Chow–Künneth Decomposition for Some Moduli Spaces[1]

Jaya NN Iyer, Stefan Müller–Stach

Received: February 2, 2008
Revised: July 7, 2008

Communicated by Thomas Peternell

Abstract. In this paper we investigate Murre's conjecture on the Chow–Künneth decomposition for universal families of smooth curves over spaces which dominate the moduli space $\mathcal{M}_g$, in genus at most 8 and show existence of a Chow–Künneth decomposition. This is done in the setting of equivariant cohomology and equivariant Chow groups to get equivariant Chow–Künneth decompositions.

2000 Mathematics Subject Classification: 14C25, 14D05, 14D20, 14D21
Keywords and Phrases: Equivariant Chow groups, orthogonal projectors.

Contents

[1]This work was supported by DFG Sonderforschungsbereich/Transregio 45.

1. Introduction

Suppose X is a nonsingular projective variety defined over the complex numbers. We consider the rational Chow group $CH^i(X)_{\mathbb{Q}} = CH^i(X) \otimes \mathbb{Q}$ of algebraic cycles of codimension i on X. The conjectures of S. Bloch and A. Beilinson predict a finite descending filtration $\{F^jCH^i(X)_{\mathbb{Q}}\}$ on $CH^i(X)_{\mathbb{Q}}$ and satisfying certain compatibility conditions. A candidate for such a filtration has been proposed by J. Murre and he has made the following conjecture [Mu2],
Murre's conjecture: The motive (X, Δ) of X has a Chow-Künneth decomposition:

$$\Delta = \sum_{i=0}^{2d} \pi_i \in CH^d(X \times X) \otimes \mathbb{Q}$$

such that π_i are orthogonal projectors, lifting the Künneth projectors in $H^{2d-i}(X) \otimes H^i(X)$. Furthermore, these algebraic projectors act trivially on the rational Chow groups in a certain range.
These projectors give a candidate for a filtration of the rational Chow groups, see §2.1.
This conjecture is known to be true for curves, surfaces and a product of a curve and surface [Mu1], [Mu3]. A variety X is known to have a Chow–Künneth decomposition if X is an abelian variety/scheme [Sh],[De-Mu], a uniruled threefold [dA-Mü1], universal families over modular varieties [Go-Mu], [GHM2] and the universal family over one Picard modular surface [MMWYK], where a partial set of projectors are found. Finite group quotients (maybe singular) of an abelian variety also satisfy the above conjecture [Ak-Jo]. Furthermore, for some varieties with a nef tangent bundle, Murre's conjecture is proved in [Iy]. A criterion for existence of such a decomposition is also given in [Sa]. Some other examples are also listed in [Gu-Pe].
Gordon-Murre-Hanamura [GHM2], [Go-Mu] obtained Chow–Künneth projectors for universal families over modular varieties. Hence it is natural to ask if the universal families over the moduli space of curves of higher genus also admit a Chow–Künneth decomposition. In this paper, we investigate the existence of Chow–Künneth decomposition for families of smooth curves over spaces which closely approximate the moduli spaces of curves $\mathcal{M}_g$ of genus at most 8, see §5.
In this example, we take into account the non-trivial action of a linear algebraic group G acting on the spaces. This gives rise to the equivariant cohomology and equivariant Chow groups, which were introduced and studied by Borel, Totaro, Edidin-Graham [Bo], [To], [Ed-Gr]. Hence it seems natural to formulate Murre's conjecture with respect to the cycle class maps between the rational equivariant Chow groups and the rational equivariant cohomology, see §4.5. Since in concrete examples, good quotients of non-compact varieties exist, it became necessary to extend Murre's conjecture for non-compact smooth varieties, by taking only the bottom weight cohomology $W_iH^i(X, \mathbb{Q})$ (see [D]), into consideration. This is weaker than the formulation done in [BE]. For our purpose though, it suffices to look at this weaker formulation. We then

construct a category of equivariant Chow motives, fixing an algebraic group G (see [dB-Az], [Ak-Jo], for a category of motives of quotient varieties, under a finite group action).
With this formalism, we show (see §5.2);

Theorem 1.1. *The equivariant Chow motive of a universal family of smooth curves $\mathcal{X} \to U$ over spaces U which dominate the moduli space of curves $\mathcal{M}_g$, for $g \leq 8$, admits an equivariant Chow–Künneth decomposition, for a suitable linear algebraic group G acting non-trivially on $\mathcal{X}$.*

Whenever smooth good quotients exist under the action of G, then the equivariant Chow-Künneth projectors actually correspond to the absolute Chow–Künneth projectors for the quotient varieties. In this way, we get orthogonal projectors for universal families over spaces which closely approximate the moduli spaces $\mathcal{M}_g$, when g is at most 8.
One would like to try to prove a Chow–Künneth decomposition for $\mathcal{M}_g$ and $\mathcal{M}_{g,n}$ (which parametrizes curves with marked points) and we consider our work a step forward. However since we only work on an open set U one has to refine projectors after taking closures a bit in a way we don't yet know.
Other examples that admit a Chow–Künneth decomposition are Fano varieties of r-dimensional planes contained in a general complete intersection in a projective space, see Corollary 5.3.
The proofs involve classification of curves in genus at most 8 by Mukai [Muk],[Muk2] with respect to embeddings as complete intersections in homogeneous spaces. This allows us to use Lefschetz theorem and construct orthogonal projectors.
Acknowledgements: The first named author thanks the Math Department of Mainz, for its hospitality during the visits in 2007 and 2008, when this work was carried out. We also thank a referee for a useful remark concerning our definition of the weight filtration.

2. Preliminaries

The category of nonsingular projective varieties over $\mathbb{C}$ will be denoted by $\mathcal{V}$. Let $CH^i(X)_\mathbb{Q} = CH^i(X) \otimes \mathbb{Q}$ denote the rational Chow group of codimension i algebraic cycles modulo rational equivalence.
Suppose $X, Y \in Ob(\mathcal{V})$ and $X = \cup X_i$ be a decomposition into connected components X_i and $d_i = \dim X_i$. Then $\mathrm{Corr}^r(X,Y) = \oplus_i CH^{d_i+r}(X_i \times Y)_\mathbb{Q}$ is called a space of correspondences of degree r from X to Y.
A category $\mathcal{M}$ of Chow motives is constructed in [Mu2]. Suppose X is a nonsingular projective variety over $\mathbb{C}$ of dimension d. Let $\Delta \subset X \times X$ be the diagonal. Consider the Künneth decomposition of the class of Δ in the Betti Cohomology:

$$[\Delta] = \oplus_{i=0}^{2d} \pi_i^{hom}$$

where $\pi_i^{hom} \in H^{2d-i}(X,\mathbb{Q}) \otimes H^i(X,\mathbb{Q})$.

