condition becomes $\lambda(L) \equiv c-1 \mod 2$ for Case (A) and $\lambda(L) \equiv c \mod 2$ for Case (B). So, Case (A) is the trivial one and corresponds to Theorem 2.3. Case (B) exactly gives Theorem 2.6 up to the identifications of line bundles explained in Appendix A. □

A Line bundles and dualizing complexes

We use Hartshorne [15] or Liu [18] as general references for algebraic geometry. We still denote by $\mathcal{S}ch$ the category of noetherian separated connected schemes (we do not need "over $\mathbb{Z}[\frac{1}{2}]$" in this appendix).

Definition A.1. Let $\tilde{\iota} : E \hookrightarrow B$ be a regular closed immersion of codimension one, with $B \in Sch$. Consider the ideal $I_E \subset \mathcal{O}_B$ defining E

$$0 \longrightarrow I_E \xrightarrow{\sigma_E} \mathcal{O}_B \longrightarrow \tilde{\iota}_* \mathcal{O}_E \longrightarrow 0\ . \tag{22}$$

By assumption, I_E is an invertible ideal, i.e. a line bundle. The *line bundle associated to* E is defined as its dual $\mathcal{O}(E) := (I_E)^\vee$, see [15, II.6.18]. We thus have by construction a global section $\sigma_E : \mathcal{O}(E)^\vee \to \mathcal{O}_B$, which vanishes exactly on E. This gives an explicit trivialization of $\mathcal{O}(E)$ outside E. On the other hand, the restriction of $\mathcal{O}(E)$ to E is the normal bundle $\mathcal{O}(E)_{|_E} \cong N_{E/B}$.

Example A.2. Let $Bl = Bl_Z(X)$ be the blow-up of X along a regular closed immersion $Z \hookrightarrow X$ as in Setup 1.1. Let $I = I_Z \subset \mathcal{O}_X$ be the sheaf of ideals defining Z. By construction of the blow-up, we have $Bl = \mathrm{Proj}(\mathcal{S})$ where $\mathcal{S}$ is the sheaf of graded $\mathcal{O}_X$-algebras

$$\mathcal{S} := \mathcal{O}_X \oplus I \oplus I^2 \oplus I^3 \oplus \cdots$$

Similarly, $E = \mathrm{Proj}(S/\mathcal{J})$ where $\mathcal{J} := I \cdot \mathcal{S} \subset \mathcal{S}$ is the sheaf of homogeneous ideals

$$\mathcal{J} = I \oplus I^2 \oplus I^3 \oplus I^4 \oplus \cdots$$

So, $E = \mathbb{P}_Z(C_{Z/X})$ is a projective bundle over Z associated to the vector bundle $C_{Z/X} = I/I^2$ which is the conormal bundle of Z in X. Associating $\mathcal{O}_{Bl}$-sheaves to graded $\mathcal{S}$-modules, the obvious exact sequence $0 \to \mathcal{J} \to \mathcal{S} \to \mathcal{S}/\mathcal{J} \to 0$ yields

$$0 \longrightarrow \widetilde{\mathcal{J}} \xrightarrow{\sigma_E} \mathcal{O}_{Bl} \longrightarrow \tilde{\iota}_* \mathcal{O}_E \longrightarrow 0\ . \tag{23}$$

Compare (22). This means that here $I_E = \widetilde{\mathcal{J}}$. But now, $\mathcal{J}$ is obviously $\mathcal{S}(1)$ truncated in non-negative degrees. Since two graded $\mathcal{S}$-modules which coincide above some degree have the same associated sheaves, we have $I_E = \widetilde{\mathcal{J}} = \widetilde{\mathcal{S}(1)} = \mathcal{O}_{Bl}(1)$. Consequently, $\mathcal{O}(E) = (I_E)^\vee = \mathcal{O}_{Bl}(-1)$. In particular, we get

$$\mathcal{O}(E)_{|_E} = \mathcal{O}_{Bl}(-1)_{|_E} = \mathcal{O}_E(-1). \tag{24}$$

