Around the Gysin Triangle II.

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Abstract. The notions of orientation and duality are well understood in algebraic topology in the framework of the stable homotopy category. In this work, we follow these lines in algebraic geometry, in the framework of motivic stable homotopy, introduced by F. Morel and V. Voevodsky. We use an axiomatic treatment which allows us to consider both mixed motives and oriented spectra over an arbitrary base scheme. In this context, we introduce the Gysin triangle and prove several formulas extending the traditional panoply of results on algebraic cycles modulo rational equivalence. We also obtain the Gysin morphism of a projective morphism and prove a duality theorem in the (relative) pure case. These constructions involve certain characteristic classes (Chern classes, fundamental classes, cobordism classes) together with their usual properties. They imply statements in motivic cohomology, algebraic K-theory (assuming the base is regular) and "abstract" algebraic cobordism as well as the dual statements in the corresponding homology theories. They apply also to ordinary cohomology theories in algebraic geometry through the notion of a mixed Weil cohomology theory, introduced by D.-C. Cisinski and the author in [CD06], notably rigid cohomology.

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Contents

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Notations

We fix a noetherian base scheme S. The schemes considered in this paper are always assumed to be finite type S-schemes. Similarly, a smooth scheme (resp. morphism of schemes) means a smooth S-scheme (resp. S-morphism of S-schemes). We eliminate the reference to the base S in all notation (e.g. $\times$, $\mathbb{P}^n$, ...)
An immersion i of schemes will be a locally closed immersion and we say i is an open (resp. closed) immersion when i is open (resp. closed). We say a morphism $f : Y \to X$ is projective[2] if Y admits a closed X-immersion into a *trivial* projective bundle over X.
Given a smooth closed subscheme Z of a scheme X, we denote by $N_Z X$ the normal vector bundle of Z in X. Recall a morphism $f : Y \to X$ of schemes is said to be transversal to Z if $T = Y \times_X Z$ is smooth and the canonical morphism $N_T Y \to T \times_Z N_Z X$ is an isomorphism.
For any scheme X, we denote by $\mathrm{Pic}(X)$ the Picard group of X.
Suppose X is a smooth scheme. Given a vector bundle E over X, we let $P = \mathbb{P}(E)$ be the projective bundle of lines in E. Let $p : P \to X$ be the canonical projection. There is a canonical line bundle λ_P on P such that $\lambda_P \subset p^{-1}(E)$. We call it the *canonical line bundle* on P. We set $\xi_P = p^{-1}(E)/\lambda_P$, called the *universal quotient bundle*. For any integer $n \geq 0$, we also use the abbreviation $\lambda_n = \lambda_{\mathbb{P}^n_S}$. We call the projective bundle $\mathbb{P}(E \oplus 1)$, with its canonical open immersion $E \to \mathbb{P}(E \oplus 1)$, the *projective completion* of E.

1. Introduction

In algebraic topology, it is well known that oriented multiplicative cohomology theories correspond to algebras over the complex cobordism spectrum **MU**. Using the stable homotopy category allows a systematic treatment of this kind of generalized cohomology theory, which are considered as oriented ring spectra.
In algebraic geometry, the motive associated to a smooth scheme plays the role of a universal cohomology theory. In this article, we unify the two approaches : on the one hand, we replace ring spectra by spectra with a structure of modules over a suitable oriented ring spectra - e.g. the spectrum **MGL** of algebraic cobordism. On the other hand, we introduce and consider formal group laws in the motivic theory, generalizing the classical point of view.
More precisely, we use an axiomatic treatment based on homotopy invariance and excision property which allows to formulate results in a triangulated category which models both stable homotopy category and mixed motives. A suitable notion of orientation is introduced which implies the existence of Chern classes together with a formal group law. This allows to prove a purity theorem which implies the existence of Gysin morphisms for closed immersions and their companion residue morphisms. We extend the definition of the Gysin morphism to the case of a projective morphism, which involves a delicate study of cobordism classes in the case of an arbitrary formal group law. This theory then implies very neatly the duality statement in the projective smooth case. Moreover, these

[2] If X admits an amble line bundle, this definition coincide with that of [EGA2].

constructions are obtained over an arbitrary base scheme, eventually singular and with unequal characteristics.
Examples are given which include triangulated mixed motives, generalizing the constructions and results of V. Voevodsky, and **MGL**-modules. Thus, this work can be applied in motivic cohomology (and motivic homology), as well as in algebraic cobordism. It also applies in homotopy algebraic K-theory[3] and some of the formulas obtained here are new in this context. It can be applied finally to classical cohomology theories through the notion of a *mixed Weil theory* introduced in [CD06]. In the case of rigid cohomology, the formulas and constructions given here generalize some of the results obtained by P. Berthelot and D. Pétrequin. Moreover, the theorems proved here are used in an essential way in [CD06].

1.1. **The axiomatic framework.** We fix a triangulated symmetric monoidal category $\mathscr{T}$, with unit $\mathbb{1}$, whose objects are simply called *motives*[4]. To any pair of smooth schemes (X, U) such that $U \subset X$ is associated a motive $M(X/U)$ functorial with respect to $U \subset X$, and a canonical distinguished triangle :

$$M(U) \to M(X) \to M(X/U) \xrightarrow{\partial} M(U)[1],$$

where we put $M(U) := M(U/\emptyset)$ and so on. The first two maps are obtained by functoriality. As usual, the Tate motive is defined to be $\mathbb{1}(1) := M(\mathbb{P}^1_S/S_\infty)[-2]$ where S_∞ is the point at infinity.
The axioms we require are, for the most common, additivity (Add), homotopy invariance (Htp), Nisnevich excision (Exc), Künneth formula for pairs of schemes (Kun) and stability (Stab) – *i.e.* invertibility of $\mathbb{1}(1)$ (see paragraph 2.1 for the precise statement). All these axioms are satisfied by the stable homotopy category of schemes of F. Morel and V. Voevodsky. However, we require a further axiom which is in fact our principal object of study, the orientation axiom (Orient) : to any line bundle L over a smooth scheme X is associated a morphism $c_1(L) : M(X) \to \mathbb{1}(1)[2]$ – the first Chern class of L – compatible with base change and constant on the isomorphism class of L/X.
The best known example of a category satisfying this set of axioms is the triangulated category of (geometric) mixed motives over S, denoted by $DM_{gm}(S)$. It is defined according to V. Voevodsky along the lines of the case of a perfect base field but replacing Zariski topology by the Nisnevich one (cf section 2.3.1). Another example can be obtained by considering the category of oriented spectra in the sense of F. Morel (see [Vez01]). However, in order to define a monoidal structure on that category, we have to consider modules over the algebraic cobordism spectrum **MGL**, in the E_∞-sense. One can see that oriented spectra are equivalent to **MGL**-modules, but the tensor product is given with respect to the **MGL**-module structure.

[3]Recall homotopy algebraic K-theory was introduced by Weibel in [Wei89]. This cohomology theory coincide with algebraic K-theory when S is regular.

[4] A correct terminology would be to call these objects *generalized triangulated motives* or *triangulated motives with coefficients* as the triangulated mixed motives defined by Voevodsky are particular examples.

Any object $\mathbb{E}$ of the triangulated category $\mathscr{T}$ defines a bigraded cohomology (resp. homology) theory on smooth schemes by the formulas

$$\mathbb{E}^{n,p}(X) = \mathrm{Hom}_{\mathscr{T}}\big(M(X), \mathbb{1}(p)[n]\big) \ \textit{resp.}\ \mathbb{E}_{n,p}(X) = \mathrm{Hom}_{\mathscr{T}}\big(\mathbb{1}(p)[n], \mathbb{E} \otimes M(X)\big).$$

As in algebraic topology, there is a rich algebraic structure on these graded groups (see section 2.2). The Künneth axiom (Kun) implies that, in the case where $\mathbb{E}$ is the unit object $\mathbb{1}$, we obtain a multiplicative cohomology theory simply denoted by H^{**}. It also implies that for any smooth scheme X, $\mathbb{E}^{**}(X)$ has a module structure over $H^{**}(X)$. More generally, if we put $A = H^{**}(S)$, called the ring of (universal) coefficients, cohomology and homology groups of the previous kind are graded A-modules.

1.2. **Central constructions.** These axioms are sufficient to establish an essential basic fact, the projective bundle theorem :
(**th. 3.2**)[5] Let X be a smooth scheme, $P \xrightarrow{p} X$ be a projective bundle of dimension n, and c be the first Chern class of the canonical line bundle. Then the map: $\sum_{0 \leq i \leq n} p_* \boxtimes c^i : M(P) \to \bigoplus_{0 \leq i \leq n} M(X)(i)[2i]$ is an isomorphism.
Remark that considering any motive $\mathbb{E}$, even without ring structure, we obtain $\mathbb{E}^{**}(P) = \mathbb{E}^{**}(X) \otimes_{H^{**}(X)} H^{**}(P)$ where tensor product is taken with respect to the $H^{**}(X)$-module structure. In the case $\mathbb{E} = \mathbb{1}$, we thus obtain the projective bundle formula for H^{**} which allows the definition of *(higher) Chern classes* following the classical method of Grothendieck :
(**def. 3.10**) For any smooth scheme X, any vector bundle E over X and any integer $i \geq 0$, $c_i(E) : M(X) \to \mathbb{1}(i)[2i]$.
Moreover, the projective bundle formula leads to the following constructions :

(i) (**3.7 & 3.8**) A *formal group law* $F(x, y)$ over A such that for any smooth scheme X which admits an ample line bundle, for any line bundles L, L' over X, the formula
$$c_1(L \otimes L') = F(c_1(L), c_1(L'))$$
is well defined and holds in the A-algebra $H^{**}(X)$.
(ii) (**def. 5.12**) For any smooth schemes X, Y and any projective morphism $f : Y \to X$ of relative dimension n, the associated *Gysin morphism* $f^* : M(X) \to M(Y)(-n)[-2n]$.
(iii) (**def. 4.6**) For any closed immersion $i : Z \to X$ of codimension n between smooth schemes, with complementary open immersion j, the Gysin triangle :
$$M(X - Z) \xrightarrow{j_*} M(X) \xrightarrow{i^*} M(Z)(n)[2n] \xrightarrow{\partial_{X,Z}} M(X - Z)[1].$$
The last morphism in this triangle is called the *residue morphism*.

The Gysin morphism permits the construction of a *duality pairing* in the pure case :
(**th. 5.23**) For any smooth projective scheme $p : X \to S$ of relative dimension n,

[5] The proof is essentially based on a very elegant lemma due to F. Morel.

with diagonal embedding $\delta : X \to X \times X$, there is a strong duality[6] (in the sense of Dold-Puppe) :

$$\text{(iv)} \qquad \begin{aligned} \mu_X &: \mathbb{1} \xrightarrow{p^*} M(X)(-n)[-2n] \xrightarrow{\delta_*} M(X)(-n)[-2n] \otimes M(X) \\ \epsilon_X &: M(X) \otimes M(X)(-n)[-2n] \xrightarrow{\delta^*} M(X) \xrightarrow{p_*} \mathbb{1}. \end{aligned}$$

In particular, the Hom object $\underline{\mathrm{Hom}}(M(X), \mathbb{1})$ is defined in the monoidal category $\mathscr{T}$ and μ_X induces a canonical duality isomorphism :

$$\underline{\mathrm{Hom}}(M(X), \mathbb{1}) \to M(X)(-n)[-2n].$$

This explicit duality allows us to recover the usual form of duality between cohomology and homology as in algebraic topology, in terms of the fundamental class of X and cap-product on one hand and in terms of the fundamental class of δ and slant product on the other hand. Moreover, considering a motive $\mathbb{E}$ with a monoid structure in $\mathscr{T}$ and such that the cohomology $\mathbb{E}^{**}$ satisfies the Künneth formula, we obtain the usual Poincaré duality theorem in terms of the trace morphism (induced by the Gysin morphism $p^* : \mathbb{1} \to M(X)(n)[2n]$) and cup-product (see **paragraph 5.24**).

Note also we deduce easily from our construction that the Gysin morphism associated to a morphism f between smooth projective schemes is the dual of f_* (**prop. 5.26**).

Remark finally that, considering any closed subscheme Z_0 of S, and taking tensor product with the motive $M(S/S - Z_0)$ in the constructions (ii), (iii) and (iv), we obtain a Gysin morphism and a Gysin triangle with *support*. For example, given a projective morphism $f : Y \to X$ as in (*ii*), $Z = X \times_S Z_0$ and $T = Y \times_S Z_0$, we obtain the morphism $M_Z(X) \to M_T(Y)(-n)[-2n]$. Similarly, if X is projective smooth of relative dimension n, $M_Z(X)$ admits a strong dual, $M_Z(X)(-n)[-2n]$. Of course, all the other formulas given below are valid for these motives with support.

1.3. **Set of formulas.** The advantage of the motivic point of view is to obtain universal formulae which imply both cohomological and homological statements, with a minimal amount of algebraic structure involved.

1.3.1. *Gysin morphism.* We prove the basic properties of the Gysin morphisms such as functoriality ($g^*f^* = (fg)^*$), compatibility with the monoidal structure $(f \times g)^* = f^* \otimes g^*$), the projection formula ($(1_{Y*} \boxtimes f_*)f^* = f^* \boxtimes 1_{X*}$) and the base change formula in the transversal case ($f^*p_* = q_*g^*$).

For the needs of the following formulae, we introduce a useful notation which appear in the article. For any smooth scheme X, any cohomology class $\alpha \in H^{n,p}(X)$ and any morphism $\phi : M(X) \to M$ in $\mathscr{T}$, we put

$$\phi \boxtimes \alpha := (\phi \otimes \alpha) \circ \delta_* : M(X) \to M(p)[n]$$

where α is considered as a morphism $M(X) \to \mathbb{1}(p)[n]$, and $\delta_* : M(X) \to M(X) \otimes M(X)$ is the morphism induced by the diagonal of X/S and by the Künneth axiom (Kun).

[6] Note we use essentially the axiom (Kun) here.

More striking are the following formulae which express the *defect* in base change formulas. Fix a commutative square of smooth schemes

$$\begin{tikzcd} T \arrow[r, "q"] \arrow[d, "g"'] \arrow[dr, phantom, "\Delta"] & Y \arrow[d, "f"] \\ Z \arrow[r, "p"'] & X \end{tikzcd} \tag{1.1}$$

which is cartesian on the underlying topological spaces, and such that p (resp. q) is projective of relative dimension n (resp. m).

Excess of intersection (**prop. 4.16**).– Suppose the square Δ is cartesian. We then define the *excess intersection bundle* ξ associated to Δ as follows. Choose a projective bundle P/X and a closed immersion $Z \xrightarrow{i} P$ over X with normal bundle $N_Z P$. Consider the pullback Q of P over Y and the normal bundle $N_Y Q$ of Y in Q. Then $\xi = N_Y Q/g^{-1}N_Z P$ is independent up to isomorphism of the choice of P and i. The rank of ξ is the integer $e = n - m$.

Then, $p^* f_* = \big(g_* \boxtimes c_e(\xi)\big) q^*$.

Ramification formula (**th. 4.26**).– Consider the square Δ and assume $n = m$. Suppose that T admits an ample line bundle and (for simplicity) that S is integral[7].

Let $T = \cup_{i \in I} T_i$ be the decomposition of T into connected components. Consider an index $i \in I$. We let p_i and g_i be the restrictions of p and g to T_i. The canonical map $T \to Z \times_X Y$ is a thickening. Thus, the connected component T_i corresponds to a unique connected component T'_i of $Z \times_X Y$. According to the classical definition, the *ramification index* of f along T_i is the geometric multiplicity $r_i \in \mathbb{N}^*$ of T'_i. We define (cf **def. 4.24**) a generalized intersection multiplicity for T_i which takes into account the formal group law F, called for this reason the *F-intersection multiplicity*. It is an element $r(T_i; f, g) \in H^{0,0}(T_i)$. We then prove the formula :

$$p^* f_* = \sum_{i \in I} \big(r(T_i; f, g) \boxtimes_{T_i} g_{i*}\big) q_i^*.$$

In general, $r(T_i; f, g) = r_i + \epsilon$ where the correction term ϵ is a function of the coefficients of F – it is zero when F is additive.

1.3.2. *Residue morphism.* A specificity of the present work is the study of the Gysin triangle, notably its boundary morphism, called the residue morphism. Consider a square Δ as in (1.1). Put $U = X - Z$, $V = Y - T$ and let $h : V \to U$ be the morphism induced by f.

We obtain the following formulas :

(1) $(j_* \boxtimes 1_{U*})\partial_{X,Z} = \partial_{X,Z} \boxtimes i_*$.
(2) For any smooth scheme Y, $\partial_{X \times Y, Z \times Y} = \partial_{X,Z} \otimes 1_{Y*}$.
(3) If f is a closed immersion, $\partial_{X-Z,Y-T}\partial_{Y,T} + \partial_{X-Y,Z-T}\partial_{Z,T} = 0$.
(4) If f is projective, $\partial_{Y,T} g^* = h^* \partial_{X,Z}$.
(5) When f is transversal to i, $h_* \partial_{Y,T} = \partial_{X,Z} g_*$.
(6) When Δ satisfies the hypothesis of *Excess of intersection*,
$h_* \partial_{Y,T} = \partial_{X,Z}\big(g_* \boxtimes c_e(\xi)\big)$.

[7] We prove in the text a stronger statement assuming only that S is reduced.

(7) When Δ satisfies the hypothesis of *Ramification formula*,
$\sum_{i \in I} h_* \partial_{Y,T_i} = \sum_{i \in I} \partial_{X,Z}\big(r(T_i; f, g) \boxtimes g_{i*}\big)$.

The differential taste of the residue morphism appears clearly in the last formula (especially in the cohomological formulation) where the multiplicity $r(T_i; f, g)$ takes into account the ramification index r_i. Even in algebraic K-theory, this formula seems to be new.

1.3.3. *Blow-up formulas.* Let X be a smooth scheme and $Z \subset X$ be a smooth closed subscheme of codimension n. Let B be the blow-up of X with center Z and consider the cartesian square
$$\begin{tikzcd} P \arrow[r, "k"] \arrow[d, "p"'] & B \arrow[d, "f"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$
. Let e be the top Chern class of the canonical quotient bundle on the projective space P/Z.

(1) (**prop. 5.38**) Let $M(P)/M(Z)$ be the kernel of the split monomorphism p_*. The morphism (k_*, f^*) induces an isomorphism :
$$M(P)/M(Z) \oplus M(X) \to M(B).$$

(2) (**prop. 5.39**) The short sequence
$$0 \to M(B) \xrightarrow{\binom{k^*}{f_*}} M(P)(1)[2] \oplus M(X) \xrightarrow{(p_* \boxtimes e, -i^*)} M(Z)(n)[2n] \to 0$$
is split exact. Moreover, $(p_* \boxtimes e, -i^*) \circ \binom{p^*}{0}$ is an isomorphism[8].

The first formula was obtained by V. Voevodsky using resolution of singularities in the case where S is the spectrum of a perfect field. The second formula is the analog of a result of W. Fulton on Chow groups (cf [Ful98, 6.7]).

1.4. **Characteristic classes.** Besides Chern classes, we can introduce the following characteristic classes in our context.

Let $i : Z \to X$ be a closed immersion of codimension n between smooth schemes, $\pi : Z \to S$ the canonical projection. We define the *fundamental class* of Z in X (**paragraph 4.14**) as the cohomology class represented by the morphism
$$\eta_X(Z) : M(X) \xrightarrow{i^*} M(Z)(n)[2n] \xrightarrow{\pi_*} \mathbb{1}(n)[2n].$$
It is a cohomology class in $H^{2n,n}(X)$ satisfying the more classical expression $\eta_X(Z) = i_*(1)$.

Considering the hypothesis of the ramification formula above, when $n = m = 1$, we obtain the enlightening formula (cf **cor. 4.28**) :
$$f^*(\eta_X(Z)) = \sum_{i \in I} [r_i]_F \cdot \eta_Y(T_i)$$
where r_i is the ramification index of f along T_i and $[r_i]_F$ is the r_i-th formal sum with respect to F applied to the cohomological class $\eta_Y(T_i)$. Indeed, the fact T admits an ample line bundle implies this class is nilpotent.

The most useful fundamental class in the article is the Thom class of a vector bundle E/X of rank n. Let $P = \mathbb{P}(E \oplus 1)$ be its projective completion and

[8] This isomorphism is the identity at least in the case when $F(x, y) = x + y$

consider the canonical section $X \xrightarrow{s} P$. The *Thom class* of E/X is $t(E) := \eta_P(X)$. By the projection formula, $s^* = p_* \boxtimes t(E)$, where $p : P \to X$ is the canonical projection. Let λ (resp. ξ) be the canonical line bundle (resp. universal quotient bundle) on P/X. We also obtain the following equalities[9] :

$$t(E) = c_n(\lambda^\vee \otimes p^{-1}E) \overset{(*)}{=} c_n(\xi) = \sum_{i=0}^{n} c_i(p^{-1}E) \cup \big(- c_1(\lambda)\big)^i.$$

This is straightforward in the case where $F(x,y) = x + y$ but more difficult in general.
We also obtain a computation which the author has not seen in the literature (even in complex cobordism). Write $F(x,y) = \sum_{i,j} a_{ij}.x^i y^j$ with $a_{ij} \in A$. Consider the diagonal embedding $\delta : \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$. Let λ_1 and λ_2 be the respective canonical line bundle on the first and second factor of $\mathbb{P}^n \times \mathbb{P}^n$. Then (**prop. 5.30**) the fundamental class of δ satisfies

$$\eta_{\mathbb{P}^n \times \mathbb{P}^n}(\mathbb{P}^n) = \sum_{0 \leq i,j \leq n} a_{1,i+j-n}.c_1(\lambda_1^\vee)^i c_1(\lambda_2^\vee)^j.$$

Another kind of characteristic classes are *cobordism classes*. Let $p : X \to S$ be a smooth projective scheme of relative dimension n. The cobordism class of X/S is the cohomology class represented by the morphism

$$[X] : \mathbb{1} \xrightarrow{p^*} M(X)(-n)[-2n] \xrightarrow{p_*} \mathbb{1}(-n)[-2n].$$

It is a class in $A^{-2n,-n}$. As an application of the previous equality, we obtain the following computation (**cor. 5.31**) :

$$[\mathbb{P}^n] = (-1)^n . \det \begin{pmatrix} 0 & \cdots & 0 & 1 & a_{1,1} \\ \vdots & \cdot^{\cdot^{\cdot}} & & & a_{1,2} \\ 0 & & & & \vdots \\ 1 & & & & \\ a_{1,1} & a_{1,2} & \cdots & & a_{1,n} \end{pmatrix}$$

which of course coincides with the expression given by the classical theorem of Myschenko in complex cobordism. In fact, our method gives a new proof of the latter theorem.

1.5. **Outline of the work.** In section 2, we give the list of axioms (cf 2.1) satisfied by the category $\mathscr{T}$ and discuss the first consequences of these. Remark an originality of our axiomatic is that we not only consider pairs of schemes but also quadruples (used in the proof of 4.32). The last subsection 2.3 gives the principle examples which satisfy the axiomatic 2.1. Section 3 contains the projective bundle theorem and its consequences, the formal group law and Chern classes.
Section 4 contains the study of the Gysin triangle. The fundamental result in this section is the purity theorem 4.3. Usually, one constructs the Thom isomorphism using the Thom class (4.4). Here however, we directly construct the former isomorphism from the projective bundle theorem and the deformation to the normal

[9]This corrects an affirmation of I. Panin in the introduction of [Pan03a, p. 268] where equality $(*)$ is said not to hold.

cone. This makes the construction more canonical – though there is a delicate choice of signs hidden (cf beginning of section 4.1) – and it thus gives a canonical Thom class. We then study the two principle subjects around the Gysin triangle : the base change formula and its defect (section 4.2 which contains notably 4.26 and 4.16 cited above) and the interaction (containing notably the functoriality of the Gysin morphism) of two Gysin triangles attached with smooth subschemes of a given smooth scheme (th. 4.32).
In section 5, we first recall the notion of *strong duality* introduced by A. Dold and D. Puppe and give some complements. Then we give the construction of the Gysin morphism in the projective case and the duality statement. The general situation is particularly complicated when the formal group law F is not the additive one, as the Gysin morphism associated to the projection p of $\mathbb{P}^n$ is not easy to handle. Our method is to exploit the strong duality on $\mathbb{P}^n$ implied by the projective bundle theorem. We show that the fundamental class of the diagonal δ of $\mathbb{P}^n/S$ determines canonically the Gysin morphism of the projection (see def. 5.6). This is due to the explicit form of the duality pairing for $\mathbb{P}^n$ cited above : the motive $M(\mathbb{P}^n)$ being strongly dualizable, one morphism of the duality pairing (μ_X, ϵ_X) determines the other; the first one is induced by δ^* and the other one by p^*. Once this fact is determined, we easily obtain all the properties required to define the Gysin morphism and then the general duality pairing. The article ends with the explicit determination of the cobordism class of $\mathbb{P}^n$ and the blow-up formulas as illustrations of the theory developed here.

1.6. **Final commentary.** In another work [Dég08], we study the Gysin triangle directly in the category of geometric mixed motives over a perfect field. In the latter, we used the isomorphism of the relevant part of motivic cohomology groups and Chow groups and prove our Gysin morphism induces the usual pushout on Chow groups via this isomorphism (cf [Dég08, 1.21]). This gives a shortcut for the definitions and propositions proved here in the particular case of motives over a perfect field. In *loc. cit.* moreover, we also use the isomorphism between the diagonal part of the motivic cohomology groups of a field L and the Milnor K-theory of L and prove our Gysin morphism induces the usual norm morphism on Milnor K-theory (cf [Dég08, 3.10]) – after a limit process, considering L as a function field.
The present work is obviously linked with the fundamental book on algebraic cobordism by Levine and Morel [LM07] (see also [Lev08b]), but here, we study oriented cohomology theories from the point of view of stable homotopy. This point of view is precisely that of [Lev08a]. It is more directly linked with the prepublication [Pan03b] of I. Panin which was mainly concerned with the construction of pushforwards in cohomology, corresponding to our Gysin morphism (see also [Smi06] and [Pim05] for extensions of this work). Our study gives a unified self-contained treatment of all these works, except that we have not considered here the theory of transfers and Chern classes with support (see [Smi06], [Lev08a, part 5]).