Definition 2.1. *The motive of X is said to have Künneth decomposition if each of the classes π_i^{hom} is algebraic, i.e., π_i^{hom} is the image of an algebraic* *cycle π_i under the cycle class map from the rational Chow groups to the Betti cohomology.*

Definition 2.2. *The motive of X is said to have a Chow–Künneth decomposition if each of the classes π_i^{hom} is algebraic and they are orthogonal projectors, i.e., $\pi_i \circ \pi_j = \delta_{i,j}\pi_i$.*

Lemma 2.3. *If X and Y have a Chow–Künneth decomposition then $X \times Y$ also has a Chow–Künneth decomposition.*

Proof. If π_i^X and π_j^Y are the Chow–Künneth components for $h(X)$ and $h(Y)$ respectively then

$$\pi_i^{X\times Y} = \sum_{p+q=i} \pi_p^X \times \pi_q^Y \in CH^*(X \times Y \times X \times Y)_{\mathbb{Q}}$$

are the Chow–Künneth components for $X \times Y$. Here the product $\pi_p^X \times \pi_q^Y$ is taken after identifying $X \times Y \times X \times Y \simeq X \times X \times Y \times Y$. $\square$

2.1. Murre's conjectures. J. Murre [Mu2], [Mu3] has made the following conjectures for any smooth projective variety X.
(A) The motive $h(X) := (X, \Delta_X)$ of X has a Chow-Künneth decomposition:

$$\Delta_X = \sum_{i=0}^{2n} \pi_i \in CH^n(X \times X) \otimes \mathbb{Q}$$

such that π_i are orthogonal projectors.
(B) The correspondences $\pi_0, \pi_1, ..., \pi_{j-1}, \pi_{2j+1}, ..., \pi_{2n}$ act as zero on $CH^j(X)\otimes \mathbb{Q}$.
(C) Suppose

$$F^r CH^j(X) \otimes \mathbb{Q} = \mathrm{Ker}\pi_{2j} \cap \mathrm{Ker}\pi_{\mathrm{2j-1}} \cap ... \cap \mathrm{Ker}\pi_{\mathrm{2j-r+1}}.$$

Then the filtration $F^\bullet$ of $CH^j(X) \otimes \mathbb{Q}$ is independent of the choice of the projectors π_i.
(D) Further, $F^1CH^i(X) \otimes \mathbb{Q} = (CH^i(X) \otimes \mathbb{Q})_{hom}$, the cycles which are homologous to zero.

In §4, we will extend (A) in the setting of equivariant Chow groups.

3. Equivariant Chow groups and equivariant Chow motives

In this section, we recall some preliminary facts on the equivariant groups to formulate Murre's conjectures for a smooth variety X of dimension d, which is equipped with an action by a linear reductive algebraic group G. The equivariant groups and their properties that we recall below were defined by Borel, Totaro, Edidin-Graham, Fulton [Bo],[To],[Ed-Gr], [Fu2].

3.1. Equivariant cohomology $H^i_G(X,\mathbb{Z})$ of X. Suppose X is a variety with an action on the left by an algebraic group G. Borel defined the equivariant cohomology $H^*_G(X)$ as follows. There is a contractible space EG on which G acts freely (on the right) with quotient $BG := EG/G$. Then form the space

$$EG \times_G X := EG \times X/(e.g,x) \sim (e,g.x).$$

In other words, $EG \times_G X$ represents the (topological) quotient stack $[X/G]$.

Definition 3.1. *The equivariant cohomology of X with respect to G is the ordinary singular cohomology of $EG \times_G X$:*

$$H^i_G(X) = H^i(EG \times_G X).$$

For the special case when X is a point, we have

$$H^i_G(point) = H^i(BG)$$

For any X, the map $X \to point$ induces a pullback map $H^i(BG) \to H^i_G(X)$. Hence the equivariant cohomology of X has the structure of a $H^i(BG)$-algebra, at least when $H^i(BG) = 0$ for odd i.

3.2. Equivariant Chow groups $CH^i_G(X)$ of X. [Ed-Gr]
As in the previous subsection, let X be a smooth variety of dimension n, equipped with a left G-action. Here G is an affine algebraic group of dimension g. Choose an l-dimensional representation V of G such that V has an open subset U on which G acts freely and whose complement has codimension more than $n-i$. The diagonal action on $X \times U$ is also free, so there is a quotient in the category of algebraic spaces. Denote this quotient by $X_G := (X \times U)/G$.

Definition 3.2. *The i-th equivariant Chow group $CH^G_i(X)$ is the usual Chow group $CH_{i+l-g}(X_G)$. The codimension i equivariant Chow group $CH^i_G(X)$ is the usual codimension i Chow group $CH^i(X_G)$.*

Note that if X has pure dimension n then

$$\begin{aligned} CH^i_G(X) &= CH^i(X_G) \\ &= CH_{n+l-g-i}(X_G) \\ &= CH^G_{n-i}(X). \end{aligned}$$

Proposition 3.3. *The equivariant Chow group $CH^G_i(X)$ is independent of the representation V, as long as $V-U$ has codimension more than $n-i$.*

Proof. See [Ed-Gr, Definition-Proposition 1]. □

If $Y \subset X$ is an m-dimensional subvariety which is invariant under the G-action, and compatible with the G-action on X, then it has a G-equivariant fundamental class $[Y]_G \in CH^G_m(X)$. Indeed, we can consider the product $(Y \times U) \subset X \times U$, where U is as above and the corresponding quotient $(Y \times U)/G$ canonically embeds into X_G. The fundamental class of $(Y \times U)/G$ defines the class $[Y]_G \in CH^G_m(X)$. More generally, if V is an l-dimensional representation of G and $S \subset X \times V$ is an $m+l$-dimensional subvariety which is invariant under the G-action, then the quotient $(S \cap (X \times U))/G \subset (X \times U)/G$ defines the G-equivariant fundamental class $[S]_G \in CH_m^G(X)$ of S.

