Secondary Invariants for Frechet Algebras and Quasihomomorphisms

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Received: March 3, 2008
Revised: June 30, 2008

Communicated by Joachim Cuntz

Abstract. A Fréchet algebra endowed with a multiplicatively convex topology has two types of invariants: homotopy invariants (topological K-theory and periodic cyclic homology) and secondary invariants (multiplicative K-theory and the non-periodic versions of cyclic homology). The aim of this paper is to establish a Riemann-Roch-Grothendieck theorem relating direct images for homotopy and secondary invariants of Fréchet m-algebras under finitely summable quasihomomorphisms.

2000 Mathematics Subject Classification: 19D55, 19K56, 46L80, 46L87.
Keywords and Phrases: K-theory, bivariant cyclic cohomology, index theory.

1 Introduction

For a noncommutative space described by an associative Fréchet algebra $\mathscr{A}$ over $\mathbb{C}$, we distinguish two types of invariants. The first type are (smooth) homotopy invariants, for example topological K-theory [27] and periodic cyclic homology [5]. The other type are secondary invariants; they are no longer stable under homotopy and carry a finer information about the "geometry" of the space $\mathscr{A}$. Typical examples of secondary invariants are algebraic K-theory [29] (which will not be used here), multiplicative K-theory [17] and the unstable versions of cyclic homology [18]. The aim of this paper is to define push-forward maps for homotopy and secondary invariants between two Fréchet algebras $\mathscr{A}$ and $\mathscr{B}$, induced by a smooth finitely summable quasihomomorphism [8]. The compatibility between the different types of invariants is expressed through a noncommutative Riemann-Roch-Grothendieck theorem (Theorem 6.3). The present p aper is the first part of a series on secondary characteristic classes. In the second part we will show how to obtain *local formulas* for push-forward maps, following a general principle inspired by renormalization which establishes the link with chiral anomalies in quantum field theory [25]; in order to keep a reasonable size to the present paper, these methods will be published in a separate survey with further examples [26].

We deal with Fréchet algebras endowed with a multiplicatively convex topology, or Fréchet m-algebras for short. These algebras can be presented as inverse limits of sequences of Banach algebras, and as a consequence many constructions valid for Banach algebras carry over Fréchet m-algebras. In particular Phillips [27] defines topological K-theory groups $K_n^{\rm top}(\mathscr{A})$ for any such algebra $\mathscr{A}$ and $n \in \mathbb{Z}$. The fundamental properties of interest for us are (smooth) homotopy invariance and Bott periodicity, i.e. $K_{n+2}^{\rm top}(\mathscr{A}) \cong K_n^{\rm top}(\mathscr{A})$. Hence there are essentially two topological K-theory groups for any Fréchet m-algebra, $K_0^{\rm top}(\mathscr{A})$ whose elements are roughly represented by idempotents in the stabilization of $\mathscr{A}$ by the algebra $\mathscr{K}$ of "smooth compact operators", and $K_1^{\rm top}(\mathscr{A})$ whose elements are represented by invertibles. Fréchet m-algebras naturally arise in many situations related to differential geometry, commutative or not, and the formulation of index problems. In the latter situation one usually encounters an algebra $\mathscr{I}$ of "finitely summable operators", for us a Fréchet m-algebra provided with a continuous trace on its p-th power for some $p \geq 1$. A typical example is the Schatten class $\mathscr{I} = \mathscr{L}^p(H)$ of p-summable operators on an infinite-dimensional separable Hilbert space H. $\mathscr{A}$ can be stabilized by the completed projective tensor product $\mathscr{I}\hat{\otimes}\mathscr{A}$ and its topological K-theory $K_n^{\rm top}(\mathscr{I}\hat{\otimes}\mathscr{A})$ is the natural receptacle for indices. Other important topological invariants of $\mathscr{A}$ (as a locally convex algebra) are provided by the periodic cyclic homology groups $HP_n(\mathscr{A})$, which is the correct version sharing the properties of smooth homotopy invariance and periodicity mod 2 with topological K-theory [5]. For any finitely summable algebra $\mathscr{I}$ the Chern character $K_n^{\rm top}(\mathscr{I}\hat{\otimes}\mathscr{A}) \to HP_n(\mathscr{A})$ allows to obtain cohomological formulations of index theorems.

If one wants to go beyond differential topology and detect *secondary* invariants as well, which are no longer stable under homotopy, one has to deal with algebraic K-theory [29] and the unstable versions of cyclic homology [18]. In principle the algebraic K-theory groups $K_n^{\rm alg}(\mathscr{A})$ defined for any $n \in \mathbb{Z}$ provide interesting secondary invariants for any ring $\mathscr{A}$, but are very hard to calculate. It is also unclear if algebraic K-theory can be linked to index theory in a way consistent with topological K-theory, and in particular if it is possible to construct direct images of algebraic K-theory classes in a reasonable context. Instead, we will generalize slightly an idea of Karoubi [16, 17] and define for any Fréchet m-algebra $\mathscr{A}$ the multiplicative K-theory groups $MK_n^{\mathscr{I}}(\mathscr{A})$, $n \in \mathbb{Z}$, indexed by a given finitely summable Fréchet m-algebra $\mathscr{I}$. Depending on the parity of the degree n, multiplicative K-theory classe s are represented by idempotents or invertibles in certain extensions of $\mathscr{I}\hat{\otimes}\mathscr{A}$, together with a transgression of their Chern character in certain quotient complexes. Multiplicative K-theory is by definition a mixture of the topological K-theory $K_n^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A})$ and the non-periodic cyclic homology $HC_n(\mathscr{A})$. It provides a "good" approximation of algebraic K-theory but is much more tractable. In addition, the Jones-Goodwillie Chern character in *negative cyclic homology* $K_n^{\mathrm{alg}}(\mathscr{A}) \to HN_n(\mathscr{A})$ factors through multiplicative K-theory. The precise relations between topological, multiplicative K-theory and the various versions of cyclic homology are encoded in a commutative diagram whose rows are long exact sequences of abelian groups

$$\begin{tikzcd} K_{n+1}^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \arrow[r] \arrow[d] & HC_{n-1}(\mathscr{A}) \arrow[r, "\delta"] \arrow[d, equal] & MK_n^{\mathscr{I}}(\mathscr{A}) \arrow[r] \arrow[d] & K_n^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \arrow[d] \\ HP_{n+1}(\mathscr{A}) \arrow[r, "S"] & HC_{n-1}(\mathscr{A}) \arrow[r, "\widetilde{B}"] & HN_n(\mathscr{A}) \arrow[r, "I"] & HP_n(\mathscr{A}) \end{tikzcd} \tag{1}$$

The particular case $\mathscr{I} = \mathbb{C}$ was already considered by Karoubi [16, 17] after the construction by Connes and Karoubi of regulator maps on algebraic K-theory [6]. The incorporation of a finitely summable algebra $\mathscr{I}$ is rather straightforward. This diagram describes the primary and secondary invariants associated to the noncommutative "manifold" $\mathscr{A}$. We mention that the restriction to Fréchet m-algebras is mainly for convenience. In principle these constructions could be extended to all locally convex algebras over $\mathbb{C}$, however the subsequent results, in particular the proof of the Riemann-Roch-Grothendieck theorem would become much more involved.

If now $\mathscr{A}$ and $\mathscr{B}$ are two Fréchet m-algebras, it is natural to consider the adequate "morphisms" mapping the primary and secondary invariants from $\mathscr{A}$ to $\mathscr{B}$. Let $\mathscr{I}$ be a p-summable Fréchet m-algebra. By analogy with Cuntz' description of bivariant K-theory for C^*-algebras [8], if $\mathscr{E} \triangleright \mathscr{I}\hat{\otimes}\mathscr{B}$ denotes a Fréchet m-algebra containing $\mathscr{I}\hat{\otimes}\mathscr{B}$ as a (not necessarily closed) two-sided ideal, we define a *p-summable quasihomomorphism* from $\mathscr{A}$ to $\mathscr{B}$ as a continuous homomorphism

$$\rho : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s\hat{\otimes}\mathscr{B} \ ,$$

where $\mathscr{E}^s$ and $\mathscr{I}^s$ are certain $\mathbb{Z}_2$-graded algebras obtained from $\mathscr{E}$ and $\mathscr{I}$ by a standard procedure. Quasihomomorphisms come equipped with a parity (even or odd) depending on the construction of $\mathscr{E}^s$ and $\mathscr{I}^s$. In general, we may suppose that the parity is p mod 2. We say that $\mathscr{I}$ is multiplicative if it is provided with a homomorphism $\boxtimes : \mathscr{I}\hat{\otimes}\mathscr{I} \to \mathscr{I}$, possibly defined up to adjoint action of multipliers on $\mathscr{I}$, and compatible with the trace. A basic example of multiplicative p-summable algebra is, once again, the Schatten class $\mathscr{L}^p(H)$. Then it is easy to show that such a quasihomomorphism induces a pushforward map in topological K-theory $\rho_! : K_n^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \to K_{n-p}^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{B})$, whose degree coincides with the parity of the quasihomomorphism. This is what one expects from bivariant K-theory and is not really new. Our goal is to extend this map to the entire diagram (1). Direct images for the unstable versions of cyclic homology are necessarily induced by a bivariant *non-periodic* cyclic cohomology class $\mathrm{ch}^p(\rho) \in HC^p(\mathscr{A}, \mathscr{B})$. This bivariant Chern character exists only under certain admissibility properties about the algebra $\mathscr{E}$ (note that it is sufficient for $\mathscr{I}$ to be $(p+1)$-summable instead of p-summable). In particular, the bivariant Chern caracter constructed by Cuntz for any quasihomomorphism in [9, 10] cannot be used here because it provides a bivariant periodic cyclic cohomology class, which does not detect the secondary invariants of $\mathscr{A}$ and $\mathscr{B}$. We give the precise definition of an admissible quasihomomorphism and construct the bivariant Chern character $\mathrm{ch}^p(\rho)$ in section 3, on the basis of previous works [23]. An analogous construction was obtained by Nistor [20, 21] or by Cuntz and Quillen [12]. However the bivariant Chern character of [23] is related to other constructions involv ing the heat operator and can be used concretely for establishing local index theorems, see for example [24]. The pushforward map in topological K-theory combined with the bivariant Chern character leads to a pushforward map in multiplicative K-theory $\rho_! : MK_n^{\mathscr{I}}(\mathscr{A}) \to MK_{n-p}^{\mathscr{I}}(\mathscr{B})$. Our first main result is the following non-commutative version of the Riemann-Roch-Grothendieck theorem (see Theorem 6.3 for a precise statement):

Theorem 1.1 *Let $\rho : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s \hat{\otimes} \mathscr{B}$ be an admissible quasihomomorphism of parity p mod 2. Suppose that $\mathscr{I}$ is $(p+1)$-summable in the even case and p-summable in the odd case. Then one has a graded-commutative diagram*

$$\begin{tikzcd}
K^{\mathrm{top}}_{n+1}(\mathscr{I}\hat{\otimes}\mathscr{A}) \arrow[r] \arrow[d, "\rho_!"] & HC_{n-1}(\mathscr{A}) \arrow[r] \arrow[d, "\mathrm{ch}^p(\rho)"] & MK^{\mathscr{I}}_n(\mathscr{A}) \arrow[r] \arrow[d, "\rho_!"] & K^{\mathrm{top}}_n(\mathscr{I}\hat{\otimes}\mathscr{A}) \arrow[d, "\rho_!"] \\
K^{\mathrm{top}}_{n+1-p}(\mathscr{I}\hat{\otimes}\mathscr{B}) \arrow[r] & HC_{n-1-p}(\mathscr{B}) \arrow[r] & MK^{\mathscr{I}}_{n-p}(\mathscr{B}) \arrow[r] & K^{\mathrm{top}}_{n-p}(\mathscr{I}\hat{\otimes}\mathscr{B})
\end{tikzcd}$$

compatible with the cyclic homology SBI exact sequences after taking the Chern characters $K^{\mathrm{top}}_*(\mathscr{I}\hat{\otimes}\cdot) \to HP_*$ and $MK^{\mathscr{I}}_* \to HN_*$.

At this point it is interesting to note that the pushforward maps $\rho_!$ and the bivariant Chern character $\mathrm{ch}^p(\rho)$ enjoy some invariance properties with respect to equivalence relations among quasihomomorphisms. Two types of equivalence relations are defined: smooth homotopy and conjugation by invertibles. The second relation corresponds to "compact perturbation" in Kasparov KK-theory for C^*-algebras [2]. In the latter situation, the M_2-stable version of conjugation essentially coincide with homotopy, at least for separable $\mathscr{A}$ and σ-unital $\mathscr{B}$. For Fréchet algebras however, M_2-stable conjugation is *strictly stronger* than homotopy as an equivalence relation. This is indeed in the context of Fréchet algebras that secondary invariants appear. The pushforward maps in topological K-theory and periodic cyclic homology are invariant under homotopy of quasihomomorphisms. The maps in multiplicative K-theory and the non-perio dic versions of cyclic homology

HC_* and HN_* are only invariant under conjugation and not homotopy. Also note that in contrast with the C^*-algebra situation, the Kasparov product of two quasihomomorphisms $\rho : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s \hat{\otimes} \mathscr{B}$ and $\rho' : \mathscr{B} \to \mathscr{F}^s \triangleright \mathscr{I}^s \hat{\otimes} \mathscr{C}$ is not defined as a quasihomomorphism from $\mathscr{A}$ to $\mathscr{C}$. The various bivariant K-theories constructed for m-algebras [9, 10] or even for general bornological algebras [11] cannot be used here, again because they are homotopy invariant by construction. We leave the construction of a bivariant K-theory compatible with secondary invariants as an open problem.

In the last part of the paper we illustrate the Riemann-Roch-Grothendieck theorem by constructing assembly maps for certain crossed product algebras. If Γ is a discrete group acting on a Fréchet m-algebra $\mathscr{A}$, under certain conditions the crossed product $\mathscr{A} \rtimes \Gamma$ is again a Fréchet m-algebra and one would like to obtain multiplicative K-theory classes out of a geometric model inspired by the Baum-Connes construction [1]. Thus let $P \stackrel{\Gamma}{\to} M$ be a principal Γ-bundle over a compact manifold M, and denote by $\mathscr{A}_P$ the algebra of smooth sections of the associated $\mathscr{A}$-bundle. If D is a K-cycle for M represented by a pseudodifferential operator, we obtain a quasihomomorphism from $\mathscr{A}_P$ to $\mathscr{A} \rtimes \Gamma$ and hence a map

$$MK_n^{\mathscr{I}}(\mathscr{A}_P) \to MK_{n-p}^{\mathscr{I}}(\mathscr{A} \rtimes \Gamma)$$

for suitable p and Schatten ideal $\mathscr{I}$. In general this map cannot exhaust the entire multiplicative K-theory of the crossed product but nevertheless interesting secondary invariants arise in this way. In the case where $\mathscr{A}$ is the algebra of smooth functions on a compact manifold, $\mathscr{A}_P$ is commutative and its secondary invaiants are closely related to (smooth) Deligne cohomology. From this point of view the pushforward map in multiplicative K-theory should be considered as a non-commutative version of "integrating Deligne classes along the fibers" of a submersion. We perform the computations for the simple example provided by the noncommutative torus.

