On the Nonexistence of Certain Morphisms from Grassmannian to Grassmannian in Characteristic 0

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Abstract. This paper proves some properties of the big Chern classes of a vector bundle on a smooth scheme over a field of characteristic 0. These properties together with the explicit computation of the big Chern classes of universal quotient bundles of Grassmannians are used to prove the main Theorems (Theorems 1,2 and 3) of this paper.

The nonexistence certain morphisms between Grassmannians over a field of characteristic 0 follows directly from these theorems. One of our theorems, for instance, states that the higher Adams operations applied to the class of a universal quotient bundle of a Grassmannian that is not a line bundle yield elements in the K-ring of the Grassmannian that are not representable as classes of genuine vector bundles. This is not true for Grassmannians over a field of characteristic p.

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1 Introduction

1.1 Motivation

Problems regarding the constraints that morphisms between homogeneous spaces must satisfy have been studied by Kapil Paranjape and V. Srinivas [7], [8]. In [7], they characterize self maps of finite degree between homogeneous spaces and prove that finite surjective morphisms from Grassmannian to Grassmannian are actually isomorphisms. In [8], they prove that if S is a smooth quadric hypersurface in $\mathbb{P}^{n+1}$, where $n = 2k+1$, and if $2^k | d$, then there exist continuous maps $f : \mathbb{P}^n \to S$ so that $f^*(\mathcal{O}_S(1)) = \mathcal{O}_{\mathbb{P}^n}(d)$. Let $G(r,n)$ denote the Grassmannian of r-dimensional quotient spaces of an n-dimensional vector space over a field of characteristic 0. In the same spirit, given an integer $p \geq 2$, one can ask questions like whether there exists a map from a Grassmannian $G(r,n)$ to another Grassmannian $G(r,M)$ so that $f^*[Q_{G(r,M)}] = \psi^p[Q_{G(r,n)}]$ where $[V]$ denotes the class of a vector bundle V in K-theory and $Q_{G(r,n)}$ and $Q_{G(r,M)}$ denote the universal quotient bundles of $G(r,n)$ and $G(r,M)$ respectively. Another question in the same spirit would be whether there exist morphisms $f : G(r,n) \to G(r-1,M)$ so that $f^*(\mathrm{ch}_l(Q)) = \mathrm{ch}_l(Q)$. The answers to the first question is in the negative for all $r \geq 2, n \geq 2r+1$ and the answer to the second question is in the negative for infinitely many r, with n assumed to be large enough. It may be noted that in these questions, our attention is not restricted solely to dominant/finite morphisms unlike in the results in [7] and [8]. Indeed, the results proven here are not obtainable by the methods of [7] and [8] as far I can see.

1.2 Statements of the results

The following theorems contain the answers obtained for the above questions. These theorems are proven in this paper. Before we proceed, we state that all varieties in this paper are smooth projective varieties over a field of characteristic 0. For any smooth projective variety X, let $K(X)$ denote the K-ring of X. For any vector bundle V on X, let $[V]$ denote the class of V in $K(X) \otimes \mathbb{Q}$.

Theorem 1. *Let Q denote the universal quotient bundle of a Grassmannian $G(r,n)$. Suppose that $r \geq 2$ and that $n \geq 2r+1$. Then, for all $p \geq 2$, the element $\psi^p[Q]$ of $K(G(r,n)) \otimes \mathbb{Q}$ is not equal to $[V]$ for any genuine vector bundle V on $G(r,n)$.*

Corollary 1. *If $f : G(r,n) \to G(r,\infty)$ is a morphism of schemes with $r \geq 2$ and $n \geq 2r+1$, then $f^*[Q_{G(r,\infty)}] \neq \psi^p[Q_{G(r,n)}]$ for any $p \geq 2$.*

Let X be a smooth variety, and let $F_r\mathrm{CH}^l(X) \otimes \mathbb{Q}$ denote the subspace of $\mathrm{CH}^l(X) \otimes \mathbb{Q}$ spanned by $\{\mathrm{ch}_l(V) | V$ a vector bundle of rank $\leq r\}$. Then, this filtration is nontrivial as a theory. Let $Q_{G(r,n)}$ denote the universal quotient bundle of $G(r,n)$, and let ch denotes the Chern character map, with ch_l denoting the degree l component of ch.

Theorem 2. *Given any natural number $l \geq 2$, there exist infinitely many natural numbers $r > 0$, and a constant C depending on l so that whenever*

$n > Cr^2 + r$,

$$\mathrm{ch}_l(Q_{G(r,n)}) \in F_r\, CH^l(G(r,n)) \otimes \mathbb{Q} \setminus F_{r-1}\, CH^l(G(r,n)) \otimes \mathbb{Q}.$$

Corollary 2. *Given any natural number $l \geq 2$, there exist infinitely many natural numbers $r > 0$, and a constant C depending on l so that whenever $n > Cr^2 + r$, and $f : G(r,n) \to G(r-1,\infty)$ is a morphism of varieties, then*

$$f^*(\mathrm{ch}_l(Q_{G(r-1,\infty)})) \neq \mathrm{ch}_l(Q_{G(r,n)}).$$

Corollary 3. *There exist infinitely many r so that if $f : G(r,n) \to G(r-1,\infty)$ is any morphism of schemes with $n > 7r^2 + r + 2$, then*

$$f^*\,\mathrm{ch}_2(Q_{G(r-1,\infty)}) = \kappa \mathrm{ch}_1(Q_{G(r,n)})^2$$

for some constant $\kappa \in \mathbb{K}$ that possibly depends on r.

Theorem 3. *If $f : G(3,6) \to G(2,\infty)$ is a morphism, then*

$$f^*(\mathrm{ch}_2(Q_{G(2,\infty)})) = \kappa \mathrm{ch}_1(Q_{G(3,6)})^2$$

for some constant $\kappa \in \mathbb{K}$.

1.3 An outline of the set up of the proofs

All these results are proven using certain facts about certain characteristic classes. These characteristic classes were discovered by M. Kapranov [6] (and independently by M.V. Nori [1]) as far as I know. In this paper, I shall show that these objects are characteristic classes that commute with Adams operations (Lemma 9 and Lemma 13 of Section 4.2 in this paper). These characteristic classes are defined as follows.

Let X be a smooth projective variety and let V be a vector bundle on X. Consider the Atiyah class

$$\theta_V \in \mathrm{H}^1(X, \mathrm{End}(V) \otimes \Omega)$$

of V. Denote the k -fold cup product of θ_V with itself by θ_V^k. Applying the composition map $\mathrm{End}(V)^{\otimes k} \to \mathrm{End}(V)$, followed by the trace map $tr : \mathrm{End}(V) \to \mathcal{O}_X$ to θ_V^k, we obtain the characteristic class

$$\mathrm{t}_k(V) \in \mathrm{H}^k(X, \Omega^{\otimes k}).$$

Note that the projection $\Omega^{\otimes k} \to \wedge^k \Omega$ when applied to $\mathrm{t}_k(V)$ gives us $k!\,\mathrm{ch}_k(V)$ where $\mathrm{ch}_k(V)$ denotes the degree k part of the Chern character of V. The classes t_k are referred to in the paper by Kapranov [1] as the *big Chern classes*. These classes and their properties are discussed in greater detail in Section 4 of this paper. The big Chern classes together give a ring homomorphism

$\oplus\,\mathrm{t}_k : K(X) \otimes \mathbb{Q} \to \oplus\,\mathrm{H}^k(X, \Omega^{\otimes k})$ where the right hand side is equipped with a commutative product that shall be described in the Section 2. The commutative ring $\oplus\,\mathrm{H}^k(X, \Omega^{\otimes k})$ shall henceforth be denoted by $\mathrm{R}(X)$. Both this product and the usual cup product in addition to some other (λ-ring) structure on this ring are preserved under pullbacks. Moreover, the two products are distinct and the Adams gradation on $\mathrm{R}(X)$ is distinct from the obvious one (unlike in the case of the usual cohomology ring). These facts place serious restrictions on what pullback maps $f^* : \mathrm{R}(X) \to \mathrm{R}(Y)$ corresponding to morphisms $f : Y \to X$ look like. An important subring of the ring $\mathrm{R}(X)$ will be calculated explicitly for the Grassmannian $G(r, n)$ at the end of Section 3.

Notation: Throughout this paper, $\mathbb{K}$ shall be used to denote the base field. We assume throughout this paper that the characteristic of $\mathbb{K}$ is zero.

1.4 Brief outlines of the proofs

1.4.1 Outline for Theorems 2 and 3

The basic idea behind the proofs of Theorem 2, Corollary 2 and Theorem 3 is the same.

If $\sigma \in S_k$ is a permutation of $\{1, \ldots, k\}$, and if $\mathcal{F}$ is a vector bundle on X, then σ gives us a homomorphism $\sigma : \mathcal{F}^{\otimes k} \to \mathcal{F}^{\otimes k}$ of $\mathcal{O}_X$ modules. If $f_1, \ldots, f_k$ are sections of $\mathcal{F}$ over an affine open subscheme $\mathrm{Spec}(U)$ of X, then

$$\sigma(f_1 \otimes \cdots \otimes f_k) = f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(k)}.$$

This gives us a right action of S_k on $\mathcal{F}^{\otimes k}$. If $\mathcal{F} = \Omega$, the cotangent bundle of X, then $\sigma : \Omega^{\otimes k} \to \Omega^{\otimes k}$ induces a map $\sigma_* : \mathrm{H}^k(X, \Omega^{\otimes k}) \to \mathrm{H}^k(X, \Omega^{\otimes k})$. Extending this action of S_k on $\mathrm{H}^k(X, \Omega^{\otimes k})$ gives us an endomorphism β_* of $\mathrm{H}^k(X, \Omega^{\otimes k})$ corresponding to each element β of the group ring $\mathbb{K}S_k$ of S_k.

To prove Corollary 2, it suffices to show that for l fixed, there exist infinitely many r such that there is some natural number k with the property that there exists an element β of $\mathbb{K}S_k$ such that

$$\beta_*\,\mathrm{t}_k(\alpha_l(Q_{G(r,n)})) \neq 0$$

and

$$\beta_*\,\mathrm{t}_k(\alpha_l(Q_{G(r-1,\infty)})) = 0.$$

Here $\alpha_l(V) = \mathrm{ch}^{-1}\,\mathrm{ch}_l(V)$ for any vector bundle V. This is enough because t_k,α_l and β_* commute with pullbacks. If Corollary 2 were to be violated with the above situation being true, we would have something that is 0 [in this case, $\beta_*\,\mathrm{t}_k(\alpha_l(Q_{G(r-1,\infty)}))$] pulling back to something that is nonzero [in this case, $\beta_*\,\mathrm{t}_k(\alpha_l(Q_{G(r,n)}))$]. This gives us a contradiction. A little more work is required to prove Theorem 2.

1.4.2 Outline for Theorem 1

The proof of Theorem 1 is in the same spirit, though much more complicated. We will define a functor of type (k,l) (or a functor of "Adams weight l") to be a map (not necessarily a ring homomorphism/abelian group homomorphism) from $K(X)\otimes\mathbb{Q}\to \mathrm{R}_k(X)$ which takes an element $x\in K(X)\otimes\mathbb{Q}$ to a linear combination of expressions of the form

$$\beta_*\left(\mathrm{t}_{\lambda_1}(\alpha_{l_1}(x))\cup\cdots\cup \mathrm{t}_{\lambda_s}(\alpha_{l_s}(x))\right)$$

where $\beta\in\mathbb{K}S_k$. If v_l is a functor of type (k,l) then v_l commutes with pullbacks and

$$v_l(\psi^p x)=p^l v_l(x).$$

Corollary 1 will be proven by showing that there is a linear dependence relation

$$\sum_l a_l v_l(Q_{G(r,n)})=0$$

for all $n\geq 2r+1$, with $v_l(Q_{G(r,n)})\neq 0$, where v_l's are functors of type $(2r,l)$. We will pick a linear dependence relation of this type of shortest length. If Corollary 1 is false, we will obtain yet another linear dependence relation $\sum_l p^l a_l v_l(Q_{G(r,n)})=0$, contradicting the fact that the chosen linear dependence relation is of shortest length. A little more work will give us Theorem 1.

Detailed proofs are given in Sections 6 and 7, but the previous sections are required to understand the set up for the proofs. An important ingredient required to flesh-out the proof outlined above is the explicit calculation of $\mathrm{t}_k(Q_{G(r,n)})$. This is done in section 5.

1.5 Remarks about possible future extensions

It can be easily shown that any linear dependence relation between functors of type (k,l) applied to the universal quotient bundle of $G(r,n)$

$$\sum_l a_l v_l(Q_{G(r,n)})=0$$

that holds for all n large enough will apply to a vector bundle of rank r on a smooth projective variety X. Thus, if we are able to prove that we have a linear dependence relation

$$\sum_l a_l v_l(Q_{G(r,n)})=0$$

for all n large enough with $v_l(V) \neq 0$ then we will be able to apply the same argument to show that in K-Theory, higher Adams operations applied to $[V]$ give us elements not expressible as the class of any genuine vector bundle.

One can try doing this for other homogenous vector bundles in the Grassmannian, and in general, other vector bundles on a G/P space arising out of P-representations, where G is a linear reductive group and P is a parabolic subgroup. This could lead to further progress towards finding the P representations that give rise to vector bundles satisfying Theorem 1. More intricate combinatorics than was used here in this paper may be required for further progress along these lines.

At first sight, it may look that theorem 2 needs to be strengthened. Indeed, on going through the proof, one feels strongly that the filtration F_r of $\mathrm{CH}^l() \otimes \mathbb{Q}$, which theorem 2 says is nontrivial as a theory, is in fact, strictly increasing as a theory. More specifically, I feel that given any $l \geq 2$ fixed, and $r \geq 2$, there exists some Grassmannian $G = G(r,n)$ so that $\mathrm{ch}_l(Q) \in F_r\mathrm{CH}^l(G) \otimes \mathbb{Q} \setminus F_{r-1}\mathrm{CH}^l(G) \otimes \mathbb{Q}$.

One approach to this question is entirely combinatorial (along the lines of the proof to theorems 2 and 3). Let V_λ denote the irreducible representation of S_k corresponding to the partition λ of k. Let $|\lambda|$ denote the number of rows in the Young diagram of λ. The combinatorial approach to this question is to try to show that for some k and a *particular* $\beta \in \mathbb{K}S_k$ depending on l and k only, the subspace spanned by the conjugates of β_{r-1} is of strictly smaller dimension than that spanned by conjugates of β_r. Here, β_i is the image of β under the projection $\mathbb{K}S_k \to \oplus_{|\lambda| \leq i} End(V_\lambda)$. Approaching this question along these lines would indeed involve algebraic combinatorics extensively.

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2 The λ-ring $\mathrm{R}(X)$

We recall that a (p,q)-shuffle is a permutation σ of $\{1, 2, \ldots, p+q\}$ such that $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(p+q)$. We denote the set of all

(p,q)-shuffles by $\mathrm{Sh}_{p,q}$ throughout the rest of this work. Also, for the rest of this work, the sign of a permutation σ shall be denoted by $\mathrm{sgn}(\sigma)$.

If $\sigma \in S_k$ is a permutation of $\{1,\dots,k\}$, and $\mathcal{F}$ a vector bundle on X, then σ gives us a homomorphism $\sigma : \mathcal{F}^{\otimes k} \to \mathcal{F}^{\otimes k}$ of $\mathcal{O}_X$ modules. If $f_1,\dots,f_k$ are sections of $\mathcal{F}$ over an affine open subscheme $\mathrm{Spec}(U)$ of X, then $\sigma(f_1 \otimes \cdots \otimes f_k) = f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(k)}$.

This gives us a right action of S_k on $\mathcal{F}^{\otimes k}$. If $\mathcal{F} = \Omega$, the cotangent bundle of X, then $\sigma : \Omega^{\otimes k} \to \Omega^{\otimes k}$ induces a map $\sigma_* : \mathrm{H}^k(X,\Omega^{\otimes k}) \to \mathrm{H}^k(X,\Omega^{\otimes k})$.

If $f : Y \to X$ is a morphism of varieties, we have a natural pullback map $\tilde{f^*} : \mathrm{H}^k(X,{\Omega_X}^{\otimes k}) \to \mathrm{H}^k(Y, f^*{\Omega_X}^{\otimes k})$. This can be composed by the map ${\iota^{\otimes k}}_* : \mathrm{H}^k(Y, f^*{\Omega_X}^{\otimes k}) \to \mathrm{H}^k(Y,{\Omega_Y}^{\otimes k})$ to define the pullback $f^* : \mathrm{H}^k(X,{\Omega_X}^{\otimes k}) \to \mathrm{H}^k(Y,{\Omega_Y}^{\otimes k})$, where $\iota : f^*\Omega_X \to \Omega_Y$. We note that

$$f^* \circ \sigma_* = \sigma_* \circ f^*.$$

If $\alpha \in \mathrm{H}^l(X,{\Omega_X}^{\otimes l})$ and $\beta \in \mathrm{H}^m(X,{\Omega_X}^{\otimes m})$, define

$$\alpha \odot \beta := \sum_{\sigma\in \mathrm{Sh}_{l,m}} \mathrm{sgn}(\sigma)\sigma_*^{-1}(\alpha \cup \beta).$$

$\odot$ gives us a product on $\oplus\, \mathrm{H}^k(X,\Omega^{\otimes k})$. Moreover,

Proposition 1. *If α and β are as in the previous paragraph, then $\alpha\odot\beta = \beta\odot\alpha$. In other words, $\odot$ equips $\mathrm{R}(X)$ with the structure of a commutative ring.*

Proof. If γ is the permutation of $\{1,\dots,k+l\}$ where $\gamma(i) = l+i$ for $1 \le i \le k$ and $\gamma(i) = i-k$ for $k = 1 \le i \le l+k$, then $\mathrm{sgn}(\gamma) = (-1)^{kl}$. Also, $\sigma \to \sigma\circ\gamma$ gives us a bijection between $\mathrm{Sh}_{l,k}$ and $\mathrm{Sh}_{k,l}$.
Thus

$$\alpha \odot \beta = \sum_{\sigma\in \mathrm{Sh}_{k,l}} \mathrm{sgn}(\sigma)\sigma_*^{-1}(\alpha\cup\beta) = \sum_{\tau\in \mathrm{Sh}_{l,k}} \mathrm{sgn}(\gamma)\,\mathrm{sgn}(\tau)(\tau\circ\gamma)_*^{-1}(\alpha\cup\beta)$$

$$= \sum_{\tau\in \mathrm{Sh}_{l,k}} \mathrm{sgn}(\tau)(\gamma^{-1}\circ\tau^{-1})_*\,\mathrm{sgn}(\gamma)(\alpha\cup\beta) = \sum_{\tau\in \mathrm{Sh}_{l,k}} \mathrm{sgn}(\tau)\tau_*^{-1}(\mathrm{sgn}(\gamma)\gamma_*^{-1}(\alpha\cup\beta))$$

$$= \sum_{\tau\in \mathrm{Sh}_{l,k}} \mathrm{sgn}(\tau)\tau_*^{-1}(\beta\cup\alpha) = \beta\odot\alpha.$$

(Note that $(\gamma^{-1}\circ\tau^{-1})_* = \tau_*^{-1}\circ\gamma_*^{-1}$ since the action of S_{k+l} on $\Omega^{\otimes k+l}$ is a right action). ☐

We recall from Fulton and Lang [9] that a special λ-ring A is a commutative ring together with operations $\psi^p : A \to A$ indexed by the natural numbers so that

a) ψ^p is a ring homomorphism for all p.
b) $\psi^p \circ \psi^q = \psi^{pq}$.
c) $\psi^1 = \mathrm{id}$.

Here, we show that $\mathrm{R}(X)$ has a special λ-ring structure (i.e, has Adams operations). This is done in Lemma 2. It will be clear from their definition that the Adams operations commute with pullbacks. *The graded tensor co-algebra $T^*\Omega$ of the cotangent bundle Ω_X is a sheaf of graded-commutative Hopf-algebras on X. The product on $T^*\Omega$ and the Adams operations on $T^*\Omega$ therefore induce corresponding operations on the cohomology ring of $T^*\Omega$. Proposition 1 in fact, proves that the ring $\mathrm{R}(X)$ is a subring of the cohomology ring of $T^*\Omega$. It turns out that the Adams operations on the cohomology of $T^*\Omega$ restrict to Adams operations on $\mathrm{R}(X)$ as well.* The rest of this section is devoted to explaining the details of the outline we have just highlighted. We begin with a digression on Hopf-algebras.