Proposition 3.4. *If $\alpha \in CH_m^G(X)$ then there exists a representation V such that $\alpha = \sum a_i[S_i]_G$, for some G-invariant subvarieties S_i of $X \times V$.*

Proof. See [Ed-Gr, Proposition 1]. □

3.3. Functoriality properties. Suppose $f : X \to Y$ is a G-equivariant morphism. Let $\mathcal{S}$ be one of the following properties of schemes or algebraic spaces: proper, flat, smooth, regular embedding or l.c.i.

Proposition 3.5. *If $f : X \to Y$ has property $\mathcal{S}$, then the induced map $f_G : X_G \to Y_G$ also has property $\mathcal{S}$.*

Proof. See [Ed-Gr, Proposition 2]. □

Proposition 3.6. *Equivariant Chow groups have the same functoriality as ordinary Chow groups for equivariant morphisms with property $\mathcal{S}$.*

Proof. See [Ed-Gr, Proposition 3]. □

If X and Y have G-actions then there are exterior products

$$CH_i^G(X) \otimes CH_j^G(Y) \to CH_{i+j}^G(X \times Y).$$

In particular, if X is smooth then there is an intersection product on the equivariant Chow groups which makes $\oplus_j CH_j^G(X)$ into a graded ring.

3.4. Cycle class maps. [Ed-Gr, §2.8]
Suppose X is a complex algebraic variety and G is a complex algebraic group. The equivariant Borel-Moore homology $H_{BM,i}^G(X)$ is the Borel-Moore homology $H_{BM,i}(X_G)$, for $X_G = X \times_G U$. This is independent of the representation as long as $V - U$ has sufficiently large codimension. This gives a cycle class map,

$$cl_i : CH_i^G(X) \to H_{BM,2i}^G(X, \mathbb{Z})$$

compatible with usual operations on equivariant Chow groups. Suppose X is smooth of dimension d then X_G is also smooth. In this case the Borel-Moore cohomology $H_{BM,2i}^G(X, \mathbb{Z})$ is dual to $H^{2d-i}(X_G) = H^{2d-i}(X \times_G U)$.
This gives the cycle class maps

$$cl^i : CH_G^i(X) \to H_G^{2i}(X, \mathbb{Z}). \tag{1}$$

There are also maps from the equivariant groups to the usual groups:

$$H_G^i(X, \mathbb{Z}) \to H^i(X, \mathbb{Z}) \tag{2}$$

and

$$CH_G^i(X) \to CH^i(X). \tag{3}$$

3.5. Weight filtration $W_{.}$ on $H^i_G(X,\mathbb{Z})$. In this paper, we assign only the bottom weight W_i of the equivariant cohomology in the simplest situation. Consider a smooth variety X equipped with a left G action as above.
We can define

$$W_iH^i_G(X,\mathbb{Q}) := W_iH^i((X\times U)/G,\mathbb{Q}),$$

for $U\subset V$ an open subset with a free G-action, where codim $V-U$ is at least $n-i$.

Lemma 3.7. *The group $W_iH^i_G(X,\mathbb{Q})$ is independent of the choice of the G-representation V as long as codim $V-U$ is at least $n-i$.*

Proof. The proof of independence of V in the case of equivariant Chow groups [Ed-Gr, Definition-Proposition 1] applies directly in the case of the bottom weight equivariant cohomology. $\square$

3.6. Equivariant Chow motives and the category of equivariant Chow motives. When G is a finite group then a category of Chow motives for (maybe singular) quotients of varieties under the G-action was constructed in [dB-Az], [Ak-Jo]. More generally, we consider the following situation, taking into account the equivariant cohomology and the equivariant rational Chow groups, which does not seem to have been considered before.
Fix an affine complex algebraic group G. Let $\mathcal{V}_G$ be the category whose objects are complex smooth projective varieties with a G-action and the morphisms are G-equivariant morphisms.
For any $X,Y,Z\in Ob(\mathcal{V}_G)$, consider the projections

$$X\times Y\times Z \xrightarrow{p_{XY}} X\times Y,$$
$$X\times Y\times Z \xrightarrow{p_{YZ}} Y\times Z,$$
$$X\times Y\times Z \xrightarrow{p_{XZ}} X\times Z.$$

which are G-equivariant.
Let d be the dimension of X. The group of correspondences from X to Y of degree r is defined as

$$\mathrm{Corr}^r_G(X\times Y) := CH^{r+d}_G(X\times Y).$$

Every G-equivariant morphism $X\to Y$ defines an element in $\mathrm{Corr}^0_G(X\times Y)$, by taking the graph cycle.
For any $f\in \mathrm{Corr}^r_G(X,Y)$ and $g\in \mathrm{Corr}^e_G(Y,Z)$ define their composition

$$g\circ f\in \mathrm{Corr}^{r+e}_G(X,Z)$$

by the prescription

$$g\circ f = p_{XZ*}(p^*_{XY}(f).p^*_{YZ}(g)).$$

This gives a linear action of correspondences on the equivariant Chow groups

$$\mathrm{Corr}^r_G(X,Y)\times CH^s_G(X)_{\mathbb{Q}} \longrightarrow CH^{r+s}_G(Y)_Q$$
$$(\gamma,\alpha)\mapsto p_{Y*}(p^*_X\alpha.\gamma)$$

for the projections $p_X: X\times Y\longrightarrow X$, $p_Y: X\times Y\longrightarrow Y$.

The category of pure equivariant G-motives with rational coefficients is denoted by $\mathcal{M}_G^+$. The objects of $\mathcal{M}_G^+$ are triples $(X,p,m)_G$, for $X \in Ob(\mathcal{V}_G)$, $p \in \mathrm{Corr}_G^0(X,X)$ is a projector, i.e., $p \circ p = p$ and $m \in \mathbb{Z}$. The morphisms between the objects $(X,p,m)_G, (Y,q,n)_G$ in $\mathcal{M}_G^+$ are given by the correspondences $f \in \mathrm{Corr}_G^{n-m}(X,Y)$ such that $f \circ p = q \circ f = f$. The composition of the morphisms is the composition of correspondences. This category is pseudoabelian and $\mathbb{Q}$-linear [Mu2]. Furthermore, it is a tensor category defined by

$$(X,p,m)_G \otimes (Y,q,n)_G = (X \times Y, p \otimes q, m+n)_G.$$

The object $(\mathrm{Spec}\,\mathbb{C}, id, 0)_G$ is the unit object and the Lefschetz motive $\mathbb{L}$ is the object $(\mathrm{Spec}\,\mathbb{C}, id, -1)_G$. Here $\mathrm{Spec}\,\mathbb{C}$ is taken with a trivial G-action. The Tate twist of a G-motive M is $M(r) := M \otimes \mathbb{L}^{\otimes -r} = (X,p,m+r)_G$.

