Rationally Connected Foliations on Surfaces

Sebastian Neumann

Received: November 21, 2008

Communicated by Thomas Peternell

Abstract. In this short note we study foliations on surfaces with rationally connected leaves. Our main result is that on a surface there exists a polarisation such that the Harder-Narasimhan filtration of the tangent bundle with respect to this polarisation yields the maximal rationally connected quotient of the surface.

2000 Mathematics Subject Classification: 14J26, 37F75

1 Introduction

Let X be a smooth projective variety over the complex numbers. In this note we are interested in foliations with rationally connected leaves. In [KSCT07] it is shown how to construct such foliations from the *Harder-Narasimhan filtration* of the tangent bundle of the variety. This construction depends heavily on a chosen polarisation, and therefore the question arises how this foliation varies with the polarisation.

There is another way to construct a fibration with rationally connected fibers, the *maximal rationally connected quotient*. This is a rational map whose fibers are rationally connected. Almost every rational curve in X lies in a fiber of this map.

We can ask if the Harder-Narasimhan filtration of the tangent bundle always induces the maximal rationally connected quotient with respect to any polarisation. The answer is negative already on surfaces as shown by an example of Thomas Eckl [Eck08].

In this note we will prove that on surfaces there always exists a polarisation such that the Harder-Narasimhan filtration yields the maximal rationally connected quotient.

ACKNOWLEDGMENTS

The author was supported in part by the Graduiertenkolleg "Globale Strukturen in Geometrie und Analysis" of the Deutsche Forschungsgemeinschaft. The result is part of the author's forthcoming Ph.D. thesis written under the supervision of Stefan Kebekus. He would like to thank Stefan Kebekus, Thomas Eckl and Sammy Barkowski for numerous discussions.
The results in this note were presented at a Workshop in Grenoble in April 2008. Similar results have been obtained independently in [SCT08].

2 PRELIMINARY RESULTS AND NOTATION

Let X be an n-dimensional projective variety over the complex numbers with an ample line bundle H. Given a torsion-free coherent sheaf $\mathcal{F}$ on X, we define the *slope of $\mathcal{F}$ with respect to H* to be

$$\mu_H(\mathcal{F}) := \frac{c_1(\mathcal{F}) \cdot H^{n-1}}{\mathrm{rk}(\mathcal{F})}.$$

We call $\mathcal{F}$ *semistable with respect to H* if for any nonzero proper subsheaf $\mathcal{G}$ of $\mathcal{F}$ we have $\mu_H(\mathcal{G}) \leq \mu_H(\mathcal{F})$.
If there exists a nonzero subsheaf $\mathcal{G} \subset \mathcal{F}$ such that $\mu_H(\mathcal{G}) > \mu_H(\mathcal{F})$, we will call $\mathcal{G}$ a *destabilizing subsheaf* of $\mathcal{F}$.

THEOREM 2.1 ([Mar80, Proposition 1.5.]). *Let $\mathcal{F}$ be a torsion-free coherent sheaf on a smooth projective variety and H be an ample line bundle on X. There exists a unique filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_k = \mathcal{F}$$

of $\mathcal{F}$ depending on H, the Harder-Narasimhan filtration *or* HN-filtration, *with the following properties:*

(i) The quotients $\mathcal{G}_i := \mathcal{F}_i/\mathcal{F}_{i-1}$ are torsion-free and semistable.

(ii) The slopes of the quotients satisfy $\mu_H(\mathcal{G}_1) > \ldots > \mu_H(\mathcal{G}_k)$.

DEFINITION 2.2. Let $\mathcal{F}$ be a torsion-free coherent sheaf on a smooth projective variety. The unique sheaf $\mathcal{F}_1$ appearing in the Harder-Narasimhan filtration of $\mathcal{F}$ is called *the maximal destabilizing subsheaf of $\mathcal{F}$*.