2.1 Adams operations on commutative Hopf-algebras

We recall that a Hopf-algebra over a field $\mathbb{K}$ of characteristic 0 is a vector space H together with maps $\mu : H \otimes H \to H$ (multiplication), $\Delta : H \to H \otimes H$ (comultiplication), $u : \mathbb{K} \to H$ (unit) and $c : H \to \mathbb{K}$ (counit) such that the six properties listed below are satisfied.
1. Multiplication is associative and comultiplication is coassociative.
2. Multiplication is a coalgebra homomorphism and comultiplication is an algebra homomorphism.
3. $\mu \circ (u \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes u) = \mathrm{id} : H \to H$.
4. $(\mathrm{id} \otimes c) \circ \Delta = (c \otimes \mathrm{id}) \circ \Delta = \mathrm{id} : H \to H$.
5. u is a coalgebra map and c is an algebra map.
6. $c \circ u = \mathrm{id} : \mathbb{K} \to \mathbb{K}$.

One can define a Hopf algebra in the category of $\mathcal{O}_X$ modules in the same spirit. It is an $\mathcal{O}_X$ module $\mathcal{H}$ together with maps of $\mathcal{O}_X$ modules $\mu : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ (multiplication), $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ (comultiplication), $u : \mathcal{O}_X \to \mathcal{H}$ (unit) and $c : \mathcal{H} \to \mathcal{O}_X$ (counit) such that
1. Multiplication is associative and comultiplication is coassociative.
2. Multiplication is a coalgebra homomorphism and comultiplication is an algebra homomorphism.
3. $\mu \circ (u \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes u) = \mathrm{id} : \mathcal{H} \to \mathcal{H}$.
4. $(\mathrm{id} \otimes c) \circ \Delta = (c \otimes \mathrm{id}) \circ \Delta = \mathrm{id} : \mathcal{H} \to \mathcal{H}$.
5. u is a coalgebra map and c is an algebra map.
6. $c \circ u = \mathrm{id} : \mathcal{O}_X \to \mathcal{O}_X$.

The Hopf algebra $\mathcal{H}$ is said to be (graded) commutative if $\mu \circ \tau = \mu$ where τ is the (signed) swap map from $\mathcal{H} \otimes \mathcal{H}$ to itself. In the graded case $\tau(a \otimes b) = (-1)^{|a||b|} b \otimes a$, where a and b are homogenous sections of $\mathcal{H}$ over an affine open subset of X. $|a|$ and $|b|$ denote the degrees of a and b respectively.

The following four facts are completely analogous to statements in section 4.5.1 of Loday [2]. The checks Loday [2] asks us to do to make these observations for the case of a commutative Hopf algebra over a field also go through in our case, that of a graded commutative Hopf algebra in the category of $\mathcal{O}_X$ modules. These checks are left to the reader as they are fairly simple.

Fact 1. If $\mathcal{H}$ is a (graded) commutative Hopf algebra in the category of $\mathcal{O}_X$ modules, we can define the convolution of two maps $f, g \in \mathrm{End}_{\mathcal{O}_X}(\mathcal{H})$ by

$$f * g = \mu \circ (f \otimes g) \circ \Delta.$$

The convolution product $*$ is an associative product on $\mathrm{End}_{\mathcal{O}_X}(\mathcal{H})$.

Fact 2. If f is an algebra morphism, then if g and h are any $\mathcal{O}_X$ linear maps,

$$f \circ (g * h) = (f \circ g) * (f \circ h).$$

Fact 3. If $\mathcal{H}$ is (graded) commutative and f and g are algebra morphisms, then $f * g$ is an algebra morphism.

Fact 4. It follows from Fact 3 that

$$\psi^k := \mathrm{id} * \cdots * \mathrm{id} \in \mathrm{End}_{\mathcal{O}_X}(\mathcal{H})$$

is an algebra morphism for all natural numbers k. It also follows from Fact 2 that

$$\psi^p \circ \psi^q = \psi^{pq}$$

for all natural numbers p, q.

Further, the following proposition, which is an extension of Proposition 4.5.3 of Loday [2] to graded commutative Hopf algebras in the category of $\mathcal{O}_X$ modules, holds as well. Since the proof of Proposition 4.5.3 of [2] given by Loday [2] goes through in this case with trivial modifications, we omit the proof of the following proposition.

Proposition 2. *If $\mathcal{H} = \oplus_{n \geq 0} \mathcal{H}_n$ is a (graded) commutative Hopf algebra in the category of $\mathcal{O}_X$ modules, then*
a) ψ^p maps $\mathcal{H}_n$ to itself for all p and n.
b) There exist elements $e_n^{(i)}$ of $\mathrm{End}_{\mathcal{O}_X}(\mathcal{H}_n)$ such that

$$\psi^k = \sum_{i=1}^{n} k^i e_n^{(i)}.$$

Further,

$$e_n^{(i)} \circ e_n^{(j)} = \delta_{ij} e_n^{(i)}$$

where δ_{ij} is the Kronecker delta.

An immediate consequence (when $k = 1$) of this proposition is that

$$\mathrm{id} = e_n^{(1)} + \cdots + e_n^{(n)}.$$

The Hopf algebra that is relevant to us is the (graded) tensor co-algebra of a vector bundle $\mathcal{F}$. Here,

$$T^*(\mathcal{F})_n = \mathcal{F}^{\otimes n}$$

$$\Delta(f_1 \otimes \cdots \otimes f_n) = \sum_{0 \leq i \leq n} f_1 \otimes \cdots \otimes f_i \bigotimes f_{i+1} \otimes \cdots \otimes f_n \in T^*(\mathcal{F}) \otimes T^*(\mathcal{F})$$

(cut coproduct) and

$$\mu(f_1 \otimes \cdots \otimes f_p \bigotimes f_{p+1} \otimes \cdots \otimes f_{p+q}) = \sum_{\sigma \in \mathrm{Sh}_{p,q}} \mathrm{sgn}(\sigma) f_{\sigma^{-1}(1)} \otimes \cdots \otimes f_{\sigma^{-1}(p+q)}$$

where f_i is a section of $\mathcal{F}$ over an affine open subscheme U of X for each i.

We note that in this case,

$$\psi^2(f_1 \otimes \cdots \otimes f_n) = \sum_{p+q=n} \sum_{\sigma \in \mathrm{Sh}_{p,q}} \mathrm{sgn}(\sigma) f_{\sigma^{-1}(1)} \otimes \cdots \otimes f_{\sigma^{-1}(n)}.$$

In this particular case, we also want to find out about the idempotents $e_n^{(i)} \in \mathrm{End}_{\mathcal{O}_X}(\mathcal{F})^{\otimes n}$. The following extension of Proposition 4.5.6 from Loday [2] is what we want. Again, since the proof given in [2] extends with trivial modifications to our case. We therefore, leave the proof of the following proposition to the reader.

Lemma 1.

$$e_n^{(i)} = \sum_{j=1}^{n} a_n^{i,j} l_n^j$$

where

$$\sum_{i=1}^{n} a_n^{i,j} X^i = \binom{X - j + n}{n}$$

and

$$l_n^j = \sum_{\sigma \in S_{n,j}} (\mathrm{sgn}\,\sigma) {\sigma_*}^{-1}.$$

Here, $S_{n,j} = \{\sigma \in S_n | card\{i | \sigma(i) > \sigma(i+1)\} = j - 1\}$.

For example, $e_n^{(n)} = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) {\sigma_*}^{-1}$.

2.2 Description of λ-ring structure on $\mathrm{R}(X)$

Consider the tensor co-algebra $T^*\Omega$. Consider the Adams operations ψ^k on $T^*\Omega$ as described in the previous subsection. Note that $\psi^k|_{\Omega^{\otimes n}}$ induces a map $\psi^k_* : \mathrm{R}_n(X) \to \mathrm{R}_n(X)$. Thus the Adams operation ψ^k induces a map $\psi^k_* : \mathrm{R}(X) \to \mathrm{R}(X)$ that is $\mathbb{K}$-linear. That $\psi^p \circ \psi^q = \psi^{pq}$ implies that $\psi^p_* \circ \psi^q_* = \psi^{pq}_*$. Define the k-th Adams operation on $\oplus \mathrm{H}^n(X, \Omega^{\otimes n})$ to be ψ^k_*. That the Adams operations so defined are ring endomorphisms of $\mathrm{R}(X)$ follows from the fact that the product in $\mathrm{R}(X)$ is induced by the product in $T^*\Omega$. We have therefore, proven the following Lemma.

Lemma 2. $\mathrm{R}(X)$ *is a special λ-ring with Adams operations ψ^p given by ψ^p_*.*

Remark. The Adams operations on $\mathrm{R}(X)$ are thus seen to be defined combinatorially.

3 The ring $\mathrm{R}(G(r,n))^{Gl(n)}$

In this section we explicitly compute an important part of $\mathrm{R}(G(r,n))$, where $G(r,n)$ is the Grassmannian of r dimensional quotients of an n-dimensional vector space. $G(r,n)$ is a homogenous space $Gl(n)/P$ where P is the appropriate parabolic subgroup of $Gl(n)$. Let N denote the unipotent normal subgroup of P.

All the vector bundles that arise during the course of stating and proving the main theorems are $Gl(n)$- equivariant. Thus, the big Chern classes of these vector bundles lie in the part of $\mathrm{R}(G(r,n))$ fixed by $Gl(n)$. If V is an n dimensional vector space, let S be the subspace of V preserved by P and Q the corresponding quotient. The cotangent bundle Ω of the $G(r,n)$ is the vector bundle arising out of the P-representation $Q^* \otimes S$ on which N acts trivially.

Convention. When we refer to Ω in the category of P-representations, we shall refer to the P representation giving rise to the cotangent bundle of $G(r,n)$.

We are now in a position to make the following four observations. Together with the step by step justifications that follow them, these observations describe the method we will use to compute $\mathrm{R}(G(r,n))^{Gl(n)}$ while rigorously justifying our computations at the same time. Observation 1 that follows is a serious statement. We devote the appendix of this paper to sketch its proof. Observations 2 and 3 are first stated "proposition style" and then followed up with proofs. Observation 4 is a sequence of four computations that is crucial to the explicit description of $\mathrm{R}(G(r,n))^{Gl(n)}$ that we provide.

Observation 1. *Let $\mathcal{S}V$ denote the vector bundle on G/P arising out of a P-representation V. Then, $\mathrm{H}^k(G/P, \mathcal{S}V)^G$ is isomorphic to $\mathrm{H}^k(P, V)$.*

Here, $\mathrm{H}^k(P,V)$ is in the category of P-modules. This statement follows from a theorem of Bott [4]. Though the base field is the field of complex numbers in [4], an extension of this result to an arbitrary base field of characteristic 0 can be shown using the method of flat descent [11] (Theorem 6 in the appendix to this paper). We sketch a proof of this fact in the appendix to this paper.

Observation 2. *In the case of a Grassmannian,*

$$\mathrm{H}^k(G/P,\mathcal{S}V)^G \cong \mathrm{H}^k(N,V)^{P/N}.$$

Proof. We have the Lyndon-Hochschild-Serre spectral sequence

$$E_2{}^{pq} = \mathrm{H}^p(P/N;\mathrm{H}^q(N;A)) \implies \mathrm{H}^{p+q}(P;A)$$

where A is any P-representation. In the case of a Grassmannian, P/N is isomorphic to $Gl(Q)\times Gl(S)$. The category of P/N -representations is semisimple, and all but the bottom row of the spectral sequence vanish. Thus in the case of a Grassmannian,

$$\mathrm{H}^k(G/P,\mathcal{S}V)^G \cong \mathrm{H}^k(N,V)^{P/N}.$$

□

Observation 3. *From now on $G = Gl(n)$ and P is a parabolic subgroup such that G/P is the Grassmannian $G(r,n)$. Let $\mathcal{N}$ denote the category of N -representations. For any P-representations V and W on which N acts trivially,*

$$\mathrm{Ext}^k_{\mathcal{N}}(W,V) \cong \mathrm{Hom}_{\mathbb{K}}(W\otimes\wedge^k\Omega,V).$$

Proof. We prove the above assertion as follows.

Step 1: Note that N is a Lie group, and in our case (that of a Grassmannian) the exponential map gives a bijection between the Lie-algebra η associated to N and N itself. The category of (finite dimensional) η representations is thus equivalent to a full subcategory of $\mathcal{N}$ in which all our N representations lie. Note that characteristic 0 is needed to formally define the exponential map and its inverse. Also, the category of η-representations is equivalent to the category of $U(\eta)$-representations, where $U(\eta)$ is the universal enveloping algebra of η. Since η is abelian, (in the case of the Grassmannian) $U(\eta) = \mathrm{Sym}^*\eta$. In what follows, we shall work in the category of $\mathrm{Sym}^*\eta$-modules.

Step 2: Consider the Ad action of P on η. The resulting P representation is the P-representation $Q^*\otimes S$ on which N acts trivially. Since co-tangent bundle

Ω of $G(r,n)$ arises out of this P-representation, we abuse notation and denote this P-representation by Ω. For the rest of this section as well as in Sections 5.2 and 5.3, Ω shall denote this P-representation from which the cotangent bundle of $G(r,n)$ arises. As vector spaces, $\eta \simeq \Omega$. As algebras,

$$U(\eta) \simeq \mathrm{Sym}^*(\Omega).$$

Step 3: Note that $\mathrm{Sym}^*(\Omega)$ acts trivially on W. In other words, $y.w = 0$ for any $w \in W$ and any y in the ideal of $\mathrm{Sym}^*(\Omega)$ generated by Ω. Therefore, a projective $\mathrm{Sym}^*(\Omega)$-module resolution of W can be obtained by taking the Koszul complex

$$\ldots \to W \otimes \wedge^k \Omega \otimes \mathrm{Sym}^* \Omega \to W \otimes \wedge^{k-1} \Omega \otimes \mathrm{Sym}^* \Omega \to \ldots \to W \otimes \mathrm{Sym}^* \Omega \to W \to 0.$$

It follows that if V is any other $\mathrm{Sym}^* \Omega$-module, then $\mathrm{Ext}^k(W,V)$ is just the k-th cohomology of the complex

$$0 \to \mathrm{Hom}(W \otimes \mathrm{Sym}^* \Omega, V) \to \ldots \to \ldots \mathrm{Hom}(W \otimes \wedge^k \Omega \otimes \mathrm{Sym}^* \Omega, V) \to \ldots.$$

If V is also a trivial $\mathrm{Sym}^* \Omega$-module, then we see that

$$\mathrm{Hom}(W \otimes \wedge^k \Omega \otimes \mathrm{Sym}^* \Omega, V) = \mathrm{Hom}_{\mathbb{K}}(W \otimes \wedge^k \Omega, V)$$

and the Koszul differential in the previous complex is 0. Thus,

$$\mathrm{Ext}^k_{\mathcal{N}}(W,V) \cong \mathrm{Hom}_{\mathbb{K}}(W \otimes \wedge^k \Omega, V).$$

□

Observation 4. *$\mathrm{R}(G(r,n))^{Gl(n)}$ is isomorphic to a quotient of the group ring $\mathbb{K}S_k$ as a $\mathbb{K}$-vector space. For the rest of this paper we identify $\mathrm{R}(G(r,n))^{Gl(n)}$ with this quotient via a particular isomorphism. An explicit step by step construction of this isomorphism is provided in paragraphs* a).-d). *below.*

a). It follows from Observation 3, Observation 2 and the fact that $P/N \cong Gl(Q) \times Gl(S)$ that

$$\mathrm{H}^k(G(r,n), \Omega^{\otimes k})^{Gl(n)} \cong \mathrm{Hom}_{\mathbb{K}}(\wedge^k \Omega, \Omega^{\otimes k})^{Gl(Q)\times Gl(S)}.$$

We recall from Weyl [10] that if V is any vector space, the map

$$\varphi_V : \mathbb{K}S_k \to \mathrm{End}_{\mathbb{K}}(V^{\otimes k})^{Gl(V)}$$

$$v_1 \otimes \cdots \otimes v_n \mapsto v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

is a surjection. It follows from this that

$$\varphi_{Q^*} \otimes \varphi_S : \mathbb{K}S_k \otimes \mathbb{K}S_k \to (\mathrm{End}_{\mathbb{K}}(Q^{*\otimes k}) \otimes \mathrm{End}_{\mathbb{K}}(S^{\otimes k}))^{Gl(Q)\times Gl(S)}$$

is surjective.

b). Let $i : \wedge^k\Omega \to \Omega^{\otimes k}$ denote the standard inclusion. Let $p : \Omega^{\otimes k} \to \Omega^{\otimes k}$ denote standard projection onto the image of i. Note that $p = \frac{1}{k!}\sum_{\omega\in S_k}\operatorname{sgn}(\omega)\omega$. If $\alpha \in (\operatorname{End}_{\mathbb{K}}(Q^{*\otimes k}) \otimes \operatorname{End}_{\mathbb{K}}(S^{\otimes k}))^{Gl(Q)\times Gl(S)}$, then $\alpha \circ i = 0$ iff $\alpha \circ p = 0$. Recall that $\Omega \cong Q^* \otimes S$. Therefore, every element in $\operatorname{Hom}_{\mathbb{K}}(\wedge^k\Omega, \Omega^{\otimes k})^{Gl(Q)\times Gl(S)}$ is the image of a linear combination of elements of the form

$$(\tau \otimes \sigma) \circ \frac{1}{k!}\sum_{\omega\in S_k}\operatorname{sgn}(\omega)(\omega \otimes \omega).$$

Also, since we are using the right action of $S_k \times S_k$ on $Q^{*\otimes k} \otimes S^{\otimes k}$,

$$\begin{aligned}(\tau \otimes \sigma) \circ \frac{1}{k!}\sum_{\omega\in S_k}\operatorname{sgn}(\omega)(\omega \otimes \omega) &= \frac{1}{k!}\sum_{\omega\in S_k}\operatorname{sgn}(\omega)(\omega \otimes \omega)(\tau \otimes \sigma)\\ &= \frac{1}{k!}\sum_{\omega\in S_k}\operatorname{sgn}(\omega\sigma)\operatorname{sgn}(\sigma^{-1})(\omega\sigma \otimes \omega\sigma)(\sigma^{-1}\tau \otimes \mathrm{id})\\ &= \frac{1}{k!}\operatorname{sgn}(\sigma)\sum_{\omega\in S_k}\operatorname{sgn}(\omega)(\omega \otimes \omega)(\sigma^{-1}\tau \otimes \mathrm{id}).\end{aligned}$$

c). Identify $\operatorname{End}_{\mathbb{K}}(\Omega^{\otimes k})$ with $(\operatorname{End}_{\mathbb{K}}(Q^{*\otimes k}) \otimes \operatorname{End}_{\mathbb{K}}(S^{\otimes k}))$ and think of $S_k \times S_k$ as acting on this with the left copy of S_k permuting the Q^* and the right copy permuting the S. Then, the map p is identified with $\frac{1}{k!}\sum_{\omega\in S_k}\operatorname{sgn}(\omega)(\omega \otimes \omega)$. It follows from the above computation that if $\sigma, \tau \in S_k$ then

$$(\sigma \otimes \tau) \circ p = \operatorname{sgn}(\sigma)(\sigma^{-1}\tau \otimes \mathrm{id}) \circ p.$$

Therefore, every element in $\operatorname{Hom}_{\mathbb{K}}(\wedge^k\Omega, \Omega^{\otimes k})^{Gl(Q)\times Gl(S)}$ is the image of a linear combination of elements of the form

$$(\sigma^{-1}\tau \otimes \mathrm{id}) \circ p.$$

It follows that as a $\mathbb{K}$-vector space, $\operatorname{Hom}_{\mathbb{K}}(\wedge^k\Omega, \Omega^{\otimes k})^{Gl(Q)\times Gl(S)}$ can be identified with a quotient of the group ring $\mathbb{K}S_k$. We shall shortly determine this quotient precisely – but not before making a final computation.

d). Identify Ω with $Q^* \otimes S$. With this identification, if $\sigma \in S_k$, the right action of σ on $\Omega^{\otimes k}$ corresponds to the right action of $\sigma \otimes \sigma$ on $Q^{*\otimes k} \otimes S^{\otimes k}$. Also, if $\beta \in \mathbb{K}S_k$, then

$$\begin{aligned}&\frac{1}{k!}\sum_{\omega\in S_k}\operatorname{sgn}(\omega)(\omega \otimes \omega)(\beta \otimes \mathrm{id})(\sigma \otimes \sigma)\\ &\qquad= \frac{1}{k!}\sum_{\omega\in S_k}\operatorname{sgn}(\omega\sigma)\operatorname{sgn}(\sigma)(\omega\sigma \otimes \omega\sigma)(\sigma^{-1}\beta\sigma \otimes \mathrm{id})\end{aligned}$$

$$= \frac{1}{k!}\,\mathrm{sgn}(\sigma)\sum_{\omega\in S_k}\mathrm{sgn}(\omega)(\omega\otimes\omega)(\sigma^{-1}\beta\sigma\otimes\mathrm{id}).$$

The main result of this section. Henceforth $\mathrm{B}(G(r,n))$ shall denote $\mathrm{R}(G(r,n))^{Gl(n)}$. Observations 1-4 above enable us to conclude that $\mathrm{B}(G(r,n))$ is isomorphic to a quotient of $\mathbb{K}S_k$ as a $\mathbb{K}$-vector space.