The paper is organized as follows. In section 2 we review the Cuntz-Quillen formulation of (bivariant) cyclic cohomology [12] in terms of quasi-free extensions for m-algebras. Nothing is new but we take the opportunity to fix the notations and recall a proof of generalized Goodwillie theorem. In section 3 we define quasihomomorphisms and construct the bivariant Chern character. The formulas are identical to those found in [23] but in addition we carefully establish their adic properties and conjugation invariance. In section 4 we recall Phillips' topological K-theory for Fréchet m-algebras, and introduce the periodic Chern character $K_n^{\mathrm{top}}(\mathscr{I} \hat{\otimes} \mathscr{A}) \to HP_n(\mathscr{A})$ when $\mathscr{I}$ is a finitely summable algebra. The essential point here is to give explicit and simple formulas for subsequent use. Section 5 is devoted to the definition of the multiplicative K-theory groups $MK_n^{\mathscr{I}}(\mathscr{A})$ and the proof of the long exact seque nce relating them with topological K-theory and cyclic homology. We also construct the negative Chern character $MK_n^{\mathscr{I}}(\mathscr{A}) \to HN_n(\mathscr{A})$ and show the compatibility with the SBI exact sequence. Direct images of topological and multiplicative K-theory under quasihomomorphisms are constructed in section 6 and the Riemann-Roch-Grothendieck theorem is proved. The example of assembly maps and crossed products is treated in section 7.

2 Cyclic homology

Cyclic homology can be defined for various classes of associative algebras over $\mathbb{C}$, in particular complete locally convex algebras. For us, a locally convex algebra $\mathscr{A}$ has a topology induced by a family of continuous seminorms $p : \mathscr{A} \to \mathbb{R}_+$, for which the multiplication $\mathscr{A} \times \mathscr{A} \to \mathscr{A}$ is jointly continuous. Hence for any seminorm p there exists a seminorm q such that $p(a_1a_2) \leq q(a_1)q(a_2)$, $\forall a_i \in \mathscr{A}$. For technical reasons however we shall restrict ourselves to *multiplicatively convex* algebras [5], whose topology is generated by a family of submultiplicative seminorms

$$p(a_1a_2) \leq p(a_1)p(a_2) \quad \forall a_i \in \mathscr{A} \ .$$

A complete multiplicatively convex algebra is called m-algebra, and may equivalently be described as a projective limit of Banach algebras. The unitalization $\mathscr{A}^+ = \mathbb{C} \oplus \mathscr{A}$ of an m-algebra $\mathscr{A}$ is again an m-algebra, for the seminorms $\tilde{p}(\lambda 1 + a) = |\lambda| + p(a)$, $\forall \lambda \in \mathbb{C}, a \in \mathscr{A}$. In the same way, if $\mathscr{B}$ is another m-algebra, the direct sum $\mathscr{A} \oplus \mathscr{B}$ is an m-algebra for the topology generated by the seminorms $(p \oplus q)(a,b) = p(a) + q(b)$, where p is a seminorm on $\mathscr{A}$ and q a seminorm on $\mathscr{B}$. Also, the algebraic tensor product $\mathscr{A} \otimes \mathscr{B}$ may be endowed with the projective topology induced by the seminorms

$$(p \otimes q)(c) = \inf \Big\{ \sum_{i=1}^n p(a_i)q(b_i) \text{ such that } c = \sum_{i=1}^n a_i \otimes b_i \ \in \mathscr{A} \otimes \mathscr{B} \Big\} \ . \quad (2)$$

The *completion* $\mathscr{A} \hat{\otimes} \mathscr{B} = \mathscr{A} \hat{\otimes}_\pi \mathscr{B}$ of the algebraic tensor product under this family of seminorms is the projective tensor product of Grothendieck [14], and is again an m-algebra.

Cyclic homology, cohomology and bivariant cyclic cohomology for m-algebras can be defined either within the cyclic bicomplex formalism of Connes [5], or the X-complex of Cuntz and Quillen [12]. We will make an extensive use of both formalisms throughout this paper. In general, we suppose that all linear maps or homomorphims between m-algebras are continuous, tensor products are completed projective tensor products, and extensions of m-algebras $0 \to \mathscr{I} \to \mathscr{R} \to \mathscr{A} \to 0$ always admit a continuous linear splitting $\sigma : \mathscr{A} \to \mathscr{R}$.

2.1 Cyclic bicomplex

Non-commutative differential forms. Let $\mathscr{A}$ be an m-algebra. The space of non-commutative differential forms over $\mathscr{A}$ is the algebraic direct sum

$\Omega\mathscr{A} = \bigoplus_{n\geq 0}\Omega^n\mathscr{A}$ of the n-forms subspaces $\Omega^n\mathscr{A} = \mathscr{A}^+\hat{\otimes}\mathscr{A}^{\hat{\otimes}n}$ for $n \geq 1$ and $\Omega^0\mathscr{A} = \mathscr{A}$, where $\mathscr{A}^+$ is the unitalization of $\mathscr{A}$. Each of the subspaces $\Omega^n\mathscr{A}$ is complete but we do not complete the direct sum. It is customary to use the differential notation $a_0da_1\dots da_n$ (resp. $da_1\dots da_n$) for the string $a_0\otimes a_1\dots\otimes a_n$ (resp. $1\otimes a_1\dots\otimes a_n$). A continuous differential $d:\Omega^n\mathscr{A}\to\Omega^{n+1}\mathscr{A}$ is uniquely specified by $d(a_0da_1\dots da_n) = da_0da_1\dots da_n$ and $d^2 = 0$. A continuous and associative product $\Omega^n\mathscr{A}\times\Omega^m\mathscr{A}\to\Omega^{n+m}\mathscr{A}$ is defined as usual and fulfills the Leibniz rule $d(\omega_1\omega_2) = d\omega_1\omega_2 + (-)^{|\omega_1|}\omega_1 d\omega_2$, where $|\omega_1|$ is the degree of ω_1. This tu rns $\Omega\mathscr{A}$ into a differential graded (DG) algebra. On $\Omega\mathscr{A}$ are defined various operators. First of all, the Hochschild boundary map $b:\Omega^{n+1}\mathscr{A}\to\Omega^n\mathscr{A}$ reads $b(\omega da) = (-)^n[\omega,a]$ for $\omega\in\Omega^n\mathscr{A}$, and $b=0$ on $\Omega^0\mathscr{A} = \mathscr{A}$. One easily shows that b is continuous and $b^2=0$, hence $\Omega\mathscr{A}$ is a complex graded over $\mathbb{N}$. The Hochschild homology of $\mathscr{A}$ (with coefficients in the bimodule $\mathscr{A}$) is the homology of this complex:

$$HH_n(\mathscr{A}) = H_n(\Omega\mathscr{A}, b)\ , \qquad \forall n\in\mathbb{N}\ . \tag{3}$$

Then the Karoubi operator $\kappa:\Omega^n\mathscr{A}\to\Omega^n\mathscr{A}$ is defined by $1-\kappa = db+bd$. Therefore κ is continuous and commutes with b and d. One has $\kappa(\omega\, da) = (-)^n da\,\omega$ for any $\omega\in\Omega^n\mathscr{A}$ and $a\in\mathscr{A}$. The last operator is Connes' $B:\Omega^n\mathscr{A}\to\Omega^{n+1}\mathscr{A}$, equal to $(1+\kappa+\ldots+\kappa^n)d$ on $\Omega^n\mathscr{A}$. It is also continuous and verifies $B^2 = 0 = Bb+bB$ and $B\kappa = \kappa B = B$. Thus $\Omega\mathscr{A}$ endowed with the two anticommuting differentials (b,B) becomes a bicomplex. It splits as a direct sum $\Omega\mathscr{A} = \Omega\mathscr{A}^+\oplus\Omega\mathscr{A}^-$ of even and odd degree differential forms, hence is a $\mathbb{Z}_2$-graded complex for the total boundary map $b+B$. However its homology is trivial [18]. The various versions of cyclic homology are defined using the natural filtrations on $\Omega\mathscr{A}$. Following Cuntz and Quillen [12], we define the *Hodge filtration* on $\Omega\mathscr{A}$ as the decreasing family of $\mathbb{Z}_2$-graded subcomplexes for the total boundary $b+B$

$$F^n\Omega\mathscr{A} = b\Omega^{n+1}\mathscr{A}\oplus\bigoplus_{k>n}\Omega^k\mathscr{A}\ , \qquad \forall n\in\mathbb{Z}\ ,$$

with the convention that $F^n\Omega\mathscr{A} = \Omega\mathscr{A}$ for $n<0$. The completion of $\Omega\mathscr{A}$ is defined as the projective limit of $\mathbb{Z}_2$-graded complexes

$$\widehat{\Omega}\mathscr{A} = \varprojlim_n \Omega\mathscr{A}/F^n\Omega\mathscr{A} = \prod_{n\geq 0}\Omega^n\mathscr{A}\ . \tag{4}$$

Hence $\widehat{\Omega}\mathscr{A} = \widehat{\Omega}^+\mathscr{A}\oplus\widehat{\Omega}^-\mathscr{A}$ is a $\mathbb{Z}_2$-graded complex endowed with the total boundary map $b+B$. It is itself filtered by the decreasing family of $\mathbb{Z}_2$-graded subcomplexes $F^n\widehat{\Omega}\mathscr{A} = \mathrm{Ker}(\widehat{\Omega}\mathscr{A}\to\Omega\mathscr{A}/F^n\Omega\mathscr{A})$, which may be written

$$F^n\widehat{\Omega}\mathscr{A} = b\Omega^{n+1}\mathscr{A}\oplus\prod_{k>n}\Omega^k\mathscr{A}\ , \qquad \forall n\in\mathbb{Z}\ . \tag{5}$$

In particular the quotient $\widehat{\Omega}\mathscr{A}/F^n\widehat{\Omega}\mathscr{A}$ is a $\mathbb{Z}_2$-graded complex isomorphic to $\Omega\mathscr{A}/F^n\Omega\mathscr{A}$, explicitly

$$\widehat{\Omega}\mathscr{A}/F^n\widehat{\Omega}\mathscr{A} = \bigoplus_{k=0}^{n-1}\Omega^k\mathscr{A} \oplus \Omega^n\mathscr{A}/b(\Omega^{n+1}\mathscr{A}) \ , \tag{6}$$

and it vanishes for $n<0$. As a topological vector space, $\widehat{\Omega}\mathscr{A}/F^n\widehat{\Omega}\mathscr{A}$ may fail to be separated because the image $b(\Omega^{n+1}\mathscr{A})$ is not closed in general.

Definition 2.1 *In any degree $n \in \mathbb{Z}$, the* periodic, non-periodic *and* negative *cyclic homologies are respectively the $(b+B)$-homologies*

$$\begin{aligned} HP_n(\mathscr{A}) &= H_{n+2\mathbb{Z}}(\widehat{\Omega}\mathscr{A}) \ , \\ HC_n(\mathscr{A}) &= H_{n+2\mathbb{Z}}(\widehat{\Omega}\mathscr{A}/F^n\widehat{\Omega}\mathscr{A}) \ , \\ HN_n(\mathscr{A}) &= H_{n+2\mathbb{Z}}(F^{n-1}\widehat{\Omega}\mathscr{A}) \ . \end{aligned} \tag{7}$$

Hence $HP_n(\mathscr{A}) \cong HP_{n+2}(\mathscr{A})$ is 2-periodic, $HC_n(\mathscr{A}) = 0$ for $n<0$ and $HN_n(\mathscr{A}) \cong HP_n(\mathscr{A})$ for $n \le 0$. By construction these cyclic homology groups fit into a long exact sequence

$$\ldots \longrightarrow HP_{n+1}(\mathscr{A}) \xrightarrow{S} HC_{n-1}(\mathscr{A}) \xrightarrow{B} HN_n(\mathscr{A}) \xrightarrow{I} HP_n(\mathscr{A}) \longrightarrow \ldots \tag{8}$$

where S is induced by projection, I by inclusion, and the connecting map corresponds to the operator B. The link between cyclic and Hochshild homology may be obtained through *non-commutative de Rham homology* [16], defined as

$$HD_n(\mathscr{A}) := H_{n+2\mathbb{Z}}(\widehat{\Omega}\mathscr{A}/F^{n+1}\widehat{\Omega}\mathscr{A}) \ , \quad \forall n \in \mathbb{Z} \ . \tag{9}$$

This yields for any $n \in \mathbb{Z}$ a short exact sequence of $\mathbb{Z}_2$-graded complexes

$$0 \longrightarrow G_n(\mathscr{A}) \longrightarrow \widehat{\Omega}\mathscr{A}/F^n\widehat{\Omega}\mathscr{A} \longrightarrow \widehat{\Omega}\mathscr{A}/F^{n-1}\widehat{\Omega}\mathscr{A} \longrightarrow 0 \ ,$$

where G_n is $\Omega^n\mathscr{A}/b\Omega^{n+1}\mathscr{A}$ in degree n mod 2, and $b\Omega^n\mathscr{A}$ in degree $n-1$ mod 2. One has $H_{n+2\mathbb{Z}}(G_n) = HH_n(\mathscr{A})$ and $H_{n-1+2\mathbb{Z}}(G_n) = 0$, so that the associated six-term cyclic exact sequence in homology reduces to

$$0 \to HD_{n-1}(\mathscr{A}) \to HC_{n-1}(\mathscr{A}) \to HH_n(\mathscr{A}) \to HC_n(\mathscr{A}) \to HD_{n-2}(\mathscr{A}) \to 0 \ ,$$

and Connes's SBI exact sequence [4] for cyclic homology is actually obtained by splicing together the above sequences for all $n \in \mathbb{Z}$:

$$\ldots \longrightarrow HC_{n+1}(\mathscr{A}) \xrightarrow{S} HC_{n-1}(\mathscr{A}) \xrightarrow{B} HH_n(\mathscr{A}) \xrightarrow{I} HC_n(\mathscr{A}) \longrightarrow \ldots \tag{10}$$

Hence the non-commutative de Rham homology group $HD_n(\mathscr{A})$ may be identified with the image of the periodicity shift $S : HC_{n+2}(\mathscr{A}) \to HC_n(\mathscr{A})$. Clearly the exact sequence (8) can be transformed to (10) by taking the natural maps $HP_n(\mathscr{A}) \to HC_n(\mathscr{A})$ and $HN_n(\mathscr{A}) \to HH_n(\mathscr{A})$.

Passing to the dual theory, let $\mathrm{Hom}(\widehat{\Omega}\mathscr{A}, \mathbb{C})$ be the $\mathbb{Z}_2$-graded complex of linear maps $\widehat{\Omega}\mathscr{A} \to \mathbb{C}$ which are continuous for the adic topology on $\widehat{\Omega}\mathscr{A}$ induced by the Hodge filtration. It is concretely described as the direct sum

$$\mathrm{Hom}(\widehat{\Omega}\mathscr{A}, \mathbb{C}) = \bigoplus_{n\geq 0} \mathrm{Hom}(\Omega^n\mathscr{A}, \mathbb{C}) \ ,$$

where $\mathrm{Hom}(\Omega^n\mathscr{A}, \mathbb{C})$ is the space of continuous linear maps $\Omega^n\mathscr{A} \to \mathbb{C}$. The space $\mathrm{Hom}(\widehat{\Omega}\mathscr{A}, \mathbb{C})$ is endowed with the transposed of the boundary operator $b + B$ on $\widehat{\Omega}\mathscr{A}$. Then the periodic cyclic cohomology of $\mathscr{A}$ is the cohomology of this complex:

$$HP^n(\mathscr{A}) = H^{n+2\mathbb{Z}}(\mathrm{Hom}(\widehat{\Omega}\mathscr{A}, \mathbb{C})) \ . \tag{11}$$

One defines analogously the non-periodic and negative cyclic cohomologies which fit into an IBS long exact sequence.

2.2 X-complex and quasi-free algebras

We now turn to the description of the X-complex. It first appeared in the coalgebra context in Quillen's work [28], and subsequently was used by Cuntz and Quillen in their formulation of cyclic homology [12]. Here we recall the X-complex construction for m-algebras.

