Andreotti–Mayer Loci and the Schottky Problem

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Abstract. We prove a lower bound for the codimension of the Andreotti-Mayer locus $N_{g,1}$ and show that the lower bound is reached only for the hyperelliptic locus in genus 4 and the Jacobian locus in genus 5. In relation with the intersection of the Andreotti-Mayer loci with the boundary of the moduli space $\mathcal{A}_g$ we study subvarieties of principally polarized abelian varieties (B, Ξ) parametrizing points b such that Ξ and the translate Ξ_b are tangentially degenerate along a variety of a given dimension.

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1. Introduction

The Schottky problem asks for a characterization of Jacobian varieties among all principally polarized abelian varieties. In other words, it asks for a description of the Jacobian locus $\mathcal{J}_g$ in the moduli space $\mathcal{A}_g$ of all principally polarized abelian varieties of given dimension g. In the 1960's Andreotti and Mayer (see [2]) pioneered an approach based on the fact that the Jacobian variety of a non-hyperelliptic (resp. hyperelliptic) curve of genus $g \geq 3$ has a singular locus of dimension $g-4$ (resp. $g-3$). They introduced the loci $N_{g,k}$ of principally polarized abelian varieties (X, Θ_X) of dimension g with a singular locus of Θ_X of dimension $\geq k$ and showed that $\mathcal{J}_g$ (resp. the hyperelliptic locus $\mathcal{H}_g$) is an irreducible component of $N_{g,g-4}$ (resp. $N_{g,g-3}$). However, in general there are more irreducible components of $N_{g,g-4}$ so that the dimension of the singular locus of Θ_X does not suffice to characterize Jacobians or hyperelliptic Jacobians. The locus $N_{g,0}$ of abelian varieties with a singular theta divisor has codimension 1 in $\mathcal{A}_g$ and in a beautiful paper (see [27]) Mumford calculated its class. But in general not much is known about these Andreotti-Mayer loci $N_{g,k}$. In particular, we do not even know their codimension. In this paper we give estimates for the codimension of these loci. These estimates are in general not sharp, but we think that the following conjecture gives the sharp bound.

Conjecture 1.1. *If $1 \leq k \leq g-3$ and if N is an irreducible component of $N_{g,k}$ whose general point corresponds to an abelian variety with endomorphism ring $\mathbb{Z}$ then* $\mathrm{codim}_{\mathcal{A}_g}(N) \geq \binom{k+2}{2}$. *Moreover, equality holds if and only if one of the following happens:*

(i) *$g = k+3$ and $N = \mathcal{H}_g$;*
(ii) *$g = k+4$ and $N = \mathcal{J}_g$.*

We give some evidence for this conjecture by proving the case $k = 1$. In our approach we need to study the behaviour of the Andreotti-Mayer loci at the boundary of the compactified moduli space. A principally polarized $(g-1)$-dimensional abelian variety (B, Ξ) parametrizes semi-abelian varieties that are extensions of B by the multiplicative group $\mathbb{G}_m$. This means that B maps to a part of the boundary of the compactified moduli space $\tilde{\mathcal{A}}_g$ and we can intersect B with the Andreotti-Mayer loci. This motivates the definition of loci $N_k(B, \Xi) \subset B$ for a principally polarized $(g-1)$-dimensional abelian variety (B, Ξ). They are formed by the points b in B such that Ξ and its translate Ξ_b are 'tangentially degenerate' (see Section 11 below) along a subvariety of dimension k. These intrinsically defined subvarieties of an abelian variety are interesting in their own right and deserve further study. The conjecture above then leads to a boundary version that gives a new conjectural answer to the Schottky problem for simple abelian varieties.

Conjecture 1.2. *Let $k \in \mathbb{Z}_{\geq 1}$. Suppose that (B, Ξ) is a simple principally polarized abelian variety of dimension g not contained in $N_{g,i}$ for all $i \geq k$. Then there is an irreducible component Z of $N_k(B, \Xi)$ with* $\mathrm{codim}_B(Z) = k+1$ *if and only if one of the following happens:*

(i) *either $g \geq 2$, $k = g-2$ and B is a hyperelliptic Jacobian,*
(ii) *or $g \geq 3$, $k = g-3$ and B is a Jacobian.*

In our approach we will use a special compactification $\tilde{\mathcal{A}}_g$ of $\mathcal{A}_g$ (see [29, 28, 5]). The points of the boundary $\partial\tilde{\mathcal{A}}_g = \tilde{\mathcal{A}}_g - \mathcal{A}_g$ correspond to suitable compactifications of g-dimensional semi-abelian varieties. We prove Conjecture 1.1 for $k = 1$ by intersecting with the boundary. For higher values of k the intersection with the boundary looks very complicated.

2. The universal theta divisor

Let $\pi : \mathcal{X}_g \to \mathcal{A}_g$ be the universal principally polarized abelian variety of relative dimension g over the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g over $\mathbb{C}$. In this paper we will work with orbifolds and we shall identify $\mathcal{X}_g$ (resp. $\mathcal{A}_g$) with the orbifold $\mathrm{Sp}(2g,\mathbb{Z}) \ltimes \mathbb{Z}^{2g} \backslash \mathbb{H}_g \times \mathbb{C}^g$ (resp. with $\mathrm{Sp}(2g,\mathbb{Z}) \backslash \mathbb{H}_g$), where

$$\mathbb{H}_g = \{(\tau_{ij}) \in \mathrm{Mat}(g \times g, \mathbb{C}) : \tau = \tau^t, \mathrm{Im}(\tau) > 0\}$$

is the usual *Siegel upper-half space* of degree g. The τ_{ij} with $1 \leq i \leq j \leq g$ are coordinates on $\mathbb{H}_g$ and we let $z_1, ..., z_g$ be coordinates on $\mathbb{C}^g$.
The *Riemann theta function* $\vartheta(\tau, z)$, given on $\mathbb{H}_g \times \mathbb{C}^g$ by

$$\vartheta(\tau, z) = \sum_{m \in \mathbb{Z}^g} e^{\pi i [m^t \tau m + 2m^t z]},$$

is a holomorphic function and its zero locus is an effective divisor $\tilde{\Theta}$ on $\mathbb{H}_g \times \mathbb{C}^g$ which descends to a divisor Θ on $\mathcal{X}_g$. If the abelian variety X is a fibre of π, then we let Θ_X be the restriction of Θ to X. Note that since $\theta(\tau, z)$ satisfies $\theta(\tau, -z) = \theta(\tau, z)$, the divisor Θ_X is symmetric, i.e., $\iota^*(\Theta_X) = \Theta_X$, where $\iota = -1_X : X \to X$ is multiplication by -1 on X. The divisor Θ_X defines the line bundle $\mathcal{O}_X(\Theta_X)$, which yields the principal polarization on X. The isomorphism class of the pair (X, Θ_X) represents a point ζ of $\mathcal{A}_g$ and we will write $\zeta = (X, \Theta_X)$. Similarly, it will be convenient to identify a point ξ of Θ with the isomorphism class of a representative triple (X, Θ_X, x), where $\zeta = (X, \Theta_X)$ represents $\pi(\xi) \in \mathcal{A}_g$ and $x \in \Theta_X$.
The tangent space to $\mathcal{X}_g$ at a point ξ, with $\pi(\xi) = \zeta$, will be identified with the tangent space $T_{X,x} \oplus T_{\mathcal{A}_g,\zeta} \cong T_{X,0} \oplus \mathrm{Sym}^2(T_{X,0})$. If $\xi = (X, \Theta_X, x)$ corresponds to the $\mathrm{Sp}(2g, \mathbb{Z}) \ltimes \mathbb{Z}^{2g}$-orbit of a point $(\tau_0, z_0) \in \mathbb{H}_g \times \mathbb{C}^g$, then the tangent space $T_{\mathcal{X}_g,\xi}$ to $\mathcal{X}_g$ at ξ can be identified with the tangent space to $\mathbb{H}_g \times \mathbb{C}^g$ at (τ_0, z_0), which in turn is naturally isomorphic to $\mathbb{C}^{g(g+1)/2+g}$, with coordinates (a_{ij}, b_ℓ) for $1 \leq i, j \leq g$ and $1 \leq \ell \leq g$ that satisfy $a_{ij} = a_{ji}$. We thus view the a_{ij}'s as coordinates on the tangent space to $\mathbb{H}_g$ at τ_0 and the b_ℓ's as coordinates on the tangent space to X or its universal cover.
An important remark is that by identifying the tangent space to $\mathcal{A}_g$ at $\zeta = (X, \Theta_X)$ with $\mathrm{Sym}^2(T_{X,0})$, we can view the projectivized tangent space $\mathbb{P}(T_{\mathcal{A}_g,\zeta}) \cong \mathbb{P}(\mathrm{Sym}^2(T_{X,0}))$ as the linear system of all quadrics in the dual of $\mathbb{P}^{g-1} = \mathbb{P}(T_{X,0})$. In particular, the matrix (a_{ij}) can be interpreted as the matrix defining a dual quadric in the space $\mathbb{P}^{g-1}$ with homogeneous coordinates $(b_1 : \ldots : b_g)$. Quite naturally, we will often use $(z_1 : \ldots : z_g)$ for the homogeneous coordinates in $\mathbb{P}^{g-1}$.
Recall that the Riemann theta function ϑ satisfies the *heat equations*

$$\frac{\partial}{\partial z_i} \frac{\partial}{\partial z_j} \vartheta = 2\pi\sqrt{-1}(1 + \delta_{ij}) \frac{\partial}{\partial \tau_{ij}} \vartheta$$

for $1 \leq i, j \leq g$, where δ_{ij} is the Kronecker delta. We shall abbreviate this equation as

$$\partial_i \partial_j \vartheta = 2\pi\sqrt{-1}(1 + \delta_{ij}) \partial_{\tau_{ij}} \vartheta,$$

where ∂_j means the partial derivative $\partial/\partial z_j$ and $\partial_{\tau_{ij}}$ the partial derivative $\partial/\partial \tau_{ij}$. One easily checks that also all derivatives of θ verify the heat equations. We refer to [39] for an algebraic interpretation of the heat equations in terms of deformation theory.
If $\xi = (X, \Theta_X, x) \in \Theta$ corresponds to the $\mathrm{Sp}(2g, \mathbb{Z}) \ltimes \mathbb{Z}^{2g}$-orbit of a point (τ_0, z_0), then the Zariski tangent space $T_{\Theta,\xi}$ to Θ at ξ is the subspace of $T_{\mathcal{X}_g,\xi} \simeq \mathbb{C}^{g(g+1)/2+g}$ defined, with the above conventions, by the linear equation

$$(1)\qquad \sum_{1\le i\le j\le g} \frac{1}{2\pi\sqrt{-1}(1+\delta_{ij})} a_{ij}\,\partial_i\partial_j\vartheta(\tau_0,z_0) + \sum_{1\le \ell\le g} b_\ell\,\partial_\ell\vartheta(\tau_0,z_0) = 0$$

in the variables (a_{ij}, b_ℓ), $1 \le i,j \le g$, $1 \le \ell \le g$. As an immediate consequence we get the result (see [36], Lemma (1.2)):

Lemma 2.1. *The point $\xi = (X, \Theta_X, x)$ is a singular point of Θ if and only if x is a point of multiplicity at least 3 for Θ_X.*

3. The locus S_g

We begin by defining a suborbifold of Θ supported on the set of points where $\pi_{|\Theta}$ fails to be of maximal rank.

Definition 3.1. The closed suborbifold S_g of Θ is defined on the universal cover $\mathbb{H}_g \times \mathbb{C}^g$ by the $g+1$ equations

$$(2)\qquad \vartheta(\tau,z) = 0, \quad \partial_j\vartheta(\tau,z) = 0, \quad j = 1,\ldots,g.$$

Lemma 2.1 implies that the support of S_g is the union of $\mathrm{Sing}(\Theta)$ and of the set of smooth points of Θ where $\pi_{|\Theta}$ fails to be of maximal rank. Set-theoretically one has

$$S_g = \{(X,\Theta_X,x) \in \Theta : x \in \mathrm{Sing}(\Theta_X)\}$$

and $\mathrm{codim}_{\mathcal{X}_g}(S_g) \le g+1$. It turns out that every irreducible component of S_g has codimension $g+1$ in $\mathcal{X}_g$ (see [8] and an unpublished preprint by Debarre [9]). We will come back to this later in §7 and §8.

With the above identification, the Zariski tangent space to S_g at a given point (X,Θ_X,x) of $\mathcal{X}_g$, corresponding to the $\mathrm{Sp}(2g,\mathbb{Z})$-orbit of a point $(\tau_0,z_0) \in \mathbb{H}_g \times \mathbb{C}^g$, is given by the $g+1$ equations

$$(3)\qquad \begin{gathered} \sum_{1\le i\le j\le g} a_{ij}\partial_{\tau_{ij}}\vartheta(\tau_0,z_0) = 0, \\ \sum_{1\le i\le j\le g} a_{ij}\partial_{\tau_{ij}}\partial_k\vartheta(\tau_0,z_0) + \sum_{1\le\ell\le g} b_\ell\partial_\ell\partial_k\vartheta(\tau_0,z_0) = 0, \qquad 1\le k\le g \end{gathered}$$

in the variables (a_{ij}, b_ℓ) with $1 \le i,j,\ell \le g$. We will use the following notation:

(a) q is the row vector of length $g(g+1)/2$, given by $(\partial_{\tau_{ij}}\theta(\tau_0,z_0))$, with lexicographically ordered entries;

(b) q_k is the row vector of length $g(g+1)/2$, given by $(\partial_{\tau_{ij}}\partial_k\theta(\tau_0,z_0))$, with lexicographically ordered entries;

(c) M is the $g\times g$-matrix $(\partial_i\partial_j\vartheta(\tau_0,z_0))_{1\le i,j\le g}$.

Then we can rewrite the equations (3) as

$$(4)\qquad a\cdot q^t = 0, \quad a\cdot q_k^t + b\cdot M_k^t = 0,\ (k=1,\ldots,g),$$

where a is the vector (a_{ij}) of length $g(g+1)/2$, with lexicographically ordered entries, b is a vector in $\mathbb{C}^g$ and M_j the j-th row of the matrix M.

In this setting, the equation (1) for the tangent space to $T_{\Theta,\xi}$ can be written as:

$$a \cdot q^t + b \cdot \partial\vartheta(\tau_0, z_0)^t = 0 \tag{5}$$

where ∂ denotes the gradient.
Suppose now the point $\xi = (X, \Theta_X, x)$ in S_g, corresponding to $(\tau_0, z_0) \in \mathbb{H}_g \times \mathbb{C}^g$ is not a point of $\mathrm{Sing}(\Theta)$. By Lemma 2.1 the matrix M is not zero and therefore we can associate to ξ a quadric Q_ξ in the projective space $\mathbb{P}(T_{X,x}) \simeq \mathbb{P}(T_{X,0}) \simeq \mathbb{P}^{g-1}$, namely the one defined by the equation

$$b \cdot M \cdot b^t = 0.$$

Recall that $b = (b_1, \dots, b_g)$ is a coordinate vector on $T_{X,0}$ and therefore $(b_1 : \dots : b_g)$ are homogeneous coordinates on $\mathbb{P}(T_{X,0})$. We will say that Q_ξ is *indeterminate* if $\xi \in \mathrm{Sing}(\Theta)$.
The vector q naturally lives in $\mathrm{Sym}^2(T_{X,0})^\vee$ and therefore, if q is not zero, the point $[q] \in \mathbb{P}(\mathrm{Sym}^2(T_{X,0})^\vee)$ determines a quadric in $\mathbb{P}^{g-1} = \mathbb{P}(T_{X,0})$. The heat equations imply that this quadric coincides with Q_ξ.
Consider the matrix defining the Zariski tangent space to S_g at a point $\xi = (X, \Theta_X, x)$. We denote by $r := r_\xi$ the corank of the quadric Q_ξ, with the convention that $r_\xi = g$ if $\xi \in \mathrm{Sing}(\Theta)$, i.e., if Q_ξ is indeterminate. If we choose coordinates on $\mathbb{C}^g$ such that the first r basis vectors generate the kernel of q then the shape of the matrix A of the system (3) is

$$A = \begin{pmatrix} q & 0_g \\ q_1 & 0_g \\ & \vdots \\ q_r & 0_g \\ * & B \end{pmatrix}, \tag{6}$$

where q and q_k are as above and B is a $(g-r) \times g$-matrix with the first r columns equal to zero and the remaining $(g-r) \times (g-r)$ matrix symmetric of maximal rank.
Next, we characterize the smooth points $\xi = (X, \Theta_X, x)$ of S_g. Before stating the result, we need one more piece of notation. Given a non-zero vector $b = (b_1, \dots, b_g) \in T_{X,0}$, we set $\partial_b = \sum_{\ell=1}^g b_\ell \partial_\ell$. Define the matrix $\partial_b M$ as the $g \times g$-matrix $(\partial_i \partial_j \partial_b \vartheta(\tau_0, z_0))_{1 \le i,j \le g}$. Then define the quadric $\partial_b Q_\xi = Q_{\xi,b}$ of $\mathbb{P}(T_{X,0})$ by the equation

$$z \cdot \partial_b M \cdot z^t = 0.$$

If $z = e_i$ is the i-th vector of the standard basis, one writes $\partial_i Q_\xi = Q_{\xi,i}$ instead of Q_{ξ,e_i} for $i = 1, \dots, g$. We will use similar notation for higher order derivatives or even for differential operators applied to a quadric.

Definition 3.2. We let $\mathcal{Q}_\xi$ be the linear system of quadrics in $\mathbb{P}(T_{X,0})$ spanned by Q_ξ and by all quadrics $Q_{\xi,b}$ with $b \in \ker(Q_\xi)$.

Since Q_ξ has corank r, the system $\mathcal{Q}_\xi$ is spanned by $r+1$ elements and therefore $\dim(\mathcal{Q}_\xi) \leq r$. This system may happen to be empty, but then Q_ξ is indeterminate, i.e., ξ lies in $\mathrm{Sing}(\Theta)$. Sometimes we will use the lower suffix x instead of ξ to denote quadrics and linear systems, e.g. we will sometimes write Q_x instead of Q_ξ, etc. By the heat equations, the linear system $\mathcal{Q}_\xi$ is the image of the vector subspace of $\mathrm{Sym}^2(T_{X,0})^\vee$ spanned by the vectors $q, q_1, \ldots, q_r$.

Proposition 3.3. *The subscheme S_g is smooth of codimension $g+1$ in $\mathcal{X}_g$ at the point $\xi = (X, \Theta_X, x)$ of S_g if and only if the following conditions hold:*

(i) $\xi \notin \mathrm{Sing}(\Theta)$*, i.e., Q_ξ is not indeterminate and of corank $r < g$;*
(ii) *the linear system $\mathcal{Q}_\xi$ has maximal dimension r; in particular, if $b_1, \ldots, b_r$ span the kernel of Q_ξ, then the $r+1$ quadrics Q_ξ, $Q_{\xi,b_1}, \ldots, Q_{\xi,b_r}$ are linearly independent.*

Proof. The subscheme S_g is smooth of codimension $g+1$ in $\mathcal{X}_g$ at ξ if and only if the matrix A appearing in (6) has maximal rank $g+1$. Since the submatrix B of A has rank $g-r$, the assertion follows. □

Corollary 3.4. *If Q_ξ is a smooth quadric, then S_g is smooth at $\xi = (X, \Theta_X, x)$.*

4. Quadrics and Cornormal Spaces

Next we study the differential of the restriction to S_g of the map $\pi : \mathcal{X}_g \to \mathcal{A}_g$ at a point $\xi = (X, \Theta_X, x) \in S_g$. We are interested in the kernel and the image of $d\pi_{|S_g,\xi}$. We can view these spaces in terms of the geometry of $\mathbb{P}^{g-1} = \mathbb{P}(T_{X,0})$ as follows:

$$\Pi_\xi = \mathbb{P}(\ker(d\pi_{|S_g,\xi})) \subseteq \mathbb{P}(T_{X,0})$$

is a linear subspace of $\mathbb{P}(T_{X,0})$ and

$$\Sigma_\xi = \mathbb{P}(\mathrm{Im}(d\pi_{|S_g,\xi})^\perp) \subseteq \mathbb{P}(\mathrm{Sym}^2(T_{X,0})^\vee)$$

is a linear system of quadrics in $\mathbb{P}(T_{X,0})$.
The following proposition is the key to our approach; we use it to view the quadrics as elements of the conormal space to our loci in the moduli space.

Proposition 4.1. *Let $\xi = (X, \Theta_X, x)$ be a point of S_g. Then:*

(i) *Π_ξ is the vertex of the quadric Q_ξ. In particular, if ξ is a singular point of Θ, then Π_ξ is the whole space $\mathbb{P}(T_{X,0})$;*
(ii) *Σ_ξ contains the linear system $\mathcal{Q}_\xi$.*

Proof. The assertions follow from the shape of the matrix A in (6). □

This proposition tells us that, given a point $\xi = (X, \Theta_X, x) \in S_g$, the map $d\pi_{|S_g,\xi}$ is not injective if and only if the quadric Q_ξ is singular.
The orbifold S_g is stratified by the corank of the matrix $(\partial_i\partial_j\theta)$.

Definition 4.2. For $0 \le k \le g$ we define $S_{g,k}$ as the closed suborbifold of S_g defined by the equations on $\mathbb{H}_g \times \mathbb{C}^g$

$$\begin{aligned} &\vartheta(\tau, z) = 0, \quad \partial_j \vartheta(\tau, z) = 0, \quad (j = 1, \dots, g), \\ &\mathrm{rk}\big((\partial_i \partial_j \vartheta(\tau, z))_{1 \le i,j \le g}\big) \le g - k. \end{aligned} \tag{7}$$

Geometrically this means that $\xi \in S_{g,k}$ if and only if $\dim(\Pi_\xi) \ge k-1$ or equivalently Q_ξ has corank at least k. We have the inclusions

$$S_g = S_{g,0} \supseteq S_{g,1} \supseteq \ldots \supseteq S_{g,g} = S_g \cap \mathrm{Sing}(\Theta)$$

and $S_{g,1}$ is the locus where the map $d\pi_{|S_g,\xi}$ is not injective. The loci $S_{g,k}$ have been considered also in [16].

We have the following dimension estimate for the $S_{g,k}$.

Proposition 4.3. *Let $1 \le k \le g-1$ and let Z be an irreducible component of $S_{g,k}$ not contained in $S_{g,k+1}$. Then we have*

$$\mathrm{codim}_{S_g}(Z) \le \binom{k+1}{2}.$$

Proof. Locally, in a neighborhood U in S_g of a point z of $Z \backslash S_{g,k+1}$ we have a morphism $f : U \to \mathcal{Q}$, where $\mathcal{Q}$ is the linear system of all quadrics in $\mathbb{P}^{g-1}$. The map f sends $\xi = (X, \Theta_X, x) \in U$ to Q_ξ. The scheme $S_{g,k}$ is the pull-back of the subscheme $\mathcal{Q}_k$ of $\mathcal{Q}$ formed by all quadrics of corank k. Since $\mathrm{codim}_{\mathcal{Q}}(\mathcal{Q}_k) = \binom{k+1}{2}$, the assertion follows. □

Using the equations (7) it is possible to make a local analysis of the schemes $S_{g,k}$, e.g. it is possible to write down equations for their Zariski tangent spaces (see §6 for the case $k = g$). This is however not particularly illuminating, and we will not dwell on this here.