We need to specify which quotient of $\mathbb{K}S_k$ gives $\mathrm{B}(G(r,n))$. Recall that the irreducible representations of S_k over $\mathbb{C}$ can be realized over $\mathbb{Q}$ and hence over any field of characteristic 0. We also recall that the irreducible representations of S_k are indexed by partitions λ of k. They are self-dual, and $V_\lambda\otimes Alt = V_{\bar\lambda}$, where $\bar\lambda$ is the partition conjugate to λ. Note that $\mathbb{K}S_k$ is isomorphic to $\oplus_\lambda \mathrm{End}(V_\lambda)$.

Notation. Let $|\lambda|$ denote the rank (number of summands) of the partition λ. Let P_r denote the projection from $\mathbb{K}S_k$ to $\oplus_{|\lambda|\le r}\mathrm{End}(V_\lambda)$ for $1\le r\le k$, and let $P_{r,n}$ denote the projection from $\mathbb{K}S_k$ to $\oplus_{|\lambda|\le r,|\bar\lambda|\le n-r}\mathrm{End}(V_\lambda)$. If n is large enough, $P_{r,n}=P_r$.

The main result in this section is the following.

Lemma 3. *1. As a vector space,*

$$\mathrm{B}(G(r,n))\cong\oplus_k P_{r,n}(\mathbb{K}S_k).$$

2. If $\sigma\in S_k$ then

$$\sigma_*P_{r,n}(\beta)=P_{r,n}(\mathrm{sgn}(\sigma)\sigma^{-1}\beta\sigma)\ \forall\ \beta\in\mathbb{K}S_k.$$

3. If $\alpha\in S_k$ and $\beta\in S_l$ then

$$P_{r,n}(\alpha)\cup P_{r,n}(\beta)=P_{r,n}(\alpha\times\beta)$$

where $\alpha\times\beta$ is thought of as an element of S_{k+l} in the obvious fashion.

The second part of this lemma follows from the paragraph d). of Observation 4 in this subsection. The following sequence of lemmas proves the remaining parts of the above lemma.

3.1 A lemma and some corollaries

Lemma 4. *Let G be a finite group and let $\chi: G\to\mathbb{C}^*$ be a 1-dimensional representation of G. Then, if $\beta\in\mathbb{C}(G)$, $\sum_{g\in G}\chi(g)(g\otimes g)(\beta\otimes\mathrm{id})=0$ in $\mathbb{C}(G\times G)=\mathbb{C}(G)\otimes\mathbb{C}(G)$ iff $\beta=0$.*

Proof. If $\beta = 0$ then clearly $\sum_{g \in G} \chi(g)(g \otimes g)(\beta \otimes \mathrm{id}) = 0$. For the implication in the opposite direction, let us see what $\sum_{g \in G} \chi(g)(g \otimes g)$ does to $\mathbb{C}(G \times G) = \oplus \operatorname{End}(V_x \otimes V_y)$ where the V_x are the irreducible representations of G. Let e_i be a basis for V_x and let f_j be a basis of V_y. Suppose that $g(e_i) = \sum_{j=1}^{j=\dim(V_x)} g_{ij}^x e_j$ and that $g(f_k) = \sum_{l=1}^{l=\dim(V_y)} g_{kl}^y f_l$ for all $i \in \{1, \dots, \dim(V_x)\}$ and for all $k \in \{1, \dots, \dim(V_y)\}$. Then,

$$\sum_g \chi(g)(g \otimes g)(e_i \otimes f_j) = \sum_g \sum_{k,l} g_{ik}{}^x g_{jl}{}^y \chi(g)(e_k \otimes f_l)$$

$$= \sum_{k,l} (e_k \otimes f_l)(\sum_g \chi(g) g_{ik}{}^x g_{jl}{}^y) = \sum_{k,l} (e_k \otimes f_l)(\sum_g g_{ik}{}^z g_{jl}{}^y)$$

where $V_z = V_x \otimes \chi$.

Note that $\sum_g (g \otimes g) \in \operatorname{End}(V_z \otimes V_y)$ is a G-module homomorphism. In fact, G acts trivially on $(\sum_g g \otimes g).(V_z \otimes V_y)$. Thus, $\frac{1}{|G|} \sum_g (g \otimes g)$ acts as a projection to the trivial part of $V_z \otimes V_y$. Note that $V_z \otimes V_y$ has a contains precisely $\langle \chi_z, \bar{\chi}_y \rangle$ copies of the trivial representation of G. In particular, it contains one copy of the trivial representation of G iff V_z and V_y are dual representations. In that case, the projection to that copy of the trivial representation is given by $v \otimes w \mapsto \frac{1}{d} w(v) \sum e_i \otimes f_i$ where d is the dimension of V_z. Here, $\{e_i\}$ is a basis for V_z and $\{f_i\}$ is the basis for V_y dual to $\{e_i\}$. This tells us that $\sum_g g_{ik}{}^z g_{jl}{}^y = \frac{|G|}{d} \delta_{y\bar{z}} \delta_{ij} \delta_{kl}$.

Therefore, in $\operatorname{End}(V_x \otimes V_y)$, if V_z is not dual to V_y, then $\sum_{g \in G} \chi(g)(g \otimes g) = 0$. Assume that V_z is dual to V_y. Let $\{e_i\}$ be a basis for V_z and let $\{f_i\}$ be the basis of V_y dual to $\{e_i\}$. If $\{\tilde{e}_i\}$ is the basis of V_x corresponding to $\{e_i\}$, then with respect to the ordered basis $\tilde{e}_1 \otimes f_1, \tilde{e}_2 \otimes f_1, \dots, \tilde{e}_d \otimes f_1, \tilde{e}_1 \otimes f_2, \dots, \tilde{e}_d \otimes f_2, \dots, \tilde{e}_1 \otimes f_d, \dots, \tilde{e}_d \otimes f_d$ of $V_x \otimes V_y$, $\frac{d}{|G|} \sum_{g \in G} \chi(g)(g \otimes g)$ corresponds to the matrix M such that $M_{ij} = 1$ if $i, j \in \{kd + k + 1 | 0 \leq k \leq d - 1\}$ and $M_{ij} = 0$ otherwise. On the other hand, $\beta \otimes \mathrm{id}$ in $\operatorname{End}(V_x \otimes V_y)$ is given by a block diagonal matrix each of whose diagonal blocks is the matrix representing β in $\operatorname{End}(V_x)$. This proves the desired lemma.

□

In fact, in the above proof, we have also proven the following lemma.

Lemma 5. *Let G be a finite group, and let $\chi : G \to \mathbb{C}^*$ be a 1-dimensional representation of G. Let V_x and V_y be irreducible representations of G such that $V_x \otimes \chi$ is dual to V_y. Then, if $\beta \in \mathbb{C}(G)$, $\sum_{g \in G} \chi(g)(g \otimes g)(\beta \otimes \mathrm{id}) = 0$ in $\operatorname{End}(V_x \otimes V_y)$ iff $\beta = 0$ in $\operatorname{End}(V_x)$.*

In our problem, the group in question is S_k. We note that these lemmas give us the precise description of $\mathrm{B}(G(r, n)$ when $\mathbb{K} = \mathbb{C}$. Let $\mathbb{S}_\lambda$ denote the Schur-functor associated with the partition λ of k. In other words, if V is any vector space $\mathbb{S}_\lambda(V) = V^{\otimes k} \otimes_{\mathbb{K}S_k} V_\lambda$ where V_λ is the irreducible representation of S_k corresponding to the partition λ. We know that if V is a vector space of rank m, $\mathbb{S}_\lambda(V) = 0$ iff λ has more than m parts. Therefore, if Q has rank r, then $\mathbb{S}_\lambda(Q) = 0$ iff $|\lambda| > r$ and $\mathbb{S}_{\bar{\lambda}}(S) = 0$ iff $|\bar{\lambda}| > n-r$. Moreover, if λ and μ are two partitions of k, then $V^{\otimes k} \otimes W^{\otimes k} \otimes_{\mathbb{K}(S_k \times S_k)} V_\lambda \otimes V_\mu = \mathbb{S}_\lambda(V) \otimes \mathbb{S}_\mu(W)$. If $\gamma \in \mathbb{K}(S_k \times S_k) \neq 0$ in $\mathrm{End}(V_\lambda \otimes V_\mu)$, then $\mathbb{K}(S_k \times S_k).\gamma$ contains $V_\lambda \otimes V_\mu$. Therefore, $V^{\otimes k} \otimes W^{\otimes k} \otimes_{\mathbb{K}(S_k \times S_k)} \gamma$ contains $\mathbb{S}_\lambda(V) \otimes \mathbb{S}_\mu(W)$. Lemma 5 therefore says the following when $\mathbb{K} = \mathbb{C}$.

Lemma 6. *If the rank of Q is r and that of S is $n-r$, then*

$$\sum_\sigma \mathrm{sgn}(\sigma)(\sigma \otimes \sigma)(\beta \otimes \mathrm{id}) = 0$$

as an element of $\mathrm{Hom}_{\mathbb{K}}(\Omega^{\otimes k}, \Omega^{\otimes k})$ *iff* $\beta = 0$ *as an element of* $\mathrm{End}(V_\lambda)$ *for all partitions λ such that $|\lambda| \leq r$ and $|\bar{\lambda}| \leq n-r$.*

Proof. Let $\gamma = \sum_\sigma \mathrm{sgn}(\sigma)(\sigma \otimes \sigma)(\beta \otimes \mathrm{id})$. Then, by Lemma 5, $\gamma = 0$ in $\mathrm{End}(V_\lambda \otimes V_\mu)$ if $\mu \neq \bar{\lambda}$. Therefore, γ kills $\mathbb{S}_\lambda(Q^*) \otimes \mathbb{S}_\mu(S)$ whenever $\mu \neq \bar{\lambda}$. On the other hand, if $\gamma \neq 0$ in $\mathrm{End}(V_\lambda \otimes V_{\bar{\lambda}})$, then $\Omega^{\otimes k}.\gamma$ contains a copy of $\mathbb{S}_\lambda(Q^*) \otimes \mathbb{S}_{\bar{\lambda}}(S)$. The desired lemma follows immediately.

□

Since the irreducible representations of S_k over $\mathbb{C}$ can be realized over $\mathbb{Q}$ and hence over any field of characteristic 0,lemmas 4,5 and 6 thus hold for $\mathbb{K}S_k$ where $\mathbb{K}$ is *any* field of characteristic 0. *This proves the first part of Lemma 3 specifying the vector space structure of* $\mathrm{B}(G(r,n))$. We have so far also identified the right S_k module structure of $\mathrm{B}(G(r,n))$. To describe the ring structure completely, we need to be able to compute cup products explicitly under this identification.

We now show how one computes the cup product of two elements $X_k \in \mathrm{Hom}_{\mathbb{K}}(\wedge^k\Omega, \Omega^{\otimes k}) \subset \mathrm{H}^k(G(r,n), \Omega^{\otimes k})$ and $Y_l \in \mathrm{Hom}_{\mathbb{K}}(\wedge^l\Omega, \Omega^{\otimes l}) \subset \mathrm{H}^l(G(r,n), \Omega^{\otimes l})$. Let $X_k = (\gamma_k \otimes \mathrm{id}) \circ i_k \in \mathrm{End}_{\mathbb{K}}(Q^{*\otimes k}) \otimes \mathrm{End}_{\mathbb{K}}(S^{\otimes k})$ and $Y_l = (\delta_l \otimes \mathrm{id}) \circ i_l \in \mathrm{End}_{\mathbb{K}}(Q^{*\otimes l}) \otimes \mathrm{End}_{\mathbb{K}}(S^{\otimes l})$ where i_k and i_l are the standard inclusions $\wedge^k\Omega \to \Omega^{\otimes k}$ and $\wedge^l\Omega \to \Omega^{\otimes l}$ respectively. $\mathrm{End}_{\mathbb{K}}(\Omega^{\otimes *})$ is identified with $\mathrm{End}_{\mathbb{K}}(Q^{*\otimes *}) \otimes \mathrm{End}_{\mathbb{K}}(S^{\otimes *})$ as usual. The following Lemma explicitly computes $X_k \cup Y_l$.

Lemma 7.

$$[(\gamma_k \otimes \mathrm{id}) \circ i_k] \cup [(\delta_l \otimes \mathrm{id}) \circ i_l] = [((\gamma_k \otimes \delta_l) \otimes \mathrm{id}) \circ i_{k+l}].$$

The element $(\gamma_k \otimes \delta_l) \in \mathbb{K}(S_k \times S_l) \subset \mathbb{K}(S_{k+l})$ where $S_k \times S_l$ is embedded in S_{k+l} in the natural way.

Before proving this Lemma, we note that *part 3 of Lemma 3 follows immediately from the above lemma.*

Proof. Let W be any $\mathbb{K}$-vector space $\operatorname{Sym}^* \Omega$ acts trivially. In other words, $y.w = 0$ for any $w \in W$ and any y in the ideal of $\operatorname{Sym}^*(\Omega)$ generated by Ω. Let $\phi \in \operatorname{End}(W)$. Let $\bar{\phi} : W \otimes \operatorname{Sym}^*(\Omega) \to W$ denote the map $\phi \otimes \eta$ where $\eta : \operatorname{Sym}^* \Omega \to \mathbb{K}$ canonical map from $\operatorname{Sym}^* \Omega$ to its quotient by the ideal generated by Ω.

Let $\alpha_j : \Omega^{\otimes j} \otimes \operatorname{Sym}^*(\Omega) \to \Omega^{\otimes j} \otimes \operatorname{Sym}^*(\Omega)$ denote the map

$$\omega_1 \otimes \cdots \otimes \omega_j \bigotimes Y \mapsto \omega_1 \otimes \cdots \otimes \omega_{j-1} \bigotimes \omega_j Y$$

for $\omega_1, \dots, \omega_j \in \Omega$ and $Y \in \operatorname{Sym}^*(\Omega)$.

Let $d : \wedge^j \Omega \otimes \operatorname{Sym}^*(\Omega) \to \wedge^{j-1} \Omega \otimes \operatorname{Sym}^*(\Omega)$ denote the Koszul differential.

Note that the following diagram commutes.

$$\begin{tikzcd}
\Omega^{\otimes j} \otimes \operatorname{Sym}^*(\Omega) \arrow[r, "\alpha_j"] \arrow[d, "i_j \otimes \mathrm{id}_{\operatorname{Sym}^*(\Omega)}"'] & \Omega^{\otimes j-1} \otimes \operatorname{Sym}^*(\Omega) \arrow[d, "i_{j-1} \otimes \mathrm{id}_{\operatorname{Sym}^*(\Omega)}"] \\
\wedge^j \Omega \otimes \operatorname{Sym}^*(\Omega) \arrow[r, "d"] & \wedge^{j-1} \Omega \otimes \operatorname{Sym}^*(\Omega)
\end{tikzcd}$$

We have the following commutative diagrams.

$$\begin{tikzcd}
0 \arrow[r] & \Omega^{\otimes k} \arrow[r] & Z_1 \arrow[r] & \dots \\
\dots \arrow[r] & \Omega^{\otimes k} \otimes \operatorname{Sym}^* \Omega \arrow[r, "\alpha_k"] \arrow[u, "\bar{\gamma}_k"] & \Omega^{\otimes k-1} \otimes \operatorname{Sym}^* \Omega \arrow[r] \arrow[u, "\theta_1"] & \dots \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
\dots \arrow[r] & Z_k \arrow[r] & \mathbb{K} \arrow[r] & 0 \\
\dots \arrow[r] \arrow[u] & \operatorname{Sym}^* \Omega \arrow[r] \arrow[u] & \mathbb{K} \arrow[r] \arrow[u, "\mathrm{id}"'] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \Omega^{\otimes l} \arrow[r] & W_l \arrow[r] & \dots \\
\dots \arrow[r] & \Omega^{\otimes l} \otimes \operatorname{Sym}^* \Omega \arrow[r, "\alpha_l"] \arrow[u, "\bar{\delta}_l"] & \Omega^{\otimes l-1} \otimes \operatorname{Sym}^* \Omega \arrow[r] \arrow[u, "\theta_2"] & \dots \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
\dots \arrow[r] & W_l \arrow[r] & \mathbb{K} \arrow[r] & 0 \\
\dots \arrow[r] \arrow[u] & \operatorname{Sym}^* \Omega \arrow[r] \arrow[u] & \mathbb{K} \arrow[r] \arrow[u, "\mathrm{id}"'] & 0
\end{tikzcd}$$

The top rows of the two commutative diagrams are exact sequences representing X_k and Y_l respectively. To compute the cup product $X_k \cup Y_l$ we only need to find vertical arrows making all squares in the following diagram commute.

$$\begin{tikzcd}
0 \arrow[r] & \Omega^{\otimes k+l} \arrow[r] & Z_1 \otimes \Omega^{\otimes l} \arrow[r] & \dots \\
\dots \arrow[r] & \Omega^{\otimes k+l} \otimes \mathrm{Sym}^* \Omega \arrow[r, "\alpha_{k+l}"] \arrow[u] & \Omega^{\otimes k+l-1} \otimes \mathrm{Sym}^* \Omega \arrow[r] \arrow[u] & \dots \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
\dots \arrow[r] & Z_k \otimes \Omega^{\otimes l} \arrow[r] & W_1 \arrow[r] & \dots \\
\dots \arrow[r] \arrow[u] & \Omega^{\otimes l} \otimes \mathrm{Sym}^* \Omega \arrow[r, "\alpha_l"'] \arrow[u] & \Omega^{\otimes l-1} \otimes \mathrm{Sym}^* \Omega \arrow[r] \arrow[u, "\theta_2"'] & \dots \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
\dots \arrow[r] & W_l \arrow[r] & \mathbb{K} \arrow[r] & 0 \\
\dots \arrow[r] \arrow[u] & \mathrm{Sym}^* \Omega \arrow[r] \arrow[u] & \mathbb{K} \arrow[r] \arrow[u, "\mathrm{id}"'] & 0
\end{tikzcd}$$

Note that the diagrams below commute.

$$\begin{tikzcd}
0 \arrow[r] & \Omega^{\otimes k+l} \arrow[r] & Z_1 \otimes \Omega^{\otimes l} \arrow[r] & \dots \\
\dots \arrow[r] & \Omega^{\otimes k+l} \otimes \mathrm{Sym}^* \Omega \arrow[r, "\alpha_{k+l}"] \arrow[u, "\gamma_k \bar{\otimes} \delta_l"] & \Omega^{\otimes k+l-1} \otimes \mathrm{Sym}^* \Omega \arrow[r] \arrow[u, "\theta_1 \otimes \delta_l"] & \dots \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
\dots \arrow[r] & Z_k \otimes \Omega^{\otimes l} \\
\dots \arrow[r] \arrow[u] & \Omega^{\otimes l} \otimes \mathrm{Sym}^* \Omega \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
Z_k \otimes \Omega^{\otimes l} \arrow[r] & \Omega^{\otimes l} \\
\Omega^{\otimes l} \otimes \mathrm{Sym}^* \Omega \arrow[r] \arrow[u, "- \otimes \delta_l"] & \Omega^{\otimes l} \arrow[u, "\delta_l"']
\end{tikzcd}$$

These diagrams prove the desired lemma.