Let $\mathscr{R}$ be an m-algebra. The space of non-commutative one-forms $\Omega^1\mathscr{R}$ is a $\mathscr{R}$-bimodule, hence we can take its quotient $\Omega^1\mathscr{R}_\natural$ by the subspace of commutators $[\mathscr{R}, \Omega^1\mathscr{R}] = b\Omega^2\mathscr{R}$. $\Omega^1\mathscr{R}_\natural$ may fail to be separated in general. However, it is automatically separated when $\mathscr{R}$ is *quasi-free*, see below. In order to avoid confusions in the subsequent notations, we always write a one-form $x_0\mathbf{d}x_1 \in \Omega^1\mathscr{R}$ with a bold $\mathbf{d}$ when dealing with the X-complex of $\mathscr{R}$. The latter is the $\mathbb{Z}_2$-graded complex [12]

$$X(\mathscr{R}): \quad \mathscr{R} \underset{\overline{b}}{\overset{\natural\mathbf{d}}{\rightleftarrows}} \Omega^1\mathscr{R}_\natural \ , \tag{12}$$

where $\mathscr{R} = X_+(\mathscr{R})$ is located in even degree and $\Omega^1\mathscr{R}_\natural = X_-(\mathscr{R})$ in odd degree. The class of the generic element $(x_0\mathbf{d}x_1 \bmod [,]) \in \Omega^1\mathscr{R}_\natural$ is usually denoted by $\natural x_0\mathbf{d}x_1$. The map $\natural\mathbf{d} : \mathscr{R} \to \Omega^1\mathscr{R}_\natural$ thus sends $x \in \mathscr{R}$ to $\natural\mathbf{d}x$. Also, the Hochschild boundary $b : \Omega^1\mathscr{R} \to \mathscr{R}$ vanishes on the commutator subspace $[\mathscr{R}, \Omega^1\mathscr{R}]$, hence passes to a well-defined map $\overline{b} : \Omega^1\mathscr{R}_\natural \to \mathscr{R}$. Explicitly the image of $\natural x_0\mathbf{d}x_1$ by $\overline{b}$ is the commutator $[x_0, x_1]$. These maps are continuous and satisfy $\natural\mathbf{d}\circ\overline{b} = 0$ and $\overline{b}\circ\natural\mathbf{d} = 0$, so that $(X(\mathscr{R}), \natural\mathbf{d}\oplus\overline{b})$ indeed defines a $\mathbb{Z}_2$-graded complex. We mention that everything can be formulated when $\mathscr{R}$ itself is a $\mathbb{Z}_2$-graded algebra: we just have to replace everywhere the ordinary commutators by graded commutators, and the differentials anticommute with elements of odd degree. In particular one gets $\overline{b}\natural x\mathbf{d}y = (-)^{|x|}[x, y]$, where $|x|$ is the

degree of x and $[x,y]$ is the graded commutator. The X-complex is obviously a functor from m-algebras to $\mathbb{Z}_2$-graded complexes: if $\rho : \mathscr{R} \to \mathscr{S}$ is a continuous homomorphism, it induces a chain map of even degree $X(\rho) : X(\mathscr{R}) \to X(\mathscr{S})$, by setting $X(\rho)(x) = \rho(x)$ and $X(\rho)(\natural x_0 \mathbf{d} x_1) = \natural\, \rho(x_0)\mathbf{d}\rho(x_1)$.
In fact the X-complex may be identified with the quotient of the $(b+B)$-complex $\widehat{\Omega}\mathscr{R}$ by the subcomplex $F^1\widehat{\Omega}\mathscr{R} = b\Omega^2\mathscr{R} \oplus \prod_{k\geq 2}\Omega^k\mathscr{R}$ of the Hodge filtration, i.e. there is an exact sequence

$$0 \to F^1\widehat{\Omega}\mathscr{R} \to \widehat{\Omega}\mathscr{R} \to X(\mathscr{R}) \to 0 \ .$$

It turns out that the X-complex is especially designed to compute the cyclic homology of algebras for which the subcomplex $F^1\widehat{\Omega}\mathscr{R}$ is contractible. This led Cuntz and Quillen to the following definition:

Definition 2.2 ([12]) *An m-algebra $\mathscr{R}$ is called* quasi-free *if there exists a continuous linear map $\phi : \mathscr{R} \to \Omega^2\mathscr{R}$ with property*

$$\phi(xy) = \phi(x)y + x\phi(y) + \mathbf{d}x\mathbf{d}y \ , \quad \forall x,y \in \mathscr{R} \ . \tag{13}$$

We refer to [12, 19] for many other equivalent definitions of quasi-free algebras. Let us just observe that a quasi-free algebra has dimension ≤ 1 with respect to Hochschild cohomology. Indeed, the map ϕ allows to contract the Hochschild complex of $\mathscr{R}$ in dimensions > 1, and this contraction carries over to the cyclic bicomplex. First, the linear map

$$\sigma : \Omega^1\mathscr{R}_\natural \to \Omega^1\mathscr{R} \ , \qquad \natural x\mathbf{d}y \mapsto x\mathbf{d}y + b(x\phi(y))$$

is well-defined because it vanishes on the commutator subspace $[\mathscr{R}, \Omega^1\mathscr{R}] = b\Omega^2\mathscr{R}$ by the algebraic property of ϕ. Hence σ is a continuous linear splitting of the exact sequence $0 \to b\Omega^2\mathscr{R} \to \Omega^1\mathscr{R} \to \Omega^1\mathscr{R}_\natural \to 0$. By the way, this implies that for a quasi-free algebra $\mathscr{R}$, the topological vector space $\Omega^1\mathscr{R}$ splits into the direct sum of two closed subspaces $b\Omega^2\mathscr{R}$ and $\Omega^1\mathscr{R}_\natural$. Then, we extend ϕ to a continuous linear map $\phi : \Omega^n\mathscr{R} \to \Omega^{n+2}\mathscr{R}$ in all degrees $n \geq 1$ by the formula

$$\phi(x_0\mathbf{d}x_1\dots\mathbf{d}x_n) = \sum_{i=0}^{n}(-)^{ni}\phi(x_i)\mathbf{d}x_{i+1}\dots\mathbf{d}x_n\mathbf{d}x_0\dots\mathbf{d}x_{i-1} \ .$$

The following proposition gives a chain map $\gamma : X(\mathscr{R}) \to \widehat{\Omega}\mathscr{R}$ which is inverse to the natural projection $\pi : \widehat{\Omega}\mathscr{R} \to X(\mathscr{R})$ up to homotopy. Remark that the infinite sum $(1-\phi)^{-1} := \sum_{n=0}^{\infty}\phi^n$ makes sense as a linear map $\mathscr{R} \to \widehat{\Omega}^+\mathscr{R}$ or $\Omega^1\mathscr{R} \to \widehat{\Omega}^-\mathscr{R}$.

Proposition 2.3 *Let $\mathscr{R}$ be a quasi-free m-algebra. Then*
i) The map $\gamma : X(\mathscr{R}) \to \widehat{\Omega}\mathscr{R}$ defined for $x,y \in \mathscr{R}$ by

$$\begin{aligned} \gamma(x) &= (1-\phi)^{-1}(x) \\ \gamma(\natural x\mathbf{d}y) &= (1-\phi)^{-1}(x\mathbf{d}y + b(x\phi(y))) \end{aligned} \tag{14}$$

is a chain map of even degree from the X-complex to the $(b+B)$-complex.
ii) Let $\pi : \widehat{\Omega}\mathscr{R} \to X(\mathscr{R})$ be the natural projection. There is a contracting homotopy of odd degree $h : \widehat{\Omega}\mathscr{R} \to \widehat{\Omega}\mathscr{R}$ such that

$$\begin{aligned} \pi \circ \gamma &= \mathrm{Id} \quad on \quad X(\mathscr{R}) \ , \\ \gamma \circ \pi &= \mathrm{Id} + [b+B, h] \quad on \quad \widehat{\Omega}\mathscr{R} \ . \end{aligned}$$

Hence $X(\mathscr{R})$ and $\widehat{\Omega}\mathscr{R}$ are homotopy equivalent.

Proof: See the proof of [22], Proposition 4.2. There the result was stated in the particular case of a tensor algebra $\mathscr{R} = T\mathscr{A}$, but the general case of a quasi-free algebra is strictly identical (with the tensor algebra the image of γ actually lands in the subcomplex $\Omega T\mathscr{A} \subset \widehat{\Omega} T\mathscr{A}$ for a judicious choice of ϕ, but for generic quasi-free algebras it is necessary to take the completion $\widehat{\Omega}\mathscr{R}$ of $\Omega\mathscr{R}$). ∎

Extensions. Let $\mathscr{A}$ be an m-algebra. By an extension of $\mathscr{A}$ we mean an exact sequence of m-algebras $0 \to \mathscr{I} \to \mathscr{R} \to \mathscr{A} \to 0$ provided with a continuous linear splitting $\mathscr{A} \to \mathscr{R}$, and the topology of the ideal $\mathscr{I}$ is induced by its inclusion in $\mathscr{R}$. Hence as a topological vector space $\mathscr{R}$ is the direct sum of the closed subspaces $\mathscr{I}$ and $\mathscr{A}$. By convention, the powers $\mathscr{I}^n$ of the ideal $\mathscr{I}$ will always denote the image in $\mathscr{R}$ of the n-th tensor power $\mathscr{I} \hat{\otimes} \ldots \hat{\otimes} \mathscr{I}$ by the multiplication map. For $n \leq 0$, we define $\mathscr{I}^0$ as the algebra $\mathscr{R}$. Now let us suppose that all the powers $\mathscr{I}^n$ are closed and direct summands in $\mathscr{R}$ (this is automatically satisfied if $\mathscr{R}$ is quasi-free). Then the quotients $\mathscr{R}/\mathscr{I}^n$ are m-algebras and give rise to an inverse system with surjective homomorphisms

$$0 \leftarrow \mathscr{A} = \mathscr{R}/\mathscr{I} \leftarrow \mathscr{R}/\mathscr{I}^2 \leftarrow \ldots \leftarrow \mathscr{R}/\mathscr{I}^n \leftarrow \ldots$$

We denote by $\widehat{\mathscr{R}} = \varprojlim_n \mathscr{R}/\mathscr{I}^n$ the projective limit and view it as a *pro-algebra* indexed by the directed set $\mathbb{Z}$ (see [19]). Since the bicomplex of non-commutative differential forms $\widehat{\Omega}\mathscr{R}$ and the X-complex $X(\mathscr{R})$ are functorial in $\mathscr{R}$, we can define $\widehat{\Omega}\widehat{\mathscr{R}}$ and $X(\widehat{\mathscr{R}})$ as the $\mathbb{Z}_2$-graded pro-complexes

$$\begin{aligned} \widehat{\Omega}\widehat{\mathscr{R}} &= \varprojlim_n \widehat{\Omega}(\mathscr{R}/\mathscr{I}^n) = \varprojlim_{m,n} \Omega(\mathscr{R}/\mathscr{I}^n)/F^m\Omega(\mathscr{R}/\mathscr{I}^n) \ , \\ X(\widehat{\mathscr{R}}) &= \varprojlim_n X(\mathscr{R}/\mathscr{I}^n) \ , \end{aligned}$$

endowed respectively with the total boundary maps $b+B$ and $\partial = \natural\mathbf{d} \oplus \overline{b}$. When $\mathscr{R}$ is quasi-free, a refinement of Proposition 2.3 yields a chain map $\gamma : X(\widehat{\mathscr{R}}) \to \widehat{\Omega}\widehat{\mathscr{R}}$ inverse to the projection $\pi : \widehat{\Omega}\widehat{\mathscr{R}} \to X(\widehat{\mathscr{R}})$ up to homotopy, which we call a *generalized Goodwillie equivalence*:

Proposition 2.4 *Let $0 \to \mathscr{I} \to \mathscr{R} \to \mathscr{A} \to 0$ be an extension of m-algebras, with $\mathscr{R}$ quasi-free. Then the chain map $\gamma : X(\mathscr{R}) \to \widehat{\Omega}\mathscr{R}$ extends to a homotopy equivalence of pro-complexes $X(\widehat{\mathscr{R}}) \to \widehat{\Omega}\widehat{\mathscr{R}}$.*

Proof: We recall the proof because it will be useful for establishing Proposition 3.10. Let us introduce the following decreasing filtration of the space $\Omega^m\mathscr{R}$ by the subspaces $H^k\Omega^m\mathscr{R}$, $k\in\mathbb{Z}$:

$$H^k\Omega^m\mathscr{R}=\sum_{k_0+\ldots+k_m\geq k}\mathscr{I}^{k_0}\mathbf{d}\mathscr{I}^{k_1}\ldots\mathbf{d}\mathscr{I}^{k_m}\ .$$

Clearly $H^{k+1}\Omega^m\mathscr{R}\subset H^k\Omega^m\mathscr{R}$, and for $k\leq 0$ $H^k\Omega^m\mathscr{R}=\Omega^m\mathscr{R}$. Morally, $H^k\Omega^m\mathscr{R}$ contains at least k powers of the ideal $\mathscr{I}$. The direct sum $\bigoplus_m H^k\Omega^m\mathscr{R}$ is stable by the operators d, b, κ, B for any k. We have to establish how k changes when the linear map $\phi:\Omega^m\mathscr{R}\to\Omega^{m+2}\mathscr{R}$ is applied. First consider $\phi:\mathscr{R}\to\Omega^2\mathscr{R}$. If $x_1,\ldots,x_k$ denote k elements in $\mathscr{R}$, one has by the algebraic property of ϕ (see [12])

$$\begin{aligned}\phi(x_1\ldots x_k)=&\sum_{i=1}^k x_1\ldots x_{i-1}\phi(x_i)x_{i+1}\ldots x_k\\&+\sum_{1\leq i<j\leq k}x_1\ldots x_{i-1}\mathbf{d}x_ix_{i+1}\ldots x_{j-1}\mathbf{d}x_jx_{j+1}\ldots x_k\ .\end{aligned}$$

Taking the elements x_i in the ideal $\mathscr{I}$ and using that $\phi(\mathscr{I})\subset\Omega^2\mathscr{R}$ yields

$$\phi(\mathscr{I}^k)\subset\sum_{i=1}^k\mathscr{I}^{i-1}\mathbf{d}\mathscr{R}\mathbf{d}\mathscr{R}\mathscr{I}^{k-i}+\sum_{1\leq i<j\leq k}\mathscr{I}^{i-1}\mathbf{d}\mathscr{I}\mathscr{I}^{j-i-1}\mathbf{d}\mathscr{I}\mathscr{I}^{k-j}\ .$$