It is useful to give an interpretation of the points $\xi = (X, \Theta_X, x) \in S_{g,k}$ in terms of singularities of the theta divisor Θ_X. Suppose that ξ is such that $\mathrm{Sing}(\Theta_X)$ contains a subscheme isomorphic to $\mathrm{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2))$ supported at x. This subscheme of X is given by a homomorphism

$$\mathcal{O}_{X,x} \to \mathbb{C}[\epsilon]/(\epsilon^2), \quad f \mapsto f(x) + \Delta^{(1)} f(x) \cdot \epsilon,$$

where $\Delta^{(1)}$ is a non-zero differential operator of order ≤ 1, hence $\Delta^{(1)} = \partial_b$, for some non-zero vector $b \in \mathbb{C}^g$. Then the condition $\mathrm{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2)) \subset \mathrm{Sing}(\Theta_X)$ is equivalent to saying that ϑ and $\partial_b \vartheta$ satisfy the equations

$$f(\tau_0, z_0) = 0, \quad \partial_j f(\tau_0, z_0) = 0, \quad 1 \le j \le g, \tag{8}$$

and this, in turn, is equivalent to the fact that the quadric Q_ξ is singular at the point $[b]$.

More generally, we have the following proposition, which explains the nature of the points in $S_{g,k}$ for $k < g$.

Proposition 4.4. *Suppose that $x \in \mathrm{Sing}(\Theta_X)$ does not lie on $\mathrm{Sing}(\Theta)$. Then $\mathrm{Sing}(\Theta_X)$ contains a scheme isomorphic to $\mathrm{Spec}(\mathbb{C}[\epsilon_1, \dots, \epsilon_k]/(\epsilon_i \epsilon_j : 1 \le i, j \le k < g))$ supported at x if and only if the quadric Q_ξ has corank $r \ge k$. Moreover, the Zariski tangent space to $\mathrm{Sing}(\Theta_X)$ at x is the kernel space of Q_ξ.*

Proof. With a suitable choice of coordinates in X, the condition that the scheme $\mathrm{Spec}(\mathbb{C}[\epsilon_1, \ldots, \epsilon_k]/(\epsilon_i\epsilon_j : 1 \le i, j \le k < g))$ is contained in $\mathrm{Sing}(\Theta_X)$ is equivalent to the fact that the functions ϑ and $\partial_i\vartheta$ for $i = 1, \ldots, k$ satisfy (8). But this the same as saying that $\partial_i\partial_j\vartheta(\tau_0, z_0)$ is zero for $i = 1, \ldots, k$, $j = 1, \ldots, g$, and the vectors e_i, $i = 1, \ldots, k$, belong to the kernel of Q_ξ. This settles the first assertion.
The scheme $\mathrm{Sing}(\Theta_X)$ is defined by the equations (2), where τ is now fixed and z is the variable. By differentiating, and using the same notation as above, we see that the equations for the Zariski tangent space to $\mathrm{Sing}(\Theta_X)$ at x are $\sum_{i=1}^g b_i\partial_i\partial_j\vartheta(\tau_0, z_0)$, $j = 1, \ldots, g$ i.e., $b \cdot M = 0$, which proves the second assertion. $\square$

5. Curvi-linear subschemes in the singular locus of theta

A 0-dimensional curvi-linear subscheme $\mathrm{Spec}(\mathbb{C}[t]/(t^{N+1})) \subset X$ of length $N+1$ supported at x is given by a homomorphism

$$\delta : \mathcal{O}_{X,x} \to \mathbb{C}[t]/(t^{N+1}), \qquad f \mapsto \sum_{j=0}^{N} \Delta^{(j)} f(x) \cdot t^j, \tag{9}$$

with $\Delta^{(j)}$ a differential operator of order $\le j$, $j = 1, \ldots, N$, with $\Delta^{(N)}$ non-zero, and $\Delta^{(0)}(f) = f(x)$. The condition that the map δ is a homomorphism is equivalent to saying that

$$\Delta^{(k)}(fg) = \sum_{r=0}^{k} \Delta^{(r)} f \cdot \Delta^{(k-r)} g, \quad k = 0, \ldots, N \tag{10}$$

for any pair (f, g) of elements of $\mathcal{O}_{X,x}$. Two such homomorphisms δ and δ' define the same subscheme if and only if they differ by composition with a automorphism of $\mathbb{C}[t]/(t^{N+1})$.

LEMMA 5.1. *The map δ defined in* (9) *is a homomorphism if and only if there exist translation invariant vector fields $D_1, \ldots, D_N$ on X such that for every $k = 1, \ldots, N$ one has*

$$\Delta^{(k)} = \sum_{h_1+2h_2+\ldots+kh_k=k>0} \frac{1}{h_1! \cdots h_k!} D_1^{h_1} \cdots D_k^{h_k}. \tag{11}$$

Moreover, two N-tuples of vector fields $(D_1, \ldots, D_N)$ and $(D'_1, \ldots, D'_N)$ determine the same 0*-dimensional curvi-linear subscheme of X of length $N+1$ supported at a given point $x \in X$ if and only if there are constants $c_1, \ldots, c_N$, with $c_1 \neq 0$, such that*

$$D'_i = \sum_{j=1}^{i} c_j^{i-j+1} D_j, \quad i = 1, \ldots, N.$$

Proof. If the differential operators $\Delta^{(k)}$, $k = 1, \ldots, N$, are as in (11), one computes that (10) holds, hence δ is a homomorphism.

As for the converse, the assertion trivially holds for $k=1$. So we proceed by induction on k. Write $\Delta^{(k)} = \sum_{i=1}^{k} D_i^{(k)}$, where $D_i^{(k)}$ is the homogeneous part of degree i, and write D_k instead of $D_1^{(k)}$. Using (10) one verifies that for every $k = 1,\dots,N$ and every positive $i \le k$ one has

$$iD_i^{(k)} = \sum_{j=1}^{k-i+1} D_j D_{i-1}^{(k-j)}.$$

Formula (11) follows by induction and easy combinatorics.
To prove the final assertion, use the fact that an automorphism of $\mathbb{C}[t]/(t^{N+1})$ is determined by the image $c_1t + c_2t^2 + \ldots + c_Nt^N$ of t, where $c_1 \neq 0$. $\square$

In formula (11) one has $h_k \le 1$. If $\Delta^{(1)} = D_1$ then $\Delta^{(2)} = \frac{1}{2}D_1^2 + D_2$, $\Delta^{(3)} = (1/3!)D_1^3 + (1/2)D_1D_2 + D_3$ etc.
Each non-zero summand in (11) is of the form $(1/h_{i_1}!\cdots h_{i_\ell}!)D_{i_1}^{h_{i_1}}\cdots D_{i_\ell}^{h_{i_\ell}}$, where $1 \le i_1 < \ldots < i_\ell \le k$, $i_1h_{i_1} + \ldots + i_\ell h_{i_\ell} = k$ and $h_{i_1},\dots,h_{i_\ell}$ are positive integers. Thus formula (11) can be written as

$$\Delta^{(k)} = \sum_{\{h_{i_1},\dots,h_{i_\ell}\}} \frac{1}{h_{i_1}!\cdots h_{i_\ell}!} D_{i_1}^{h_{i_1}}\cdots D_{i_\ell}^{h_{i_\ell}}, \tag{12}$$

where the subscript $\{h_{i_1},\dots,h_{i_\ell}\}$ means that the sum is taken over all ℓ-tuples of positive integers $(h_{i_1},\dots,h_{i_\ell})$ with $1 \le i_1 < \cdots < i_\ell \le k$ and $i_1h_{i_1} + \cdots + i_\ell h_{i_\ell} = k$.

Remark 5.2. Let $x \in X$ correspond to the pair (τ_0, z_0). The differential operators $\Delta^{(k)}$, $k = 1,\dots,N$, defined as in (11) or (12) have the following property: if f is a regular function such that $\Delta^{(i)}f$ satisfies (8) for all $i = 0,\dots,k-1$, then one has $\Delta^{(k)}f(\tau_0,z_0) = 0$.

We want now to express the conditions in order that a 0-dimensional curvilinear subscheme of X of length $N+1$ supported at a given point $x \in X$ corresponding to the pair (τ_0,z_0) and determined by a given N-tuple of vector fields $(D_1,\dots,D_N)$ lies in $\mathrm{Sing}(\Theta_X)$. To do so, we keep the notation we introduced above.
Let us write $D_i = \sum_{\ell=1}^{g} \eta_{i\ell}\partial_\ell$, so that D_i corresponds to the vector $\eta_i = (\eta_{i1},\dots,\eta_{ig})$. As before we denote by M the matrix $(\partial_i\partial_j\theta(\tau_0,z_0))$.

Proposition 5.3. *The 0-dimensional curvi-linear subscheme R of X of length $N+1$, supported at the point $x \in X$ corresponding to the pair (τ_0,z_0) and determined by the N-tuple of vector fields $(D_1,\dots,D_N)$ lies in* $\mathrm{Sing}(\Theta_X)$ *if and only if $x \in \mathrm{Sing}(\Theta_X)$ and moreover for each $k = 1,\dots,N$ one has*

$$\sum_{\{h_{i_1},\dots,h_{i_\ell}\}} \frac{1}{h_{i_1}!\cdots h_{i_\ell}!} \eta_{i_\ell} \cdot \partial_{\eta_{i_1}}^{h_{i_1}} \cdots \partial_{\eta_{i_\ell}}^{h_{i_\ell}-1} M = 0, \tag{13}$$

where the sum is taken over all ℓ-tuples of positive integers $(h_{i_1},\dots,h_{i_\ell})$ with $1 \le i_1 < \cdots < i_\ell \le k$ and $i_1h_{i_1} + \cdots + i_\ell h_{i_\ell} = k$.

Proof. The scheme R is contained in $\mathrm{Sing}(\Theta_X)$ if and only if one has

$$\Delta^{(k)}\theta(\tau_0, z_0) = 0, \quad \partial_j \Delta^{(k)}\theta(\tau_0, z_0) = 0 \qquad k = 0, \dots, N, \quad j = 1, \dots, g.$$

By Remark 5.2 this is equivalent to

$$\theta(\tau_0, z_0) = 0, \quad \partial_j \Delta^{(k)}\theta(\tau_0, z_0) = 0 \qquad k = 0, \dots, N, \quad j = 1, \dots, g.$$

The assertion follows by the expression (12) of the operators $\Delta^{(k)}$. □

For instance, consider the scheme R_1, supported at $x \in \mathrm{Sing}(\Theta_X)$, corresponding to the vector field D_1. Then R_1 is contained in $\mathrm{Sing}(\Theta_X)$ if and only if

$$\eta_1 \cdot M = 0. \tag{14}$$

This agrees with Proposition 4.4. If R_2 is the scheme supported at x and corresponding to the pair of vector fields (D_1, D_2), then R_2 is contained in $\mathrm{Sing}(\Theta_X)$ if and only if, besides (14) one has also

$$(1/2)\eta_1 \cdot \partial_{\eta_1} M + \eta_2 \cdot M = 0. \tag{15}$$

Next, consider the scheme R_3 supported at x and corresponding to the triple of vector fields (D_1, D_2, D_3). Then R_3 is contained in $\mathrm{Sing}(\Theta_X)$ if and only if, besides (14) and (15) one has also

$$(1/3!)\eta_1 \cdot \partial^2_{\eta_1} M + (1/2)\eta_2 \cdot \partial_{\eta_1} M + \eta_3 \cdot M = 0 \tag{16}$$

and so on. Observe that (13) can be written in more than one way. For example $\eta_2 \cdot \partial_{\eta_1} M = \eta_1 \cdot \partial_{\eta_2} M$ so that (16) could also be written as

$$(1/3!)\eta_1 \cdot \partial^2_{\eta_1} M + (1/2)\eta_1 \cdot \partial_{\eta_2} M + \eta_3 \cdot M = 0.$$

So far we have been working in a fixed abelian variety X. One can remove this restriction by working on S_g and by letting the vector fields $D_1, \dots, D_N$ vary with X, which means that we let the vectors η_i depend on the variables τ_{ij}. Then the equations (13) define a subscheme $S_g(D)$ of $\mathrm{Sing}(\Theta)$ which, as a set, is the locus of all points $\xi = (X, \Theta_X, x) \in S_g$ such that $\mathrm{Sing}(\Theta_X)$ contains a curvi-linear scheme of length $N+1$ supported at x, corresponding to the N-tuple of vector fields $D = (D_1, \dots, D_N)$, computed on X.

One can compute the Zariski tangent space to $S_g(D)$ at a point $\xi = (X, \Theta_X, x)$ in the same way, and with the same notation, as in §3. This gives in general a complicated set of equations. However we indicate one case in which one can draw substantial information from such a computation. Consider indeed the case in which $D_1 = \ldots = D_N \neq 0$, and call b the corresponding tangent vector to X at the origin, depending on the the variables τ_{ij}. In this case we use the notation $D_{b,N} = (D_1, \dots, D_N)$ and we denote by $R_{x,b,N}$ the corresponding curvi-linear scheme supported at x. For a given such $D = (D_1, \dots, D_N)$, consider the linear system of quadrics

$$\Sigma_\xi(D) = \mathbb{P}(\mathrm{Im}(d\pi_{|S_g(D),\xi})^\perp)$$

in $\mathbb{P}(T_{X,0})$. One has again an interpretation of these quadrics in terms of the normal space:

PROPOSITION 5.4. *In the above setting, the space $\Sigma_\xi(D_{b,N})$ contains the quadrics $Q_\xi, \partial_b Q_\xi, \dots, \partial_b^N Q_\xi$.*

Proof. The equations (13) take now the form

$$\theta(\tau, z) = 0,\ \partial_i\theta(\tau, z) = 0, \quad i = 1, \dots, g$$
$$b \cdot M = b \cdot \partial_b M = \cdots = b \cdot \partial_b^{N-1} M = 0.$$

By differentiating the assertion immediately follows. $\square$

6. Higher multiplicity points of the theta divisor

We now study the case of higher order singularities on the theta divisor. For a multi-index $I = (i_1, \dots, i_g)$ with $i_1, \dots, i_g$ non-negative integers we set $z^I = z_1^{i_1} \cdots z_g^{i_g}$ and denote by ∂_I the operator $\partial_1^{i_1} \cdots \partial_g^{i_g}$. Moreover, we let $|I| = \sum_{\ell=1}^g i_\ell$, which is the *length* of I and equals the *order* of the operator ∂_I.

DEFINITION 6.1. For a positive integer r we let $S_g^{(r)}$ be the subscheme of $\mathcal{X}_g$ which is defined on $\mathbb{H}_g \times \mathbb{C}^g$ by the equations

$$\partial_I\vartheta(\tau, z) = 0, \qquad |I| = 0, \dots, r-1. \tag{17}$$

One has the chain of subschemes

$$\ldots \subseteq S_g^{(r)} \subseteq \ldots \subseteq S_g^{(3)} \subseteq S_g^{(2)} = S_g \subset S_g^{(1)} = \Theta$$

and as a set $S_g^{(r)} = \{(X, \Theta_X, x) \in \Theta : x \text{ has multiplicity} \geq r \text{ for } \Theta_X\}$. One denotes by $\mathrm{Sing}^{(r)}(\Theta_X)$ the subscheme of $\mathrm{Sing}(\Theta_X)$ formed by all points of multiplicity at least r. One knows that $S_g^{(r)} = \emptyset$ as soon as $r > g$ (see [37]). We can compute the Zariski tangent space to $S_g^{(r)}$ at a point $\xi = (X, \Theta_X, x)$ in the same vein, and with the same notation, as in §3. Taking into account that θ and all its derivatives verify the heat equations, we find the equations by replacing in (3) the term $\theta(\tau_0, z_0)$ by $\partial_I\theta(\tau_0, z_0)$.
As in §3, we wish to give some geometrical interpretation. For instance, we have the following lemma which partially extends Lemma 2.1 or 3.3.

LEMMA 6.2. *For every positive integer r the scheme $S_g^{(r+2)}$ is contained in the singular locus of $S_g^{(r)}$.*

Next we are interested in the differential of the restriction of the map $\pi : \mathcal{X}_g \to \mathcal{A}_g$ to $S_g^{(r)}$ at a point $\xi = (X, \Theta_X, x)$ which does not belong to $S_g^{(r+1)}$. This means that Θ_X has a point of multiplicity exactly r at x. If we assume, as we may, that x is the origin of X , i.e. $z_0 = 0$, then the Taylor expansion of θ has the form

$$\vartheta = \sum_{i=r}^{\infty} \vartheta_i,$$

where ϑ_i is a homogeneous polynomial of degree i in the variables $z_1, \dots, z_g$ and

$$\theta_r = \sum_{I=(i_1,\dots,i_g),|I|=r} \frac{1}{i_1! \cdots i_g!} \partial_I\theta(\tau_0, z_0) z^I$$

is not identically zero. The equation $\theta_r = 0$ defines a hypersurface TC_ξ of degree r in $\mathbb{P}^{g-1} = \mathbb{P}(T_{X,0})$, which is the *tangent cone* to Θ_X at x.
We will denote by $\mathrm{Vert}(TC_\xi)$ the *vertex* of TC_ξ, i.e., the subspace of $\mathbb{P}^{g-1}$ which is the locus of points of multiplicity r of TC_ξ. Note that it may be empty. In case $r = 2$, the tangent cone TC_ξ is the quadric Q_ξ introduced in §3 and $\mathrm{Vert}(TC_\xi)$ is its vertex Π_ξ.
More generally, for every $s \geq r$, one can define the subscheme $TC_\xi^{(s)} = TC_x^{(s)}$ of $\mathbb{P}^{g-1} = \mathbb{P}(T_{X,0})$ defined by the equations

$$\theta_r = \ldots = \theta_s = 0,$$

which is called the *asymptotic cone* of order s to Θ_X at x.
Fix a multi-index $J = (j_1, \ldots, j_g)$ of length $r-2$. For any pair (h,k) with $1 \leq h,k \leq g$, let $J_{(h,k)}$ be the multi-index of length r obtained from J by first increasing by 1 the index j_h and then by 1 the index j_k (that is, by 2 if they coincide). Consider then the quadric Q_ξ^J in $\mathbb{P}^{g-1} = \mathbb{P}(T_{X,0})$ defined by the equation

$$q_\xi^J(z) := \sum_{1 \leq h,k \leq g} \partial_{J_{(h,k)}} \theta(\tau_0, z_0) z_h z_k = 0$$

with the usual convention that the quadric is indeterminate if the left-hand-side is identically zero. This is a *polar quadric* of TC_ξ, namely it is obtained from TC_ξ by iterated operations of polarization. Moreover all polar quadrics are in the span $\langle Q_\xi^J, |J| = r-2\rangle$. We will denote by $\mathcal{Q}_\xi^{(r)}$ the span of all quadrics Q_ξ^J, $|J| = r-2$ and $\partial_b Q_\xi^J$, $|J| = r-2$, with equation

$$\sum_{1 \leq i,j \leq g} \partial_b \partial_{J_{(h,k)}} \theta(\tau_0, z_0) z_i z_j = 0,$$

for every non-zero vector $b \in \mathbb{C}^g$ such that $[b] \in \mathrm{Vert}(TC_\xi)$.
We are now interested in the kernel and the image of $d\pi_{|S_g^{(r)},\xi}$. Equivalently we may consider the linear system of quadrics $\Sigma_\xi^{(r)} = \mathbb{P}(\mathrm{Im}(d\pi_{|S_g^{(r)},\xi})^\perp)$, and the subspace $\Pi_\xi^{(r)} = \mathbb{P}(\ker(d\pi_{|S_g^{(r)},\xi}))$ of $\mathbb{P}(T_{X,0})$. The following proposition partly extends Proposition 4.1 and 4.4 and its proof is similar.

Proposition 6.3. *Let $\xi = (X, \Theta_X, x)$ be a point of $S_g^{(r)}$. Then:*

(i) $\Pi_\xi^{(r)} = \mathrm{Vert}(TC_\xi)$. *In particular, if $\xi \in S_g^{(r+1)}$, then $\Pi_\xi^{(r)}$ is the whole space $\mathbb{P}(T_{X,0})$;*
(ii) $\Sigma_\xi^{(r)}$ *contains the linear system $\mathcal{Q}_\xi^{(r)}$.*

Remark 6.4. As a consequence, just like in Proposition 4.4, one sees that for $\xi = (X, \Theta_X, x)$ the Zariski tangent space to $\mathrm{Sing}^{(r)}(\Theta_X)$ at x is contained in $\mathrm{Vert}(TC_\xi)$.

As an application, we have:

Proposition 6.5. *Let $\xi = (X, \Theta_X, x)$ be a point of a component Z of $S_g^{(3)}$ such that TC_ξ is not a cone. Then $\dim(\mathcal{Q}_\xi^{(3)}) = g-1$ and therefore the codimension of the image of Z in $\mathcal{A}_g$ is at least g.*

Proof. Since TC_ξ is a not a cone its polar quadrics are linearly independent. □

The following example shows that the above bound is sharp for $g = 5$.

Example 6.6. Consider the locus $\mathcal{C}$ of intermediate Jacobians of cubic threefolds in $\mathcal{A}_5$. Note that $\dim(\mathcal{A}_5) = 15$ and $\dim(\mathcal{C}) = 10$. These have (at least) an isolated triple point on their theta divisor whose tangent cone gives back the cubic threefold. The locus $\mathcal{C}$ is dominated by an irreducible component of $S_5^{(3)}$ for which the estimate given in Proposition 6.5 is sharp. Cf. [6] where Casalaina-Martin proves that the locus of intermediate Jacobians of cubic threefolds is an irreducible component of the locus of principally polarized abelian varieties of dimension 5 with a point of multiplicity ≥ 3.

7. The Andreotti–Mayer loci

Andreotti and Mayer consider in $\mathcal{A}_g$ the algebraic sets of principally polarized abelian varieties X with a locus of singular points on Θ_X of dimension at least k. More generally, we are interested in the locus of principally polarized abelian varieties possessing a k-dimensional locus of singular points of multiplicity r on the theta divisor. To define these loci scheme-theoretically we consider the morphism $\pi : \mathcal{X}_g \to \mathcal{A}_g$ and the quasi-coherent sheaf on $\mathcal{A}_g$

$$\mathcal{F}_k^{(r)} = \bigoplus_{i=k}^{g-2} R^i \pi_* \mathcal{O}_{S_g^{(r)}}.$$

Definition 7.1. For integers k and r with $0 \leq k \leq g-2$ and $2 \leq r \leq g$ we define $N_{g,k,r}$ as the support of $\mathcal{F}_k^{(r)}$. We also set $M_{g,k,r} = \pi^{-1}(N_{g,k,r})$, a subscheme of both $S_{g,k}$ and $S_g^{(r)}$. We write $N_{g,k}$ and $M_{g,k}$ for $N_{g,k,2}$ and $M_{g,k,2}$.

The schemes $N_{g,k}$ are the so-called *Andreotti–Mayer loci* in $\mathcal{A}_g$, which were introduced in a somewhat different way in [2].
Note that $N_{g,k,r}$ is locally defined by an annihilator ideal and so carries the structure of subscheme. Corollary 8.12 below and results by Debarre [11] (see §19) imply that the scheme structure at a general point of $N_{g,0}$ defined above coincides with the one considered by Mumford in [27].
We now want to see that as a set $N_{g,k,r}$ is the locus of points corresponding to (X, Θ_X) such that $\mathrm{Sing}(\Theta_X)$ has an irreducible component of dimension $\geq k$ of points of multiplicity $\geq r$ for Θ_X.