□

3.2 An example.

Lemma 3 tells us that if $X = G(\infty, \infty) = \varinjlim G(r, \infty)$ then $\mathrm{R}(X) = \oplus_k \mathbb{K} S_k$ with $\sigma_* \alpha = \mathrm{sgn}(\sigma) \sigma^{-1} \alpha \sigma$ for all $\sigma \in S_k, \alpha \in \mathbb{K} S_k$. Thus, by Lemma 3 and

Proposition 1, if $\alpha \in S_k$, and $\beta \in S_l$, then $\alpha \odot \beta = \sum_{\sigma \in \mathrm{Sh}_{k,l}} \sigma(\alpha \times \beta)\sigma^{-1}$. In other words, $\mathrm{R}(X)$ is the commutative algebra generated by symbols x_γ for all $\gamma \in S_k$, for all k modulo the relations $x_\alpha x_\beta = \sum_{\sigma \in \mathrm{Sh}_{k,l}} x_{\sigma(\alpha\times\beta)\sigma^{-1}}$. This can be seen to be larger than the usual cohomology ring of this space.

4 The big Chern Classes t_k and a ring homomorphism from $K(X) \otimes \mathbb{Q}$ to $\mathrm{R}(X)$

Let V be a locally free coherent sheaf on a scheme X/S with X smooth over S. An algebraic connection on V is defined as an $\mathcal{O}_S$ linear sheaf homomorphism $D : V \to \Omega_{X/S} \bigotimes_{\mathcal{O}_X} V$ satisfying the Leibniz rule, i.e,

$$D(fv) = df \otimes v + fDv \ \forall \ f \in \Gamma(U, \mathcal{O}_X) \ , \ v \in \Gamma(U, V),$$

for every U open in X. Note that a connection on V by itself is not $\mathcal{O}_X$ linear. However, if D_1 and D_2 are two connections on $V|_U$ with $U \subseteq X$ open, then $D_1 - D_2 \in \Gamma(U, \mathrm{End}(V) \otimes \Omega_{X/S})$.

For each open $U \subseteq X$, let $C_V(U)$ denote the set of connections on $V|_U$. This gives us a sheaf of sets on X on which $\mathrm{End}(V) \otimes_{\mathcal{O}_X} \Omega_{X/S}$ acts simply transitively. Consider a covering of X by open affines U_i such that V is trivial on U_i, and pick an element $D_i \in C_V(U_i) \ \forall \ i$ (D_i exists as $d^n : \mathcal{O}_X^n \to \Omega_X^n$ is a connection and thus gives a connection on $V|_{U_i} \cong \mathcal{O}_X^n$, where n is the rank of V). The D_i together give rise to a well defined element $\theta_V \in \mathrm{H}^1(X, \mathrm{End}(V) \otimes \Omega)$.

Lemma 8. *$\theta_{V\otimes W} = A_V + B_W$, where A_V and B_W are the elements in $\mathrm{H}^1(X, \mathrm{End}(V) \otimes \mathrm{End}(W) \otimes \Omega)$ induced from θ_V and θ_W respectively by the maps $\mathrm{End}(V) \to \mathrm{End}(V) \otimes \mathrm{End}(W)$ ($m \mapsto m \otimes \mathrm{id}_W$) and $\mathrm{End}(W) \to \mathrm{End}(V) \otimes \mathrm{End}(W)$, ($m' \mapsto \mathrm{id}_V \otimes m'$) respectively.*

Corollary 4. *$\theta_{V\otimes V}$ is induced from θ_V by the map $\mathrm{End}(V) \to \mathrm{End}(V) \otimes \mathrm{End}(V)$, ($m \mapsto m \otimes \mathrm{id}_V + \mathrm{id}_V \otimes m$).*

Proof. Since V and W are locally free, we can cover X by open sets U_i so that V and W are free over U_i for each i. Let $D_i \in C_V(U_i)$, and $E_i \in C_W(U_i)$ for each i. The desired result follows from the fact that $\mathrm{id}_V \otimes E_i + D_i \otimes \mathrm{id}_W \in C_{(V\otimes W)}(U_i)$. $\square$

4.1 The big Chern Classes t_k

Given any two locally free coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on X, one has a cup product $\cup : \mathrm{H}^i(X, \mathcal{F}) \otimes \mathrm{H}^j(X, \mathcal{G}) \to \mathrm{H}^{i+j}(X, \mathcal{F} \otimes \mathcal{G})$. Hence, we can consider the cup product of θ_V with itself k times -

$$\theta_V \cup \cdots \cup \theta_V =: \theta_V^k \in \mathrm{H}^k(X, \mathrm{End}(V)^{\otimes k} \bigotimes \Omega^{\otimes k}).$$

The composition map $\varphi : \mathrm{End}(V)^{\otimes k} \to \mathrm{End}(V)$ induces a map $\varphi_* : \mathrm{H}^k(X, \mathrm{End}(V)^{\otimes k} \otimes \Omega^{\otimes k}) \to \mathrm{H}^k(X, \mathrm{End}(V) \otimes \Omega^{\otimes k})$. Let $\widetilde{\mathrm{t}_k(V)} := \varphi_* \theta_V^k$. The trace map $tr : \mathrm{End}(V) \to \mathcal{O}_X$ is $\mathcal{O}_X$ -linear and induces $tr_* : \mathrm{H}^k(X, \mathrm{End}(V) \otimes \Omega^{\otimes k}) \to \mathrm{H}^k(X, \Omega^{\otimes k})$. By definition, $\mathrm{t}_k(V) := tr_* \widetilde{\mathrm{t}_k(V)}$. The classes t_k are referred to in Kapranov [6] as the *big Chern classes*. The projection $\Omega^{\otimes k} \to \wedge^k \Omega$ when applied to $\mathrm{t}_k(V)$ gives us $k!\,\mathrm{ch}_k(V)$ where $\mathrm{ch}_k(V)$ is the degree k part of the Chern character of V. The appropriate reference for the construction of the Atiyah class and the construction of the components of the Chern character as done here is Atiyah [12].

4.2 Basic properties of the big Chern classes

Firstly, t_k is a characteristic class. In other words,

Lemma 9. *If $0 \to V' \to V \to V'' \to 0$ is an exact sequence of locally free coherent sheaves on X, then $\mathrm{t}_k(V) = \mathrm{t}_k(V') + \mathrm{t}_k(V'')$.*

Proof. Let V,V' and V'' be as in the statement of this lemma. We first prove this lemma for the case when k = 1. Consider a cover of X by affine open sets U_i such that V and V' are trivial over the U_i. On each U_i, choose a connection D_i, so that the restriction $D_i|V'$ of D_i to V' is a connection on V'. In other words, $D_i(\Gamma(U_i, V')) \subset \Gamma(U_i, \Omega \otimes V')$. On the other hand,for each $U \subset X$ open, one can consider the $\mathbb{K}$-vector space $C_{V,V'}(U)$ of connections on $V|U$ that give rise to a connection on $V'|U$. Note that the difference between any two elements of $C_{V,V'}(U)$ is an element of $\Gamma(U, \mathcal{P} \otimes \Omega)$, which acts simply transitively on $C_{V,V'}(U)$. Here, $\mathcal{P}$ is the subsheaf of sections of $\mathrm{End}(V)$ that preserve V'.

Let $C_V(U_i)$ denote the space of connections on $V|U_i$. Thinking of the $\Pi_i D_i$ as an element of $\Pi_i C_V(U_i)$ we see that the Cech 1-cocycle $\Pi_{i<j}(D_i - D_j)$ of $\Pi_{i<j}\Gamma(U_i \cap U_j, \mathrm{End}(V) \otimes \Omega)$ yields the Atiyah class θ_V of V in $\mathrm{H}^1(X, \mathrm{End}(V) \otimes \Omega)$. On the other hand, when the D_i are thought of as elements of $C_{V,V'}(U_i)$, they similarly give rise to an element $\theta_{V,V'}$ of $\mathrm{H}^1(X, \mathcal{P} \otimes \Omega)$. If $i : P \to \mathrm{End}(V)$ is the natural inclusion, then clearly, $(i \otimes id)_* \theta_{V,V'} = \theta_V$. We shall denote $(i \otimes id)$ by i henceforth. Note that $tr \circ i = tr$. Hence, $tr_* \theta_{V,V'} = tr_* \theta_V = t_1(V)$. On the other hand, restriction to V' gives us a map $p_1 : P \to \mathrm{End}(V')$. Then $p_{1*}\theta_{V,V'}$ is the cohomology class obtained by looking at $D_i|V'$ as elements of $C_{V'}(Ui)$ which is $\theta_{V'}$. We also have a projection $p_2 : P \to \mathrm{End}(V'')$. Note that since the D_i are connections on V that restrict to connections on V', they induce connections on V'' (all restricted to U_i) which we will again denote by D_i. Note that $p_{2*}\theta_{V,V'}$ is the cohomology class obtained by thinking of D_i as elements of $C_{V''}(U_i)$, i.e, $\theta_{V''}$. Now, $tr|_{\mathcal{P}} = tr \circ p_1 + tr \circ p_2$. This proves the lemma for k = 1.

Let $\theta^k_{V,V'} := \theta_{V,V'} \cup \cdots \cup \theta_{V,V'} \in \mathrm{H}^k(X, \mathcal{P}^{\otimes k} \bigotimes \Omega^{\otimes k})$. Let $\varphi : \mathcal{P}^{\otimes k} \to \mathcal{P}$ denote the composition map. Let $\widetilde{\mathrm{t}_k(V,V')} := \varphi_* \theta^k(V,V') \in \mathrm{H}^k(X, \mathcal{P} \otimes \Omega^{\otimes k})$. The following observations prove the lemma in general.

1. $i_* \widetilde{\mathrm{t}_k(V,V')} = \widetilde{\mathrm{t}_k(V)}$. This follows from the commutativity of the following diagram.

$$\begin{array}{ccc}
\mathcal{P}^{\otimes k} & \xrightarrow{i^{\otimes k}} & \mathrm{End}(V)^{\otimes k} \\
\Big\downarrow \varphi & & \varphi \Big\downarrow \\
\mathcal{P} & \xrightarrow{i} & \mathrm{End}(V)
\end{array}$$

2. $p_{1*} \widetilde{\mathrm{t}_k(V,V')} = \widetilde{\mathrm{t}_k(V')}$ and $p_{2*} \widetilde{\mathrm{t}_k(V,V')} = \widetilde{\mathrm{t}_k(V'')}$. This is because the two diagrams below commute.

$$\begin{array}{ccc}
\mathcal{P}^{\otimes k} & \xrightarrow{p_1{}^{\otimes k}} & \mathrm{End}(V')^{\otimes k} \\
\Big\downarrow \varphi & & \varphi \Big\downarrow \\
\mathcal{P} & \xrightarrow{p_1} & \mathrm{End}(V')
\end{array}$$

$$\begin{array}{ccc}
\mathcal{P}^{\otimes k} & \xrightarrow{p_2{}^{\otimes k}} & \mathrm{End}(V'')^{\otimes k} \\
\Big\downarrow \varphi & & \varphi \Big\downarrow \\
\mathcal{P} & \xrightarrow{p_2} & \mathrm{End}(V'')
\end{array}$$

From this and the additivity of trace, we see that $\mathrm{t}_k(V) = \mathrm{t}_k(V') + \mathrm{t}_k(V'')$.

$\square$

Lemma 10. *If $f : Y \to X$ is a morphism of varieties and V is a vector bundle on X, then $t_k(f^*V) = f^* t_k(V)$.*

Lemma 11. *If $V = V' \oplus V''$ as $\mathcal{O}_X$-modules and p_1 and p_2 are the natural projections $\mathrm{End}(V) \to \mathrm{End}(V')$ and $\mathrm{End}(V) \to \mathrm{End}(V'')$ respectively, then $p_{1*} \widetilde{\mathrm{t}_k(V)} = \widetilde{\mathrm{t}_k(V')}$ and $p_{2*} \widetilde{\mathrm{t}_k(V)} = \widetilde{\mathrm{t}_k(V'')}$.*

Lemmas 10 and 11 are fairly straightforward to verify and we shall skip their verification. Another important property that we prove here is that $\oplus\, \mathrm{t}_k : K(X) \otimes \mathbb{Q} \to \mathrm{R}(X)$ is a ring homomorphism.

Lemma 12. *If V and W are two locally free coherent sheaves on X, then,*

$$\mathrm{t}_k(V \otimes W) = \sum_{l+m=k} \mathrm{t}_l(V) \odot \mathrm{t}_m(W)$$

where $\odot$ is the product $\mathrm{H}^l(X,\Omega^{\otimes l})\otimes\mathrm{H}^m(X,\Omega^{\otimes m})\to\mathrm{H}^k(X,\Omega^{\otimes k})$ appearing in Proposition 1. In other words, $\oplus\,\mathrm{t}_k : K(X)\otimes\mathbb{Q}\to\mathrm{R}(X)$ is a ring homomorphism.

Proof. We know that $\theta_{V\otimes W}=\theta_V\otimes\mathrm{id}_W+\mathrm{id}_V\otimes\theta_W$. Therefore,

$$\theta_{V\otimes W}{}^k=(A_V+B_W)\cup\cdots\cup(A_V+B_W)$$

where $A_V=\theta_V\otimes\mathrm{id}_W$ and $B_W=\mathrm{id}_V\otimes\theta_W$. Thus,

$$\theta_{V\otimes W}{}^k=(A_V+B_W)^k=\sum_{l+m=k}\sum_{\sigma\in\mathrm{Sh}_{l,m}}\mathrm{sgn}(\sigma){\sigma^{-1}}_*({A_V}^l\cup{B_W}^m).$$

Here, a given permutation $\mu\in S_k$ acts on $\mathrm{End}(V\otimes W)^{\otimes k}\bigotimes\Omega^{\otimes k}$ by

$$v_1\otimes\cdots\otimes v_k\bigotimes w_1\otimes\cdots\otimes w_k\mapsto v_{\mu(1)}\otimes\cdots\otimes v_{\mu(k)}\bigotimes w_{\mu(1)}\otimes\cdots\otimes w_{\mu(k)}$$

and therefore induces a map from $\mathrm{H}^k(X,\mathrm{End}(V\otimes W)^{\otimes k}\otimes\Omega^{\otimes k})$ to itself.

To verify that

$$(A_V+B_W)^k=\sum_{l+m=k}\sum_{\sigma\in\mathrm{Sh}_{l,m}}\mathrm{sgn}(\sigma){\sigma^{-1}}_*({A_V}^l\cup{B_W}^m),$$

note that in $(A_V+B_W)^k$, terms having l A_V's cupped with m B_W's are in one-one correspondence with sequences $b_1<\cdots<b_m$, $b_i\in\{1,2,3,\dots,l+m\}\ \forall i$ (the b_i's being the positions of the B_W's). Such sequences are in $1-1$ correspondence with (l,m) shuffles. The sequence $\mathrm{B}:=b_1,..,b_m$ corresponds to the (l,m)-shuffle σ_B such that $\sigma_\mathrm{B}(l+i)=b_i, 1\le i\le m$. Note that $\mathrm{sgn}(\sigma_\mathrm{B}){\sigma_\mathrm{B}}_*^{-1}{A_V}^l\cup{B_W}^m$ is exactly the term in $(A_V+B_W)^k$ where the B_W's are in positions $b_1,\dots,b_m$. The lemma is now proven by recognizing that $tr_*\circ\varphi_*\sigma_*^{-1}({A_V}^l\cup{B_W}^m)=\sigma_*^{-1}\,\mathrm{t}_l(V)\cup\mathrm{t}_m(W)$ if σ is any (l,m)-shuffle. This is because the inverse of an (l,m)-shuffle does not change the order of composition among the $\mathrm{End}(V)$-terms and among the $\mathrm{End}(W)$ terms respectively. ☐

Not only that, the ring homomorphism $\oplus\,\mathrm{t}_k$ is also a homomorphism of special λ-rings. In other words, the big Chern classes commute with Adams operations. Indeed, the following lemma proves this fact. Note that in any special λ-ring A, the eigenspace corresponding to the eigenvalue p^l of the Adams operation ψ^p coincides with that corresponding to the eigenvalue 2^l of the operation ψ^2 for any $p\ge 1$. Therefore, to verify that $\oplus\,\mathrm{t}_k$ commutes with the Adams operations, it suffices to verify that $\oplus\,\mathrm{t}_k$ commutes with ψ^2. This is done in the lemma below.

Lemma 13. $\mathrm{t}_k(\psi^2V)=\psi^2\,\mathrm{t}_k(V)$.

Proof. By the corollary to Lemma 8 (Corollary 4), $\theta_{V\otimes V}$ is induced from θ_V by the map $\beta : \mathrm{End}(V) \to \mathrm{End}(V)$ given by $m \to m \otimes \mathrm{id}_V + \mathrm{id}_V \otimes m$ i.e, $\theta_{V\otimes V} = \beta_* \theta_V$. Therefore,

$$\theta_{V\otimes V}{}^k = \beta_*\theta_V \cup \cdots \cup \beta_*\theta_V = (\beta \otimes \cdots \otimes \beta)_* \theta_V{}^k.$$

By abuse of notation, we shall refer to $\beta \otimes \cdots \otimes \beta$ as β. Then, $\theta^k_{V\otimes V} = \beta_* \theta^k_V$, where $\beta : \mathrm{End}(V)^{\otimes k} \to \mathrm{End}(V)^{\otimes k}$ is given by

$$m_1 \otimes \cdots \otimes m_k \mapsto \bigotimes_{i=1}^k (m_i \otimes \mathrm{id}_V + \mathrm{id}_V \otimes m_i).$$

Further, a direct computation shows that if W is a vector space over a field F, with $charF \neq 2$, $W \otimes W = \mathrm{Sym}^2 W \oplus \wedge^2 W$. Let p_1 and p_2 denote the resulting projections from $\mathrm{End}(W) \otimes \mathrm{End}(W) = \mathrm{End}(W \otimes W)$ onto $\mathrm{End}(\mathrm{Sym}^2 W)$ and $\mathrm{End}(\wedge^2 W)$ respectively. If $M, N \in \mathrm{End}(W)$, then

$$tr(p_1(M \otimes N)) - tr(p_2(M \otimes N)) = tr(M \circ N).$$

By this fact, and Lemma 11, we see that

$$\begin{aligned} \mathrm{t}_k(\psi^2 V) = \mathrm{t}_k(\mathrm{Sym}^2 V) - \mathrm{t}_k(\wedge^2 V) &= tr_* p_{1*} \mathrm{t}_k(\widetilde{V \otimes V}) - tr_* p_{2*} \mathrm{t}_k(\widetilde{V \otimes V}) \\ &= tr_* \alpha_* \mathrm{t}_k(\widetilde{V \otimes V}) \end{aligned}$$

where $\alpha : \mathrm{End}(V) \otimes \mathrm{End}(V) \to \mathrm{End}(V)$ is the composition map.