DEFINITION 2.3. Let $\mathcal{F}$ be a coherent torsion-free sheaf on a smooth projective variety with Harder-Narasimhan filtration

$$0 = \mathcal{F}_0 \subset \ldots \subset \mathcal{F}_k = \mathcal{F}$$

with respect to an ample line bundle H. If the slope of the quotient $\mathcal{F}_i/\mathcal{F}_{i-1}$ is positive with respect to H, then $\mathcal{F}_i$ is called *positive with respect to H*.

Remark 2.4. Note that the construction of the Harder-Narasimhan filtration naturally extends to $\mathbb{Q}$- and $\mathbb{R}$-divisors, i.e. we do not need to assume that the chosen polarisation is integral.

Obviously, the Harder-Narasimhan filtration depends only on the numerical class of the chosen ample bundle. In particular it makes sense to ask how the filtration of a given sheaf depends on the ample bundle sitting in the finite dimensional vector space of all divisors modulo numerical equivalence.
We can now state an important result originally formulated by Miyaoka and explicitly shown in [KSCT07]. For a survey on these and related results we refer the reader to [KSC06].

Theorem 2.5 ([KSCT07, Theorem 1]). *Let X be a smooth projective variety and let*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_k = TX$$

be the Harder-Narasimhan filtration of the tangent bundle with respect to a polarisation H. Write $\mu_i := \mu_H(\mathcal{F}_i/\mathcal{F}_{i-1})$ for the slopes of the quotients. Assume $\mu_1 > 0$ and set $m := \max\{i \in \mathbb{N} | \mu_i > 0\}$. Then each $\mathcal{F}_i$ with $i \leq m$ is a foliation, i.e. a saturated subsheaf of the tangent bundle closed under Lie bracket. Furthermore the leaves of these foliations are algebraic and for general $x \in X$ the closure of the leaf through x is rationally connected.

Let X be a smooth projective variety and assume the conditions of Theorem (2.5) are fulfilled. Thus we obtain foliations $\mathcal{F}_1, \ldots, \mathcal{F}_k$ with algebraic and rationally connected leaves. By setting

$$\begin{array}{cccc} q_i : & X & \dashrightarrow & \mathrm{Im}(q_i) \subset \mathrm{Chow}(X) \\ & x & \mapsto & \mathcal{F}_i\text{-leaf through } x \end{array}$$

we obtain a rational map, such that the closure of the general fibre is rationally connected, see [KSCT07] Section 7.
There is another map with this property called the *maximal rationally connected quotient*, or *MRC-quotient*, for short based on a construction by Campana [Cam81] [Cam94] and Kollár-Miyaoka-Mori [KMM92], see also [Kol96, Chapter IV, Theorem 5.2].

Theorem 2.6 ([KMM92, Theorem 2.7.]). *Let X be a smooth projective variety. There exists a variety Z and a rational map $\phi : X \dashrightarrow Z$ with the following properties:*

- *the fibers of ϕ are rationally connected,*
- *a very general fiber of ϕ is an equivalence class with respect to rational connectivity and*
- *up to birational equivalence the map ϕ and the variety Z are unique.*

In this paper we ask if the Harder-Narasimhan filtration with respect to a certain polarisation yields the MRC-quotient. We will give a positive answer for surfaces in the next section.

3 Rationally Connected Foliations on Surfaces and the MRC-quotient

In this section X denotes a smooth projective surface over the the field of complex numbers.
We want to investigate the regions in the ample cone which induce the same HN-filtration. More precisely we divide the ample cone into parts, so that in each part we get the same HN-filtration of the tangent bundle. With this at hand we are able to show that the MRC-quotient comes from the Harder-Narasimhan filtration of the tangent bundle with respect to a certain polarisation.
In order to compute the HN-filtration of the tangent bundle on surfaces, we only have to search for a destabilizing subsheaf whose quotient is torsion-free. This is formulated in the next lemma.