Lemma 7.2. *Let X be an abelian variety of dimension g and $W \subset X$ an irreducible reduced subvariety of dimension n and let ω_W be its dualizing sheaf. Then $H^0(W, \omega_W) \neq (0)$.*

Proof. Let $f : W' \to W$ be the normalization of W. We first claim the inequality $h^0(W', \omega_{W'}) \leq h^0(W, \omega_W)$. To see this, note that by [24], p. 48 ff (see also [17], Exerc. 6.10, p. 239, 7.2, p. 249), there exists a map $f_*\omega_{W'} \to \mathcal{H}om(f_*\mathcal{O}_{W'}, \omega_W)$, hence a map

$$H^0(W', \omega_{W'}) \to H^0(W, \mathcal{H}om(f_*\mathcal{O}_{W'}, \omega_W)).$$

Now $\mathcal{O}_W \to f_*\mathcal{O}_{W'}$ is an injection and therefore $H^0(W, \mathcal{H}om(f_*\mathcal{O}_{W'}, \omega_W)$ maps to $H^0(W, \mathcal{H}om(\mathcal{O}_W, \omega_W))$ and we thus get a map $H^0(W', \omega_{W'}) \to H^0(W, \omega_W)$ which is injective as one sees by looking at the smooth part of W.
Let $\tilde{W}$ be a desingularization of W'. According to [22] we have $h^0(\tilde{W}, \Omega^n_{\tilde{W}}) \leq h^0(W', \omega_{W'})$. Since $\tilde{W}$ maps to X we have $h^0(\tilde{W}, \Omega^1_{\tilde{W}}) \geq n$. If $h^0(\tilde{W}, \Omega^n_{\tilde{W}})$ were 0 then $\wedge^n H^0(\tilde{W}, \Omega^1_{\tilde{W}}) \to H^0(\tilde{W}, \Omega^n_{\tilde{W}})$ would be the zero map contradicting the fact that W has dimension n. $\square$

Corollary 7.3. *We have* $(X, \Theta_X) \in N_{g,k,r}$ *if and only if* $\dim(\mathrm{Sing}^{(r)}(\Theta_X)) \geq k$.

Proof. By the previous lemma and Serre duality for a reduced irreducible subvariety W of dimension m in X it follows that $H^m(W, \mathcal{O}_W) \neq (0)$ and we know $H^k(W, \mathcal{O}_W) = (0)$ for $k > m$. This implies the corollary. $\square$

There are the inclusions

$$N_{g,k,r} \subseteq N_{g,k,r-1}, \quad N_{g,k,r} \subseteq N_{g,k-1,r}.$$

If $p = (n_1, \ldots, n_r)$ with $1 \leq n_1 \leq \ldots \leq n_r < g$ and $n_1 + \ldots + n_r = g$ is a partition of g we write $\mathcal{A}_{g,p}$ for the suborbifold (or substack) of $\mathcal{A}_g$ corresponding to principally polarized abelian varieties that are a product of r principally polarized abelian varieties of dimensions $n_1, \ldots, n_r$. We write $r(p) = r$ for the length of the partition and write $\mathcal{A}_{g,[r]}$ for the suborbifold $\cup_{r(p)=r}\mathcal{A}_{g,p}$ of $\mathcal{A}_g$ corresponding to pairs (X, Θ_X) isomorphic as a polarized abelian variety to the product of r principally polarized abelian varieties. One has the stratification

$$\mathcal{A}_{g,[g]} \subset \mathcal{A}_{g,[g-1]} \subset \cdots \subset \mathcal{A}_{g,[2]}.$$

We will denote by:
i) $\Pi_g = \cup_{r\geq 2}\mathcal{A}_{g,r} = \mathcal{A}_{g,[2]}$ the locus of *decomposable* principally polarized abelian varieties;
ii) $\mathcal{A}_g^{(\mathrm{ns})}$ the locus of classes of *non-simple abelian varieties*, i.e., of principally polarized abelian varieties of dimension g which are isogenous to a product of abelian varieties of dimension smaller that g;
iii) $\mathcal{A}_g^{\mathrm{End}\neq\mathbb{Z}}$ the locus of classes of *singular abelian varieties*, i.e., of principally polarized abelian varieties whose endomorphism ring is larger than $\mathbb{Z}$.

Remark 7.4. Note the inclusions $\Pi_g \subset \mathcal{A}_g^{(\mathrm{ns})} \subset \mathcal{A}_g^{\mathrm{End}\neq\mathbb{Z}}$. The locus Π_g is reducible with irreducible components $\mathcal{A}_{g,p}$ with p running through the partitions $g = (i, g-i)$ of g for $1 \leq i \leq g/2$ and we have $\mathrm{codim}_{\mathcal{A}_g}(\mathcal{A}_{g,(i,g-i)}) = i(g-i)$. In contrast to this $\mathcal{A}_g^{\mathrm{End}\neq\mathbb{Z}}$ and $\mathcal{A}_g^{(\mathrm{ns})}$ are the union of infinitely countably many irreducible closed subsets of $\mathcal{A}_g$ of codimension at least $g-1$, the minimum codimension being achieved for families of abelian varieties that are isogenous to products of an elliptic curve with an abelian variety of dimension $g-1$ (compare with [7]).

We recall a result from [20] and the main result from [12].

Theorem 7.5. *For every integer r with $2 \leq r \leq g$ one has:*

(i) $N_{g,k,r} = \emptyset$ *if* $k > g-r$;

(ii) $N_{g,g-r,r} = \mathcal{A}_{g,[r]}$, *i.e.,* $(X, \Theta_X) \in N_{g,g-r,r}$ *is an r-fold product.*

Hence, for every integer r such that $2 \leq r \leq g$, one has the stratification

$$N_{g,0,r} \supset N_{g,1,r} \supset \ldots \supset N_{g,k,r} \supset \ldots \supset N_{g,g-r,r} = \mathcal{A}_{g,[r]},$$

whereas for every integer k such that $0 \leq k \leq g-2$, one has the stratification

$$N_{g,k,2} \supset N_{g,k,3} \supset \ldots \supset N_{g,k,r} \supset \ldots \supset N_{g,k,g-k} = \mathcal{A}_{g,[g-k]}.$$

8. Lower bounds for the codimension of Andreotti-Mayer loci

The results in the previous sections give information about the Zariski tangent spaces to these loci and this will allow us to prove bounds on the dimension of the Andreotti–Mayer loci, which is our main objective in this paper.
We start with the results on tangent spaces. We need some notation.

Definition 8.1. Let $\zeta = (X, \Theta_X)$ represent a point in $N_{g,k,r}$. By $\mathcal{L}_{g,k,r}(\zeta)$ we denote the linear system of quadrics $\mathbb{P}(T^{\perp}_{N_{g,k,r},\zeta})$, where $T_{N_{g,k,r},\zeta}$ is the Zariski tangent space and where we view $\mathbb{P}(T_{\mathcal{A}_{g,\zeta}})$ as a space of quadrics as in Section 2. As usual, we may drop the index r if $r=2$ and write $\mathcal{L}_{g,k}(\zeta)$ for $\mathcal{L}_{g,k,2}(\zeta)$.

Notice that

$$\dim_{\zeta}(N_{g,k,r}) \leq \binom{g+1}{2} - \dim(\mathcal{L}_{g,k}(\zeta)) + 1.$$

Definition 8.2. For $\zeta = (X, \Theta_X) \in N_{g,k,r}$ we denote by $\mathrm{Sing}^{(k,r)}(\Theta_X)$ the locally closed subset

$$\mathrm{Sing}^{(k,r)}(\Theta_X) = \{x \in \mathrm{Sing}(\Theta_X) : \dim_x(\mathrm{Sing}^{(r)}(\Theta_X)) \geq k\}.$$

Moreover, we define $\mathcal{Q}^{(k,r)}_{\zeta}$ to be the linear system of quadrics in $\mathbb{P}^{g-1} = \mathbb{P}(T_{X,0})$ spanned by the union of all linear systems $\mathcal{Q}^{(r)}_{\xi}$ with $\xi = (X, \Theta_X, x)$ and $x \in \mathrm{Sing}^{(k,r)}(\Theta_X)$.

Propositions 4.1 and 6.3 imply the following basic tool for giving upper bounds on the dimension of the Andreotti–Mayer loci.

Proposition 8.3. *Let N be an irreducible component of $N_{g,k,r}$ with its reduced structure. If $\zeta = (X, \Theta_X)$ is a general point of N then the projectivized conormal space to N at ζ, viewed as a subspace of $\mathbb{P}(\mathrm{Sym}^2(T^{\vee}_{X,0}))$, contains the linear system $\mathcal{Q}^{(k,r)}_{\zeta}$.*

Proof. Let M be an irreducible component of π^*N in $S^{(r)}_{g,k}$. If ξ is smooth point of M then the image of the Zariski tangent space to M at ξ under $d\pi$ is orthogonal to $\mathcal{Q}^{(r)}_\xi$ for all $x \in \mathrm{Sing}^{k,r}(\Theta_X)$. Since we work in characteristic 0 the map $d\pi$ is surjective on the tangent spaces for general points $m \in M$ and $\pi(m) \in N$. Therefore the result follows from Propositions 4.1 and 6.3. $\square$

We need a couple of preliminary results. First we state a well-known fact, which can be proved easily by a dimension count.

Lemma 8.4. *Every hypersurface of degree $d \leq 2n-3$ in $\mathbb{P}^n$ with $n \geq 2$ contains a line.*

Next we prove the following:

Lemma 8.5. *Let V in $\mathbb{P}^s$ be a hypersurface of degree $d \geq 3$. If all polar quadrics of V coincide, then V is a hyperplane H counted with multiplicity d, and the polar quadrics coincide with $2H$.*

Proof. If $d = 3$ the assertion follows from general properties of duality (see [41], p. 215) or from an easy calculation.
If $d > 3$, then the result, applied to the cubic polars of V, tells us that all these cubic polars are equal to $3H$, where H is a fixed hyperplane. This immediately implies the assertion. $\square$

The next result has been announced in [8].

Theorem 8.6. *Let $g \geq 4$ and let N be an irreducible component of $N_{g,k}$ not contained in $N_{g,k+1}$. Then:*

(i) *for every positive integer $k \leq g-3$, one has $\mathrm{codim}_{\mathcal{A}_g}(N) \geq k+2$, whereas $\mathrm{codim}_{\mathcal{A}_g}(N) = g-1$ if $k = g-2$;*
(ii) *if N is contained in $N_{g,k,r}$ with $r \geq 3$, then $\mathrm{codim}_{\mathcal{A}_g}(N) \geq k+3$;*
(iii) *if $g-4 \geq k \geq g/3$, then $\mathrm{codim}_{\mathcal{A}_g}(N) \geq k+3$.*

Proof. By Theorem 7.5 and Remark 7.4, we may assume $k < g-2$. By definition, there is some irreducible component M of $\pi^{-1}_{S_g}(N)$ with $\dim(M) = \dim(N)+k$ which dominates N via π. We can take a general point $(X, \Theta_X, z) \in M$ so that $\zeta = (X, \Theta_X)$ is a general point in N. By Remark 7.4 we may assume X is simple.
Let R be the unique k-dimensional component of $\pi^{-1}_{|S_g}(\zeta)$ containing (X, Θ_X, z). Its general point is of the form $\xi = (X, \Theta_X, x)$ with x the general point of the unique k-dimensional component of $\mathrm{Sing}(\Theta_X)$ containing z, and x has multiplicity r on Θ_X. By abusing notation, we may still denote this component by R. Proposition 6.3 implies that the linear system of quadrics $\mathbb{P}(T^\perp_{N,\zeta})$ contains all polar quadrics of TC_ξ with $\xi = (X, \Theta_X, x) \in R$.
Thus we have a rational map

$$\phi : (\mathbb{P}^{g-1})^{r-2} \times R \dashrightarrow \mathcal{Q}^{(r)}_\zeta$$

which sends the general point $(b, \xi) := (b_1, \ldots, b_{r-2}, \xi)$ to the polar quadric $Q_{b,\xi}$ of TC_ξ with respect to $b_1, \ldots, b_{r-2}$. It is useful to remark that the quadric

$Q_{b,\xi}$ is a cone with vertex containing the projectivized Zariski tangent space to R at x (see Remark 6.4).

Claim 8.7 (The finiteness property). *For each $b \in (\mathbb{P}^{g-1})^{r-2}$ the map ϕ restricted to $\{b\} \times R$ has finite fibres.*

Proof of the claim. Suppose the assertion is not true. Then there is an irreducible curve $Z \subseteq R$ such that, for $\xi = (X, \Theta_X, x)$ corresponding to the general point in Z, one has $Q_{b,\xi} = Q$. Set $\Pi = \mathrm{Vert}(Q)$, which is a proper subspace of $\mathbb{P}^{g-1}$.
Consider the *Gauss map*

$$\gamma = \gamma_Z : Z \dashrightarrow \mathbb{P}^{g-1} = \mathbb{P}(T_{X,0})$$

which associates to a smooth point of Z its projectivized tangent space. Then Proposition 6.3 implies that $\gamma(\xi) \in R \subseteq \mathrm{Vert}(Q_\xi) = \Pi$. Thus $\gamma(Z)$ is degenerate in $\mathbb{P}^{g-1}$ and this yields that X is non-simple, cf. [31]. This is a contradiction which proves the claim. $\square$

Claim 8.7 implies that the image of the map ϕ has dimension at least k, hence $\mathrm{codim}_{\mathcal{A}_g}(N_{g,k}) \geq k+1$. To do better we need the following information.

Claim 8.8 (The non-degeneracy property). *The image of the map ϕ does not contain any line.*

Proof of the claim. Suppose the claim is false. Take a line L in the image of the map ϕ, and let $\mathcal{L}$ be the corresponding pencil of quadrics. By Proposition 6.3 and Remark 6.4, the general quadric in $\mathcal{L}$ has rank $\rho \leq g-k$. Then part (i) of Segre's Theorem 21.2 in §21 below implies that the Gauss image $\gamma(Z)$ of any irreducible component of the curve $Z = \phi^{-1}(L)$ is degenerate. This again leads to a contradiction. This proves the claim. $\square$

Claim 8.8 now implies that the image of ϕ spans a linear space of dimension at least $k+1$, hence (i) follows.
To prove part (ii) we now want to prove that $\dim(\mathcal{Q}_\zeta^{(r)}) > k+1$. Remember that the image of ϕ has dimension at least k by Claim 8.7. If the image has dimension at least $k+1$, then by Claim 8.8 it cannot be a projective space and therefore $\dim(\mathcal{Q}_\zeta^{(r)}) > k+1$. So we can assume that the image has dimension k. Therefore each component of the fibre F_Q over a general point Q in the image has dimension $(g-1)(r-2)$.
Consider now the projection of F_Q to R. If the image is positive-dimensional then there is a curve Z in R such that the image of the Gauss map of Z is contained in the vertex of Q. Then X is non-simple, a contradiction (see the proof of Claim 8.7).
Therefore the image of F_Q on R is constant, equal to a point ξ, hence $F_Q = (\mathbb{P}^{g-1})^{r-2} \times \{\xi\}$. By Lemma 8.5 there is a hyperplane H_ξ such that $TC_\xi = rH_\xi$, and $Q = 2H_\xi$. Therefore we have a rational map

$$\psi : R \dashrightarrow \mathbb{P}^{g-1\vee}$$

sending ξ to H_ξ. We notice that the image of ϕ is then equal to the 2-Veronese image if the image of ψ.
By Claim 8.7, the map ψ has finite fibres. By an argument as in Claim 8.8, we see that the image of ψ does not contain a line, hence is not a linear space. Thus it spans a space of dimension $s \geq k+1$. Then its 2-Veronese image, which is the image of ϕ spans a space of dimension at least $2s \geq 2k+2 > k+2$.
To prove part (iii), by part (ii), we can assume $r = 2$. It suffices to show that $\dim(\mathcal{Q}_\zeta) > k+1$, where $\mathcal{Q}_\zeta := \mathcal{Q}_\zeta^{(2)}$. Suppose instead that $\dim(\mathcal{Q}_\zeta) = k+1$ and set $\Sigma = \phi(R)$. We have to distinguish two cases:

(a) not every quadric in $\mathcal{Q}_\zeta$ is singular;
(b) the general quadric in $\mathcal{Q}_\zeta$ is singular, of rank $g - \rho < g$.

In case (a), consider the discriminant $\Delta \subset \mathcal{Q}_\zeta$, i.e. the scheme of singular quadrics in $\mathcal{Q}_\zeta$. This is a hypersurface of degree g, which, by Proposition 4.4, contains Σ with multiplicity at least k. Thus $\deg(\Sigma) \leq g/k \leq 3$ and Σ contains some line, so that we have the corresponding pencil $\mathcal{L}$ of singular quadrics. By Claim 8.8, one arrives at a contradiction.
Now we treat case (b). Let $g - h$ be the rank of the general quadric in Σ. One has $g - h \leq g - k$, hence $k \leq h$. Moreover one has $g - h \leq g - \rho$, i.e. $\rho \leq h$. Suppose first $\rho = h$, hence $\rho \geq k$. Let s be the dimension of the subspace

$$\Pi := \bigcap_{Q \in \mathcal{Q}_\zeta, \mathrm{rk}(Q) = g-\rho} \mathrm{Vert}(Q).$$

By applying part (iii) of Segre's Theorem 21.2 to a general pencil contained in $\mathcal{Q}_\zeta$, we deduce that

$$3\rho \leq g + 2s + 2. \tag{18}$$

Claim 8.9. *One has $s < r - k$.*

Proof. Suppose $s \geq \rho - k$. If Π' is a general subspace of Π of dimension $\rho - k$, then its intersection with $\mathbb{P}(T_{R,x})$, where $x \in R$ is a general point, is not empty. Since $\rho - k < g - k$ and X is simple, this is a contradiction (see [31], Lemma II.12). □

By (18) and Claim 8.9 we deduce that $\rho + 2k \leq g$ and therefore $3k \leq g$, a contradiction.
Suppose now $\rho < h$. Then part (iv) of Segre's Theorem 21.2 yields

$$\deg(\Sigma) \leq \frac{g - 2 - \rho}{h - \rho}.$$

The right-hand-side is an increasing function of ρ, thus $\deg(\Sigma) \leq g - h - 1 \leq 2k - 1$ because $g \leq 3k \leq 2k + h$. By Lemma 8.4 the locus Σ contains a line, and we can conclude as in the proof of the non-degeneracy property 8.8.

□

The following corollary was proved independently by Debarre [9] and includes a basic result by Mumford [27].

Corollary 8.10. *Let $g \geq 4$. Then:*

(i) *every irreducible component S of S_g has codimension $g+1$ in $\mathcal{X}_g$, hence S_g is locally a complete intersection in $\mathcal{X}_g$;*

(ii) *if $\xi = (X, \Theta_X, x)$ is a general point of S, then either Θ_X has isolated singularities or (X, Θ_X) is a product of an elliptic curve with a principally polarized abelian variety of dimension $g-1$;*

(iii) *every irreducible component N of $N_{g,0}$ has codimension 1 in $\mathcal{A}_g$;*

(iv) *if $(X, \Theta_X) \in N$ is a general element of an irreducible component N of $N_{g,0}$, then Θ_X has isolated singularities. Moreover for every point $x \in \mathrm{Sing}(\Theta_X)$, the quadric Q_x is smooth and independent of x.*

Proof. Take an irreducible component S of S_g and let N be its image via the map $\pi : \mathcal{X}_g \to \mathcal{A}_g$. Then there is a maximal integer $k \leq g-2$ such that N is contained in $N_{g,k}$. Suppose first that $1 \leq k \leq g-3$. Then by Theorem 8.6 the codimension of N in $\mathcal{A}_g$ is at least $k+2$. This implies that the codimension of S in $\mathcal{X}_g$ is at least $g+2$, which is impossible because S_g is locally defined in $\mathcal{X}_g$ by $g+1$ equations.
Then either $k = 0$ or $k = g-2$. In the former case S maps to an irreducible component N of $N_{g,0}$ which is a proper subvariety of codimension 1 in $\mathcal{A}_g$.
If $k = g-2$, then by Theorem 7.5 the polarized abelian variety (X, Θ_X) is a product of principally polarized abelian varieties. The resulting abelian variety has a vanishing thetanull and so N is contained in the divisor of a modular form (a product of thetanulls, cf. e.g. [27], p. 370) which is a component of $N_{g,0}$. Assertions (i)–(iii) follow by a dimension count (see Remark 7.4). Part (iv) follows by Propositions 4.1 and 8.3. □

Finally, we show a basic property of S_g.

Theorem 8.11. *The locus S_g is reduced.*

Proof. The assertion is well-known for $g \leq 3$, using the theory of curves. We may assume $g \geq 4$.
Let S be an irreducible component of S_g, let $N = \pi(S)$ and $k \leq g-2$ the maximal integer such that N is contained in $N_{g,k}$. As in the proof of Corollary 8.10, one has either $k = 0$ or $k = g-2$. Assume first $k = 0$, and let $\xi = (X, \Theta_X, x) \in S$ be a general point. We are going to prove that $\mathrm{Sing}(\Theta_X)$ is reduced of dimension 0.
First we prove that x has multiplicity 2 for Θ_X. Suppose this is not the case and x has multiplicity $r \geq 3$. Since $N_{g,0}$ has codimension 1 all polar quadrics of TC_ξ are the same quadric, say Q (see Proposition 6.3 and 8.3). By Lemma 8.5 the tangent cone TC_ξ is a hyperplane H with multiplicity r. Again by Proposition 6.3 all the derivatives of Q with respect to points $b \in H$ coincide with Q. By Proposition 5.3 the scheme $D_{b,2}$ supported at x is contained in $\mathrm{Sing}(\Theta_X)$. By taking into account Proposition 5.4 and repeating the same argument we see that this subscheme can be indefinitely extended to a 1-dimensional subscheme, containing x and contained in $\mathrm{Sing}(\Theta_X)$. This

implies that the corresponding component of $N_{g,0}$ is contained in $N_{g,1}$, which is not possible since the codimension of $N_{g,1}$ is at least 3.
If x has multiplicity 2 the same argument shows that the quadric Q_ξ is smooth. By Corollary 3.4, ξ is a smooth point of S_g and this proves the assertion.
Suppose now that $k = g-2$. Then by Theorem 7.5 N is contained in the locus of products $\mathcal{A}_{g,(1,g-1)}$, and for dimension reasons it is equal to it and then the result follows from a local analysis with theta functions. □

The following corollary is due to Debarre ([11]).

Corollary 8.12. *If (X, Θ_X) is a general point in a component of $N_{g,0}$ then Θ_X has finitely many double points with the same tangent cone which is a smooth quadric.*

9. A conjecture

As shown in [8], part (i) of Theorem 8.6 is sharp for $k = 1$ and $g = 4$ and 5. However, as indicated in [8], it is never sharp for $k = 1$ and $g \geq 6$, or for $k \geq 2$. In [8] we made the following conjecture, which is somehow a natural completion of Andreotti–Mayer's viewpoint in [2] on the Schottky problem.
Recall the *Torelli morphism* $t_g : \mathcal{M}_g \to \mathcal{A}_g$ which maps the isomorphism class of a curve C to the isomorphism class of its principally polarized Jacobian $(J(C), \Theta_C)$. As a map of orbifolds it is of degree 2 for $g \geq 3$ since the general abelian variety has an automorphism group of order 2 and the general curve one of order 1. We denote by $\mathcal{J}_g$ the *jacobian locus* in $\mathcal{A}_g$, i.e., the Zariski closure of $t_g(\mathcal{M}_g)$ in $\mathcal{A}_g$ and by $\mathcal{H}_g$ the *hyperelliptic locus* in $\mathcal{A}_g$, that is, the Zariski closure in $\mathcal{A}_g$ of $t_g(H_g)$, where H_g is the closed subset of $\mathcal{M}_g$ consisting of the isomorphism classes of the hyperelliptic curves. By Torelli's theorem we have $\dim(\mathcal{J}_g) = 3g-3$ and $\dim(\mathcal{H}_g) = 2g-1$ for $g \geq 2$.