Therefore $\phi(\mathscr{I}^k)\subset H^{k-1}\Omega^2\mathscr{R}$ for any k. Now from the definition of ϕ on $\Omega^m\mathscr{R}$, one has

$$\phi(\mathscr{I}^{k_0}\mathbf{d}\mathscr{I}^{k_1}\ldots\mathbf{d}\mathscr{I}^{k_m})\subset\sum_{l=0}^m\phi(\mathscr{I}^{k_l})\mathbf{d}\mathscr{I}^{k_{l+1}}\ldots\mathbf{d}\mathscr{I}^{k_{l-1}}\subset H^{k-1}\Omega^{m+2}\mathscr{R}$$

whenever $k=k_0+\ldots+k_m$, hence $\phi(H^k\Omega^m\mathscr{R})\subset H^{k-1}\Omega^{m+2}\mathscr{R}$. Now let us evaluate the even part of the chain map $\gamma:\mathscr{R}\to\widehat{\Omega}^+\mathscr{R}$. The part of γ landing in $\Omega^{2m}\mathscr{R}$ is the m-th power ϕ^m. One gets $\phi^m(\mathscr{I}^k)\subset H^{k-m}\Omega^{2m}\mathscr{R}$, hence ϕ^m sends the quotient algebra $\mathscr{R}/\mathscr{I}^k$ to $\Omega^{2m}(\mathscr{R}/\mathscr{I}^n)$ provided $(1+2m)n\leq k-m$ (indeed $1+2m$ is the maximal number of factors $\mathscr{R}$ in the tensor product $\Omega^{2m}\mathscr{R}$). Passing to the projective limits, ϕ^m induces a well-defined map $\widehat{\mathscr{R}}\to\Omega^{2m}\widehat{\mathscr{R}}$, and summing over all degrees $2m$ yields $\gamma:\widehat{\mathscr{R}}\to\widehat{\Omega}^+\widehat{\mathscr{R}}$.
Let us turn to the odd part of the chain map $\gamma:\Omega^1\mathscr{R}_\natural\to\widehat{\Omega}^-\mathscr{R}$. By construction, it is the composition of the linear map $\sigma:\Omega^1\mathscr{R}_\natural\to\Omega^1\mathscr{R}$ with all the powers $\phi^m:\Omega^1\mathscr{R}\to\Omega^{2m+1}\mathscr{R}$. One has $\natural(\mathscr{I}^k\mathbf{d}\mathscr{R}+\mathscr{R}\mathbf{d}(\mathscr{I}^k))\subset\natural(\mathscr{I}^k\mathbf{d}\mathscr{R}+\mathscr{I}^{k-1}\mathbf{d}\mathscr{I})$, and by the definition of σ,

$$\begin{aligned}\sigma\natural(\mathscr{I}^k\mathbf{d}\mathscr{R}+\mathscr{I}^{k-1}\mathbf{d}\mathscr{I})\ &\subset\ \mathscr{I}^k\mathbf{d}\mathscr{R}+\mathscr{I}^{k-1}\mathbf{d}\mathscr{I}+b(\mathscr{I}^k\phi(\mathscr{R})+\mathscr{I}^{k-1}\phi(\mathscr{I}))\\&\subset\ H^k\Omega^1\mathscr{R}+bH^{k-1}\Omega^2\mathscr{R}\ \subset\ H^{k-1}\Omega^1\mathscr{R}\ .\end{aligned}$$

Therefore $(\phi^m \circ \sigma)\natural(\mathscr{I}^k\mathbf{d}\mathscr{R} + \mathscr{I}^{k-1}\mathbf{d}\mathscr{I}) \subset H^{k-m-1}\Omega^{2m+1}\mathscr{R}$. Since $\mathscr{R}$ is the direct sum of $\mathscr{R}/\mathscr{I}^k$ and $\mathscr{I}^k$ as a topological vector space, the quotient $\Omega^1(\mathscr{R}/\mathscr{I}^k)_\natural$ coincides with $\Omega^1\mathscr{R}/(\mathscr{I}^k\mathbf{d}\mathscr{R} + \mathscr{R}\mathbf{d}(\mathscr{I}^k) + [\mathscr{R},\Omega^1\mathscr{R}])$, and the map $\phi^m \circ \sigma : \Omega^1(\mathscr{R}/\mathscr{I}^k)_\natural \to \Omega^{2m+1}(\mathscr{R}/\mathscr{I}^n)$ is well-defined provided $(2m+2)n \leq k-m-1$. Thus passing to the projective limits induces a map $\Omega^1\widehat{\mathscr{R}}_\natural \to \Omega^{2m+1}\widehat{\mathscr{R}}$, and summing over m yields $\gamma : \Omega^1\widehat{\mathscr{R}}_\natural \to \widehat{\Omega}^-\widehat{\mathscr{R}}$.
Finally, the contracting homotopy h of Proposition 2.3 is also constructed from ϕ (see [22] Proposition 4.2), hence extends to a contracting homotopy $h : \widehat{\Omega}\widehat{\mathscr{R}} \to \widehat{\Omega}\widehat{\mathscr{R}}$. The relations $\pi\circ\gamma = \mathrm{Id}$ on $X(\widehat{\mathscr{R}})$ and $\gamma\circ\pi = \mathrm{Id} + [b+B,h]$ on $\widehat{\Omega}\widehat{\mathscr{R}}$ follow immediately. ∎

Adic filtration. Suppose that $\mathscr{I}$ is a (not necessarily closed) two-sided ideal in $\mathscr{R}$, provided with its own topology of m-algebra for which the inclusion $\mathscr{I} \to \mathscr{R}$ is continuous and the multiplication map $\mathscr{R}^+ \times \mathscr{I} \times \mathscr{R}^+ \to \mathscr{I}$ is jointly continuous. As usual we define the powers $\mathscr{I}^n$ as the two-sided ideals corresponding to the image in $\mathscr{R}$ of the n-fold tensor products $\mathscr{I}\hat{\otimes}\ldots\hat{\otimes}\mathscr{I}$ under multiplication. Following [12], we introduce the adic filtration of $X(\mathscr{R})$ by the subcomplexes

$$\begin{array}{rcl} F^{2n}_{\mathscr{I}}X(\mathscr{R}) & : & \mathscr{I}^{n+1} + [\mathscr{I}^n,\mathscr{R}] \rightleftarrows \natural\mathscr{I}^n\mathbf{d}\mathscr{R} \\ F^{2n+1}_{\mathscr{I}}X(\mathscr{R}) & : & \mathscr{I}^{n+1} \rightleftarrows \natural(\mathscr{I}^{n+1}\mathbf{d}\mathscr{R} + \mathscr{I}^n\mathbf{d}\mathscr{I})\ , \end{array} \tag{15}$$

where the commutator $[\mathscr{I}^n,\mathscr{R}]$ is by definition the image of $\mathscr{I}^n\mathbf{d}\mathscr{R}$ under the Hochschild operator b, and $\mathscr{I}^n$ is defined as the unitalized algebra $\mathscr{R}^+$ for $n \leq 0$. This is a decreasing filtration because $F^{n+1}_{\mathscr{I}}X(\mathscr{R}) \subset F^n_{\mathscr{I}}X(\mathscr{R})$, and for $n < 0$ one has $F^n_{\mathscr{I}}X(\mathscr{R}) = X(\mathscr{R})$. Denote by $X_n(\mathscr{R},\mathscr{I}) = X(\mathscr{R})/F^n_{\mathscr{I}}X(\mathscr{R})$ the quotient complex. It is generally not separated. One gets in this way an inverse system of $\mathbb{Z}_2$-graded complexes $\{X_n(\mathscr{R},\mathscr{I})\}_{n\in\mathbb{Z}}$ with projective limit $\widehat{X}(\mathscr{R},\mathscr{I})$.

Now suppose that we start from an extension of m-algebras $0 \to \mathscr{I} \to \mathscr{R} \to \mathscr{A} \to 0$ with continuous linear splitting, and assume that any power $\mathscr{I}^n$ is a direct summand in $\mathscr{R}$. Then, the sequence $X_n(\mathscr{R},\mathscr{I})$ is related to the X-complexes of the quotient m-algebras $\mathscr{R}/\mathscr{I}^n$:

$$\begin{array}{l} 0 \leftarrow X_0(\mathscr{R},\mathscr{I}) = \mathscr{A}/[\mathscr{A},\mathscr{A}] \leftarrow X_1(\mathscr{R},\mathscr{I}) = X(\mathscr{A}) \leftarrow \ldots \\ \ldots \leftarrow X(\mathscr{R}/\mathscr{I}^{n-1}) \leftarrow X_{2n-1}(\mathscr{R},\mathscr{I}) \leftarrow X_{2n}(\mathscr{R},\mathscr{I}) \leftarrow X(\mathscr{R}/\mathscr{I}^n) \leftarrow \ldots \end{array}$$

Hence the projective limit of the system $\{X_n(\mathscr{R},\mathscr{I})\}_{n\in\mathbb{Z}}$ is isomorphic to the X-complex of the pro-algebra $\widehat{\mathscr{R}}$:

$$\widehat{X}(\mathscr{R},\mathscr{I}) = \varprojlim_n X_n(\mathscr{R},\mathscr{I}) = \varprojlim_n X(\mathscr{R}/\mathscr{I}^n) = X(\widehat{\mathscr{R}})\ . \tag{16}$$

The pro-complex $\widehat{X}(\mathscr{R},\mathscr{I})$ is naturally filtered by the family of subcomplexes $F^n\widehat{X}(\mathscr{R},\mathscr{I}) = \mathrm{Ker}(\widehat{X} \to X_n)$. If $0 \to \mathscr{J} \to \mathscr{S} \to \mathscr{B} \to 0$ is another extension of m-algebras with continuous linear splitting, then the space of linear maps between the two pro-complexes $\widehat{X}(\mathscr{R},\mathscr{I})$ and $\widehat{X}(\mathscr{S},\mathscr{J})$, or between $\widehat{X}$ and $\widehat{X}'$ for short, is given by

$$\mathrm{Hom}(\widehat{X},\widehat{X}') = \varprojlim_m \Big(\varinjlim_n \mathrm{Hom}(X_n, X'_m) \Big) \ , \tag{17}$$

where $\mathrm{Hom}(X_n, X'_m)$ is the space of continuous linear maps between the $\mathbb{Z}_2$-graded complexes $X_n(\mathscr{R},\mathscr{I})$ and $X_m(\mathscr{S},\mathscr{J})$. Thus $\mathrm{Hom}(\widehat{X},\widehat{X}')$ is a $\mathbb{Z}_2$-graded complex. It corresponds to the space of linear maps $\{f : \widehat{X} \to \widehat{X}' \mid \forall k, \exists n : \ f(F^n\widehat{X}) \subset F^k\widehat{X}'\}$; the boundary of an element f of parity $|f|$ is given by the graded commutator $\partial \circ f - (-)^{|f|} f \circ \partial$ with the bounary maps $\partial = \natural\mathbf{d}\oplus\overline{b}$ on $\widehat{X}$ and $\widehat{X}'$. $\mathrm{Hom}(\widehat{X},\widehat{X}')$ itself is filtered by the subcomplexes of linear maps of order $\leq n$ for any $n \in \mathbb{N}$:

$$\mathrm{Hom}^n(\widehat{X},\widehat{X}') = \{f : \widehat{X} \to \widehat{X}' \mid \forall k,\ f(F^{k+n}\widehat{X}) \subset F^k\widehat{X}'\} \ . \tag{18}$$

These Hom-complexes will be used in the various definitions of bivariant cyclic cohomology, once the relation between the adic filtration over the X-complex of a quasi-free algebra $\mathscr{R}$ and the Hodge filtration of the cyclic bicomplex over the quotient algebra $\mathscr{A} = \mathscr{R}/\mathscr{I}$ is established.

2.3 The tensor algebra

Taking $\mathscr{R}$ as the tensor algebra of an m-algebra $\mathscr{A}$ provides the link with cyclic homology [9, 12]. The (non-unital) tensor algebra $T\mathscr{A}$ is the completion of the algebraic direct sum $\bigoplus_{n\geq 1} \mathscr{A}^{\hat{\otimes} n}$ with respect to the family of seminorms

$$\widehat{p} = \bigoplus_{n\geq 1} p^{\otimes n} = p \oplus (p\otimes p) \oplus (p\otimes p\otimes p) \oplus \dots \ ,$$

where p runs through all the submultiplicative seminorms on $\mathscr{A}$. Of course $p^{\otimes n}$ is the projective seminorm on $\mathscr{A}^{\otimes n}$ defined by a generalization of (2). These seminorms are submultiplicative with respect to the tensor product $\mathscr{A}^{\hat{\otimes} n} \times \mathscr{A}^{\hat{\otimes} m} \to \mathscr{A}^{\hat{\otimes} n+m}$ and therefore the completion $T\mathscr{A}$ is an m-algebra. It is free, hence quasi-free: a linear map $\phi : T\mathscr{A} \to \Omega^2 T\mathscr{A}$ with the property $\phi(xy) = \phi(x)y + x\phi(y) + \mathbf{d}x\mathbf{d}y$ may be canonically constructed by setting $\phi(a) = 0$ on the generators $a \in \mathscr{A} \subset T\mathscr{A}$, and then recursively $\phi(a_1 \otimes a_2) = \mathbf{d}a_1\mathbf{d}a_2$, $\phi(a_1\otimes a_2\otimes a_3) = (\mathbf{d}a_1\mathbf{d}a_2)a_3 + \mathbf{d}(a_1\otimes a_2)\mathbf{d}a_3$, and so on...

The multiplication map $T\mathscr{A} \to \mathscr{A}$, sending $a_1\otimes\ldots\otimes a_n$ to the product $a_1\ldots a_n$, is continuous and we denote by $J\mathscr{A}$ its kernel. Since the inclusion $\sigma_{\mathscr{A}} : \mathscr{A} \to T\mathscr{A}$ is a continuous linear splitting of the multiplication map, the two-sided ideal $J\mathscr{A}$ is a direct summand in $T\mathscr{A}$. This implies a linearly split quasi-free extension $0 \to J\mathscr{A} \to T\mathscr{A} \to \mathscr{A} \to 0$. It is the universal free extension of $\mathscr{A}$ in the following sense: let $0 \to \mathscr{I} \to \mathscr{R} \to \mathscr{A} \to 0$ be any other extension

($\mathscr{R}$ is not necessarily quasi-free), provided with a continuous linear splitting $\sigma : \mathscr{A} \to \mathscr{R}$. Then one gets a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & J\mathscr{A} \arrow[r] \arrow[d, "\rho_*"'] & T\mathscr{A} \arrow[r] \arrow[d, "\rho_*"'] & \mathscr{A} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathscr{I} \arrow[r] & \mathscr{R} \arrow[r] & \mathscr{A} \arrow[r] \arrow[l, dotted, bend right, "\sigma"'] & 0 \end{tikzcd}$$

where $\rho_* : T\mathscr{A} \to \mathscr{R}$ is the continuous algebra homomorphism obtained by setting $\rho_*(a) = \sigma(a)$ on the generators $a \in \mathscr{A} \subset T\mathscr{A}$. Moreover, the homomorphism ρ_* is independent of the linear splitting σ up to homotopy (two splittings can always be connected by a linear homotopy).
As remarked by Cuntz and Quillen [12], the tensor algebra is closely related to a deformation of the algebra of even-degree noncommutative differential forms $\Omega^+\mathscr{A}$. Endow the space $\Omega^+\mathscr{A}$ with the *Fedosov product*

$$\omega_1 \odot \omega_2 := \omega_1\omega_2 - d\omega_1 d\omega_2 \ , \quad \omega_i \in \Omega^+\mathscr{A} \ . \tag{19}$$

Then $(\Omega^+\mathscr{A}, \odot)$ is a dense subalgebra of $T\mathscr{A}$, with the explicit correspondence

$$\Omega^+\mathscr{A} \ni a_0 da_1 \ldots da_{2n} \longleftrightarrow a_0 \otimes \omega(a_1, a_2) \otimes \ldots \otimes \omega(a_{2n-1}, a_{2n}) \in T\mathscr{A} \ .$$

It turns out that the Fedosov product $\odot$ extends to the projective limit $\widehat{\Omega}^+\mathscr{A}$ and the latter is isomorphic to the pro-algebra

$$\widehat{T}\mathscr{A} = \varprojlim_n T\mathscr{A}/(J\mathscr{A})^n \ . \tag{20}$$