Definition 3.8. *The theory of equivariant Chow motives (*[Sc]*) provides a functor*

$$h : \mathcal{V}_G \longrightarrow \mathcal{M}_G^+.$$

For each $X \in Ob(\mathcal{V}_G)$ the object $h(X) = (X, \Delta, 0)_G$ is called the equivariant Chow motive of X. Here Δ is the class of the diagonal in $CH^*(X \times X)_{\mathbb{Q}}$, which is G-invariant for the diagonal action on $X \times X$ and hence lies in $Corr_G^0(X,X) = CH_G^*(X \times X)_{\mathbb{Q}}$.

4. Murre's conjectures for the equivariant Chow motives

Suppose X is a complex smooth variety of dimension d, equipped with a G-action. Consider the product variety $X \times X$ together with the diagonal action of the group G.
The cycle class map

$$cl^d : CH^d(X \times X)_{\mathbb{Q}} \to H^{2d}(X \times X, \mathbb{Q}). \tag{4}$$

actually maps to the weight $2d$ piece $W_{2d}H^{2d}(X \times X, \mathbb{Q})$ of the ordinary cohomology group.
Applying this to the spaces $X \times U$, for open subset $U \subset V$ as in §3.2, (4) holds for the equivariant groups as well and there are cycle class maps:

$$cl^d : CH_G^d(X \times X)_{\mathbb{Q}} \to W_{2d}H_G^{2d}(X \times X, \mathbb{Q}). \tag{5}$$

Lemma 4.1. *The image of the diagonal cycle $[\Delta_X]$ under the cycle class map cl^d lies in the subspace*

$$\bigoplus_i W_{2d-i}H_G^{2d-i}(X) \otimes W_iH_G^i(X)$$

of $W_{2d}H_G^{2d}(X \times X, \mathbb{Q})$.

Proof. First we prove the assertion for the ordinary cohomology of non-compact smooth varieties and next apply it to the product spaces $X \times U$, which is equipped with a free G-action and the quotient space X_G.

If X is a compact smooth variety then we notice that the weight $2d$ piece coincides with the cohomology group $H^{2d}(X \times X, \mathbb{Q})$ and by the Künneth formula for products the statement follows in the usual cohomology. Suppose X is not compact. Using (4), notice that the image of the diagonal cycle $[\Delta_X]$ lies in $W_{2d}H^{2d}(X \times X, \mathbb{Q})$. Choose a smooth compactification $\overline{X}$ of X and consider the commutative diagram:

$$\begin{array}{ccc} \bigoplus_i H^{2d-i}(\overline{X}) \otimes H^i(\overline{X}) & \stackrel{\simeq}{\to} & H^{2d}(\overline{X} \times \overline{X}, \mathbb{Q}) \\ \downarrow & & \downarrow \\ \bigoplus_i W_{2d-i}H^{2d-i}(X) \otimes W_iH^i(X) & \stackrel{k}{\to} & W_{2d}H^{2d}(X \times X, \mathbb{Q}). \end{array}$$

The vertical arrows are surjective maps, defined by the localization. Hence the map k is surjective. The injectivity follows because this is the Künneth product map, restricted to the bottom weight cohomology. This shows that k is an isomorphism.
In particular, the isomorphism k can be applied to the bottom weights of the ordinary cohomology groups of the smooth variety $X \times U$, for any open subset $U \subset V$ of large complementary codimension and V is a G-representation. But this is essentially the bottom weight of the equivariant cohomology group of X. To conclude, we need to observe that the diagonal cycle $[\Delta_X]$ is G-invariant.

$\square$

Denote the decomposition of the G-invariant diagonal cycle

$$\Delta_X = \oplus_{i=0}^{2d} \pi_i^G \in W_{2d}H_G^{2d}(X \times X, \mathbb{Q}) \tag{6}$$

such that π_i^G lies in the space $W_{2d-i}H_G^{2d-i}(X) \otimes W_iH_G^i(X)$.
We defined the equivariant Chow motive of a smooth projective variety with a G-action in §3.6. We extend the notion of orthogonal projectors on a smooth variety equipped with a G-action, as follows.

Definition 4.2. *Suppose X is a smooth variety equipped with a G-action. The equivariant Chow motive $(X, \Delta_X)_G$ of X is said to have an* equivariant Künneth decomposition *if the classes π_i^G are algebraic, i.e., they have a lift in the equivariant Chow group $CH_G^d(X \times X)_{\mathbb{Q}}$. Furthermore, if X admits a smooth compactification $X \subset \overline{X}$ such that the action of G extends on $\overline{X}$ and the Künneth projectors extend to orthogonal projectors on $\overline{X}$ then we say that X has an* equivariant Chow–Künneth decomposition*.*

Remark 4.3. *When G is a linear algebraic group, using the results of Sumihiro* [Su]*, Bierstone-Milman* [Bi-Mi, Theorem 13.2]*, Reichstein-Youssin* [Re-Yo]*, one can always choose a smooth compactification $\overline{X} \supset X$ such that action of G extends to $\overline{X}$. Since any affine algebraic group is linear, we can always find smooth G-equivariant compactifications in our set-up.*

Suppose X is a smooth variety with a free G-action so that we can form the quotient variety $Y := X/G$. Using [Ed-Gr], we have the identification of the rational Chow groups

$$CH^*(Y)_{\mathbb{Q}} \,=\, CH^*_G(X)_{\mathbb{Q}}$$

and

$$CH^*(Y\times Y)_{\mathbb{Q}} \,=\, CH^*_G(X\times X)_{\mathbb{Q}}.$$

Furthermore, these identifications respect the ring structure on the above rational Chow groups. A similar identification also holds for the rational cohomology groups. In view of this, we make the following definition.