The final work we would like to mention is the thesis of J. Ayoub on cross functors ([Ayo07]). In fact, it is now folklore that the six functor formalism yields a construction of the Gysin morphism. In the work of Ayoub however, the questions of orientability are not treated. In particular, the Gysin morphism we obtain takes value in a certain Thom space. To obtain the Gysin morphism in the usual form, we have to consider the Thom isomorphism introduced here. Moreover, we do not need the localization property in our study whereas it is essentially used in the formalism of cross functors. This is a strong property which is not known in general for triangulated mixed motives. Finally, the interest of this article relies in the study of the defect of the base change formula which is not covered by the six functor formalism.

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2. The general setting : homotopy oriented triangulated systems

2.1. **Axioms and notations.** Let $\mathscr{D}$ be the category whose objects are the cartesian squares

$$(*)\qquad \begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\Delta"] & V \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

made of immersions between smooth schemes. The morphisms in $\mathscr{D}$ are the evident commutative cubes. We will define the *transpose* of the square Δ, denoted by Δ', as the square

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\Delta'"] & U \arrow[d] \\ V \arrow[r] & X \end{tikzcd}$$

made of the same immersions. This defines an endofunctor of $\mathscr{D}$.
In all this work, we consider a triangulated symmetric monoidal category $(\mathscr{T}, \otimes, \mathbb{1})$ together with a covariant functor $M : \mathscr{D} \to \mathscr{T}$. Objects of $\mathscr{T}$ are called *premotives*. Considering a square as in $(*)$, we adopt the suggesting notation

$$M\left(\frac{X/U}{V/W}\right) = M(\Delta).$$

We simplify this notation in the following two cases :

(1) If $V = W = \emptyset$, we put $M(X/U) = M(\Delta)$.
(2) If $U = V = W = \emptyset$, we put $M(X) = M(\Delta)$.

We call *closed pair* any pair (X, Z) of schemes such that X is smooth and Z is a closed (not necessarily smooth) subscheme of X. As usual, we define the premotive of X with support in Z as $M_Z(X) = M(X/X - Z)$.
A *pointed scheme* is a scheme X together with an S-point $x : S \to X$. When X is smooth, the reduced premotive associated with (X, x) will be $\tilde{M}(X, x) = M(S \xrightarrow{x} X)$. Let $n > 0$ be an integer. We will always assume the smooth scheme $\mathbb{P}^n_S$ is pointed by the infinity. We define the *Tate twist* as the premotive $\mathbb{1}(1) = \tilde{M}(\mathbb{P}^1_S)[-2]$ of $\mathscr{T}$.

2.1. We suppose the functor M satisfies the following axioms :

(Add) For any finite family of smooth schemes $(X_i)_{i\in I}$,
$M(\sqcup_{i\in I} X_i) = \oplus_{i\in I} M(X_i)$.

(Htp) For any smooth scheme X, the canonical projection of the affine line induces an isomorphism $M(\mathbb{A}^1_X) \to M(X)$.

(Exc) Let (X, Z) be a closed pair and $f : V \to X$ be an étale morphism. Put $T = f^{-1}(Z)$ and suppose the map $T_{red} \to Z_{red}$ obtained by restriction of f is an isomorphism. Then the induced morphism $\phi : M_T(V) \to M_Z(X)$ is an isomorphism.

(Stab) The Tate premotive $\mathbb{1}(1)$ admits an inverse for the tensor product denoted by $\mathbb{1}(-1)$.

(Loc) For any square Δ as in $(*)$, a morphism $\partial_\Delta : M\left(\frac{X/U}{V/W}\right) \to M(V/W)[1]$ is given natural in Δ and such that the sequence of morphisms

$$M(V/W) \to M(X/U) \to M\left(\frac{X/U}{V/W}\right) \xrightarrow{\partial_\Delta} M(V/W)[1]$$

made of the evident arrows is a distinguished triangle in $\mathscr{T}$.

(Sym) Let Δ be a square as in $(*)$ and consider its transpose Δ'. There is given a morphism $\epsilon_\Delta : M\left(\frac{X/U}{V/W}\right) \to M\left(\frac{X/V}{U/W}\right)$ natural in Δ.
If in the square Δ, $V = W = \emptyset$, we put

$$\partial_{X/U} = \partial_{\Delta'} \circ \epsilon_\Delta : M(X/U) \to M(U)[1].$$

We ask the following coherence properties :
(a) $\epsilon_{\Delta'} \circ \epsilon_\Delta = 1$.
(b) If $\Delta = \Delta'$ then $\epsilon_\Delta = 1$.
(c) The following diagram is anti-commutative :

$$\begin{tikzcd} M\left(\frac{X/U}{V/W}\right) \arrow[r, "\epsilon_\Delta"] \arrow[d, "\partial_\Delta"'] & M\left(\frac{X/V}{U/W}\right) \arrow[r, "\partial_{\Delta'}"] & M(U/W)[1] \arrow[d, "\partial_{U/W}[1]"] \\ M(V/W)[1] \arrow[rr, "\partial_{V/W}[1]"] & & M(W)[2]. \end{tikzcd}$$

(Kun) (a) For any open immersions $U \to X$ and $V \to Y$ of smooth schemes, there are canonical isomorphisms:
$M(X/U) \otimes M(Y/V) = M(X \times Y/X \times V \cup U \times Y)$, $\quad M(S) = \mathbb{1}$
satisfying the coherence conditions of a monoidal functor.

(b) Let X and Y be smooth schemes and $U \to X$ be an open immersion. Then, $\partial_{X\times Y/U\times Y} = \partial_{X/U} \otimes 1_{Y*}$ through the preceding canonical isomorphism.

(Orient) For any smooth scheme X, there is an application, called the *orientation*,

$$c_1 : \mathrm{Pic}(X) \to \mathrm{Hom}_{\mathscr{T}}(M(X), \mathbb{1}(1)[2])$$

which is functorial in X and such that the class $c_1(\lambda_1) : M(\mathbb{P}^1_S) \to \mathbb{1}(1)[2]$ is the canonical projection.

For any integer $n \in \mathbb{N}$, we let $\mathbb{1}(n)$ (resp. $\mathbb{1}(-n)$) be the n-th tensor power of $\mathbb{1}(1)$ (resp. $\mathbb{1}(-1)$). Moreover, for an integer $n \in \mathbb{Z}$ and a premotive $\mathbb{E}$, we put $\mathbb{E}(n) = \mathbb{E} \otimes \mathbb{1}(n)$.

2.2. Using the excision axiom (Exc) and an easy noetherian induction, we obtain from the homotopy axiom (Htp) the following stronger result :

(Htp') For any fiber bundle E over a smooth scheme X, the morphism induced by the canonical projection $M(E) \to M(X)$ is an isomorphism.

We further obtain the following interesting property :

(Add') Let X be a smooth scheme and Z, T be disjoint closed subschemes of X. Then the canonical map $M_{Z\sqcup T}(X) \to M_Z(X) \oplus M_T(X)$ induced by naturality is an isomorphism.

Indeed, using (Loc) with $V = X - T$, $W = X - (Z \sqcup T)$ and $U = W$, we get a distinguished triangle

$$M_Z(V) \to M_{Z\sqcup T}(X) \xrightarrow{\pi} M\left(\frac{X/W}{V/W}\right) \to M_Z(V)[1].$$

Using (Exc), we obtain $M_Z(V) = M_Z(X)$. The natural map $M_{Z\sqcup T}(X) \to M_Z(X)$ induces a retraction of the first arrow. Moreover, we get $M\left(\frac{X/W}{V/W}\right) = M_T(X)$ from the symmetry axiom (Sym). Note that we need (Sym)(b) and the naturality of ϵ_Δ to identify π with the natural map $M_{Z\sqcup T}(X) \to M_T(X)$.

Remark 2.3. *About the axioms.—*

(1) There is a stronger form of the excision axiom (Exc) usually called the Brown-Gersten property (or distinguished triangle). In the situation of axiom (Exc), with $U = X - Z$ and $W = V - T$, we consider the cone in the sense of [Nee01] of the morphism of distinguished triangles

$$\begin{tikzcd} M(W) \arrow[r] \arrow[d] & M(V) \arrow[r] \arrow[d] & M(V/W) \arrow[r] \arrow[d] & M(V)[1] \arrow[d] \\ M(U) \arrow[r] & M(X) \arrow[r] & M(X/U) \arrow[r] & M(U)[1] \end{tikzcd}$$

This is a *candidate triangle* in the sense of *op. cit.* of the form

$$M(W) \to M(U) \oplus M(V) \to M(X) \to M(W).$$

Thus, in our abstract setting, it is not necessarily a distinguished triangle. We call (BG) the hypothesis that in every such situation, the candidate triangle obtained above is a distinguished triangle. We will not need the hypothesis (BG) ; however, in the applications, it is always true and the reader may use this stronger form for simplification.

(2) We can replace axiom (Kun)(a) by a weaker one

(wKun) The restriction of M to the category of pairs of schemes (X, U) is a *lax* monoidal symmetric functor.

(Kun)(b) is then replaced by an obvious coherence property of the boundary operator in (Loc). This hypothesis is sufficient for the needs of the article with a notable exception of the duality pairing 5.23. For example, if one wants to work with cohomology theories directly, one has to use rather this axiom, replace $\mathscr{T}$ by an abelian category and "distinguished triangle" by "long exact sequence" everywhere. The arguments given here covers equally this situation, except for the general duality pairing.

(3) The symmetry axiom (Sym) encodes a part of a richer structure which possess the usual examples (all the ones considered in section 2.3). This is the structure of a derivator as the object $M(\Delta)$ may be seen as a homotopy colimit. The coherence axioms which appear in (Sym) are very natural from this point of view.

Definition 2.4. Let $\mathbb{E}$ be a premotive. For any smooth scheme X and any couple $(n, p) \in \mathbb{Z} \times \mathbb{Z}$, we define respectively the cohomology and the homology groups of X with coefficient in $\mathbb{E}$ as

$$\mathbb{E}^{n,p}(X) = \mathrm{Hom}_{\mathscr{T}}\big(M(X), \mathbb{E}(p)[n]\big),$$
$$\text{resp. } \mathbb{E}_{n,p}(X) = \mathrm{Hom}_{\mathscr{T}}\big(\mathbb{1}(p)[n], \mathbb{E} \otimes M(X)\big).$$

We refer to the corresponding bigraded cohomology group (resp. homology group) by $\mathbb{E}^{**}(X)$ (resp. $\mathbb{E}_{**}(X)$). The first index is usually refered to as the cohomological (resp. homological) *degree* and the second one as the cohomological (resp. homological) *twist*. We also define the module of coefficients attached to $\mathbb{E}$ as $\mathbb{E}^{**} = \mathbb{E}^{**}(S)$.
When $\mathbb{E} = \mathbb{1}$, we use the notations $H^{**}(X)$ (resp. $H_{**}(X)$) for the cohomology (resp. homology) with coefficients in $\mathbb{1}$. Finally, we simply put $A = H^{**}(S)$.

Remark that, from axiom (Kun)(a), A is a bigraded ring. Moreover, using the axiom (Stab), $A = H_{**}(S)$. Thus, there are two bigraduations on A, one cohomological and the other homological, and the two are exchanged as usual by a change of sign. The tensor product of morphisms in $\mathscr{T}$ induces a structure of left bigraded A-module on $\mathbb{E}^{**}(X)$ (resp. $\mathbb{E}_{**}(X)$). There is a lot more algebraic structures on these bigraded groups that we have gathered in section 2.2.
The axiom (Orient) gives a natural transformation

$$c_1 : \mathrm{Pic} \to H^{2,1}$$

of presheaves of *sets* on $\mathscr{S}m_S$, or in other words, an *orientation* on the fundamental cohomology $H^{*,*}$ assocciated with the functor M. In our setting, cohomology classes are morphisms in $\mathscr{T}$: for any element $L \in \mathrm{Pic}(X)$, we view $c_1(L)$ both as a cohomology class, the *first Chern class*, and as a morphism in $\mathscr{T}$.

Remark 2.5. In the previous definition, we can replace the premotive $M(X)$ by any premotive $\mathcal{M}$. This allows to define as usual the cohomology/homology of an (arbitrary) pair (X, U) made by a smooth scheme X and a smooth subscheme U of X. Particular cases of this general definition is the cohomology/homology of a smooth scheme X with support in a closed subscheme Z and the reduced cohomology/homology associated with a pointed smooth scheme.

2.2. **Products.** Let X be a smooth scheme and $\delta : X \to X \times X$ its associated diagonal embedding. Using axiom (Kun)(a) and functoriality, we get a morphism $\delta'_* : M(X) \to M(X) \otimes M(X)$. Given two morphisms $x : M(X) \to \mathbb{E}$ and $y : M(X) \to \mathbb{F}$ in $\mathscr{T}$, we can define a product

$$x \boxtimes y = (x \otimes y) \circ \delta'_* : M(X) \to \mathbb{E} \otimes \mathbb{F}.$$

2.6. By analogy with topology, we will call *ringed premotive* any premotive $\mathbb{E}$ equipped with a commutative monoid structure in the symmetric monoidal category $\mathscr{T}$. This means we have a product map $\mu : \mathbb{E} \otimes \mathbb{E} \to \mathbb{E}$ and a unit map $\eta : \mathbb{1} \to \mathbb{E}$ satisfying the formal properties of a commutative monoid.
For any smooth scheme X and any couple of integer $(n, p) \in \mathbb{Z}^2$, the unit map induces morphisms

$$\begin{aligned}\varphi_X :& H^{n,p}(X) \to \mathbb{E}^{n,p}(X)\\ \psi_X :& H_{n,p}(X) \to \mathbb{E}_{n,p}(X)\end{aligned}$$

which we call the *regulator maps*.
Giving such a ringed premotive $\mathbb{E}$, we define[10] the following products :

- *Exterior products* :

$$\begin{aligned}\mathbb{E}^{n,p}(X) \otimes \mathbb{E}^{m,q}(Y) &\to \mathbb{E}^{n+m,p+q}(X \times Y),\\ (x, y) &\mapsto x \times y := \mu \circ x \otimes y\\ \mathbb{E}_{n,p}(X) \otimes \mathbb{E}_{m,q}(Y) &\to \mathbb{E}_{n+m,p+q}(X \times Y),\\ (x, y) &\mapsto x \times y := (\mu \otimes 1_{X \times Y*}) \circ (x \otimes y)\end{aligned}$$

- *Cup-product* :

$$\mathbb{E}^{n,p}(X) \otimes \mathbb{E}^{m,q}(X) \to \mathbb{E}^{n+m,p+q}(X), (x, x') \mapsto x \cup x' := \mu \circ (x \boxtimes x').$$

  Then $\mathbb{E}^{**}$ is a bigraded ring and $\mathbb{E}^{**}(X)$ is a bigraded $\mathbb{E}^{**}$-algebra. Moreover, $\mathbb{E}^{**}$ is a bigraded A-algebra and the regulator map is a morphism of bigraded A-algebra.
- *Slant products*[11] :

$$\begin{aligned}\mathbb{E}^{n,p}(X \times Y) \otimes \mathbb{E}_{m,q}(X) &\to \mathbb{E}^{n-m,p-q}(Y),\\ (w, x) &\mapsto w/x := \mu \circ (1_{\mathbb{E}} \otimes w) \circ (x \otimes 1_{Y*})\\ \mathbb{E}^{n,p}(X) \otimes \mathbb{E}_{m,q}(X \times Y) &\to \mathbb{E}_{m-n,q-p}(Y),\\ (x, w) &\mapsto x \backslash w := (\mu \otimes 1_{Y*}) \circ (x \otimes 1_{\mathbb{E}} \otimes 1_{Y*}) \circ w.\end{aligned}$$

[10] We do not indicate the commutativity isomorphisms for the tensor product and the twists in the formulas to make them shorter.

[11] For the first slant product defined here, we took a slightly different covention than [Swi02, 13.50(ii)] in order to obtain formula (5.3). Of course, the two conventions coincide up to the isomorphism $X \times Y \simeq Y \times X$.

- *Cap-product* :

$$\mathbb{E}^{n,p}(X) \otimes \mathbb{E}_{m,q}(X) \to \mathbb{E}_{m-n,q-p}(X), (x, x') \mapsto x \cap x' := x \backslash \big((1_{\mathbb{E}} \otimes \delta_*) \circ x'\big).$$

- *Kronecker product* :

$$\mathbb{E}^{n,p}(X) \otimes \mathbb{E}_{m,q}(X) \to A^{n-m,p-q}, (x, x') \mapsto \langle x, x' \rangle := x/x'$$

where y is identified to a homology class in $\mathbb{E}_{m,q}(S \times X)$.

The regulator maps (cohomological and homological) are compatible with these products in the obvious way.

Remark 2.7. These products satisfy a lot of formal properties. We will not use them in this text but we refer the interested reader to [Swi02, chap. 13] for more details (see more precisely 13.57, 13.61, 13.62).

2.8. We can extend the definition of these products to the cohomology of an open pair (X, U). We refer the reader to *loc. cit.* for this extension but we give details for the cup-product in the case of cohomology with supports as this will be used in the sequel.
Let X be a smooth scheme and Z, T be two closed subschemes of X. Then the diagonal embedding of X/S induces using once again axiom (Kun)(a) a morphism $\delta''_* : M_{Z \cap T}(X) \to M_Z(X) \otimes M_T(X)$. This allows to define a product of motives with support. Given two morphisms $x : M_Z(X) \to \mathbb{E}$ and $y : M_T(X) \to \mathbb{F}$ in $\mathscr{T}$, we define

$$x \boxtimes y = (x \otimes y) \circ \delta''_* : M_{Z \cap T}(X) \to \mathbb{E} \otimes \mathbb{F}.$$

In cohomology, we also define the cup-product with support :

$$\mathbb{E}^{n,p}_Z(X) \otimes E^{m,q}_T(X) \to \mathbb{E}^{n+m,p+q}_{Z \cap T}(X), (x, y) \mapsto x \cup_{Z,T} y = \mu \circ (x \otimes y) \circ \delta''_*.$$

Note that considering the canonical morphism $\nu_{X,W} : \mathbb{E}^{n,p}_W(X) \to \mathbb{E}^{n,p}(X)$, for any closed subscheme W of X, we obtain easily :

$$\nu_{X,Z}(x) \cup \nu_{X,T}(y) = \nu_{X,Z \cap T}(x \cup_{Z,T} y). \tag{2.1}$$

2.9. Suppose now that $\mathbb{E}$ has no ring structure. It nethertheless always has a module structure over the ringed premotive $\mathbb{1}$ – given by the structural map (isomorphism) $\eta : \mathbb{1} \otimes \mathbb{E} \to \mathbb{E}$.
This induces in particular a structure of left $H^{**}(X)$-module on $\mathbb{E}^{**}(X)$ for any smooth scheme X. Moreover, it allows to extend the definition of slant products and cap products. Explicitely, this gives in simplified terms :

- *Slant products* :

$$H^{n,p}(X \times Y) \otimes \mathbb{E}_{m,q}(X) \to \mathbb{E}^{n-m,p-q}(Y),$$
$$(w, y) \mapsto w/y := \eta \circ (1_{\mathbb{E}} \otimes w) \circ (x \otimes 1_{Y*})$$

- *Cap-products* :

$$\mathbb{E}^{n,p}(X) \otimes H_{m,q}(X) \to \mathbb{E}_{m-n,q-p}(X), (x, x') \mapsto x \cap x' := (x \otimes 1_{X*}) \circ \delta_* \circ x'.$$

These generalized products will be used at the end of the article to formulate duality with coefficients in $\mathbb{E}$ (cf paragraph 5.24).

Note finally that, analog to the cap-product, we have a $H^{**}(X)$-module structure on $E_{**}(X)$ that can be used to describe the projective bundle formula in $\mathbb{E}$-homology (cf formula (2) of 3.4).

2.3. **Examples.**

2.3.1. *Motives.* Suppose S is a regular scheme. Below, we give the full construction of the category of geometric motives of Voevodsky over S, and indicate how to check the axioms of 2.1. Note however we will give a full construction of this category, together with the category of motivic complexes and spectra, over any noetherian base S in [CD07]. Here, the reader can find all the details for the proof of the axioms 2.1 (especially axiom (Orient)).
For any smooth schemes X and Y, we let $c_S(X,Y)$ be the abelian group of cycles in $X \times_S Y$ whose support is finite equidimensional over X. As shown in [Dég07, sec. 4.1.2], this defines the morphisms of a category denoted by $\mathscr{S}m^{\mathrm{cor}}_S$. The category $\mathscr{S}m^{\mathrm{cor}}_S$ is obviously additive. It has a symmetric monoidal structure defined by the cartesian product on schemes and by the exterior product of cycles on morphisms.
Following Voevodsky, we define the category of effective geometric motives $DM^{eff}_{gm}(S)$ as the pseudo-abelian envelope[12] of the Verdier triangulated quotient

$$\mathrm{K}^b(\mathscr{S}m^{\mathrm{cor}}_S)/\mathscr{T}$$

where $\mathrm{K}^b(\mathscr{S}m^{\mathrm{cor}}_S)$ is the category of bounded complexes up to chain homotopy equivalence and $\mathscr{T}$ is the thick subcategory generated by the following complexes :

(1) For any smooth scheme X,

$$\dots 0 \to \mathbb{A}^1_X \xrightarrow{p} X \to 0 \dots$$

with p the canonical projection.

(2) For any cartesian square of smooth schemes

$$\begin{tikzcd} W \arrow[r, "k"] \arrow[d, "g"'] & V \arrow[d, "f"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

such that j is an open immersion, f is étale and the induced morphism $f^{-1}(X-U)_{red} \to (X-U)_{red}$ is an isomorphism,

$$\dots 0 \to W \xrightarrow{\binom{g}{-k}} U \oplus V \xrightarrow{(j,f)} X \to 0 \dots \tag{2.2}$$

Consider a cartesian square of immersions

$$\begin{tikzcd} W \arrow[r, "k"] \arrow[d, "g"'] \arrow[dr, phantom, "\Delta"] & V \arrow[d, "f"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

[12] Recall that according to the result of [BS01], the pseudo-abelian envelope of a triangulated category is still triangulated

This defines a morphism of complexes in $\mathscr{S}m_S^{\mathrm{cor}}$:

$$\psi : \left\{ \begin{tikzcd} \dots 0 \arrow[r] & W \arrow[r, "k"] \arrow[d, "g"'] & V \arrow[r] \arrow[d, "f"] & 0 \dots \\ \dots 0 \arrow[r] & U \arrow[r, "j"] & X \arrow[r] & 0 \dots \end{tikzcd} \right.$$

We let $M(\Delta)$ be the cone of ψ and see it as an object of $DM_{gm}^{eff}(S)$. To fix the convention, we define this cone as the triangle (2.2) above. With this convention, we define ϵ_Δ as the following morphism :

$$\begin{tikzcd} \dots 0 \arrow[r] & W \arrow[r] \arrow[d, "-1"'] & U \oplus V \arrow[r] \arrow[d, "\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}"] & X \arrow[r] \arrow[d, "+1"] & 0 \dots \\ \dots 0 \arrow[r] & W \arrow[r] & V \oplus U \arrow[r] & X \arrow[r] & 0 \dots \end{tikzcd}$$

The reader can now check easily that the resulting functor $M : \mathscr{D} \to DM_{gm}^{eff}(S)$, satisfies all the axioms of 2.1 except (Stab) and (Orient). We let $\mathbb{Z} = M(S)$ be the unit object for the monoidal structure of $DM_{gm}^{eff}(S)$.
To force axiom (Stab), we formally invert the motive $\mathbb{Z}(1)$ in the monoidal category $DM_{gm}^{eff}(S)$. This defines the *triangulated category of (geometric) motives* denoted by $DM_{gm}(S)$. Remark that according to the proof of [Voe02, lem. 4.8], the cyclic permutation of the factors of $\mathbb{Z}(3)$ is the identity. This implies the monoidal structure on $DM_{gm}^{eff}(S)$ induces a unique monoidal structure on $DM_{gm}(S)$ such that the obvious triangulated functor $DM_{gm}^{eff}(S) \to DM_{gm}(S)$ is monoidal. Now, the functor $M : \mathscr{D} \to DM_{gm}(S)$ still satisfies all axioms of 2.1 mentioned above but also axiom (Stab).
To check the axiom (Orient), it is sufficient to construct a natural application

$$\mathrm{Pic}(X) \to \mathrm{Hom}_{DM_{gm}^{eff}(S)}(M(X), \mathbb{Z}(1)[2]).$$

We indicate how to obtain this map. Note moreover that, from the following construction, it is a morphism of abelian group.
Still following Voevodsky, we have defined in [Dég07] the abelian category of sheaves with transfers over S, denoted by $Sh(\mathscr{S}m_S^{\mathrm{cor}})$. We define the cateogy $DM^{eff}(S)$ of motivic complexes as the $\mathbb{A}^1$-localization of the derived category of $Sh(\mathscr{S}m_S^{\mathrm{cor}})$. The Yoneda embedding $\mathscr{S}m_S^{\mathrm{cor}} \to Sh(\mathscr{S}m_S^{\mathrm{cor}})$ sends smooth schemes to *free* abelian groups. For this reason, the canonical functor

$$DM_{gm}^{eff}(S) \to DM^{eff}(S)$$

is fully faithful. Let $\mathbb{G}_m$ be the sheaf with transfers which associates to a smooth scheme its group of invertible (global) functions. Following Suslin and Voevodsky (cf also [Dég05, 2.2.4]), we construct a morphism in $DM^{eff}(S)$:

$$\mathbb{G}_m \to M(\mathbb{G}_m) = \mathbb{Z} \oplus \mathbb{Z}(1)[1]$$

This allows to define the required morphism :

$$\begin{aligned} \mathrm{Pic}(X) = H^1_{\mathrm{Nis}}(X; \mathbb{G}_m) &\simeq \mathrm{Hom}_{DM^{eff}(S)}(M(X), \mathbb{G}_m[1]) \\ &\to \mathrm{Hom}_{DM^{eff}(S)}(M(X), \mathbb{Z}(1)[2]) \simeq \mathrm{Hom}_{DM_{gm}^{eff}(S)}(M(X), \mathbb{Z}(1)[2]). \end{aligned}$$

The first isomorphism uses that the sheaf $\mathbb{G}_m$ is $\mathbb{A}^1$-local and that the functor forgetting transfers is exact (cf [Dég07, prop. 2.9]).