Let $\varphi : \mathrm{End}(V \otimes V)^{\otimes k} \to \mathrm{End}(V \otimes V)$ be the composition map. Observe that $\alpha \circ \varphi \circ \beta : \mathrm{End}(V)^{\otimes k} \to \mathrm{End}(V)$ is the map given by

$$m_1 \otimes \cdots \otimes m_k \mapsto \sum_{p+q=k} \sum_{\sigma \in \mathrm{Sh}_{p,q}} m_{\sigma(1)} \circ \cdots \circ m_{\sigma(k)}$$

($\circ$ denoting the usual matrix multiplication on the right hand side of the last equation). Consider the map $\gamma : \mathrm{End}(V)^{\otimes k} \to \mathrm{End}(V)^{\otimes k}$ given by

$$m_1 \otimes \cdots \otimes m_k \mapsto \sum_{p+q=k} \sum_{\sigma \in \mathrm{Sh}_{p,q}} m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(k)}.$$

Then, we see that

$$tr_* \circ \varphi_* \circ \gamma_* \theta_V{}^k = tr_* \circ \alpha_* \mathrm{t}_k(\widetilde{V \otimes V}) = \mathrm{t}_k(\psi^2 V).$$

Also observe that $\psi^2 \, \mathrm{t}_k(V) = tr_* \varphi_* \psi^2_* \theta_V{}^k$ since the following diagram commutes.

$$\begin{array}{ccc} \mathrm{End}(V)^{\otimes k} \otimes \Omega^{\otimes k} & \xrightarrow{\mathrm{id} \otimes \psi^2} & \mathrm{End}(V)^{\otimes k} \otimes \Omega^{\otimes k} \\ {\scriptstyle tr\circ(\varphi\otimes\mathrm{id})}\downarrow & & \downarrow{\scriptstyle tr\circ(\varphi\otimes\mathrm{id})} \\ \Omega^{\otimes k} & \xrightarrow{\psi^2} & \Omega^{\otimes k} \end{array}$$

Here, ψ^2_* on $\mathrm{H}^k(X, \mathrm{End}(V)^{\otimes k} \otimes \Omega^{\otimes k})$ is by definition induced on co-homology by the endomorphism $\mathrm{id} \otimes \psi^2$ of $\mathrm{End}(V)^{\otimes k} \otimes \Omega^{\otimes k}$. Thus, the following lemma remains to be proven.

□

Lemma 14. $\gamma_* {\theta_V}^k = \psi^2_* {\theta_V}^k$

Proof. Note that the cup-product is anti-commutative. Therefore, if $\sigma \in S_k$, then the map given by

$$\sigma : m_1 \otimes \cdots \otimes m_k \bigotimes v_1 \otimes \cdots \otimes v_k \mapsto \mathrm{sgn}(\sigma) m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(k)} \bigotimes v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$$

preserves ${\theta_V}^k$.
If $\sigma \in S_k$ let $\sigma \otimes \mathrm{id}$ denote the endomorphism of $\mathrm{End}(V)^{\otimes k} \otimes \Omega^{\otimes k}$ such that

$$m_1 \otimes \cdots \otimes m_k \bigotimes v_1 \otimes \cdots \otimes v_k \mapsto m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(k)} \bigotimes v_1 \otimes \cdots \otimes v_k.$$

Similarly, let $\mathrm{id} \otimes \sigma$ denote the endomorphism of $\mathrm{End}(V)^{\otimes k} \otimes \Omega^{\otimes k}$ such that

$$m_1 \otimes \cdots \otimes m_k \bigotimes v_1 \otimes \cdots \otimes v_k \mapsto m_1 \otimes \cdots \otimes m_k \bigotimes v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}.$$

It now suffices to note that

$$\begin{aligned}
\gamma = \sum_{p+q=k} \sum_{\sigma \in \mathrm{Sh}_{p,q}} \sigma \otimes \mathrm{id} &= \sum_{p+q=k} \sum_{\sigma \in \mathrm{Sh}_{p,q}} \mathrm{sgn}(\sigma)(\mathrm{id} \otimes \sigma^{-1}) \circ (\sigma) \\
\implies \gamma_* \theta_V^k &= \sum_{p+q=k} \sum_{\sigma \in \mathrm{Sh}_{p,q}} \mathrm{sgn}(\sigma)(\mathrm{id} \otimes \sigma^{-1})_* \circ (\sigma)_* \theta_V^k \\
&= \sum_{p+q=k} \sum_{\sigma \in \mathrm{Sh}_{p,q}} \mathrm{sgn}(\sigma)(\mathrm{id} \otimes \sigma^{-1})_* \theta_V^k \\
&= \psi^2_* \theta_V^k.
\end{aligned}$$

□

Recalling that $\alpha_l(V) = \mathrm{ch}^{-1}(\mathrm{ch}_l(V))$, where ch is the Chern character map, we now have the following corollary of Lemma 13 below.

Corollary 5. $\mathrm{t}_k(\alpha_l(V)) = {e_k}^{(l)}_* \mathrm{t}_k(V)$ *where* ${e_k}^{(l)}$ *is the idempotent described in Lemma 1.*

Proof. Note that $\psi^2 = \sum e^{(l)} 2^l$. The fact that the ${e_k}^{(l)}$ are mutually orthogonal idempotents adding upto id tells us that $\psi^2 \circ {e_k}^{(l)} = 2^l {e_k}^{(l)}$. Therefore, $\psi^2 \mathrm{t}_k(V) = \sum 2^l {e_k}^{(l)}_* \mathrm{t}_k(V) = \mathrm{t}_k(\psi^2 V) = \mathrm{t}_k(\sum 2^l \alpha_l(V)) = \sum 2^l \mathrm{t}_k(\alpha_l(V))$. Since eigenvectors corresponding to different eigenvalues of a linear operator on a finite dimensional vector space over a field of characteristic 0 are linearly independent, the desired result follows. □

Remark. More conceptually, if TV is the graded tensor algebra over a vector space V, (with usual tensor product giving the multiplication, and coproduct dictated by the fact that $V \subset TV$ are primitive elements), then T^*V is the graded Hopf algebra dual to TV. The map $\psi^2 = \mu \circ \Delta : T^*V \to T^*V$ has as its dual the map $\mu \circ \Delta : TV \to TV$. The 2^l-eigenspace of this map is seen to be $''\operatorname{Sym}^l(L(V))''$. Thus, the 2^l-eigenspace of $\psi^2 : T^*V \to T^*V$ is dual to the space $''\operatorname{Sym}^l(L(V))''$. Thus, $\mathrm{t}_k(\alpha_l(V))$ lands in k-cohomology with coefficients in a space dual to $''\operatorname{Sym}^l(L(\Omega))''$. Moreover, the last corollary explicitly describes the projector that gives $\mathrm{t}_k(\alpha_l(V))$ from $\mathrm{t}_k(V)$ as the action on $\mathrm{t}_k(V)$ of a certain idempotent in $\mathbb{K}(S_k)$. Thus, one can recover $\mathrm{t}_k(\alpha_l(V))$ from $\mathrm{t}_k(V)$ combinatorially.

5 Calculating $\mathrm{t}_k(Q)$, Q the universal quotient bundle of a Grassmannian $G(r,n)$

We remark that $Q_{G(r,n)}$ is often denoted by just Q in this and subsequent sections. The Grassmannian whose universal quotient bundle we are referring to is usually clear by the context.

5.1 Alternative construction for $\widetilde{\mathrm{t}_k(V)}$ and $\mathrm{t}_k(V)$

Let V be a locally free coherent sheaf on a (separated) scheme X/S. It is a fact that θ_V is the element in $\operatorname{Ext}^1(V, V \otimes \Omega) \cong \mathrm{H}^1(X, \operatorname{End}(V) \otimes \Omega)$ corresponding to the exact sequence $0 \to V \otimes \Omega \to J_1(V) \to V \to 0$ where $J_1(V)$ is the first jet bundle of V. Suppose that $\alpha \in \mathrm{H}^i(X, \mathcal{F}) = \operatorname{Ext}^i(\mathcal{O}_X, \mathcal{F})$ is given by an exact sequence

$$0 \to \mathcal{F} \to Y_1 \to \ldots \to Y_i \to \mathcal{O}_X \to 0$$

and that $\beta \in \mathrm{H}^j(X, \mathcal{G}) = \operatorname{Ext}^j(\mathcal{O}_X, \mathcal{G})$ is given by an exact sequence

$$0 \to \mathcal{G} \to Z_1 \to \ldots \to Z_j \to \mathcal{O}_X \to 0.$$

Let $\alpha * \beta$ be the element in $\mathrm{H}^{i+j}(X, \mathcal{F} \otimes \mathcal{G}) = \operatorname{Ext}^{i+j}(\mathcal{O}_X, \mathcal{F} \otimes \mathcal{G})$ defined by the exact sequence which is the tensor product of the exact sequences representing α and β respectively. We note that the product

$$* : \mathrm{H}^i(X, \mathcal{F}) \otimes \mathrm{H}^j(X, \mathcal{G}) \to \mathrm{H}^{i+j}(X, \mathcal{F} \otimes \mathcal{G})$$

$$\alpha \otimes \beta \mapsto \alpha * \beta$$

has the linearity and anticommutativity properties required of the cup product. Since all the cohomology classes we are dealing with are represented by exact sequences of $\mathcal{O}_X$-modules, we can define the cup product to be the product $*$. With this definition of the cup product, it will follow that $\widetilde{\mathrm{t}_k(V)} \in \operatorname{Ext}^k(V, V \otimes \Omega^{\otimes k})$ is given by $(\theta_V \otimes {\mathrm{id}_\Omega}^{k-1}) \circ \cdots \circ \theta_V$ where $\circ$ denotes the Yoneda product and θ_V is treated as an element in $\operatorname{Ext}^1(V, V \otimes \Omega)$.

5.2 Computation of $\widetilde{\mathrm{t}_1(Q)}$

Recall that Ω is identified with $Q^* \otimes S$. Let $\Delta : S \to Q \otimes \Omega$ be the map whose dual $\Delta^* : Q^* \otimes Q \otimes S^* \to S^*$ is $ev \otimes \mathrm{id}_{S^*}$, where $ev : Q^* \otimes Q \to \mathbb{K}$ is the evaluation map. Also, $ev \otimes \mathrm{id}_S$ is a map from $Q \otimes \Omega$ to S.

Lemma 15. *The element of* $\mathrm{End}_{\mathbb{K}}(Q \otimes \Omega)$ *representing* θ_Q *is* $\Delta \circ (ev \otimes \mathrm{id}_S)$.

Proof. We note that the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r, "\varphi"] \arrow[d, "\Delta"'] & V \arrow[r, "\epsilon"] \arrow[d] & Q \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & Q \otimes Q^* \otimes S \arrow[r] & J_1(Q) \arrow[r, "\gamma"] & Q \arrow[r] & 0
\end{tikzcd}$$

The bottom row of this diagram is the exact sequence giving θ_V. By the universal property of push-forwards, we see that the following diagram commutes (F denotes the pushforward $V \amalg_S Q^* \otimes Q \otimes S$).

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r, "\varphi"] \arrow[d, "\Delta"'] & V \arrow[r, "\epsilon"] \arrow[d] & Q \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & Q^* \otimes Q \otimes S \arrow[r] \arrow[d, "\mathrm{id}"] & F \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \\
0 \arrow[r] & Q^* \otimes Q \otimes S \arrow[r] & J_1(Q) \arrow[r] & Q \arrow[r] & 0
\end{tikzcd}$$

Therefore, θ_Q can be represented by the second row of the above diagram in $\mathrm{Ext}^1(Q, Q \otimes \Omega)$. Observe, however, that every arrow in this exact sequence is a P-module homomorphism (of course, $Q^* \otimes Q \otimes S$, V and therefore, F are all P-modules). Thus θ_Q can be represented by an exact sequence in the category of P-representations. It follows that for all $k \geq 1$, $\widetilde{\mathrm{t}_k(Q)}$ and $\mathrm{t}_k(Q)$ can be represented by exact sequences in the category of P-representations. Therefore, to find θ_Q, we need to find arrows α and β so that all squares in the following diagram commute.

$$\begin{tikzcd}
0 \arrow[r] & Q^* \otimes Q \otimes S \arrow[r] & F \arrow[r] & Q \arrow[r] & 0 \\
0 \arrow[r] & S \arrow[r, "\varphi"] \arrow[u, "\Delta"] & V \arrow[r, "\epsilon"] \arrow[u] & Q \arrow[r] \arrow[u, "\mathrm{id}"'] & 0 \\
\dots \arrow[r] & Q \otimes \Omega \otimes \mathrm{Sym}^* \Omega \arrow[r] \arrow[u, "\alpha"] & Q \otimes \mathrm{Sym}^* \Omega \arrow[r] \arrow[u, "\beta"] & Q \arrow[r] \arrow[u, "\mathrm{id}"'] & 0
\end{tikzcd}$$

Observe that $\Omega = \mathrm{Hom}_{\mathbb{K}}(Q, S) \subseteq \mathrm{End}(V)$ (here, we have chosen a $\mathbb{K}$-vector space splitting $0 \to S \to V \leftrightarrows Q \to 0$. Choosing such a splitting describes

Ω as the subspace of elements in $\mathrm{End}(V)$ consisting of matrices whose "upper right block" is the only nonzero block. Note that the product of two such matrices is 0. Thus, any element of $\mathrm{Sym}^*\Omega$ can be thought of as an element of $\mathrm{Hom}(Q,V) \subset \mathrm{End}(V)$. In this scheme of things, we choose β to be the natural evaluation map, and α the restriction of β to $Q \otimes \Omega \otimes \mathrm{Sym}^*\Omega$. Note that β and α are $\mathrm{Sym}^*\Omega$-module homomorphisms by construction. Note that $\alpha : Q \otimes \Omega \otimes \mathrm{Sym}^*\Omega$ is the $\mathrm{Sym}^*\Omega$-module homomorphism induced by $\tilde{\alpha} := ev \in \mathrm{Hom}_{\mathbb{K}}(Q \otimes \Omega, S)$, where ev is the natural evaluation map. It follows that as an element in $\mathrm{Hom}_{\mathbb{K}}(Q \otimes \Omega, Q \otimes \Omega)$, θ_Q is given by $\Delta \circ (ev \otimes \mathrm{id}_S)$.

$\square$

Let $\{e_i\}, 1 \le i \le r$ be a basis for Q. Let $\{f_i\}$ be the basis of Q^* dual to $\{e_i\}$. Let $\{u_i\}, 1 \le i \le n-r$ be a basis for S, and $\{v_i\}$ the basis for S^* dual to $\{u_i\}$. The following is a restatement of Lemma 15.

Lemma 16. *With the notation just fixed, as an element of* $\mathrm{End}_{\mathbb{K}}(Q \otimes \Omega) \doteq \mathrm{End}(Q) \otimes \mathrm{End}(\Omega) \doteq Q^* \otimes Q \bigotimes Q \otimes S^* \bigotimes Q^* \otimes S$,

$$\theta_Q = \sum_{l_1,m_1,r_1} f_{m_1} \otimes e_{l_1} \bigotimes e_{m_1} \otimes v_{r_1} \bigotimes f_{l_1} \otimes u_{r_1}$$

(l_1, m_1 running from 1 to r, r_1 running from 1 to $n-r$).

Proof. $ev(e_i \otimes f_j \otimes u_k) = \delta_{ij} u_k$ and $\Delta(u_k) = \sum_{l=1}^r e_l \otimes f_l \otimes u_k$. Therefore, $\theta_Q(e_i \otimes f_j \otimes u_k) = \delta_{ij} \sum_{l=1}^r e_l \otimes f_l \otimes u_k$. On the other hand,

$$f_{m_1} \otimes e_{l_1} \bigotimes e_{m_1} \otimes v_{r_1} \bigotimes f_{l_1} \otimes u_{r_1} (e_i \otimes f_j \otimes u_k) = \delta_{im_1}\delta_{jm_1}\delta_{kr_1} e_{l_1} \otimes f_{l_1} \otimes u_{r_1}.$$

This is nonzero iff $i = j = m_1$ and $k = r_1$. This proves the desired result. $\square$

5.3 Computing $\widetilde{\mathrm{t}_k(Q)}$ for $k > 1$

This is done inductively. The method by which Yoneda products are computed is very similar to the cup product computation in the previous section. We therefore omit the details and state the key results.

If $i : \wedge^k \Omega \to \Omega^{\otimes k}$ is the natural inclusion, $\widetilde{\mathrm{t}_k(Q)}$ is given by $\gamma_k \circ i$ where $\gamma_k \in \mathrm{End}_{\mathbb{K}}(Q \otimes \Omega^{\otimes k})$ is as described in the following lemma.

Lemma 17. *Identifying* $\mathrm{End}_{\mathbb{K}}(Q \otimes \Omega^{\otimes k})$ *with* $\mathrm{End}_{\mathbb{K}}(Q) \bigotimes \Omega^{*\otimes k} \bigotimes \Omega^{\otimes k}$, *we have*

$$\gamma_k = \sum_{\substack{l_1,\dots,l_k \\ m_1,\dots,m_k \\ r_1,\dots,r_k}} \left(\begin{array}{c} (f_{m_1} \otimes e_{l_1}) \circ \cdots \circ (f_{m_k} \otimes e_{l_k}) \bigotimes (e_{m_1} \otimes v_{r_1}) \otimes \cdots \otimes (e_{m_k} \otimes v_{r_k}) \\ \bigotimes (f_{l_1} \otimes u_{r_1}) \otimes \cdots \otimes (f_{l_k} \otimes u_{r_k}) \end{array} \right).$$

Here, the $l_i, 1 \le i \le k$ and the $m_i, 1 \le i \le k$ run from 1 to r , while the $r_i, 1 \le i \le k$ run from 1 to $n-r$.

Having computed $\widetilde{\mathrm{t}_k(Q)}$ we compute $\mathrm{t}_k(Q)$. For this, we note that $\mathrm{t}_k(Q) = (tr \otimes \mathrm{id})_* \widetilde{\mathrm{t}_k(Q)}$ where $\widetilde{\mathrm{t}_k(Q)} \in \mathrm{End}(Q) \otimes \mathrm{Hom}_{\mathbb{K}}(\wedge^k \Omega, \Omega^{\otimes k})$ and $tr : \mathrm{End}(Q) \to \mathbb{K}$ is the trace map. Calculating $\mathrm{t}_k(Q)$ is then easy. In the formula in the previous lemma, we see that

$$(f_{m_1} \otimes e_{l_1}) \circ \cdots \circ (f_{m_k} \otimes e_{l_k})(e_i) = \delta_{im_k} \delta_{l_k m_{k-1}} \ldots \delta_{l_2 m_1} e_{l_1}.$$

From this, we see that $(f_{m_1} \otimes e_{l_1}) \circ \cdots \circ (f_{m_k} \otimes e_{l_k})$ has trace 1 iff $m_k = l_1, l_k = m_{k-1}, \ldots, l_2 = m_1$ and has trace 0 otherwise. From this it follows that if $i : \wedge^k \Omega \to \Omega^{\otimes k}$ is the natural inclusion, $\mathrm{t}_k(Q)$ is given by $\mu_k \circ i$ where $\mu_k \in \mathrm{Hom}_{\mathbb{K}}(\Omega^{\otimes k}, \Omega^{\otimes k})$ is as described in the following lemma.

Lemma 18. *Identifying* $\mathrm{End}_{\mathbb{K}}(\Omega^{\otimes k})$ *with* $\Omega^{*\otimes k} \bigotimes \Omega^{\otimes k}$ *we have*

$$\mu_k = \sum_{\substack{l_1,\ldots,l_k \\ r_1,\ldots,r_k}} (e_{l_2} \otimes v_{r_1}) \otimes \cdots \otimes (e_{l_k} \otimes v_{r_{k-1}}) \otimes (e_{l_1} \otimes v_{r_k}) \bigotimes (f_{l_1} \otimes u_{r_1}) \otimes \cdots \otimes (f_{l_k} \otimes u_{r_k})$$

$$= \sum_{\substack{m_1,\ldots,m_k \\ r_1,\ldots,r_k}} (e_{m_1} \otimes v_{r_1}) \otimes \cdots \otimes (e_{m_k} \otimes v_{r_k}) \bigotimes (f_{m_k} \otimes u_{r_1}) \otimes (f_{m_1} \otimes u_{r_2}) \otimes \cdots \otimes (f_{m_{k-1}} \otimes u_{r_k}).$$

As a consequence, the basis element $f_{i_1} \otimes \cdots \otimes f_{i_k} \bigotimes u_{j_1} \otimes \cdots \otimes u_{j_k}$ of $\Omega^{\otimes k}$ is mapped by $\mathrm{t}_k(Q)$ to $f_{i_k} \otimes f_{i_1} \otimes \cdots \otimes f_{i_{k-1}} \bigotimes u_{j_1} \otimes \cdots \otimes u_{j_k}$. Therefore, if we identify $\mathrm{End}_{\mathbb{K}}(\Omega^{\otimes k})$ with $Q^{*\otimes k} \otimes S^{\otimes k}$, $\mathrm{t}_k(Q)$ can be thought of as $(k\ k-1\ k-2\ ..\ 2\ 1) \otimes \mathrm{id}_{S^{\otimes k}}$ where $(k\ k-1\ k-2\ ..\ 2\ 1)$ is the k-cycle acting on $Q^{*\otimes k}$ by the usual action of S_k on $V^{\otimes k}$ for a vector space V. We denote this k-cycle by τ_k.