Conjecture 9.1. *If $1 \leq k \leq g-3$ and if N is an irreducible component of $N_{g,k}$ not contained in $\mathcal{A}_g^{\mathrm{End}\neq\mathbb{Z}}$, then $\operatorname{codim}_{\mathcal{A}_g}(N) \geq \binom{k+2}{2}$. Moreover, equality holds if and only if one of the following happens:*

(i) *$g = k+3$ and $N = \mathcal{H}_g$;*
(ii) *$g = k+4$ and $N = \mathcal{J}_g$.*

By work of Beauville [4] and Debarre [11] the conjecture is true for $g = 4$ and $g = 5$. Debarre [10, 11] gave examples of components of $N_{g,k}$ for which the bound in Conjecture 9.1 for the codimension in $\mathcal{A}_g$ fails, but they are contained in $\mathcal{A}_g^{\mathrm{End}\neq\mathbb{Z}}$, since the corresponding abelian varieties are isogenous to products. Our main objective in this paper will be to prove the conjecture for $k = 1$.

Remark 9.2. The question about the dimension of the Andreotti–Mayer loci is related to the one about the loci $S_{g,k}$ introduced in §3. Note that $M_{g,k} = \pi^*(N_{g,k})$ is a subscheme of $S_{g,k}$. Let N be an irreducible component of $N_{g,k}$ not contained in $N_{g,k+1}$, let M be the irreducible component of $M_{g,k}$ dominating N and let Z be an irreducible component of $S_{g,k}$ containing M. We now give a heuristic argument. Recalling Proposition 4.3, we can consider Z to be *well-behaved* if $\operatorname{codim}_{S_g}(Z) = \binom{k+1}{2}$. Since M is contained in Z, one has also $\operatorname{codim}_{S_g}(M) \geq \binom{k+1}{2}$ and therefore $\operatorname{codim}_{\mathcal{A}_g}(N) \geq \binom{k+2}{2}$, which is the first assertion in Conjecture 9.1. In this setting, the equality holds if and only if Z is well-behaved and $M = Z$.

On the other hand, since M is got from Z by imposing further restrictions, one could expect that M is, in general, *strictly* contained in Z and therefore that $\operatorname{codim}_{\mathcal{A}_g}(N) > \binom{k+2}{2}$.

In this circle of ideas, it is natural to ask if $\mathcal{J}_g$ [resp. $\mathcal{H}_g$] is dominated by a well-behaved component of $S_{g,g-4}$ [resp. $S_{g,g-3}$]. This is clearly the case if $g = 4$.

A second, related, question is whether $\mathcal{J}_g$ [resp. $\mathcal{H}_g$] is contained in some irreducible subvariety of codimension $c < \binom{g-2}{2}$ [resp. $c < \binom{g-1}{2}$] in $\mathcal{A}_g$, whose general point corresponds to a principally polarized abelian variety (X, Θ_X) with $\operatorname{Sing}(\Theta_X)$ containing a subscheme isomorphic to $\operatorname{Spec}(\mathbb{C}[\epsilon_1, \ldots, \epsilon_k]/(\epsilon_i\epsilon_j \colon 1 \leq i, j \leq k))$ with $k = g - 4$ [resp. $k = g - 3$].

One might be tempted to believe that an affirmative answer to the first question implies a negative answer to the second. This is not the case. Indeed $\mathcal{H}_4$ is contained in the locus of Jacobians of curves with an effective even theta characteristic. In this case $S_{4,1}$ has two well-behaved irreducible components of dimension 8, one dominating $\mathcal{H}_4$ with fibres of dimension 1, the other one dominating $\mathcal{J}_4 \cap \theta_{4,0}$, where $\theta_{4,0}$ is the *theta-null locus*, see §19 and §10 below and [16]. These two components intersect along a 7-dimensional locus in S_4 which dominates $\mathcal{H}_4$.

Note that it is not always the case that a component of $S_{g,k}$ is well-behaved. For example, there is an irreducible component of $S_{g,g-2}$, the one dominating $\mathcal{A}_{g,(1,g-1)}$, which is also an irreducible component of S_g.

10. An Example for Genus $g = 4$

In this section we illustrate the fact that the quadrics associated to the singularities of the theta divisor may provide more information than obtained above. The locus S_4 in the universal family $\mathcal{X}_4$ consists of three irreducible components: i) one dominating $\theta_{0,4}$, the locus of abelian varieties with a vanishing theta-null; we call it $\mathcal{A}$; ii) one dominating the Jacobian locus $\mathcal{J}_4$; we call it $\mathcal{B}$; iii) one dominating $\mathcal{A}_{4,(3,1)}$, the locus of products of an elliptic curve with a principally polarized abelian variety of dimension 3.

The components $\mathcal{A}$ and $\mathcal{B}$ have codimension 5 in $\mathcal{X}_4$ and they intersect along a locus $\mathcal{C}$ of codimension 6. The image of $\mathcal{C}$ in $\mathcal{A}_4$ is the locus of Jacobians with a half-canonical g^1_3. The quadric associated to the singular point of order 2 of X in Θ_X has corank 1 at a general point. We refer to [16] for a characterization of the intersection $\theta_{0,4} \cap \mathcal{J}_4$.

Proposition 10.1. *If* $\xi = (X, \Theta_X, x)$ *is a point of* $\mathcal{C}$ *then* $\operatorname{Sing}(\Theta_X)$ *contains a scheme of length* 3 *at* x.

Proof. The scheme $\operatorname{Sing}(\Theta_X)$ contains a scheme $\operatorname{Spec}(\mathbb{C}[\epsilon])$ at x, say corresponding to the tangent vector $D_1 \in \ker q_x$, cf. Section 5. In order that ξ be a singular point of S_4, the quadrics q and q_1 associated to x and D_1 (cf. equation (6) and Proposition 3.3) must be linearly dependent. By Proposition 5.3 this implies that $\mathrm{Sing}(\Theta_X)$ contains a scheme of length 3 at x. $\square$

Proposition 10.2. *The two components $\theta_{0,4}$ and $\mathcal{J}_4$ of $N_{4,0}$ in $\mathcal{A}_4$ are smooth at a non-empty open subset of their intersection and are tangent there.*

Proof. Let $\xi = (X, \Theta_X, x)$ be a general point of $\mathcal{C}$ so that the singular point x of Θ_X defines a quadric q_x of rank 3. Set $\eta = (X, \Theta_X) \in \mathcal{A}_4$. From the equations (3) and (6), one deduces that the tangent cone to $N_{4,0}$ at η is supported on $q_x^{\perp}$. Suppose, as we may by applying a suitable translation, that x is the origin 0 in X. Then $\theta_{0,4}$ is locally defined by the equation $\theta(\tau, 0)$ in $\mathcal{A}_4$ (see [16], §2). Hence $\theta_{0,4}$ is smooth at η with tangent space $q_x^{\perp}$. Now the jacobian locus $\mathcal{J}_4$ is also smooth at η by the injectivity of the differential of Torelli's morphism at non-hyperelliptic curves (see [15]) and the assertion follows. $\square$

11. The Gauss map and tangencies of theta divisors

In this section we shall study the situation where a number of translates of the theta divisor of a principally polarized abelian variety are 'tangentially degenerate,' that is, are smooth but with linearly dependent tangent spaces or singular at prescribed points.

Let (B, Ξ) be a polarized abelian variety of dimension g, where Ξ is an effective divisor on B. As usual we let $B = \mathbb{C}^g/\Lambda$, with Λ a $2g$-dimensional lattice, and $p : \mathbb{C}^g \to B$ be the projection. So we have coordinates $z = (z_1, \ldots, z_g)$ in $\mathbb{C}^g$ and therefore on B, and we can keep part of the conventions and the notation used so far. Let $\xi = \xi(z)$ be the Riemann theta function whose divisor on $\mathbb{C}^g$ descends to Ξ via p.

If Ξ is reduced, then the Gauss map of (B, Ξ) is the morphism

$$\gamma = \gamma_\Xi \colon \Xi - \mathrm{Sing}(\Xi) \to \mathbb{P}(T^\vee_{B,0}), \quad x \mapsto \mathbb{P}(t_{-x}(T_{\Xi,\xi})),$$

where $t_{-x}(T_{\Xi,x})$ is the tangent space to Ξ at ξ translated to the origin. If $\Xi_b = t_b(\Xi)$ is the translate of Ξ by the point $b \in B$ defined by the equation $\xi(z - b) = 0$ then for $x \in \Xi - \mathrm{Sing}(\Xi)$ the origin is a smooth point for Ξ_{-x} and $\gamma(x) = \mathbb{P}(T_{\Xi_{-x},0})$.

As usual we have natural homogeneous coordinates $(z_1 : \ldots : z_g)$ in $\mathbb{P}(T_{B,0}) = \mathbb{P}^{g-1}$ and therefore dual coordinates in the dual projective space $\mathbb{P}(T^\vee_{B,0}) = \mathbb{P}^{g-1\vee}$. Then the expression of γ in coordinates equals $\gamma(p(z)) = (\partial_1\xi(z) : \ldots : \partial_g\xi(z))$ with $\partial_i = \partial/\partial z_i$.

The following lemma is well-known.

Lemma 11.1. *Let (B, Ξ) be a simple abelian variety of dimension g and Ξ an irreducible effective divisor on B. Then the map γ_Ξ has finite fibres. Moreover, for a smooth point $x \in \Xi$ there are only finitely many $b \in B$ such that Ξ_b is smooth at x and tangent to Ξ there.*

Proof. Suppose γ_Ξ does not have finite fibres. Then there is an irreducible curve C of positive geometric genus contained in the smooth locus of Ξ which is contracted by γ_Ξ to a point of $\mathbb{P}^{g-1\vee}$ corresponding to a hyperplane $\Pi \subset \mathbb{P}^{g-1}$. Then the image of the Gauss map γ_C of C lies in Π. This implies that C does not generate B and contradicts the fact that B is simple. As to the second statement, if it does not hold there exists an irreducible curve C such that for all $b \in C$ the divisor Ξ_b is smooth at x and tangent to Ξ. Then the curve $C' = \{x - b : b \in C\}$ is contained in Ξ and contracted by γ_Ξ. This contradicts the fact that γ_Ξ has finite fibres as we just proved. □

Definition 11.2. Let h be a natural number with $1 \leq h \leq g-1$. The subscheme $T_h(B,\Xi)$ of $B \times B^h$ is defined in $\mathbb{C}^g \times (\mathbb{C}^g)^h$ with coordinates $(z, u_1, \ldots, u_h)$ by the equations

$$\xi(z) = 0, \quad \xi(z-u_1) = 0, \quad \ldots \quad , \xi(z-u_h) = 0,$$

$$\mathrm{rk}\begin{pmatrix} \partial_1\xi(z) & \cdots & \partial_g\xi(z) \\ \partial_1\xi(z-u_1) & \cdots & \partial_g\xi(z-u_1) \\ & \vdots & \\ \partial_1\xi(z-u_h) & \cdots & \partial_g\xi(z-u_h) \end{pmatrix} \leq h. \tag{19}$$

The projection to the first factor induces a morphism $p_1 : T_h(B,\Xi) \to \Xi$. Note that for $i = 1, \ldots, h$ the variety E_i of codimension $h+1 \leq g$ in $B \times B^h$ defined by the equations

$$\xi(z) = 0, \quad \xi(z-u_j) = 0 \quad (j \neq i), \quad \text{and} \quad u_i = 0,$$

is contained in $T_h(B,\Xi)$. Moreover, the expected codimension of the irreducible components of $T_h(B,\Xi)$ is $g+1$. We will say that an irreducible component T of $T_h(B,\Xi)$ is *regular* if:

(i) $T \neq E_i$ for $i = 1, \ldots, h$;
(ii) on a non-empty open subset of T all the rows of the matrix in (19) are non-zero.

In particular, if T is regular then $p_1(T) \not\subseteq \mathrm{Sing}(\Xi)$.

Proposition 11.3. *If B is simple each regular component of $T_h(B,\Xi)$ has dimension $hg-1$.*

Proof. Let us first assume $h = 1$ and denote a component of $T_h(B,\Xi)$ by T. By composing p_1 with the Gauss map $\gamma = \gamma_\Xi$, we obtain a rational map $\phi : T \dashrightarrow \mathbb{P}(T_{B,0}^\vee)$. We shall prove that ϕ has finite fibres. Let v be a point in the image of ϕ coming from a point in the open subset as in Definition 11.2, (ii). By Lemma 11.1 there are only finitely many smooth points $z_1, \ldots, z_a \in \Xi$ such that $\gamma(z_i) = v$ for $i = 1, \ldots, a$. For each $1 \leq i \leq a$ we consider the theta divisor defined by $\xi(z_i - u) = 0$. Again by Lemma 11.1 there are only finitely many points $u_{i1}, \ldots, u_{i\ell_i}$ in it such that (z_i, u_{ij}) may give rise to a point on T. So ϕ has finite fibres. Thus it is dominant and T has dimension $g-1$.
Now we prove the assertion by induction on h. Consider the projection $q : T \to B \times B^{h-1}$ by forgetting the last factor B. If the image T' of T is contained in $T_{h-1}(B,\Xi)$, then it is contained in a regular component of $T_{h-1}(B,\Xi)$, hence

by induction the codimension of T is at least $g+1$, while we know from the equations that it is at most $g+1$, and the assertion follows.
Suppose T' is not contained in $T_{h-1}(B,\Xi)$. Let $\mathbb{U}(h-1,g-1) \to \mathbb{G}(h-1,g-1)$ be the universal bundle over the Grassmannian of $(h-1)$-planes in $\mathbb{P}^{g-1}$. Then we define a rational map $\psi : T \dashrightarrow \mathbb{U}(h-1,g-1)$ in the following way. If $(z,u_1,\ldots,u_h)$ is a general point in T, then by the assumption that $T' \not\subseteq T_{h-1}(B,\Xi)$ the hyperplanes $\gamma(z),\gamma(z-u_1),\ldots,\gamma(z-u_{h-1})$ are linearly independent and we define

$$\psi(z,u_1,\ldots,u_h) = (\langle\gamma(z),\gamma(z-u_1),\ldots,\gamma(z-u_{h-1})\rangle,\gamma(z-u_h)).$$

We claim that Lemma 11.1 implies that the general fibre of ψ has dimension $\leq h(h-1)$. Indeed, in a fibre of ψ the points $\gamma(z),\gamma(z-u_1),\ldots,\gamma(z-u_{h-1})$ vary in a $(h-1)$-dimensional space giving at most $h(h-1)$ parameters and if $\gamma(z),\gamma(z-u_1),\ldots,\gamma(z-u_h)$ are fixed there are for $(z,u_1,\ldots,u_h)$ only finitely many possibilities. Thus $\dim(T)$ is bounded from above by

$$\dim(\mathbb{U}(h-1,g-1)) + h(h-1) = h(g-h) + (h-1) + h(h-1) = hg-1,$$

and this proves the assertion. $\square$

We will also consider the closed subscheme $T_h^0(B,\Xi)$ of $T_h(B,\Xi)$ which is defined in $\mathbb{C}^g \times (\mathbb{C}^g)^h$ by the equations

$$\xi(z)=0, \quad \xi(z-u_i)=0, \quad i=1,\ldots,h$$

and

$$\mathrm{rk}\begin{pmatrix} \partial_1\xi(z) & \cdots & \partial_g\xi(z) \\ \partial_1\xi(z-u_1) & \cdots & \partial_g\xi(z-u_1) \\ & \vdots & \\ \partial_1\xi(z-u_h) & \cdots & \partial_g\xi(z-u_h) \end{pmatrix} = 0.$$

Finally we will consider the closed subset $\mathcal{T}_h(X,B)$ which is the union of $T_h^0(X,B)$ and of all regular components of $T_h(B,\Xi)$. Look at the projection

$$p = p_2 : \mathcal{T}(X,B) \to B^h.$$

Definition 11.4. We define $N_{0,h}(B,\Xi)$ to be the image of $\mathcal{T}_h(X,B)$ under the map p. More generally, for each integer k we define

$$N_{k,h}(B,\Xi) := \overline{\{u=(u_1,\ldots,u_h)\in B^h \colon u_1,\ldots,u_h \neq 0, \quad \dim(p_2^{-1}(u)) \geq k\}}.$$

Roughly speaking, $N_{k,h}(B,\Xi)$ is the closure of the set of all $(u_1,\ldots,u_h)\in B^h$ such that Ξ contains an irreducible subvariety V of dimension $n \geq k$ and for all $z \in V$ either:

(a) $z, z-u_1,\ldots,z-u_h$ are smooth points of Ξ and $\gamma(z),\gamma(z-u_1),\ldots,\gamma(z-u_h)$ are linearly dependent, or
(b) all the points z, $z-u_1,\ldots,z-u_h$ are contained in $\mathrm{Sing}(\Xi)$.

In case (a) we say that the divisors $\Xi,\Xi_{u_1},\ldots,\Xi_{u_h}$, all passing through z with multiplicity one, are *tangentially degenerate* at z.

In case $h = 1$ we have only two divisors which are just *tangent* at z. We will drop the index h if $h = 1$. Thus, $N_0(B, \Xi)$ is the set of all u such that Ξ and Ξ_u are either tangent, or both singular, somewhere. Note that, if Ξ is symmetric, i.e., if $\xi(z)$ is even, as happens for the Riemann theta function, then $N_0(B, \Xi)$ contains the divisorial component $2\Xi := \{2\xi\colon \xi \in \Xi\}$, since $\gamma(\xi) = \gamma(-\xi)$ for all smooth points $\xi \in \Xi$.
One has the following result by Mumford (see [27], Proposition 3.2).

Theorem 11.5. *If (B, Ξ) is a principally polarized abelian variety of dimension g with Ξ smooth, then $N_0(B, \Xi)$ is a divisor on B algebraically equivalent to $\frac{(g+2)!}{6}\,\Xi$.*

As we will see later, the following result is related to Conjecture 9.1.

Proposition 11.6. *Suppose that (B, Ξ) is a simple principally polarized abelian variety of dimension g. Assume that $(B, \Xi) \notin N_{g,k}$. Then for $k \geq 0$ and $1 \leq h \leq g-1$ and for every irreducible component Z of $N_{k,h}(B, \Xi)$ one has* $\operatorname{codim}_{B^h}(Z) \geq k+1$.

Proof. By the definition of $N_{k,h}(B, \Xi)$ and by the fact that $\mathrm{Sing}(\Xi)$ has dimension $< k$ an irreducible component of $N_{k,h}(B, \Xi)$ can only be contained in the image of a regular component of $T_h(B, \Xi)$. The assertion follows by Proposition 11.3 and the fact that the fibres of p_2 on a regular component have dimension $\geq k$. □

12. Properties of the loci $N_{k,h}(B, \Xi)$

We will prove now a more precise result in the spirit of Proposition 11.6.

Proposition 12.1. *Suppose that (B, Ξ) is a simple, principally polarized abelian variety of dimension g. Assume that $N_{k,h}(B, \Xi)$ is positive-dimensional for some $h \geq 1$ and $k \geq 1$. Then $(B, \Xi) \in N_{g,k-1}$.*

Proof. We may assume $(B, \Xi) \notin N_{g,k}$, otherwise there is nothing to prove. We may also assume $h \geq 1$ is minimal under the hypothesis that $N_{k,h}(B, \Xi)$ has positive dimension.
Let $(u_1, \ldots, u_h)$ be a point in $N_{k,h}(B, \Xi)$, so that $\Xi_0 := \Xi$ and $\Xi_i := \Xi_{u_i}, i = 1, \ldots, h$, are tangentially degenerate along a non-empty subset V^0 of an irreducible subvariety V of B of dimension k such that Ξ_i for $i = 0, \ldots, h$ are smooth at all points $v \in V^0$.
For $j = 0, \ldots, h$ the element $s_j^{(i)} := \partial_i \xi(z - u_j)$ (with the convention that $u_0 = 0$) is a section of $\mathcal{O}(\Xi_j)$ when restricted to V since Ξ_j contains V. We know that for given j not all $s_j^{(i)}$ are identically zero on V^0. Our assumptions on the tangential degenerateness and minimality tell us that there exist non-zero rational functions a_j such that we have a non-trivial relation

$$\sum_{j=0}^{h} a_j s_j^{(i)} = 0 \quad \text{for} \quad i = 1, \ldots, g. \tag{20}$$

Suppose that the a_j are regular on all of a desingularization $f : W \to V$ of V. Then they are constant and the relation $\sum a_j s_j^{(i)} = 0$ holds on the whole of W. By writing relation (20) in different patches which trivialize the involved line bundles, and comparing them, we see that, if the transition functions are not all proportional, then we can shorten the original relation by subtracting two of them. This would contradict the minimality assumption. Therefore, we have that all of the divisors $f^*(\Xi - \Xi_{u_j})_{|V}$ with $j = 1, \dots h$ are linearly equivalent. Since $u = (u_1, \dots, u_h)$ varies in a subvariety of positive dimension this implies that the map $\mathrm{Pic}^0(B) \to \mathrm{Pic}(W)$ has a positive-dimensional kernel, and this is impossible since B is simple. So there exists an index j such that the function a_j has poles on a divisor Z_j of W. Note now that a point which is non-singular for all the divisors Ξ_j, $j = 1, \dots, h$, is certainly not a pole for the functions a_j. Therefore, for each $j = 1, \dots, h$, there an ℓ depending of j such that Z_j is contained in the divisorial part of the scheme $f^*(\mathrm{Sing}(\Xi_\ell))$. Moreover, since Z_j moves in a linear system on W, it cannot be contracted by the birational morphism f. This proves that Ξ_ℓ is singular along a variety of dimension $k-1$ contained in V, which proves the assertion. $\square$

Corollary 12.2. *Suppose that (B, Ξ) is a simple, principally polarized abelian variety of dimension g. Assume that $(B, \Xi) \notin N_{g,0}$, that is, Ξ is smooth. Then every irreducible component Z of $N_{0,h}(B, \Xi)$ is a divisor of B^h.*

Proof. Each irreducible component Z of $N_{0,h}(B, \Xi)$ is dominated by a regular component T of $T_h(B, \Xi)$, which has dimension $hg - 1$ by Proposition 11.3. The map $p : T \to Z$ is finite by Corollary 11.1. The assertion follows. $\square$

Remark 12.3. If we have two divisors Ξ_0, Ξ_1 which are tangentially degenerate along an irreducible k-dimensional variety V whose general point is smooth for both Ξ_0, Ξ_1, then Ξ_0, Ξ_1 are *both* singular along some $(k-1)$-dimensional variety contained in V. This can be easily proved by looking at the relation (20) in this case, and noting that the polar divisor Z_j is contained in $f^*(\mathrm{Sing}(\Xi_j))$, $j = 0, 1$.

Remark 12.4. Suppose $(B, \Xi) \notin N_{g,0}$. Then $N_0(B, \Xi)$ is described by all differences of pairs of points of Ξ having the same Gauss image.

Suppose $(B, \Xi) \in N_{g,0} - N_{g,1}$ and assume $\{x, -x\} = \mathrm{Sing}(\Xi)$ have multiplicity 2 and the quadric $Q_x = Q_{-x}$ is smooth. It may or may not be the case that $b = -2x \in N_1(B, \Xi)$. In any case, we claim that $N_1(B, \Xi) - \{-2x\}$ is contained in the set of all differences of points in $\gamma_\Xi^{-1}(Q_x^*)$ with x. Let us give a sketch of this assertion, which provides, in this case, a different argument for the proof of Proposition 11.6.

If $b \in N_1(B, \Xi) - \{-2x\}$, there is a curve $C \subset \Xi$ such that for $t \in C$ general $\gamma_\Xi(t) = \gamma_\Xi(t + b)$. Along C the divisors Ξ and Ξ_b are tangent, hence the curve contains x (see Remark 12.3). Note that the curve C is smooth at x. Indeed, locally at x, the divisor Ξ is a quadric cone of rank $g - 1$ in $\mathbb{C}^g$ with vertex x, whereas Ξ_b is a hyperplane through x, and they can only be tangent along a line.