2.3.2. *Stable homotopy exact functors.* In this example, S is any noetherian scheme. For any smooth scheme X, we let X_+ be the pointed sheaf of sets on $\mathscr{S}m_S$ represented by X with a (disjoint) base point added.
Consider an immersion $U \to X$ of smooth schemes. We let X/U be the pointed sheaf of sets which is the cokernel of the pointed map $U_+ \to X_+$.
Suppose moreover given a square Δ as in $(*)$. Then we obtain an induced morphism of pointed sheaves of sets $V/W \to X/U$ which is injective. We let $\frac{X/U}{V/W}$ be the cokernel of this monomorphism. Thus, we obtain a cofiber sequence in $\mathscr{H}_\bullet(S)$

$$V/W \to X/U \to \frac{X/U}{V/W} \xrightarrow{\partial_\Delta} S^1_s \wedge V/W.$$

Moreover, the functor

$$\mathscr{D} \to \mathscr{H}_\bullet(S), \Delta \mapsto \frac{X/U}{V/W}$$

satisfies axioms (Add), (Htp), (Exc) and (Kun) from [MV99].
Consider now the stable homotopy category of schemes $S\mathscr{H}(S)$ (cf [Jar00]) together with the infinite suspension functor

$$\Sigma^\infty : \mathscr{H}_\bullet(S) \to S\mathscr{H}(S).$$

The category $S\mathscr{H}(S)$ is a triangulated symmetric monoidal category. The canonical functor $\mathscr{D} \to S\mathscr{H}(S)$ satisfies all the axioms of 2.1 except axiom (Orient). In fact, (Loc) and (Sym) follows easily from the definitions and (Stab) was forced in the construction of $S\mathscr{H}(S)$.
Suppose we are given a triangulated symmetric monoidal category $\mathscr{T}$ together with a triangulated symmetric monoidal functor

$$R : S\mathscr{H}(S) \to \mathscr{T}.$$

This induces a canonical functor

$$M : \mathscr{D} \to \mathscr{T}, \frac{X/U}{V/W} \mapsto M\left(\frac{X/U}{V/W}\right) := R\left(\Sigma^\infty\left(\frac{X/U}{V/W}\right)\right)$$

and $(M, \mathscr{T})$ satisfies formally all the axioms 2.1 except (Orient).
Let $B\mathbb{G}_m$ be the classifying space of $\mathbb{G}_m$ defined in [MV99, section 4]. It is an object of the simplicial homotopy category $\mathscr{H}^s_\bullet(S)$ and from *loc. cit.*, proposition 1.16,

$$Pic(X) = \mathrm{Hom}_{\mathscr{H}^s_\bullet(S)}(X_+, B\mathbb{G}_m).$$

Let $\pi : \mathscr{H}^s_\bullet(S) \to \mathscr{H}_\bullet(S)$ be the canonical $\mathbb{A}^1$-localisation functor. Applying proposition 3.7 of *loc. cit.*, $\pi(B\mathbb{G}_m) = \mathbb{P}^\infty$ where $\mathbb{P}^\infty$ is the tower of pointed schemes

$$\mathbb{P}^1 \to .. \to \mathbb{P}^n \xrightarrow{\iota_n} \mathbb{P}^{n+1} \to ...$$

made of the inclusions onto the corresponding hyperplane at infinity. We let $M(\mathbb{P}^\infty)$ (resp. $\tilde{M}(\mathbb{P}^\infty)$) be the ind-object of $\mathscr{T}$ obtained by applying M (resp. $\tilde{M}$) on each degree of the tower above.

Using this, we can define an application

$$\begin{aligned}\rho_X : &Pic(X) = \mathrm{Hom}_{\mathscr{H}^s_\bullet(S)}(X_+, B\mathbb{G}_m)\\ &\to \mathrm{Hom}_{\mathscr{H}_\bullet(S)}(X_+, \pi(B\mathbb{G}_m)) = \mathrm{Hom}_{\mathscr{H}_\bullet(S)}(X_+, \mathbb{P}^\infty)\\ &\to \mathrm{Hom}_{\mathscr{T}}\Big(M(X), \tilde{M}(\mathbb{P}^\infty)\Big)\end{aligned}$$

where the last group of morphisms denotes by abuse of notations the group of morphisms in the category of ind-objects of $\mathscr{T}$ – and similarly in what follows.

Remark 2.10. Note that the sequence $(\lambda_n)_{n\in\mathbb{N}}$ of line bundles is sent by $\rho_{\mathbb{P}^\infty}$ to the canonical projection $M(\mathbb{P}^\infty) \to \tilde{M}(\mathbb{P}^\infty)$ – this follows from the construction of the isomorphism of *loc. cit.*, prop. 1.16.
Recall that $\mathbb{1}(1)[2] = \tilde{M}(\mathbb{P}^1)$ in $\mathscr{T}$. Let $\pi : M(\mathbb{P}^1) \to \tilde{M}(\mathbb{P}^1)$ be the canonical projection and $\iota : \mathbb{P}^1 \to \mathbb{P}^\infty$ be the canonical morphism of pointed ind-schemes. We introduce the following two sets :

- (S_1) The transformations $c_1 : \mathrm{Pic}(X) \to \mathrm{Hom}_{\mathscr{T}}(M(X), \mathbb{1}(1)[2])$ natural in the smooth scheme X such that $c_1(\lambda_1) = \pi$.
- (S_2) The morphisms $c_1' : \tilde{M}(\mathbb{P}^\infty) \to \tilde{M}(\mathbb{P}^1)$ such that $c_1' \circ \iota_* = 1$.

We define the following applications :

1. $\varphi : (S_1) \to (S_2)$.
   Consider an element c_1 of (S_1). The collection $\big(c_1(\lambda_n)\big)_{n\in\mathbb{N}}$ defines a morphism $M(\mathbb{P}^\infty) \to \tilde{M}(\mathbb{P}^1)$. Moreover, the restriction of this latter morphism $\tilde{M}(\mathbb{P}^\infty) \to \tilde{M}(\mathbb{P}^1)$ is obviously an element of (S_2), denoted by $\varphi(c_1)$.
2. $\psi : (S_2) \to (S_1)$.
   Let c_1' be an element of (S_2). For any smooth scheme X, we define

$$\psi(c_1') : \mathrm{Pic}(X) \xrightarrow{\rho_X} \mathrm{Hom}_{\mathscr{T}}\Big(M(X), \tilde{M}(\mathbb{P}^\infty)\Big) \xrightarrow{c'_{1*}} \mathrm{Hom}_{\mathscr{T}}\Big(M(X), \tilde{M}(\mathbb{P}^1)\Big).$$

Using remark 2.10, we check easily that $\psi(c_1')$ belongs to (S_1).

The following lemma is obvious from these definitions :

Lemma 2.11. *Given, the hypothesis and definitions above,* $\varphi \circ \psi = 1$.

Thus, an element of (S_2) determines canonically an element of (S_1). This gives a way to check the axiom (Orient) for a functor R as above. Moreover, we will see below (cf paragraph 3.7) that given an element of (S_1), we obtain a canonical isomorphism $H^{**}(\mathbb{P}^\infty) = A[[t]]$ of bigraded algebra, t having bidegree $(2,1)$. Then elements of (S_2) are in bijection with the set of generators of the the bigraded algebra $H^{**}(\mathbb{P}^\infty)$. Thus in this case, elements of (S_2) are equivalent to *orientations* of the cohomology H^{**} in the classical sense of algebraic topology.

Example 2.12. (1) Let $S = \mathrm{Spec}(k)$ be the spectrum of a field, or more generally any regular scheme. In [CD06, 2.1.4], D.C. Cisinski and the author introduce the notion of mixed Weil theory (and more generally of stable theory) as axioms for cohomology theories on smooth S-schemes which extends the classical axioms of Weil. Examples of such cohomology theories are algebraic De Rham cohomology if k has characteristic 0, rigid cohomology if k has caracteristic p and étale l-adic cohomology in any case,

l being invertible in k (cf part 3 of *loc. cit.*). To a mixed Weil theory (or more generally a stable theory) is associated a commutative ring spectrum (cf *loc. cit.* 2.1.5) and a triangulated closed symmetric monoidal category $D_{\mathbb{A}^1}(S, \mathcal{E})$ – which is obtained by localization of a derived category. By construction (see *loc. cit.* (1.5.3.1)), we have a triangulated monoidal symmetric functor

$$S\mathscr{H}(S) \to D_{\mathbb{A}^1}(S, \mathcal{E}).$$

In *loc. cit.* 2.2.9, we associate a canonical element of the set (S_2) for this functor. Thus the resulting functor $\mathscr{D} \to D_{\mathbb{A}^1}(S, \mathcal{E})$ satisfies all the axioms of 2.1.

(2) Consider a noetherian scheme S and the model category of symmetric T-spectra Sp_S over S defined by R. Jardine in [Jar00]. It is a cofibrantly generated, symmetric monoidal model category which satisfies the monoid axiom of [SS00, 3.1] (cf [Jar00, 4.19] for this latter fact).

A commutative monoid $\mathbb{E}$ in the category Sp_S will be called a (homotopy) *coherent* ring spectrum. Given such a ring spectrum, according to [SS00, 4.1(2)], the category of $\mathbb{E}$-modules in the symmetric monoidal category Sp_S carries a structure of a cofibrantly generated, symmetric monoidal model category such that the pair of adjoint functors $(F, \mathcal{O})$ given by the free $\mathbb{E}$-module functor and the obvious forgetful functor is a Quillen adjunction. We denote by $S\mathscr{H}(S; \mathbb{E})$ the associated homotopy category and consider the left derived free $\mathbb{E}$-module functor

$$S\mathscr{H}(S) \to S\mathscr{H}(S; \mathbb{E}).$$

It is a triangulated symmetric monoidal functor. Then, as indicated in the previous remark, an element of (S_2) relative to this functor is equivalent to an orientation on the ring spectrum $\mathbb{E}$ in the classical sense (see [Vez01, 3.1]).

The basic example of such a ring spectrum is the cobordism ring spectrum **MGL**. Indeed, **MGL** has a structure of a coherent ring spectrum in our sense and is evidently oriented (see [PPR07, 1.2.3 and 2.1] for details). Thus the homotopy category $S\mathscr{H}(S; \mathbf{MGL})$ of **MGL**-modules satisfies the axioms 2.1.

Another example is given by the spectrum **BGL** introduced by Voevodsky in [Voe98, par. 6.2]. According to *loc. cit.*, th. 6.9, it represents the homotopy invariant algebraic K-theory defined by Weibel (cf [Wei89]). However, it is not at all clear to get a coherent structure on the ring spectrum **BGL** with the definition given in *loc. cit.* To obtain such a coherent ring structure on **BGL** we invoke a recent result of Gepner and Snaith which construct a coherent ring spectrum homotopy equivalent to **BGL** in [DV07, 5.9].

3. Chern classes

3.1. **The projective bundle theorem.** Let X be a smooth scheme and P be a projective bundle over X of rank n. We denote by $p : P \to X$ the canonical projection and by λ the canonical line bundle on P. Put $c = c_1(\lambda) : M(P) \to \mathbb{1}(1)[2]$. We can define a canonical map :

$$\epsilon_P := \sum_{0 \leq i \leq n} p_* \boxtimes c^i : M(P) \to \bigoplus_{0 \leq i \leq n} M(X)(i)[2i]$$

Consider moreover an open subscheme $U \subset X$, $P_U = P \times_X U$. We let $\pi : P/P_U \to X/U$ be the canonical projection and $\nu : P/P_U \to (P \times P)/(P \times P_U)$ the morphism induced by the diagonal embedding and the graph of the immersion $P_U \to P$. Using the product of motives with support (cf 2.8), we also define a canonical map :

$$\epsilon_{P/P_U} := \sum_{0 \leq i \leq n} \pi_* \boxtimes c^i : M(P/P_U) \to \bigoplus_{0 \leq i \leq n} M(X/U)(i)[2i]$$

Lemma 3.1. *Using the above notations, the following diagram is commutative :*

$$\begin{tikzcd}
M(P_U) \arrow[r] \arrow[d, "\epsilon_{P_U}"'] \arrow[dr, phantom, "(1)"] & M(P) \arrow[r] \arrow[d, "\epsilon_P"'] \arrow[dr, phantom, "(2)"] & M(P/P_U) \arrow[r] \arrow[d, "\epsilon_{P/P_U}"'] \arrow[dr, phantom, "(3)"] & M(P_U)[1] \arrow[d, "\epsilon_{P_U}"] \\
\bigoplus_i M(U)(i)[2i] \arrow[r] & \bigoplus_i M(X)(i)[2i] \arrow[r] & \bigoplus_i M(X/U)(i)[2i] \arrow[r] & \bigoplus_i M(U)(i)[2i+1]
\end{tikzcd}$$

where the top (resp. bottom) line is the distinguished triangle (resp. sum of distinguished triangles) obtained using (Loc) *(resp. and tensoring with* $\mathbb{1}(i)[2i]$*).*

Proof. Coming back to the definition of product and product with supports, squares (1) and (2) are commutative by functoriality of M. For square (3), besides this functoriality, we have to use axiom (Kun)(b). □

Theorem 3.2. *With the above hypothesis and notations, the morphism* $\epsilon_P : M(P) \to \bigoplus_{0 \leq i \leq n} M(X)(i)[2i]$ *is an isomorphism in* $\mathscr{T}$.

Proof. Consider an open cover $X = U \cup V$, $W = U \cap V$. Assume that ϵ_{P_U}, ϵ_{P_V} and ϵ_{P_W} are isomorphisms. Then according to the previous lemma, ϵ_{P_V/P_W} is an isomorphism. Using the compatibility of the first Chern class with pullback, we obtain a commutative diagram

$$\begin{tikzcd}
M(P_V/P_W) \arrow[r] \arrow[d, "\epsilon_{P_V/P_W}"'] & M(P/P_U) \arrow[d, "\epsilon_{P/P_U}"] \\
\bigoplus_i M(V/W)(i)[2i] \arrow[r] & \bigoplus_i M(X/U)(i)[2i]
\end{tikzcd}$$

where the horizontal maps are obtained by functoriality. According to axiom (Exc), these maps are isomorphisms which implies ϵ_{P/P_U} is an isomorphism. Applying ance again the previous lemma, we deduce that ϵ_P is an isomorphism.

This reasoning shows that we can argue locally on X and assume P is trivializable as a projective bundle over X. Then, as the map depends only on the isomorphism class of the projective bundle P, we can assume $P = \mathbb{P}^n_X$. Finally, by property (Kun)(a), $\epsilon_{\mathbb{P}^n_X} = M(X) \otimes \epsilon_{\mathbb{P}^n}$ and we can assume $X = S$. Put simply $\epsilon_n = \epsilon_{\mathbb{P}^n}$.

For $n = 0$, the statement is trivial. Assume $n > 0$. Recall we consider the scheme $\mathbb{P}^n$ pointed by the infinite point. The morphism ϵ_n induces a map $\tilde{M}(\mathbb{P}^n) \to \oplus_{0<i\leq n} \tilde{M}(\mathbb{P}^1)^{\otimes, i}$ still denoted by ϵ_n and we have to prove this later is an isomorphism. Put $c_{1,n} = c_1(\lambda_n)$ for any integer $n \geq 0$.

The canonical inclusion $\mathbb{P}^{n-1} \to \mathbb{P}^n - \{0\}$ is the zero section of a vector bundle. For any integer $i \in [1,n]$, we put $U_i = \{(x_1,...,x_n) \in \mathbb{A}^n \mid x_i \neq 0\}$ considered as an open subscheme of $\mathbb{A}^n$. We obtain the canonical isomorphism denoted by τ_n :

$$\begin{aligned} M(\mathbb{P}^n/\mathbb{P}^{n-1}) &\overset{(1)}{\simeq} M(\mathbb{P}^n/\mathbb{P}^n - \{0\}) \overset{(2)}{\simeq} M(\mathbb{A}^n/\mathbb{A}^n - \{0\}) \\ &= M(\mathbb{A}^n/\cup_i U_i) \overset{(3)}{=} M(\mathbb{A}^1/\mathbb{A}^1 - \{0\})^{\otimes,n} \overset{(4)}{\simeq} \tilde{M}(\mathbb{P}^1)^{\otimes,n} \end{aligned}$$

where (1) follows from (Htp) and (Loc), (2) from (Exc), (3) from (Kun)(a) and (4) from (Exc), (Htp) and (Loc).
Consider the following diagram

$$\begin{tikzcd} \tilde{M}(\mathbb{P}^{n-1}) \arrow[rr, "\iota_{n-1*}"] \arrow[d, "\epsilon_{n-1}"'] \arrow[drr, phantom, "(a)"] & & \tilde{M}(\mathbb{P}^n) \arrow[r, "\pi_n"] \arrow[d, "\epsilon_n"'] \arrow[dr, phantom, "(b)"] & M(\mathbb{P}^n/\mathbb{P}^{n-1}) \arrow[d, "\tau_n"] \\ \oplus_{0<i<n}\tilde{M}(\mathbb{P}^1)^{\otimes,i} \arrow[rr] & & \oplus_{0<i\leq n}\tilde{M}(\mathbb{P}^1)^{\otimes,i} \arrow[r] & \tilde{M}(\mathbb{P}^1)^{\otimes,n} \end{tikzcd} \tag{3.1}$$

where ι_{n-1} is the canonical inclusion, π_n is the obvious morphism obtained by functoriality in $\mathscr{D}$, and the bottom line is made up of the evident split distinguished triangle. We prove by induction on $n > 0$ the following statement :

(3.2)
- (i) ι_{n-1*} is a split monomorphism.
- (ii) $c^n_{1,n-1} = 0$ which means square (a) is commutative.
- (iii) $c^n_{1,n} = \tau_n \pi_n$ which means square (b) is commutative.
- (iv) ϵ_n is an isomorphism.

For $n = 1$, this is obvious as (iii) is a part of axiom (Orient).
The induction relies on the following lemma due to Morel.

Lemma 3.3. *Let $\delta_n : \mathbb{P}^n \to (\mathbb{P}^n)^n$ be the iterated n-th diagonal of $\mathbb{P}^n/S$ and denote by $\delta_{n*} : \tilde{M}(\mathbb{P}^n) \to \tilde{M}(\mathbb{P}^n)^{\otimes,n}$ the morphism induced by δ_n and axiom* (Kun)(a). *Let $\iota_{1,n} : \mathbb{P}^1 \to \mathbb{P}^n$ be the canonical inclusion.*
Then the following square commutes :

$$\begin{tikzcd} \tilde{M}(\mathbb{P}^n) \arrow[r, "\delta_{n*}"] \arrow[d, "\pi_n"'] & \tilde{M}(\mathbb{P}^n)^{\otimes,n} \\ M(\mathbb{P}^n/\mathbb{P}^{n-1}) \arrow[r, "\tau_n"] & \tilde{M}(\mathbb{P}^1)^{\otimes,n}. \arrow[u, "(\iota_{1,n*})^{\otimes,n}"'] \end{tikzcd}$$

Consider an integer $i \in [1,n]$ and let $\bar{U}_i$ be the open subscheme of $\mathbb{P}^n$ made of points $(x_1 : ... : x_n : x_{n+1})$ such that $x_i \neq 0$ and put $\Omega_i = \mathbb{P}^{i-1} \times U_i \times \mathbb{P}^{n-i}$.
We consider the following commutative diagram :

$$\begin{tikzcd} \tilde{M}(\mathbb{P}^n) \arrow[r, "(1)"] \arrow[d, "\pi_n"] & M\big((\mathbb{P}^n)^n/\cup_i \Omega_i\big) & \\ M(\mathbb{P}^n/\mathbb{P}^{n-1}) \arrow[r, "\sim"] & M\big(\mathbb{P}^n/\cup_i \bar{U}_i\big) \arrow[u] & M(\mathbb{A}^n/\cup_i U_i) \arrow[l, "\sim"'] \arrow[ul, "(2)"'] \end{tikzcd}$$

where the map (1) is induced by δ_n, the maps on the lower horizontal line are isomorphisms given respectively by the inclusions $\mathbb{P}^{n-1} \subset \cup_i \bar{U}_i$ and $U_i \subset \bar{U}_i$.
Consequently, the map (2) is induced by the restriction of δ_n. However, this map is

$\mathbb{A}^1$-homotopic to the product $\iota^{(1)} \times ... \times \iota^{(n)}$ where $\iota^{(i)} : \mathbb{A}^1 \to \mathbb{P}^n$ is the embedding defined by $\iota^{(i)}(x) = (x_1 : ... : x_{n+1})$ with $x_j = 0$ if $j \notin \{i, n+1\}$, $x_i = x$, $x_{n+1} = 1$. It follows from property (Htp) and (Kun)(a) that the map (2) is equal to the morphism

$$M(\mathbb{A}^1/\mathbb{A}^1 - \{0\})^{\otimes, n} \xrightarrow{\iota_*^{(1)} \otimes ... \otimes \iota_*^{(n)}} M(\mathbb{P}^n/\bar{U}_1) \otimes ... \otimes M(\mathbb{P}^n/\bar{U}_n).$$

Note finally the scheme $\bar{U}_i \simeq \mathbb{A}^n$ is contractible and, from property (Htp), the corresponding map $\iota_*^{(i)} : M(\mathbb{A}^1/\mathbb{A}^1 - \{0\}) \to \tilde{M}(\mathbb{P}^n)$ does not depend on the integer i. Thus the preceding commutative diagram together with the identifications just described allows to conclude.

With that lemma in hand, we conclude as follows. Suppose the property (3.2) is true for $n-1$.

The composite map $\left(\sum_{0<i<n} p_* \boxtimes c_n^i\right) \circ \iota_{n-1}$ is equal to ϵ_{n-1} as $c_n \circ \iota_{n-1*} = c_{n-1}$. This shows (3.2)(i). Then, the preceding lemma implies properties (ii) and (iii). Now, using (Loc) and (Sym), the upper horizontal line of diagram (3.1) is a split distinguished triangle which concludes. □

Using axiom (Stab), we obtain the following corollary :

Corollary 3.4. *Consider the hypothesis and notations of the previous theorem. Then $H^{**}(P)$ is a free $H^{**}(X)$-module with base $1, ..., c^n$.*
Let $\mathbb{E}$ be a motive.

(1) *The map*

$$\mathbb{E}^{**}(X) \otimes_{H^{**}(X)} H^{**}(P) \to \mathbb{E}^{**}(P), x \otimes \lambda \to \lambda.p^*(x)$$

is an isomorphism. If moreover $\mathbb{E}$ has a ringed motive structure, it is an isomorphism of $\mathbb{E}^{**}(X)$-algebra.

(2) *Considering the $H^{**}(X)$-module structure on $\mathbb{E}_{**}(X)$ (cf the end of 2.9), the map*

$$\mathbb{E}_{**}(P) \to \bigoplus_{0 \le i \le n} \mathbb{E}_{**}(X), \varphi \mapsto \sum_i c^i \cap p_*(\varphi)$$

is an isomorphism.

Remark 3.5. It can be seen actually that the first assertion of this corollary is equivalent to the fact $H^{n,m}(M(X)(r)) = H^{n,m-r}(M(X))$ which is a weak form of the stability axiom (Stab).

A corollary of the projective bundle theorem is the following result, classical in topology and first exploited in the homotopy category of schemes by Morel :

Corollary 3.6. *Consider the permutation isomorphism $\eta : \mathbb{1}(1) \otimes \mathbb{1}(1) \to \mathbb{1}(1) \otimes \mathbb{1}(1)$ in the symmetric monoidal category $\mathscr{T}$. Then $\eta = 1$.*
Let $\mathbb{E}$ be a ringed motive and X be a smooth scheme.
For any $x \in \mathbb{E}^{n,p}(X)$ and $y \in \mathbb{E}^{m,q}(X)$, $x \cup y = (-1)^{nm}.y \cup x$.

Proof. In general, for $x \in \mathbb{E}^{n,p}(X)$ and $y \in \mathbb{E}^{m,q}(X)$, we have $x \cup y = (-1)^{nm}\eta^{pq}.y \cup x$. In particular, when $X = \mathbb{P}^2$ and $c = c_1(\lambda_2)$, we get $c^2 = \eta.c^2$. This implies $\eta = 1$ from the previous corollary and the other assertion follows. □

3.2. **The associated formal group law.**

3.7. Put $H^{**}(\mathbb{P}^\infty) = \varprojlim_{n>0} H^{**}(\mathbb{P}^n)$. Then corollary 3.4 together with the relation (3.2)(ii) implies $H^{**}(\mathbb{P}^\infty) = A[[c]]$, free ring of power series over A with generator $c = (c_{1,n})_{n>0}$ of degree $(2,1)$. Moreover, $H^{**}(\mathbb{P}^\infty \times \mathbb{P}^\infty) = A[[x,y]]$.
Consider the Segre embeddings $\sigma_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{n+m+nm}$ for $(n,m) \in \mathbb{N}^2$ and the induced map on ind-schemes $\sigma : \mathbb{P}^\infty \times \mathbb{P}^\infty \to \mathbb{P}^\infty$. Then the map $\sigma^* : H^{**}(\mathbb{P}^\infty) \to H^{**}(\mathbb{P}^\infty \times \mathbb{P}^\infty)$ corresponds to a power series

$$F = \sum_{i,j} a_{ij}.x^i y^j \in A[[x,y]]$$

which according to the classical situation[13] in algebraic topology is a commutative formal group law :

$$F(x,0) = x, F(x,y) = F(y,x), F(x,F(y,z)) = F(F(x,y),z).$$

For any $(i,j) \in \mathbb{N}^2$, the element $a_{i,j} \in A$ is of homological degree $(2(i+j-1), i+j-1)$ and the first two relations above are equivalent to

$$a_{0,1} = 1, a_{0,i} = 0 \text{ if } i \neq 1, a_{i,j} = a_{j,i}.$$

Recall also there is a formal inverse associated to F, that is a formal power series $m \in A[[x]]$ such that $F(x, m(x)) = 0$. We can find the notation $x +_F y = F(x,y)$ in the litterature. For an integer $n \geq 0$, we put $[n]_F \cdot x = x +_F \ldots +_F x$, that is the power series in x equal to the formal n-th addition of x with itself. These notations will be fixed through the rest of the article.