Let $P_{r,n}$ be as in Lemma 3. By Lemma 18 and the above paragraph,

Lemma 18'.
$$\mathrm{t}_k(Q) = P_{r,n}(\tau_k).$$

6 Proofs of Theorems 2 and 3

We recall that $S_{k,j}$ denotes the set $S_{k,j} = \{\sigma \in S_k | card\{i | \sigma(i) > \sigma(i+1)\} = j-1\}$, i.e, the set of permutations of $\{1, \ldots, k\}$ with $j-1$ descents. By part 2 of Lemma 3, if $\sum a_\sigma \operatorname{sgn}(\sigma)\sigma \in \mathbb{K}(S_k)$, we have

$$\sum a_\sigma \operatorname{sgn}(\sigma)\sigma_*(\mathrm{t}_k(Q)) = P_{r,n}(\sum a_\sigma \sigma^{-1} \tau_k \sigma).$$

The following lemma now follows immediately from Corollary 5.

Lemma 19.
$$\mathrm{t}_k(\alpha_l(Q)) = P_{r,n}(\sum_{j=1}^{n} \sum_{\sigma \in S_{k,j}} (a_k^{l,j} \sigma \tau_k \sigma^{-1}))$$

A remark and some notation. $\sum_{j=1}^{k}\sum_{\sigma\in S_{k,j}}\mathrm{sgn}(\sigma)a_k^{l,j}\sigma^{-1}$ is the operator $e_k^{(l)}$ for the graded commutative Hopf-algebra T^*V. In fact, $\sum_{j=1}^{k}\sum_{\sigma\in S_{k,j}}a_k^{l,j}\sigma$ is the operator $e_k^{(l)}$ for the co-commutative ordinary Hopf-algebra TV. We henceforth denote this idempotent by $\tilde{e}_k^{(l)}$. Let $*$ denote the conjugation action of $\mathbb{K}S_k$ on itself. If $a\in S_k$ and $b\in\mathbb{K}S_k$ then $a*b=aba^{-1}$ and $(\sum c_g g)*h=\sum c_g ghg^{-1}, h\in\mathbb{K}S_k$. Then, Lemma 19 can be concisely restated as

$$\mathrm{t}_k(\alpha_l(Q)) = P_{r,n}(\tilde{e}_k^{(l)}*\tau_k).$$

Note that $*$ is a left action.

6.1 Proofs of Corollary 2 and Corollary 3

Recall the definitions of the projections P_r and $P_{r,n}$ from Section 3. Assume for now that n is large enough so that $P_r=P_{r,n}$ for all values of k that we shall use. Let $I(k,r,l)$ denote the annihilator in $\mathbb{K}S_k$ of $\mathrm{t}_k(\alpha_l(Q))$. By Lemma 3 and Lemma 19 this is precisely the subspace

$$I(k,r,l)=\{\sum_g c_g\,\mathrm{sgn}(g)g|P_r((\sum_g c_g g^{-1})*\tilde{e}_k^{(l)}*\tau_k)=0\}.$$

If $\langle\alpha\rangle$ denotes the subspace of $\mathbb{K}S_k$ spanned by conjugates of α by elements of S_k where $\alpha\in\mathbb{K}S_k$, then

$$\dim(I(k,r,l))=\dim(\langle\tilde{e}_k^{(l)}*\tau_k\rangle)-\dim(\langle P_r(\tilde{e}_k^{(l)}*\tau_k)\rangle).$$

Note that since P_{r-1} factors through P_r, $I(k,r,l)\subseteq I(k,r-1,l)$. It follows that this inclusion is strict if

$$\dim(\langle P_r(\tilde{e}_k^{(l)}*\tau_k)\rangle)>\dim(\langle P_{r-1}(\tilde{e}_k^{(l)}*\tau_k)\rangle).$$

We will prove the following lemma.

Lemma 20. *For a fixed l, there exists a constant C and infinitely many r such that there exists a $k<Cr^2$ so that*

$$\dim(\langle P_r(\tilde{e}_k^{(l)}*\tau_k)\rangle)>\dim(\langle P_{r-1}(\tilde{e}_k^{(l)}*\tau_k)\rangle).$$

Note that in such a situation, if $n>Cr^2+r$ then $P_r=P_{r,n}$ as projection operators on $\mathbb{K}S_k$. We can then pick an element β in $\mathbb{K}S_k$ such that $\beta_*\,\mathrm{t}_k(\alpha_l(Q_{G(r-1,\infty)}))=0$ and $\beta_*\,\mathrm{t}_k(\alpha_l(Q_{G(r,n)}))\neq 0$. If Corollary 2 were false there would be a morphism $f:G(r,n)\to G(r-1,\infty)$ so that $f^*\left(\beta_*\,\mathrm{t}_k(\alpha_l(Q_{G(r-1,\infty)}))\right)=\beta_*\,\mathrm{t}_k(\alpha_l(Q_{G(r,n)}))$. This gives us a contradiction. Therefore, Corollary 2 follows immediately from Lemma 20.

We will prove Lemma 20 by a simple counting argument. We however, need the following lemma.

Lemma 21.
$$\tilde{e}_k^{(l)} * \tau_k = \tilde{e}_{k-1}^{(l-1)} * \tau_k$$
where $S_{k-1} \subset S_k$ is embedded as the subgroup fixing k.

Proof. Let α be a permutation of $\{1, 2, 3, \ldots, k-1\}$ with $j-1$ descents. Then, among the permutations $\alpha, \alpha\tau_k, \ldots, \alpha{\tau_k}^{k-1}$, we see that j of the permutations have $j-1$ descents, while the remaining $k-j$ have j descents. For, $\alpha{\tau_k}^i$ has j descents or $j-1$ descents depending on whether $\alpha(k-i) < \alpha(k-i+1)$ or not, for $2 \leq i \leq k-1$. For $j-1$ such i, $\alpha(k-i) > \alpha(k-i+1)$ (corresponding to the descents of α). These $j-1$ elements together with α have $j-1$ descents. The remaining $k-j$ permutations have j descents. As ${\tau_k}^i \tau_k {\tau_k}^{-i} = \tau_k$, the coefficient of $\alpha\tau_k\alpha^{-1}$ in $\tilde{e}_k^{(l)} * \tau_k$ is given by $ja_k^{l,j} + (k-j)a_k^{l,j+1}$, since among the elements $\alpha, \alpha\tau_k, \ldots, \alpha{\tau_k}^{k-1}$, those with $j-1$ descents contribute $a_k^{l,j}$ and those with j descents contribute $a_k^{l,j+1}$ to the coefficient of $\alpha\tau_k\alpha^{-1}$ in $\tilde{e}_k^{(l)} * \tau_k$. The desired lemma follows from observing that $ja_k^{l,j} + (k-j)a_k^{l,j+1} = ja_{k-1}^{l-1,j}$, since $j\binom{X-j+k}{k} + (k-j)\binom{X-j-1-k}{k} = X\binom{X-j-1-k}{k-1}$.

□

Proof. (Proof of Lemma 20). Suppose we have shown that there exists a constant C such that for a fixed l and r,

$$\dim(\langle \tilde{e}_k^{(l)} * \tau_k \rangle) > \dim(\langle P_r(\tilde{e}_k^{(l)} * \tau_k) \rangle)$$

if $k \geq Cr^2$. Then,

$$\text{there exists } s \geq r \text{ so that } \dim(\langle P_s(\tilde{e}_k^{(l)} * \tau_k) \rangle) < \dim(\langle P_{s+1}(\tilde{e}_k^{(l)} * \tau_k) \rangle).$$

Therefore, for any l and r, there exists $s \geq r$ so that

$$\dim(\langle P_s(\tilde{e}_k^{(l)} * \tau_k) \rangle) < \dim(\langle P_{s+1}(\tilde{e}_k^{(l)} * \tau_k) \rangle).$$

With l,r and s as above, pick $k = Cr^2$. Then $k < C(s+1)^2$ as well. This proves the lemma provided we actually show that there exists a constant C such that for a fixed l and r,

$$\dim(\langle \tilde{e}_k^{(l)} * \tau_k \rangle) > \dim(\langle P_r(\tilde{e}_k^{(l)} * \tau_k) \rangle)$$

whenever $k > Cr^2$. This is what we will do now.

1. Observe that the stabilizer of τ_k under conjugation is the cyclic subgroup generated by τ_k. Thus, S_{k-1} acts freely on the conjugates of τ_k and $\beta * \tau_k = 0$ for some $\beta \in \mathbb{K}S_{k-1}$ iff $\beta = 0$. It follows from this remark and the Lemma 21 that $\dim(\langle \tilde{e}_k^{(l)} * \tau_k \rangle)$ is the dimension of the representation $\mathbb{K}S_{k-1}.\tilde{e}_{k-1}^{(l-1)}$ of $\mathbb{K}S_{k-1}$. By exercise 4.5 in Loday[2] that this space has dimension equal to the coefficient of q^{l-1} in $q(q+1)\ldots(q+k-2)$.

2. On the other hand, look at $\dim(\oplus_{|\lambda|\leq r}\operatorname{End}(V_\lambda)$ for a fixed r. Note that if $\lambda : k = \lambda_1 + \cdots + \lambda_{r'}$ is a partition of k, and if Π denotes the product of the hook lengths of the Young diagram corresponding to λ, then $\dim(V_\lambda) = \frac{k!}{\Pi} \leq \frac{k!}{\lambda_1!\lambda_2!\ldots\lambda_{r'}!}$. Thus, $\dim(\operatorname{End}(V_\lambda)) \leq \left(\frac{k!}{\lambda_1!\lambda_2!\ldots\lambda_{r'}!}\right)^2$. Hence,

$$\dim(\oplus_{|\lambda|\leq r}\operatorname{End}(V_\lambda)) \leq \sum_{\substack{\lambda_1+\cdots+\lambda_r=k\\ \lambda_i\geq 0}} \left(\frac{k!}{\lambda_1!\lambda_2!\ldots\lambda_r!}\right)^2$$

$$\leq \Big(\sum_{\substack{\lambda_1+\cdots+\lambda_r=k\\ \lambda_i\geq 0}} \frac{k!}{\lambda_1!\lambda_2!\ldots\lambda_r!}\Big)^2 = r^{2k}.$$

Therefore, for a fixed r,

$$\dim(\langle P_r(\tilde{e}_k^{(l)} * \tau_k)\rangle) \leq \dim(\oplus_{|\lambda|\leq r}\operatorname{End}(V_\lambda)) \leq r^{2k}.$$

On the other hand,

$$\dim(\langle \tilde{e}_k^{(l)} * \tau_k\rangle) = \text{ coefficient of } q^{l-1} \text{ in } q(q+1)\ldots(q+k-2) \geq \frac{(k-2)!}{(l-2)!}.$$

We need to find k large enough so that $\frac{(k-2)!}{(l-2)!} > r^{2k}$. To see this we need to find k large enough so that

$$\ln((k-2)!) - \ln((l-2)!) > 2k\ln r.$$

Note that

$$\ln((k-2)!) > (k-2)\ln(k-2) - (k-3).$$

We therefore, only need to find k large enough so that

$$(k-2)\ln(k-2) > k-3+\ln((l-2)!) + (k-2)\ln(r^2) + 2\ln(r^2).$$

Put $D = \ln(r^4(l-2)!)$. We then need k so that

$$(k-2)\ln(k-2) > k-3+D+(k-2)\ln(r^2).$$

Certainly, there exists $N \in \mathbb{N}$ so that $N(k-2) > (k-3)+D$ To see this, note that we can pick $N > D+1$ if $k > 3$ for instance. In fact, picking $N > 5+\ln((l-2)!)$ works as well. The latter choice of N is independent of r. If $k-2 > e^N r^2$, then we see that

$$(k-2)\ln(k-2) > k-3+D+(k-2)\ln(r^2).$$

Certainly, $k > e^{N+1}r^2$ would do for our purposes.

Thus, if l and r are fixed, we have shown that there is a constant C so that when $k > Cr^2$, then

$$\dim(\langle \tilde{e}_k^{(l)} * \tau_k \rangle) > \dim(\langle P_r(\tilde{e}_k^{(l)} * \tau_k)\rangle).$$

If $l = 2$, in particular, we need

$$(k-2)\ln(k-2) > k - 3 + (k-2)\ln(r^2) + 2\ln(r^2)$$

We see that this happens if $k - 2 > 7r^2$.

□

This completes the proof of Corollary 2. In addition, we have shown in Lemma 20 and hence in Corollary 2 that if $l = 2$, $C = 7$ works.

To complete the proof of Corollary 3, we make some observations.

Observation 1. *By Lemma 21,*

$$\tau_k = \sum_{l\geq 2} \tilde{e}_{k-1}^{(l-1)} * \tau_k = \sum_{l \geq 2} \tilde{e}_k^{(l)} * \tau_k$$

$$\implies \mathrm{t}_k(Q) = \sum_{l\geq 2} \mathrm{t}_k(\alpha_l(Q)) \implies \mathrm{t}_k(\alpha_1(Q)) = 0 \ \forall k \geq 2.$$

Observation 2. *Since* $\oplus\, \mathrm{t}_k : K(X) \otimes \mathbb{Q} \to \oplus\, \mathrm{H}^k(X, \Omega^{\otimes k})$ *is a ring homomorphism, is follows that*

$$\mathrm{t}_k(\alpha_1(Q)^2) = 0$$

if $k \neq 2$.

If $f : G(s+1, N) \to G(s, M)$ is a morphism, then one sees that

$$f^*(\alpha_2(Q')) = A\alpha_1(Q)^2 + B\alpha_2(Q)$$

where Q and Q' are the universal quotient bundles of $G(s+1, N)$ and $G(s, M)$ respectively. By Observation 2,

$$\mathrm{t}_k\left(f^*\left(\alpha_2(Q_{G(s,M)})\right)\right) = B\,\mathrm{t}_k(\alpha_2(Q_{G(s+1,N)})).$$

If $B \neq 0$, one sees that $I(k, s, 2) \subseteq I(k, s+1, 2)$ (a contradiction). This finally proves Corollary 3.

To prove Theorem 2, we need the following lemma.

Lemma 22. *X a smooth (projective) scheme. Suppose that $[V] \in K(X)$ is given by $[V] = \sum a_i[V_i]$, where V_i's are of rank $\leq r$. Then, $I(k, r, l)$ annihilates $\mathrm{t}_k(\alpha_l([V]))$.*

Proof. There exists $N \in \mathbb{N}$ so that for each $m > N$ there exist surjections $\mathbb{G}_i \to V_i(m)$ where $\mathbb{G}_i$ is a free $\mathcal{O}_X$ module for each i. Let K_i denote the rank of $\mathbb{G}_i$. This is equivalent to saying that for each i there exists a morphism $f_i : X \to G(rank(V_i), K_i)$ so that $V_i(m) = {f_i}^* Q_i$, Q_i being the universal quotient bundle of $G(rank(V_i), K_i)$. Thus for each i, $I(k,r,l)$ kills $\mathrm{t}_k\left(\alpha_l(V_i \otimes \mathcal{O}(m))\right)$ for each $m > N$.

To prove this lemma, it suffices to show that $I(k,r,l)$ kills $\mathrm{t}_k(\alpha_l(V_i))$ for each i. For this, we note that $\oplus\, \mathrm{t}_k(\mathcal{O}(1)) = e^{\mathrm{t}_1(\alpha_1(\mathcal{O}_1))}$, with the understanding that $\mathrm{t}_1(\alpha_1(\mathcal{O}_1))^{D+1} = 0$ where D is the dimension of the ambient projective space. Thus, $\oplus\, \mathrm{t}_k(\mathcal{O}(m)) = e^{m\, \mathrm{t}_1(\alpha_1(\mathcal{O}_1))}$. Since the Vandermonde determinant $\Delta(N+1, .., N+D+1) \neq 0$, we can find a linear combination W of $\mathcal{O}(N+1), \dots, \mathcal{O}(N+D+1)$ so that $\mathrm{t}_k(W) = 0$ for every $k \geq 1$ and $\mathrm{t}_0(W) = 1$. Clearly, $\mathrm{t}_k(\alpha_l(V_i \otimes W)) = \mathrm{t}_k(\alpha_l(V_i))$ is killed by $I(k,r,l)$.

$\square$

Proof of Theorem 2. Lemma 20 implies that given any fixed $l \geq 2$, there exists a constant C such that there exist infinitely many r such that given any $n > Cr^2 + r$,

$$I(k,r,l) \subsetneq I(k,r-1,l).$$

Lemma 22 implies that $I(k,r-1,l)$ annihilates $\mathrm{t}_k(x)$ for any element x of $F_{r-1}\mathrm{CH}^l(Q_{G(r,n)}) \otimes \mathbb{Q}$. Theorem 2 now follows immediately from the fact that $I(k,r,l)$ is the annihilator of $\mathrm{t}_k(\alpha_l(Q_{G(r,n)}))$ by definition.

6.2 Outline of proof of Theorem 3

Originally, the hope was for a stronger result saying that for fixed l and r, there exists a k satisfying $I(k,r,l) \subsetneq I(k,r-1,l)$. In fact, there was the hope of being able to show that $I(2r,r,l) \subsetneq I(2r,r-1,l)$. This would have shown that there is no morphism $f : G(r,2r) \to G(r-1,M)$ so that $f^*(\alpha_l(Q')) = \alpha_l(Q)$. We have so far been unable to do this in general. However, we have found (by means of a computer program) that $I(6,3,2) \subsetneq I(6,2,2)$ thus proving that if $f : G(3,6) \to G(2,M)$ is a morphism, then $f^*(\alpha_2(Q')) = C\alpha_1(Q)^2$. This we do by showing that $\oplus_{|\lambda|=3}\mathrm{End}(V_\lambda)$ contains an irreducible representation V_μ of S_6 not contained in $\oplus_{|\lambda|\leq 2}\mathrm{End}(V_\lambda)$, and that if π_μ denotes the projection from $\mathbb{K}S_k$ to V_μ, then $\pi_\mu * \tilde{e}_6^{(2)} * \tau_6 \neq 0$. This is achieved using a Mathematica program.

7 Proof of Theorem 1

7.1 A certain decomposition of $\mathbb{K}S_k$

Observe that $\mathbb{K}S_k = \oplus W_\lambda$ where W_λ is the $\mathbb{K}$-span of elements of S_k in the conjugacy class corresponding to the partition λ. We shall break each of the spaces W_λ further into a direct sum of $\mathbb{K}$-vector spaces in a specific manner. The significance of the new decomposition shall become clear as we proceed. First, let us decompose the conjugacy class $C_{(k)}$ which is the conjugacy class of the cycle τ_k. Note that $\tau_k = \sum_{l\geq 2} \tilde{e}_k^{(l)} * \tau_k$ and that $\tilde{e}_k^{(l)} \tilde{e}_k^{(l')} = \delta_{ll'} \tilde{e}_k^{(l)}$. Define operators Π_l on $C_{(k)}$ by $\Pi_l(\beta\tau_k\beta^{-1}) = \beta * (\tilde{e}_k^{(l)} * \tau_k)$ for $\beta \in S_k$ and extend this by linearity to $C_{(k)}$. Note that $\sum_{l\geq 2} \Pi_l(\beta * \tau_k) = \beta * \tau_k$. First, we need to check that we actually have a well defined operator here. It suffices to show that if $\beta, \gamma \in S_k$ with $\beta * \tau_k = \gamma * \tau_k$ then $\Pi_l(\beta * \tau_k) = \Pi_l(\gamma * \tau_k)$. In other words, we need to show that $\beta * (\tilde{e}_k^{(l)} * \tau_k) = \gamma * (\tilde{e}_k^{(l)} * \tau_k)$ which is equivalent to showing that $(\beta^{-1}\gamma) * (\tilde{e}_k^{(l)} * \tau_k) = \tilde{e}_k^{(l)} * \tau_k$. But $\beta * \tau_k = \gamma * \tau_k$ iff $\beta^{-1}\gamma = \tau_k^s$ for some s. Therefore, the fact that Π_l is well defined follows from the following lemma.