Thus it makes sense to consider the image of x for γ_C, which is a point on Q_x. The point $x+b \in C+b \subset \Xi$ is smooth for Ξ and $\gamma(x+b)$ is clearly tangent to Q_x at $\gamma_C(x)$.

13. On N_{g-2} and N_{g-3} for Jacobians

The following result shows that the bound in Proposition 11.6 is sharp in the case $h=1$. Recall that a curve C is called *bielliptic* if it is a double cover of an elliptic curve.

Proposition 13.1. *Let C be a smooth, irreducible projective curve of genus g and let $(J=J(C),\Theta_C)$ be its principally polarized Jacobian. Then one has:*

(i) *$\{\mathcal{O}_C(p-q)\in J : p,q\in C\} \subseteq N_{g-3}(J,\Theta_C)$;*

(ii) *either C is bielliptic or the equality holds in (i);*

(iii) *if C is hyperelliptic, then*

$$\{\mathcal{O}_C(p-q)\in J : p,q\in C, h^0(C,\mathcal{O}_C(p+q))=2\} = N_{g-2}(J,\Theta_C).$$

Proof. We begin with (i). We assume that C is not hyperelliptic; the hyperelliptic case is similar and can be left to the reader. We may identify C with its canonical image in $\mathbb{P}^{g-1}$. Moreover, we identify J with $\mathrm{Pic}^{g-1}(C)$ and $\Theta_C \subset \mathrm{Pic}^{g-1}(C)$ with the set of effective divisor classes of degree $g-1$. Then the Gauss map $\gamma_C := \gamma_{\Theta_C}$ can be geometrically described as the map which sends the class of an effective divisor D of degree $g-1$ such that $h^0(C,\mathcal{O}_C(D))=1$ to the hyperplane in $\mathbb{P}^{g-1}$ spanned by D (see [15], p. 360).
Take two distinct points p_1,p_2 on C. Then $|\omega_C(-p_1-p_2)|$ is a linear series of degree $2g-4$ and dimension $g-3$. For $i=1,2$ we let V_i be the subvariety of Θ_C which is the Zariski closure of the set of all divisor classes of the type $E+p_i$, where E is a divisor of degree $g-2$ contained in some divisor of $|\omega_C(-p_1-p_2)|$. Clearly $\dim(V_i)=g-3$, hence V_i is not contained in $\mathrm{Sing}(\Theta_C)$ which is of dimension $g-4$. If u denotes the divisor class p_2-p_1 then $x\mapsto x+u$ defines an isomorphism $V_1\cong V_2$. Moreover, for x in an non-empty open subset of V_1 we have $\gamma_C(x)=\gamma_C(x+u)$. This proves (i).
Conversely, assume there is a point $u\in \mathrm{Pic}^0(C)-\{0\}$, and a pair of irreducible subvarieties V_1,V_2 of Θ_C of dimension $g-3$ such that $x\mapsto x+u$ gives a birational map from V_1 to V_2. For x in a non-empty open subset $U\subset V_1$ we have $\gamma_C(x)=\gamma_C(x+u)$. If D and D' are the effective divisors of degree $g-1$ on C corresponding to x and $x+u$ and if E is the greatest common divisor of D and D' then by the geometric interpretation of the Gauss map γ_C there is an effective divisor F with $\deg(E)=\deg(F)$ such that $D+D'-E+F\equiv K_C$. Thus $(2D-E)+F\equiv K_C-u$.
Consider the linear series $|K_C-u|$, which is a g^{g-2}_{2g-2}. If this linear series has a base point q, then there is a point p such that $K_C-u-q\equiv K_C-p$, i.e. $D'-D\equiv u\equiv p-q$, proving (ii). So we may assume that $|K_C-u|$ has no base point. If C is not bielliptic, then $|K_C-u|$ determines a birational map of C to a curve in $\mathbb{P}^{g-2}$. On the other hand it contains the $(g-3)$-dimensional family of divisors of the form $(2D-E)+F$, which are singular along the divisor

$D - E$. This is only possible if $\deg(D - E) = 1$, i.e., $u \equiv D' - D \equiv p - q$, with $p, q \in C$. But in this case $|K_C - u|$ has the base point q, a contradiction. This proves (ii).

Assume now C is hyperelliptic. Let $p_1 + p_2$ be an effective divisor of the g^1_2 on C with $p_1 \neq p_2$. Then $|\omega_C(-p_1 - p_2)|$ is a linear series of degree $2g - 4$ and dimension $g - 2$. For $i = 1, 2$ we let V_i be the subvariety of Θ_C which is the Zariski closure of the set of all divisors classes of the type $E + p_i$, where E is a divisor of degree $g - 2$ contained in a divisor of $|\omega_C(-p_1 - p_2)|$. The variety V_i has dimension $g - 2$ and is not contained in $\mathrm{Sing}(\Theta_C)$, which is of dimension $g - 3$. The translation over u induces an isomorphism $V_1 \cong V_2$ and for x in a non-empty subset U of V_1 we have $\gamma_C(x) = \gamma_C(x + u)$. Hence the left-hand-side in (iii) is contained in $N_{g-2}(J, \Theta_C)$.

Finally, assume there is a point $u \in \mathrm{Pic}^0(C) - \{0\}$, and a pair of irreducible subvarieties V, V' of Θ_C of dimension $g - 2$ such that translation by u gives a rational map $V \dashrightarrow V'$ with $\gamma_C(x) = \gamma_C(x + u)$ on a non-empty open subset of V. Let D, D' be the effective divisors of degree $g - 1$ on C corresponding to x and $x + u$. As in the proof of part (ii), let E be the greatest common divisor of D and D'. In the present situation the linear series $|K_C - u|$ of dimension $g - 2$ contains the $(g - 2)$-dimensional family of divisors of the form $2D - E$, which are singular along the divisor $D - E$. This means that $2(D - E)$ is in the base locus of $|K_C - u|$. This is only possible if $D = E + q$ for some point $q \in C$, and $K_C - u - 2q \equiv (g - 2)(p + q)$, where p is conjugated to q under the hyperelliptic involution. In conclusion, we have $u \equiv p - q$ and the equality in (iii) follows. □

Remark 13.2. The hypothesis that C is not bielliptic is essential in (ii) of Proposition 13.1. Let in fact C be a non-hyperelliptic bielliptic curve which is canonically embedded in $\mathbb{P}^{g-1}$. Let $f : C \to E$ be the bielliptic covering. One has $f_*\mathcal{O}_C \simeq \mathcal{O}_E \oplus \xi^\vee$, with $\xi^{\oplus 2} \simeq \mathcal{O}_E(B)$, where B is the branch divisor of f. Let $u \in \mathrm{Pic}^0(E) - \{0\}$ be a general point, which we can consider as a non-trivial element in $\mathrm{Pic}^0(C)$ via the inclusion $f^* : \mathrm{Pic}^0(E) \to \mathrm{Pic}^0(C)$ Note that f is ramified, hence f^* is injective. We want to show that $u \in N_{g-3}(J, \Theta_C)$, proving that equality does not hold in (i) in this case.

The canonical image of C is contained in a cone X with vertex a point $v \in \mathbb{P}^{g-1}$ over the curve E embedded in $\mathbb{P}^{g-2}$ as a curve of degree $g - 1$ via the linear system $|\xi|$. Let us consider the subvariety W of $|\xi|$ consisting of all divisors $M \in |\xi|$ such that there is a subdivisor $p + q$ of M with $p, q \in E$ and $p - q \equiv u$. It is easily seen that W is irreducible of dimension $g - 3$.

Notice that for M general in W one has $M = p + q + N$, with N effective of degree $g - 3$. Therefore we may write $K_C \equiv f^*(M) \equiv f^*(p) + f^*(q) + F + F'$, where F, F' are disjoint, effective divisors of degree $g - 3$ which are exchanged by the bielliptic involution.

We let V be the $(g - 3)$-dimensional subvariety of Θ_C described by all classes of divisors D of degree $g - 1$ on C of the form $D = f^*(p) + F$, as M varies in W. Then $D + u \equiv D' := f^*(q) + F$ and D and D' span the same hyperplane through v in $\mathbb{P}^{g-1}$. Therefore, if $x \in V$ is the point corresponding to D, one has $\gamma_C(x) = \gamma_C(x+u)$. This proves that $u \in N_{g-3}(J, \Theta_C)$.

14. A boundary version of the Conjecture

We will now formulate a conjecture. As we will see later, it can be considered as a *boundary version* of Conjecture 9.1 (see also Proposition 11.6).

Conjecture 14.1. *Suppose that (B, Ξ) is a simple principally polarized abelian variety of dimension g. Assume that $(B, \Xi) \notin N_{g,i}$ for all $i \geq k \geq 1$. Then there is an irreducible component Z of $N_k(B, \Xi)$ with* $\mathrm{codim}_B(Z) = k+1$ *if and only if one of the following happens:*

(i) *either $g \geq 2$, $k = g-2$ and B is a hyperelliptic Jacobian,*
(ii) *or $g \geq 3$, $k = g-3$ and B is a Jacobian.*

One implication in this conjecture holds by Proposition 13.1. Note that the conjecture would give an answer for simple abelian varieties to the Schottky problem that asks for a characterization of Jacobian varieties among all principally polarized abelian varieties. For related interesting questions, see [30].

15. Semi-abelian Varieties of Torus Rank One

Let (B, Ξ) be a principally polarized abelian variety of dimension $g-1$. The polarization Ξ gives rise to the isomorphism

$$\phi_\Xi : B \to \hat{B} = \mathrm{Pic}^0(B),\ b \to \mathcal{O}_B(\Xi - \Xi_b).$$

and we shall identify B and $\hat{B}$ via this isomorphism. Thus an element $b \in \hat{B} \cong B$ determines a line bundle $L = L_b = \mathcal{O}_B(\Xi - \Xi_b)$ with trivial first Chern class. We can associate to L a *semi-abelian variety* $X = X_{B,b}$, namely the $\mathbb{G}_m$-bundle over B defined by L which is an algebraic group since it coincides with the theta group $\mathcal{G}_b := \mathcal{G}(L)$ of L (cf. [25], p. 221). This gives the well-known equivalence between $\hat{B}$ and $\mathrm{Ext}(B, \mathbb{G}_m)$, the group of extension classes of B with $\mathbb{G}_m$ in the category of algebraic groups (see [35], p. 184).
Both the line bundle L and the $\mathbb{G}_m$-bundle X determine a $\mathbb{P}^1$-bundle $\mathbb{P} = \mathbb{P}(L \oplus \mathcal{O}_B)$ over B with projection $\pi : \mathbb{P} \to B$ and two sections over B, say s_0 and s_∞ given by the projections $L \oplus \mathcal{O}_B \to L$ and $L \oplus \mathcal{O}_B \to \mathcal{O}_B$. If we set $P_0 = s_0(B)$ and $P_\infty = s_\infty(B)$ then we can identify $\mathbb{P} - P_0 - P_\infty$ with X. By [17], Proposition 2.6 on page 371 we have $\mathcal{O}_\mathbb{P}(1) \cong \mathcal{O}_\mathbb{P}(P_0)$. We can complete X by considering the non-normal variety $\overline{X} = \overline{X}_{B,b}$ obtained by glueing P_0 and P_∞ by a translation over $b \in B \cong \hat{B}$. On $\mathbb{P}$ we have the linear equivalence $P_0 - P_\infty \equiv \pi^{-1}(\Xi - \Xi_b)$. We set $E := P_\infty + \pi^{-1}(\Xi)$ and put $M_\mathbb{P} = \mathcal{O}_\mathbb{P}(E)$. This line bundle restricts to $\mathcal{O}_B(\Xi)$ on P_0 and to $\mathcal{O}_B(\Xi_b)$ on P_∞, and thus descends to a line bundle $M = M_{\overline{X}}$ on $\overline{X}$. We have $\pi_*(\mathcal{O}_\mathbb{P}(E)) = \mathcal{O}_B(\Xi) \oplus \mathcal{O}_B(\Xi_b)$ and $H^0(\mathbb{P}, M_\mathbb{P})$ is generated by two sections with divisors $P_\infty + \pi^{-1}(\Xi)$ and $P_0 + \pi^{-1}(\Xi_b)$. One concludes that $H^0(\overline{X}, M)$ corresponds to the sections of $M_\mathbb{P}$ such that translation over b carries its restriction to P_0 to the restriction

to P_∞. It follows that $h^0(\overline{X}, M) = 1$ with effective divisor $\overline{\Xi}$, which is called the *generalized theta divisor* on $\overline{X}$.
Analytically we can describe a section of $\mathcal{O}_{\bar{X}}(\overline{\Xi})$ on the universal cover $\mathbb{C}\times\mathbb{C}^{g-1}$ by a function

$$\xi(\tau, z) + u\,\xi(\tau, z-\omega),$$

where $\omega \in \mathbb{C}^{g-1}$ represents $b \in B = \mathbb{C}^{g-1}/\mathbb{Z}^{g-1} + \tau\mathbb{Z}^{g-1}$, $\xi(\tau, z)$ is Riemann's theta function for B and $u = \exp(2\pi i\zeta)$ is the coordinate on $\mathbb{C}^*$. This is called the *generalized theta function* of $\overline{X}$.
Let D be the Weil divisor on $\overline{X}$ that is the image of P_0 (or, what is the same, of P_∞). We consider a locally free subsheaf $T_{\rm vert}$ of the tangent sheaf to $\overline{X}$, namely the dual of the sheaf $\Omega^1(\log D)$ of rank g. If $d \in D$ is a point such that on the normalization $\mathbb{P}$ near the two preimages $z_1, \ldots, z_{g-1}, u$ and $z_1+b, \ldots, z_{g-1}+b, v$ are local coordinates such that $u = 0$ defines P_0 (resp. $v = 0$ defines P_∞) with $uv = 1$ then $T_{\rm vert}$ is generated by $\partial/\partial z_1, \ldots, \partial/\partial z_{g-1}, u\partial/\partial u - v\partial/\partial v$. Here $z_1, \ldots, z_{g-1}$ are coordinates on B. We interpret local sections of $T_{\rm vert}$ as derivations. In particular, if an effective Cartier divisor Y of $\overline{X}$ has local equation $f = 0$, then for each local section ∂ of $T_{\rm vert}$, the restriction to Y of ∂f is a local section of $\mathcal{O}_{\overline{X}}(Y)/\mathcal{O}_{\overline{X}}$. Then the subscheme $\mathrm{Sing}_{\rm vert}(\overline{\Xi})$ of $\overline{\Xi}$ is locally defined by the g equations

$$\partial_i f = 0 \quad \text{modulo } f \text{ in } \mathcal{O}_{\overline{X}}(\overline{\Xi})/\mathcal{O}_{\overline{X}} \tag{21}$$

with $f = 0$ a local equation of $\overline{\Xi}$ and ∂_i local generators of $T_{\rm vert}$.
The equations for $\mathrm{Sing}_{\rm vert}(\overline{\Xi})$ on X are thus given by

$$\begin{aligned} \xi(\tau, z) + u\,\xi(\tau, z-\omega) &= 0, \\ \xi(\tau, z-\omega) &= 0, \\ \partial_i\xi(\tau, z) + u\partial_i\xi(\tau, z-\omega) &= 0, \quad (1 \le i \le g-1). \end{aligned}$$

The points in $\mathrm{Sing}_{\rm vert}(\overline{\Xi})$ are of two sorts depending on whether they lie on the double locus D of $\overline{X}$ or not. The singular points of $\mathrm{Sing}(\overline{\Xi})$ on $X = \overline{X} - D$ are the points (z, u), with $u \neq 0$, which are zeros of $\xi(\tau, z)$ and $\xi(\tau, z-\omega)$ and such that $\gamma(\tau, z) = -u\gamma(\tau, z-\omega)$. That is, geometrically, these correspond under the projection on B to the points on B where Ξ and Ξ_b are tangent to each other. To describe the singular points of $\mathrm{Sing}(\overline{\Xi})$ on D, we consider the composition

$$\phi\colon B \cong P_0 \to \mathbb{P} \xrightarrow{\nu} \overline{X},$$

where ν is the normalisation. Then we have $\phi^{-1}(\mathrm{Sing}_{\rm vert}(\overline{\Xi})) = \mathrm{Sing}(\Xi)$ and the same if we identify B with P_∞.
Points of $\mathrm{Sing}_{\rm vert}(\overline{\Xi})$ determine again quadrics in $\mathbb{P}^{g-1}$ as follows. Note that the projective space $\mathbb{P}(T_{X,0})$ contains a point P_b corresponding to the tangent space $T_{\mathbb{G}_m} \subset T_X$ of the algebraic torus $\mathbb{G}_m$ at the origin. Recall that we write $\gamma(\tau, z)$ for the row vector

$$\gamma(\tau, z) = (\partial_1\xi, \ldots, \partial_{g-1}\xi)(\tau, z).$$

Then a singular point determines a, possibly indeterminate, quadric defined by the matrix

$$\begin{pmatrix} 0 & \gamma(\tau, z-\omega) \\ \gamma(\tau, z-\omega)^t & M \end{pmatrix} \tag{22}$$

with M the $(g-1)\times(g-1)$ matrix $(\partial/\partial\tau_{ij}\xi(\tau,z) + u\partial/\partial\tau_{ij}\xi(\tau,z-\omega))$. Note that we have $\gamma(\tau,z) = -u\gamma(\tau,z-\omega)$. The quadric passes through the point P_b. For a point on D the quadric is a cone with vertex P_b over a quadric in $\mathbb{P}^{g-2}$ given by M.

Remark 15.1. The above considerations show that a point in $\mathrm{Sing}_{\mathrm{vert}}(\overline{\Xi})$ has to be regarded as a point of multiplicity larger than 2 if the matrix (22) vanishes identically. This can happen only if z and $z+\omega$ are both singular for Ξ.

16. Standard Compactifications of Semi-Abelian Varieties

Let (B,Ξ) be a principally polarized abelian variety. We assume now that $\dim(B) = g-r$ with $r \geq 1$ and extend the considerations of the previous section.

The extensions of B by $\mathbb{G}_m^r$ are parametrized by $\mathrm{Ext}^1(B,\mathbb{G}_m^r) \cong \hat{B}^r$. To a point $(b_1,\ldots,b_r) \in \hat{B}^r$ one associates the $\mathbb{G}_m^r$-extension $X = X_b$ obtained as the fibre product of theta groups $\mathcal{G}_{b_1} \times_B \cdots \times_B \mathcal{G}_{b_r}$.

One of the type of degenerations of abelian varieties that we shall encounter are special compactifications of semi-abelian varieties. We shall call them *standard compactifications of torus rank r*. Let $b = (b_1,\ldots,b_r) \in \hat{B}^r$. The algebraic group $X = X_b$ sits in a $\mathbb{P}^1 \times_B \cdots \times_B \mathbb{P}^1$-bundle $\pi : \mathbb{P} \to B$ that is obtained as the fibre product over B of the $\mathbb{P}^1$-bundles $P_{b_i} = \mathbb{P}(L_{b_i} \oplus \mathcal{O}_B)$. The complement $\mathbb{P} - X$ is a union of $2r$ divisors $\sum_{i=1}^r \Pi_0^{(i)} + \Pi_\infty^{(i)}$, where $\Pi_0^{(i)}$ (resp. $\Pi_\infty^{(i)}$) is given by taking 0 (resp. ∞) in the i-th fibre coordinate, with projections $\pi_{i,0}, \pi_{i,\infty}$ to B.

We now define a non-normal variety obtained from $\mathbb{P}$ by glueing $\Pi_0^{(i)}$ with $\Pi_\infty^{(i)}$ for $i = 1,\ldots,r$. This identification depends on a $r \times r$-matrix $T = (t_{ij})$ with entries from $\mathbb{G}_m$ such that $t_{ii} = 1$ and $t_{ij} = t_{ji}^{-1}$. Let $s_0^{(i)} : B \to P_{b_i}$ (resp. $s_\infty^{(i)}$) be the zero-section (infinity section) of P_{b_i}. We glue the point

$$(\beta, x_1, \ldots, x_{i-1}, s_0^{(i)}(\beta), x_{i+1}, \ldots, x_r)$$

on $\Pi_0^{(i)}$ with the point

$$(\beta + b_i, t_{i,1}x_1, \ldots, t_{i,i-1}x_{i-1}, s_\infty^{(i)}(\beta), t_{i,i+1}x_{i+1}, \ldots, t_{i,r}x_r)$$

on $\Pi_\infty^{(i)}$. We denote the resulting variety by $\overline{X}$. It depends on the parameters $b \in \hat{B}^r$ and $t \in \mathrm{Mat}(r \times r, \mathbb{G}_m)$.

We have the linear equivalences $\Pi_0^{(i)} - \Pi_\infty^{(i)} \equiv \pi^*(\Xi - \Xi_{b_i})$. We set $E = \Pi_\infty + \pi^*(\Xi) = \sum \Pi_\infty^{(i)} + \pi^*(\Xi)$ and $E_i = \Pi_\infty - \Pi_\infty^{(i)}$ and $M := M_{\mathbb{P}} = \mathcal{O}_{\mathbb{P}}(E)$. This line bundle restricts to $\mathcal{O}_{\Pi_0^{(i)}}(E_i + \pi_{i,0}^*(\Xi))$ on $\Pi_0^{(i)}$ and to $\mathcal{O}_{\Pi_\infty^{(i)}}(E_i + \pi_{i,\infty}^*(\Xi_{b_i}))$

on $\Pi_\infty^{(i)}$. Thus, by the definition of the glueing, M descends to a line bundle $\overline{M} := M_{\overline{X}}$ on $\overline{X}$. We have

$$\begin{aligned}\pi_*(M) &= \big(\otimes_{i=1}^r (\mathcal{O}_B \oplus L_i^{-1})\big) \otimes \mathcal{O}_B(\Xi) \\ &\cong \oplus_{k=1}^r \big(\oplus_{0\leq i_1<\ldots<i_k\leq r} \mathcal{O}_B(\Xi_{b_{i_1}+\ldots+b_{i_k}})\big).\end{aligned}$$

Hence we have $h^0(\mathbb{P}, M) = 2^r$. As in the preceding section one sees that only a 1-dimensional space of sections descends to sections of $\overline{M}$ on $\overline{X}$. In terms of coordinates $(\zeta_1, \ldots, \zeta_r, z_1, \ldots, z_{g-r})$ on the universal cover of X, where $(z_1, \ldots, z_{g-r}) \in \mathbb{C}^{g-r}$ are coordinates on the universal cover of B, a non-zero section of $\overline{M}$ is given by

$$\sum_{I\subseteq\{1,\ldots,r\}} u_I\, t_I\, \xi(\tau, z - \omega_I),$$

where I runs through the subsets of $\{1, \ldots, r\}$, $u_I = \prod_{i\in I} u_i$ with $u_i = \exp(2\pi\zeta_i)$, $t_I = \prod_{i,j\in I, i<j} t_{ij}$, $b_I = \sum_{i\in I} b_i$ and $\omega_I \in \mathbb{C}^{g-r}$ represents $b_I \in B$. This is the *generalized theta function* of $\overline{X}$, whose zero locus is the *generalized theta divisor* $\overline{\Xi}$ of $\overline{X}$.

Next we look at the singular points of $\overline{\Xi}$. All points in $\overline{\Xi}\cap D$ are singular points of $\overline{\Xi}$. However, just as in the rank one case in the preceding section we will in general disregard these singularities of $\overline{\Xi}$, and we will only look at the so-called *vertical singularities*, which we are going to define now (cf. [27], §2).