Proposition 3.8. *Let X be a smooth scheme.*

(1) *For any line bundle L/X, the class $c_1(L)$ is nilpotent in $H^{**}(X)$.*
(2) *Suppose X admits an ample line bundle. For any line bundles L, L' over X,*

$$c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2)) \in H^{2,1}(X).$$

Proof. For the first point, we first remark the question is local in X. As X is noetherian, we are reduced by induction to consider an open covering $X = U \cup V$, such that $c_1(L_U)$ (resp. $c_1(L_V)$) is nilpotent in $H^{**}(U)$ (resp. $H^{**}(V)$) where L_U (resp. L_V) is the restrion of L to U (resp. V). Let n (resp. m) be the order of nilpotency of $c_1(L_U)$ (resp. $c_1(L_V)$). Let $Z = X - U$ (resp. $T = X - V$) and consider the canonical morphism $\nu_{X,W} : H^{**}_W(X) \to H^{**}(X)$ for $W = Z, T$. From axiom (Loc), there exists a class a (resp. b) in $H^{**}_Z(X)$ (resp. $H^{**}_T(X)$ such that $a = c_1(L)^n$ (resp. $b = c_1(L)^m$). As $Z \cap T = \emptyset$, axiom (Loc) implies $a \cup_{Z,T} b = 0$. Thus, relation (2.1) implies $c_1(L)^{n+m} = 0$ as wanted.
The first point follows, as λ is locally trivial and the Chern class of a trivial line bundle is 0 by definition.
For the second point, the assumption implies there is a torsor $\pi : X' \to X$ under a vector bundle over X such that X' is affine. From axioms (Htp') and (Exc), we

[13] Recall these properties follows from the fact that the coefficients a_{ij} for $i \leq n$, $j \leq m$ are determined by the map $\sigma_{n,m}$. The reader can find a more detailed proof in [LM07], proof of cor. 10.6.

obtain that $\pi_* : M(X') \to M(X)$ is an isomorphism. Thus we are reduced to the case where X is affine.
Then, the line bundle L is generated by its section (cf [EGA2, 5.1.2,e]), which means there is a closed immersion $L \xrightarrow{\iota} \mathbb{A}^{n+1}_X$ where $n+1$ is the cardinal of a generating family. In particular, we get a morphism

$$f : X \simeq \mathbb{P}(L) \xrightarrow{\iota} \mathbb{P}^n_X \to \mathbb{P}^n$$

with the property that $f^{-1}(\lambda_n) = L$. In the same way, we can find a morphism $g : X \to \mathbb{P}^m$ such that $g^{-1}(\lambda_m) = L'$. We consider the morphism

$$\varphi : X \to X \times X \xrightarrow{f \times g} \mathbb{P}^n \times \mathbb{P}^m \xrightarrow{\sigma_{n,m}} \mathbb{P}^{nm+n+m}.$$

By construction, $\varphi^{-1}(\lambda_{nm+n+m}) = L \otimes L'$ and this concludes, computing in two ways the Chern class of this line bundle. □

Consider a ringed motive $\mathbb{E}$ with regulator map $\varphi : H^{**} \to \mathbb{E}^{**}$.
The map $\sigma^* : \mathbb{E}^{**}(\mathbb{P}^\infty) \to \mathbb{E}^{**}(\mathbb{P}^\infty \times \mathbb{P}^\infty)$ defines a formal group law $F_\mathbb{E}$ with coefficients in $\mathbb{E}^{**}$ and $F_\mathbb{E} = \sum_{i,j} \varphi_S(a_{i,j}).x^i y^j$. Thus the regulator map induces a morphism of formal group law $(A, F) \to (\mathbb{E}^{**}, F_\mathbb{E})$.

Remark 3.9. In case F is the additive formal group law, $F(x, y) = x + y$, for any ringed motive $\mathbb{E}$, $F_\mathbb{E}$ is the additive formal group law. This is the case for example if $\mathscr{T} = DM_{gm}(S)$ or $\mathscr{T}$ is the category of modules over a mixed Weil theory.
When F is the universal multiplicative formal group law $F = x + y + \beta.xy$, the obstruction for $F_\mathbb{E}$ to be additive is the element $\varphi(\beta)$.

3.3. **Higher Chern classes.** We now follow the classical approach of Grothendieck to define higher Chern classes. Consider a vector bundle E of rank $n > 0$ over a smooth scheme X. Let λ (resp. p) be the canonical invertible sheaf (resp. projection) of the projective bundle $\mathbb{P}(E)/X$. From corollary 3.4, there are unique classes $c_i(E) \in H^{2i,i}(X)$ for $i = 0, ..., n$, such that

$$\sum_{i=0}^{n} p^*(c_i(E)) \cup \big(- c_1(\lambda)\big)^{n-i} = 0 \tag{3.3}$$

and $c_0(E) = 1$.

Definition 3.10. With the above notations, we call $c_i(E)$ the i-th Chern class of E. We also put $c_i(E) = 0$ for any integer $i > n$.

Remark 3.11. In the case $n = 1$, due to our choice of conventions, $\lambda = p^{-1}(E)$. The previous relation is not a definition, but a tautology. This enlighten particularly our choice of sign in the previous relation. Besides, when $c_1(\lambda^\vee) = -c_1(\lambda)$ (in particular when the formal group law F is additive), relation (3.3) agrees precisely with that of [Gro58].

Remark 3.12. Considering any ringed motive $\mathbb{E}$, with regulator map $\varphi : H \to \mathbb{E}$, $\varphi \circ c_i$ defines Chern classes for cohomology with coefficients in $\mathbb{E}$. When no ringed structure is given on $\mathbb{E}$, we still get an action of the former Chern classes on the $\mathbb{E}$-cohomology using the action of the cohomology theory H (cf 2.9).

The Chern classes are obviously functorial with respect to pullback and invariant under isomorphism of vector bundles. They also satisfy the Whitney sum formula ; we recall the proof to the reader as it uses the axiom (Kun)(a) in an essential way.

Lemma 3.13. *Let X be a smooth scheme and consider an exact sequence of vector bundles over X :*

$$0 \to E' \to E \to E'' \to 0$$

Then for any $k \in \mathbb{N}$, $c_k(E) = \sum_{i+j=k} c_i(E') \cup c_j(F'')$.

Proof. By compatibility of Chern classes with pullback we can assume the sequence above is split. Let n (resp. m) be the rank of E'/X (resp. E''/X). Put $P = \mathbb{P}(E)$ and consider $c \in H^{2,1}(P)$ (resp. $p : P \to X$) the first Chern class of the canonical line bundle on (resp. canonical projection of) P/X.
Put $a = \sum_{i=0}^{n} p^*(c_i(E')).c^{n-i}$ and $b = \sum_{j=0}^{m} p^*(c_j(E'')).c^{m-j}$ as cohomology classes in $H^{**}(P)$. We have to prove $a \cup b = 0$.
Consider the canonical embeddings $i : \mathbb{P}(E') \to P$ and $j : P - \mathbb{P}(E'') \to P$. Then $i^*(a) = 0$ which implies by property (Htp') that $j^*(a) = 0$. Thus there exists $a' \in H^{*,*}_{\mathbb{P}(E'')}(P)$ such that $a = \nu_F(a')$ where $\nu_F : H^{*,*}_{\mathbb{P}(E'')}(P) \to H^{**}(P)$ is the canonical morphism. Similarly, there exists $b' \in H^{*,*}_{\mathbb{P}(E')}(P)$ such that $b = \nu_E(b')$ where $\nu_E : H^{*,*}_{\mathbb{P}(E')}(P) \to H^{**}(P)$ is the canonical morphism. Then, relation (2.1) allows[14] to conclude because $\mathbb{P}(E') \cap \mathbb{P}(E'') = \emptyset$ in P and $H^{*,*}_{\emptyset}(P) = 0$ from property (Loc). □

Remark 3.14. Suppose X admits an ample line bundle and consider a vector bundle E/X. As a corollary of the first point of proposition 3.8 and the usual splitting principle, we obtain that the class $c_n(E)$ is nilpotent in $H^{**}(X)$ for any integer $n \geq 0$.

4. The Gysin triangle

In this section, we consider closed pairs (X, Z) – recall X is assumed to be smooth and Z is a closed subscheme of X. We say (X, Z) is *smooth* (resp. of *codimension* n) if Z is smooth (resp. has everywhere codimension n in X). A *morphism* of closed pair $(f, g) : (Y, T) \to (X, Z)$ is a commutative square

$$\begin{tikzcd} T \arrow[r] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ Z \arrow[r] & X \end{tikzcd}$$

which is cartesian on the underlying topological space. This means the canonical embedding $T \to Z \times_X Y$ is a thickening. We say the morphism is *cartesian* if the square is cartesian.
The premotive $M_Z(X)$ is functorial with respect to morphisms of closed pairs.

[14] This is where axiom (Kun)(a) is used.

4.1. **Purity isomorphism.** Consider a projective bundle over a smooth scheme X of rank n. For any integer $0 \leq r \leq n$, we will consider the embedding[15]

$$\mathfrak{l}_r(P) : M(X)(r)[2r] \xrightarrow{(-1)^r} \bigoplus_{0\leq i\leq n} M(X)(i)[2i] \xrightarrow{\epsilon_{P/X}^{-1}} M(P).$$

where the first map is the canonical embedding time $(-1)^r$ and the second one is induced by the isomorphism of theorem 3.2.

4.1. Consider a smooth closed pair (X, Z). Let $N_Z X$ (resp. $B_Z X$) be the normal bundle (resp. blow-up) of (X, Z) and $P_Z X$ be the projective completion of $N_Z X$. We denote by $B_Z(\mathbb{A}^1_X)$ the blow-up of $\mathbb{A}^1_X$ with center $\{0\} \times Z$. It contains as a closed subscheme the trivial blow-up $\mathbb{A}^1_Z = B_Z(\mathbb{A}^1_Z)$. We consider the closed pair $(B_Z(\mathbb{A}^1_X), \mathbb{A}^1_Z)$ over $\mathbb{A}^1$. Its fiber over 1 is the closed pair (X, Z) and its fiber over 0 is $(B_Z X \cup P_Z X, Z)$. Thus we can consider the following deformation diagram :

$$(X, Z) \xrightarrow{\bar{\sigma}_1} (B_Z(\mathbb{A}^1_X), \mathbb{A}^1_Z) \xleftarrow{\bar{\sigma}_0} (P_Z X, Z). \tag{4.1}$$

We will also consider the open subscheme $D_Z X = B_Z(\mathbb{A}^1_X) - B_Z X$, which still contains $\mathbb{A}^1_Z$ as a closed subscheme. The previous diagram then gives by restriction a second deformation diagram :

$$(X, Z) \xrightarrow{\sigma_1} (D_Z X, \mathbb{A}^1_Z) \xleftarrow{\sigma_0} (N_Z X, Z). \tag{4.2}$$

Note these two deformation diagrams are functorial in (X, Z) with respect to cartesian morphisms of closed pairs.

Remark 4.2. As we will see in the followings, one of the advantage to consider the deformation space $D_Z X$ is that, when X is a vector bundle over Z and the embedding $Z \subset X$ is the 0-section, we can define a canonical isomorphism $D_Z X \simeq \mathbb{A}^1 \times X$. In fact, when $X = \operatorname{Spec}(A)$ and $Z = \operatorname{Spec}(A/I)$, $D_Z X = \operatorname{Spec}(\oplus_{n\in\mathbb{Z}} I^n.t^{-n})$ with the convention that for $n < 0$, $I^n = A$ (t is an indeterminate). Thus, if $A = A_0[x_1, ..., x_n]$, $I = (x_1, ..., x_n)$, we get an isomorphism defined on the affine level by

$$A[t', x'_1, ..x'_n] \to \oplus_{n\in\mathbb{Z}} I^n.t^{-n}, t' \mapsto t, x'_i \mapsto t^{-1}x_i.$$

This isomorphism is independant on the regular sequence parametrizing I. Thus, in the case when X is an arbitrary vector bundle, we can glue the isomorphisms obtained by choosing local parametrizations.

Proposition 4.3. *Let n be a natural integer.*
There exists a unique family of isomorphisms of the form

$$\mathfrak{p}_{(X,Z)} : M_Z(X) \to M(Z)(n)[2n]$$

indexed by smooth closed pairs of codimension n such that :

[15] The change of sign which appears in this formula amounts to take $-c$ instead of c as a generator of the algebra $H^{**}(P)$.

(1) *for every cartesian morphism $(f,g) : (Y,T) \to (X,Z)$ of smooth closed pairs of codimension n, the following diagram is commutative :*

$$\begin{tikzcd} M_T(Y) \arrow[r, "(f,g)_*"] \arrow[d, "\mathfrak{p}_{(Y,T)}"'] & M_Z(X) \arrow[d, "\mathfrak{p}_{(X,Z)}"] \\ M(T)(n)[2n] \arrow[r, "g_*(n)[2n]"] & M(Z)(n)[2n]. \end{tikzcd}$$

(2) *Let X be a smooth scheme, E be a vector bundle over X of rank n. Put $P = \mathbb{P}(E \oplus 1)$. Consider the closed pair (P,X) corresponding to the canonical section of P/X. Then $\mathfrak{p}_{(P,X)}$ is the inverse of the following composition*

$$M(X)(n)[2n] \xrightarrow{\mathfrak{l}_n(P)} M(P) \xrightarrow{\pi} M_X(P)$$

where the second arrow is obtained by functoriality in $\mathscr{D}$.

Proof. Uniqueness : Consider a smooth closed pair (X,Z) of codimension n. Applying property (1) above to the deformation diagram (4.1), we obtain the following commutative diagram :

$$\begin{tikzcd} M_Z(X) \arrow[r, "\bar\sigma_{1*}"] \arrow[d, "\mathfrak{p}_{(X,Z)}"'] & M_{\mathbb{A}^1_Z}(B_Z(\mathbb{A}^1_X)) \arrow[d, "\mathfrak{p}_{(B_Z(\mathbb{A}^1_X),\mathbb{A}^1_Z)}"] & M_Z(P_ZX) \arrow[l, "\bar\sigma_{0*}"'] \arrow[d, "\mathfrak{p}_{(P_ZX,Z)}"] \\ M(Z)(n)[2n] \arrow[r, "s_{1*}(n)[2n]"] & M(\mathbb{A}^1_Z)(n)[2n] & M(Z)(n)[2n] \arrow[l, "s_{0*}(n)[2n]"'] \end{tikzcd}$$

The morphisms $s_0, s_1 : Z \to \mathbb{A}^1_Z$ are respectively the zero section and the unit section of $\mathbb{A}^1_Z/Z$. Using axiom (Htp), $s_{0*} = s_{1*}$. Thus in the above diagram, all morphisms are isomorphisms. Now, property (2) stated previously determines uniquely $\mathfrak{p}_{(P_ZX,Z)}$, thus $\mathfrak{p}_{(X,Z)}$ is also uniquely determined.
Existence : Consider property (2). Let $i : \mathbb{P}(E) \to P$ be the canonical embedding. Its corestriction $i' : \mathbb{P}(E) \to P - X$ is the zero section of a vector bundle, thus it induces an isomorphism on premotives from property (Htp'). By (Loc), we then obtain the distinguished triangle :

$$M(\mathbb{P}(E)) \xrightarrow{i_*} M(P) \xrightarrow{\pi} M_X(P) \xrightarrow{+1}$$

We easily obtain $\mathfrak{l}_r(\mathbb{P}(E)) \circ i_* = \mathfrak{l}_r(P)$ for any integer $r < n$. Thus the composite $\mathfrak{l}_n(P) \circ \pi$ is an isomorphism as required. We put: $\mathfrak{p}_{(P,X)} = \big(\mathfrak{l}_n(P) \circ \pi\big)^{-1}$.
Considering the proof of uniqueness, we have to show that $\bar\sigma_{0*}$ and $\bar\sigma_{1*}$ are isomorphisms. Considering the excision axiom (Exc), this is equivalent to prove the morphisms

$$M_Z(X) \xrightarrow{\sigma_{1*}} M_{\mathbb{A}^1_Z}(D_Z(X)) \xleftarrow{\sigma_{0*}} M_Z(N_ZX)$$

induced by diagram (4.2) are isomorphisms. In the case $X = \mathbb{A}^n_Z$ and the inclusion $Z \subset X$ is the 0-section, the result follows from remark 4.2 and axiom (Htp).
We can argue locally for the Zariski topology on X. In fact, consider an open cover $X = U \cup V$, $W = U \cap V$, such that the case of $(U, Z \cap U)$, $(V, Z \cap V)$ and $(W, Z \cap W)$ are known. Using axiom (Sym), (Exc) and (Loc), the canonical map

$$M\left(\frac{V/V - Z \cap V}{W/W - Z \cap W}\right) \to M\left(\frac{X/X - Z}{U/U - Z \cap U}\right)$$

is an isomorphism, and the same is true when we replace (X, Z) by $(D_Z X, \mathbb{A}^1_Z)$. This fact, together with the above three assumptions and axiom (Loc), allows to obtain the result for $(D_Z X, \mathbb{A}^1_Z)$.
Thus we can assume there exists a parametrisation of the closed pair (X, Z), that is to say a cartesian morphism $(f, g) : (X, Z) \to (\mathbb{A}^{d+n}_S, \mathbb{A}^d_S)$ such that f is étale. Consider the pullback square

$$\begin{tikzcd} X' \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[d, "f"] \\ \mathbb{A}^n_Z \arrow[r, "1\times g"] & \mathbb{A}^{n+d}_S. \end{tikzcd}$$

There is an obvious closed immersion $Z \to X'$ and its image is contained in $q^{-1}(Z)$. As q is étale, Z is a direct factor of $q^{-1}(Z)$. Put $W = q^{-1}(Z) - Z$ and $\Omega = X' - W$. Thus Ω is an open subscheme of X', and the reader can check that p and q induce cartesian étale morphisms

$$(X, Z) \leftarrow (\Omega, Z) \to (\mathbb{A}^n_Z, Z).$$

The functorialty of (4.2) and axiom (Exc) allow to conclude in view of the previous case.
To sum up, the purity isomorphism $\mathfrak{p}_{(X,Z)}$ is defined as the composite

$$M_Z X \xrightarrow{\bar\sigma_{0*}} M_{\mathbb{A}^1_Z}\big(B_Z(\mathbb{A}^1_X)\big) \xrightarrow{\bar\sigma_{1*}^{-1}} M_Z(P_Z X) \xrightarrow{\mathfrak{p}_{(Z, P_Z X)}} M(Z)(n)[2n].$$

We finally have to check the coherence of this definition in the case of the closed pair (P, X), $P = \mathbb{P}(E \oplus 1)$, appearing in property (2). Explicitely, we have to check that in this case $\bar\sigma_{1*}^{-1} \circ \bar\sigma_{0*} = 1$. This is easily seen considering the commutative diagram :

$$\begin{tikzcd} M_X(P) \arrow[r, "\bar\sigma_{1*}"] & M_{\mathbb{A}^1_X}(B_Z(\mathbb{A}^1_X)) & M_X(P) \arrow[l, "\bar\sigma_{0*}"'] \\ M_X(E) \arrow[u] \arrow[r, "\sigma_{1*}"] & M_{\mathbb{A}^1_X}(D_X E) \arrow[u] & M_X(E). \arrow[u] \arrow[l, "\sigma_{0*}"'] \end{tikzcd}$$

We have identified the projective normal bundle of (P, X) (resp. the normal bundle of (E, X)) with P (resp. E). According to remark 4.2, there is a canonical isomorphism $D_X E \simeq \mathbb{A}^1 \times E$ through which σ_0 (resp. σ_1) corresponds to the zero (resp. unit) section. The homotopy axiom (Htp) allows to conclude. □

4.4. Let X be a smooth scheme, E be a vector bundle over X of rank n and put $P = \mathbb{P}(E \oplus 1)$. Let λ be the canonical line bundle on P, and $p : P \to X$ be the canonical projection. We define the *Thom class* of E/X as the cohomology class

$$t(E) = \sum_{i=0}^{n} p^*(c_i(E)) \cup (-c_1(\lambda))^{n-i}$$

in $H^{2n}(P)$. This is in fact a morphism $M(P) \to \mathbb{1}(n)[2n]$ whose restriction to $M(\mathbb{P}(E))$ is zero. This implies the morphism

$$p_* \boxtimes t(E) : M(P) \to M(X)(n)[2n]$$

factors as a morphism $M_X(P) \to M(X)(n)[2n]$ and this latter is equal to $\mathfrak{p}_{(P,X)}$. Indeed, $p_* \boxtimes t(E)$ is a split epimorphism with splitting $\mathfrak{l}_n(P)$.

We introduce the *Thom premotive*[16] as $M\mathrm{Th}(E) := M_X(E)$ - remark it is functorial with respect to monomorphisms of vector bundles. Using property (Exc), the natural morphism $M\mathrm{Th}(E) \to M_X(P)$ is an isomorphism. As a consequence, the morphism $p_* \boxtimes t(E)$ induces an isomorphism $M\mathrm{Th}(E) : M\mathrm{Th}(E) \to M(X)(n)[2n]$ which is precisely the purity isomorphism $\mathfrak{p}_{(E,X)}$. In the litterature, this arrow is called the *Thom isomorphism*.

Remark 4.5. Recall the universal quotient bundle ξ on P is defined by the exact sequence
$$0 \to \lambda \to p^{-1}(E \oplus 1) \to \xi \to 0.$$
Thus the Whitney sum formula 3.13 gives: $t(E) = c_n(\xi)$.

Definition 4.6. Let (X, Z) be a smooth closed pair of codimension n. Put $U = X - Z$ and consider the obvious immersions $i : Z \to X$ and $j : U \to X$. Considering the notations of the previous proposition, we call $\mathfrak{p}_{(X,Z)}$ the *purity isomorphism* associated with (X, Z). Using this isomorphism together with property (Loc) we obtain a distinguished triangle
$$M(X-Z) \xrightarrow{j_*} M(X) \xrightarrow{i^*} M(Z)(n)[2n] \xrightarrow{\partial_{X,Z}} M(X-Z)[1]$$
called the *Gysin triangle*. The morphism i^* (resp. $\partial_{X,Z}$) is called the *Gysin morphism* (resp. *residue morphism*) associated with (X, Z).

Example 4.7. Let X be a smooth scheme and E/X be a vector bundle of rank n. Put $P = \mathbb{P}(E \oplus 1)$ and consider the canonical section $s : X \to P$ of P/X. Then property (2) of proposition 4.3 implies $s^* \circ \mathfrak{l}_n(P) = 1 :$ the Gysin triangle of (P, X) is split and $\partial_{P,X} = 0$. Moreover, remark 4.4 and the previous definition implies that
$$s^* = p_* \boxtimes t(E).$$

4.2. Base change.

Definition 4.8. Let (X, Z) (resp. (Y, T)) be a smooth closed pair of codimension n (resp. m). Let $(f, g) : (Y, T) \to (X, Z)$ be a morphism of closed pairs. We define the morphism $(f, g)_! : M(T)(m)[2m] \to M(Z)(n)[2n]$ by the equality $(f, g)_! := \mathfrak{p}_{(X,Z)} \circ (f, g)_* \circ \mathfrak{p}^{-1}_{(Y,T)}$.

Thus we obtain a commutative diagram

$$\begin{tikzcd}
M(Y-T) \arrow[r, "l_*"] \arrow[d, "h_*"'] & M(Y) \arrow[r, "k^*"] \arrow[d, "f_*"'] & M(T)(n)[2n] \arrow[r, "\partial_{Y,T}"] \arrow[d, "(f,g)_!"] & M(Y-T)[1] \arrow[d, "h_*[1]"] \\
M(X-Z) \arrow[r, "j_*"] & M(X) \arrow[r, "i^*"] & M(Z)(n)[2n] \arrow[r, "\partial_{X,Z}"] & M(X-Z)[1]
\end{tikzcd} \tag{4.3}$$

where i, j, k, l are the obvious immersions and h is the restriction of f.
In what follows, we will compute the morphism $(f, g)_!$ in various cases. The commutativity of the second square will give us *refined projection formulas*. The new thing in our study is that any such formula corresponds to another formula involving residue morphisms as we see by considering the third commutative square.

[16]Analog of the Thom space in algebraic topology.

Remark 4.9. The notation $(f,g)_!$ is to be compared with the notation of [Ful98] for the "refined Gysin morphism". In fact, the reader will notice that in the case of motivic cohomology, our formulas extend the formulas of Fulton to the case of arbitrary weights (and arbitrary base). Be careful however that our Gysin morphism $i^* : M(X) \to M(Z)(n)[2n]$ corresponds to the usual pushout on Chow groups (cf [Dég08][1.21]). The Gysin morphism considered by Fulton is induced by the usual functoriality of motives. This fact can be understand if we thought of Chow groups over a field studied by Fulton as motivic homology with compact support.