Moreover, $\widehat{\Omega}\mathscr{A}$ and $X(\widehat{T}\mathscr{A}) = \widehat{X}(T\mathscr{A}, J\mathscr{A})$ are isomorphic as $\mathbb{Z}_2$-graded pro-vector spaces [12], and this isomorphism identifies the Hodge filtration $F^n\widehat{\Omega}\mathscr{A}$ with the adic filtration $F^n\widehat{X}(T\mathscr{A}, J\mathscr{A})$. By a fundamental result of Cuntz and Quillen, all these identifications are *homotopy equivalences* of pro-complexes, i.e. the boundary $b + B$ on $\widehat{\Omega}\mathscr{A}$ corresponds to the boundary $\natural\mathbf{d} \oplus \overline{b}$ on $\widehat{X}(T\mathscr{A}, J\mathscr{A})$ up to homotopy and rescaling (see [12]). Hence the periodic and negative cyclic homologies of $\mathscr{A}$ may be computed respectively by $\widehat{X}(T\mathscr{A}, J\mathscr{A})$ and $F^n\widehat{X}(T\mathscr{A}, J\mathscr{A})$. Also, the non-periodic cyclic homology of $\mathscr{A}$ may be computed by the quotient complex $X_n(T\mathscr{A}, J\mathscr{A})$ which is homotopy equivalent to the complex $\widehat{\Omega}\mathscr{A}/F^n\widehat{\Omega}\mathscr{A}$. More generally the same result holds if tensor algebra $T\mathscr{A}$ is replaced by any quasi-free extension of $\mathscr{A}$. Indeed if $0 \to \mathscr{I} \to \mathscr{R} \to \mathscr{A} \to 0$ is a quasi-free extension with c ontinuous linear splitting, the classifying homomorphism $\rho_* : T\mathscr{A} \to \mathscr{R}$ induces a map of pro-complexes $X(\rho_*) : \widehat{X}(T\mathscr{A}, J\mathscr{A}) \to \widehat{X}(\mathscr{R}, \mathscr{I})$ compatible with the adic filtrations induced by the ideals $J\mathscr{A}$ and $\mathscr{I}$. It turns out to be a homotopy equivalence, irrespective to the choice of $\mathscr{R}$:

Theorem 2.5 (Cuntz-Quillen [12]) *For any linearly split extension of m-algebras* $0 \to \mathscr{I} \to \mathscr{R} \to \mathscr{A} \to 0$ *with* $\mathscr{R}$ *quasi-free, one has isomorphisms*

$$\begin{aligned} HP_n(\mathscr{A}) &= H_{n+2\mathbb{Z}}(\widehat{X}(\mathscr{R},\mathscr{I}))\ , \\ HC_n(\mathscr{A}) &= H_{n+2\mathbb{Z}}(X_n(\mathscr{R},\mathscr{I}))\ , \\ HD_n(\mathscr{A}) &= H_{n+2\mathbb{Z}}(X_{n+1}(\mathscr{R},\mathscr{I}))\ , \\ HN_n(\mathscr{A}) &= H_{n+2\mathbb{Z}}(F^{n-1}\widehat{X}(\mathscr{R},\mathscr{I}))\ . \end{aligned} \tag{21}$$

These filtrations also allow to define various versions of bivariant cyclic cohomology, which may be formulated either within the X-complex framework or by means of the $(b+B)$-complex of differential forms.

Definition 2.6 ([12]) *Let* $\mathscr{A}$ *and* $\mathscr{B}$ *be m-algebras, and choose arbitrary (linearly split) quasi-free extensions* $0 \to \mathscr{I} \to \mathscr{R} \to \mathscr{A} \to 0$ *and* $0 \to \mathscr{J} \to \mathscr{S} \to \mathscr{B} \to 0$. *The bivariant periodic cyclic cohomology of* $\mathscr{A}$ *and* $\mathscr{B}$ *is the homology of the* $\mathbb{Z}_2$*-graded complex (17) of linear maps between the pro-complexes* $\widehat{X}(\mathscr{R},\mathscr{I})$ *and* $\widehat{X}(\mathscr{S},\mathscr{J})$*:*

$$HP^n(\mathscr{A},\mathscr{B}) = H_{n+2\mathbb{Z}}\big(\mathrm{Hom}(\widehat{X}(\mathscr{R},\mathscr{I}),\widehat{X}(\mathscr{S},\mathscr{J}))\big)\ ,\quad \forall n\in\mathbb{Z}\ . \tag{22}$$

For any $n\in\mathbb{Z}$*, the non-periodic cyclic cohomology group* $HC^n(\mathscr{A},\mathscr{B})$ *of degree* n *is the homology, in degree* $n \bmod 2$*, of the* $\mathbb{Z}_2$*-graded subcomplex (18) of linear maps of order* $\leq n$*:*

$$HC^n(\mathscr{A},\mathscr{B}) = H_{n+2\mathbb{Z}}\big(\mathrm{Hom}^n(\widehat{X}(\mathscr{R},\mathscr{I}),\widehat{X}(\mathscr{S},\mathscr{J}))\big)\ . \tag{23}$$

The embedding $\mathrm{Hom}^n \hookrightarrow \mathrm{Hom}^{n+2}$ *induces, for any* n*, the* S*-operation in bivariant cyclic cohomology* $S: HC^n(\mathscr{A},\mathscr{B}) \to HC^{n+2}(\mathscr{A},\mathscr{B})$*, and* $\mathrm{Hom}^n \hookrightarrow \mathrm{Hom}$ *yields a natural map* $HC^n(\mathscr{A},\mathscr{B}) \to HP^n(\mathscr{A},\mathscr{B})$*.*

Of course the bivariant periodic theory has period two: $HP^{n+2} = HP^n$. Let us look at particular cases. The algebra $\mathbb{C}$ is quasi-free hence $\widehat{X}(T\mathbb{C},J\mathbb{C})$ is homotopically equivalent to $X(\mathbb{C}) : \mathbb{C} \rightleftarrows 0$, and the periodic cyclic homology of $\mathbb{C}$ is simply $HP_0(\mathbb{C}) = \mathbb{C}$ and $HP_1(\mathbb{C}) = 0$. This implies that for any m-algebra $\mathscr{A}$, we get the usual isomorphisms $HP^n(\mathbb{C},\mathscr{A}) \cong HP_{-n}(\mathscr{A})$ and $HP^n(\mathscr{A},\mathbb{C}) \cong HP^n(\mathscr{A})$ in any degree n. For the non-periodic theory, one has the isomorphism $HC^n(\mathbb{C},\mathscr{A}) \cong HN_{-n}(\mathscr{A})$ with negative cyclic homology, and $HC^n(\mathscr{A},\mathbb{C}) \cong HC^n(\mathscr{A})$ is the non-periodic cyclic cohomology of Connes [4]. Finally, since any class $\varphi \in HC^p(\mathscr{A},\mathscr{B})$ is represented by a chain map sending the subcomplex $F^n\widehat{X}(T\mathscr{A},J\mathscr{A})$ to $F^{n-p}\widehat{X}(T\mathscr{B},J\mathscr{B})$ for any $n\in\mathbb{Z}$, it is not difficult to check that φ induces a transformation of degree $-p$ between the SBI exact sequences for $\mathscr{A}$ and $\mathscr{B}$, i.e. a graded-commutative diagram

$$\begin{tikzcd} HP_{n+1}(\mathscr{A}) \arrow[r, "S"] \arrow[d, "\varphi"] & HC_{n-1}(\mathscr{A}) \arrow[r, "B"] \arrow[d, "\varphi"] & HN_n(\mathscr{A}) \arrow[r, "I"] \arrow[d, "\varphi"] & HP_n(\mathscr{A}) \arrow[d, "\varphi"] \\ HP_{n-p+1}(\mathscr{B}) \arrow[r, "S"] & HC_{n-p-1}(\mathscr{B}) \arrow[r, "B"] & HN_{n-p}(\mathscr{B}) \arrow[r, "I"] & HP_{n-p}(\mathscr{B}) \end{tikzcd}$$

The graded-commutativity comes from the fact that the middle square is actually *anti*commutative when φ is of odd degree, for in this case the connecting morphism B anticommutes with the chain map representing φ.

3 Quasihomomorphisms and Chern character

In this section we define quasihomomorphisms for *metrizable* (or Fréchet) m-algebras and construct a bivariant Chern character. The topology of a Fréchet m-algebra is defined by a countable family of submultiplicative seminorms, and can alternatively be considered as the projective limit of a sequence of Banach algebras [27]. In particular, the projective tensor product of two Fréchet m-algebras is again a Fréchet m-algebra.

We say that a Fréchet m-algebra $\mathscr{I}$ is p-summable (with $p \geq 1$ an integer), if there is a continuous trace $\mathrm{Tr} : \mathscr{I}^p \to \mathbb{C}$ on the pth power of $\mathscr{I}$. Recall that by definition, $\mathscr{I}^p$ is the image in $\mathscr{I}$ of the p-th completed tensor product $\mathscr{I}\hat{\otimes}\ldots\hat{\otimes}\mathscr{I}$ under the multiplication map. Hence the trace is understood as a continuous linear map $\mathscr{I}\hat{\otimes}\ldots\hat{\otimes}\mathscr{I} \to \mathbb{C}$, and the tracial property means that it vanishes on the image of $1-\lambda$, where the operator λ is the backward cyclic permutation $\lambda(i_1 \otimes \ldots \otimes i_p) = i_p \otimes i_1 \ldots \otimes i_{p-1}$. In the low degree $p = 1$ we interpret the trace just as a linear map $\mathscr{I} \to \mathbb{C}$ vanishing on the subspace of commutators $[\mathscr{I}, \mathscr{I}] := b\Omega^1\mathscr{I}$.

Now consider any Fréchet m-algebra $\mathscr{B}$ and form the completed tensor product $\mathscr{I}\hat{\otimes}\mathscr{B}$. Suppose that $\mathscr{E}$ is a Fréchet m-algebra containing $\mathscr{I}\hat{\otimes}\mathscr{B}$ as a (not necessarily closed) two-sided ideal, in the sense that the inclusion $\mathscr{I}\hat{\otimes}\mathscr{B} \to \mathscr{E}$ is continuous. The left and right multiplication maps $\mathscr{E}\times\mathscr{I}\hat{\otimes}\mathscr{B}\times\mathscr{E} \to \mathscr{I}\hat{\otimes}\mathscr{B}$ are then automatically jointly continuous (see [7]). As in [13], we define the semi-direct sum $\mathscr{E} \ltimes \mathscr{I}\hat{\otimes}\mathscr{B}$ as the algebra modeled on the vector space $\mathscr{E} \oplus \mathscr{I}\hat{\otimes}\mathscr{B}$, where the product is such that as many elements as possible are put in the summand $\mathscr{I}\hat{\otimes}\mathscr{B}$. The semi-direct sum is a Fréchet algebra but it may fail to be multiplicatively convex in general. *The situation when $\mathscr{E} \ltimes \mathscr{I}\hat{\otimes}\mathscr{B}$ is a Fréchet m-algebra* will be depicted as

$$\mathscr{E} \triangleright \mathscr{I}\hat{\otimes}\mathscr{B} \tag{24}$$

to stress the analogy with [8]. The definition of quasihomomorphisms involves a $\mathbb{Z}_2$-graded version of $\mathscr{E} \triangleright \mathscr{I}\hat{\otimes}\mathscr{B}$, depending only on a choice of parity (even or odd). It is constructed as follows:

1) Even case: Define $\mathscr{E}^s_+$ as the Fréchet m-algebra $\mathscr{E} \ltimes \mathscr{I}\hat{\otimes}\mathscr{B}$. It is endowed with a linear action of the group $\mathbb{Z}_2$ by automorphisms: the image of an element $(a, b) \in \mathscr{E} \oplus \mathscr{I}\hat{\otimes}\mathscr{B}$ under the generator F of the group is $(a+b, -b)$. We define the $\mathbb{Z}_2$-graded algebra $\mathscr{E}^s$ as the crossed product $\mathscr{E}^s_+ \rtimes \mathbb{Z}_2$. Hence $\mathscr{E}^s$ splits as the direct sum $\mathscr{E}^s_+ \oplus \mathscr{E}^s_-$ where $\mathscr{E}^s_+$ is the subalgebra of even degree elements and $\mathscr{E}^s_- = F\mathscr{E}^s_+$ is the odd subspace.

This definition is rather abstract but there is a concrete description of $\mathscr{E}^s$

in terms of 2×2 matrices. Consider $M_2(\mathscr{E}) = M_2(\mathbb{C})\hat{\otimes}\mathscr{E}$ as a $\mathbb{Z}_2$-graded algebra with grading operator $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Thus diagonal elements are of even degree and off-diagonal elements are odd. $\mathscr{E}^s$ can be identified with a (non-closed) subalgebra of $M_2(\mathbb{C})\hat{\otimes}\mathscr{E}$ in the following way. Any element $x+Fy \in \mathscr{E}^s$ may be decomposed to its even and odd parts $x, y \in \mathscr{E} \oplus \mathscr{I}\hat{\otimes}\mathscr{B}$, with $x = (a,b)$ and $y = (c,d)$. Then $x + Fy$ is represented by the matrix

$$x + Fy = \begin{pmatrix} a+b & c \\ c+d & a \end{pmatrix} \quad \text{with} \quad a, c \in \mathscr{E} \ , \ b, d \in \mathscr{I}\hat{\otimes}\mathscr{B} \ .$$

The action of $\mathbb{Z}_2$ on $\mathscr{E}^s_+$ is implemented by the adjoint action of the following odd-degree multiplier of $M_2(\mathscr{E})$:

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{C}) \ , \quad F^2 = 1 \ . \tag{25}$$

Denote by $\mathscr{I}^s = \mathscr{I}^s_+ \oplus \mathscr{I}^s_-$ the $\mathbb{Z}_2$-graded algebra $M_2(\mathbb{C})\hat{\otimes}\mathscr{I}$, with $\mathscr{I}^s_+$ the subalgebra of diagonal elements and $\mathscr{I}^s_-$ the off-diagonal subspace. We thus have an inclusion of $\mathscr{I}^s\hat{\otimes}\mathscr{B}$ as a (non-closed) two-sided ideal in $\mathscr{E}^s$, with $\mathscr{E}^s \triangleright \mathscr{I}^s\hat{\otimes}\mathscr{B}$. The commutator $[F, \mathscr{E}^s_+]$ is contained in $\mathscr{I}^s_-\hat{\otimes}\mathscr{B}$. Finally, we denote by tr_s the supertrace of even degree on $M_2(\mathbb{C})$:

$$\mathrm{tr}_s : M_2(\mathbb{C}) \to \mathbb{C} \ , \qquad \mathrm{tr}_s \begin{pmatrix} a' & c \\ c' & a \end{pmatrix} = a' - a \ .$$

2) Odd case: Now regard $M_2(\mathscr{E})$ as a trivially graded algebra and define $\mathscr{E}^s_+$ as the (non-closed) subalgebra

$$\mathscr{E}^s_+ = \begin{pmatrix} \mathscr{E} & \mathscr{I}\hat{\otimes}\mathscr{B} \\ \mathscr{I}\hat{\otimes}\mathscr{B} & \mathscr{E} \end{pmatrix} \tag{26}$$

provided with its own topology of complete m-algebra. Let $C_1 = \mathbb{C} \oplus \varepsilon\mathbb{C}$ be the complex Clifford algebra of the one-dimensional euclidian space. C_1 is the $\mathbb{Z}_2$-graded algebra generated by the unit $1 \in \mathbb{C}$ in degree zero and ε in degree one with $\varepsilon^2 = 1$. We define the $\mathbb{Z}_2$-graded algebra $\mathscr{E}^s$ as the tensor product $C_1\hat{\otimes}\mathscr{E}^s_+$. Hence $\mathscr{E}^s = \mathscr{E}^s_+ \oplus \mathscr{E}^s_-$ where $\mathscr{E}^s_+$ is the subalgebra of even degree and $\mathscr{E}^s_- = \varepsilon\mathscr{E}^s_+$ is the odd subspace. Similarly, define $\mathscr{I}^s = M_2(C_1)\hat{\otimes}\mathscr{I} = \mathscr{I}^s_+ \oplus \mathscr{I}^s_-$. Then $\mathscr{I}^s\hat{\otimes}\mathscr{B}$ is a (non-closed) two-sided ideal of $\mathscr{E}^s$ and one has $\mathscr{E}^s \triangleright \mathscr{I}^s\hat{\otimes}\mathscr{B}$. The matrix

$$F = \varepsilon \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in M_2(C_1) \ , \quad F^2 = 1 \tag{27}$$

is an odd multiplier of $\mathscr{E}^s$ and the commutator $[F, \mathscr{E}^s_+]$ is contained in $\mathscr{I}^s_-\hat{\otimes}\mathscr{B}$. Finally, we define a supertrace tr_s of odd degree on C_1 by sending the generators 1 to 0 and ε to $\pm\sqrt{2i}$. The normalization $\pm\sqrt{2i}$ is chosen for compatibility with Bott periodicity, see [22]. We will choose conventionally the "sign" as $-\sqrt{2i}$ in order to simplify the subsequent formulas. One thus has

$$\mathrm{tr}_s : M_2(C_1) \to \mathbb{C} \ , \quad \mathrm{tr}_s \begin{pmatrix} a + \varepsilon a' & b + \varepsilon b' \\ c + \varepsilon c' & d + \varepsilon d' \end{pmatrix} = -\sqrt{2i}\,(a' + d') \ .$$

The objects F, $\mathscr{E}^s$ and $\mathscr{I}^s$ are defined in such a way that we can handle the even and odd case simultaneously. This allows to give the following synthetic definition of quasihomomorphisms.