Proposition A.3 (Picard group in codimension one). *Let $B \in \mathcal{S}ch$ be a scheme and $\tilde{\iota} : E \hookrightarrow B$ be a regular closed immersion of codimension one of an irreducible subscheme $E \in \mathcal{S}ch$ with open complement $\tilde{\upsilon} : U \hookrightarrow B$. We then have a complex*

$$\mathbb{Z} \longrightarrow \mathrm{Pic}(B) \xrightarrow{\tilde{\upsilon}^*} \mathrm{Pic}(U) \tag{25}$$

where the first map sends 1 *to the line bundle* $\mathcal{O}(E)$ *associated to* E. *This complex is exact if* B *is normal, and* $\tilde{\upsilon}^*$ *is surjective when* B *is furthermore regular. It is also exact when* B *is the blow-up of a normal scheme* X *along a regular embedding.*

Proof. (25) is a complex since $\mathcal{O}(E)$ is trivial on U. When B is normal, $\mathrm{Pic}(B)$ injects in the group $\mathrm{Cl}(B)$ of Weil divisors (see [18, 7.1.19 and 7.2.14 (c)]), for which the same sequence holds by [15, Prop. II.6.5]. Exactness of (25) then follows by diagram chase. The surjectivity of $\tilde{\upsilon}^*$ when B is regular follows from [15, Prop. II.6.7 (c)]. When B is the blow-up of X along Z, we can assume that $\mathrm{codim}_X(Z) \geq 2$ by the previous point. Then, the result again follows by diagram chase, using that $\mathrm{Pic}(B) = \mathrm{Pic}(X) \oplus \mathbb{Z}$, as proved in Proposition A.6 (i) below. ☐

Remark A.4. Note that the blow-up of a normal scheme along a regular closed embedding isn't necessarily normal if the subscheme is not reduced. For example, take $X = \mathbb{A}^2 = \mathrm{Spec}(k[x, y])$ and Z defined by the equations $x^2 = y^2 = 0$. Then, Bl is the subscheme of $\mathbb{A}^2 \times \mathbb{P}^1$ defined by the equations $x^2 v = y^2 u$ where $[u : v]$ are homogeneous coordinates for $\mathbb{P}^1$ and it is easy to check that the whole exceptional fiber is singular. Thus Bl is not normal (not even regular in codimension one).

Proposition A.5 (Picard group of a projective bundle). *Let* $X \in Sch$ *be a (connected) scheme and* $\mathcal{F}$ *a vector bundle over* X. *We consider the projective bundle* $\mathbb{P}_X(\mathcal{F})$ *associated to* $\mathcal{F}$. *Its Picard group is* $\mathrm{Pic}(X) \oplus \mathbb{Z}$ *where* $\mathbb{Z}$ *is generated by* $\mathcal{O}(-1)$ *and* $\mathrm{Pic}(X)$ *comes from the pull-back from* X.

Proof. Surjectivity of $\mathrm{Pic}(X) \oplus \mathbb{Z} \to \mathrm{Pic}(\mathbb{P}_X(\mathcal{F}))$ is a formal consequence of Quillen's formula [23, Prop. 4.3] for the K-theory of a projective bundle. Indeed, the determinant map $K_0 \to \mathrm{Pic}$ is surjective with an obvious set theoretic section and can easily be computed on each component of Quillen's formula. Injectivity is obtained by pulling back to the fiber of a point for the $\mathbb{Z}$ component, and by the projection formula for the remaining $\mathrm{Pic}(X)$ component. ☐

Proposition A.6 (Picard group of a blow-up). *Under Setup 1.1, we have:*

(i) *The Picard group of* $Bl = Bl_Z(X)$ *is isomorphic to* $\mathrm{Pic}(X) \oplus \mathbb{Z}$ *where the direct summand* $\mathrm{Pic}(X)$ *comes from the pull-back* π^* *and* $\mathbb{Z}$ *is generated by the class of the exceptional divisor* $\mathcal{O}(E) = \mathcal{O}_{Bl}(-1)$.

(ii) *If* X *is normal, the map* $\upsilon^* : \mathrm{Pic}(X) \to \mathrm{Pic}(U)$ *is injective. If* X *is regular it is an isomorphism.*

(iii) *The exceptional fiber* E *is the projective bundle* $\mathbb{P}(C_{Z/X})$ *over* Z *and its Picard group is therefore* $\mathrm{Pic}(Z) \oplus \mathbb{Z}$ *where* $\mathbb{Z}$ *is generated by* $\mathcal{O}_E(-1)$.