4.2.1. *The transversal case.*

Proposition 4.10. *Consider the hypothesis of definition 4.8.*
Suppose (f,g) is cartesian and $n = m$. Then $(f,g)_! = g_*(n)[2n]$.

Proof. Diagram (4.1) is functorial with respect to cartesian morphism. Let $p : P_TY \to P_ZX$ be the morphism induced by (f,g) on the projective completions of the normal bundles. Through the morphisms $\bar\sigma_{0*}$ and $\bar\sigma_{1*}$ for the closed pairs (X,Z) and (Y,T), the morphism $(f,g)_*$ is isomorphic to

$$(p,g)_* : M(P_TY,T) \to M(P_ZX,Z).$$

As $n = m$ and $Y = X \times_Z T$, one has $P_TY = P_ZX \times_Z T$. Using the compatibility of the projective bundle isomorphism with base change, we see that the following diagram commutes

$$\begin{tikzcd} M(T)(n)[2n] \arrow[r, "\mathfrak{l}_n(P_TY)"] \arrow[d, "g_*(n)[2n]"'] & M(P_TY) \arrow[d, "p_*"] \\ M(Z)(n)[2n] \arrow[r, "\mathfrak{l}_n(P_ZX)"] & M(P_ZX) \end{tikzcd}$$

which concludes in view of the property (2) in proposition 4.3. □

Corollary 4.11. *Consider a smooth closed pair (X,Z) of codimension n and $i : Z \to X$ the corresponding immersion. Put $U = X - Z$.*
Then $(1_{Z*} \boxtimes i_*) \circ i^* = i^* \boxtimes 1_{X*}$ as a morphism $M(X) \to M(Z \times X)(n)[2n]$,
and $(j_* \boxtimes 1_{U*}) \circ \partial_{X,Z} = \partial_{X,Z} \boxtimes i_*$ as a morphism $M(Z)(n)[2n] \to M(U \times X)[1]$.

Proof. We consider the cartesian square

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "\gamma_i"'] & X \arrow[d, "\delta_X"] \\ Z \times X \arrow[r, "i \times 1_X"] & X \times X \end{tikzcd}$$

where δ_X is the diagonal embedding of X/S. The two formulas then follow from the previous proposition applied to the morphism of closed pairs $(\delta_X, \gamma_i) : (X,Z) \to (X \times X, Z \times X)$ with the help of the following elementary lemma :

Lemma 4.12. *Let (X,Z) be a smooth closed pair of codimension n and Y be a smooth scheme.*
Then $(i \times 1_Y)^* = i^* \otimes 1_{Y*}$ and $\partial_{X \times Y, Z \times Y} = \partial_{X,Z} \otimes 1_{Y*}$.

Using axiom (Kun)(a) and (Kun)(b), the lemma is reduced to prove that $\mathfrak{p}_{(X\times Y,Z\times Y)} = \mathfrak{p}_{(X,Z)} \otimes Y$. From the construction of the purity isomorphism, we are reduced to show that for a projective bundle P/X, $\epsilon_{P\times Y} = \epsilon_P \otimes 1_{X*}$ using the notations of theorem 3.2. This last equality follows finally from axiom (Kun)(a) and the functoriality of the first Chern class in axiom (Orient). □

Remark 4.13. (1) In the formula of this lemma, there is hidden a permutation isomorphism for the tensor product. In this paper, we will not need to care about this isomorphism. However, in some cases, it may result in a change of sign (see [Dég05], rem. 2.6.2).
(2) Considering a ringed premotive $\mathbb{E}$, the previous corollary gives the usual projection formula for i : for any $z \in \mathbb{E}^{**}(Z)$ and any $x \in \mathbb{E}^{**}(X)$, $i_*(z \cup i^*(x)) = i_*(z) \cup x$.

4.14. Let (X,Z) be a smooth closed pair of codimension n, $i : Z \to X$ the corresponding closed immersion. Following Grothendieck (see [Gro58]), we define the *fundamental class* of Z in X as the cohomology class $\eta_X(Z) = i_*(1)$ in $H^{2n,n}(X)$. As a morphism, it is equal to the composite

$$M(X) \xrightarrow{i^*} M(Z)(n)[2n] \xrightarrow{\pi_{Z*}} \mathbb{1}(n)[2n]$$

where $\pi_Z : Z \to S$ is the structural morphism of Z/S.
Suppose that i admits a retraction $p : X \to Z$. Then corollary 3.10 gives the following computation[17] of the Gysin morphism :

$$i^* = p_* \boxtimes \eta_X(Z). \tag{4.4}$$

Suppose given a vector bundle E/X and put $P = \mathbb{P}(E \oplus 1)$. Applying example 4.7, we get

$$\eta_P(X) = t(E)$$

where X is embedded in P through the canonical section. Indeed example 4.7 is a particular case of the formula (4.4).
More generally, we can define the *localised fundamental class* of Z in X as the cohomology class $\bar{\eta}_X(Z) \in H^{2n,n}_Z(X)$ equal to the composite

$$M_Z(X) \xrightarrow{\mathfrak{p}_{(X,Z)}} M(Z)(n)[2n] \xrightarrow{\pi_{Z*}} \mathbb{1}(n)[2n].$$

Considering the canonical morphism $\nu_{X,Z} : H^{2n,n}_Z(X) \to H^{2n,n}(X)$, we have tautologically $\nu_{X,Z}(\bar{\eta}_X(Z)) = \eta_X(Z)$.
For any vector bundle E/X of rank n, $P = \mathbb{P}(E \oplus 1)$, the *localised Thom class* $\bar{t}(E) = \bar{\eta}_P(X)$ is uniquely determined by the Thom class $t(E)$. Usually, $\bar{t}(E)$ is considered as an element of $H^{2n,n}_X(E)$ using axiom (Exc).

As a last application of the previous corollary, let us remark the following :

Corollary 4.15. *Let (X,Z) be a smooth closed pair of codimension m, and P be a projective bundle of rank n over X.*

[17] Considered in cohomology, this is a well known formula.

Then for any integer $r \in [0,n]$, the following diagram is commutative :

$$\begin{tikzcd}
M(P_V) \arrow[r, "\nu_*"] \arrow[d, "p_{V*} \boxtimes c_1(\lambda_V)^r"'] & M(P) \arrow[r, "\iota^*"] \arrow[d, "p_* \boxtimes c_1(\lambda)^r"'] & M(P_Z)(m)[2m] \arrow[r, "\partial_\iota"] \arrow[d, "p_{Z*} \boxtimes c_1(\lambda_Z)^r"'] & M(P_V)[1] \arrow[d, "p_{V*}[1] \boxtimes c_1(\lambda_V)^r"'] \\
M(V)(r)[2r] \arrow[r, "j_*"] & M(Y)(r)[2r] \arrow[r, "i^*"] & M(Z)(r+m)[2(r+m)] \arrow[r, "\partial_i"] & M(V)(r)[2r+1].
\end{tikzcd}$$

In particular, the Gysin triangle is compatible with the projective bundle isomorphisms and with the induced embeddings $\mathfrak{l}_r(P_?)$.

4.2.2. *The excess intersection case.* Remark that in the hypothesis of definition 4.8, we have a canonical closed immersion

$$N_T Y \xrightarrow{\nu} g^*(N_Z X).$$

In particular, we have necessarily the inequality $n \geq m$.

Proposition 4.16. *Consider the hypothesis of definition 4.8. Suppose (f,g) is cartesian.*
Put $e = n - m$ and consider $\xi = g^{-1}(N_Z X)/N_T Y$, quotient vector bundle over T.
Then $(f,g)_! = (g_* \boxtimes_T c_e(\xi))\,(m)[2m]$.

Remark 4.17. The integer e is usually called the *excess of intersection*, and ξ the *excess intersection bundle*.

Proof. The morphism (f,g) induces the following composite morphism on normal bundles :

$$N_T Y \xrightarrow{\nu} g^{-1}(N_Z X) \xrightarrow{g'} N_Z X.$$

Thus, considering now the functoriality of diagram (4.2) with respect to the cartesian morphism (f,g), we obtain $(f,g)_! = (\nu, 1_T)_!(g',g)_!$. From proposition 4.10, $(g',g)_! = g_*(n)[2n]$. We conclude using the following lemma :

Lemma 4.18. *Let E and F be vector bundles over a smooth scheme T of respective rank n and m. Consider a monomorphism $\nu : F \to E$ of vector bundles and put $e = n - m$.*
Then $(\nu, 1_T)_! = \big(1_{T*} \boxtimes c_e(E/F)\big)(m)[2m]$.

To prove the lemma, we use the description of $\mathfrak{p}_{(F,T)}$ and $\mathfrak{p}_{(E,T)}$ using the Thom class (cf 4.4). Let P, Q and $\bar{\nu} : Q \to P$ be the respective projective completions of E, F and ν. Let $p : P \to T$ and $q : Q \to T$ be the canonical projections. We are reduced to prove the relation $\bar{\nu}^*(t(E)) = (q^* c_e(E/F)) \cup t(F)$ in $H^{2n,n}(Q)$.
From remark 4.5, we get $t(E) = c_n(\xi_P)$ (resp. $t(F) = c_m(\xi_Q)$) where ξ_P (resp. ξ_Q) is the universal quotient bundle on P (resp. Q). Thus, the relation follows from the Whitney sum formula 3.13 and the following exact sequence of vector bundles over Q :

$$0 \to \xi_Q \to \bar{\nu}^{-1}\xi_P \to q^{-1}(E/F) \to 0.$$

□

Corollary 4.19. *Let (X, Z) be a smooth closed pair of codimension n. Then :*
(1) $i^* i_* = 1_Z \boxtimes c_n(N_Z X)$ as a morphism $M(Z) \to M(Z)(n)[2n]$.
(2) $\partial_{X,Z} \circ (1_Z \boxtimes c_n(N_Z X)) = 0$.

This follows from the previous proposition applied with $(f, g) = (i, 1_Z)$. We usually refer to the first formula as the *self-intersection formula*.

4.20. Consider a vector bundle E over a smooth scheme X of rank n. Let $E^\times$ be the complement of the zero section in E and $\pi : E^\times \to X$ be the obvious projection. Then using property (Htp') and the previous corollary, we obtain from the Gysin triangle for (E, X) the following distinguished triangle

$$M(E^\times) \xrightarrow{\pi_*} M(X) \xrightarrow{1_X \boxtimes c_n(E)} M(X)(n)[2n] \xrightarrow{\partial_{E,X}} M(E^\times)[1]$$

which we shall call the *Euler distinguished triangle*. Indeed, in cohomology with coefficients in a ringed premotive $\mathbb{E}$, it corresponds to a long exact sequence where one of the arrow is the cup product by $c_n(E)$.
As a corollary of the self-intersection formula 4.19, we obtain the following tool to compute fundamental classes which generalises in our setting a theorem of Grothendieck (cf [Gro58, th. 2]).

Corollary 4.21. *Consider a smooth closed pair (X, Z) of codimension n. Let i be the corresponding closed immersion and $\eta_X(Z) = i_*(1) \in H^{2n,n}(X)$ be the fundamental class of Z in X (cf 4.14).*
Suppose there exists a vector bundle E on X and a section s of E/X such that s is transversal to the zero section s_0 of E and $Z = s^{-1}(s_0(X))$.
Then, $\eta_X(Z) = c_n(E)$.

It simply follows from corollary 4.19 applied to s_0 together with proposition 4.10 applied to the following transversal square :

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ \downarrow & & \downarrow{\scriptstyle s_0} \\ X & \xrightarrow{s} & E. \end{array}$$

Example 4.22. Let E be a vector bundle of rank n over a smooth scheme X. Put $P = \mathbb{P}(E \oplus 1)$ and consider $p : P \to X$ (resp. $s : X \to P$, λ) the canonical projection (resp. section, line bundle) of P/X. Consider finally the vector bundle $F = \lambda^\vee \otimes p^{-1}(E)$ over P. The sequence of morphisms of vector bundles over P,

$$\lambda \to p^{-1}(E \oplus 1) \to p^{-1}(E)$$

gives a section σ of F/P. We check easily it is transversal to the zero section and we have $\sigma^{-1}(0) = X$, while the embedding $\sigma^{-1}(0) \to P$ is s. Thus we obtain from the previous corollary $\eta_X(P) = c_n(F)$. Considering paragraph 4.14 and remark 4.5 we thus obtain three expressions of the fundamental class of X in P :

$$t(E) = c_n\big(p^{-1}(E \oplus 1)/\lambda\big) = c_n\big(\lambda^\vee \otimes p^{-1}(E)\big).$$

Note the last equality, though obvious in the case where F is the additive formal group, is not evident to check directly in the general case. However, we left as an exercice to the reader to check it using the inverse series of the formal group law F in the case of a line bundle. This implies the general case by the splitting principle.

4.2.3. *The ramified case.* In this section, we study the case of a morphism $(f,g) : (Y,T) \to (X,Z)$ of smooth closed pairs of same codimension n. This corresponds to the *proper case* in the operation of pullback of Z along f. We put $T' = Z \times_X Y$ and consider the canonical thickening $T' \to T$ induced by (f,g).
We first need an assumtion. Let $T' = \bigcup_{i \in I} T'_i$ be the decomposition into connected components. For any $i \in I$, we also consider the decomposition $T'_i = \bigcup_{j \in J_i} T'_{ij}$ into irreducible components. Put $T_{ij} = T'_{ij} \times_{T'} T$. As $T \to T'$ is a thickening, the geometric multiplicity $m(T'_{ij})$ of T'_{ij} is an integral multiple of the geometric multiplicity $m(T_{ij})$ of T_{ij}. We introduce the following condition on the morphism (f,g):

(Special) For any $i \in I$, there exists an integer $r_i \geq 0$ such that for any $j \in J_i$, $m(T'_{ij}) = r_i.m(T_{ij})$.

The integer r_i will be called the *ramification index* of f along T_i.

Remark 4.23. When S is irreducible, this condition is always fulfilled. When S is integral, T'_i is irreducible and the integer r_i is nothing else than the geometric multiplicity of T'_i.

Under this assumption, we define intersection multiplicities which take into account the formal group law F introduced in paragraph 3.7.
Let B be the blow-up of $\mathbb{A}^1_X$ with center $\{0\} \times Z$, and P its exceptional divisor. Put $C = B \times_X Y$, and for any $i \in I$, $Q_i = P \times_T T_i$. Remark that Q_i/T_i admits a canonical section s_i. We denote by L_i the line bundle over T_i obtain by the pullback of the normal bundle $N_{Q_i}(C)$ along s_i. We consider the localised Thom class $\bar{t}(L_i) \in H^{2,1}_{T_i}(L_i)$ (cf 4.14); we recall it is sent to 1 by the purity isomorphism $\mathfrak{p}^*_{(L_i,T_i)} : H^{2,1}_{T_i}(L_i) \to H^{0,0}(T_i)$.
Note that, according to remark 3.14, the Thom class $t(L_i)$ is nilpotent. Thus, the same is true for $\bar{t}(L_i)$. In particular, we can apply the power series $[r_i]_F$ (see paragraph 3.7) to the element $\bar{t}(L_i)$ of the A-algebra $H^{**}_{T_i}(L_i)$. This defines an element $[r_i]_F \cdot \bar{t}(L_i) \in H^{**}_{T_i}(L_i)$ of bidegree $(2,1)$.

Definition 4.24. Consider a morphism $(f,g) : (Y,T) \to (X,Z)$ which satisfies the condition (Special). Assume T admits an ample line bundle.
We consider the notations introduced above. For any $i \in I$, we define the *F-intersection multiplicity* of T_i in $f^{-1}(Z)$ as the element

$$r(T_i; f, g) = \mathfrak{p}^*_{(L_i,T_i)}\big([r_i]_F \cdot \bar{t}(L)\big) \in H^{0,0}(T_i)$$

where r_i is the ramification index of f along T_i.

A straightforward check shows the F-intersection multiplicities are compatible with flat base change. When the formal group law F is additive, we easily get that $r(T_i; f, g) = r_i$.
In the codimension $n = 1$ case, we can also consider the localised fundamental class $\bar{\eta}_Y(T_i) \in H^{2,1}_{T_i}(Y)$ introduced in paragraph 4.14. It corresponds to the localised Thom class $\bar{t}(N_{T_i}(Y))$ under the isomorphisms given by the deformation diagram

(4.2). Thus applying remark 3.14 as above, we obtain that the class $\bar{\eta}_Y(T_i)$ is nilpotent. In particular, we can consider the class $[r_i]_F \cdot \bar{\eta}_Y(T_i) \in H^{2,1}_{T_i}(Y)$ obtained by applying the power series $[r_i]_F$ of 3.7. We then obtain a natural expression of the F-intersection multiplicity :

Lemma 4.25. *Consider the hypothesis and assumptions of the previous definition and assume $n = 1$. Let $\bar{\eta}_Y(T_i) \in H^{2,1}_{T_i}(Y)$ be the localised fundamental class of T_i in Y (cf paragraph 4.14) and $\mathfrak{p}^*_{(Y,T_i)} : H^{2,1}_{T_i}(Y) \to H^{0,0}(T_i)$ be the purity isomorphism in cohomology.*
Then, $r(T_i; f, g) = \mathfrak{p}^*_{(Y,T_i)}\big([r_i]_F \cdot \bar{\eta}_Y(T_i)\big)$.

Proof. We may assume T is connected. Thus $I = \{i\}$ and we put $L = L_i$, $r = r_i$ with the notations of the previous definition. As $n = 1$, the zero section of $\mathbb{A}^1_X/X$ induces the following transversal square

$$\begin{tikzcd} Z = \mathbb{P}(N_Z X) \arrow[r, "s"] \arrow[d] & \mathbb{P}(N_Z X \oplus 1) = P \arrow[d] \\ X = B_Z X \arrow[r] & B_Z(\mathbb{A}^1_X) = B \end{tikzcd}$$

which, after pullback above Y gives a cartesian square, still transversal, $\begin{tikzcd} T \arrow[r, "t"] \arrow[d] & Q \arrow[d] \\ Y \arrow[r] & C \end{tikzcd}$

with t the canonical section of Q/T. Thus we get :

$$\begin{aligned} \mathfrak{p}^*_{(L,T)}([r]_F \cdot \bar{t}(L)) &= t^* \mathfrak{p}^*_{(N_Q C,Q)}([r]_F \cdot \bar{t}(N_Q C)) = t^* \mathfrak{p}^*_{(C,Q)}([r]_F \cdot \bar{\eta}_C(Q)) \\ &= \mathfrak{p}^*_{(Y,T)}([r]_F \cdot \bar{\eta}_Y(T)) \end{aligned}$$

where the last equality follows from the transversal square above and proposition 4.10 whereas the other equalities follow from the definitions. □

Before stating the main result of this section, we need to recall an extension of the functoriality of the deformation diagram (4.2) to certain morphisms of closed pairs (see also [Dég03, proof of 3.3]).
Consider a morphism $(f, g) : (Y, T) \to (X, Z)$ of smooth closed pairs of codimension 1. Let $\mathcal{I}$ (resp. $\mathcal{J}$, $\mathcal{J}'$) be the ideal defining Z in X (resp. T in Y, T' in Y). The map f induces a morphism $\varphi : \mathcal{I} \to f_* \mathcal{J}'$ of sheaves over X.
We consider the second deformation space $D_Z X = B_Z(\mathbb{A}^1_X) - B_Z X$ as in 4.1. An easy computation shows

$$D_Z X = \mathrm{Spec}_X \left(\bigoplus_{n \in \mathbb{Z}} \mathcal{I}^n . u^{-n} \right)$$

where $\mathcal{I}^n = \mathcal{O}_X$ for $n < 0$, and u is an indeterminate.
Assume[18] $\underline{\mathcal{J}' = \mathcal{J}^r}$. Then we can define a morphism of sheaves of rings over X :

$$\bigoplus_{n \in \mathbb{Z}} \mathcal{I}^n . u^{-n} \to \bigoplus_{n \in \mathbb{Z}} f_*(\mathcal{J}'^n) . u^{-n} \to \bigoplus_{m \in \mathbb{Z}} f_*(\mathcal{J}^m) . v^{-m}$$

[18] As the immersion $T \to Y$ is regular, this can happen only in the codimension 1 case. Note it implies (f, g) satisfies the condition (Special) and the ramification indexes are all equal to r.

where the first arrow is induced by φ and the second is the obvious inclusion which maps u to v^r as $\mathcal{J}' = \mathcal{J}^r$.
Taking the spectrum of these morphisms over X, we get a morphism

$$\rho_r(f,g) : D_T Y \to D_Z X$$

of schemes over $\mathbb{A}^1$. The fibre of $\rho_r(f,g)$ over 1 is simply f and one can check that its fiber over 0 is a composite morphism

$$\sigma_r(f,g) : N_T Y \xrightarrow{\nu} g^* N_Z X \xrightarrow{\mu} N_Z X$$

such that μ is induced by g and ν is a homogenous morphism of degree r. Thus, considering the respective deformation diagrams (4.2) for (X,Z) and (Y,T) we obtain a commutative diagram of closed pairs

$$\begin{tikzcd}(Y,T) \arrow[r, "\sigma'_1"] \arrow[d, "(f,g)"'] & (D_T Y, \mathbb{A}^1_T) \arrow[d, dashed, "(\rho_r(f,g),1\times g)"] & (N_T Y, T) \arrow[l, "\sigma'_0"'] \arrow[d, "(\sigma_r(f,g),g)"] \\ (X,Z) \arrow[r, "\sigma_1"] & (D_Z X, \mathbb{A}^1_Z) & (N_Z X, Z). \arrow[l, "\sigma_0"']\end{tikzcd} \tag{4.5}$$

Theorem 4.26. *Let $(f,g) : (Y,T) \to (X,Z)$ be a morphism of smooth closed pairs of codimension n. We assume T admits an ample line bundle and (f,g) satisfies condition (Special).*
Then

$$(f,g)_! = \sum_{i \in I} r(T_i; f,g) \boxtimes_T g_{i*}$$

where $T = \bigcup_{i \in I} T_i$ is the decomposition into connected component, $g_i = g|_{T_i}$ and $r(T_i; f,g)$ is the F-intersection multiplicity of T_i in $f^{-1}(Z)$.

Proof. Using axiom (Add'), we can assume T is connected.
We first reduce to the codimension $n = 1$ case. Consider the blow-up $B = B_Z(\mathbb{A}^1_X)$ and its exceptional divisor $P = \mathbb{P}(N_Z X \oplus 1)$. Consider also the cartesian morphism $(p,q) : (B,P) \to (X,Z)$. If we put $B_Y = B \times_X Y$, $Q = P \times_Z T$, we obtain the following commutative diagram of morphisms closed pairs :

$$\begin{tikzcd}(B_Y, Q) \arrow[r, "(f',g')"] \arrow[d, "(\pi',q)"'] & (B,P) \arrow[d, "(\pi,p)"] \\ (Y,T) \arrow[r, "(f,g)"] & (X,Z).\end{tikzcd}$$

By definition, $(f,g)_!(\pi',q)_! = (\pi,p)_!(f',g')_!$.
Note that (π,p) and (π',q) are cartesians. We can apply proposition 4.16 to (π,p) : the excess intersection bundle is the universal quotient bundle ξ_0 on P and $(\pi,p)_! = p_* \boxtimes c_n(\xi_0)$. Thus, according to remark 4.5 and paragraph 4.14, $(\pi,p)_! = s^*$ where $s : X \to P$ is the canonical section.
Similarly, if we put $\xi = {g'}^{-1}(\xi_0)$, we get $(\pi',q)_! = q_* \boxtimes c_n(\xi) = t^*$ with $t : T \to Q$ the canonical section. Note this latter morphism is a split epimorphism with splitting $\mathfrak{l}_n(Q)$. Thus we get

$$(f,g)_! = s^* \circ (f',g')_! \circ \mathfrak{l}_n(Q).$$

Remark that $Q = P \times_B B_Y$. Thus the morphism (f', g') of smooth closed pairs of codimension 1 satifies the condition (Special) and the ramification indexes of f along T and f' along Q are equal. Assume $(f', g') = r(Q; f', g') \boxtimes g'_*$. According to the expression above, we get

$$(f,g)_! = \big(r(Q;f',g') \boxtimes s^* g'_*\big) \circ \mathfrak{l}_n(Q) \stackrel{(1)}{=} \big(r(Q;f',g') \boxtimes g_* t^*\big) \circ \mathfrak{l}_n(Q)$$
$$\stackrel{(2)}{=} \big((r(Q;f',g') \circ t_*) \boxtimes g_*\big) t^* \circ \mathfrak{l}_n(Q) = (r(Q;f',g') \circ t_*) \boxtimes g_*.$$

where equality (1) follows from the projection formula of proposition 4.10 and equality (2) from the other projection formula of corollary 4.11. From definition 4.24, the reader can now easily check the equality of the cohomological classes $t^*[r(Q; f', g')] = r(T; f, g)$.
Thus we are reduced to the case $n = 1$, T still being connected. Let r be the ramification index of f along T. Let $\mathcal{J}$ (resp. $\mathcal{J}'$) be the ideal sheaf of T (resp. T') in Y. As $Z \to X$ and $T \to Y$ are regular immersions of a divisor, we see that necessarily, $\mathcal{J}' = \mathcal{J}^r$. Considering now diagram (4.5), we obtain that $(f, g)_! = (\sigma_r(f, g), g)_!$. In view of the factorization of the morphism $\sigma_r(f, g)$, we then are reduced to the following lemma :

Lemma 4.27. *Let T be a smooth scheme which admits an ample line bundle. Consider a line bundle N over T and $N^{\otimes r}$ be its r-th tensor power over T.*
Let $\nu : N \to N^{\otimes r}$ be the obvious homogenous morphism of degree r, and $(\nu, 1_T) : (N, T) \to (N^{\otimes r}, T)$ be the corresponding morphism of closed pairs.
Then $(\nu, 1_T)_! = \rho \otimes 1_{T*}$ where ρ is the unique element of $H^{00}(T)$ such that $[r]_F \cdot t(N) = \rho.t(N)$.