Definition 3.1 *Let $\mathscr{A}$, $\mathscr{B}$, $\mathscr{I}$, $\mathscr{E}$ be Fréchet m-algebras. Assume that $\mathscr{I}$ is p-summable and $\mathscr{E} \triangleright \mathscr{I}\hat{\otimes}\mathscr{B}$. A* quasihomomorphism *from $\mathscr{A}$ to $\mathscr{B}$ is a continuous homomorphism*

$$\rho : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s\hat{\otimes}\mathscr{B} \tag{28}$$

sending $\mathscr{A}$ to the even degree subalgebra $\mathscr{E}^s_+$. The quasihomomorphism comes equipped with a degree (even or odd) depending on the degree chosen for the above construction of $\mathscr{E}^s$. In particular, the linear map $a \in \mathscr{A} \mapsto [F, \rho(a)] \in \mathscr{I}^s_-\hat{\otimes}\mathscr{B}$ is continuous.

In other words, a quasihomomorphism of even degree $\rho = \left(\begin{smallmatrix} \rho_+ & 0 \\ 0 & \rho_- \end{smallmatrix}\right)$ is a pair of homomorphisms $(\rho_+, \rho_-) : \mathscr{A} \rightrightarrows \mathscr{E}$ such that the difference $\rho_+(a) - \rho_-(a)$ lies in the ideal $\mathscr{I}\hat{\otimes}\mathscr{B}$ for any $a \in \mathscr{A}$. A quasihomomorphism of odd degree is a homomorphism $\rho : \mathscr{A} \to M_2(\mathscr{E})$ such that the off-diagonal elements land in $\mathscr{I}\hat{\otimes}\mathscr{B}$.

The Cuntz-Quillen formalism for bivariant cyclic cohomology $HC^n(\mathscr{A}, \mathscr{B})$ requires to work with quasi-free extensions of $\mathscr{A}$ and $\mathscr{B}$. Hence let us suppose that we choose such extensions of m-algebras

$$0 \to \mathscr{G} \to \mathscr{F} \to \mathscr{A} \to 0 \ , \qquad 0 \to \mathscr{J} \to \mathscr{R} \to \mathscr{B} \to 0 \ ,$$

with $\mathscr{F}$ and $\mathscr{R}$ quasi-free. We always take $\mathscr{F} = T\mathscr{A}$ as the universal free extension of $\mathscr{A}$, but we leave the possibility to take any quasi-free extension $\mathscr{R}$ for $\mathscr{B}$ since the tensor algebra $T\mathscr{B}$ will not be an optimal choice in general. The first step toward the bivariant Chern character is to lift a given quasihomomorphism $\rho : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s\hat{\otimes}\mathscr{B}$ to a quasihomomorphism from $\mathscr{F}$ to $\mathscr{R}$, compatible with the filtrations by the ideals $\mathscr{G} \subset \mathscr{F}$, $\mathscr{J} \subset \mathscr{R}$. This requires to fix some admissibility conditions on the intermediate algebra $\mathscr{E}$:

Definition 3.2 *Let $0 \to \mathscr{J} \to \mathscr{R} \to \mathscr{B} \to 0$ be a quasi-free extension of $\mathscr{B}$, and let $\mathscr{I}$ be p-summable with trace $\mathrm{Tr} : \mathscr{I}^p \to \mathbb{C}$. We say that $\mathscr{E} \triangleright \mathscr{I}\hat{\otimes}\mathscr{B}$ is provided with an* $\mathscr{R}$-admissible extension *if there are algebras $\mathscr{M} \triangleright \mathscr{I}\hat{\otimes}\mathscr{R}$ and $\mathscr{N} \triangleright \mathscr{I}\hat{\otimes}\mathscr{J}$ and a commutative diagram of extensions*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{N} & \longrightarrow & \mathscr{M} & \longrightarrow & \mathscr{E} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathscr{I}\hat{\otimes}\mathscr{J} & \longrightarrow & \mathscr{I}\hat{\otimes}\mathscr{R} & \longrightarrow & \mathscr{I}\hat{\otimes}\mathscr{B} & \longrightarrow & 0 \end{array} \tag{29}$$

with the following properties:

i) Any power $\mathscr{N}^n$ is a direct summand in $\mathscr{M}$ (as a topological vector space);

ii) For any degree $n \geq \max(1, p-1)$, the linear map $(\mathscr{I}\hat{\otimes}\mathscr{R})^n \mathbf{d}(\mathscr{I}\hat{\otimes}\mathscr{R}) \to \Omega^1\mathscr{R}_\natural$ induced by the trace $\mathscr{I}^{n+1} \to \mathbb{C}$ factors through the quotient

$$\natural(\mathscr{I}\hat{\otimes}\mathscr{R})^n \mathbf{d}(\mathscr{I}\hat{\otimes}\mathscr{R}) = (\mathscr{I}\hat{\otimes}\mathscr{R})^n \mathbf{d}(\mathscr{I}\hat{\otimes}\mathscr{R}) \bmod [\mathscr{M}, \Omega^1\mathscr{M}] \ ,$$

and the chain map $\mathrm{Tr} : F^{2n+1}_{\mathscr{I}\hat{\otimes}\mathscr{R}} X(\mathscr{M}) \to X(\mathscr{R})$ thus obtained is of order zero with respect to the adic filtration induced by the ideals $\mathscr{N} \subset \mathscr{M}$ and $\mathscr{J} \subset \mathscr{R}$.

In the following we will say that $\mathscr{E}$ is $\mathscr{R}$-admissible, keeping in mind that the extension $\mathscr{M}$ is given. Condition *i)* is automatically satisfied for example if $\mathscr{M}$ is a quasi-free algebra (this will not always be the case). The chain map Tr of condition *ii)* is constructed as follows. For $n \geq 1$ one has the inclusion $\natural(\mathscr{I}\hat{\otimes}\mathscr{R})^{n+1}\mathbf{d}\mathscr{M} \subset \natural(\mathscr{I}\hat{\otimes}\mathscr{R})^n \mathbf{d}(\mathscr{I}\hat{\otimes}\mathscr{R})$, so that the subcomplex of the $\mathscr{I}\hat{\otimes}\mathscr{R}$-adic filtration reads

$$F^{2n+1}_{\mathscr{I}\hat{\otimes}\mathscr{R}} X(\mathscr{M}) \ : \ (\mathscr{I}\hat{\otimes}\mathscr{R})^{n+1} \rightleftarrows \natural(\mathscr{I}\hat{\otimes}\mathscr{R})^n \mathbf{d}(\mathscr{I}\hat{\otimes}\mathscr{R}) \ .$$

Then, the trace $\mathscr{I}^{n+1} \to \mathbb{C}$ induces a partial trace $(\mathscr{I}\hat{\otimes}\mathscr{R})^{n+1} \to \mathscr{R}$. The latter combined with $\natural(\mathscr{I}\hat{\otimes}\mathscr{R})^n \mathbf{d}(\mathscr{I}\hat{\otimes}\mathscr{R}) \to \Omega^1\mathscr{R}_\natural$ yields a linear map in any degree $n \geq \max(1, p-1)$

$$\mathrm{Tr} : F^{2n+1}_{\mathscr{I}\hat{\otimes}\mathscr{R}} X(\mathscr{M}) \to X(\mathscr{R}) \ , \tag{30}$$

compatible with the inclusions $F^{2n+3}_{\mathscr{I}\hat{\otimes}\mathscr{R}} X(\mathscr{M}) \subset F^{2n+1}_{\mathscr{I}\hat{\otimes}\mathscr{R}} X(\mathscr{M})$. The trace over $\mathscr{I}^{n+1}$ ensures that (30) is a chain map. It is not obvious, however, that it is automatically of degree zero with respect to the $\mathscr{N}$-adic and $\mathscr{J}$-adic filtrations, i.e. that the intersection $F^{2n+1}_{\mathscr{I}\hat{\otimes}\mathscr{R}} X(\mathscr{M}) \cap F^k_{\mathscr{N}} X(\mathscr{M})$ is mapped to $F^k_{\mathscr{J}} X(\mathscr{R})$ for any $k \in \mathbb{Z}$. This should be imposed as a condition.
Remark that the case $p = 1$, $n = 0$ is pathological, since there is no canonical way to map the space $\natural((\mathscr{I}\hat{\otimes}\mathscr{R})\mathbf{d}\mathscr{M} + \mathscr{M}^+\mathbf{d}(\mathscr{I}\hat{\otimes}\mathscr{R}))$ to $\Omega^1\mathscr{R}_\natural$ using only the trace over $\mathscr{I}$. In this situation, it seems preferable to impose directly the existence of a chain map $\mathrm{Tr} : F^1_{\mathscr{I}\hat{\otimes}\mathscr{R}} X(\mathscr{M}) \to X(\mathscr{R})$ in the definition of admissibility.

Example 3.3 When $\mathscr{A}$ is arbitrary and $\mathscr{B} = \mathbb{C}$, a p-summable quasihomomorphism represents a K-homology class of $\mathscr{A}$ in the sense of [4, 5]. Here we take $\mathscr{I} = \mathscr{L}^p(H)$ as the Schatten ideal of p-summable operators on a separable infinite-dimensional Hilbert space H. Recall that $\mathscr{I}$ is a two-sided ideal in the algebra of all bounded operators $\mathscr{L} = \mathscr{L}(H)$. $\mathscr{I}$ is a Banach algebra for the norm $\|x\|_p = (\mathrm{Tr}(|x|^p))^{1/p}$, $\mathscr{L}$ is provided with the operator norm, and the products $\mathscr{I} \times \mathscr{L} \times \mathscr{I} \to \mathscr{I}$ are jointly continuous. Since $\mathscr{L}$ and $\mathscr{I}$ are Banach algebras, the semi-direct sum $\mathscr{L} \ltimes \mathscr{I}$ is automatically a Banach algebra and we can write $\mathscr{L} \triangleright \mathscr{I}$. A p-summable K-homology class of even degree is represented by a pair of continuous homomorphisms $(\rho_+, \rho_-) : \mathscr{A} \rightrightarrows \mathscr{L}$ such that the difference $\rho_+ - \rho_-$ lands to $\mathscr{I}$. We get in this way an even degree quasihomomorph ism $\rho : \mathscr{A} \to \mathscr{L}^s \triangleright \mathscr{I}^s$. Here it is important to note that by a slight modification of the intermediate algebra $\mathscr{L}$, it is always possible to consider $\mathscr{I}$ as a closed ideal [13]. Indeed if we define $\mathscr{E}$ as the Banach algebra

$$\mathscr{E} = \mathscr{L} \ltimes \mathscr{I}$$

then one clearly has $\mathscr{E} \triangleright \mathscr{I}$ and $\mathscr{I}$ is closed in $\mathscr{E}$ by construction. The pair of homomorphisms $(\rho_+, \rho_-) : \mathscr{A} \rightrightarrows \mathscr{L}$ may be replaced with a new pair $(\rho'_+, \rho'_-) : \mathscr{A} \rightrightarrows \mathscr{E}$ by setting $\rho'_+(a) = (\rho_-(a), \rho_+(a) - \rho_-(a))$ and $\rho'_-(a) = (\rho_-(a), 0)$ in $\mathscr{L} \oplus \mathscr{I}$. The above K-homology class is then represented by the new quasihomomorphism $\rho' : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s$.
In the odd case, a p-summable K-homology class is represented by a continuous homomorphism $\rho : \mathscr{A} \to \left(\begin{smallmatrix} \mathscr{L} & \mathscr{I} \\ \mathscr{I} & \mathscr{L} \end{smallmatrix}\right)$, which can be equivalently described as a homomorphism $\rho' : \mathscr{A} \to M_2(\mathscr{E})$ with off-diagonal elements in $\mathscr{I}$.
Concerning cyclic homology, the algebra $\mathbb{C}$ is quasi-free, hence the quasi-free extension $\mathscr{R} = \mathbb{C}$ and $\mathscr{J} = 0$ computes the cyclic homology of $\mathbb{C}$. Therefore by choosing $\mathscr{M} = \mathscr{E}$ and $\mathscr{N} = 0$, the algebra $\mathscr{E} \triangleright \mathscr{I}$ is $\mathbb{C}$-admissible (condition *ii)* is trivial since $\Omega^1\mathbb{C}_\natural = 0$).

Example 3.4 More generally, if $\mathscr{I}$ is a p-summable Fréchet m-algebra contained as a (not necessarily closed) two-sided ideal in a unital Fréchet m-algebra $\mathscr{L}$, with $\mathscr{L} \triangleright \mathscr{I}$, a p-summable quasihomomorphism from $\mathscr{A}$ to $\mathscr{B}$ could be constructed from the generic intermediate algebra $\mathscr{E} = \mathscr{L} \hat{\otimes} \mathscr{B}$, provided that the map $\mathscr{I} \hat{\otimes} \mathscr{B} \to \mathscr{E}$ is injective. If $0 \to \mathscr{J} \to \mathscr{R} \to \mathscr{B} \to 0$ is a quasi-free extension of $\mathscr{B}$, the choice $\mathscr{M} = \mathscr{L} \hat{\otimes} \mathscr{R}$ and $\mathscr{N} = \mathscr{L} \hat{\otimes} \mathscr{J}$ shows that $\mathscr{E}$ is $\mathscr{R}$-admissible provided that the maps $\mathscr{I} \hat{\otimes} \mathscr{R} \to \mathscr{M}$ and $\mathscr{I} \hat{\otimes} \mathscr{J} \to \mathscr{N}$ are injective. In fact it is easy to get rid of these injectivity conditions by redefining the algebra

$$\mathscr{E} = (\mathscr{L} \ltimes \mathscr{I}) \hat{\otimes} \mathscr{B} \ ,$$

which contains $\mathscr{I} \hat{\otimes} \mathscr{B}$ as a closed ideal. Then $\mathscr{E}$ becomes automatically $\mathscr{R}$-admissible by taking $\mathscr{M} = (\mathscr{L} \ltimes \mathscr{I}) \hat{\otimes} \mathscr{R}$ and $\mathscr{N} = (\mathscr{L} \ltimes \mathscr{I}) \hat{\otimes} \mathscr{J}$ (remark that $(\mathscr{L} \ltimes \mathscr{I})^n = \mathscr{L} \ltimes \mathscr{I}$ for any n because $\mathscr{L}$ is unital, hence $\mathscr{N}^n$ is a direct summand in $\mathscr{M}$). The chain map $\mathrm{Tr} : F^{2n+1}_{\mathscr{I} \hat{\otimes} \mathscr{R}} X(\mathscr{M}) \to X(\mathscr{R})$ is obtained by multiplying all the factors in $\mathscr{L}$ and $\mathscr{I}$, and taking the trace on $\mathscr{I}^{n+1}$, while the compatibility between the $\mathscr{N}$-adic and $\mathscr{J}$-adic filtrations is obvious. Although interesting examples arise under this form (see section 7), the algebra $\mathscr{E}$ cannot be decomposed into a tensor product with $\mathscr{B}$ in all situations.