Definition 4.4. *Suppose X is a smooth variety with a G-action and G acts freely on X. Denote the quotient space $Y := X/G$. The absolute Chow–Künneth decomposition of Y is defined to be the equivariant Chow–Künneth decomposition of X.*

We can now extend Murre's conjecture to smooth varieties with a G-action, as follows.

Conjecture 4.5. *Suppose X is a smooth variety with a G-action. Then X has an equivariant Chow–Künneth decomposition.*

In particular, if the action of G is trivial then we can extend Murre's conjecture to a (not necessarily compact) smooth variety, by taking only the bottom weight cohomology $W_iH^i(X)$ of the ordinary cohomology. This is weaker than obtaining projectors for the ordinary cohomology. We remark a projector π_1 in the case of quasi–projective varieties has been constructed by Bloch and Esnault [BE].

5. Families of curves

Our goal in this paper is to find an (explicit) absolute Chow–Künneth decomposition for the universal families of curves over close approximations of the moduli space of smooth curves of small genus. We begin with the following situation which motivates the statements on universal curves.

Lemma 5.1. *Any smooth hypersurface $X \subset \mathbb{P}^n$ of degree d has an absolute Chow–Künneth decomposition. If $L \subset X$ is any line, then the blow-up $X' \to X$ also has a Chow–Künneth decomposition.*

Proof. Notice that the cohomology of X is algebraic except in the middle dimension $H^{n-1}(X,\mathbb{Q})$. By the Lefschetz Hyperplane section theorem, the algebraic cohomology $H^{2j}(X,Q)$, $j\neq n-1$, is generated by the hyperplane section H^j. So the projectors are simply

$$\pi_r := \frac{1}{d}.H^{n-1-r}\times H^r \,\in\, CH^{n-1}(X\times X)_{\mathbb{Q}}$$

for $r\neq n-1$. We can now take $\pi_{n-1} := \Delta_X - \sum_{r,r\neq n-1}\pi_r$. This gives a complete set of orthogonal projectors and a Chow–Künneth decomposition for X. Since $X'\to X$ is a blow-up along a line, the new cohomology is again algebraic, by the blow-up formula. Similarly we get a Chow–Künneth decomposition for X' (see also [dA-Mü2, Lemma 2] for blow-ups). $\square$

The above lemma can be generalized to the following situation.

Lemma 5.2. *Suppose Y is a smooth projective variety of dimension r over $\mathbb{C}$ which has only algebraic cohomology groups $H^i(Y)$ for all $0 \le i \le m$ for some $m < r$. Then we can construct orthogonal projectors*

$$\pi_0, \pi_1, ..., \pi_m, \pi_{2r-m}, \pi_{2r-m+1}, ..., \pi_{2r}$$

in the usual Chow group $CH^r(Y \times Y)_{\mathbb{Q}}$, and where π_{2i} acts as $\delta_{i,p}$ on $H^{2p}(Y)$ and $\pi_{2i-1} = 0$. Moreover, if there is an affine complex algebraic group G acting on Y, then we can lift the above projectors in the equivariant Chow group $CH^r_G(Y \times Y)_{\mathbb{Q}}$ as orthogonal projectors.

Proof. See also [dA-Mü1, dA-Mü2]. Let $H^{2p}(Y)$ be generated by cohomology classes of cycles $C_1, \ldots, C_s$ and $H^{2r-2p}(Y)$ be generated by cohomology classes of cycles $D_1, \ldots, D_s$. We denote by M the intersection matrix with entries

$$M_{ij} = C_i \cdot D_j \in \mathbb{Z}.$$

After base change and passing to $\mathbb{Q}$–coefficients we may assume that M is diagonal, since the cup–product $H^{2p}(Y, \mathbb{Q}) \otimes H^{2r-2p}(Y, \mathbb{Q}) \to \mathbb{Q}$ is non–degenerate. We define the projector π_{2p} as

$$\pi_{2p} = \sum_{k=1}^{s} \frac{1}{M_{kk}} D_k \times C_k.$$

It is easy to check that $\pi_{2p*}(C_k) = D_k$. Define π_{2r-2p} as the adjoint, i.e., transpose of π_{2p}. Via the Gram–Schmidt process from linear algebra we can successively make all projectors orthogonal. □

Suppose $X \subset \mathbb{P}^n$ is a smooth complete intersection of multidegree $d_1 \le d_2 \le ... \le d_s$. Let $F_r(X)$ be the variety of r-dimensional planes contained in X. Let $\delta := \min\{(r+1)(n-r) - \binom{d+r}{r}, n - 2r - s\}$.

Corollary 5.3. *If X is general then $F_r(X)$ is a smooth projective variety of dimension δ and it has an absolute Chow–Künneth decomposition.*

Proof. The first assertion on the smoothness of the variety $F_r(X)$ is well–known, see [Al-Kl], [ELV], [De-Ma]. For the second assertion, notice that $F_r(X)$ is a subvariety of the Grassmanian $G(r, \mathbb{P}^n)$ and is the zero set of a section of a vector bundle. Indeed, let S be the tautological bundle on $G(r, \mathbb{P}^n)$. Then a section of $\oplus_{i=1}^s Sym^{d_i} H^0(\mathbb{P}^n, \mathcal{O}(1))$ induces a section of the vector bundle $\oplus_{i=1}^s Sym^{d_i} S^*$ on $G(r, \mathbb{P}^n)$. Thus, $F_r(X)$ is the zero locus of the section of the $\bigoplus_{i=1}^s Sym^{d_i} S^*$ induced by the equations defining the complete intersection X. A Lefschetz theorem is proved in [De-Ma, Theorem 3.4]:

$$H^i(G(r, \mathbb{P}^n), \mathbb{Q}) \to H^i(F_r(X), \mathbb{Q})$$

is bijective, for $i \le \delta - 1$. We can apply Lemma 5.2 to get the orthogonal projectors in all degrees except in the middle dimension. The projector corresponding to the middle dimension can be gotten by subtracting the sum of these projectors from the diagonal class.