Put $P = \mathbb{P}(N \oplus 1)$, $P' = \mathbb{P}(N^{\otimes r} \oplus 1)$ and consider the projective completion $\bar{\nu} : P \to P'$ of ν. Let λ (resp. λ') be the canonical line bundle and p (resp. p') be the canonical projection of P/T (resp. P'/T). An easy computation shows that $\bar{\nu}^*(\lambda') = \lambda^{\otimes r}$. Recall from 4.22 that the Thom class of N (resp. $L^{\otimes r}$) is equal to $t(N) = c_1(\lambda^\vee \otimes p^{-1}N)$ (resp. $t(N^{\otimes r}) = c_1(\lambda'^\vee \otimes p'^{-1}N^{\otimes r})$). Thus, from the second point of proposition 3.8, $\bar{\nu}^* t(N^{\otimes r}) = [r]_F \cdot t(N)$. This latter class is zero on $P - T$, thus we get the relation $[r]_F \cdot t(N) = \rho.t(N)$ in $H^{2,1}(P)$. The conclusion now follows according to the computation of the Thom isomorphism 4.4.
To finish the proof with that lemma, we remark that $\rho = r(T; \nu, 1_T) = r(T; f, g)$. □

Corollary 4.28. *Let $(f, g) : (Y, T) \to (X, Z)$ be a morphism of smooth closed pairs of codimension 1. We assume T admits an ample line bundle and (f, g) satisfies condition (Special). Let $(T_i)_{i\in I}$ be the connected components of T, and $r_i \in \mathbb{N}$ be the ramification index of f along T_i.*
Then, for any $i \in I$, the fundamental class $\eta_Y(T_i)$ is nilpotent and

$$f^*(\eta_X(Z)) = \sum_{i\in I} [r_i]_F \cdot \eta_T(T_i)$$

where $[r_i]_F$ is the power series equal to the r_i-th formal sum with respect to the formal group law F.

Proof. Let $i : Z \to X$ and $j_i : T_i \to Y$ be the canonical immersions. We simply put $\rho_i = r(T_i; f, g) \in H^{0,0}(T_i)$. In cohomology, the preceding theorem applied to (f, g) gives the relation

$$f^* i_*(z) = \sum_{i \in I} j_{i*}(\rho_i \boxtimes g_i^*(z))$$

for $z \in H^{**}(Z)$. Applied with $z = 1$, this gives $f^*(\eta_X(Z)) = \sum_i j_{i*}(\rho_i)$. Recall from lemma 4.25 that $\rho_i = \mathfrak{p}^*_{(Y,T_i)}([r_i]_F \cdot \bar{\eta}_Y(T_i))$. Tautologically, the composition $j_{i*}\mathfrak{p}^*_{(Y,T_i)}$ is equal to the canonical morphism $H^{**}_{T_i}(Y) \to H^{**}(Y)$ simply obtained by functoriality. For conclusion, it is sufficient to recall this latter is a morphism of A-algebra (cf paragraph 2.8). □

Remark 4.29. In the previous corollary, the integers r_i can be understood as follows: locally, Z is parametrized by a S-regular function $a : X \to \mathbb{A}^1$. Then, (f, g) is special if $a \circ f$ can be written locally $u. \prod_{i \in I} b_i^{r_i}$ where u is a unit and $b_i : Y \to \mathbb{A}^1$ is a S-regular function parametrizing T_i – this expression should remain the same when we change any of the parameters b_i or a.

4.3. **Crossing Gysin triangles.** The following lemma will be the key point of the main result of this section. Though it will appear finally as a particular case, we begin by proving it to enlighten the proof of theorem 4.32.

Lemma 4.30. *Let Z be a smooth scheme, E and E' be vector bundles over Z of respective ranks n and m. Put $Q = \mathbb{P}(E \oplus 1)$, $Q' = \mathbb{P}(E' \oplus 1)$ and $P = Q \times_Z Q'$. Consider the fundamental class (see paragraph 4.14) $\eta_P(Z)$ (resp. $\eta_P(Q)$, $\eta_P(Q')$) of the canonical embedding of Z (resp. Q, Q') in P, as an element of $H^{**}(P)$. Then $\eta_P(Z) = \eta_P(Q) \cup \eta_P(Q')$.*

Proof. Put $d = n + m$. Let $\bar{\eta}_P(Z)$ be the localised fundamental class of Z in P (cf paragraph 4.14). Consider the deformation diagram (4.1) for the closed pair (P, Z), with $B = B_Z(\mathbb{A}^1_P)$:

$$(P, Z) \xrightarrow{\bar{\sigma}_1} (B, \mathbb{A}^1_Z) \xleftarrow{\bar{\sigma}_0} (P, Z).$$

As $\bar{\sigma}_0^*$ and $\bar{\sigma}_1^*$ are isomorphisms, $\bar{\eta}_P(Z)$ is uniquely determined by the class $\bar{t} = \bar{\sigma}_1^*(\bar{\eta}_P(Z))$ and $\bar{t}$ is uniquely determined by the fact that $\bar{\sigma}_0^*(\bar{t})$ corresponds to the Thom class $t(E \oplus E')$ in $H^{2d,d}(P)$.
Consider the divisor $D = B_Z(\mathbb{A}^1 \times \mathbb{P}(E) \times \mathbb{P}(E' \oplus 1))$ (resp. $D' = B_Z(\mathbb{A}^1 \times \mathbb{P}(E \oplus 1) \times \mathbb{P}(E'))$ in B and the class $c = -c_1(D)$ (resp. $c' = -c_1(D')$) in $H^{2,1}(B)$. Let π be the canonical projection of P/X. We define a cohomology class in $H^{2,1}(B)$:

$$t = \left(\sum_{0 \le i \le n} \pi^*(c_i(E)) \cup c^{n-i} \right) \cup \left(\sum_{0 \le j \le m} \pi^*(c_j(E')) \cup c'^{m-j} \right).$$

Then t vanishes on $B - \mathbb{A}^1_Z$ and, by construction, its pullback by $\bar{\sigma}_0$ is equal to $t(E \oplus E')$. Thus t corresponds to the class $\bar{t}$ mentionned above, through the map $H^{2d,d}_{\mathbb{A}^1_Z}(B) \to H^{2d,d}(B)$. The computation of its pullback by $\bar{\sigma}_1$ gives the desired formula. □

Remark 4.31. Another way to obtain this lemma is to apply corollary 4.21 with $X = P$ and $E = \xi \times_Z \xi'$ where ξ (resp. ξ') is the universal quotient bundle of Q (resp. Q') — compare with remark 4.5.

Theorem 4.32. *Consider a cartesian square of smooth schemes*
$$\begin{tikzcd} Z \arrow[r, "k"] \arrow[d, "l"'] & Y' \arrow[d, "j"] \\ Y \arrow[r, "i"] & X \end{tikzcd}$$
such that i,j,k,l are closed immersions of respective pure codimension n, m, s, t. We put $d = n+s = m+t$ and consider the closed immersion $h : (Y-Z) \longrightarrow (X-Y')$ induced by i.
Then, in the following diagram :

$$\begin{tikzcd} M(X) \arrow[r, "j^*"] \arrow[d, "i^*"'] \arrow[dr, phantom, "(1)"] & M(Y')(m)[2m] \arrow[r, "\partial_{X,Y'}"] \arrow[d, "k^*"] \arrow[dr, phantom, "(2)"] & M(X-Y')[1] \arrow[d, "h^*"] \\ M(Y)(n)[2n] \arrow[r, "l^*"'] & M(Z)(d)[2d] \arrow[r, "\partial_{Y,Z}"] \arrow[d, "\partial_{Y',Z}"'] \arrow[dr, phantom, "(3)"] & M(Y-Z)(n)[2n+1] \arrow[d, "\partial_{X-Y',Y-Z}"] \\ & M(Y-Z)(m)[2m+1] \arrow[r, "\partial_{X-Y,Y-Z}"'] & M(X-Y\cup Y')[2] \end{tikzcd}$$

squares (1) and (2) are commutative and square (3) is anti-commutative.

Proof. Put $Y'' = Y \cup Y'$. Using axiom (Loc) and (Sym)(c), we obtain the following diagram :

$$(\mathcal{D}) : \begin{tikzcd} M(X-Y'') \arrow[r] \arrow[d] & M(X-Y) \arrow[r] \arrow[d] & M\left(\frac{X-Y}{X-Y''}\right) \arrow[r] \arrow[d] & M(X-Y'')[1] \arrow[d] \\ M(X-Y') \arrow[r] \arrow[d] & M(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "(1)"] & M\left(\frac{X}{X-Y'}\right) \arrow[r] \arrow[d] \arrow[dr, phantom, "(2)"] & M(X-Y')[1] \arrow[d] \\ M\left(\frac{X-Y'}{X-Y''}\right) \arrow[r] \arrow[d] & M\left(\frac{X}{X-Y}\right) \arrow[r] \arrow[d] & M\left(\frac{X/X-Y}{X-Y'/X-Y''}\right) \arrow[r] \arrow[d] \arrow[dr, phantom, "(3)"] & M\left(\frac{X-Y'}{X-Y''}\right)[1] \arrow[d] \\ M(X-Y'')[1] \arrow[r] & M(X-Y)[1] \arrow[r] & M\left(\frac{X-Y}{X-Y''}\right)[1] \arrow[r] & M(X-Y'')[2], \end{tikzcd}$$

in which any line or any row is a distinguished triangle, every square is commutative except square (3) which is anticommutative.
We put $M(X;Y,Y') = M\left(\frac{X/X-Y}{X-Y'/X-Y''}\right)$ for short. The proof will consist in constructing a purity isomorphism $\mathfrak{p}_{(X;Y,Y')} : M(X;Y,Y') \to M(Z)(d)[2d]$ which satisfies the following properties :

(i) *Functoriality :* The morphism $\mathfrak{p}_{(X;Y,Y')}$ is functorial with respect to morphisms in X which are transversal to Y, Y' and Z respectively.
(ii) *Symmetry :* The following diagram is commutative :

$$\begin{tikzcd} M(X;Y,Y') \arrow[rr, "\epsilon"] \arrow[dr, "\mathfrak{p}_{(X;Y,Y')}"'] & & M(X;Y',Y) \arrow[dl, "\mathfrak{p}_{(X;Y',Y)}"] \\ & M(Z)(d)[2d], & \end{tikzcd}$$

where ϵ is the isomorphism given in axiom (Sym).

(iii) *Compatibility :* The following diagram is commutative :

$$\begin{tikzcd}
M(\frac{X-Y'}{X-\Omega}) \arrow[r] \arrow[d, "\mathfrak{p}_{(X-Y',Y-Z)}"'] & M(\frac{X}{X-Y}) \arrow[r] \arrow[d, "\mathfrak{p}_{(X,Y)}"'] & M(X;Y,Y') \arrow[r] \arrow[d, "\mathfrak{p}_{(X;Y,Y')}"'] & M(\frac{X-Y'}{X-\Omega})[1] \arrow[d, "\mathfrak{p}_{(X-Y',Y-Z)}"'] \\
M(Y-Z)(n)[2n] \arrow[r] & M(Y)(n)[2n] \arrow[r, "j^*"] & M(Z)(d)[2d] \arrow[r, "\partial_{Y,Z}"] & M(Y-Z)(n)[2n+1]
\end{tikzcd}$$

With this isomorphism, we can deduce the three relations of the theorem by considering squares (1), (2), (3) in the above diagram when we apply the evident purity isomorphisms where we can. We then are reduced to construct the isomorphism and to prove the above relations. The difficult one is the second relation because we have to show that two isomorphisms in a triangulated category are equal. This forces to be very precise in the construction of the isomorphism.

We use a construction analog to the construction of the purity isomorphism in proposition 4.3. The first deformation space (cf paragraph 4.1) for the pair (X,Y) is $B = B_Y(\mathbb{A}^1_X)$. We let $P = P_Y X$ be the projective completion of the normal bundle of (X,Y). Consider also the closed pair $(U,V) = (X-Y', Y-Z)$. The analog deformation space for (U,V) is $B_U = B \times_X U$ and the projective completion of its normal bundle is $P_V = P \times_Y V$.

The deformation diagrams (4.1) for (X,Y) and (U,V) induce the following morphisms

$$M(X;Y,Y') = M\left(\frac{X/X-Y}{U/U-V}\right) \xrightarrow{\bar\sigma_{1*}} M\left(\frac{B/B-\mathbb{A}^1_Y}{B_U/B_U-\mathbb{A}^1_V}\right) \xleftarrow{\bar\sigma_{0*}} M\left(\frac{P/P-Y}{P_V/P_V-V}\right)$$

and the axiom (Loc) together with the purity theorem 4.3 shows $\bar\sigma_{0*}$ and $\bar\sigma_{1*}$ are isomorphisms.

Using the compatibility of the Gysin triangle with the projective bundle isomorphism (cf corollary 4.15), we obtain a commutative diagram :

$$\begin{tikzcd}
M(P_V/P_V-V) \arrow[r] & M(P/P-U) \arrow[r] & M\left(\frac{P/P-Y}{P_V/P_V-V}\right) \arrow[r, "+1"] & {} \\
M(P_V) \arrow[r] \arrow[u] \arrow[d, equal] & M(P) \arrow[r] \arrow[u] \arrow[d, equal] & M(\frac{P}{P_V}) \arrow[r, "+1"] \arrow[u] & {} \\
M(P_V) \arrow[r] & M(P) \arrow[r] & M(P_Z)(s)[2s] \arrow[r, "+1"] \arrow[u, "\mathfrak{p}^{-1}_{(P,P_Z)}"'] & {} \\
M(Y-Z)(n)[2n] \arrow[r] \arrow[u, "\mathfrak{l}_n(P_V)"] & M(Y)(n)[2n] \arrow[r, "j^*"] \arrow[u, "\mathfrak{l}_n(P)"] & M(Z)(d)[2d] \arrow[r, "+1"] \arrow[u, "\mathfrak{l}_n(P_Z)(s)[2s]"'] & {}
\end{tikzcd}$$

The composite of the vertical maps thus gives a morphism of triangles. Using property (2) of proposition 4.3, the first two maps of this morphism are isomorphisms and so is the third. This last isomorphism together with the maps $\bar\sigma_{1*}$ and $\bar\sigma_{0*}$ gives the desired isomorphism $\mathfrak{p}_{(X;Y,Y')}$.

Note that property (iii) is obvious by construction. Property (i) is easily obtained as in proposition 4.10.

Thus we have only to prove property (ii). First of all, we remark that the previous construction implies immediately the commutativity of the diagram :

$$\begin{tikzcd}
M(X;Y,Y') \arrow[rr, "\alpha_{(X;Y,Y')}"] \arrow[dr, "\mathfrak{p}_{(X;Y,Y')}"'] & & M(X;Y,Z) \arrow[dl, "\mathfrak{p}_{(X;Y,Z)}"] \\
& M(Z)(d)[2d], &
\end{tikzcd}$$

where $\alpha_{(X;Y,Y')}$ is induced by the evident open immersions.
Consider the following map

$$\beta_{(X;Y,Y')} : M_Z(X) \xrightarrow{\pi_{(X,Y,Z)}} M(X;Y,Z) \xrightarrow{\alpha^{-1}_{(X;Y,Y')}} M(X;Y,Y')$$

where $\pi_{(X,Y,Z)}$ is obtained by functoriality as usual – it is an isomorphism from axioms (Loc) and (Sym). Using the coherence axiom (Sym)(b), one checks that the following diagram is commutative

$$\begin{tikzcd}
& M_Z(X) \arrow[dl, "\beta_{(X;Y,Y')}"'] \arrow[dr, "\beta_{(X;Y',Y)}"] & \\
M(X;Y,Y') \arrow[rr, "\epsilon"] & & M(X;Y',Y).
\end{tikzcd}$$

Thus, it will be sufficient to prove the commutativity of the following diagram :

$$\begin{tikzcd}
M_Z(X) \arrow[rr, "\pi_{(X,Y,Z)}"] \arrow[dr, "\mathfrak{p}_{(X,Z)}"'] & (*) & M(X;Y,Z) \arrow[dl, "\mathfrak{p}_{(X;Y,Z)}"] \\
& M(Z)(d)[2d]. &
\end{tikzcd}$$

In the remainings of the proof, we consider the triples of smooth schemes (X',Y',Z') such that $Z' \subset Y' \subset X'$ are closed subschemes. A morphism of triples $(f,g,h) : (X'',Y'',Z'') \to (X',Y',Z')$ is a morphism of schemes $f : X'' \to X'$ which is transversal to Y' and Z', and such that $Y'' = f^{-}(Y')$, $Z'' = f^{-1}(Z')$. Using the functoriality of $\mathfrak{p}_{(X;Y,Z)}$, we remark that diagram $(*)$ is natural with respect to morphisms of triples.
We use the notations of paragraph 4.1. We also put $B(X',Z') := B_{Z'}(X')$, for a closed pair (X',Z'), and so on for the other schemes depending on a closed pair, to clarify the following considerations. We consider the evident closed pair $(D_Z X, D_Z X|_Y)$ and we put $D(X,Y,Z) = D(D_Z X, D_Z X|_Y)$. This scheme is in fact fibered over $\mathbb{A}^2$. The fiber over $(1,1)$ is X and the fiber over $(0,0)$ is $B(B_Z X \cup P_Z X, B_Z X|_Y \cup P_Z X|_Y)$. In particular, the $(0,0)$-fiber contains the scheme $P(P_Z X, P_Z Y)$.
We now put : $D = D(X,Y,Z)$, $D' = D(Y,Y,Z)$. Remark that $D(Z,Z,Z) = \mathbb{A}^2_Z$. Similarly, we put $P = P(P_Z X, P_Z Y)$, $Q = P_Z Y$. Remark finally that if we consider $Q' = P_Y X|_Z$, then[19] $P = Q \times_Z Q'$.
From the above description of fibers, we obtain a deformation diagram of triples :

$$(X,Y,Z) \to (D,D',\mathbb{A}^2_Z) \leftarrow (E,G,Z).$$

[19]This is equivalent to the canonical isomorphism $N(N_Z X, N_Z Y) = N_Z Y \oplus N_Y X|_Z$.

Note that these morphisms are on the smaller closed subscheme the $(0,0)$-section and $(1,1)$-section of $\mathbb{A}^2_Z$ over Z, denoted respectively by s_1 and s_0. Now we apply these morphisms to diagram $(*)$ obtaining the following commutative diagram :

$$\begin{tikzcd}
M_Z(X) \arrow[rr] \arrow[dd, "\mathfrak{p}_{(X,Z)}"'] \arrow[dr, "\pi_{X,Y,Z}"] & & M_{\mathbb{A}^2_Z}(D) \arrow[dd, "\mathfrak{p}_{(D,\mathbb{A}^2_Z)}"'] \arrow[dr] & & M_Z(P) \arrow[ll] \arrow[dd, "\mathfrak{p}_{(P,Z)}"'] \arrow[dr, "\pi_{P,Q,Z}"] & \\
& M(X;Y,Z) \arrow[rr] \arrow[dl, "\mathfrak{p}_{(X,Y,Z)}"] & & M(D;D',\mathbb{A}^2_Z) \arrow[dl, "\mathfrak{p}_{(D,D',Z)}"] & & M(P;Q,Z) \arrow[ll] \arrow[dl, "\mathfrak{p}_{(P,Q,Z)}"] \\
M(Z)(d)[2d] \arrow[rr, "s_{1*}"'] & & M(\mathbb{A}^2_Z)(d)[2d] & & M(Z)(d)[2d]. \arrow[ll, "s_{0*}"] &
\end{tikzcd}$$

The square parts of this prism are commutative. As morphisms s_{1*} and s_{0*} are isomorphisms, the commutativity of the left triangle is equivalent to the commutativity of the right one.
Thus, we are reduced to the case of the smooth triple (P,Q,Z). Now, using the canonical split epimorphism $M(P) \to M(P/P-Z)$, we are reduced to prove the commutativity of the diagram :

$$\begin{tikzcd}
M(P) \arrow[r] \arrow[d, "i^*"'] & M\left(\frac{P/P-Q}{P-Z/P-Q}\right) \arrow[dl, "\mathfrak{p}_{(P,Q,Z)}"] \\
M(Z)(d)[2d] &
\end{tikzcd}$$

where $i : Z \to P$ denotes the canonical closed immersion.
Using property (iii) of the isomorphism $\mathfrak{p}_{(P,Q,Z)}$, we are finally reduced to prove the commutativity of the triangle

$$\begin{tikzcd}
M(P) \arrow[rr, "j^*"] \arrow[dr, "i^*"'] & & M(Q)(n)[2n] \arrow[dl, "k^*"] \\
& M(Z)(d)[2d] &
\end{tikzcd}$$

where we considered $Z \xrightarrow{k} Q \xrightarrow{j} P$ the canonical closed embeddings. This now simply follows from paragraph 4.14 and lemma 4.30. $\square$

As a corollary (apply commutativity of square (1) in the case $Y' = Z$), we get the functoriality of the Gysin morphism of a closed immersion :

Corollary 4.33. *Let $Z \xrightarrow{l} Y \xrightarrow{i} X$ be closed immersions between smooth schemes of respective pure codimension n and m.*
Then, $l^* \circ i^* = (i \circ l)^*$ as a morphism $M(X) \to M(Z)(n+m)[2(n+m)]$.

A corollary of this result, using lemma 4.12, is the compatibility of the Gysin morphism with products :

Corollary 4.34. *Consider a closed immersion $i : Z \to X$ (resp. $k : T \to Y$) between smooth schemes of pure codimension n (resp. m).*
Then $(i \times k)^* = i^* \otimes k^*$ as a morphism[20] :

$$M(X) \otimes M(Y) \to M(Z) \otimes M(T)(n+m)[2(n+m)].$$

[20] When we identify $M(Z)(n)[2n] \otimes M(T)(m)[2m]$ with $M(Z) \otimes M(T)(n+m)[2(n+m)]$ through the canonical isomorphism.

Remark 4.35. In the hypothesis of the previous corollary, we obtain in terms of fundamental classes :

$$\eta_{X\times Y}(Z\times T)=\eta_X(Z)\otimes\eta_Y(T).$$

We also obtain a result of intersection of fundamental classes in the case of smooth cycles :

Corollary 4.36. *Let X be a smooth scheme, Z and T be smooth closed subschemes of X. We assume that :*

1. *The intersection of Z and T in X is proper.*
2. *There is a closed subscheme W in $Z\cap T$ which is smooth, homeomorphic to $Z\cap T$ and admits an ample line bundle.*
3. *The induced morphism of closed pairs $(T,W)\to(X,Z)$ satisfies condition (Special).*

Let $\nu_{X,W}:H^{**}_W(X)\to H^{**}(X)$ be the canonical morphism. According to (Add′), $H^{**}_W(X)=\bigoplus_{i\in I}H^{**}_{W_i}(X)$ where $(W_i)_{i\in I}$ are the connected components of W. For any $i\in I$, we can consider the localised fundamental class $\bar\eta_X(W_i)$ as an element of $H^{**}_W(X)$ (see paragraph 4.14). We let $\rho_i\in H^{0,0}(W_i)$ be the F-intersection multiplicity of W_i in $Z\cap T$ (see definition 4.24). Then,

$$\eta_X(Z)\cup\eta_X(T)=\nu_{X,W}\Big(\sum\nolimits_{i\in I}\rho_i.\bar\eta_X(W_i)\Big)$$

using the $H^{0,0}(W)$-module structure of $H^{**}_W(X)$ obtained through the purity isomorphism.

Proof. We apply theorem 4.26 to the obvious square :

$$\begin{array}{ccc} W & \xrightarrow{\nu'} & T \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{\nu} & X. \end{array}$$

For any $i\in I$, we let ν'_i (resp. μ_i) be the immersion of W_i in T (resp. X). We thus obtain the formula in $H^{**}(T)$: $f^*\nu_*(1)=\sum_{i\in I}\nu'_{i*}(\rho_i)$.
Applying f_* to this formula and using corollary 4.11 for the left hand side, corollary 4.33 for the right hand side, we obtain $\eta_X(Z)\boxtimes\eta_X(T)=\sum_{i\in I}\mu_{i*}(\rho_i)$. By the very definition now, $\mu_{i*}(\rho_i)=\nu_{X,W}(\rho_i.\bar\eta_X(W_i))$. □

5. Duality and Gysin morphism

5.1. **Preliminaries.** For the rest of the section, we fix a monoidal category $\mathscr{C}$ with unit $\mathbb{1}$.

Definition 5.1. Let M an object of $\mathscr{C}$.
We say M is *strongly dualizable* if the following conditions are fulfilled :

1. The functor $M\otimes .$ admits a right adjoint $\underline{\mathrm{Hom}}(M,.)$.
2. For any object N of $\mathscr{C}$, consider the map

   $$M\otimes\underline{\mathrm{Hom}}(M,\mathbb{1})\otimes N\xrightarrow{ad\otimes 1_N}N$$

   induced by the evident adjunction morphism. Then the adjoint map

   $$\underline{\mathrm{Hom}}(M,\mathbb{1})\otimes N\to\underline{\mathrm{Hom}}(M,N)$$

is an isomorphism.

This definition coincides with definition 1.2 of [DP80]. Obviously, strongly dualizable objects are stable by finite sums and tensor product. Remark also that any invertible object of $\mathscr{C}$ for the tensor product is *a fortiori* strongly dualizable.

Definition 5.2. Consider an object M of $\mathscr{C}$.
A *strong dual* of M is an object $M^\vee$ of $\mathscr{C}$ and two morphisms $\mu : M \otimes M^\vee \to \mathbb{1}$, $\epsilon : \mathbb{1} \to M^\vee \otimes M$ such that the following composites

$$(i) \quad M \xrightarrow{1\otimes\epsilon} M \otimes M^\vee \otimes M \xrightarrow{\mu\otimes 1} M$$

$$(ii) \quad M^\vee \xrightarrow{\epsilon\otimes 1} M^\vee \otimes M \otimes M^\vee \xrightarrow{1\otimes\mu} M^\vee$$

are the identity morphisms.