Example 3.5 An important example where $\mathscr{E}$ cannot be taken in the previous form is provided by the Bott element of the real line. Here $\mathscr{A} = \mathbb{C}$ and $\mathscr{B} = C^\infty(0,1)$ is the algebra of smooth functions $f : [0,1] \to \mathbb{C}$ such that f and all its derivatives vanish at the endpoints 0 and 1. Take $\mathscr{I} = \mathbb{C}$ as a 1-summable algebra, and $\mathscr{E} = C^\infty[0,1]$ is the algebra of smooth functions $f : [0,1] \to \mathbb{C}$ with the derivatives vanishing at the endpoints, while f itself takes arbitrary values at 0 and 1. $\mathscr{B}$ and $\mathscr{E}$ provided with their usual Fréchet topology are m-algebras, and one has $\mathscr{E} \triangleright \mathscr{B}$. The Bott element is represented by the quasihomomorphism of *odd degree*

$$\rho : \mathbb{C} \to \mathscr{E}^s \triangleright \mathscr{I}^s \hat{\otimes} \mathscr{B} \ ,$$

where $\mathscr{I}^s = M_2(C_1)$ and $\mathscr{E}^s \subset M_2(C_1)\hat{\otimes}\mathscr{E}$ by construction. The homomorphism $\rho : \mathbb{C} \to \mathscr{E}^s_+$ is built from an arbitrary real-valued function $\xi \in \mathscr{E}$ such that $\xi(0) = 0$, $\xi(1) = \pi/2$, and sends the unit $e \in \mathbb{C}$ to the matrix

$$\rho(e) = R^{-1} \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} R \ , \qquad R = \begin{pmatrix} \cos\xi & \sin\xi \\ -\sin\xi & \cos\xi \end{pmatrix} \ .$$

The algebra $\mathscr{B}$ is quasi-free hence we can choose $\mathscr{R} = \mathscr{B}$, $\mathscr{J} = 0$ as quasi-free extension. The cyclic homology of $\mathscr{B}$ is therefore computed by $X(\mathscr{B})$. Moreover, setting $\mathscr{M} = \mathscr{E}$ and $\mathscr{N} = 0$ shows that $\mathscr{E}$ is $\mathscr{B}$-admissible. Indeed, $\Omega^1\mathscr{B}_\natural$ is contained in the space $\Omega^1(0,1)$ of ordinary (commutative) complex-valued smooth one-forms over $[0,1]$ vanishing at the endpoints with all their derivatives. The chain map $\mathrm{Tr} : F^{2n+1}_{\mathscr{B}} X(\mathscr{E}) \to X(\mathscr{B})$ is thus well-defined for any $n \geq 1$, and just amounts to project noncommutative forms over $\mathscr{E}$ to ordinary (commutative) differential forms over $[0,1]$.

We shall now construct the bivariant Chern character of a given p-summable quasihomomorphism $\rho : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s \hat{\otimes} \mathscr{B}$. We take the universal free extension $0 \to J\mathscr{A} \to T\mathscr{A} \to \mathscr{A} \to 0$ for $\mathscr{A}$, and choose some quasi-free extension $0 \to \mathscr{J} \to \mathscr{R} \to \mathscr{B} \to 0$ for $\mathscr{B}$ with the property that the algebra $\mathscr{E} \triangleright \mathscr{I} \hat{\otimes} \mathscr{B}$ is $\mathscr{R}$-admissible. The bivariant Chern character should be represented by a chain map between the complexes $X(T\mathscr{A})$ and $X(\mathscr{R})$, compatible with the adic filtrations induced by the ideals $J\mathscr{A}$ and $\mathscr{J}$ (section 2). Our task is thus to lift the quasihomomorphism to the quasi-free algebras $T\mathscr{A}$ and $\mathscr{R}$. First, the admissibility condition provides a diagram of extensions (29). From $\mathscr{M} \triangleright \mathscr{I} \hat{\otimes} \mathscr{R}$ define the $\mathbb{Z}_2$-graded algebra $\mathscr{M}^s$ in complete analogy with $\mathscr{E}^s$: depending on the degree of the quasihomomorphism, $\mathscr{M}^s$ is a subalgebra of $M_2(\mathbb{C})\hat{\otimes}\mathscr{M}$ (even case) or $M_2(C_1)\hat{\otimes}\mathscr{M}$ (odd case), with commutator property $[F, \mathscr{M}^s_+] \subset \mathscr{I}^s_- \hat{\otimes} \mathscr{R}$. Also, from $\mathscr{N} \triangleright \mathscr{I} \hat{\otimes} \mathscr{J}$ define $\mathscr{N}^s$ as the $\mathbb{Z}_2$-graded subalgebra of $M_2(\mathbb{C})\hat{\otimes}\mathscr{N}$ or $M_2(C_1)\hat{\otimes}\mathscr{N}$ with commutator $[F, \mathscr{N}^s_+] \subset \mathscr{I}^s_- \hat{\otimes} \mathscr{J}$. The algebras $\mathscr{E}^s$, $\mathscr{M}^s$ and $\mathscr{N}^s$ are gifted with a differential of odd degree induced by the graded commutator $[F,\ ]$ (its square vanishes because $F^2 = 1$). Then we get an extension of $\mathbb{Z}_2$-graded differential algebras

$$0 \to \mathscr{N}^s \to \mathscr{M}^s \to \mathscr{E}^s \to 0 \ .$$

The restriction to the even-degree subalgebras yields an extension of trivially graded algebras $0 \to \mathscr{N}^s_+ \to \mathscr{M}^s_+ \to \mathscr{E}^s_+ \to 0$, split by a continuous linear map $\sigma : \mathscr{E}^s_+ \to \mathscr{M}^s_+$ (recall the splitting is our basic hypothesis about extensions of m-algebras). The universal properties of the tensor algebra $T\mathscr{A}$ then allows to extend the homomorphism $\rho : \mathscr{A} \to \mathscr{E}^s_+$ to a continuous homomorphism $\rho_* : T\mathscr{A} \to \mathscr{M}^s_+$ by setting $\rho_*(a_1 \otimes \ldots \otimes a_n) = \sigma\rho(a_1) \otimes \ldots \otimes \sigma\rho(a_n)$:

$$\begin{tikzcd} 0 \arrow[r] & J\mathscr{A} \arrow[r] \arrow[d, "\rho_*"'] & T\mathscr{A} \arrow[r] \arrow[d, "\rho_*"'] & \mathscr{A} \arrow[r] \arrow[d, "\rho"] & 0 \\ 0 \arrow[r] & \mathscr{N}^s_+ \arrow[r] & \mathscr{M}^s_+ \arrow[r] & \mathscr{E}^s_+ \arrow[r] \arrow[l, dotted, bend right, "\sigma"'] & 0 \end{tikzcd} \tag{31}$$

A priori ρ_* depends on the choice of linear splitting σ, but in a way which will not affect the cohomology class of the bivariant Chern character. This construction may be depicted in terms of a p-summable quasihomomorphism $\rho_* : T\mathscr{A} \to \mathscr{M}^s \triangleright \mathscr{I}^s \hat{\otimes} \mathscr{R}$, compatible with the adic filtration by the ideals in the sense that $J\mathscr{A}$ is mapped to $\mathscr{N}^s \triangleright \mathscr{I}^s \hat{\otimes} \mathscr{J}$. Hence, ρ_* extends to a quasihomomorphism of pro-algebras

$$\rho_* : \widehat{T}\mathscr{A} \to \widehat{\mathscr{M}^s} \triangleright \mathscr{I}^s \hat{\otimes} \widehat{\mathscr{R}} \ , \tag{32}$$

where $\widehat{T}\mathscr{A}$, $\widehat{\mathscr{M}^s}$ and $\widehat{\mathscr{R}}$ are the adic completions of $T\mathscr{A}$, $\mathscr{M}^s$ and $\mathscr{R}$ with respect to the ideals $J\mathscr{A}$, $\mathscr{N}^s$ and $\mathscr{J}$. Next, depending on the degree of the quasihomomorphism, the even supertrace $\mathrm{tr}_s : M_2(\mathbb{C}) \to \mathbb{C}$ or the odd supertrace $\mathrm{tr}_s : M_2(C_1) \to \mathbb{C}$ yields a chain map $X(\mathscr{M}^s) \to X(\mathscr{M})$ by setting $\alpha x \mapsto \mathrm{tr}_s(\alpha)x$ and $\natural\, \alpha x \mathbf{d}(\beta y) \mapsto \pm \mathrm{tr}_s(\alpha\beta) \natural x \mathbf{d} y$ for any $x, y \in \mathscr{M}$ and $\alpha, \beta \in M_2(\mathbb{C})$ or $M_2(C_1)$. The sign $\pm$ is the parity of the matrix β, which has to move across the differential $\mathbf{d}$. Hence composing with the chain map $\mathrm{Tr} : F^{2n+1}_{\mathscr{I}\hat{\otimes}\mathscr{R}} X(\mathscr{M}) \to X(\mathscr{R})$ guaranteed by the admissibility condition, we obtain for any integer $n \geq \max(1, p-1)$ a *supertrace* chain map

$$\tau : F^{2n+1}_{\mathscr{I}^s\hat{\otimes}\mathscr{R}} X(\mathscr{M}^s) \xrightarrow{\mathrm{tr}_s} F^{2n+1}_{\mathscr{I}\hat{\otimes}\mathscr{R}} X(\mathscr{M}) \xrightarrow{\mathrm{Tr}} X(\mathscr{R}) \ , \tag{33}$$

of order zero with respect to the $\mathscr{N}^s$-adic filtration on $X(\mathscr{M}^s)$ and the $\mathscr{J}$-adic filtration on $X(\mathscr{R})$. The parity of τ corresponds to the parity of the quasihomomorphism. This allows to construct a chain map $\widehat{\chi}^n : \widehat{\Omega}\mathscr{M}^s_+ \to X(\mathscr{R})$ from the $(b+B)$-complex of the algebra $\mathscr{M}^s_+$, in any degree $n \geq p$ having the same parity as the supertrace τ. Observe that the linear map $x \in \mathscr{M}^s_+ \mapsto [F, x] \in \mathscr{I}^s_- \hat{\otimes} \mathscr{R}$ is continuous by construction.

Proposition 3.6 *Let $\rho : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s \hat{\otimes} \mathscr{B}$ be a p-summable quasihomomorphism of parity p* mod 2, *with $\mathscr{R}$-admissible algebra $\mathscr{E}$. Given any integer $n \geq p$ of the same parity, consider two linear maps $\widehat{\chi}^n_0 : \Omega^n \mathscr{M}^s_+ \to \mathscr{R}$ and $\widehat{\chi}^n_1 : \Omega^{n+1} \mathscr{M}^s_+ \to \Omega^1 \mathscr{R}_\natural$ defined by*

$$\widehat{\chi}^n_0(x_0 \mathbf{d}x_1 \ldots \mathbf{d}x_n) = (-)^n \frac{\Gamma(1+\frac{n}{2})}{(n+1)!} \sum_{\lambda \in S_{n+1}} \varepsilon(\lambda)\, \tau(x_{\lambda(0)}[F, x_{\lambda(1)}] \ldots [F, x_{\lambda(n)}]) \tag{34}$$

$$\widehat{\chi}_1^n(x_0\mathbf{d}x_1\dots\mathbf{d}x_{n+1}) = (-)^n\frac{\Gamma(1+\frac{n}{2})}{(n+1)!}\sum_{i=1}^{n+1}\tau\natural(x_0[F,x_1]\dots\mathbf{d}x_i\dots[F,x_{n+1}])$$

where S_{n+1} is the cyclic permutation group of $n+1$ elements and ε is the signature. Then $\widehat{\chi}_0^n$ and $\widehat{\chi}_1^n$ define together a chain map $\widehat{\chi}^n : \widehat{\Omega}\mathscr{M}_+^s \to X(\mathscr{R})$ of parity n mod 2, i.e. fulfill the relations

$$\widehat{\chi}_0^n B = 0\ ,\quad \natural\mathbf{d}\widehat{\chi}_0^n - (-)^n\widehat{\chi}_1^n B = 0\ ,\quad \overline{b}\widehat{\chi}_1^n - (-)^n\widehat{\chi}_0^n b = 0\ ,\quad \widehat{\chi}_1^n b = 0\ . \tag{35}$$

Moreover $\widehat{\chi}^n$ is invariant under the Karoubi operator κ acting on $\Omega^n\mathscr{M}_+^s$ and $\Omega^{n+1}\mathscr{M}_+^s$.