(iv) *The pull-back* $\tilde{\iota}^* : \mathrm{Pic}(Bl) \to \mathrm{Pic}(E)$ *maps* $[\mathcal{O}(E)] \in \mathrm{Pic}(Bl)$ *to* $[\mathcal{O}_E(-1)]$.

Under these identifications, Diagram (1) *induces the following pull-back maps on Picard groups:*

$$\begin{tikzcd}
\mathrm{Pic}(Z) \arrow[d, "\binom{\mathrm{id}}{0}"'] & \mathrm{Pic}(X) \arrow[l, "\iota^*"'] \arrow[r, "\upsilon^*"] \arrow[d, "\binom{\mathrm{id}}{0}"] & \mathrm{Pic}(U)\,. \\
\mathrm{Pic}(Z)\oplus\mathbb{Z} & \mathrm{Pic}(X)\oplus\mathbb{Z} \arrow[l, "\left(\begin{smallmatrix}\iota^* & 0\\ 0 & \mathrm{id}\end{smallmatrix}\right)"'] \arrow[ur, "(\upsilon^*\ 0)"'] &
\end{tikzcd}$$

Proof. By Example A.2, we get (iv) and we can deduce (iii) from Proposition A.5. To prove (ii), use that for X normal (resp. regular) $\mathrm{Pic}(X)$ injects into (resp. is isomorphic to) the group $\mathrm{Cl}(X)$ of Weil divisors classes (see [18, 7.1.19 and 7.2.14 (c), resp. 7.2.16]), and that $\mathrm{Cl}(X) = \mathrm{Cl}(U)$ since $\mathrm{codim}_X(Z) \geq 2$. Finally, for (i), consider the commutative diagram

$$\begin{tikzcd}
\mathrm{D_{perf}}(Z) \arrow[d, "\mathrm{L}\tilde{\pi}^*"'] & \mathrm{D_{perf}}(X) \arrow[l, "\mathrm{L}\iota^*"'] \arrow[d, "\mathrm{L}\pi^*"] \\
\mathrm{D_{perf}}(E) & \mathrm{D_{perf}}(Bl) \arrow[l, "\mathrm{L}\tilde{\iota}^*"']
\end{tikzcd} \tag{26}$$

of induced functors on the derived categories of perfect complexes. We will use:

Fact 1: The tensor triangulated functors $\mathrm{L}\pi^*$ and $\mathrm{L}\tilde{\pi}^*$ are fully faithful with left inverse $\mathrm{R}\,\pi_*$ and $\mathrm{R}\,\tilde{\pi}_*$ respectively, see Thomason [24, Lemme 2.3].

Fact 2: If $M \in \mathrm{D_{perf}}(Bl)$ is such that $\mathrm{L}\tilde{\iota}^*(M) \simeq \mathrm{L}\tilde{\pi}^*(N)$ for some $N \in \mathrm{D_{perf}}(Z)$, then $M \simeq \mathrm{L}\pi^*(L)$ for some $L \in \mathrm{D_{perf}}(X)$, which must then be $\mathrm{R}\,\pi_*(M)$ by Fact 1. This follows from [11, Prop. 1.5]. (In their notation, our assumption implies that M is zero in all successive quotients $\mathrm{D}^{i+1}_{\mathrm{perf}}(Bl)/\,\mathrm{D}^{i}_{\mathrm{perf}}(Bl)$ hence belongs to $\mathrm{D}^0_{\mathrm{perf}}(Bl)$.)