The conditions of the definition imply that $M^\vee \otimes .$ is right adjoint to $M \otimes .$ and the natural transformations $\epsilon \otimes .$ and $\mu \otimes .$ are the adjunction transformations. Moreover, M is strongly dualizable as condition (2) of the first definition simply follows from the structural isomorphism $(M^\vee \otimes \mathbb{1}) \otimes N \simeq M^\vee \otimes N$ (see also [DP80, 1.3]).
Remark we also obtain that $. \otimes M^\vee$ is left adjoint to $. \otimes M$ with natural transformation $. \otimes \mu$ and $. \otimes \epsilon$. This gives the following reciprocal isomorphisms which we describe for future needs :

$$(5.1) \quad \begin{aligned} &\mathrm{Hom}_{\mathscr{C}}(M^\vee, \mathbb{E}) \to \mathrm{Hom}_{\mathscr{C}}(\mathbb{1}, \mathbb{E} \otimes M), \varphi \mapsto (\varphi \otimes 1_M) \circ \epsilon \\ &\mathrm{Hom}_{\mathscr{C}}(\mathbb{1}, \mathbb{E} \otimes M) \to \mathrm{Hom}_{\mathscr{C}}(M^\vee, \mathbb{E}), \psi \mapsto (1_{\mathbb{E}} \otimes \mu) \circ (\psi \otimes 1_{M^\vee}), \end{aligned}$$

where $\mathbb{E}$ is any object of $\mathscr{C}$.
The following lemma gives some precisions on the relation between "strongly dualizable" and "strong dual" :

Lemma 5.3. *Consider a strongly dualizable object M of $\mathscr{C}$. Let $M^\vee$ be an object of $\mathscr{C}$.*
Consider the following sets :

1. *Couples of morphisms $\mu : M \otimes M^\vee \to \mathbb{1}$ and $\epsilon : M^\vee \otimes M \to \mathbb{1}$ such that $(M^\vee, \mu, \epsilon)$ is a strong dual of M.*
2. *Morphisms $\mu : M \otimes M^\vee \to \mathbb{1}$ such that the adjoint map $\phi : M^\vee \to \underline{\mathrm{Hom}}(M, \mathbb{1})$ is an isomorphism.*

We associate to any morphism μ in (2) the following composite

$$\epsilon_\mu : \mathbb{1} \xrightarrow{ad'} \underline{\mathrm{Hom}}(M, M) \to \underline{\mathrm{Hom}}(M, \mathbb{1}) \otimes M \xrightarrow{\phi^{-1}\otimes 1} M^\vee \otimes M$$

where the first map is the evident adjunction morphism and the second one is induced by the isomorphism obtained by the property of the strongly dualizable object M.
Then (μ, ϵ_μ) is an element of (1) and the application

$$(2) \to (1), \mu \mapsto (\mu, \epsilon_\mu)$$

is a bijection.

We left the easy check to the reader.

Definition 5.4. Let M (resp. N) be an object of $\mathscr{C}$ and $(M^\vee, \mu_M, \epsilon_M)$ (resp. $(N^\vee, \mu_N, \epsilon_N)$) be a strong dual of M (resp. N).
For any morphism $f : M \to N$, we define the transpose morphism of f (with respect to the chosen strong duals) as the composite

$$ {}^{\mathrm{t}}f : N^\vee \xrightarrow{\mu_M \otimes 1} M^\vee \otimes M \otimes N^\vee \xrightarrow{1 \otimes f \otimes 1} M^\vee \otimes N \otimes N^\vee \xrightarrow{1 \otimes \epsilon_N} M^\vee. $$

Remark that the morphism ${}^{\mathrm{t}}f$ in the previous definition is characterized by either one of the next two properties :

(i) The following diagram is commutative :

$$\begin{tikzcd} M \otimes N^\vee \arrow[r, "f \otimes 1"] \arrow[d, "1 \otimes {}^{\mathrm{t}}f"'] & N \otimes N^\vee \arrow[d, "\mu_N"] \\ M \otimes M^\vee \arrow[r, "\mu_M"] & \mathbb{1}. \end{tikzcd}$$

(ii) The following diagram is commutative :

$$\begin{tikzcd} N^\vee \arrow[r, "{}^{\mathrm{t}}f"] \arrow[d] & M^\vee \arrow[d] \\ \underline{\mathrm{Hom}}(N, \mathbb{1}) \arrow[r, "\underline{\mathrm{Hom}}(f, \mathbb{1})"] & \underline{\mathrm{Hom}}(M, \mathbb{1}) \end{tikzcd}$$

where the vertical maps are induced by adjunction from μ_N and μ_M – cf lemma 5.3.

5.2. **The projective bundle case.** Fix an integer $n \geq 0$. Using the projective bundle theorem 3.2 and axiom (Stab), we obtain that the motive $M(\mathbb{P}^n)$ is strongly dualizable, as a finite sum of invertible motives.
Let λ_n be the canonical line bundle on $\mathbb{P}^n$, $c' = c_1(\lambda_n)$. From the projective bundle theorem 3.2, c' is a generator of the A-algebra $H^{**}(\mathbb{P}^n)$. Let $c = c_1(\lambda_n^\vee)$. According to paragraph 3.7, $c = m(c') = -c' \mod c'^2$ where m is the inverse series associated to the formal group law F. Thus, the class c is still a generator of $H^{**}(\mathbb{P}^n)$ and also satisfies the relation $c^{n+1} = 0$. In all this section on duality, we systematically use this generator.
We consider the following morphism

$$ \mu_n : M(\mathbb{P}^n) \otimes M(\mathbb{P}^n)(-n)[-2n] \xrightarrow{\delta^*} M(\mathbb{P}^n) \xrightarrow{p_*} \mathbb{1} $$

where $p : \mathbb{P}^n \to S$ is the canonical projection and $\delta : \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$ the diagonal embedding of $\mathbb{P}^n/S$.
If we consider this morphism as a cohomological class in $H^{2n,n}(\mathbb{P}^n \times \mathbb{P}^n)$, it is the fundamental class $\eta_{\mathbb{P}^n \times \mathbb{P}^n}(\mathbb{P}^n) = \delta_*(1) \in H^{2n,n}(\mathbb{P}^n \times \mathbb{P}^n)$ of the diagonal. Using the projective bundle theorem 3.2, it can be written

$$ \eta_{\mathbb{P}^n \times \mathbb{P}^n}(\mathbb{P}^n) = \sum_{0 \leq i,j \leq n} \eta_{i,j}^{(n)}.c^i \cup d^j $$

where $\eta_{i,j}^{(n)}$ is an element in $A^{2(n-i-j),n-i-j}$ and c (resp. d) is the first Chern class of the canonical dual line bundle on the first (resp. second) factor of $\mathbb{P}^n \times \mathbb{P}^n$.
We define the $(n+1)$-dimensional square matrix $M_n = \big(\eta_{i,j}^{(n)}\big)_{0 \leq i,j \leq n}$ over the bigraded ring A. Note that M_n is symmetric. Remark finally that the morphism

induced by adjunction from μ_n gives by another application of theorem 3.2 a morphism

$$\bigoplus_{i=0}^{n} \mathbb{1}(n-i)[2(n-i)] \to \bigoplus_{j=0}^{n} \mathbb{1}(j)[2j]$$

whose matrix is precisely M_n.

Lemma 5.5. *For any integer $i \geq 0$, put $\eta_i = \eta_{ii}^{(i)} \in A^{-2i,-i}$. The matrix M_n has the form*

$$\begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & \iddots & \eta_1 \\ 0 & \iddots & \iddots & \vdots \\ 1 & \eta_1 & \cdots & \eta_n \end{pmatrix}$$

Proof. First remark the lemma is clear when $n = 0$.
Consider the canonical embedding $\sigma : \mathbb{P}^n \to \mathbb{P}^{n+1}$. We apply the excess intersection formula 4.16 in the case of the following square

$$\begin{tikzcd} \mathbb{P}^n \arrow[r, "\delta"] \arrow[d, "\sigma"'] & \mathbb{P}^n \times \mathbb{P}^n \arrow[d, "\sigma\times\sigma"] \\ \mathbb{P}^{n+1} \arrow[r, "\delta'"] & \mathbb{P}^{n+1} \times \mathbb{P}^{n+1}. \end{tikzcd}$$

In this case, the excess of codimension is 1 and the excess intersection bundle on $\mathbb{P}^n$ is the canonical dual line bundle $\lambda_n^\vee$. Proposition 4.16 then gives the formula $(\sigma \times \sigma)^*(\delta'_*(1)) = \delta_*(c_1(\lambda_n^\vee))$.
The projection on the first factor $p_1 : \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ gives a retraction of δ, and consequently, $\delta_*(c_1(\lambda_n^\vee)) = c \cup \delta_*(1)$. Thus the previous relation reads :

$$\sum_{0 \leq i,j \leq n} \eta_{i,j}^{(n+1)}.c^i \cup d^j = \sum_{0 \leq i,j \leq n} \eta_{i,j}^{(n)}.c^{i+1} \cup d^j$$

with the notations which precede the lemma. This in turn gives the relations

$$\begin{cases} \eta_{0,j}^{(n+1)} = 0 & \text{if } 0 \leq j \leq n, \\ \eta_{i,j}^{(n+1)} = \eta_{i-1,j}^{(n)} & \text{if } 0 < i \leq n \text{ and } 0 \leq j \leq n, \end{cases}$$

which allow to conclude by induction on the integer n. $\square$

As a corollary, we obtain from lemma 5.3 that $\mu_n : M(\mathbb{P}^n) \otimes M(\mathbb{P}^n)(-n)[-2n] \to \mathbb{1}$ turns $M(\mathbb{P}^n)(-n)[-2n]$ into a strong dual of $M(\mathbb{P}^n)$.

Definition 5.6. We define the Gysin morphism $p^* : \mathbb{1} \to M(\mathbb{P}^n)(-n)[-2n]$ associated to the projection $p : \mathbb{P}^n \to S$ as the transpose of the morphism $p_* : M(\mathbb{P}^n) \to \mathbb{1}$ with respect to the strong duality on $M(\mathbb{P}^n)$ induced by μ_n.
Moreover, for any smooth scheme X, considering the projection $p_X : \mathbb{P}^n_X \to X$, we define the Gysin morphism associated to p_X as the morphism

$$p_X^* := 1 \otimes p^* : M(X) \to M(\mathbb{P}^n_X)(-n)[-2n].$$

Using property (ii) after definition 5.4, we obtain the following way to compute p^*. Consider the inverse matrix

$$M_n^{-1} = \begin{pmatrix} \eta'_n & \cdots & \eta'_1 & 1 \\ \vdots & \diagup & \diagup & 0 \\ \eta'_1 & \diagup & \diagup & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$

where $\eta'_i \in A^{-2i,-i}$ is given by the determinant of the matrix obtained by removing line 0 and column $n-i$ from M_n times $(-1)^i$. Then

$$p^* : \mathbb{1} \to \bigoplus_{i=0}^{n} \mathbb{1}(i-n)[2(i-n)]$$

is given by the vector $\begin{pmatrix} \eta'_n \\ \vdots \\ \eta'_1 \\ 1 \end{pmatrix}$.

Note we have the fundamental relation in $A^{-2n,-n}$:

$$\sum_{i=0}^{n} \eta_i \cup \eta'_{n-i} = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{5.2}$$

Remark 5.7. The coefficients η_i and η'_i will be determined in proposition 5.30 and corollary 5.31.

5.3. **The Gysin morphism associated to a projective morphism.**

5.3.1. *Preliminary lemmas.*

Lemma 5.8. *Fix a couple of integers $n, m \in \mathbb{N}$ and a smooth scheme X. Consider the projection morphisms*

$$\begin{array}{ccc} \mathbb{P}^n_X \times_X \mathbb{P}^m_X & \xrightarrow{q'} & \mathbb{P}^n_X \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbb{P}^m_X & \xrightarrow{q} & X. \end{array}$$

Then ${q'}^* p^* = {p'}^* q^*$.

Obvious from definition 5.6.

Lemma 5.9. *Consider a closed immersion $i : Z \to X$ between smooth schemes and an integer $n \geq 0$. Consider the pullback square*

$$\begin{array}{ccc} \mathbb{P}^n_Z & \xrightarrow{l} & \mathbb{P}^n_X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow{i} & X. \end{array}$$

Then $l^* p^* = q^* i^*$.

It follows easily from definition 5.6 and lemma 4.12.

Lemma 5.10. *Consider and integer $n \geq 0$ and a smooth scheme X. Consider the canonical projection $p : \mathbb{P}^n_X \to X$.*
Then for any section $s : X \to \mathbb{P}^n_X$ of p, we have : $s^*p^* = 1$.

Proof. Recall from paragraph 4.14 that $s^* = p_* \boxtimes_{\mathbb{P}^n_X} (\pi_{X*}s^*)$ where $\pi_X : X \to S$ is the structural morphism of X/S. We easily obtain the following relation :

$$(p_* \boxtimes_{\mathbb{P}^n_X} 1) \circ p^* = 1 \boxtimes_X p^*.$$

Thus : $s^*p^* = 1 \boxtimes_X (\pi_{X*}s^*p^*)$.
As s is a section of p, it can be written $s = \nu \times 1_X$ for a closed immersion $\nu : X \to \mathbb{P}^n_S$. Consider the following cartesian squares :

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "\nu"'] & \mathbb{P}^n_S \times X \arrow[r, "p"] \arrow[d, "1_{\mathbb{P}^n_S} \times \nu"] & X \arrow[d, "\nu"] \\ \mathbb{P}^n_S \arrow[r, "\delta"] & \mathbb{P}^n_S \times \mathbb{P}^n_S \arrow[r, "\pi"] & \mathbb{P}^n_S \end{tikzcd}$$

where δ is the diagonal embedding and π the canonical projection on the second factor. Using the projection formula for each square – for the first square, this is 4.10, for the second square, it follows easily from definition 5.6 – we obtain : $\nu_*s^*p^* = \delta^*\pi^*\nu_*$.
As $\pi_{X*} = \pi_{\mathbb{P}^n_S*}\nu_*$, we thus are reduced to prove $\delta^*\pi^* = 1$. To conclude, the reader has the choice :

(1) A direct computation shows that the matrix of π^* (resp. δ^*), through the projective bundle isomorphism 3.2, is

$$\left(\delta_i^k.\eta'_{n-j}\right)_{(j,k)\in[0,n]^2,\ i\in[0,n]}$$
$$\text{resp. } \left(\eta_{j+k-l-n}\right)_{l\in[0,n],\ (j,k)\in[0,n]^2}.$$

The fundamental relation (5.2) allows to conclude.
(2) Use definition 5.4 to compute $\pi^* = 1 \otimes p^*$ in terms of the duality pairing $(\mu_n, \epsilon_{\mu_n})$ (cf lemma 5.3). Apply the projection formula 4.10 to compute directly $\delta^*\pi^*$; the second relation of definition 5.2 concludes.
(3) Prove $\delta^* = {}^t(\delta_*)$ using characterization (i) after definition 5.4 (and the usual projection formula 4.10).

□

5.3.2. *Definition.*

Lemma 5.11. *Consider a commutative diagram :*

$$\begin{tikzcd} & \mathbb{P}^n_X \arrow[dr, "p"] & \\ Y \arrow[ur, "k"] \arrow[dr, "i"'] & & X \\ & \mathbb{P}^m_X \arrow[ur, "q"'] & \end{tikzcd}$$

where i (resp. k) is a closed immersion of codimension r (resp. s) and p (resp. q) is the canonical projection. Then, $k^*p^* = i^*q^*$.

Proof. Let us introduce the following morphisms :

$$\begin{tikzcd}
 & & \mathbb{P}^n_X \arrow[dr, "p"] & \\
Y \arrow[urr, bend left, "k"] \arrow[r, "\nu"] \arrow[drr, bend right, "i"'] & \mathbb{P}^n_X \times_X \mathbb{P}^m_X \arrow[ur, "q'"] \arrow[dr, "p'"] & & X. \\
 & & \mathbb{P}^m_X \arrow[ur, "q"'] &
\end{tikzcd}$$

Applying lemma 5.8, we are reduced to prove $k^* = \nu^* {q'}^*$ and $i^* = \nu^* {p'}^*$. In other words, we are reduced to the case $m = 0$ and $q = 1_X$.
In this case, we introduce the following morphisms :

$$\begin{tikzcd}
Y \arrow[drr, bend left, "k"] \arrow[dr, "s"] \arrow[ddr, equal] & & \\
 & \mathbb{P}^n_Y \arrow[r, "l"'] \arrow[d, "q"] & \mathbb{P}^n_X \arrow[d, "p"] \\
 & Y \arrow[r, "i"] & X
\end{tikzcd}$$

Then the lemma follows from lemma 5.9, lemma 5.10 and corollary 4.33. □

Consider smooth schemes X and Y and a projective morphism $f : Y \to X$ of codimension d. Consider an arbitrary factorization $Y \xrightarrow{i} \mathbb{P}^n_X \xrightarrow{p} X$ of f into a closed immersion of codimension $d+n$ and the canonical projection. The preceding lemma shows that the composite morphism

$$M(X) \xrightarrow{p^*} M(\mathbb{P}^n_X)(-n)[-2n] \xrightarrow{i^*} M(Y)(d)[2d]$$

is independent of the chosen factorization.

Definition 5.12. Considering the above notations, we define the Gysin morphism associated to f as the morphism

$$f^* := i^* p^* : M(X) \to M(Y)(d)[2d].$$

5.3.3. *Properties.*

5.13. Let us first remark that, as a corollary of 4.34, we obtain : $(f \times g)^* = f^* \otimes g^*$ for any projective morphisms f and g.

Proposition 5.14. *Consider projective morphisms $Z \xrightarrow{g} Y \xrightarrow{f} X$ between smooth schemes.*
Then $g^* f^* = (fg)^*$.

Proof. We choose a factorization $Y \xrightarrow{i} \mathbb{P}^n_X \xrightarrow{p} X$ (resp. $Z \xrightarrow{j} \mathbb{P}^m_X \xrightarrow{q} X$) of f (resp. fg) and we introduce the diagram

$$\begin{tikzcd}
 & & & \mathbb{P}^m_X \arrow[dddrrr, bend left, "q"] & & & \\
 & & & \mathbb{P}^n_X \times_X \mathbb{P}^m_X \arrow[u, "p'"] \arrow[drr, "q'"] & & & \\
 & \mathbb{P}^m_Y \arrow[urr, "i'"] \arrow[drr, "q''"] & & & & \mathbb{P}^n_X \arrow[dr, "p"] & \\
Z \arrow[ur, "k"] \arrow[rrr, "g"] \arrow[uuurrr, bend left, "j"] & & & Y \arrow[urr, "i"] \arrow[rrr, "f"] & & & X.
\end{tikzcd}$$

in which p' is deduced from p by base change and so on for q' and q''.
Then, by using the factorizations given in the preceding diagram, the proposition follows directly using 5.9, 5.8, 4.33 and finally 5.11. $\square$

Proposition 5.15. *Consider a commutative square of smooth schemes*

$$\begin{tikzcd} T \arrow[r, "k"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

such that i is a closed immersion and f is a projective morphism. Let h be the pullback of f on $X - Z$. Let n, m, s, t be the respective codimension of i, k, f, g. Note that $n + s = m + t$ and put $d = n + s$.
Then the following square is commutative :

$$\begin{tikzcd} M(T)(d)[2d] \arrow[r, "\partial_{Y,T}"] & M(Y-T)(s)[2s+1] \\ M(Z)(n)[2n] \arrow[r, "\partial_{X,Z}"] \arrow[u, "g^*"] & M(X-Z)[1] \arrow[u, "h^*"'] \end{tikzcd}$$

Proof. By construction of the Gysin morphism, we have only to consider the case where f is the projection of a projective bundle or a closed immersion. It follows from lemma 4.12 in the first case and from theorem 4.32 in the second. $\square$

Remark 5.16. Applying the two preceding propositions and case (i) of the following proposition, we obtain that the Gysin triangle is functorial with respect to the Gysin morphism of a projective morphism in the case of a cartesian square as in the preceding statement.

Proposition 5.17. *Consider a cartesian square of smooth schemes*

$$\begin{tikzcd} Y' \arrow[r, "g"] \arrow[d, "q"'] & X' \arrow[d, "p"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

such that f (resp. g) is a projective morphism of codimension n (resp. m). Note that necessarily, $n \geq m$.

(i) *Suppose $n = m$ and $Y \times_X X'$ is smooth (*i.e. *$Y' = Y \times_X X'$).*
   Then $f^* p_* = q_* g^*$.
(ii) *Suppose $Y \times_X X'$ is smooth and $n > m$. Put $e = n - m$. We attach to the above square a vector bundle ξ of rank e called the excess intersection bundle : choose a projective bundle P/X and a factorization $Y \xrightarrow{i} P \xrightarrow{p} X$ of f into a closed immersion followed by the canonical projection. We obtain a canonical embedding $N_{Y'}(P \times_X X') \to q^* N_Y(P)$ and denote by ξ the quotient bundle over Y'. This definition is independent of the choice of the factorization as shown in* [Ful98]*, proof of prop. 6.6.*
   Then, $f^* p_* = \big(q_* \boxtimes c_e(\xi)\big) \circ g^*$.

Proof. In each case, we reduce to the corresponding assertion for a closed immersion (4.10, 4.16 and 4.26) by choosing a factorization of f into a closed immersion followed by a projection and by considering its pullback on X'. Indeed, the assertion (i) when f is the projection of a projective bundle is trivial. $\square$

We obtain finally the analog of corollary 4.11 :

Corollary 5.18. *Let $f : Y \to X$ be a projective morphism between smooth scheme of pure codimension d.*
Then $(1_{Y*} \boxtimes f_*) \circ f^* = f^* \boxtimes 1_{X*}$ as a morphism $M(X) \to M(Y \times X)(d)[2d]$.

The proof is the same as for 4.11 using assertion (i) of the proposition above and the formula of 5.13.

5.19. We now consider the analog of the ramification formula 4.26. Consider a commutative square of smooth schemes

$$\begin{tikzcd} T \arrow[r, "q"] \arrow[d, "g"'] \arrow[dr, phantom, "\Delta"] & Y \arrow[d, "f"] \\ Z \arrow[r, "p"'] & X \end{tikzcd}$$

which is cartesian on the underlying topological spaces and such that p and q are projective morphisms of codimension n. We assume T admits an ample line bundle.
Put $T' = T \times_X Y$ and note the morphism $T \to T'$ induce by Δ is a thickening. Let $T' = \bigcup_{i \in I} T'_i$ (resp. $T'_i = \bigcup_{j \in J_j} T'_{ij}$) be the decomposition into connected (resp. irreducible) components. Put $T_i = T'_i \times_{T'} T$ and $T_{ij} = T'_{ij} \times_{T'} T$. We introduce the following condition on Δ :

(Special) For any $i \in I$, there exists an integer $r_i \geq 0$ such that for any $j \in J_i$, $m(T'_{ij}) = r_i.m(T_{ij})$.

In this case, the integer r_i will be called the *ramification index* of f along T_i.
Consider a factorization $Z \xrightarrow{i} P \xrightarrow{\pi} X$ of p into a closed immersion and the projection of a projective bundle. We put $Q = P \times_X Y$ and consider the obvious morphism of closed pairs $(h, g) : (Q, T) \to (P, Z)$. Of course, Δ satisfies (Special) if and only if (h, g) satisfies (Special). Moreover, for any $i \in I$, the element $r(T_i; h, g)$ is independent of the chosen factorization. Indeed, taking into account the compatibility of F-intersection multiplicity with flat base change, this boils down to the following lemma :

Lemma 5.20. *Consider a commutative diagram of smooth schemes*

$$\begin{tikzcd} & T' \arrow[dl, "t"'] \arrow[d, "g'"] \arrow[drr] & & \\ T \arrow[rrr] \arrow[d, "g"'] & Z' \arrow[dl, "s"'] \arrow[drr] & & Y \arrow[d, "f"] \\ Z \arrow[rrr] & & & X \end{tikzcd}$$

such that T and T' are connected and admits an ample line bundle, $t = s \times_Z T$ and (f, g) (resp. (f, g')) is a morphism of smooth closed pairs satisfying condition (Special) with ramification index r.
Then, $r(T'; f, g') = t^* r(T; f, g) \in H^{0,0}(T')$.

Proof. Consider the blow-up $B = B_Z(\mathbb{A}^1_X)$ (resp. $B' = B_{Z'}(\mathbb{A}^1_X)$) and its exceptional divisor P (resp. P'). As $Z' \subset Z$, we get a cartesian transversal square, together with its pullback over Y

$$\begin{tikzcd} P' \arrow[r] \arrow[d] & B', \arrow[d] \\ P \arrow[r] & B. \end{tikzcd} \quad \text{pullback above } Y: \quad \begin{tikzcd} Q' \arrow[r] \arrow[d] & C'. \arrow[d] \\ Q \arrow[r] & C \end{tikzcd}$$

The second square is still transversal. Put $L = N_Q C|_Z$ and $L' = N_{Q'}(C')|_{Z'}$. Thus, $L' = L|_{Z'}$. According to this equality, the lemma follows from the definition of F-intersection multiplicities. □

Definition 5.21. Consider the notations and hypothesis of 5.19, assuming the square Δ satisfies condition (Special). For any $i \in I$, we define the F-intersection multiplicity $r(T_i; \Delta)$ of T_i in the pullback square Δ as the well defined element $r(T_i; h, g)$ according to the notations above.

The following proposition is now a corollary of 4.26 :

Proposition 5.22. *Consider the hypothesis and notations of the preceding definition. Put $g_i = g|_{T_i}$ and $q_i = q|_{T_i}$.*
Then, $p^* f_* = \sum_{i \in I} \big(r(T_i; \Delta) \boxtimes g_{i*}\big) q_i^*$.