Lemma 23.
$$\tau_k^s * (\tilde{e}_k^{(l)} * \tau_k) = \tilde{e}_k^{(l)} * \tau_k$$
for any integer s.

Proof. This really follows from the fact that for any smooth scheme X, and for any vector bundle V on X,

$$\mathrm{sgn}(\tau_k)\tau_{k*}\,\mathrm{t}_k(V) = \mathrm{t}_k(V).$$

After all, $\mathrm{sgn}(\tau_k)\tau_{k*}{\theta_V}^k = {\theta_V}^k$ (by the properties of the cup product). Hence,

$$tr_*\varphi_*\,\mathrm{sgn}(\tau_k)\tau_{k*}{\theta_V}^k = tr_*\varphi_*{\theta_V}^k$$

where $\varphi : \mathrm{End}(V)^{\otimes k} \to \mathrm{End}(V)$ is k-fold composition. The right hand side of this equation is $\mathrm{t}_k(V)$ by definition. The left hand side is $\mathrm{sgn}(\tau_k)\tau_{k*}\,\mathrm{t}_k(V)$ since
$$tr \circ \varphi \circ \tau_k = \tau_k \circ tr \circ \varphi.$$
This tells us that $\mathrm{sgn}(\tau_k^s)\tau_{k*}^s\,\mathrm{t}_k(V) = \mathrm{t}_k(V)$. To finish the proof of the lemma, we observe that by Lemma 19,

$$\tau_k^s * (\tilde{e}_k^{(l)} * \tau_k) = \mathrm{sgn}(\tau_k^s)\tau_{k*}^s\,\mathrm{t}_k(\alpha_l(Q'))$$

and that
$$\tilde{e}_k^{(l)} * \tau_k = \mathrm{t}_k(\alpha_l(Q'))$$
where Q' is the universal quotient bundle of the Grassmannian $G(r', 2r')$ with r' chosen to be greater than k. □

The other detail to be verified is the fact that the operators Π_l are mutually orthogonal projections. For this, we see that

$$\Pi_l(\beta * \tau_k) = \beta * (\tilde{e}_k^{(l)} * \tau_k) = (\beta\tilde{e}_k^{(l)}) * \tau_k \implies \Pi_l \circ \Pi_m(\beta * \tau_k)$$
$$= (\beta\tilde{e}_k^{(m)}\tilde{e}_k^{(l)}) * \tau_k = (\beta\delta_{lm}\tilde{e}_k^{(l)}) * \tau_k.$$

We therefore, have a direct sum decomposition $W_{(k)} = \oplus_{l\geq 2}\Pi_l(W_{(k)})$.

We now proceed to breakup W_λ into a direct sum of $\mathbb{K}$-vector spaces in an analogous manner. Note that C_λ is the conjugacy class of $\tau_\lambda := \tau_{\lambda_1}\tau_{\lambda_2}\dots\tau_{\lambda_s}$ where the partition λ is given by $\lambda : k = \lambda_1 + .. + \lambda_s$, the λ_i 's arranged in decreasing order and where τ_{λ_i} is the cycle $(\lambda_1+\cdots+\lambda_i, \lambda_1+\cdots+\lambda_i-1, \dots, \lambda_1+\cdots+\lambda_{i-1})$ which is after all the cycle τ_{λ_i} embedded in S_k under the composition $S_{\lambda_i} \subset S_{\lambda_1}\times\cdots\times S_{\lambda_s} \subset S_k$. Call the map $S_{\lambda_1}\times\cdots\times S_{\lambda_s} \subset S_k$ as φ. Note that φ extends to a $\mathbb{K}$-algebra homomorphism $\varphi : \mathbb{K}(S_{\lambda_1}\times\cdots\times S_{\lambda_s}) \to \mathbb{K}(S_k)$. Identify $\mathbb{K}(S_{\lambda_1})\otimes\cdots\otimes\mathbb{K}(S_{\lambda_s})$ with $\mathbb{K}(S_{\lambda_1}\times\cdots\times S_{\lambda_s})$ and consider $(\tilde{e}^{(l_1)}_{\lambda_1}\otimes\cdots\otimes\tilde{e}^{(l_s)}_{\lambda_s})*\tau_\lambda$. By this we mean that we are looking at $\tilde{e}^{(l_1)}_{\lambda_1}\otimes\cdots\otimes\tilde{e}^{(l_s)}_{\lambda_s}$ as an element of $\mathbb{K}S_k$ through the homomorphism φ. We now make the following observations that give a step by step, explicit construction of the decomposition of $\mathbb{K}S_k$ that we are interested in.

Observation 1. *The elements $\tilde{e}^{(l_1)}_{\lambda_1}\otimes\cdots\otimes\tilde{e}^{(l_s)}_{\lambda_s}$ are mutually orthogonal idempotents in $\mathbb{K}(S_k)$ adding up to* id. This follows from the fact that the above statement is true in $\mathbb{K}(S_{\lambda_1}\times\cdots\times S_{\lambda_s})$.

Observation 2. *As $\tau_\lambda = \tau_{\lambda_1}\otimes\cdots\otimes\tau_{\lambda_s}$,*

$$(\tilde{e}^{(l_1)}_{\lambda_1}\otimes\cdots\otimes\tilde{e}^{(l_s)}_{\lambda_s})*\tau_\lambda = (\tilde{e}^{(l_1)}_{\lambda_1}*\tau_{\lambda_1})\otimes\cdots\otimes(\tilde{e}^{(l_s)}_{\lambda_s}*\tau_{\lambda_s})$$

It follows that if for some i, $\lambda_i \geq 2$ and $l_i = 1$, then

$$(\tilde{e}^{(l_1)}_{\lambda_1}\otimes\cdots\otimes\tilde{e}^{(l_s)}_{\lambda_s})*\tau_\lambda = 0.$$

Observation 3. *Let*

$$\tilde{e}^{(l)}_\lambda := \sum_{l_1+\cdots+l_s=l}\tilde{e}^{(l_1)}_{\lambda_1}\otimes\cdots\otimes\tilde{e}^{(l_s)}_{\lambda_s}.$$

Then $\tilde{e}^{(l)}_\lambda$ is an idempotent with

$$\tilde{e}^{(l)}_\lambda.(\tilde{e}^{(l_1)}_{\lambda_1}\otimes\cdots\otimes\tilde{e}^{(l_s)}_{\lambda_s}) = (\tilde{e}^{(l_1)}_{\lambda_1}\otimes\cdots\otimes\tilde{e}^{(l_s)}_{\lambda_s})$$

if $l_1+\cdots+l_s = l$ and

$$\tilde{e}^{(l)}_\lambda.(\tilde{e}^{(l_1)}_{\lambda_1}\otimes\cdots\otimes\tilde{e}^{(l_s)}_{\lambda_s}) = 0$$

otherwise.

Let Π_l be defined by $\Pi_l(\beta*\tau_\lambda) = (\beta\tilde{e}^{(l)}_\lambda)*\tau_\lambda$ for every $\beta\in C_\lambda$. We then have

Lemma 24. *The Π_l are well-defined mutually orthogonal projection operators on W_λ.*

Proof. Note that it suffices to show that if γ is a permutation in the stabilizer of τ_λ under conjugation, then $\gamma * (\tilde{e}_\lambda^{(l)} * \tau_\lambda) = \tilde{e}_\lambda^{(l)} * \tau_\lambda$. Note that if γ stabilizes τ_λ under conjugation, then γ is of the form $\zeta(\tau_{\lambda_1}^{r_1} \otimes \cdots \otimes \tau_{\lambda_s}^{r_s})$ where ζ permutes blocks of equal lengths among $[1, \ldots, \lambda_1], [\lambda_1 + 1, \ldots, \lambda_1 + \lambda_2], \ldots, [\lambda_1 + \cdots + \lambda_{s-1} + 1, \ldots, k]$ while preserving order within such blocks. Now we need to show that $\gamma * (\tilde{e}_\lambda^{(l)} * \tau_\lambda) = \tilde{e}_\lambda^{(l)} * \tau_\lambda$. Observe that

$$(\tau_{\lambda_1}^{r_1} \otimes \cdots \otimes \tau_{\lambda_s}^{r_s}) * (\tilde{e}_{\lambda_1}^{(l_1)} \otimes \cdots \otimes \tilde{e}_{\lambda_s}^{(l_s)}) * \tau_\lambda = (\tau_{\lambda_1}^{r_1} * \tilde{e}_{\lambda_1}^{(l_1)} * \tau_{\lambda_1}) \otimes \cdots \otimes (\tau_{\lambda_s}^{r_s} * \tilde{e}_{\lambda_s}^{(l_s)} * \tau_{\lambda_s})$$

$$= (\tilde{e}_{\lambda_1}^{(l_1)} \otimes \cdots \otimes \tilde{e}_{\lambda_s}^{(l_s)}) * \tau_\lambda$$

(the last equality by Lemma 23). So, we only need to show that

$$\zeta * \tilde{e}_\lambda^{(l)} * \tau_\lambda = \tilde{e}_\lambda^{(l)} * \tau_\lambda.$$

But this is true since ζ induces a permutation ζ' of $1, 2, .., s$ and we see that $\zeta.(\tilde{e}_{\lambda_1}^{(l_1)} \otimes \cdots \otimes \tilde{e}_{\lambda_s}^{(l_s)}) = (\tilde{e}_{\lambda_{\zeta'(1)}}^{(l_{\zeta'(1)})} \otimes \cdots \otimes \tilde{e}_{\lambda_{\zeta'(s)}}^{(l_{\zeta'(s)})})$. □

Observation 4. *It now follows from this and the fact that the* Π_l *are mutually orthogonal idempotents adding upto* id *that*

$$W_\lambda = \oplus \Pi_l(W_\lambda).$$

Also, Observation 2 tells us that $\Pi_1(W_\lambda) = 0$ *and that* $\Pi_2(W_\lambda) = 0$ *if* $\lambda \neq (k)$. *Therefore, this direct sum decomposition runs over* $l \geq 2$. *Combining this with the decomposition* $\mathbb{K}S_k = \oplus_\lambda W_\lambda$, *we see that*

$$\mathbb{K}S_k = \oplus_\lambda \oplus_{l \geq 2} \Pi_l(W_\lambda) = \oplus_{l \geq 2} \Pi_l(\mathbb{K}S_k).$$

7.2 Proof of Corollary 1

Definition :Define an *elementary functor of type* (k, l) to be a map v (not necessarily linear) from $K(X) \otimes \mathbb{Q}$ to $\mathrm{R}_k(X)$ such that

$$w(x) = \beta_* \, \mathrm{t}_{\lambda_1}(\alpha_{l_1}(x)) \cup \cdots \cup \mathrm{t}_{\lambda_s}(\alpha_{l_s}(x))$$

for some $\beta \in \mathbb{K}S_k$, some s-tuple $(\lambda_1, .., \lambda_s)$ of non-negative integers adding up to k and some s-tuple $(l_1, \ldots, l_s)$ of non-negative integers adding up to l.

Define a *functor of type* (k, l) to be a map from $K(X) \otimes \mathbb{Q}$ to $\mathrm{R}_k(X)$ given by a "linear combination of elementary functors of type (k, l)". In other words, a functor of type (k, l) is a map v from $K(X) \otimes \mathbb{Q}$ to $\mathrm{R}_k(X)$ such that

$$v(x) = \sum_{j=1}^{j=p} c_j w_j(x)$$

where $p \in \mathbb{N}$, and $w_1, .., w_p$ are elementary functors of type (k, l).

Define a *vector of type* (k, l) in $P_{r,n}(\mathbb{K}S_k)$ to be an element of the form $v(Q)$, where v is a functor of type (k, l) and Q is the universal quotient bundle of the Grassmannian $G(r, n)$.

Note that if v is a functor of type (k, l), then

$$v(\psi^p x) = p^l v(x)$$

for any $x \in K(X) \otimes \mathbb{Q}$. Also note that functors of type (k, l) respect pullbacks.

We now try to understand what the decomposition of $\mathbb{K}S_k$ given in the Section 7.1 means. Lemma 19 together with Lemma 3 part 3 tells us that

$$\mathrm{t}_{\lambda_1}(\alpha_{l_1}(Q_{G(r,n)})) \cup \cdots \cup \mathrm{t}_{\lambda_s}(\alpha_{l_s}(Q_{G(r,n)})) = P_{r,n}(\tilde{e}_\lambda^{(l)} * \tau_\lambda).$$

Also, by Lemma 3 part 2

$$\mathrm{sgn}(\beta)\beta_*^{-1}\, \mathrm{t}_{\lambda_1}(\alpha_{l_1}(Q_{G(r,n)})) \cup \cdots \cup \mathrm{t}_{\lambda_s}(\alpha_{l_s}(Q_{G(r,n)})) = P_{r,n}(\beta * \tilde{e}_\lambda^{(l)} * \tau_\lambda).$$

Let $l = \sum_i l_i$. Thus the space spanned by

$$\{\beta_*\, \mathrm{t}_{\lambda_1}(\alpha_{l_1}(Q_{G(r,n)})) \cup \cdots \cup \mathrm{t}_{\lambda_s}(\alpha_{l_s}(Q_{G(r,n)})) \mid \sum_i l_i = l, \sum_i \lambda_i = k\},$$

which is $P_{r,n}(\Pi_l(\mathbb{K}S_k))$, is precisely the space of vectors of type (k, l).

If both r and $n - r$ are larger than k, then $P_{r,n} = \mathrm{id}$. What we did in Section 7.1 shows that in this case, $\mathbb{K}S_k$ decomposes into the direct sum of the spaces $\Pi_l(\mathbb{K}S_k)$. The space $\Pi_l(\mathbb{K}S_k)$ is stable under conjugation and is the space of vectors of type (k, l). However, if k is not too large, something very interesting happens primarily because the projection $P_{n,r}$ "behaves badly" with the projections Π_l. Let $n \geq 2r + 1$ and let $k = 2r$. Then, $P_{r,n} = P_r$. Also, $\mathrm{t}_j(Q_{G(r,n)}) = \mathrm{t}_j(Q_{G(r,M)})$ for every $M \geq n$ and every $j \leq k$. It follows that $v_l(Q_{G(r,n)}) = v_l(Q_{G(r,M)})$ for all $M \geq n$ if v_l is any functor of type $(2r, l)$. Let Q denote $Q_{G(r,n)}$. The following claim holds in this situation.

Claim: There exists a nontrivial linear dependence relation of the form

$$\sum_l v_l(Q) = 0$$

such that v_l is a functor of type $(2r, l)$ for each l.

The above claim is proven in Section 7.3. This leads to Corollary 1 as follows. Choose a shortest nontrivial linear dependence relation of the form

$$\sum_l v_l(Q) = 0$$

with v_l a functor of type $(2r, l)$. Then, suppose that there exists a map $f : G(r, n) \to G(r, M)$ with $f^*([Q_{G(r,M)}]) = \psi^p[Q_{G(r,n)}]$, we can assume without loss of generality that $M \geq n$. Thus,

$$0 = f^*(\sum_l v_l(Q_{G(r,M)})) = \sum_l v_l(f^* Q_{G(r,M)}) = \sum_l v_l(\psi^p Q) = \sum_l p^l v_l(Q).$$

Since $p \geq 2$, comparing this linear dependence relation with the previous one would enable us to extract a linear dependence relation of the same form but of shorter length than the one we began with. This yields a contradiction.

The proof of theorem 1 requires a little more work which we do in Section 7.4.

7.3 A linear dependence relation between functors of type $(2r, l)$

First, we observe that if V is a vector space with $V = V_1 \oplus V_2$ and also $V = \oplus W_i$, with p_i being the projections to V_i and π_i being the projections to W_i, then

$$\dim\ p_1(W_1) + \cdots + \dim\ p_1(W_m) \geq \dim\ V_1.$$

To see this, suppose that equality holds. Then,

$$\dim\ p_1(W_i) = \dim\ W_i - \dim\ W_i \cap V_2$$

$$\implies \dim\ W_1 \cap V_2 + \cdots + \dim\ W_m \cap V_2 = \dim\ V_2.$$

From this, we see that $\pi_i(V_2) = W_i \cap V_2$ for all $i \in \{1, 2, .., , \}$. In particular, if $\pi_i(V_2) \neq W_i \cap V_2$, then

$$\dim\ p_1(W_1) + \cdots + \dim\ p_1(W_m) > \dim\ V_1.$$

Having said this, we will prove that for $V = \mathbb{K}S_{2r}$ ($V = V_1 \oplus V_2$ where $V_1 = \oplus_{|\lambda| \leq r} \operatorname{End}(V_\lambda)$ and $V_2 = \oplus_{|\lambda| > r} \operatorname{End}(V_\lambda)$ also $V = \oplus_{l \geq 2} \Pi_l(V)$)

$$\Pi_2(V_2) \neq \Pi_2(V) \cap V_2.$$

This will prove that

$$\sum_{l \geq 2} \dim\ P_r(\Pi_l(V)) > \dim\ V_1.$$

Observation 4 of Section 7.1 tells us that $\Pi_2(V) = \Pi_2(W_{(2r)})$. Any element in this space is a linear combination of conjugates of τ_{2r}. It follows that if such a linear combination is nonzero in $\operatorname{End}(V_\lambda)$ it is also nonzero as an element of $\operatorname{End}(V_{\bar{\lambda}})$, where $\bar{\lambda}$ is the partition conjugate to λ. Thus $\Pi_{(2r)}(V) \cap V_2 = 0$. It therefore, suffices to prove that $\Pi_2(V_2) \neq 0$.

Lemma 25. *To prove that* $\Pi_2(V_2) \neq 0$*, it suffices to show that*

$$\Pi_2((1\ 2r)\sum_{g \in C_\mu} g) \neq 0$$

where $(1\ 2r)$ *is the transposition interchanging* 1 *with* $2r$ *and* μ *is some partition among* $\{(2r-1,1),\ldots,(r,r)\}$*.*

Proof. Consider the matrix $M = (\chi_\lambda(C_\mu))$ where λ runs over all partitions of $2r$ that satisfy $\lambda \geq (r,r)$ (recall that there is a lexicographic ordering among the partitions,enabling one to compare them), and $\mu \in \{(2r-1,1),\ldots,(r,r)\}$. Note that if λ is such a partition and $\lambda \neq (r,r)$ then $\lambda_1 \geq r+1$. We claim that M is of rank r. To prove this, it suffices to show that N is of rank r where $N = (\psi_\lambda(C_\mu))$, where

$$\psi_\lambda = Ind_{S_\lambda}^{S_{2r}}(triv) = \chi_\lambda + \sum_{\mu > \lambda} K_{\mu\lambda}\chi_\mu.$$

However,

$$\psi_\lambda(C_\mu) = \frac{1}{|C_\mu|}[S_{2r} : S_\lambda]|C_\mu \cap S_\lambda|.$$

Therefore, $\psi_\lambda(C_\mu) = 0$ if $\mu > \lambda$. This lexicographic order is a total order. Consider the restriction of N to the rows given by the partitions in $\{(2r-1,1),\ldots,(r,r)\}$. This restriction of N is then a lower triangular matrix with nonzero diagonal entries if the rows are arranged in the correct order (since $\psi_\lambda(C_\lambda) \neq 0$). It follows that N and therefore, M are matrices of rank r.