Hence $\mathrm{Pic}(X)\oplus\mathbb{Z} \to \mathrm{Pic}(Bl)$ is injective: If L is a line bundle on X and $n \in \mathbb{Z}$ are such that $\mathrm{L}\pi^*(L)\otimes\mathcal{O}_{Bl}(n)$ is trivial then we get $n = 0$ by restricting to E and applying (iii), and we get $L \simeq \mathrm{R}\,\pi_*\,\mathrm{L}\pi^*L \simeq \mathrm{R}\,\pi_*\mathcal{O}_{Bl} \simeq \mathrm{R}\,\pi_*\,\mathrm{L}\pi^*\mathcal{O}_X \simeq \mathcal{O}_X$ by Fact 1. So, let us check surjectivity of $\mathrm{Pic}(X)\oplus\mathbb{Z} \to \mathrm{Pic}(Bl)$. Let M be a line bundle on Bl. Using (iii) again and twisting with $\mathcal{O}_{Bl}(n)$ if necessary, we can assume that $\mathrm{L}\tilde{\iota}^*(M)$ is isomorphic to $\mathrm{L}\tilde{\pi}^*N = \tilde{\pi}^*N$ for some line bundle N on Z. By Fact 2, there exists $L \in \mathrm{D_{perf}}(X)$ such that $\mathrm{L}\pi^*(L) \simeq M$. It now suffices to check that this $L \in \mathrm{D_{perf}}(X)$ is a line bundle. The natural (evaluation) map $L^\vee \otimes L \to \mathcal{O}_X$ is an isomorphism, since it is so after applying the fully faithful tensor functor $\mathrm{L}\pi^* : \mathrm{D_{perf}}(X) \to \mathrm{D_{perf}}(Bl)$. So $L \in \mathrm{D_{perf}}(X)$ is an invertible object, hence it is the m^{th} suspension of a line bundle for $m \in \mathbb{Z}$, see [7, Prop. 6.4]. Using (26), one checks by restricting to Z that $m = 0$, i.e. L is a line bundle. ∎

* * *

We now discuss dualizing complexes and relative canonical bundles. First of all, we mention the essential uniqueness of dualizing complexes on a scheme.

Lemma A.7. *Let $X \in \mathcal{S}ch$ be a scheme admitting a dualizing complex K_X. Then:*

(i) For any line bundle L and any integer i, the complex $K_X \otimes L[i]$ is also a dualizing complex and any dualizing complex on X is of this form.

(ii) If $K_X \otimes L[i] \simeq K_X$ in the derived category of X, for some line bundle L and some integer i, then $L \simeq \mathcal{O}_X$ and $i = 0$.

In other words, the set of isomorphism classes of dualizing complexes on X is a principal homogeneous space under the action of $\mathrm{Pic}(X) \oplus \mathbb{Z}$.

Proof. For (i), see [19, Lemma 3.9]. Let us prove (ii). We have the isomorphisms

$$\begin{aligned}\mathcal{O}_X \xrightarrow{\sim} \mathrm{RHom}(K_X, K_X) &\simeq \mathrm{RHom}(K_X, K_X \otimes L[n]) \\ &\simeq \mathrm{RHom}(K_X, K_X) \otimes L[n] \xleftarrow{\sim} L[n]\end{aligned}$$

in the coherent derived category. The first and last ones hold by [19, Prop. 3.6]. We thus obtain an isomorphism $\mathcal{O}_X \simeq L[n]$ in the derived category of perfect complexes (it is a full subcategory of the coherent one). This forces $n = 0$ and the existence of an honest isomorphism of sheaves $\mathcal{O}_X \simeq L$, see [7, Prop. 6.4] if necessary. □

We now use the notion of local complete intersection (l.c.i.) morphism, that is, a morphism which is locally a regular embedding followed by a smooth morphism, see [18, § 6.3.2]. The advantage of such morphisms $f : X' \to X$ is that $f^!$ is just $\mathrm{L}f^*$ twisted by a line bundle ω_f and shifted by the relative dimension $\dim(f)$.

Proposition A.8. *Let $f : X' \to X$ be an l.c.i. morphism with $X, X' \in \mathcal{S}ch$. Assume that f is proper. Then $f^!(\mathcal{O}_X)$ is a shifted line bundle $\omega_f[\dim(f)]$ and there exists a natural isomorphism $f^!(\mathcal{O}_X) \otimes \mathrm{L}f^*(-) \xrightarrow{\sim} f^!(-)$. In particular, $f^!$ preserves the subcategory $\mathrm{D}_{\mathrm{perf}}$ of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}$.*

Proof. There is always a natural morphism $f^!(\mathcal{O}_X) \otimes \mathrm{L}f^*(-) \to f^!(-)$. One shows that it is an isomorphism and that $f^!(\mathcal{O}_X)$ is a line bundle directly from the definition, since both these facts can be checked locally, are stable by composition and are true for (closed) regular immersions and smooth morphisms by Hartshorne [14, Ch. III]. The subcategory $\mathrm{D}_{\mathrm{perf}}$ is then preserved since both $\mathrm{L}f^*$ and tensoring by a line bundle preserve it. □

The above proposition reduces the description of $f^!$ to that of the line bundle ω_f.

Proposition A.9. *In the following cases, we have concrete descriptions of* ω_f.