LEMMA 3.1. *Let X be a smooth projective surface. If $\mathcal{F} \subset TX$ is a destabilizing subsheaf with respect to a polarisation such that $TX/\mathcal{F}$ is torsion-free, then the Harder-Narasimhan filtration is given by $0 \subset \mathcal{F} \subset TX$.*

Proof. Let H be a polarisation and $\mathcal{F}$ a destabilizing subsheaf of TX with respect to H. Consider the exact sequence

$$0 \to \mathcal{F} \to TX \to TX/\mathcal{F} \to 0.$$

Using that the rank and the first Chern class are additive in short exact sequences, we obtain

$$\mu_H(TX) = \frac{1}{2}\mu_H(TX/\mathcal{F}) + \frac{1}{2}\mu_H(\mathcal{F}).$$

Since $\mu_H(\mathcal{F}) > \mu_H(TX)$, we therefore have $\mu_H(\mathcal{F}) > \mu_H(TX/\mathcal{F})$. That is,

$$0 \subset \mathcal{F} \subset TX$$

satisfies the properties of the Harder-Narasimhan filtration and by the uniqueness of the HN-filtration we are done. ☐

NOTATION 3.2. We write $N^1(X)$ for the Néron-Severi group and $N^1_{\mathbb{Q}}(X)$ (resp. $N^1_{\mathbb{R}}(X)$) for the vector space of $\mathbb{Q}$–divisors (resp. $\mathbb{R}$–divisors) modulo numerical equivalence on X. The convex cone of all ample $\mathbb{R}$–divisors in $N^1_{\mathbb{R}}(X)$ is denoted by $\mathrm{Amp}_{\mathbb{R}}(X)$.

Now we define the regions in $\mathrm{Amp}_{\mathbb{R}}(X)$ we are interested in. Let $H \in N^1_{\mathbb{R}}(X)$ be an ample bundle. If TX is not semistable with respect to H, let $\mathcal{F}$ be the maximal destabilizing subsheaf of TX with respect to H, i.e. the Harder-Narasimhan filtration of TX with respect to H is given by $0 \subset \mathcal{F} \subset TX$. We call

$$\Delta_H := \left\{ \tilde{H} \in \mathrm{Amp}_{\mathbb{R}}(X) \mid \big(c_1(\mathcal{F}) - \frac{1}{2}c_1(TX)\big) \cdot \tilde{H} > 0 \right\}$$

the *destabilizing chamber with respect to H*.

Remark 3.3. By Lemma (3.1) the condition $(c_1(\mathcal{F}) - \frac{1}{2}c_1(TX)) \cdot \tilde{H} > 0$ ensures that for all polarisations in Δ_H we get the same HN-filtration, namely $0 \subset \mathcal{F} \subset TX$. So we have indeed defined the regions in the ample cone, in which the Harder-Narasimhan filtration of the tangent bundle remains constant.

Note that if the tangent bundle is semistable with respect to a certain polarisation, then we get a chamber such that for all polarisations in this chamber TX is semistable. This region is called the *semistable chamber.*
Concerning the structure of these chambers we prove the following lemma.

Lemma 3.4. *Let X be a smooth projective surface. We have:*

(i) The destabilizing chambers and the semistable chamber are convex cones in $\mathrm{Amp}_{\mathbb{R}}(X)$.

(ii) The semistable chamber is closed in $\mathrm{Amp}_{\mathbb{R}}(X)$.

(iii) The destabilizing chambers are open in $\mathrm{Amp}_{\mathbb{R}}(X)$.

(iv) The destabilizing chambers and the semistable chamber give a decomposition of the ample cone, i.e. the union of all chambers is the ample cone and the chambers are pairwise disjoint.