The locally free subsheaf T_{vert} of rank g of the tangent sheaf $T_{\overline{X}}$ is the dual of $\Omega^1(\log D)$. Its pull back to $\mathbb{P}$ is generated, in the (u, z)-coordinates, by the differential operators $u_i\partial/\partial u_i - v_i\partial/\partial v_i$ with $u_iv_i = 1$ for $i = 1, \ldots, r$ and $\partial_j = \partial/\partial z_j$ with $j = 1, \ldots, g - r$. We interpret local sections of T_{vert} as derivations as above and define the scheme $\mathrm{Sing}_{\mathrm{vert}}(\overline{\Xi})$ of *vertical singular points* of $\overline{\Xi}$ as the subscheme of $\overline{\Xi}$ defined by the equations (21) with $f = 0$ a local equation of $\overline{\Xi}$ for all local sections $\partial \in T_{\mathrm{vert}}$. This is independent of the choice of a local equation.

Lemma 16.1. *Let $(\overline{X}, \overline{\Xi})$ be a standard compactification of a semi-abelian variety X of torus rank r with abelian part (B, Ξ). If $\dim(\mathrm{Sing}_{\mathrm{vert}}(\overline{\Xi})) \geq 1$ then $(B, \Xi) \in N_{g-r,0}$.*

Proof. The compactification $\overline{X}$ is a stratified space and the (closed) strata are (standard) compactifications of semi-abelian extensions of (B, Ξ) of torus rank s with $0 \leq s \leq r$. In view of the relations $\sum_I u_I t_I \partial_i\xi(z - \omega_I) = 0$ the vertical singularities of $\overline{\Xi}$ correspond to points where 2^s translates of Ξ are tangentially degenerate. Therefore we have $\dim(N_{1,h}(B, \Xi)) \geq 1$ with $h = 2^s$. By Lemma 12.1 it follows that $(B, \Xi) \in N_{g-r,0}$. □

17. Semi-abelian varieties of torus rank two

In the compactification of the moduli space of principally polarized abelian varieties of dimension g we shall encounter two types of degenerations of torus rank 2. The first of these is a standard compactification introduced above and its normalization is a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over a principally polarized abelian variety of dimension $g-2$. For such a standard compactification the equations for $\mathrm{Sing}_{\mathrm{vert}}(\Xi)$ are given in terms of the (u,z)-coordinates by the system (we write t instead of $t_{1,2}$; note that $t \neq 0$)

$$\begin{aligned} \xi(z) - tu_1u_2\xi(z-\omega_1-\omega_2) =& 0,\\ u_1\xi(z-\omega_1) + tu_1u_2\xi(z-\omega_1-\omega_2) =& 0,\\ u_2\xi(z-\omega_2) + tu_1u_2\xi(z-\omega_1-\omega_2) =& 0,\\ \partial_i\xi(z) + u_1\partial_i\xi(z-\omega_1) + u_2\partial_i\xi(z-\omega_2) + tu_1u_2\partial_i\xi(z-\omega_1-\omega_2) =& 0,\\ i = 1,\dots,g-2.& \end{aligned}$$

From this and the analogous equations in the v-coordinates (with $u_iv_i = 1$) we see that the vertical singular points of $\overline{\Xi}$ are essentially of three types:
(i) A point $x \in D$ that is the image via $\varphi : X \to \overline{X}$ of a point in $\Pi_{1,0} \cap \Pi_{2,0} \cong B$, (i.e. in the (u,z) coordinates one has $u_1 = u_2 = 0$) is a vertical singular point of $\overline{\Xi}$ if and only if it corresponds to a singular point of Ξ on $\Pi_{1,0} \cap \Pi_{2,0} \cong B$ and to a singular point of $\Xi_{b_1+b_2}$ on $\Pi_{1,\infty} \cap \Pi_{2,\infty} \cong B$.
(ii) A point $x \in D$ which is the image via φ of a point in $\Pi_{j,0}$ but not of a point in $\Pi_{3-j,0}$, (i.e., in the (u,z) coordinates one has $u_i = 0, u_{3-j} \neq 0$, for a $j = 1,2$) is a vertical singular point of $\overline{\Xi}$ if and only if

$$\begin{aligned} \xi(\tau,z) = 0, \quad \xi(\tau, z-\omega_{3-j}) =& 0,\\ \partial_i\xi(\tau,z) + u_{3-j}\partial_i\xi(\tau,z-\omega_{3-j}) =& 0, \quad i = 1,\dots,g-2. \end{aligned}$$

i.e. if and only if z and $z-b$ are in Ξ and $\gamma_\Xi(z) = \gamma_\Xi(z-b)$.
(iii) A point $x \notin D$, (i.e. in the (u,z) coordinates one has $u_1 \neq 0 \neq u_2$) is a vertical singularity if and only if z is a singular point of the divisor $H \in |2\Xi_{b_1+b_2}|$ defined by the equation

$$\xi(\tau,z-\omega_1)\xi(\tau,z-\omega_2) = t\,\xi(\tau,z)\xi(\tau,z-\omega_1-\omega_2).$$

By the way, this occurs even in case (ii) above. Note also that, by the above equations, the existence of a vertical singularity implies that the theta divisors Ξ, Ξ_{b_1}, Ξ_{b_2} and $\Xi_{b_1+b_2}$ are tangentially degenerate at some point x of B, i.e., $z \in N_{0,3}(B,\Xi)$.
We call *of type* (i), (ii) or (iii) the singular points of $\overline{\Xi}$ according to whether cases (i), (ii) or case (iii) occur.

Remark 17.1. A point x in $\mathrm{Sing}_{\mathrm{vert}}(\overline{\Xi})$ again determines a quadric Q_x in $\mathbb{P}^{g-1}$. It is useful to remark that:
(a) in case (i) the quadric Q_x is a cone with vertex the line $L_{\mathbf{b}} := \langle P_{b_1}, P_{b_2}\rangle$ given by the tangent space to the $\mathbb{G}_m$-part over the quadric Q_z in $\mathbb{P}^{g-3}$ which corresponds to the singular point z of Ξ;
(b) in case (ii), say we are at a point with $u_1 = 0, u_2 \neq 0$. Then Q_x is a cone with vertex P_{b_1} over the quadric in the hyperplane $u_1 = 0$ with matrix

$$\begin{pmatrix} 0 & -\gamma(\tau,z)^t \\ -\gamma(\tau,z) & M \end{pmatrix}$$

with $\gamma(\tau,z) = (\partial_1\xi, \dots, \partial_{g-2}\xi)(\tau,z)$ and the matrix M is given by

$$M = (\partial/\partial\tau_{ij}\xi(\tau,z) + u\partial/\partial\tau_{ij}\xi(\tau,z-\omega))_{1\leq i,j\leq g-2}.$$

In §15 we saw that all rank 1 compactifications of $\mathbb{G}_m$-extensions of a principally polarized abelian variety B form a compact family $\hat{B}$. This is no longer the case in the higher rank case. This is where semi-abelian varieties of *non-standard type* come into the picture. This will depend on choices. It is good to see this in some detail in the rank 2 case.

Given a principally polarized abelian variety (B,Ξ) of dimension $g-2$, all standard rank 2 compactifications of (B,Ξ) are of the form $(\overline{X},\overline{\Xi})$ with $\overline{X} = \overline{X}_{B,b,t}$ with $b = (b_1,b_2) \in B\times B$ and $t \in \mathbb{C}^*$. Thus the parameter space may be identified (up to dividing by automorphisms) with the total space of the Poincaré bundle $\mathcal{P} \to B\times B$ minus the 0-section P_0. It is then natural to compactify this by looking at the associated $\mathbb{P}^1$-bundle and by glueing on it the 0-section P_0 with the infinity section P_∞. This in fact works and it is explained in [26], §7, and in [29]. We describe next the new objects that arise. We denote by L_i the line bundle associated to b_i, for $i=1,2$. We consider again the $\mathbb{P}^1\times\mathbb{P}^1$-bundle P on B as in §16 and in the glueing operation described in §16, we let $t = t_{12}$ tend to 0 (or equivalently to ∞). Letting $t\to\infty$, one contracts $\Pi_{1,0}$ and $\Pi_{2,0}$ to the section $A = \Pi_{1,\infty}\cap\Pi_{2,\infty} \cong B$, and $\Pi_{1,\infty}$ and $\Pi_{2,\infty}$ to the section $\Delta = \Pi_{1,0}\cap\Pi_{2,0}\cong B$. In order to properly describe the glueing process, we have first to blow up the two sections A and Δ in P. Let us do that. Let $w : \tilde{P} \to P$ be the blow-up, on which we have the following divisors: α is the exceptional divisor over A and δ is the exceptional divisor over Δ; $\beta,\gamma,\epsilon,\zeta$ are the proper transforms on $\tilde{P}$ of $\Pi_{1,\infty},\Pi_{2,0},\Pi_{1,0},\Pi_{2,\infty}$, respectively. We will abuse notation and denote by the same letters the restrictions of these divisors on the general fibre Φ of $\tilde{P}$ over B, which is a $\mathbb{P}^1\times\mathbb{P}^1$ blown up at two points, hence a $\mathbb{P}^2$ blown up at three points. Note that $\alpha,\beta,\gamma,\delta,\epsilon,\zeta$ are $\mathbb{P}^1$-bundles over B and one has

$$\begin{array}{lccc} (23) & \alpha \cong \mathbb{P}(L_1^\vee\oplus L_2^\vee), & \gamma\cong\mathbb{P}(L_2\oplus\mathcal{O}_B), & \epsilon\cong\mathbb{P}(L_1\oplus\mathcal{O}_B), \\ & \delta\cong\mathbb{P}(L_1\oplus L_2), & \zeta\cong\mathbb{P}((L_1\otimes L_2)\oplus L_2), & \beta\cong\mathbb{P}((L_1\otimes L_2)\oplus L_1). \end{array}$$

At this point one could be tempted to suitably glue α with γ and ϵ and δ with β and ζ. This however, as (23) shows, does not work. The right construction is instead the following.

One considers two $\mathbb{P}^2$-bundles $\phi_i : P_i^\sharp \to B$, $i=1,2$, associated to the vector bundles $L_1\oplus L_2$ and $L_1^\vee\oplus L_2^\vee$ on B, i.e.,

$$P_1^\sharp = \mathbb{P}(L_1^\vee\oplus L_2^\vee\oplus\mathcal{O}_B),\quad P_2^\sharp = \mathbb{P}(L_1\oplus L_2\oplus\mathcal{O}_B).$$

There are three relevant $\mathbb{P}^1$-subbundles of the bundles $P_i^\sharp$, $i=1,2$, namely

$$\begin{array}{llll} (24) & \bar\alpha = \mathbb{P}(L_1^\vee\oplus L_2^\vee), \quad \bar\gamma = \mathbb{P}(L_2^\vee\oplus\mathcal{O}_B), \quad \bar\epsilon = \mathbb{P}(L_1^\vee\oplus\mathcal{O}_B) & \text{in} & P_1^\sharp \\ & \bar\delta = \mathbb{P}(L_1\oplus L_2), \quad \bar\zeta = \mathbb{P}(L_1\oplus\mathcal{O}_B), \quad \bar\beta = \mathbb{P}(L_2\oplus\mathcal{O}_B) & \text{in} & P_2^\sharp. \end{array}$$

As (23) and (24) show, we can glue P with $P_1^\sharp$ and $P_2^\sharp$ in such a way that α and δ are respectively glued to $\bar\alpha$ and $\bar\delta$; ϵ is glued to $\bar\epsilon$ and β to $\bar\beta$ with a shift by $-b_1$, and ζ to $\bar{\zeta}$ and γ to $\bar{\gamma}$ with a shift by $-b_2$. The resulting variety is denoted by $\overline{X} = \overline{X}_{B,b}$. As usual, we will denote by D its singular locus. On $\tilde{P}$ we have the line bundle

$$\tilde{M} = w^*\mathcal{O}_P(\Pi_{1,\infty} + \Pi_{2,\infty}) \otimes \mathcal{O}_{\tilde{P}}(\Xi - \alpha),$$

where we write L instead of $w^*(\pi^*(L))$ for a line bundle, or divisor, L on B. With similar notation, one has

$$\begin{aligned} \tilde{M} \cong \mathcal{O}_{\tilde{P}}(\alpha + \beta + \zeta + \Xi) &\cong \mathcal{O}_{\tilde{P}}(\delta + \epsilon + \zeta + \Xi_{b_1}) \\ &\cong \mathcal{O}_{\tilde{P}}(\delta + \gamma + \beta + \Xi_{b_2}). \end{aligned} \tag{25}$$

One has $\mathcal{O}_{P_1^\sharp}(1) = \mathcal{O}_{P_1^\sharp}(\bar{\alpha})$ and the following linear equivalences

$$\bar{\alpha} - \bar{\gamma} \equiv L_2^\vee, \quad \bar{\alpha} - \bar{\epsilon} \equiv L_1^\vee \tag{26}$$

where again we write L instead of $\phi_i^*(L)$, $i = 1, 2$, for a line bundle, or divisor, L on B (see again [17], Proposition 2.6, p. 371). From (25) and (26) one deduces that $\tilde{M}$ glues with the line bundle $M_1^\sharp = \mathcal{O}_{P_1^\sharp}(\bar{\alpha} + \Xi)$ and with the line bundle $M_2^\sharp = \mathcal{O}_{P_2^\sharp}(\Xi)$, to give a line bundle $\overline{M}$ on $\overline{X}$. In the obvious coordinates $(u, z) = ((u_1, u_2), (z_1, \ldots, z_{g-2}))$, which can be considered as coordinates on $\tilde{P} - (\alpha \cup \cdots \cup \zeta)$, the sections of $\tilde{M}$ can be expressed as

$$a\,\xi(\tau, z) + a_1 u_1\,\xi(\tau, z - \omega_1) + a_2 u_2\,\xi(\tau, z - \omega_2)$$

with a, a_1, a_2 complex numbers. By taking into account the glueing conditions, one sees that only a 1-dimensional subspace V of $H^0(\tilde{P}, \tilde{M})$ gives rise to a space of sections of $H^0(\overline{X}, \overline{M})$; V is generated by

$$\xi(\tau, z) + u_1\xi(\tau, z - \omega_1) + u_2\xi(\tau, z - \omega_2). \tag{27}$$

in the (u, z)-coordinates. Note that (27) is just obtained from the generalized theta function in 16 by letting $t = t_{12}$ tend to 0.

In conclusion, one has $h^0(\overline{X}, \overline{M}) = 1$, hence there is a unique effective divisor $\overline{\Xi} = \overline{\Xi}$ which is the zero locus of a non-zero section of $H^0(\overline{X}, \overline{M})$.

As in the standard case, we parametrize an open subset of $\mathrm{Pic}^0(\overline{\Xi})$ with points in the union of $P - \bigcup_{i=1,2;h=1\infty} \Pi_{i,h}$ with $P_1^\sharp - (\bar{\alpha} \cup \bar{\gamma} \cup \bar{\epsilon})$ and $P_1^\sharp - (\bar{\delta} \cup \bar{\zeta} \cup \bar{\beta})$.

We can define the *vertical singularities* of the divisor $\overline{\Xi}$, whose equations, in the (u, z) coordinates, take the form

(28)
$$\begin{aligned} \xi(\tau, z) = 0, \quad u_1\xi(\tau, z - \omega_1) = 0, \quad u_2\xi(\tau, z - \omega_2) &= 0, \\ \partial_i\xi(\tau, z) + u_1\partial_i\xi(\tau, z - \omega_1) + u_2\partial_i\xi(\tau, z - \omega_2) &= 0, \quad i = 1, \ldots, g-2. \end{aligned}$$

Again the vertical singular points of $\overline{\Xi}$ are essentially of three types:

(i) Consider a point $x \in D$ which is the image via $\varphi : X \to \overline{X}$ of a point in $\Pi_{1,0} \cap \Pi_{2,0} \cong B$, i.e. in the (u, z) coordinates one has $u_1 = u_2 = 0$. Then this is a vertical singular point of $\overline{\Xi}$ if and only if it corresponds to a singular point of Ξ on $\Pi_{1,0} \cap \Pi_{2,0} \cong B$ and to a singular point of $\Xi_{b_1+b_2}$ on $\Pi_{1,\infty} \cap \Pi_{2,\infty} \cong B$.

(ii) Consider a point $x \in D$ which is the image via φ of a point in $\Pi_{i,0}$ but not of a point in $\Pi_{3-i,0}$, i.e. in the (u,z) coordinates one has $u_i = 0, u_{3-i} \neq 0$, for an $i = 1, 2$. If $i = 1$, this is a vertical singular point of $\overline{\Xi}$ if and only if

$$\begin{aligned} \xi(\tau, z) = 0, \quad \xi(\tau, z - \omega_2) =& 0, \\ \partial_i \xi(\tau, z) + u_2\, \partial_i \xi(\tau, z - \omega_2) =& 0, \quad i = 1, \dots, g-2. \end{aligned}$$

i.e., if and only if z and $z - b_2$ are in Ξ and $\gamma_\Xi(z) = \gamma_\Xi(z - b_2)$. Thus points of this type correspond to points in $N_0(B, \Xi)$.
(iii) Consider a point $x \notin D$, i.e., in the (u,z) coordinates one has $u_1 \neq 0 \neq u_2$. Then equations (28) mean that z corresponds to a point in $\Xi \cap \Xi_{b_1} \cap \Xi_{b_2}$ where $\Xi, \Xi_{b_1}, \Xi_{b_2}$ are tangentially degenerate. In other words points of this type correspond to points in $N_{0,2}(B, \Xi)$.

Remark 17.2. A point x in $\mathrm{Sing}_{\mathrm{vert}}(\overline{\Xi})$ determines a quadric Q_x in $\mathbb{P}^{g-1}$. Remark 17.1 is still valid here.

A variant of this second type of rank-2 degeneration is obtained as follows. Given a $\mathbb{G}_m^2$-extension $X = X_b$ of B determined by a point $b = (b_1, b_2) \in \hat{B}^2$ we consider two $\mathbb{P}^2$-bundles $\mathbb{P}$ and $\mathbb{P}'$ over B:

$$\mathbb{P} = \mathbb{P}(\mathcal{O}_B \oplus L_1 \oplus L_2) \qquad \text{and} \qquad \mathbb{P}' = \mathbb{P}(L_2 \oplus L_1 \oplus (L_1 \otimes L_2)),$$

where we write as before L_i for L_{b_i}. We can glue these over the common $\mathbb{P}^1$-subbundle $\mathbb{P}(L_1 \oplus L_2)$. Then we glue the $\mathbb{P}^1$-subbundle $\mathbb{P}(\mathcal{O}_B \oplus L_1)$ of $\mathbb{P}$ with the $\mathbb{P}^1$-subbundle $\mathbb{P}(L_2 \oplus (L_1 \otimes L_2))$ of $\mathbb{P}'$ via a shift over b_2. Similarly, we glue the $\mathbb{P}^1$-subbundle $\mathbb{P}(\mathcal{O}_B \oplus L_2)$ of $\mathbb{P}$ with the $\mathbb{P}^1$-subbundle $\mathbb{P}(L_1 \oplus (L_1 \otimes L_2))$ of $\mathbb{P}'$ via a shift over b_1. In this way we obtain a non-normal variety over B. Both $\mathbb{P}$ and $\mathbb{P}'$ come with a relatively ample bundle $\mathcal{O}_{\mathbb{P}}(1)$ and $\mathcal{O}_{\mathbb{P}'}(1)$. On $\mathbb{P}$ we have the linear equivalences

$$\Pi_1 + \pi^*(\Xi_{b_1}) \equiv \Pi_2 + \pi^*(\Xi_{b_2}) \equiv \Pi_3 + \pi^*(\Xi_{b_1+b_2}).$$

with $\Pi_i = \mathbb{P}(\mathcal{O}_B \oplus L_i)$ for $i = 1, 2$ and $\Pi_3 = \mathbb{P}(L_1 \oplus L_2)$. We let M be the line bundle $\mathcal{O}(\Pi_3 + \pi^*(\Xi_{b_1+b_2}))$ on $\mathbb{P}$ and M' the line bundle $\mathcal{O}(\Pi_3' + \pi^*(\Xi_{b_1+b_2}))$ on $\mathbb{P}'$, where Π_3' is the bundle $\mathbb{P}(L_1 \oplus L_2)$. This descends to a line bundle $\overline{M}$ on $\overline{X}$. This line bundle has a 1-dimensional space of sections generated by

$$\theta(\tau, z) = \xi(\tau, z) + u_1\, \xi(\tau, z_1 - \omega_1) + u_2\, \xi(\tau, z - \omega_2)$$

in suitable affine coordinates (u_1, u_2) on $\mathbb{P}^2$. Again the vertical singular points of $\overline{\Xi}$ are essentially of three types:
(i) A point $x \in D$ which is the image via $\varphi : X \to \overline{X}$ of a point in $\Pi_1 \cap \Pi_2 = \mathbb{P}(\mathcal{O}_B) \cong B$ is a vertical singularity if it corresponds to a singularity on Ξ, to a singularity on Ξ_{b_1} on $\Pi_1 \cap \Pi_3$ and a singularity on Ξ_{b_2} on $\Pi_2 \cap \Pi_3$;
(ii) A point $x \in D$ which is the image via φ of a point on one Π_3 but not of a point in Π_1 or Π_2 is a vertical singular point of $\overline{\Xi}$ if and only if $x \in \Xi_{b_1} \cap \Xi_{b_2}$ and $\gamma_{\Xi_{b_1}}(x) = \gamma_{\Xi_{b_2}}(x)$. Thus points of this type correspond to points in $N_0(B, \Xi)$. Something similar happens for the points on exactly one of Π_1 or Π_2.

(iii) A point $x \notin D$ is a vertical singularity if $x \in \Xi \cap \Xi_{b_1} \cap \Xi_{b_2}$ and $\Xi, \Xi_{b_1}, \Xi_{b_2}$ are tangentially degenerate at x. In other words points of this type correspond to points in $N_{0,2}(B, \Xi)$.

We see that the compactification depends on a choice, but in both cases we can deal explicitly with the singularities of the theta divisors.

Remark 17.3. The variant just given corresponds to a tesselation of $\mathbb{R}^2$ given by the lines $x = a$, $y = b$ and $x + y = c$ with $a, b, c \in \mathbb{Z}$. To a triangle with integral vertices $(n, m), (n+1, m)$ and $(n, m+1)$ (resp. $(n-1, m)$, $(n, m-1)$ and (n, m)) we associate the $\mathbb{P}^2$-bundle $\mathbb{P}(L_{nb_1+mb_2} \oplus L_{(n+1)b_1+mb_2} \oplus L_{nb_1+(m+1)b_2})$ (resp. $\mathbb{P}(L_{(n-1)b_1+mb_2} \oplus L_{nb_1+(m-1)b_2} \oplus L_{nb_1+mb_2})$ and we glue the bundles belonging to adjacent triangles over the $\mathbb{P}^1$-bundle defined by the common edge. Then the generator τ_1 (resp. τ_2) of $\mathbb{Z}^2$ acts by glueing the $\mathbb{P}^1$-bundle associated to an edge to the $\mathbb{P}^1$-bundle associated to the translate by $x \mapsto x+1$ (resp. $y \mapsto y + 1$) of this edge using a translation over b_2 (resp. b_1). The quotient under $\mathbb{Z}^2$ is the non-normal variety we just constructed. Also the earlier compactifications thus correspond to tesselations (see [26], §7).