□

Corollary 5.4. *Suppose $X \subset \mathbb{P}^n$ is a smooth projective variety of dimension d. Let $r = 2d - n$. Then we can construct orthogonal projectors*

$$\pi_0, \pi_1, \ldots, \pi_r, \pi_{2d-r}, \pi_{2d-r+1}, \ldots, \pi_{2d}.$$

Proof. Barth [Ba] has proved a Lefschetz theorem for higher codimensional subvarieties in projective spaces:

$$H^i(\mathbb{P}^n, \mathbb{Q}) \to H^i(X, \mathbb{Q})$$

is bijective if $i \leq 2d - n$ and is injective if $i = 2d - n + 1$. The claim now follows from Lemma 5.2. □

Remark 5.5. *The above corollary says that if we can embed a variety X in a low dimensional projective space then we get at least a partial set of orthogonal projectors. A conjecture of Hartshorne's says that any codimension two subvariety of $\mathbb{P}^n$ for $n \geq 6$ is a complete intersection. This gives more examples for subvarieties with several algebraic cohomology groups.*

5.1. Chow–Künneth decomposition for the universal plane curve. We want to find explicit equivariant Chow–Künneth projectors for the universal plane curve of degree d. Let $d \geq 1$ and consider the linear system $\mathbb{P} = |\mathcal{O}_{\mathbb{P}^2}(d)|$ and the universal plane curve

$$\begin{array}{ccc} \mathcal{C} & \subset & \mathbb{P}^2 \times \mathbb{P} \\ \downarrow & & \\ \mathbb{P}. & & \end{array}$$

Furthermore, we notice that the general linear group $G := GL_3(\mathbb{C})$ acts on $\mathbb{P}^2$ and hence acts on the projective space $\mathbb{P} = |\mathcal{O}_{\mathbb{P}^2}(d)|$. This gives an action on the product space $\mathbb{P}^2 \times \mathbb{P}$ and leaves the universal smooth plane curve $\mathcal{C} \subset \mathbb{P}^2 \times \mathbb{P}$ invariant under the G-action.

Lemma 5.6. *The variety $\mathcal{C}$ has an absolute Chow–Künneth decomposition and an absolute equivariant Chow–Künneth decomposition.*

Proof. We observe that $\mathcal{C} \subset \mathbb{P}^2 \times \mathbb{P}$ is a smooth hypersurface of bi-degree $(d, 1)$ with variables in $\mathbb{P}^2$ whose coefficients are polynomial functions on $\mathbb{P}$. Notice that $\mathbb{P}^2 \times \mathbb{P}$ has a Chow–Künneth decomposition and Lefschetz theorems hold for the embedding $\mathcal{C} \subset \mathbb{P}^2 \times \mathbb{P}$, since $\mathcal{O}(d, 1)$ is very ample. Now we can repeat the arguments from Lemma 5.2 to get an absolute Chow–Künneth decomposition and absolute equivariant Chow–Künneth decomposition, for the variety $\mathcal{C}$. □

5.2. Families of curves contained in homogeneous spaces. We notice that when $d = 3$ in the previous subsection, the family of plane cubics restricted to the loci of stable curves is a complete family of genus one stable curves. If $d \geq 4$, then the above family of plane curves is no longer a complete family of genus g curves. Hence to find families which closely approximate over the moduli spaces of stable curves, we need to look for curves embedded as complete intersections in other simpler looking varieties. For this purpose, we look at the curves embedded in special Fano varieties of small genus $g \leq 8$, which was studied by S. Mukai [Muk], [Muk2], [Muk3], [Muk5] and Ide-Mukai [IdMuk]. We recall the main result that we need.

Theorem 5.7. *Suppose C is a generic curve of genus $g \leq 8$. Then C is a complete intersection in a smooth projective variety which has only algebraic cohomology.*

Proof. This is proved in [Muk], [Muk2], [Muk3], [IdMuk] and [Muk5]. The below classification is for the generic curve.
When $g \leq 5$ then it is well-known that the generic curve is a linear section of a Grassmanian.
When $g = 6$ then a curve has finitely many g^1_4 if and only if it is a complete intersection of a Grassmanian and a smooth quadric , see [Muk3, Theorem 5.2].
When $g = 7$ then a curve is a linear section of a 10-dimensional spinor variety $X \subset \mathbb{P}^{15}$ if and only if it is non-tetragonal, see [Muk3, Main theorem].
When $g = 8$ then it is classically known that the generic curve is a linear section of the grassmanian $G(2, 6)$ in its Plücker embedding.

□

Suppose $\mathbb{P}(g)$ is the parameter space of linear sections of a Grassmanian or of a spinor variety, which depends on the genus, as in the proof of above Theorem 5.7. $\mathbb{P}(g)$ is a product of projective spaces on which an algebraic group G (copies of PGL_N) acts. Generic curves are isomorphic, if they are in the same orbit of G.

Proposition 5.8. *Suppose $\mathbb{P}(g)$ is as above, for $g \leq 8$. Then there is a universal curve*

$$\mathcal{C}_g \to \mathbb{P}(g)$$

such that the classifying (rational) map $\mathbb{P}(g) \to \mathcal{M}_g$ is dominant. The smooth projective variety $\mathcal{C}_g$ has an absolute Chow–Künneth decomposition and an absolute equivariant Chow-Künneth decomposition for the natural G–action mentioned above.

Proof. The first assertion follows from Theorem 5.7. For the second assertion notice that the universal curve, when $g \leq 8$, is a complete intersection in $\mathbb{P}(g) \times V$ where V is either a Grassmanian or a spinor variety, which are homogeneous varieties. In other words, $\mathcal{C}_g$ is a complete intersection in a space which has only algebraic cohomology. Hence, by Lemma 5.2, $\mathcal{C}_g$ has orthogonal projectors $\pi_0, \pi_1, ..., \pi_m, \pi_{2r-m}, \pi_{2r-m+1}, ..., \pi_{2r}$, where $r := \dim \mathcal{C}_g$ and $m = \dim \mathcal{C}_g - 1$, using Lefschetz hyplerplane section theorem. Taking $\pi_{m+1} = \Delta_{\mathcal{C}_g} - \sum_{i \neq m+1} \pi_i$, gives an absolute Chow–Künneth decomposition for $\mathcal{C}_g$. Now a homogeneous variety looks like $V = G/P$ where G is an (linear) algebraic group and P is a parabolic subgroup. Hence the group G acts on the variety V. This induces an action on the linear system $\mathbb{P}(g)$ and hence G acts on the ambient variety $\mathbb{P}(g) \times V$ and leaves the universal curve $\mathcal{C}_g$ invariant. Hence we can again apply Lemma 5.2 to obtain absolute equivariant Chow–Künneth decomposition for $\mathcal{C}_g$. □

Consider the universal family of curves $\mathcal{C}_g \to \mathbb{P}(g)$ as obtained above, which are equipped with an action of a linear algebraic group G.
Suppose there is an open subset $U_g \subset \mathbb{P}(g)$, with the universal family $\mathcal{C}_{U_g} \to U_g$, on which G acts freely to form a good quotient family

$$Y_g := \mathcal{C}_{U_g}/G \to S_g := U_g/G.$$

Notice that the classifying map $S_g \to \mathcal{M}_g$ is dominant.