(i) *When* $f : X' \to X$ *is smooth and proper,* $\omega_f \simeq \det(\Omega^1_{X'/X})$ *is the determinant of the sheaf of differentials. In particular, when* f *is the projection of a projective bundle* $\mathbb{P}(\mathcal{F})$ *to its base, where* $\mathcal{F}$ *is a vector bundle of rank* r*, then* $\omega_f \simeq f^*(\det\mathcal{F}) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{F})}(-r)$.

(ii) *When* $f : X' \hookrightarrow X$ *is a regular closed immersion,* $\omega_f \simeq \det(N_{X'/X})$ *is the determinant of the normal bundle. In particular when* $f : E \hookrightarrow B$ *is the inclusion of a prime divisor (Def. A.1), we have* $\omega_f \simeq \mathcal{O}(E)_{|E}$.

Proof. See [25, Prop. 1 and Thm. 3]. See alternatively [18, § 6.4.2]. ☐

Remark A.10. All morphisms along which we consider push-forward in this article are l.c.i. It might not be obvious for $\pi : Bl \to X$ but this follows from [1, VII 1.8 p. 424] (it is locally of the form mentioned there). So, ω_π is also a line bundle. Let us now describe the relative canonical line bundles in terms of $\omega_\iota = \det(N_{Z/X})$.

Proposition A.11. *With the notation of Setup 1.1, we have*

(i) $\omega_{\tilde\iota} = \mathcal{O}(E)_{|E} = \mathcal{O}_E(-1)$

(ii) $\omega_{\tilde\pi} = \tilde\pi^*\omega_\iota^\vee \otimes \mathcal{O}(E)^{\otimes c}_{|E} = \tilde\pi^*\omega_\iota^\vee \otimes \mathcal{O}_E(-c)$

(iii) $\omega_\pi = \mathcal{O}(E)^{\otimes(c-1)} = \mathcal{O}_{Bl}(1-c)$.

By Proposition A.8, it implies that we have

(iv) $\tilde\iota^!(-) = \mathcal{O}(E)_{|E} \otimes \mathrm{L}\tilde\iota^*(-)[-1] = \mathcal{O}_E(-1) \otimes \mathrm{L}\tilde\iota^*(-)[-1]$

(v) $\tilde\pi^!(-) = \tilde\pi^*\omega_\iota^\vee \otimes \mathcal{O}(E)^{\otimes c}_{|E} \otimes \mathrm{L}\tilde\pi^*(-)[c-1] = \tilde\pi^*\omega_\iota^\vee \otimes \mathcal{O}_E(-c) \otimes \mathrm{L}\tilde\pi^*(-)[c-1]$

(vi) $\pi^!(-) = \mathcal{O}(E)^{\otimes(c-1)} \otimes \mathrm{L}\pi^*(-) = \mathcal{O}_{Bl}(1-c) \otimes \mathrm{L}\pi^*(-)$.

Proof. Points (i) and (ii) follow from Proposition A.9 (ii) and (i), respectively. They imply (iv) and (v). To prove point (iii) let us first observe that the exact sequence (22) gives rise to an exact triangle

$$\mathcal{O}_{Bl}(l+1) \to \mathcal{O}_{Bl}(l) \to \mathrm{R}\,\tilde\iota_*(\mathcal{O}_E(l)) \to \mathcal{O}_{Bl}(l+1)[1]$$

in $\mathrm{D}_{\mathrm{perf}}(Bl)$ for any $l \in \mathbb{Z}$. Applying $\mathrm{R}\,\pi_*$ to this triangle and using that

$$\mathrm{R}\,\pi_*\,\mathrm{R}\,\tilde\iota_*\mathcal{O}_E(l) = \mathrm{R}\,\iota_*\,\mathrm{R}\,\tilde\pi_*\mathcal{O}_E(l) = 0 \quad \text{for} \quad -c < l < 0$$