5.4. **The duality pairing.** Let X/S be a smooth projective scheme of pure dimension n. Let $p : X \to S$ (resp. $\delta : X \to X \times X$) be its structural morphism (resp. its diagonal embedding).
Then we define morphisms

$$\mu_X : \mathbb{1} \xrightarrow{p^*} M(X)(-n)[-2n] \xrightarrow{\delta_*} M(X \times X)(-n)[-2n] = M(X)(-n)[-2n] \otimes M(X)$$

$$\epsilon_X : M(X) \otimes M(X)(-n)[-2n] \xrightarrow{\delta^*} M(X) \xrightarrow{p_*} \mathbb{1}.$$

The following result is now a formality :

Theorem 5.23. *Consider the notations above.*
Then $\big(M(X)(-n)[-2n], \mu_X, \epsilon_X\big)$ is a strong dual of $M(X)$.

Proof. Each identity of definition 5.2 is an easy application of 5.13, 5.17(i) (the usual projection formula) and proposition 5.14. □

5.24. *Applications* : Consider the notations of the previous proposition and let $\mathbb{E}$ be a motive.

(1) We define the fundamental class $\tau_X \in H_{2n,n}(X)$ of X as the element

$$p^* : \mathbb{1} \to M(X)(-n)[-2n].$$

We also consider $\eta \in H^{2n,n}(X \times X)$ the fundamental class of the diagonal δ.

Then the isomorphisms of (5.1) with $M = M(X)$ gives exactly, considering the definitions of cap-product and slant product (cf 2.9), the following reciprocal isomorphisms :

$$\begin{aligned} \mathbb{E}^{r,p}(X) &\leftrightarrows \mathbb{E}_{2n-r,p-n}(X) \\ x &\mapsto x \cap \tau_X \\ \eta/y &\leftarrow\!\shortmid y. \end{aligned} \tag{5.3}$$

This is the *Poincaré duality isomorphism*, as it appears in algebraic topology (cf [Ada74], [Swi02, 14.41, 14.42]). To the knowledge of the author, the first appearance of this precise form of duality in algebraic geometry is in [PY02].

(2) Suppose $\mathbb{E}$ is a ringed motive. In this case, the regulator maps

$$\varphi_X : H^{2n,n}(X) \to \mathbb{E}^{2n,n}(X)$$
$$\psi_X : H_{2n,n}(X) \to \mathbb{E}_{2n,n}(X)$$

allow to define the fundamental class of X (resp. the fundamental class of the diagonal) with coefficients in $\mathbb{E}$ as the image $\psi_X(\tau_X)$ (resp. $\varphi_X(\eta)$) of the corresponding class with coefficients in H. Moreover, we can obviously express the isomorphisms above with this classes (cf number (1) above), obtaining a Poincaré duality purely in terms of the cohomology theory $\mathbb{E}^{**}$.

(3) Suppose $\mathbb{E}$ is a ringed motive.

The morphism

$$p_* : \mathbb{E}^{**}(X) \to A$$

induced by the Gysin morphism of p is usually called the *trace morphism* (relative to S).

We suppose the cohomology $\mathbb{E}^{**}$ satisfies the following Künneth property : for any motives $M, N, P \in \{\mathbb{1}, M(X), M(X)(-n)[-2n]\}$, the pairing

$$\mathbb{E}^{**}(M) \otimes_A \mathbb{E}^{**}(N) \otimes_A \mathbb{E}^{**}(P) \to \mathbb{E}^{**}(M \otimes N \otimes P)$$

is an isomorphism.

Then it follows formally that

$$\big(\mathbb{E}^{**}(M(X)(-n)[-2n]), \mathbb{E}^{**}(\mu), \mathbb{E}^{**}(\epsilon)\big)$$

is a strong dual of $\mathbb{E}^{**}(M(X))$ in the category of graded A-modules.

More concretely, the pairing (induced by $\mathbb{E}^{**}(\mu)$)

$$\mathbb{E}^{**}(X) \otimes_A \mathbb{E}^{**}(X) \to A, x \otimes y \mapsto p_*(x \cup y)$$

is a perfect pairing of graded A-modules. This is usually called the *Poincaré duality pairing*[21] for the cohomology theory $\mathbb{E}^{**}$.

Note it implies that $E^{**}(X)$ is a projective finitely generated graded A-module (see [DP80, 1.4]).

Example 5.25. The conditions of point (3) are fulfilled when X is a Grassmanian scheme over S, or more generally a cellular variety over S, without any assumption on $\mathbb{E}$. In [CD06], we study cohomology theories $\mathbb{E}^{**}$ which satisfies the Künneth formula.

[21] This way of deducing the Poincaré duality pairing from the abstract duality of theorem 5.23 and the Künneth formula was explained to me by D.C.Cisinski.

The Gysin morphism determine the duality pairing defined above. Reciprocally, this duality determines the Gysin morphism as shown in the next proposition.

Proposition 5.26. *Let $f : Y \to X$ be a morphism between smooth projective S-schemes. Suppose X (resp. Y) is of constant relative dimension n (resp. m) over S.*
Then
$$f^* = {}^t(f_*)(-n)[-2n]$$
where the transpose morphism on the right hand side is taken with respect to the strong duals of $M(X)$ and $M(Y)$ obtained in the previous theorem.

Proof. Consider the structural projections $p : X \to S$, $q : Y \to S$ and the diagonal embeddings $\delta_X : X \to X \times X$, $\delta_Y : Y \to Y \times Y$. Let n be the dimension of X Put $M(X)^\vee = M(X)(-n)[-2n]$ and $M(Y)^\vee = M(Y)(-m)[-2m]$.
According to the first point which follows definition 5.4, we have to prove the following square is commutative :

$$\begin{tikzcd} M(Y) \otimes M(X)^\vee \arrow[r, "f_* \otimes 1"] \arrow[d, "1 \otimes f^*"'] & M(X) \otimes M(X)^\vee \arrow[d, "p_* \delta_X^*"] \\ M(Y) \otimes M(Y)^\vee \arrow[r, "q_* \delta_Y^*"] & \mathbb{1}. \end{tikzcd}$$

We introduce the following cartesian square :

$$\begin{tikzcd} Y \arrow[r, "\gamma"] \arrow[d, "f"'] & Y \times X \arrow[d, "f \times 1"] \\ X \arrow[r, "\delta_X"] & X \times X \end{tikzcd}$$

Note that $f_* \otimes 1 = (f \times 1)_*$ and $1 \otimes f^* = (1 \times f)^*$ (cf 5.13). The result follows from the computation :

$$p_* \delta_X^* (f \times 1)_* = p_* f_* \gamma^* = q_* \delta_Y^* (1 \times f)^*$$

which uses 5.17(i) and 5.14. $\square$

5.5. **Two illustrations.**

5.27. *Cobordism classes.—*

Definition 5.28. Let X be a smooth projective scheme of pure dimension n. Let $p : X \to S$ be its structural projection.
We define the cobordism class of X/S as the element of A, of (cohomological) degree $(-2n, -n)$,

$$[X] = \mathbb{1} \xrightarrow{p^*} M(X)(-n)[-2n] \xrightarrow{p_*} \mathbb{1}(-n)[-2n].$$

In other words, $[X] = p_*(1)$ as a cohomological class. Note that according to definition 5.6 and what follows it, we obtain that $[\mathbb{P}^n] = \eta'_n$. Note also that $[X \sqcup Y] = [X] + [Y]$ (from axiom (Add)) and $[X \times_S Y] = [X] \cup [Y]$ (from 5.13).

Example 5.29. Consider a factorization $X \xrightarrow{i} \mathbb{P}^N \xrightarrow{\pi} S$ of the morphism p into a closed immersion followed by the canonical projection. Let $c = N - n$ be the codimension of i. Let $\eta_{\mathbb{P}^N}(X) \in H^{2c,c}(\mathbb{P}^N)$ be the fundamental class associated to the embedding i. Then from corollary 4.33, $[X] = p_*(\eta_{\mathbb{P}^N}(X))$ (as a cohomological class).

Thus, to compute this cobordism class, we can use the projective bundle theorem, which implies we can write $\eta_{\mathbb{P}^N}(X) = \sum_{i=0}^N x_i.c^i$ where c is the Chern class of the dual canonical line bundle, and x_i is an element of A. Then,

$$[X] = \sum_{i=0}^N x_i \cup [\mathbb{P}^{N-i}]$$

as $p_*(c^i) = [\mathbb{P}^{N-i}]$ according to definition 5.6.

We want to compute now the cobordism class $[\mathbb{P}^n]$. Let $\delta : \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$ be the diagonal embedding. According to definition 5.6, we have to compute

$$\delta_*(1) = \sum_{0 \leq i,j \leq n} \eta_{i+j-n}.c^i \cup d^j \tag{5.4}$$

with the notations preceding lemma 5.5.

Let $p_1, p_2 : \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ be the projections respectively on the first and second factor. Let λ (resp. ξ) be the canonical line bundle (resp. quotient bundle) on $\mathbb{P}^n$. Consider the vector bundles $\lambda^{(i)} = p_i^{-1}(\lambda)$ and $\xi^{(i)} = p_i^{-1}(\xi)$ for $i = 1, 2$. In the preceding expression, $c = c_1(\lambda_1^\vee)$ and $d = c_1(\lambda_2^\vee)$. Put $E = \underline{\mathrm{Hom}}(\lambda_1, \xi_2) = \lambda_1^\vee \otimes \xi_2$. We get a section s of this vector bundle considering the canonical morphism

$$\lambda_1 \to \mathbb{A}^{n+1} \times \mathbb{P}^n = \mathbb{P}^n \times \mathbb{A}^{n+1} \to \xi_2.$$

It is well known (see [PSP]) that $\delta(\mathbb{P}^n)$ is the subscheme defined by $s = 0$. Thus according to corollary 4.21, $\delta_*(1) = c_n(E) = c_n(\lambda_1^\vee \otimes \xi_2)$. From this expression, we obtain easily :

(1) *Additive case* : When the formal group law is additive[22] (*i.e.* $F(x, y) = x + y$), according to a well known formula (cf [Ful98, ex. 3.2.2]),

$$c_n(\lambda_1 \otimes \xi_2) = \sum_{i=0}^n c_1(\lambda_1)^i \cup c_{n-i}(\xi_2) = \sum_{i=0}^n c^i \cup d^{n-i}.$$

Thus, $[\mathbb{P}^n] = 0$ if $n > 0$.

(2) *Case* $n = 1$: As $c^2 = d^2 = 0$, we simply obtain :
$c_1(\lambda_1 \otimes \xi_2) = F(c_1(\lambda_1), c_1(\xi_2)) = c + d + a_{1,1}.c \cup d.$
Thus $\eta_1 = a_{1,1}$ which implies $[\mathbb{P}^1] = -a_{1,1}$.

In the general case, we obtain the following computation :

Proposition 5.30. *With the notations introduced above,*

$$\delta_*(1) = \sum_{0 \leq i,j \leq n} a_{1,i+j-n}.c^i \cup d^j.$$

Proof. Consider the ind-scheme $\mathbb{P}^\infty \times \mathbb{P}^n$ and the embedding $\tau : \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^\infty \times \mathbb{P}^n$. Let $\tilde{p}_1$ (resp. p_2) be the projection on the first (resp. second) factor of $\mathbb{P}^\infty \times \mathbb{P}^n$. Put $\tilde{\lambda}_1 = \tilde{p}_1^{-1}(\lambda)$, $\lambda_2 = p_2^{-1}(\lambda)$ and $\xi_2 = p_2^{-1}(\xi)$. Thus, with a little abuse of notation, $c_n(\lambda_1 \otimes \xi_2) = \tau^* c_n\left(\tilde{\lambda}_1 \otimes \xi_2\right)$.

[22] This is the case for the category $DM(S)$.

According to the definition of ξ, we consider the short exact sequence :

$$0 \to \tilde{\lambda}_1^\vee \otimes \lambda_2 \to \tilde{\lambda}_1^\vee \otimes \mathbb{A}^{n+1} \to \tilde{\lambda}_1^\vee \otimes \xi_2 \to 0.$$

From the Whitney sum formula 3.13, we thus obtain the relation :

$$c_{n+1}(\tilde{\lambda}_1^\vee \otimes \mathbb{A}^{n+1}) = c_1(\tilde{\lambda}_1^\vee \otimes \lambda_2) \cup c_n(\tilde{\lambda}_1^\vee \otimes \xi_2).$$

Put $\tilde{c} = c_1(\tilde{\lambda}_1)$, $d = c_1(\lambda_2)$ as cohomology classes in $B = H^{**}(\mathbb{P}^\infty \times \mathbb{P}^n)$. Note moreover the A-algebra B is equal to $(A[d]/d^{n+1})[[\tilde{c}]]$. In terms of the fundamental group law F and its inverse power series m, the preceding relation reads[23] $\tilde{c}^{n+1} = F(\tilde{c}, m(d)) \cup c_n(\tilde{\lambda}_1^\vee \otimes \xi_2)$.
We have to prove :

$$c_n(\tilde{\lambda}_1^\vee \otimes \xi_2) = \sum_{0 \leq i,j \leq n} a_{1,i+j-n}.\tilde{c}^i \cup d^j \mod \tilde{c}^{n+1}.$$

Let $m(x) = \sum_{k>0} m_k.x^k$ (thus $m_1 = -1, m_2 = a_{1,1}$, etc). For any integers l, s, we put

$$M_{l,s} = \sum_{\substack{k_1+\ldots+k_l=s \\ k_1,\ldots,k_l>0}} m_{k_1}...m_{k_l}$$

when $(l,s) \neq (0,0)$, and $M_{0,0} = 1$. Thus, $F(\tilde{c}, m(d)) = \sum_{k,l,s} a_{k,l} M_{l,s}.\tilde{c}^k \cup d^s$. In particular, $F(\tilde{c}, m(d)) = u.\tilde{c} + v$ where u is invertible in B and v is nilpotent. This implies $F(\tilde{c}, m(d))$ is a non zero divisor in B and we are reduced to prove :

$$F(\tilde{c}, m(d)) \cup \sum_{0 \leq i,j \leq n} a_{1,i+j-n}.\tilde{c}^i \cup d^j = 0 \mod \tilde{c}^{n+1}.$$

The left hand side can be expanded (modulo $\tilde{c}^{n+1}$) as the sum :

$$\sum_{0 \leq u,v \leq n} \left(\sum_{k,l,s} a_{k,l} M_{l,s} a_{1,u+v-n-k-s} \right) .\tilde{c}^u \cup d^v.$$

Finally, for any integers $u, v \in [0, n]$, the coefficient of $\tilde{c}^u \cup d^v$ in the preceding sum can be written

$$\sum_w \left(\sum_{k,l} a_{k,l} M_{l,w-k} \right) a_{1,u+v-n-w}.$$

This is zero according to the relation $F(x, m(x)) = 0$. □

From definition 5.6, the previous proposition reads $\eta_i = a_{1,i}$. As a corollary (cf relation (5.2)), we recover the classical Myschenko theorem together with a nice expression of $[\mathbb{P}^n]$ as a determinant :

[23] This expression for computing $\delta_*(1)$ was also obtained in [Pan03b].

Corollary 5.31. (1) *For any integer $n \geq 0$,*

$$[\mathbb{P}^n] = (-1)^n.\det\begin{pmatrix} 0 & \cdots & 0 & 1 & a_{1,1} \\ \vdots & & & & a_{1,2} \\ 0 & & & & \vdots \\ 1 & & & & \\ a_{1,1} & a_{1,2} & \cdots & & a_{1,n} \end{pmatrix}.$$

(2) *For any integer $n > 0$, $\sum_{0 \leq i \leq n} a_{1,i}.[\mathbb{P}^{n-i}] = 0$.*

The usual formulation of the relations given in (2) uses the series $p(x) = \sum_i [\mathbb{P}^i].x^i$ and $\omega(x) = \frac{\partial F}{\partial y}(x, 0)$. It reads $p(x) = \omega(x)^{-1}$.

Remark 5.32. An interesting problem is to extend this computation to the case of an arbitrary projective bundle $\mathbb{P}(E)$. We hope the fundamental class $\eta_{\mathbb{P}(E)\times\mathbb{P}(E)}(\mathbb{P}(E))$ as an explicit description in terms of the coefficients $a_{1,i}$ and the Chern classes of E which would give an expression of $[\mathbb{P}(E)]$ as a determinant analog of the above. This will give a counter-part to a classical formula of Quillen.

5.33. *Blow-up formulas.—*

Proposition 5.34. *Let (X, Z) be a smooth closed pair and B be the blow-up of X with center Z. Let $f : B \to X$ be the canonical projection.*
Then, $f_* f^* = 1$.

Proof. Let s_1 (resp. s_0, π) be the unit section (resp. zero section, canonical projection) of $\mathbb{A}^1_X/X$. Let B' be the blow-up of $\mathbb{A}^1_X$ with center $0 \times Z$. We consider the following cartesian square :

$$\begin{tikzcd} X \arrow[r, "\bar{\sigma}_1"] \arrow[d, equal] & B' \arrow[d, "f'"] \\ X \arrow[r, "s_1"] & \mathbb{A}^1_X. \end{tikzcd}$$

From the projection formula 5.17(i), we obtain $f'^* s_{1*} = \bar{\sigma}_{1*}$ which implies $f'^* = \bar{\sigma}_{1*}\pi_*$ by the axiom (Htp).
Thus we deduce easily : $f'_* f'^* = f'_* \bar{\sigma}_{1*}\pi_* = s_{1*}\pi_* = 1$.
Finally we consider the cartesian diagram :

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "\nu"'] & X \arrow[d, "s_0"] \\ B' \arrow[r, "f'"] & \mathbb{A}^1_X \end{tikzcd}$$

Using once again the projection formula *loc. cit.* we get : $f'_* f'^* s_{0*} = s_{0*} f_* f^*$. This concludes using axiom (Htp). □

Lemma 5.35. *Let P/X be a projective bundle over a smooth scheme X of pure dimension d. Let ξ be the canonical quotient bundle of P/X and put $e = c_d(\xi)$ seen as a morphism $M(P) \to \mathbb{1}(d)[2d]$.*
Then, $(p_* \boxtimes e) \circ p^*$ is an isomorphism.

Proof. Using the projection formula 5.18, we have to prove that the cohomological class $p_*(e) \in H^{00}(X)$ is a unit. By compatibility of this class with base change and invariance under isomorphisms of projective bundles, we reduce to the case of $P = \mathbb{P}^d_S$. Let $s : S \to \mathbb{P}^d_S$ be the canonical section. Then, $e = s_*(1)$ (cf remark 4.5 combined with example 4.7). Thus, following lemma 5.10, $p_*(e) = 1$. □

Remark 5.36. In the case of an additive formal group law, we can easily see that $p_*(e) = 1$ for any projective bundle P/X which implies the composite isomorphism of the lemma is just the identity.

5.37. Let X be a smooth scheme, Z be a smooth closed subscheme of X of pure codimension n. Let B be the blow-up of X with center Z and P be the exceptional divisor. Consider the cartesian squares :

$$\begin{tikzcd} P \arrow[r, "k"] \arrow[d, "p"'] & B \arrow[d, "f"] & B-P \arrow[l, "l"'] \arrow[d, "h"] \\ Z \arrow[r, "i"] & X & X-Z. \arrow[l, "j"'] \end{tikzcd}$$

We let λ (resp. $\xi = p^{-1}(N_Z X)/\lambda$) be the canonical line bundle (resp. quotient bundle) on $P = \mathbb{P}(N_Z X)$ and we put : $e = c_{n-1}(\xi)$.

Proposition 5.38. *Using the notations above, the short sequence*

$$0 \to M(P) \xrightarrow{\binom{p_*}{k_*}} M(Z) \oplus M(B) \xrightarrow{(-i_*, f_*)} M(X) \to 0$$

is split exact with splitting $\binom{0}{f^*}$.
By abuse of notation, we denote by $M(P/Z)$ the kernel[24] *of the split monomorphism p_* and let $\tilde{k}_* : M(P/Z) \to M(B)$ be the morphism induced by k_*. Then, we obtain an isomorphism*

$$M(P/Z) \oplus M(X) \xrightarrow{(\tilde{k}_*, f^*)} M(B).$$

Proof. The previous short sequence is obviously a complex. The fact $\binom{0}{f^*}$ is a splitting is proposition 5.34.
We directly prove the last assertion of the proposition which then concludes. Consider the following diagram :

$$\begin{tikzcd} M(X-Z) \arrow[r, "\binom{0}{j_*}"] \arrow[d, "h^*"'] \arrow[dr, phantom, "(1)"] & \begin{matrix} M(P/Z) \\ \oplus \\ M(X) \end{matrix} \arrow[r, "\begin{pmatrix} 1 & 0 \\ 0 & i^* \end{pmatrix}"] \arrow[d, "(\tilde{k}_*, f^*)"] \arrow[dr, phantom, "(2)"] & \begin{matrix} M(P/Z) \\ \oplus \\ M(Z)(n)[2n] \end{matrix} \arrow[r, "(0, \partial_{X,Z})"] \arrow[d, "(k^* \tilde{k}_*, p^*)"] \arrow[dr, phantom, "(3)"] & M(X-Z)[1] \arrow[d, "h^*"] \\ M(B-P) \arrow[r, "l_*"'] & M(B) \arrow[r, "k^*"'] & M(P)(1)[2] \arrow[r, "\partial_{B,P}"'] & M(B-P)[1] \end{tikzcd}$$

The two horizontal lines are distinguished triangles. It is commutative : for square (1), use the projection formula 5.17(i), for square (2), the functoriality of the Gysin

[24] If we had a splitting $s : Z \to P$ of p, this will be the motive associated to the immersion s.

morphism 5.14, for square (3), the compatibility of residues and Gysin morphism 5.15 and the defining property of the residue $\partial_{B,P}$.
As h is an isomorphism, we are reduced to prove $(k^*\tilde{k}_*, p^*)$ is an isomorphism.
The normal bundle of $k : P \to B$ is the canonical line bundle λ. Thus, from the self-intersection formula 4.19, $k^*k_* = 1_{P*} \boxtimes c$ with $c = c_1(\lambda)$. The remaining assertion is local in X so that we can assume that $N_Z X$ is trivializable. Finally, we compute easily the matrix of

$$(k^*k_*, p^*) : \oplus_{i=0}^{n-1} M(Z)(i)[2i] \oplus M(Z)(n)[2n] \to \oplus_{i=0}^{n-1} M(Z)(i+1)[2i+1]$$

obtained through the projective bundle isomorphism 3.2 :

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & [\mathbb{P}^n] \\ \vdots & & \ddots & \ddots & \vdots & [\mathbb{P}^{n-1}] \\ & & & \ddots & 0 & \vdots \\ & & & & 1 & \\ 0 & \cdots & & \cdots & 0 & 1 \end{pmatrix}.$$

As the matrix of $(k^*\tilde{k}_*, p^*)$ is obtained from the above one removing the first column, it is obviously invertible. $\square$

Proposition 5.39. *Consider the notations 5.37. The short sequence*

$$0 \to M(B) \xrightarrow{\binom{k^*}{f_*}} M(P)(1)[2] \oplus M(X) \xrightarrow{(p_* \boxtimes e, -i^*)} M(Z)(n)[2n] \to 0$$

is split exact with pseudo-splitting $\binom{p^*}{0}$.
Let C be the cokernel of the split mono $p^* : M(Z)(n-1)[2n-2] \to M(P)$ and $\tilde{k}^* : M(B) \to C(1)[2]$ the morphism induced by k^*. Then the following morphism is an isomorphism :

$$M(B) \xrightarrow{\binom{\tilde{k}^*}{f_*}} C(1)[2] \oplus M(X).$$

Remark 5.40. This second blow-up formula is a generalization of [Ful98, 6.7(a)]. In case X and Z are projective smooth, it is simply the dual statement of the previous proposition using 5.26. More precisely, from 5.38 (resp. 5.39) the morphism

$$\begin{pmatrix} k_* & p_* \\ f^* & 0 \end{pmatrix} \text{ (resp. } \begin{pmatrix} k^* & f_* \\ p^* & 0 \end{pmatrix})$$

is an isomorphism. These two matrices are dual.

Proof. The above sequence is a complex from the excess intersection formula 4.16 applied to the morphism (f, p). The pseudo-splitting of this sequence is exactly lemma 5.35. We thus are reduced to the last assertion.
Let $\pi : M(P)(1)[2] \to C(1)[2]$ be the canonical projection. Consider the following

diagram :

$$\begin{tikzcd}[column sep=large]
M(B-P) \arrow[r, "l_*"] \arrow[d, "h^*"'] \arrow[dr, phantom, "(1)"] & M(B) \arrow[r, "k^*"] \arrow[d, "\binom{\tilde{k}^*}{f_*}"'] \arrow[dr, phantom, "(2)"] & M(P)(1)[2] \arrow[r, "\partial_{B,P}"] \arrow[d, "\binom{\pi}{p_* \boxtimes e}"] \arrow[dr, phantom, "(3)"] & M(B-P)[1] \arrow[d, "h^*"] \\
M(X-Z) \arrow[r, "\binom{0}{j_*}"'] & \begin{matrix} C(1)[2] \\ \oplus \\ M(X) \end{matrix} \arrow[r, "\begin{pmatrix} 1 & 0 \\ 0 & i^* \end{pmatrix}"'] & \begin{matrix} C(1)[2] \\ \oplus \\ M(Z)(n)[2n] \end{matrix} \arrow[r, "(0, \partial_{X,Z})"'] & M(X-Z)[1]
\end{tikzcd}$$

The horizontal lines are distinguished triangles. The diagram is commutative : (1) follows from definitions, (2) is a consequence of the excess intersection formula 4.16 for (f, p) and (3) is a consequence of the same formula, considered for residues. Finally, we are reduced to prove that $\binom{\pi}{p_* \boxtimes e}$ is an isomorphism. But $\mathrm{coKer}(p^*) \simeq \mathrm{Ker}(p_* \boxtimes e)$ by a canonical isomorphism so that the latter morphism is simply the decomposition isomorphism associated to the split epimorphism $p_* \boxtimes e$. $\square$

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