18. Compactification of $\mathcal{A}_g$

In order to study the Andreotti-Mayer loci we need to compactify $\mathcal{A}_g$. The moduli space $\mathcal{A}_g$ admits a 'minimal' compactification, the *Satake compactification* constructed by Satake and Baily-Borel in characteristic 0 and by Faltings-Chai over the integers (see [32], [13]). This compactification $\mathcal{A}_g^*$ is an orbifold or stack which admits a stratification

$$\mathcal{A}_g^* = \sqcup_{i=0}^{g} \mathcal{A}_i$$

and the closure of $\mathcal{A}_m$ in $\mathcal{A}_g^*$ is $\mathcal{A}_m^* = \sqcup_{i=0}^{m} \mathcal{A}_i$. This compactification is highly singular for $g \geq 2$. Smooth compactifications can be constructed by the theory developed by Mumford and his co-workers in characteristic 0 and by Faltings-Chai in general. These compactifications depend on combinatorial data. We shall use the *Voronoi compactification* $\tilde{\mathcal{A}}_g = \tilde{\mathcal{A}}_g^{\mathrm{Vor}}$ as described by Namikawa, Nakamura and Alexeev (see [29, 28, 5]). This compactification is a smooth orbifold with a natural map $q : \tilde{\mathcal{A}}_g \to \mathcal{A}_g^*$. It has the stratification induced by that of $\mathcal{A}_g^*$: the stratum

$$\mathcal{A}_g^{(r)} = q^{-1}(\mathcal{A}_{g-r})$$

is called the stratum of *quasi-abelian varieties of torus rank r*.

In §20 we also shall use another compactification, the *perfect cone compactification*, cf. [29, 34]. It also has a map to the Satake compactification denoted by $q : \tilde{\mathcal{A}}_g^{\mathrm{pc}} \to \mathcal{A}_g^*$; sometimes we shall denote $\tilde{\mathcal{A}}_g^{\mathrm{pc}}$ simply by $\tilde{\mathcal{A}}_g$ in order to avoid introducing more notation. It has the following properties: i) the boundary is an irreducible $\mathbb{Q}$-Cartier divisor, ii) the general point of the boundary is smooth, iii) the codimension of $\mathcal{A}_g^{(r)} = q^{-1}(\mathcal{A}_{g-r})$ equals r, iv) there is a dense open subset U of $\mathcal{A}_g^{(\leq 4)}$ and a family of compactified semi-abelian varieties $\mathcal{X} \to U$ extending the universal compactified semi-abelian variety over $\mathcal{A}_g^{\leq 1}$ such that

the standard compactifications of §16 form a dense open subset of $\mathcal{A}_g^{(r)}$. We point out that in this case for $r \leq 4$ the general fibre of $q : \mathcal{A}_g^{(r)} \to \mathcal{A}_{g-r}$ has dimension $gr - r(r+1)/2$, thus $\dim(\mathcal{A}_g^{(r))}) = g(g+1)/2 - r$.
We define for our chosen compactification $\tilde{\mathcal{A}}_g$ the boundary as $\partial\tilde{\mathcal{A}}_g := \mathcal{A}_g^{(\geq 1)}$. Moreover we set $\mathcal{A}_g^{(\leq r)} := \tilde{\mathcal{A}}_g - \mathcal{A}_g^{(\geq r+1)}$.
For the Voronoi compactification the fibres of the map $q : \mathcal{A}_g^{(\leq r)} \to \mathcal{A}_g^*$ are well-behaved if $r \leq 4$. Indeed, the points of $\mathcal{A}'_g := \mathcal{A}_g^{(\leq 4)}$ correspond to so-called *stable* quasi-abelian varieties which are compactifications of semi-abelian varieties, which can be explicitely described (see [29, 28, 5] and §§15, 17 for torus ranks 1 and 2). Thus one can define the vertical singular locus and the Andreotti-Mayer loci on the partial compactification $\mathcal{A}'_g$ (see Remark 18.1 below). For higher torus rank the situation is more complicated. For instance the fibres of $q : \mathcal{A}_g^{(\geq r)} \to \mathcal{A}_{g-r}^*$ might be non-reduced. However we will not need $r > 4$ here.
An alternative approach might be to use the idea of Alexeev and Nakamura (cf. [1], [28], [5]) who have constructed canonical limits for 1-dimensional families of abelian varieties equipped with principal theta divisors.
The stable quasi-abelian varieties that occur in $\mathcal{A}'_g$ for torus rank 1 and 2 are exactly those described in Section 15 and 17. For torus rank 3 these are described by the tesselations of $\mathbb{R}^3$ occuring on p. 188 of [28], cf. also Remark 17.3. The open stratum of $\mathcal{A}_g^{(3)}$ over $\mathcal{A}_{g-3}$ corresponds to the standard compactifications (see §16), obtained by glueing six $\mathbb{P}^3$-bundles over a $(g-3)$-dimensional abelian variety B generalizing the construction for torus rank 2 where two $\mathbb{P}^2$-bundles were glued. These closed strata correspond to degenerations of the matrix T of the glueing data on which the standard compactifications depend (see §16). For instance the codimension 3 stratum corresponds to the fact that in T two of the three elements above the main diagonal tend to zero (or to ∞).

Remark 18.1. As we remarked before, for stable quasi-abelian varieties corresponding to points $(\overline{X}, \overline{\Xi})$ of $\mathcal{A}'_g$ one can define the vertical singularities $\mathrm{Sing}_{\mathrm{vert}}(\overline{\Xi})$ using $\Omega^1(\log D)$ as in the previous sections. One checks that for these compactifications the analogue of Lemma 16.1 still holds.

We will need the following result from [14].

Theorem 18.2. *Let Z be an irreducible, closed subvariety of $\tilde{\mathcal{A}}_g$. Then $Z \cap \partial\tilde{\mathcal{A}}_g$ is not empty as soon as* $\mathrm{codim}_{\tilde{\mathcal{A}}_g}(Z) < g$.

19. The Andreotti–Mayer loci and the boundary

We are working with a fixed compactification $\tilde{\mathcal{A}}_g = \tilde{\mathcal{A}}_g^{\mathrm{Vor}}$ and, as indicated in §18 above, we can define the Andreotti–Mayer loci over the part $\mathcal{A}'_g = \mathcal{A}_g^{(\leq 4)}$ of $\tilde{\mathcal{A}}_g$. We have $N_{g,k}$ as a subscheme of $\mathcal{A}_g$ and we define $\tilde{N}_{g,k}$ as the schematic closure. The support of $\tilde{N}_{g,k}$ contains the set of points corresponding to pairs $(\overline{X}, \overline{\Xi})$ such that $\mathrm{Sing}_{\mathrm{vert}}(\overline{X})$ has a component of dimension at least k (see [27]).

It is interesting to look at the case $k = 0$, which has been worked out by Mumford [27] and Debarre [11]. In this case $\tilde{N}_{g,0}$ is a divisor and by Theorem 18.2, every irreducible component N of this divisor intersects $\partial\tilde{\mathcal{A}}_g$. Let M be an irreducible component of $N \cap \partial\tilde{\mathcal{A}}_g$, which has dimension $\binom{g+1}{2} - 2$.

First of all, notice that M cannot be equal to $\mathcal{A}_g^{(\geq 2)}$. This follows by the results in §17 and by Propositions 11.6 and 12.1. More generally, in the same way, one proves that M cannot contain $\mathcal{A}_g^{(r)}$ for any $r = 2, 3$ and 4.

Hence M intersects $\mathcal{A}_g^{(1)}$ in a non-empty open set of M, i.e., the intersection with the boundary has points corresponding to semi-abelian varieties of torus rank 1. If M does not dominate $\mathcal{A}_{g-1}$ via the map q, then each fibre must have dimension $g-1$. By Proposition 11.6 this implies that M dominates $N_{0,g-1}$.

If M dominates $\mathcal{A}_{g-1}$ via q, the fibre of $q_{|M}$ over a general point $(B, \Xi) \in \mathcal{A}_{g-1}$ is $N_0(B, \Xi)$.

Recall now that Debarre proves in [11] that $N_{0,g}$ has two irreducible components, one of which is the so-called *theta-null* component $\theta_{0,g}$: the general abelian variety (X, Θ_X) in $\theta_{0,g}$, with Θ_X symmetric, is such that Θ_X has a unique double point which is a 2-torsion point of X lying on Θ_X.

Let $M_{0,g}$ be the other component. The general abelian variety (X, Θ_X) in $M_{0,g}$, with Θ_X symmetric, is such that Θ_X has exactly two double points x and $-x$. It is useful to recall that, by Corollary 8.12, at a general point of either one of these component of $N_{g,0}$, the tangent cone to the theta divisor at the singular points is a smooth quadric.

The component $\theta_{0,g}$ intersects the boundary in two components, $\theta'_{0,g}, \theta''_{0,g}$, one dominating $\theta_{0,g-1}$, the other dominating $\mathcal{A}_{g-1}$ with fibre over the general point $(B, \Xi) \in \mathcal{A}_{g-1}$ given by the component 2Ξ of $N_0(B, \Xi)$ (see §11). Also $M_{0,g}$ intersects the boundary in two irreducible divisors $M'_{0,g}, M''_{0,g}$. The former is irreducible and dominates $M_{0,g-1}$, the latter dominates $\mathcal{A}_{g-1}$ with fibre over the general point $(B, \Xi) \in \mathcal{A}_{g-1}$ given by the components of $N_0(B, \Xi)$ different from 2Ξ.

The main ingredient for Debarre's proof of the irreducibility of $M_{0,g}$ is a monodromy argument which implies that, if (B, Ξ) is a general principally polarized abelian variety of dimension g, then $N_0(B, \Xi)$ consists of only two irreducible components.

Remark 19.1. Let (B, Θ_B) be a general element in $\theta_{0,g}$ and let $(\overline{X}, \overline{\Xi})$ be a semi-abelian variety of torus rank one with abelian part B. Then there are no points in $\mathrm{Sing}_{\mathrm{vert}}(\overline{\Xi})$ with multiplicity larger than 2. This follows from the fact that Θ_B has a unique singular point and by Remark 15.1.

We finish this section with the following result which will be useful later on. It uses the notion of asymptotic cone given in §6.

Proposition 19.2. *One has:*

(i) *let $g \geq 3$, let (B, Ξ) be a general point of $\theta_{0,g}$ and let x be the singular point of Ξ. Then the asymptotic cone $TC_\xi^{(4)}$ is strictly contained in the quadric tangent cone Q_x;*

(ii) *let $g \geq 4$, let (B, Ξ) be a general point of $M_{0,g}$ and let $x, -x$ be the singular points of Ξ. Then the asymptotic cone $TC_{\xi}^{(3)}$ is strictly contained in the quadric tangent cone $Q_x = Q_{-x}$.*

Proof. Degenerate to the jacobian locus and apply the results from [18, 19]. □

20. The Conjecture for $N_{1,g}$

In this section we prove Conjecture 9.1 for $k = 1$. We consider an irreducible component N of $\tilde{N}_{g,1}$ which is of codimension 3. The first observation is that the assumption about the codimension of N implies that the generic principally polarized abelian variety is simple since by Remark 7.4 every irreducible component of $\mathcal{A}_g^{(\mathrm{ns})}$ has codimension $\geq g-1 > 3$ if we assume $g \geq 5$.

Proposition 20.1. *Let $g \geq 6$ and let N be an irreducible component of $\tilde{N}_{g,1}$ which is of codimension 3 in $\tilde{\mathcal{A}}_g$. Then N intersects the stratum $\mathcal{A}_g^{(1)}$.*

Proof. We begin by remarking that N cannot be complete in $\mathcal{A}_g$ in view of Theorem 18.2. Therefore N intersects $\partial\tilde{\mathcal{A}}_g$. Here we shall use the perfect cone compactification, see §18. Since $\partial\tilde{\mathcal{A}}_g$ is a divisor in $\tilde{\mathcal{A}}_g$ it intersects N in codimension one. Let M be an irreducible component of $N \cap \partial\tilde{\mathcal{A}}_g$. It is our intention to prove that M has a non-empty intersection with $\mathcal{A}_g^{(1)}$.
Suppose that $M \subseteq \mathcal{A}_g^{(\geq 4)}$. For dimension reasons we have $M = \mathcal{A}_g^{(\geq 4)}$. Since we are using a compactification $\tilde{\mathcal{A}}_g$ such that the general point of $\mathcal{A}_g^{(4)}$ corresponds to a standard compactification $(\overline{X}, \overline{\Xi})$ of torus rank 4 with abelian part $(B, \Xi) \in \mathcal{A}_{g-4}$ we deduce from Lemma 16.1 and Remark 18.1 that if $\dim(\mathrm{Sing}_{\mathrm{vert}}(\overline{\Xi})) \geq 1$ then $(B, \Xi) \in N_{g-4,0}$. But for $g \geq 5$ the locus $N_{g-4,0}$ is a divisor in $\mathcal{A}_{g-4}$ and we obtain the inequality $\mathrm{codim}_{\partial\tilde{\mathcal{A}}_g}(M) \geq \mathrm{codim}_{\partial\tilde{\mathcal{A}}_g}(\mathcal{A}_g^{(4)}) + 1 \geq 5$, a contradiction. Therefore we can assume that $M \cap \mathcal{A}_g^{(\leq 3)} \neq \emptyset$.
Suppose that $M \cap \mathcal{A}_g^{(3)}$ has codimension 1 in $\mathcal{A}_g^{(3)}$. Then either M maps dominantly to $\tilde{\mathcal{A}}_{g-3}$ via the map $q : \mathcal{A}_g^{(3)} \dashrightarrow \tilde{\mathcal{A}}_{g-3}$ and M intersects the general fibre F of q in a divisor, or M maps to a divisor in $\mathcal{A}_{g-3}$ under q with full fibres F contained in M.
The former case is impossible by Proposition 12.1. In the latter case for a general (B, Ξ) in $q(M)$ all the quasi-abelian varieties $(\overline{X}, \overline{\Xi})$ in the fibre F over (B, Ξ) must have a 1-dimensional vertical singular locus of $\overline{\Xi}$. Note that $(\overline{X}, \overline{\Xi})$ corresponds to a standard compactification as considered in §16. By the discussion given in §16 and by Proposition 11.6, we see that $q(M)$ has to be contained in $N_{g-3,1}$, against Theorem 8.6. We thus conclude that $M \cap \mathcal{A}_g^{(\leq 2)} \neq \emptyset$.
Suppose that $M \cap \mathcal{A}_g^{(\geq 2)} \neq \emptyset$. Then $M \cap \mathcal{A}_g^{(2)}$ has codimension 2 in $\mathcal{A}_g^{(2)}$. As above we have that $q(M)$ is contained in $N_{g-2,0}$.
Suppose first $q(M)$ is dense in a component of $N_{g-2,0}$. If (B, Ξ) is a general element of $q(M)$, then M intersects the fibre of $q : \mathcal{A}_g^{(2)} \dashrightarrow \tilde{\mathcal{A}}_{g-2}$ over (B, Ξ) in

codimension one. This gives a contradiction by the analysis §17 and Proposition 11.6.
Suppose that $q(M)$ is not dense in a component of $N_{g-2,0}$. If (B,Ξ) is a general element of $q(M)$, then M contains the full fibre of $q:\mathcal{A}_g^{(2)}\to\mathcal{A}_{g-2}$ over (B,Ξ). By taking into account the analysis §17 and Proposition 11.6, this implies $q(M)$ contained in $N_{g-2,1}$, giving again a contradiction.
This proves that $M\cap\mathcal{A}_g^{(\leq 1)}\neq\emptyset$. □

From now on we use again the Voronoi compactification. Let $g\geq 4$ and let N be an irreducible component of $N_{g,1}$ of codimension 3 in $\mathcal{A}_g$. As in the proof above we denote by M an irreducible component of the intersection of the closure of N in $\tilde{\mathcal{A}}_g$ with the boundary $\partial\tilde{\mathcal{A}}_g$. According to Lemma 20.1 the morphism $q:\tilde{\mathcal{A}}_g\to\mathcal{A}_g^*$ induces a rational map $\alpha: M \dashrightarrow \tilde{\mathcal{A}}_{g-1}$, whose image is not contained in $\partial\tilde{\mathcal{A}}_{g-1}$.

Lemma 20.2. *In the above setting, the Zariski closure of $q(M)$ in $\tilde{\mathcal{A}}_{g-1}$ is:*

(i) *either an irreducible component N_1 of $\tilde{N}_{g-1,1}$ of codimension 3 in $\tilde{\mathcal{A}}_{g-1}$;*
(ii) *or an irreducible component N_0 of $\tilde{N}_{g-1,0}$ and in this case:*
   (a) *if $\eta=(B,\Xi)\in N_0$ is a general point, then the closure of $q^{-1}(\eta)$ in B is an irreducible component of $N_1(B,\Xi)$ of codimension 2 in B;*
   (b) *if $\xi=(\overline{X},\overline{\Xi})\in M$ is a general point, then $\mathrm{Sing}_{\mathrm{vert}}(\overline{\Xi})$ meets the singular locus D of $\overline{X}$ in one or two points, whose associated quadric has corank 1.*

Proof. If $q(M)\subseteq N_{g-1,1}$, then Theorem 8.6 implies $3\leq\mathrm{codim}_{\mathcal{A}_{g-1}}(q(M))\leq\mathrm{codim}_{\partial\tilde{\mathcal{A}}_g}(M)=3$ and the closure of $q(M)$ must an irreducible component of $\tilde{N}_{g-1,1}$. If $q(M)\not\subseteq N_{g-1,1}$ then by Proposition 12.1 we have that $q(M)\subseteq N_{g-1,0}$ and the fibre $q^{-1}(B,\Xi)\subseteq N_1(B,\Xi)$. By Proposition 11.6 we have $\mathrm{codim}_B(N_1(B,\Xi))\geq 2$ and since $N_{g-1,0}$ is a divisor in $\mathcal{A}_{g-1}$ we see that (iia) follows. The last statement (iib) follows from Remark 12.3, the analysis in Section 17, the description of $N_{g,0}$ by Mumford and Debarre (see [27], [11], and §19) and Corollary 8.12. □

We are now ready for the proof of the conjecture for $N_{g,1}$.

Theorem 20.3. *Let $g\geq 4$. Then the codimension of an irreducible component N of $N_{g,1}$ in $\mathcal{A}_g$ is at least 3 with equality if and only if:*

(i) *$g=4$ and either $N=\mathcal{H}_4$ is the hyperelliptic locus or $N=\mathcal{A}_{4,(1,3)}$;*
(ii) *$g=5$ and $N=\mathcal{J}_5$ is the jacobian locus.*

Proof. By Theorem 8.6, the codimension of N is at least 3. Suppose that N has codimension 3. It is well-known that the assertion holds true for $g=4$ and 5 (see [4], [10], [8]). We may thus assume $g\geq 6$ and proceed by induction.
Let $\zeta=(X,\Theta_X)$ be a general point of N, so that X is simple (see Remark 7.4). Let S be a 1-dimensional component of $\mathrm{Sing}(\Theta_X)$. We can assume that the

class of S in X is a multiple $m\gamma_X$ of the *minimal class* $\gamma_X = \Theta_X^{g-1}/(g-1)! \in H^2(X,\mathbb{Z})$. If not so, then $\mathrm{End}(X) \neq \mathbb{Z}$ and this implies that $\mathrm{codim}_{\mathcal{A}_g}(N) \geq g-1$ (see Remark 7.4).
By Theorem 8.6, the general point in S is a double point for Θ_X. We let R be the curve in S_g whose general point is $\xi = (X, \Theta_X, x)$, with $x \in S$ a general point. Note that R is birationally equivalent to S. Let $\mathcal{Q}$ be the linear system of all quadrics in $\mathbb{P}(T_{X,0})$. One has the map

$$\phi : \xi \in R \dashrightarrow Q_\xi \in \mathcal{Q}.$$

As in the proof of Theorem 8.6, the map ϕ is not constant. Let $\mathcal{Q}_R$ be the span of the image of ϕ. As in the proof of Theorem 8.6, one has $\dim(\mathcal{Q}_R) \geq 2$. By Proposition 8.3, $\mathcal{Q}_R$ is contained in the linear system $\mathcal{N}_{g,1}(\zeta)$ (see §7 for the definition), which has dimension at most 2 since $\mathrm{codim}_{\mathcal{A}_g}(N) = 3$. Thus $\mathcal{Q}_R = \mathcal{N}_{g,1}(\zeta)$ has dimension 2.
By Lemma 20.1, the closure of N in $\mathcal{A}_g^{(\leq 1)}$ has non-empty intersection with the boundary. As in the proof of Lemma 20.1, we let M be an irreducible component of the intersection of the closure of N in $\mathcal{A}_g^{(\leq 1)}$ with the boundary. Consider the rational map $\alpha : M \dashrightarrow \mathcal{A}_{g-1}$ and the closure of the image $\alpha(M)$, for which we have the possibilities described in Lemma 20.2.

Claim 20.4. *Possibility (i) in Lemma 20.2 does not occur.*

Proof of the claim. By induction, one reduces to the case $g = 6$ and $\alpha(M) = \mathcal{J}_5$. Let $(\overline{X}, \overline{\Xi}) \in M$ be a general point. Then $(\overline{X}, \overline{\Xi})$ is a general rank one extension of the Jacobian $(J(C), \Theta_C)$ of a general curve C of genus 5. Note that if $x \in J(C)$ corresponds to the extension, then Θ_C and $x + \Theta_C$ are not tangentially degenerate (see [17], Thm. 10.8, p. 273). Then the analysis of §15 implies that the vertical singular locus S_0 of $\overline{\Xi}$ sits on the singular locus $D \cong J(C)$ of $\overline{X}$ and it is isomorphic to $S_C = \mathrm{Sing}(\Theta_C)$ with cohomology class $\Theta^4/12$ (see [3]). Thus $\overline{\Xi} \cdot S_0 = \Theta_C \cdot S_C = 10$. Hence, if $\zeta = (X, \Theta_X)$ is a general point of N, then $\mathrm{Sing}(\Theta_X)$ is a curve S such that $\Theta_X \cdot S = 10$. On the other hand S is homologous to $m\gamma_X$ and one has $10 = m\Theta_X \cdot \gamma_X = 6m$, a contradiction. $\square$

Claim 20.4 shows that only possibility (ii) in Lemma 20.2 can occur. In particular, by (iib) of Lemma 20.2, for $\xi = (X, \Theta_X, x)$ general in R, the quadric Q_ξ has corank 1. Let $v_\xi \in \mathbb{P}^{g-1}$ be the vertex of Q_ξ. Remember that R is birational to S. Hence, by Proposition 4.4, the map

$$\gamma : \xi \in R \dashrightarrow v_\xi \in \mathbb{P}^{g-1}$$

can be regarded as the Gauss map γ_S of S.