Corollary 5.9. *The smooth variety Y_g has an absolute Chow–Künneth decomposition.*

Proof. Consider the localization sequence, for the embedding $j : \mathcal{C}_{U_g} \times \mathcal{C}_{U_g} \hookrightarrow \mathcal{C}_g \times \mathcal{C}_g$,

$$CH^d_G(\mathcal{C}_g \times \mathcal{C}_g)_{\mathbb{Q}} \xrightarrow{j^*} CH^d_G(\mathcal{C}_{U_g} \times \mathcal{C}_{U_g})_{\mathbb{Q}} \to 0.$$

Here d is the dimension of $\mathcal{C}_g$. Then the map j^* is an equivariant ring homomorphism and transforms orthogonal projectors to orthogonal projectors. Similarly there is a commuting diagram between the equivariant cohomologies:

$$\begin{array}{ccc} \bigoplus_i H^{2d-i}_G(\mathcal{C}_g) \otimes H^i_G(\mathcal{C}_g) & \stackrel{\simeq}{\to} & H^{2d}_G(\mathcal{C}_g, \mathbb{Q}) \\ \downarrow & & \downarrow \\ \bigoplus_i W_{2d-i} H^{2d-i}_G(\mathcal{C}_{U_g}) \otimes W_i H^i_G(\mathcal{C}_{U_g}) & \stackrel{\simeq}{\to} & W_{2d} H^{2d}_G(\mathcal{C}_{U_g}, \mathbb{Q}) \end{array}$$

The vertical arrows are surjective maps mapping onto the bottom weights of the equivariant cohomology groups. By Proposition 5.8, the variety $\mathcal{C}_g$ has an absolute equivariant Chow–Künneth decomposition. Hence the images of the equivariant Chow–Künneth projectors for the complete smooth variety $\mathcal{C}_g$, under the morphism j^* give equivariant Chow–Künneth projectors for the smooth variety $\mathcal{C}_{U_g}$.
Using [Ed-Gr], we have the identification of the rational Chow groups

$$CH^*(Y_g)_{\mathbb{Q}} = CH^*_G(\mathcal{C}_{U_g})_{\mathbb{Q}}$$

and

$$CH^*(Y_g \times Y_g)_{\mathbb{Q}} = CH^*_G(\mathcal{C}_{U_g} \times \mathcal{C}_{U_g})_{\mathbb{Q}}.$$

Furthermore, this respects the ring structure on the above rational Chow groups. A similar identification also holds for the rational cohomology groups. This means that the equivariant Chow–Künneth projectors for the variety $\mathcal{C}_{U_g}$

correspond to a complete set of absolute Chow–Künneth projectors for the quotient variety Y_g. □

Remark 5.10. *Since Mukai has a similar classification for the non-generic curves in genus* ≤ 8*, one can obtain absolute equivariant Chow–Künneth decomposition for these special families of smooth curves, by applying the proof of Proposition 5.8. There is also a classification for* $K3$*-surfaces and in many cases the generic* $K3$*-surface is obtained as a linear section of a Grassmanian* [Muk]. *Hence we can apply the above results to families of* $K3$*-surfaces over spaces which dominate the moduli space of* $K3$*-surfaces.*

References

[Al-Kl] A.B. Altman, S.L. Kleiman, *Foundations of the theory of Fano schemes.* Compositio Math. 34 (1977), no. 1, 3–47.

[Ak-Jo] R. Akhtar, R. Joshua, *Künneth decomposition for quotient varieties*, Indag. Math. (N.S.) 17 (2006), no. 3, 319–344.

[Ba] W. Barth *Transplanting cohomology classes in complex-projective space.* Amer. J. Math. 92 1970 951–967.

[BBD] A.A. Beilinson, J. Bernstein, P. Deligne, *Faisceaux pervers* (French) [Perverse sheaves], Analysis and topology on singular spaces, I (Luminy, 1981), 5–171, Astérisque, 100, Soc. Math. France, Paris, 1982.

[Bi-Mi] E. Bierstone, P.D. Milman, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant.* Invent. Math. 128 (1997), no. 2, 207–302.

[BE] S. Bloch, H. Esnault, *Künneth projectors for open varieties*, preprint arXiv:math/0502447.

[Bo] A. Borel *Linear algebraic groups.* Second edition. Graduate Texts in Mathematics, 126. Springer-Verlag, New York, 1991. xii+288 pp.

[CH] A. Corti, M. Hanamura, *Motivic decomposition and intersection Chow groups. I*, Duke Math. J. 103 (2000), no. 3, 459–522.

[De-Ma] O. Debarre, L. Manivel, *Sur la variètè des espaces linéaires contenus dans une intersection complète.* (French) [The variety of linear spaces contained in a complete intersection] Math. Ann. 312 (1998), no. 3, 549–574.

[dA-Mü1] P. del Angel, S. Müller–Stach, *Motives of uniruled* 3*-folds*, Compositio Math. 112 (1998), no. 1, 1–16.

[dA-Mü2] P. del Angel, S. Müller-Stach, *On Chow motives of 3-folds*, Trans. Amer. Math. Soc. 352 (2000), no. 4, 1623–1633.

[dB-Az] S. del Baño Rollin, V. Navarro Aznar, *On the motive of a quotient variety.* Dedicated to the memory of Fernando Serrano. Collect. Math. 49 (1998), no. 2-3, 203–226.

[D] P. Deligne, *Thèorie de Hodge. III*, Inst. Hautes Ètudes Sci. Publ. Math. No. 44 (1974), 5–77.

[De-Mu] Ch. Deninger, J. Murre, *Motivic decomposition of abelian schemes and the Fourier transform*, J. Reine Angew. Math. 422 (1991), 201–219.