(by [13, 2.1.15]), we obtain by induction that $\mathrm{R}\,\pi_*\mathcal{O}_{Bl}(l) = \mathrm{R}\,\pi_*\mathcal{O}_{Bl} = \mathcal{O}_X$ for $-c < l \le 0$. In particular $\mathrm{R}\,\pi_*\mathcal{O}_{Bl}(1-c) = \mathcal{O}_X$. We now use the filtration of [11, Prop. 1.5]. Let us show that $\pi^!(\mathcal{O}_X) \otimes \mathcal{O}_{Bl}(c-1)$ is in $\mathrm{D}^0_{\mathrm{perf}}(Bl)$. By loc. cit. it suffices to show that $\mathrm{L}\tilde{\iota}^*(\pi^!(\mathcal{O}_X)\otimes\mathcal{O}_{Bl}(c-1))$ is in $\mathrm{D}^0_{\mathrm{perf}}(E)$. It follows from the sequence of isomorphisms

$$\mathrm{L}\tilde{\iota}^*(\pi^!(\mathcal{O}_X)\otimes\mathcal{O}_{Bl}(c-1))\simeq \mathrm{L}\tilde{\iota}^*\pi^!(\mathcal{O}_X)\otimes\mathcal{O}_E(c-1)\overset{\text{(iv)}}{\simeq}\tilde{\iota}^!\pi^!(\mathcal{O}_X)\otimes\mathcal{O}_E(c)[1]$$
$$\simeq\tilde{\pi}^!\iota^!(\mathcal{O}_X)\otimes\mathcal{O}_E(c)[1]\overset{\text{A.8}}{\simeq}\tilde{\pi}^!(\omega_\iota)\otimes\mathcal{O}_E(c)[-c+1]\overset{\text{(v)}}{\simeq}\mathcal{O}_E\,.$$

Since $L:=\pi^!(\mathcal{O}_X)\otimes\mathcal{O}_{Bl}(c-1)$ is in $\mathrm{D}^0_{\mathrm{perf}}(Bl)$, it is of the form $\mathrm{L}\pi^*M$ for $M=\mathrm{R}\,\pi_*L$ (by Fact 1 in the proof of Prop. A.6) which we compute by duality:

$$\mathrm{R}\,\pi_*(\pi^!(\mathcal{O}_X)\otimes\mathcal{O}_{Bl}(c-1))\simeq\mathrm{R}\,\pi_*\mathrm{RHom}(\mathcal{O}_{Bl},\pi^!(\mathcal{O}_X)\otimes\mathcal{O}_{Bl}(c-1))\simeq$$
$$\simeq\mathrm{R}\,\pi_*\mathrm{RHom}(\mathcal{O}_{Bl}(1-c),\pi^!(\mathcal{O}_X))\overset{(\dagger)}{\simeq}\mathrm{RHom}(\mathrm{R}\,\pi_*\mathcal{O}_{Bl}(1-c),\mathcal{O}_X)\overset{(\star)}{\simeq}$$
$$\simeq\mathrm{RHom}(\mathcal{O}_X,\mathcal{O}_X)\simeq\mathcal{O}_X$$

where $(\star)$ is by the computation at the beginning of the proof, $(\dagger)$ is the duality isomorphism and all other isomorphisms are obtained as consequences of the monoidal structure on the $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}$ involved (see [10] and [8] for details). Hence, $\pi^!(O_X)\simeq\mathcal{O}_{Bl}(1-c)$ as announced. This proves (iii) and thus (vi). ☐

Finally, we also use dualizing complexes in the context of an $\mathbb{A}^*$-bundle $U\to Y$, i.e. a morphism that is locally of the form $\mathbb{A}^n_Y\to Y$ (and is in particular flat).

Lemma A.12. *Let $\alpha: U\to Y$ be an $\mathbb{A}^*$-bundle. Assume that Y admits a dualizing complex K_Y. Then $\mathrm{L}\alpha^*(K_Y)=\alpha^*(K_Y)$ is a dualizing complex on U.*

Proof. This can be checked locally, so we can assume α decomposes as $\alpha=f\circ u$ for an open immersion $u:\mathbb{A}^n_Y\hookrightarrow\mathbb{P}^n_Y$ followed by the structural projection $f:\mathbb{P}^n_Y\to Y$. Note that α, u and f are all flat. We have by Propositions A.8 and A.9 (i)

$$u^*f^!K_Y[n]=u^*(\mathcal{O}(-n-1)\otimes f^*K_Y)\simeq u^*f^*K_Y=\alpha^*K_Y$$

where the second equality comes from the triviality of $\mathcal{O}(-n-1)$ on $\mathbb{A}^n_Y$. Now $u^*f^!K_Y[n]$ is dualizing because proper morphisms, open immersions and shifting preserve dualizing complexes. ☐

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