Proof. The convexity property of both the semistable chamber and the destabilizing chamber follows directly from the linearity of the intersection product. Statement (iii) is a direct consequence of the continuity of the intersection product, since for a maximal destabilizing subsheaf $\mathcal{F} \subset TX$ the condition

$$\big(c_1(\mathcal{F}) - \frac{1}{2}c_1(TX)\big) \cdot H > 0$$

is an open condition.
To prove (iv) note that by definition of the chambers, each polarisation appears in at least one chamber. Since for a given polarisation the associated maximal destabilizing subsheaf of TX is unique, the polarisation appears in exactly one chamber.
Statement (ii) is a direct consequence of (iii) and (iv). ☐

In the proof of our main result, we will use the following corollary.

Corollary 3.5. *Let X be a smooth projective surface. Let ℓ be a line segment in $\mathrm{Amp}_{\mathbb{R}}(X)$, such that ℓ does not intersect the semistable chamber. Then ℓ is contained in a single destabilizing chamber.*

Proof. Assume ℓ intersects at least two destabilizing chambers. By Lemma (3.4) we get a partition of ℓ into disjoint open sets. This is impossible because ℓ is connected. ☐

To prove semistability of the tangent bundle on certain surfaces having many automorphisms, we will give a useful lemma. Let $\sigma \in \mathrm{Aut}(X)$ and $\mathcal{F} \subset TX$. By means of the differential of σ, we can identify TX and σ^*TX. Thus we can interpret $\sigma^*(\mathcal{F})$ as a subsheaf of TX. For instance, if $p \in X$ and $\mathcal{F} := TX \otimes \mathcal{I}_p$, then $\sigma^*(\mathcal{F})$ is identified with $TX \otimes \mathcal{I}_{\sigma^{-1}(p)} \subset TX$.

LEMMA 3.6. *Let X be a smooth projective surface and let $\sigma \in \mathrm{Aut}^0(X)$. Let $\mathcal{F}$ be the maximal destabilizing subsheaf of TX with respect to some polarisation. We then have $\sigma^*\mathcal{F} = \mathcal{F}$. In particular: If $\mathcal{F}$ is a foliation then the automorphism σ maps each leaf of $\mathcal{F}$ to another leaf of $\mathcal{F}$.*

Proof. Let $H \in \mathrm{Amp}_{\mathbb{R}}(X)$ and let $\mathcal{F}$ be the maximal destabilizing subsheaf of TX with respect to H. We compute the slope of $\sigma^*(\mathcal{F}) \subset TX$:

$$\begin{array}{rcl} \mu_H\big(\sigma^*(\mathcal{F})\big) & = & H \cdot \big(c_1(\sigma^*(\mathcal{F}))\big) \\ & = & H \cdot \sigma^*(c_1(\mathcal{F})) \\ & = & H \cdot c_1(\mathcal{F}) \\ & > & \frac{1}{2} c_1(TX) \cdot H. \end{array}$$

We give an explanation of the third equality. Recall that the group of automorphisms acts on the Néron-Severi group. Since $N^1(X)$ is discrete, $\mathrm{Aut}^0(X)$ acts trivially on $N^1(X)$, i.e. $\sigma^*(c_1(\mathcal{F})) = c_1(\mathcal{F})$.
We therefore have shown that $\sigma^*(\mathcal{F})$ is a destabilizing subsheaf of TX. By Lemma (3.1) and the uniqueness of the maximal destabilizing subsheaf of TX, we conclude that $\sigma^*\mathcal{F} = \mathcal{F}$. ☐