We further claim that if we restrict M to rows corresponding to $\lambda > (r,r)$, we still get a matrix of rank r. To see this, we need to show that for some scalars a_λ,

$$\chi_{(r,r)}(C_\mu) = \sum_{\lambda > (r,r)} a_\lambda \chi_\lambda(C_\mu)$$

for all $\mu \in \{(2r-1,1),\ldots,(r,r)\}$. For this, it is enough to show that

$$\psi_{(r,r)}(C_\mu) = \sum_{\lambda > (r,r)} b_\lambda \psi_\lambda(C_\mu)$$

for all $\mu \in \{(2r-1,1),\ldots,(r,r)\}$, for some scalars b_λ. In fact, we claim that there are scalars b_i,$0 \leq i \leq r-1$, so that

$$\psi_{(r,r)}(C_\mu) = \sum_{0 \leq i \leq r-1} b_i \psi_{(2r-i,i)}(C_\mu).$$

Note that $|C_{(2r-s,s)} \cap S_{(2r-t,t)}| = 0$ if $s \neq t$ and both are nonzero. Also note that $\psi_{(2r)}(C_{(r,r)}) \neq 0$. Thus the vector $(\psi_{(2r)}(C_\mu)), \mu \in \{(2r-1,1),\ldots,(r,r)\}$

is given by $(a_1,..,a_r)$, where $a_r \neq 0$. The vector $\psi_{(2r-s,s)}(C_\mu), \mu \in \{(2r-1,1),\dots,(r,r)\}$ is given by $(0,..,0,d_s,\dots,0)$, $d_s \neq 0$ for $1 \leq s \leq r-1$. Thus,

$$\psi_{(2r)}(C_\mu) - \sum \frac{a_s}{d_s}\psi_{(2r-s,s)}(C_\mu) = (0,..,0,a_r)$$

which is a nonzero multiple of $\psi_{(r,r)}(C_\mu)$. This shows that the matrix $M = \chi_\lambda(C_\mu)$ where $\lambda > (r,r)$ and $\mu \in \{(2r-1,1),\dots,(r,r)\}$ is of rank r. Since $\chi_{\bar{\lambda}} = \chi_\lambda.\,\mathrm{sgn}$, and $|\lambda| \geq r+1$ iff $\lambda > (r,r)$, the matrix $M' = \chi_\lambda(C_\mu)$ where $|\lambda| \geq r+1$ and $\mu \in \{(2r-1,1),\dots,(r,r)\}$ is obtained from M by multiplying some columns by -1 and is therefore of rank r.

Now suppose that $\Pi_2((1\ 2r)\sum_{g\in C_{(2r-s,s)}} g) \neq 0$ for some $1 \leq s \leq r$. Since M' is of rank r, we can find a linear combination of rows of M' that gives us the vector e_s i.e, $\sum_{|\lambda|>r+1} a_\lambda\chi_\lambda(C_\mu) = 0$ if $\mu \neq (2r-s,s)$ and $\sum_{|\lambda|>r+1} a_\lambda\chi_\lambda(C_\mu) = 1$ if $\mu = (2r-s,s)$. So,

$$\Pi_2((1\ 2r)(\sum_{\substack{g\in S_{2r}\\ |\lambda|>r+1}} a_\lambda\chi_\lambda(g)g)) = \Pi_2((1\ 2r)\sum_{g\in C_{(2r-s,s)}} g) \neq 0.$$

The first equality is because only the $2r$ cycles contribute to $\Pi_2(V)$. Note that since $\sum \chi_\lambda(g)g \in \mathrm{End}(V_\lambda)$ it follows that

$$(\sum_{\substack{g\in S_{2r}\\ |\lambda|>r+1}} a_\lambda\chi_\lambda(g)g) \in V_2$$

and hence

$$(1\ 2r)(\sum_{\substack{g\in S_{2r}\\ |\lambda|>r+1}} a_\lambda\chi_\lambda(g)g) \in V_2.$$

It follows that $\Pi_2(V_2) \neq 0$. $\square$

Lemma 26. *For some s, $1 \leq s \leq r$, we have* $\Pi_2((1\ 2r)\sum_{g\in C_{(2r-s,s)}} g) \neq 0$.

Proof. Every $2r$ cycle that arises in $(1\ 2r)\sum_{g\in C_{(2r-s,s)}} g$ arises with coefficient 1. We therefore need to identify the $2r$ cycles that do arise. They are those of the form $(1\ a_2\ ..\ a_s\ 2r\ a_{s+2}\ \dots)$ or $(1\ a_2\ \dots\ a_{2r-s}\ 2r \dots)$. For this proof, denote the subgroup of S_{2r} fixing the elements i and j by $S(i,j)$ for any $1 \leq i < j \leq 2r$. We note that

$$(1\ 2r)\sum_{g\in C_{(2r-s,s)}} g$$

$$= \sum_{\alpha\in S(1,2r)} \alpha * (2r\ 2r-s\ 2r-s-1....1\ 2r-1\ 2r-2\dots\ 2r-s+1)$$

$$+\alpha * (2r\ s\ s-1\dots\ 1\ 2r-1..\ s+1)$$

$$\begin{aligned} &= \sum_{\alpha\in S(1,2r)} \alpha * (\tau_{2r-1}^{s-1} + \tau_{2r-1}^{2r-s-1}) * \tau_{2r} \\ &= (\tau_{2r-1}^{-1}(\sum_{\beta\in S(2r-1,2r)} \beta)\tau_{2r-1}) * (\tau_{2r-1}^{s-1} + \tau_{2r-1}^{2r-s-1}) * \tau_{2r} \\ &= (\tau_{2r-1}^{-1} \sum_{\beta\in S(2r-1,2r)} \beta) * (\tau_{2r-1}^{s} + \tau_{2r-1}^{2r-s}) * \tau_{2r}. \end{aligned}$$

Therefore,

$$\begin{aligned} \Pi_2((1\ 2r) \sum_{g\in C_{(2r-s,s)}} g) &= (\tau_{2r-1}^{-1} \sum_{\beta\in S(2r-1,2r)} \beta) * (\tau_{2r-1}^{s} + \tau_{2r-1}^{2r-s}) * (\tilde{e}_{2r}^{(2)} * \tau_{2r}) \\ &= (\tau_{2r-1}^{-1} \sum_{\beta\in S(2r-1,2r)} \beta) * (\tau_{2r-1}^{s} + \tau_{2r-1}^{2r-s}) * (\tilde{e}_{2r-1}^{(1)} * \tau_{2r}), \end{aligned}$$

the last equality following from Lemma 21.

For this proof, denote the subgroup of S_{2r-1} fixing the element i by $S(i)$ for any $1 \le i \le 2r-1$. It therefore, suffices to show that

$$(\tau_{2r-1}^{-1} \sum_{\beta\in S(2r-1)} \beta)(\tau_{2r-1}^{s} + \tau_{2r-1}^{2r-s})(\tilde{e}_{2r-1}^{(1)}) \neq 0$$

for some s, $1 \le s \le r$. It therefore, suffices to show that

$$W_s := (\sum_{\beta\in S(2r-1)} \beta)(\tau_{2r-1}^{s} + \tau_{2r-1}^{2r-s})(\tilde{e}_{2r-1}^{(1)}) \neq 0$$

for some s, $1 \le s \le r$. Consider a vector space V of finite dimension, and let u and v be two basis vectors of V. We will show that the right action of W_s on $u^{\otimes 2r-2} \otimes v$ is nonzero. Note that

$$\begin{aligned} \frac{1}{(2r-2)!}(u^{\otimes 2r-2} \otimes v)W_s &= (u^{\otimes 2r-2} \otimes v)(\tau_{2r-1}^{s} + \tau_{2r-1}^{2r-s})\tilde{e}_{2r-1}^{(1)} \\ &= (u^{\otimes s-1} \otimes v \otimes u^{\otimes 2r-1-s} + u^{\otimes 2r-1-s} \otimes v \otimes u^{\otimes s-1})\tilde{e}_{2r-1}^{(1)}. \end{aligned}$$

Therefore, it is enough to show that

$$(u^{\otimes s-1} \otimes v \otimes u^{\otimes 2r-1-s} + u^{\otimes 2r-1-s} \otimes v \otimes u^{\otimes s-1})\tilde{e}_{2r-1}^{(1)} \neq 0$$

for some s, $1 \le s \le r$. For this, we note that

$$0 \neq ad(u)^{2r-2}(v) = (l_u - r_u)^{2r-2}(v) = \sum_i \binom{2r-2}{i} u^{\otimes i} \otimes v \otimes u^{2r-2-i}.$$

Now, $ad(u)^{2r-2}(v)$ is an element of the free Lie algebra generated by V. The idempotent $\tilde{e}_{2r-1}^{(1)}$ therefore acts as the identity on this vector, which is a linear combination of $(u^{\otimes s-1} \otimes v \otimes u^{\otimes 2r-1-s} + u^{\otimes 2r-1-s} \otimes v \otimes u^{\otimes s-1})$ where s runs from 1 to r.

□

7.4 Final step to the proof of Theorem 1

Suppose that $[\psi^p Q] = [Y]$ for some genuine vector bundle Y. Then Y is of rank r, and for all sufficiently large m, $Y \otimes \mathcal{O}(m)$ is a quotient of ${\mathcal{O}_G}^s$ for some s. It follows that $Y \otimes \mathcal{O}(m) = f^* Q'$ for some morphism $f : G(r,n) \to G(r,n')$, where Q' is the universal quotient bundle of $G(r,n')$. Without loss of generality we may assume that $n' \geq 2r+1$. Let Q denote the universal quotient bundle of $G(r,n)$. As in Section 7.2, choose a shortest linear dependence relation of the form

$$\sum_l v_l(Q) = 0$$

where v_l is a functor of type $(2r, l)$.

Then, $\sum_l v_l(Q') = 0$. Since the v_l's respect pullbacks,

$$\sum_l v_l(Y \otimes \mathcal{O}(m)) = 0$$

for all sufficiently large m. Note that $\oplus\, \mathrm{t}_k(\mathcal{O}(m)) = exp(\mathrm{t}_1(\alpha_1(\mathcal{O}(1))))$. Therefore,

$$\mathrm{t}_{\lambda_i}(\alpha_{l_i}(Y \otimes \mathcal{O}(m))) = \mathrm{t}_{\lambda_s}(\alpha_{l_s}(Y) + m\alpha_{l_s-1}(Y)\alpha_1(\mathcal{O}(1)) + \dots).$$

Therefore,

$$v_l(Y \otimes \mathcal{O}(m)) = v_l(Y) + m.A_1(Y) + \cdots + m^s A_s(Y)$$

for all l with $A_i(Y) \in \mathrm{R}(G(r,n))$. In other words, $v_l(Y \otimes \mathcal{O}(m))$ is a polynomial in m with coefficients in $\mathrm{R}(G(r,n))$ whose constant term is $v_l(Y)$. It follows that $\sum_l v_l(Y \otimes \mathcal{O}(m))$ is a polynomial in m with coefficients in $\mathrm{R}(G(r,n))$ whose constant term is $\sum_l v_l(Y)$. The fact that $\sum_l v_l(Y \otimes \mathcal{O}(m))$ vanishes for all sufficiently large m implies that $\sum_l v_l(Y) = 0$. Thus,

$$\sum_l v_l(\psi^p Q) = \sum_l p^l v_l(Q) = 0$$

as well. As in Section 7.2, since $p \geq 2$, this together with the linear dependence relation $\sum_l v_l(Q) = 0$ yields a linear dependence relation of the same form but of shorter length, thereby giving a contradiction. This finally proves Theorem 1.

Appendix

This appendix if for sketching a proof of Observation 1 of Section 3. This material is by and large reproduced from notes by Jinhyun Park [13] of a course taught by Madhav Nori at the University of Chicago in Fall 2004.

Recall that given a morphism $f : Y \to X$ of schemes, a sheaf $\mathcal{F}$ on Y is said to have descent data if it satisfies the following three properties.

$[D_1]$. Given any two morphisms $g_1, g_2 : Z \to Y$ such that $f \circ g_1 = f \circ g_2$, there is an isomorphism $c(g_1, g_2) : g_1^*\mathcal{F} \cong g_2^*\mathcal{F}$.
$[D_2]$. (Functoriality). Given any morphisms $h : W \to Z$ and $g_1, g_2 : Z \to Y$ such that $f \circ g_1 = f \circ g_2$, the following diagram commutes.

$$\begin{array}{ccc} h^* \circ g_1^*\mathcal{F} & \xrightarrow{h^*c(g_1,g_2)} & h^* \circ g_2^*\mathcal{F} \\ \simeq \big\downarrow & & \big\downarrow \simeq \\ (g_1 \circ h)^*\mathcal{F} & \xrightarrow{c(g_1 \circ h, g_2 \circ h)} & (g_2 \circ h)^*\mathcal{F} \end{array}$$

$[D_3]$. Given any three morphisms $g_1, g_2, g_3 : Z \to Y$ such that $f \circ g_1 = f \circ g_2 = f \circ g_3$ the following diagram commutes.

$$\begin{array}{ccc} g_1^*\mathcal{F} & \xrightarrow{c(g_1,g_2)} & g_2^*\mathcal{F} \\ {\scriptstyle c(g_1,g_3)} \big\downarrow & & \big\downarrow {\scriptstyle c(g_2,g_3)} \\ g_3^*\mathcal{F} & \xrightarrow{\mathrm{id}} & g_3^*\mathcal{F} \end{array}$$

We now recall a theorem of Grothendieck [15].

Theorem 4. *Let $f : Y \to X$ be a flat surjective morphism of schemes. There is an equivalence of categories*

$$\{\textit{Quasicoherent sheaves on } X\} \longleftrightarrow \{\textit{quasicoherent sheaves on } Y \textit{ with descent data}\}$$

$$\mathcal{G} \mapsto f^*\mathcal{G}.$$

The following construction due to Grothendieck [15] gives the inverse to the above equivalence of categories.

Construction 1. Let $\mathcal{F}$ be a quasicoherent sheaf on Y with descent data. Note that for every open $U \subset X$, $\mathcal{F}|_{f^{-1}(U)}$ is a quasicoherent sheaf with descent data for the morphism $f|_{f^{-1}(U)} : f^{-1}(U) \to U$. Let $\overline{\mathcal{F}}$ denote the sheafification of the presheaf

$$U \mapsto \{s \in \Gamma(f^{-1}(U), \mathcal{F}) \mid c(g_1, g_2)g_1^*s = g_2^*s \text{ for all } g_1, g_2 : Z \to f^{-1}(U)\}.$$

The inverse to the equivalence of categories in Theorem 4 is given by $\mathcal{F} \mapsto \overline{\mathcal{F}}$.
For example, $\overline{\mathcal{O}_Y} = \mathcal{O}_X$.
Let P be an affine group scheme over $\mathbb{K}$. Let $f : Y \to X$ be a principal P-bundle on X. Then, descent data for f on a sheaf $\mathcal{F}$ is indeed equivalent to a P-action on $\mathcal{F}$. Theorem 4 therefore implies the following theorem.

Theorem 5. *Let $f : Y \to X$ be a principal P-bundle. There is an equivalence of categories*

$$\{\text{ Quasicoherent sheaves on } X\} \longleftrightarrow \{\text{Quasicoherent sheaves on } Y \text{ with } P \text{ action}\}$$

$$\mathcal{G} \mapsto f^*\mathcal{G}.$$

Corollary 6. *The functor*

$$F : \{\text{ } P\text{-representations}\} \longrightarrow \{\text{locally free Quasicoherent sheaves on } X\}$$

$$F(V) = \overline{\mathcal{O}_Y \otimes_{\mathbb{K}} V}$$

is an exact functor commuting with $\otimes$.

Proof. $\mathcal{O}_Y$ is naturally a P-sheaf on Y. A representation V of P is a H-sheaf on Spec $\mathbb{K}$. Therefore, $\mathcal{O}_Y \otimes_{\mathbb{K}} V$ is a P-sheaf on $Y = Y \times_{\mathrm{Spec}\ \mathbb{K}} \mathrm{Spec}\ \mathbb{K}$. By Theorem 5, $F(V)$ is a quasicoherent sheaf on X. Clearly, $F(V)$ is locally free. It can also be verified without difficulty that $V \mapsto \mathcal{O}_Y \otimes_{\mathbb{K}} V$ is an exact functor commuting with $\otimes$. Since the functor from Theorem 5 is an equivalence of categories, the desired corollary follows. □

We can now sketch the proof of the following theorem. Let G be a affine algebraic group and let P be a closed subgroup of G. Let $\mathcal{P}$ denote the category of P-representations. With these assumptions, we have the following theorem of Bott [4]. This theorem has been referred to in Section 3 as Observation 1.

Theorem 6. *Let G be reductive. If $\mathbb{K}$ is regarded as the trivial P-representation,*

$$H^i(G/P, F(V))^G \simeq \mathit{Ext}^i_{\mathcal{P}}(\mathbb{K}, V).$$

Proof. For any $V \in \mathcal{P}$, let $T^i(V) = \mathrm{H}^i(G/P, F(V))^G$. We shall show that in the language of Grothendieck [14], $T^0(V) = \mathrm{Hom}_{\mathcal{P}}(\mathbb{K}, V)$ and $T^i(V) = R^iT^0(V)$. This will prove the desired theorem. To do this, we need to verify the following list of properties.

(a) $T^i : \mathcal{P} \to \mathbb{K}-$ vector spaces is a functor.

(b) Given a short exact sequence $0 \to V' \to V \to V'' \to 0$ in $\mathcal{P}$, there is a long exact sequence

$$\dots \xrightarrow{\delta} T^i(V') \longrightarrow T^i(V) \longrightarrow T^i(V'') \xrightarrow{\delta} T^{i+1}V' \longrightarrow \dots$$

The given short exact sequence gives a long exact sequence $\mathrm{H}^i(G/P, -)$. Now, for any exact sequence $W' \to W \to W''$ of G-representations, the sequence ${W'}^G \to W^G \to {W''}^G$ is exact. This verifies (b).

(c) The data in (b) is functorial.

(d) $T^0(V) = V^P$.

(e) (effaceability) For all $i > 0$, for all $\alpha \in T^i(V)$, there is a monomorphism

$j : V \to W$ in $\mathcal{P}$ such that $T^i(j)(\alpha) = 0$.

We check (e), the only nontrivial assertion above. Put $W = \Gamma(P, \mathcal{O}_P)$. Then, $F(W) = f_*\mathcal{O}_G$ where $f : G \to G/P$ is the natural morphism. Note that G is affine and f ia an affine morphism. Therefore, for any quasicoherent sheaf $\mathcal{F}$ on G, $\mathrm{H}^i(G, \mathcal{F}) = 0$ for every $i > 0$ and $R^i f_*\mathcal{F} = 0$ for every $i > 0$. The Leray spectral sequence then tell us that $\mathrm{H}^i(G/P, f_*\mathcal{F}) \simeq \mathrm{H}^i(G, \mathcal{F}) = 0$ for all $i > 0$. In particular, $\mathrm{H}^i(G/P, F(W)) = 0$ for every $i > 0$. Let V be any P-representation. We have an isomorphism

$$\mathrm{Hom}_{\mathcal{P}}(V, \Gamma(P, \mathcal{O}_P)) \simeq V^* \tag{1}$$

$$L \mapsto \mathrm{ev}_{\mathrm{id}} \circ L.$$

Here, $\mathrm{ev}_{\mathrm{id}} \circ L$ is the composite

$$V \xrightarrow{L} \Gamma(P, \mathcal{O}_P) \xrightarrow{\mathrm{ev}_{\mathrm{id}}} \mathbb{K}\cdot$$

Denote the inverse of the isomorphism (1) by S. Choose linear functionals $u_1, \dots, u_i, \dots$ on V such that $\cap \mathrm{Ker}(u_i) = 0$. Then, $S(u_i) \in \mathrm{Hom}_{\mathcal{P}}(V, \Gamma(P, \mathcal{O}_P))$. Clearly, the morphism $\oplus_i S(u_i) : V \to \oplus_i \Gamma(P, \mathcal{O}_P)$ is a monomorphism in $\mathcal{P}$. Further, $T^p(\oplus_i \Gamma(P, \mathcal{O}_P)) = 0$ whenever $p > 0$ since we just showed that $T^p(\Gamma(P, \mathcal{O}_P)) = 0$ whenever $p > 0$. This completes the verification of (e) and therefore, the proof of the desired theorem.

□

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