Claim 20.5. *If the general quadric in the linear system $\mathcal{Q}_R$ is singular, then for $\xi = (X, \Theta_X, x)$ general in R, the vertex v_ξ of Q_ξ is contained in the asymptotic cone $TC_\xi^{(4)}$.*

Proof of the claim. Suppose the general quadric in $\mathcal{Q}_R$ is singular. Then the general quadric in $\mathcal{Q}_R$ has corank 1 (see Lemma 20.2, (iib)) and, by Bertini's theorem, its vertex lies in the base locus of $\mathcal{Q}_R$. In particular, for $\xi = (X, \Theta_X, x)$ general in R, the vertex v_ξ of the quadric Q_ξ lies in all the quadrics of $\mathcal{Q}_R$. Choose a local parametrization $x = x(t)$ of S around a general point of it, with t varying in a disc Δ. Then $\xi(t) = (X, \Theta_X, x(t)) \in R$ and we set $Q_{\xi(t)} := Q_t$, its equation being

$$\sum_{ij} \partial_i \partial_j \theta(x(t)) z_i z_j = 0,$$

where we set $\theta(z) := \theta(\tau, z)$ for the theta function of X. The main remark is that all the subsequent derivatives of Q_t with respect to t lie in $\mathcal{Q}_R$ and actually Q_t and its first two derivatives Q_t' and Q_t'' span $\mathcal{Q}_R$, because $\dim(\mathcal{Q}_R) = 2$. Hence Bertini's theorem implies that $x' := x'(s)$ sits on all these quadrics for t and s general in Δ. The equations of Q_t' and Q_t'' are respectively

$$\sum_{ijh} \partial_i \partial_j \partial_h \theta(x(t)) x_h'(t) z_i z_j = 0$$

$$\sum_{ijhk} \partial_i \partial_j \partial_h \partial_k \theta(x(t)) x_h'(t) x_k'(t) z_i z_j + \sum_{ijh} \partial_i \partial_j \partial_h \theta(x(t)) x_h''(t) z_i z_j = 0$$

Thus we have the relations

(29)

$$\sum_{ij} \partial_i \partial_j \theta(x(t)) x_i'(s) x_j'(t) = 0, \qquad \sum_{ijh} \partial_i \partial_j \partial_h \theta(x(t)) x_h'(t) x_i'(s) x_j'(s) = 0$$

$$\sum_{ijhk} \partial_i \partial_j \partial_h \partial_k \theta(x(t)) x_h'(t) x_k'(t) x_i'(s) x_j'(s) + \sum_{ijh} \partial_i \partial_j \partial_h \theta(x(t)) x_h''(t) x_i'(s) x_j'(s) = 0$$

identically in $s, t \in \Delta$. The first of these relations says that the tangent hyperplane to Q_t at $x'(t)$ contains the vertex $x'(s)$. From the second relation we have

$$\sum_{ijh} \partial_i \partial_j \partial_h \theta(x(t)) x_h'(t) x_i'(t) x_j'(t) = 0 \tag{30}$$

which shows that v_ξ is contained in the asymptotic cone $TC_\xi^{(3)}$. By differentiating (30), one finds

$$\sum_{ijhk} \partial_i \partial_j \partial_h \partial_k \theta(x(t)) x_h'(t) x_k'(t) x_i'(t) x_j'(t) + 3 \sum_{ijh} \partial_i \partial_j \partial_h \theta(x(t)) x_h''(t) x_i'(t) x_j'(t) = 0.$$

By comparing with (29) for $s = t$, we deduce that

$$\sum_{ijhk} \partial_i \partial_j \partial_h \partial_k \theta(x(t)) x_h'(t) x_k'(t) x_i'(t) x_j'(t) = 0$$

which proves that $v_\xi \in TC_\xi^{(4)}$. □

The crucial step in our proof is the following claim.

Claim 20.6. *The general quadric in the linear system $\mathcal{Q}_R$ is non singular.*

Proof of the claim. Suppose this is not the case. Again we consider an irreducible component M of $\bar{N} \cap \partial \tilde{\mathcal{A}}'_g$ and let $(\overline{X}, \overline{\Xi})$ be a general point of M. By Claim 20.4 and Lemma 20.2, $(\overline{X}, \overline{\Xi})$ is a rank 1 extension of (B, Ξ) corresponding to a general point in a component of $N_{g-1,0}$, with extension datum b varying in a codimension 2 component of $N_1(B, \Xi)$. We let S_0 be the vertical singular locus of $\overline{\Xi}$. By the analysis of §15, this corresponds to a contact curve $C := C_b$ of Ξ with Ξ_b, which contains the singular points of both Ξ and Ξ_b (see Remark 12.3). This means that we have a point on S_0, corresponding to a singular point x of Ξ, where the tangent cone is the cone over Q_x with vertex the point of $\mathbb{P}^{g-1}$ corresponding to b (see S 15).
Now we note that C is smooth at x. Indeed locally around x, the divisor Ξ looks like a quadric cone of corank 1 in $\mathbb{P}^{g-2}$ and Ξ_b looks like a hyperplane, which touches it along a curve. This implies that C locally at x looks like a line along which a hyperplane touches a quadric cone of corank 1 (see Remark 12.4).
Hence, the Gauss image $x_b := \gamma_{C_b}(x)$ lies in Q_x and actually, by Claim 20.5, the point x_b lies in the asymptotic cone $TC_x^{(4)}$.
The Gauss map $\gamma_\Xi : \Xi \dashrightarrow \mathbb{P}^{g-2}$ of Ξ has an indeterminacy point at x and $-x$, which can be resolved by blowing up x and $-x$, since we may assume Ξ to be symmetric. Let $p : \tilde{\Xi} \to \Xi$ be the blow-up and let $\tilde{\gamma}_\Xi$ be the morphism which coincides with $\gamma_\Xi \circ p : \tilde{\Xi} \dashrightarrow \mathbb{P}^{g-2}$ on an open subset. The exceptional divisor at x and $-x$ is isomorphic to Q_x and $\tilde{\gamma}_\Xi(x_b)$ is the tangent hyperplane to Q_x at x_b. The tangency property of Ξ and Ξ_b along C implies that the tangent hyperplane to Q_x at x_b coincides with $\gamma_{\Xi_b}(x)$.
Now we let b vary in a component Z of $N_1(B, \Xi)$ of dimension $g-3$, so that we have a rational map

$$f : Z \dashrightarrow Q_x, \quad b \mapsto x_b.$$

Note that Q_x also has dimension $g-3$ and we claim that f has finite fibres, hence it is dominant. If not, we would have an irreducible curve Γ in Z such that for all $b \in \Gamma$, x_b stays fixed. But then for the general $b \in \Gamma$, the divisor Ξ_b has a fixed tangent hyperplane at x, a contradiction by Lemma 11.1.
On the other hand, by Claim 20.5 and by part (i) of Proposition 19.2, one has that f cannot be dominant, a contradiction. □

By Lemma 20.6, the curve $\phi(R) := \Sigma$ is an irreducible component of the discriminant $\Delta \subset \mathcal{Q}_R$ of singular quadrics in $\mathcal{Q}_R \cong \mathbb{P}^2$. Note that Δ has degree g in $\mathbb{P}^2$. The map ϕ has degree at least 2 since we may assume Θ_X to be symmetric, hence it factors through the multiplication by -1_X on X.

Claim 20.7. *The map $\phi : R \to \Sigma$ has degree 2.*

Proof. Let $d \geq 2$ be the degree of the map. Then for $\xi = (X, \Theta_X, x)$ general in R, we have distinct points $\xi_i = (X, \Theta_X, x_i) \in R$, with $\xi = \xi_1$, such that all the quadrics Q_{ξ_i}, $i = 1, \dots, d$, coincide with the quadric $Q = Q_\xi$. If for $i = 1, \dots, d$ we let η_i be the tangent vector to S at x_i, we have that $\eta_1 = \cdots = \eta_d$ and

$\partial_{\eta_i} Q_{\xi_i}$, for $i = 1, \dots, d$, coincide with the quadric $Q_1 = \partial_{\eta_1} Q$, which is linearly independent from Q. The analysis we made in §3 shows that S_g is smooth at each of the points ξ_i, $i = 1, \dots, d$, and the tangent space there is determined by Q and Q_{η_1}. This implies that a general deformation of $\xi = (X, \Theta_X, x)$ inside S_g carries with it a deformation of each of the points ξ_i $(i = 1, \dots, d)$ in S_g, because the involved quadrics are the same at these points. This yields that a general element of an irreducible component of $N_{g,0}$ containing N has at least d singular points. By Debarre's result in [11] (see §19), one has $d = 2$, proving our claim. □

Claim 20.8. *The map ϕ is a morphism.*

Proof. To prove the claim, it suffices to show that N is not contained in $N_{g,0,r}$, with $r \geq 3$. In order to prove this, one verifies that, for a general point $(\overline{X}, \overline{\Xi})$ in a component M of the intersection of N with the boundary, there are no points of multiplicity $r \geq 3$ in $\mathrm{Sing}_{\mathrm{vert}}(\overline{\Xi})$. Recall that, by Proposition 20.1, we may assume that $(\overline{X}, \overline{\Xi})$ is a semi-abelian variety of torus rank 1, with abelian part $\eta = (B, \Xi)$. Moreover, by Claim 20.4, only case (ii) of Lemma 20.2 can occur. Therefore we may assume that η is either a general point of $\theta_{0,g-1}$ or a general point of $M_{0,g-1}$ and the extension corresponds to a general point in an irreducible component of $N_1(B, \Xi)$ which has dimension $g - 3 > 0$. By Remark 15.1 we see that no triple points can occur on $\mathrm{Sing}_{\mathrm{vert}}(\overline{\Xi})$. □

Note now that the morphism ϕ is defined on $R \cong S$ by sections of $\mathcal{O}_S(\Theta_X)$, since the points of S verify the equations (2) and, if $\xi = (X, \Theta_X, x) \in R$, the entries of the matrix of Q_ξ are $\partial_i \partial_j \theta(\tau, z)$, where z corresponds to x. We deduce from $\deg(\Delta) = g$ and from Claim 20.7, that

$$S \cdot \Theta_X \leq 2g. \tag{31}$$

As we assumed at the beginning of the proof, the class of S in X is a multiple $m\gamma_X$ of the minimal class. In view of (31), we find $m \leq 2$. The Matsusaka-Ran criterion [31] and a result of Welters [40] imply that (X, Θ_X) is either a Jacobian or a Prym variety or depends on less than $3g$ parameters. Since $g \geq 6$ this is not possible in view of the dimensions. This ends the proof. □

Remark 20.9. A. Verra communicated to us an interesting example of an irreducible component M of codimension 6 of $N_{6,1}$ contained in the Prym locus. We briefly sketch, without entering in any detail, its construction and properties. Let C be the normalization of a general curve of type $(4, 4)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ with two nodes on a line of type $(0, 1)$, so that C has genus $g = 7$. Let d, t be the linear series formed by pull-back divisors on C of the rulings of type $(1, 0), (0, 1)$ respectively. Consider a non trivial, unramified double cover $f : \tilde{C} \to C$ of C and let (P, Ξ) be the corresponding Prym variety. Then Ξ has a 1-dimensional unstable singular locus R (see [23]), homologically equivalent to twice the minimal class, described by all classes in $\mathrm{Pic}^{12}(\tilde{C})$ of divisors of the form $f^*(d) + M$, with $f_*(M) \in t$. One proves that the map ϕ described in the proof of Theorem 20.3 sends R to a plane sextic of genus 7 which is tetragonally associated to C (see [23] for the tetragonal construction). The divisor Ξ has 24 further isolated singular points, which are pairwise exchanged by the multiplication by -1. One shows that the corresponding 12 tangent cones span the same linear system $\mathcal{Q}$ of dimension 2 spanned by $\phi(R)$. The linear system $\mathcal{Q}$ is the tangent space to the Prym locus $\mathcal{P}_6$ at (P, Ξ), which is therefore a smooth point for $\mathcal{P}_6$. By contrast, M is a non-reduced component of $N_{6,1}$ of codimension 6 such that the projectivized normal space $\mathcal{Q}$ at a general point has dimension 2 rather than 5. This shows that the hypotheses of Theorem 20.3 cannot be relaxed by assuming only that the projectivized normal space to M at a general point has dimension 2.

21. Appendix: A Result on Pencils of Quadrics

One of the ingredients of the proof Theorem 8.6 is a classical result of Corrado Segre from [33] on pencils of quadrics.
First we recall the following:

Proposition 21.1. *Let $\mathcal{L}$ be a pencil of quadrics in $\mathbb{P}^n$ with $n \geq 1$ whose general member is smooth. Then:*

- (i) *the number of singular quadrics $Q \in \mathcal{L}$ is $n+1$, where each such quadric Q has to be counted with a suitable multiplicity $\mu(Q) \geq n + 1 - \mathrm{rk}(Q)$;*
- (ii) *for a singular quadric $Q \in \mathcal{L}$ one has $\mu(Q) \geq 2$ if and only if either $\mathrm{rk}(Q) < n$ or the singular point of Q is also a base point of $\mathcal{L}$;*
- (iii) *for a singular quadric $Q \in \mathcal{L}$ with rank n one has $\mu(Q) = 2$ if and only if any other quadric $Q' \in \mathcal{L}$ is smooth at p and the tangent hyperplane to Q' at p is not tangent to Q along a line.*

Proof. Consider the linear system $\mathcal{Q}_n$ of dimension $n(n+3)/2$ all quadrics in $\mathbb{P}^n$. Inside $\mathcal{Q}_n$ we have the discriminant locus Δ_n of singular quadrics, which is a hypersurface of degree $n+1$, defined by setting the determinant of a general quadric equal to zero. The differentiation rule for determinants implies that the locus $\Delta_{n,r}$ of quadrics of rank $r < n+1$ has multiplicity $n+1-r$ for Δ_n. By intersecting Δ_n with the line corresponding to $\mathcal{L}$ we have (i).
As for assertion (ii), we may assume $\mathrm{rk}(Q) = n$, so that Q has a unique double point p, which we may suppose to be the point $(1, 0, \ldots, 0)$. Thus the matrix of Q is of the form

$$\begin{pmatrix} 0 & 0_n \\ 0_n^t & A \end{pmatrix}$$

where $0_n \in \mathbb{C}^n$ is the zero vector and A is a symmetric matrix of order n and maximal rank. Let Q' be another quadric in $\mathcal{L}$, with matrix

$$\begin{pmatrix} \beta & b \\ b^t & B \end{pmatrix}$$

with $\beta \in \mathbb{C}$, $b = (b_1, \ldots, b_n) \in \mathbb{C}^n$ and B is a symmetrix matrix of order n. By intersecting $\mathcal{L}$ with Δ_n, we find the equation

$$\det\begin{pmatrix} t\beta & tb \\ tb^t & A+tB \end{pmatrix} = 0. \tag{32}$$

The constant term in the left-hand-side is 0. The coefficient of the linear term is

$$\det\begin{pmatrix} \beta & b \\ 0_n^t & A \end{pmatrix} = \beta\det(A)$$

which proves (ii).
Let us prove (iii). Suppose $\mathrm{rk}(Q) = n$, so that Q has a unique double point p, which is a base point of $\mathcal{L}$. Again we may suppose p is the point $(1:0:\ldots:0)$ and we can keep the above notation and continue the above analysis. The left-hand-side in (32) is

$$t^2\det\begin{pmatrix} 0 & b \\ b^t & A+tB \end{pmatrix} = 0$$

hence the coefficient of the third order term is

$$\det\begin{pmatrix} 0 & b \\ b^t & A \end{pmatrix}. \tag{33}$$

One has $\mu(Q) = 2$ if and only if this determinant is not zero, hence $b \neq 0$, which is equivalent to saying that all quadrics in the pencil different from Q are smooth at p. Note that there is a vector $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$ such that $b = c \cdot A$. Now the determinant in (33) vanishes if and only if $c \cdot b^t = 0$, i.e. $c \cdot A \cdot c^t = 0$. This means that the line L joining p with $(0 : c_1 : \ldots : c_n)$ sits on Q and that the tangent hyperplane to Q' at p, which has equation $b_1x_1 + \cdot + b_nx_n = 0$, is tangent to Q along r. □

Next we prove Segre's theorem.

Theorem 21.2. *Let $\mathcal{L}$ be a linear pencil of singular quadrics in $\mathbb{P}^n$ with $n \geq 2$ whose general member Q has rank $n+1-r$, i.e.* $\mathrm{Vert}(Q) \cong \mathbb{P}^{r-1}$. *We assume that* $\mathrm{Vert}(Q)$ *is not constant when Q varies in $\mathcal{L}$ with rank $n+1-r$. Then:*

(i) *the Zariski closure*

$$V_{\mathcal{L}} = \overline{\Big(\bigcup_{Q\in\mathcal{L},\,\mathrm{rk}(Q)=n+1-r} \mathrm{Vert}(Q)\Big)}$$

is a variety of dimension r spanning a linear subspace Π of dimension m in $\mathbb{P}^n$ with $r \leq m \leq (n+r-1)/2$;

(ii) *$V_{\mathcal{L}}$ is a variety of minimal degree $m-r+1$ in Π;*

(iii) *if*

$$\dim\Big(\bigcap_{Q\in\mathcal{L},\,\mathrm{rk}(Q)=n+1-r} \mathrm{Vert}(Q)\Big) = s$$

then $r \leq (n+2s+3)/3$;

(iv) *the number of quadrics $Q \in \mathcal{L}$ of rank* $\mathrm{rk}(Q) < n+1-r$ *is $n+r-2m-1 \leq n-r-1$, where each such quadric Q has to be counted with a suitable multiplicity* $\nu(Q) \geq n+1-r-\mathrm{rk}(Q)$.

Proof. We start with the proof of part (i). Notice that, by iteratedly restricting to a general hyperplane, we can reduce to the case $r = 1$.

In this case $V_{\mathcal{L}}$ is a rational curve which, by Bertini's theorem, is contained in the base locus B of $\mathcal{L}$. Let p, q be general points on it and let L be the line joining them. There is a quadric $Q_p \in \mathcal{L}$ with vertex at p. Hence Q_p contains L. Similarly there is a different quadric Q_q with vertex at q, and it also contains L. Since Q_p and Q_q span $\mathcal{L}$, we see that L is contained in B, i.e., the secant variety to $V_{\mathcal{L}}$ is contained in B. Take now three general points p, q, r on $V_{\mathcal{L}}$. Since the lines pq, pr, qr are contained in B also the plane spanned by p, q, r is contained in B. Continuing this way, we see that $\Pi = \langle V_{\mathcal{L}} \rangle$ is contained in B. Since the general quadric in $\mathcal{L}$ has rank n (recall we are assuming $r = 1$ now), the maximal dimension of subspaces on it is $n/2$. Thus $\dim(\Pi) \leq n/2$ which proves part (i).

Also for (ii) we can reduce ourselves to the case $r = 1$, in which we have to prove that $V_{\mathcal{L}}$ is a rational normal curve in $\Pi = \langle V_{\mathcal{L}} \rangle$. Set $\dim(\Pi) = m$.

Let $p \in V_{\mathcal{L}}$ be a general point. The polar hyperplane π_p of p with respect to $Q \in \mathcal{L}$ does not depend on Q, since there is a quadric in $\mathcal{L}$ which is singular at p (see the proof of Proposition 21.1). Note that π_p has to contain all vertices of the quadrics in $\mathcal{L}$, hence it contains Π. By the linearity of polarity, we have that polarity with respect to all quadrics in $\mathcal{L}$ is constant along Π and for a general point $x \in \Pi$, the polar hyperplane π_x with respect to all quadrics in $\mathcal{L}$ contains Π. Furthermore the linear system of hyperplanes $\mathcal{P} = \{\pi_x\}_{x \in \Pi}$ has dimension $m - 1$.

Now, let $p \in V_{\mathcal{L}}$ be a general point and let Q_p be the unique quadric in $\mathcal{L}$ with a double point at p. We denote by $\mathrm{Star}(p)$ the $\mathbb{P}^{m-1}$ of all lines in Π containing p. Let $\pi \in \mathcal{P}$ be a general hyperplane, which is tangent to Q_p along a line L containing p. Moreover L sits in Π, because this is the case if $\pi = \pi_q$ with q another general point on $V_{\mathcal{L}}$, in which case L is the line $\langle p, q \rangle$. Thus we have a linear map $\phi_p : \mathcal{P} \to \mathrm{Star}(p)$, which is clearly injective and therefore an isomorphism.

Fix now another general point q on $V_{\mathcal{L}}$. The two maps ϕ_p and ϕ_q determine a linear isomorphism $\phi : \mathrm{Star}(p) \to \mathrm{Star}(q)$. Note that L meets $\phi(L)$ if and only if $L \cap \phi(L)$ is a point of $V_{\mathcal{L}}$. This implies that $V_{\mathcal{L}}$ is a rational normal curve in Π, proving part (ii).

Let us prove part (iii). It suffices to prove the assertion if $s = -1$. The variety $V_{\mathcal{L}}$ is swept out by a 1-dimensional family of projective spaces of dimension $r - 1$, i.e., the vertices of the quadrics in $\mathcal{L}$. Under the assumption $s = -1$ no two of these vertices can intersect. Thus we must have $2(r - 1) < m$. Using part (i), the assertion follows.

Finally we come to part (iv). Let us restrict $\mathcal{L}$ to a general subspace Λ of dimension $n - r$. We get a pencil $\bar{\mathcal{L}}$ of quadrics in Λ whose general member is smooth.

We get a singular quadric in $\bar{\mathcal{L}}$ when we intersect Λ with a quadric in $\mathcal{L}$ whose vertex intersects Λ. We claim that this is the only possibility for getting a singular quadric in $\bar{\mathcal{L}}$. Indeed, let $Q \in \mathcal{L}$ and suppose that its intersection

$\bar{Q} \in \bar{\mathcal{L}}$ with Λ is singular at $p \in \Lambda$, but Q is not singular at p. Then Λ is tangent to Q at p and therefore also intersects the vertex of Q.
In conclusion we have only two possibilities for getting singular quadrics in $\bar{\mathcal{L}}$:

(a) there is quadric of rank $n+1-r$ in $\mathcal{L}$ whose vertex intersects Λ;
(b) there is quadric of rank $n+1-h < n+1-r$ in $\mathcal{L}$ giving rise to a quadric of rank $n-h$ in $\bar{\mathcal{L}}$.

Case (a) occurs as many times as the degree of $V_{\mathcal{L}}$, that is, $m-r+1$ times. According to part (ii) of Proposition 21.1, each quadric Q in case (a) contributes with multiplicity at least 2 in the counting of singular quadrics in $\bar{\mathcal{L}}$. We claim that, because of the generality of Λ, this multiplicity is exactly 2. To prove this, by part (iii) of Proposition 21.1, we will prove that for each quadric Q in case (a), with vertex $p \in \Lambda$ and for any other quadric $Q' \in \mathcal{L}$, the tangent hyperplane π to Q' at p is not tangent to Q along a line contained in Λ. To see this we can, by first cutting with a general subspace of dimension $n-r+1$ through Λ, reduce ourselves to the case $r=1$, in which $V_{\mathcal{L}}$ is a rational normal curve. Choose then a general point $q \in V_{\mathcal{L}}$ and let $Q' = Q_q$ be the unique quadric in $\mathcal{L}$ with a double point at q. The hyperplane π is tangent to Q_q along the line $\langle p, q\rangle$. This implies that π is tangent to Q only along the tangent line L_p to $V_{\mathcal{L}}$ at p (see the proof of part ii)). By the generality assumption, Λ is not tangent to $V_{\mathcal{L}}$ at p. Thus the assertion follows.
As for quadrics in case (b), again by part (i) of Proposition 21.1, each such quadric contributes to the same count with multiplicity $h-r$. Since, by part (i) of Proposition 21.1, the number of singular quadrics in $\bar{\mathcal{L}}$, counted with appropriate multiplicity, is $n-r+1$, the assertion follows. □

One has the following consequence:

Corollary 21.3. *Let $\mathcal{L}$ be a linear pencil of quadrics in $\mathbb{P}^n$ with $n \geq 2$. Then the general member $Q \in \mathcal{L}$ has rank $n+1-r$ (i.e.,* $\mathrm{Vert}(Q) \cong \mathbb{P}^{r-1}$*) if and only if the base locus of $\mathcal{L}$ contains a linear subspace Π of dimension m with $r \leq m \leq (n+r-1)/2$, along which all the quadrics in $\mathcal{L}$ have a common tangent subspace of dimension $n+r-m-1$. In this case Π is the span of the variety $V_{\mathcal{L}}$.*

Proof. As usual it suffices to prove the assertion for $r=1$. If the general quadric $Q \in \mathcal{L}$ has rank $\mathrm{rk}(Q) = n$, the assertion follows from the proof of Theorem 21.2. The converse is trivial, since a smooth quadric in $\mathbb{P}^n$ has a tangent subspace of dimension $n-m-1$, and not larger, along a subspace of dimension m. □

These results imply the existence of *canonical forms* for pencils of singular quadrics, originally due to Weierstrass [38] and Kronecker [21]. This is explained in some detail in [33], §§20-25, and we will not dwell on this here.
It would be desirable to have an extension of the results in this Appendix to higher-dimensional linear families of quadrics.

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