[Del-Mu] P. Deligne, D. Mumford, *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. No. 36 1969 75–109.

[Di-Ha] S. Diaz, J. Harris, *Geometry of the Severi variety.* Trans. Amer. Math. Soc. 309 (1988), no. 1, 1–34.

[Ed-Gr] D. Edidin, W. Graham, *Equivariant intersection theory.* Invent. Math. 131 (1998), no. 3, 595–634.

[ELV] H. Esnault, M. Levine, E. Viehweg, *Chow groups of projective varieties of very small degree.* Duke Math. J. 87 (1997), no. 1, 29–58.

[Fu] W. Fulton, *Intersection theory*, Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., 2. Springer-Verlag, Berlin, 1998. xiv+470 pp.

[Fu2] W. Fulton, *Equivariant cohomology in Algebraic geometry*, http://www.math.lsa.umich.edu/∼ dandersn/eilenberg/index.html.

[Go-Mu] B. Gordon, J. P. Murre, *Chow motives of elliptic modular threefolds*, J. Reine Angew. Math. 514 (1999), 145–164.

[GHM1] B. Gordon, M. Hanamura, J. P. Murre, *Relative Chow-Künneth projectors for modular varieties*, J. Reine Angew. Math. 558 (2003), 1–14.

[GHM2] B. Gordon, M. Hanamura, J. P. Murre, *Absolute Chow-Künneth projectors for modular varieties*, J. Reine Angew. Math. 580 (2005), 139–155.

[Gu-Pe] V. Guletskiĭ, C. Pedrini, *Finite-dimensional motives and the conjectures of Beilinson and Murre* Special issue in honor of Hyman Bass on his seventieth birthday. Part III. K-Theory 30 (2003), no. 3, 243–263.

[Ha] J. Harris, *On the Severi problem*, Invent. Math. 84 (1986), no. 3, 445–461.

[Iy] J. N. Iyer, *Murre's conjectures and explicit Chow Künneth projectors for varieties with a nef tangent bundle*, to appear in Trans. of Amer. Math. Soc.

[Ja] U. Jannsen, *Motivic sheaves and filtrations on Chow groups*, Motives (Seattle, WA, 1991), 245–302, Proc. Sympos. Pure Math., 55, Part 1, Amer. Math. Soc., Providence, RI, 1994.

[Kol] J. Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3., 32. Springer-Verlag, Berlin, 1996. viii+320 pp.

[KMM] J. Kollár, Y. Miyaoka, S. Mori, *Rationally connected varieties*, Jour. Alg. Geom. 1 (1992) 429–448.

[Man] Yu. Manin, *Correspondences, motifs and monoidal transformations* (in Russian), Mat. Sb. (N.S.) 77 (119) (1968), 475–507.

[MMWYK] A. Miller, S. Müller-Stach, S. Wortmann, Y.H. Yang, K. Zuo. *Chow-Künneth decomposition for universal families over Picard modular surfaces*, arXiv:math.AG/0505017.

[Muk] S. Mukai, *Biregular classification of Fano 3-folds and Fano manifolds of coindex 3.* Proc. Nat. Acad. Sci. U.S.A. 86 (1989), no. 9, 3000–3002.

[Muk2] S. Mukai, *Curves and Grassmannians.* Algebraic geometry and related topics (Inchon, 1992), 19–40, Conf. Proc. Lecture Notes Algebraic Geom., I, Int. Press, Cambridge, MA, 1993

[Muk3] S. Mukai, *Curves and symmetric spaces. I.* Amer. J. Math. 117 (1995), no. 6, 1627–1644.

[IdMuk] M. Ide, S. Mukai, *Canonical curves of genus eight.* Proc. Japan Acad. Ser. A Math. Sci. 79 (2003), no. 3, 59–64.

[Muk5] S. Mukai, *Curves, $K3$ surfaces and Fano 3-folds of genus ≤ 10.*, Algebraic geometry and commutative algebra, Vol. I, 357–377, Kinokuniya, Tokyo, 1988.

[Md] Mumford, D. *Projective invariants of projective structures and applications* 1963 Proc. Internat. Congr. Mathematicians (Stockholm, 1962) pp. 526–530 Inst. Mittag-Leffler, Djursholm.

[Md2] D. Mumford, *Geometric invariant theory.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Band 34 Springer-Verlag, Berlin-New York 1965 vi+145 pp.

[Mu1] J. P. Murre, *On the motive of an algebraic surface*, J. Reine Angew. Math. 409 (1990), 190–204.

[Mu2] J. P. Murre, *On a conjectural filtration on the Chow groups of an algebraic variety. I. The general conjectures and some examples*, Indag. Math. (N.S.) 4 (1993), no. 2, 177–188.

[Mu3] J. P. Murre, *On a conjectural filtration on the Chow groups of an algebraic variety. II. Verification of the conjectures for threefolds which are the product on a surface and a curve*, Indag. Math. (N.S.) 4 (1993), no. 2, 189–201.

[Re-Yo] Z. Reichstein, B. Youssin, *Equivariant resolution of points of indeterminacy.* Proc. Amer. Math. Soc. 130 (2002), no. 8, 2183–2187.

[Sa] M. Saito, *Chow–Künneth decomposition for varieties with low cohomological level*, arXiv:math.AG/0604254.

[Sc] A. J. Scholl, *Classical motives*, Motives (Seattle, WA, 1991), 163–187, Proc. Sympos. Pure Math., 55, Part 1, Amer. Math. Soc., Providence, RI, 1994.

[Sh] A. M. Shermenev, *The motive of an abelian variety*, Funct. Analysis, 8 (1974), 55–61.

[Su] Sumihiro, H. *Equivariant completion.* J. Math. Kyoto Univ. 14 (1974), 1–28.

[To] B. Totaro, *The Chow ring of a classifying space.* Algebraic K-theory (Seattle, WA, 1997), 249–281, Proc. Sympos. Pure Math., 67, Amer. Math. Soc., Providence, RI, 1999.

Jaya NN Iyer
The Institute of
Mathematical Sciences,
CIT Campus
Taramani
Chennai 600113
India
jniyer@imsc.res.in

Stefan Müller–Stach
Mathematisches Institut der
Johannes Gutenberg Universität
Mainz
Staudingerweg 9
55099 Mainz
Germany
mueller-stach@mathematik.uni-mainz.de