EXAMPLE 3.7. *Hirzebruch Surfaces*
Let Σ_n be the n-th Hirzebruch surface and let $\pi : \Sigma_n \longrightarrow \mathbb{P}^1$ be the projection onto the projective line. We denote the fiber under the projection by f and the distinguished section with selfintersection $-n$ by C_0. Recall (see [Har77], chapter V.2) that $N^1_{\mathbb{R}}(\Sigma_n) =< C_0, f >$ and a divisor $D \equiv_{num} aC_0 + bf$ is ample if and only if $a > 0$ and $b > an$. The canonical bundle is given by $-K_{\Sigma_n} = 2C_0 + (2+n)f$. The relative tangent bundle of π is a natural candidate for a destabilizing subbundle. We have the sequence

$$0 \to T_{\Sigma_n/\mathbb{P}^1} \to T\Sigma_n \to \pi^*T\mathbb{P}^1 \to 0$$

Let $H := xC_0 + yf$ be a polarisation. Then one can compute that $T_{\Sigma_n/\mathbb{P}^1}$ is destabilizing if and only if $-2x - nx + 2y > 0$. In particular we compute for $n \geq 2$:

$$-2x - nx + 2y > -2x - nx + 2nx = -2x + nx \geq 0.$$

Therefore, for $n \geq 2$ the HN-filtration is given by

$$0 \subset T_{X/\mathbb{P}^1} \subset TX$$

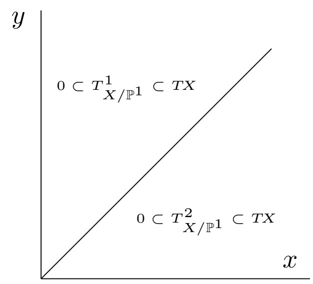

Figure 1: The ample cone of $X = \Sigma_0$ and the chamber structure. Here $T^1_{X/\mathbb{P}^1}$ and $T^2_{X/\mathbb{P}^1}$ denote the relative tangent bundle of the first and second projection.

for all polarisations. In other words we obtain only one destabilizing chamber. For $n = 0$ we have $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and we get three chambers. The two destabilizing chambers correspond to the two relative tangent bundles of the projections. They are cut out by the inequalities $x > y$ and $x < y$. There is a chamber of semistability, which is determined by the equation $x = y$.
For $n = 1$ we see that for $x > \frac{3}{2}y$ the relative tangent bundle is destabilizing. Since Σ_1 is the projective plane blown up at a point p, the group of automorphisms is the automorphism group of the projective plane leaving p fixed. The destabilizing foliation corresponds to the radial foliation through p in the plane. So if there were another foliation $\mathcal{F}$ coming from the Harder-Narasimhan filtration of $T\Sigma_1$, we could deform the leaves with these automorphisms. Then we would again obtain leaves of this foliation by Lemma (3.6). So unless $\mathcal{F}$ is the foliation given by the relative tangent bundle of the projection morphism, we could deform each leaf of $\mathcal{F}$ while leaving a point on the leaf not lying on C_0 fixed. Thus the foliation induced by $\mathcal{F}$ would have singularities on a dense open subset of Σ_1 which is absurd. So the tangent bundle is semistable for $x \leq \frac{3}{2}y$.

Now we want to answer the question if there always exists a polarisation, such that the Harder-Narasimhan filtration gives rise to the MRC-quotient.

Theorem 3.8. *Let X be a uniruled projective surface. Then there exists a polarisation, such that the maximal rationally connected quotient of X is given by the foliation associated to highest positive term in the Harder-Narasimhan filtration with respect to this polarisation.*

Proof. To start, observe that there is always a polarisation A such that $c_1(TX) \cdot A > 0$. Indeed, there exists a free rational curve $f : \mathbb{P}^1 \to X$. See [Deb01, Corollary 4.11] for a proof of the existence of such a curve. Writing

$$f^*(TX) = \mathcal{O}(a_1) \oplus \mathcal{O}(a_2)$$

with $a_1 + a_2 \geq 2$, we compute

$$-K_X \cdot f_* \mathbb{P}^1 = a_1 + a_2 \geq 2.$$

Write $\ell := f_* \mathbb{P}^1$ for this curve. Since ℓ is movable, it is in particular nef. So for an ample class H, the class $\ell + \epsilon H$ will be ample. Thus for sufficiently small ϵ the class $\ell + \epsilon H$ will intersect $-K_X$ positively.
First let us assume that X is not rationally connected. As we have just seen, we can find a polarisation H with $c_1(TX) \cdot H > 0$. There exists a destabilizing subsheaf $\mathcal{F}$ of TX, since otherwise X would be rationally connected by Theorem (2.5). Furthermore the slope of $\mathcal{F}$ has to be bigger than $c_1(TX) \cdot H$ and therefore positive. So this sheaf will give a foliation with rationally connected leaves and hence the maximal rationally connected quotient.
Now we consider the case where X is rationally connected. We then fix a very free rational curve ℓ on X. For a proof of the existence of a very free rational curve see [Deb01, Corollary 4.17]. This means that $TX|_\ell$ is ample. So we know that each quotient of $TX|_\ell$ has strictly positive degree.
Since ℓ is movable, it is in particular nef. Let H be an ample class. Because ℓ is nef, we know that $H_\epsilon := \ell + \epsilon H$ is ample in $N^1_{\mathbb{Q}}(X)$ for any $\epsilon > 0$. Observe that $c_1(TX) \cdot H_\epsilon > 0$ for sufficiently small ϵ, say for $0 < \epsilon < \epsilon_0$. If TX is semistable with respect to a certain polarisation H_ϵ with $0 < \epsilon < \epsilon_0$, the claim follows since TX has positive slope and induces a trivial foliation which gives the rationally connected quotient. If TX is not semistable for all polarisations H_ϵ with $0 < \epsilon < \epsilon_0$, let $\mathcal{F}_\epsilon$ be the maximal destabilizing subsheaf of TX with respect to H_ϵ. Because of Corollary (3.5) the ray H_ϵ stays in one destabilizing chamber and Remark (3.3) ensures that $\mathcal{F} := \mathcal{F}_\epsilon$ remains constant.
Now it is clear that for sufficiently small ϵ both the slope of $\mathcal{F}$ and the slope of $TX/\mathcal{F}$ will be positive with respect to H_ϵ. Therefore the HN-filtration of TX with respect to H_ϵ yields the maximal rationally connected quotient. □

REFERENCES

[Cam81] Frédéric Campana. Coréduction algébrique d'un espace analytique faiblement kählérien compact. *Invent. Math.*, 63(2):187–223, 1981.

[Cam94] Frédéric Campana. Remarques sur le revêtement universel des variétés kählériennes compactes. *Bull. Soc. Math. France*, 122(2):255–284, 1994.

[Deb01] Olivier Debarre. *Higher-dimensional algebraic geometry*. Universitext. Springer-Verlag, New York, 2001.

[Eck08] Thomas Eckl. Lower bounds for Seshadri constants. *Math. Nachr.*, 281(8):1119–1128, 2008.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[KMM92] János Kollár, Yoichi Miyaoka, and Shigefumi Mori. Rationally connected varieties. *J. Algebraic Geom.*, 1(3):429–448, 1992.

[Kol96] János Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 1996.

[KSC06] Stefan Kebekus and Luis Solá Conde. Existence of rational curves on algebraic varieties, minimal rational tangents, and applications. In *Global aspects of complex geometry*, pages 359–416. Springer, Berlin, 2006.

[KSCT07] Stefan Kebekus, Luis Solá Conde, and Matei Toma. Rationally connected foliations after Bogomolov and McQuillan. *J. Algebraic Geom.*, 16(1):65–81, 2007.

[Mar80] Masaki Maruyama. Boundedness of semistable sheaves of small ranks. *Nagoya Math. J.*, 78:65–94, 1980.

[SCT08] Luis Solá Conde and Matei Toma. Maximally rationally connected fibrations and movable curves. *preprint, arXiv:0811.2141v2 [math.AG]*, 2008.

Sebastian Neumann
Albert-Ludwigs-Universität
Freiburg
Mathematisches Institut
Eckerstrasse 1
D-79104 Freiburg