Proof: This follows from purely algebraic manipulations, using the following general properties:
- The graded commutator $[F,\ ]$ is a differential and $\tau([F,\ ])=0$;
- $\mathbf{d}F=0$ so that $[F,\ ]$ and $\mathbf{d}$ are anticommuting differentials;
- $\tau\natural$ is a supertrace.
The computation is lengthy but straightforward. ∎

Thus we have attached to a p-summable quasihomomorphism $\rho : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s\hat{\otimes}\mathscr{B}$ of parity p mod 2 a sequence of cocycles $\widehat{\chi}^n$ $(n\geq p)$ of the same parity in the $\mathbb{Z}_2$-graded complex $\mathrm{Hom}(\widehat{\Omega}\mathscr{M}_+^s, X(\mathscr{R}))$. They are in fact all cohomologous, and the proposition below gives an explicit transgression formula in terms of the *eta-cochain*:

Proposition 3.7 *Let $\rho : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s\hat{\otimes}\mathscr{B}$ be a p-summable quasihomomorphism of parity p mod 2, with $\mathscr{R}$-admissible algebra $\mathscr{E}$. Given any integer $n\geq p+1$ of parity opposite to p, consider two linar maps $\widehat{\eta}_0^n : \Omega^n\mathscr{M}_+^s \to \mathscr{R}$ and $\widehat{\eta}_1^n : \Omega^{n+1}\mathscr{M}_+^s \to \Omega^1\mathscr{R}_\natural$ defined by*

$$\begin{aligned}\widehat{\eta}_0^n(x_0\mathbf{d}x_1\dots\mathbf{d}x_n) &= \frac{\Gamma(\frac{n+1}{2})}{(n+1)!}\,\frac{1}{2}\tau\Big(Fx_0[F,x_1]\dots[F,x_n] + \\ &\quad \sum_{i=1}^{n}(-)^{ni}[F,x_i]\dots[F,x_n]Fx_0[F,x_1]\dots[F,x_{i-1}]\Big)\end{aligned}$$

$$\begin{aligned}&\widehat{\eta}_1^n(x_0\mathbf{d}x_1\dots\mathbf{d}x_{n+1}) = \\ &\quad \frac{\Gamma(\frac{n+1}{2})}{(n+2)!}\sum_{i=1}^{n+1}\frac{1}{2}\tau\natural(ix_0F+(n+2-i)Fx_0)[F,x_1]\dots\mathbf{d}x_i\dots[F,x_{n+1}]\ .\end{aligned} \tag{36}$$

Then $\widehat{\eta}_0^n$ and $\widehat{\eta}_1^n$ define together a cochain $\widehat{\eta}^n\in\mathrm{Hom}(\widehat{\Omega}\mathscr{M}_+^s, X(\mathscr{R}))$ of parity n mod 2, whose coboundary equals the difference of cocycles

$$\widehat{\chi}^{n-1} - \widehat{\chi}^{n+1} = (\natural\mathbf{d}\oplus\overline{b})\widehat{\eta}^n - (-)^n\widehat{\eta}^n(b+B)\ .$$

Expressed in terms of components this amounts to the identities

$$\begin{array}{rclr} \widehat{\chi}_0^{n-1} & = & -(-)^n \widehat{\eta}_0^n B \ , & \bar{b}\widehat{\eta}_1^n - (-)^n (\widehat{\eta}_0^n b + \widehat{\eta}_0^{n+2} B) = 0 \ , \\ \widehat{\chi}_1^{n-1} & = & \natural \mathbf{d} \widehat{\eta}_0^n - (-)^n \widehat{\eta}_1^n B \ , & \natural \mathbf{d} \widehat{\eta}_0^{n+2} - (-)^n (\widehat{\eta}_1^n b + \widehat{\eta}_1^{n+2} B) = 0 \ . \end{array} \tag{37}$$

Proof: Direct computation. ∎

Remark 3.8 Using a trick of Connes [4], we may replace the chain map τ by $\tau' = \frac{1}{2}\tau(F[F,\ ])$. This allows to improve the summability condition by requiring the quasihomomorphism to be only $(p+1)$-summable instead of p-summable, while the condition on the degree remains $n \geq p$ for $\widehat{\chi}^n$ and $n \geq p+1$ for $\widehat{\eta}^n$. It is traightforward to write down the new formulas for $\widehat{\chi}^n$ and observe that it involves exactly $n+1$ commutators $[F, x]$. These formulas were actually obtained in [23] in a more general setting where we allow $\mathbf{d}F \neq 0$.

Definition 3.9 *The bivariant Chern character of the quasihomomorphism $\rho : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s \hat{\otimes} \mathscr{B}$ is represented in any degree $n \geq p$ by the composition of chain maps*

$$\mathrm{ch}^n(\rho) : X(T\mathscr{A}) \xrightarrow{\gamma} \widehat{\Omega} T\mathscr{A} \xrightarrow{\rho_*} \widehat{\Omega} \mathscr{M}_+^s \xrightarrow{\widehat{\chi}^n} X(\mathscr{R}) \ , \tag{38}$$

where $\gamma : X(T\mathscr{A}) \to \widehat{\Omega} T\mathscr{A}$ is the Goodwillie equivalence constructed in section 2 for any quasi-free algebra and $\rho_* : T\mathscr{A} \to \mathscr{M}_+^s$ is the classifying homomorphism. In the same way, define a transgressed cochain in $\mathrm{Hom}(X(T\mathscr{A}), X(\mathscr{R}))$ *by means of the eta-cochain in any degree:*

$$\not{c}\mathrm{h}^n(\rho) : X(T\mathscr{A}) \xrightarrow{\gamma} \widehat{\Omega} T\mathscr{A} \xrightarrow{\rho_*} \widehat{\Omega} \mathscr{M}_+^s \xrightarrow{\widehat{\eta}^n} X(\mathscr{R}) \ . \tag{39}$$

It fulfills the transgression property $\mathrm{ch}^n(\rho) - \mathrm{ch}^{n+2}(\rho) = [\partial, \not{c}\mathrm{h}^{n+1}(\rho)]$ *where ∂ is the X-complex boundary map.*

Recall that $\gamma(x) = (1-\phi)^{-1}(x)$ and $\gamma(\natural x \mathbf{d} y) = (1-\phi)^{-1}(x\mathbf{d}y + b(x\phi(y)))$ for any $x, y \in T\mathscr{A}$, where the map $\phi : \Omega^n T\mathscr{A} \to \Omega^{n+2} T\mathscr{A}$ is uniquely defined from its restriction to zero-forms. Its existence is guaranteed by the fact that $T\mathscr{A}$ is a free algebra. Several choices are possible, but conventionally we always take $\phi : T\mathscr{A} \to \Omega^2 T\mathscr{A}$ as the canonical map obtained by setting $\phi(a) = 0$ on the generators $a \in \mathscr{A} \subset T\mathscr{A}$, and then extended to all $T\mathscr{A}$ by the algebraic property $\phi(xy) = \phi(x)y + x\phi(y) + \mathbf{d}x\mathbf{d}y$.
Of course $\mathrm{ch}^n(\rho)$ and $\not{c}\mathrm{h}^n(\rho)$ are not very interesting a priori, because the X-complex of the non-completed tensor algebra $T\mathscr{A}$ is contractible. However, taking into account the adic filtrations induced by the ideals $J\mathscr{A} \subset T\mathscr{A}$ and $\mathscr{J} \subset \mathscr{R}$ yields non-trivial bivariant objects. By virtue of Remark 3.8 we suppose from now on that $\mathscr{I}$ is $(p+1)$-summable.

Proposition 3.10 *Let $\rho : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s \hat{\otimes} \mathscr{B}$ be a $(p+1)$-summable quasihomomorphism of parity p mod 2 with $\mathscr{R}$-admissible extension $\mathscr{E}$, and let $n \geq p$ be an integer of the same parity. The composites $\widehat{\chi}^n \rho_* \gamma$ and $\widehat{\eta}^{n+1} \rho_* \gamma$ are linear maps $X(T\mathscr{A}) \to X(\mathscr{R})$ verifying the adic properties*

$$\begin{aligned} \widehat{\chi}^n \rho_* \gamma &: \quad F^k_{J\mathscr{A}} X(T\mathscr{A}) \to F^{k-n}_{\mathscr{J}} X(\mathscr{R}) \ , \\ \widehat{\eta}^{n+1} \rho_* \gamma &: \quad F^k_{J\mathscr{A}} X(T\mathscr{A}) \to F^{k-n-2}_{\mathscr{J}} X(\mathscr{R}) \ , \end{aligned}$$

for any $k \in \mathbb{Z}$. Consequently the composite $\mathrm{ch}^n(\rho) = \widehat{\chi}^n \rho_* \gamma$ defines a cocycle of parity n mod 2 in the complex $\mathrm{Hom}^n(\widehat{X}(T\mathscr{A}, J\mathscr{A}), \widehat{X}(\mathscr{R}, \mathscr{J}))$ and the Chern character is a bivariant cyclic cohomology class of degree n:

$$\mathrm{ch}^n(\rho) \in HC^n(\mathscr{A}, \mathscr{B}) \ , \quad \forall n \geq p \ . \tag{40}$$

Moreover, the transgression relation $\mathrm{ch}^n(\rho) - \mathrm{ch}^{n+2}(\rho) = [\partial, \not{\mathrm{c}}\mathrm{h}^{n+1}(\rho)]$ holds in the complex $\mathrm{Hom}^{n+2}(\widehat{X}(T\mathscr{A}, J\mathscr{A}), \widehat{X}(\mathscr{R}, \mathscr{J}))$, which implies

$$\mathrm{ch}^{n+2}(\rho) \equiv S\mathrm{ch}^n(\rho) \ \mathit{in} \ HC^{n+2}(\mathscr{A}, \mathscr{B}) \ . \tag{41}$$

In particular the cocycles $\mathrm{ch}^n(\rho)$ for different n define the same periodic cyclic cohomology class $\mathrm{ch}(\rho) \in HP^n(\mathscr{A}, \mathscr{B})$.

Proof: Let us denote by $0 \to \mathscr{G} \to \mathscr{F} \to \mathscr{A} \to 0$ the universal extension $0 \to J\mathscr{A} \to T\mathscr{A} \to \mathscr{A} \to 0$. Recall that $\mathscr{M}^s$ and its ideal $\mathscr{N}^s$ are $\mathbb{Z}_2$-graded differential algebras on which the graded commutator $[F, \ ]$ acts as a differential of odd degree. Moreover the commutation relations $[F, \mathscr{M}^s_+] \subset \mathscr{I}^s_- \hat{\otimes} \mathscr{R}$ and $[F, \mathscr{N}^s_+] \subset \mathscr{I}^s_- \hat{\otimes} \mathscr{J}$ hold. Now, we have to investigate the adic behaviour of the Goodwillie equivalence $\gamma : X(\mathscr{F}) \to \widehat{\Omega}\mathscr{F}$ with respect to the filtration $F^k_{\mathscr{G}} X(\mathscr{F})$. The first step in that direction was actually done in the proof of Proposition 2.4, where the following filtration of the subspaces $\Omega^n \mathscr{F}$ was introduced:

$$H^k \Omega^n \mathscr{F} = \sum_{k_0 + \ldots + k_n \geq k} \mathscr{G}^{k_0} \mathbf{d}\mathscr{G}^{k_1} \ldots \mathbf{d}\mathscr{G}^{k_n} \ \subset \Omega^n \mathscr{F} \ .$$

Let us look at the image of the latter filtration under the maps $\widehat{\chi}^n \rho_*$ and $\widehat{\eta}^n \rho_* : \widehat{\Omega}\mathscr{F} \to X(\mathscr{R})$ given by Eqs. (34, 36). We know that the homomorphism $\rho_* : \mathscr{F} \to \mathscr{M}^s_+$ respects the ideals $\mathscr{G}$ and $\mathscr{N}^s_+$. Hence if $x_0, \ldots, x_n$ denote $n+1$ elements in $\mathscr{G}^{k_0}, \ldots, \mathscr{G}^{k_n}$ respectively, with $k_0 + \ldots + k_n \geq k$, then $x_{\lambda(0)}[F, x_{\lambda(1)}] \ldots [F, x_{\lambda(n)}] \in (\mathscr{N}^s)^k$ for any permutation $\lambda \in S_{n+1}$. Hence applying the supertrace τ, which is a chain map of order zero with respect to the $\mathscr{N}^s$-adic and $\mathscr{J}$-adic filtrations on $X(\mathscr{M}^s)$ and $X(\mathscr{R})$, yields (from now on we omit to write the homomorphism ρ_*)

$$\widehat{\chi}^n_0(H^k \Omega^n \mathscr{F}) \subset \mathscr{J}^k \ . \tag{42}$$

In the same way, for $n+1$ elements $x_0, \ldots, x_{n+1}$ in $\mathscr{G}^{k_0}, \ldots, \mathscr{G}^{k_{n+1}}$, the one-form $\natural x_0[F, x_1] \ldots \mathbf{d}x_i \ldots [F, x_{n+1}]$ involves $k_0 + \ldots + k_{n+1} \geq k$ powers of the ideal $\mathscr{N}^s$, hence lies in the subspace $\natural((\mathscr{N}^s)^k\mathbf{d}\mathscr{M}^s + (\mathscr{N}^s)^{k-1}\mathbf{d}\mathscr{N}^s)$. Thus applying the supertrace τ one gets

$$\widehat{\chi}_1^n(H^k\Omega^{n+1}\mathscr{F}) \subset \natural(\mathscr{J}^k\mathbf{d}\mathscr{R} + \mathscr{J}^{k-1}\mathbf{d}\mathscr{J})\ . \tag{43}$$

Proceeding in exactly the same fashion with the maps $\widehat{\eta}_0^n : \Omega^n\mathscr{F} \to \mathscr{R}$ and $\widehat{\eta}_1^n : \Omega^{n+1}\mathscr{F} \to \Omega^1\mathscr{R}_\natural$, it is clear that

$$\begin{aligned} \widehat{\eta}_0^n(H^k\Omega^n\mathscr{F}) &\subset \mathscr{J}^k\ . & (44)\\ \widehat{\eta}_1^n(H^k\Omega^{n+1}\mathscr{F}) &\subset \natural(\mathscr{J}^k\mathbf{d}\mathscr{R} + \mathscr{J}^{k-1}\mathbf{d}\mathscr{J})\ . & (45) \end{aligned}$$

However these estimates are not optimal concerning the component $\widehat{\chi}_1^n$. We need a refinement of the H-filtration. For any $k \in \mathbb{Z}$, $n \geq 0$, let us define the subspaces

$$G^k\Omega^n\mathscr{F} = \sum_{k_0+\ldots+k_n\geq k} \mathscr{G}^{k_0}(\mathbf{d}\mathscr{F})\mathscr{G}^{k_1}(\mathbf{d}\mathscr{F})\ldots\mathscr{G}^{k_{n-1}}(\mathbf{d}\mathscr{F})\mathscr{G}^{k_n} + H^{k+1}\Omega^n\mathscr{F}\ .$$

Then for fixed n, $G^*\Omega^n\mathscr{F}$ is a decreasing filtration of $\Omega^n\mathscr{F}$, and by convention $G^k\Omega^n\mathscr{F} = \Omega^n\mathscr{F}$ for $k \leq 0$. One has $G^k\Omega^n\mathscr{F} \subset H^k\Omega^n\mathscr{F}$. Now observe the following. Since $[F,\ ]$ and $\mathbf{d}$ are derivations, the map $x_0\mathbf{d}x_1\ldots\mathbf{d}x_i\ldots\mathbf{d}x_{n+1} \mapsto \natural x_0[F,x_1]\ldots\mathbf{d}x_i\ldots[F,x_{n+1}]$ has the property that

$$\begin{aligned} &\mathscr{G}^{k_0}(\mathbf{d}\mathscr{F})\mathscr{G}^{k_1}(\mathbf{d}\mathscr{F})\ldots(\mathbf{d}\mathscr{F})\mathscr{G}^{k_i}\ldots(\mathbf{d}\mathscr{F})\mathscr{G}^{k_{n+1}} \to\\ &\quad \natural(\mathscr{N}^s)^{k_0}[F,\mathscr{M}^s](\mathscr{N}^s)^{k_1}[F,\mathscr{M}^s]\ldots(\mathbf{d}\mathscr{M}^s)(\mathscr{N}^s)^{k_i}\ldots[F,\mathscr{M}^s](\mathscr{N}^s)^{k_{n+1}}\\ &\quad \subset \natural(\mathscr{N}^s)^k\mathbf{d}\mathscr{M}^s\ , \end{aligned}$$

and because $\widehat{\chi}_1^n(H^{k+1}\Omega^{n+1}\mathscr{F}) \subset \natural(\mathscr{J}^{k+1}\mathbf{d}\mathscr{R} + \mathscr{J}^k\mathbf{d}\mathscr{J}) \subset \natural\mathscr{J}^k\mathbf{d}\mathscr{R}$, one gets the crucial estimate

$$\widehat{\chi}_1^n(G^k\Omega^{n+1}\mathscr{F}) \subset \natural\mathscr{J}^k\mathbf{d}\mathscr{R}\ . \tag{46}$$

Now we have to understand the way γ sends the X-complex filtration

$$\begin{aligned} F_{\mathscr{G}}^{2k}X(\mathscr{F}) &: \mathscr{G}^{k+1} + [\mathscr{G}^k,\mathscr{F}] \rightleftarrows \natural\mathscr{G}^k\mathbf{d}\mathscr{F}\\ F_{\mathscr{G}}^{2k+1}X(\mathscr{F}) &: \mathscr{G}^{k+1} \rightleftarrows \natural(\mathscr{G}^{k+1}\mathbf{d}\mathscr{F} + \mathscr{G}^k\mathbf{d}\mathscr{G})\ , \end{aligned}$$

to the filtration $G^*\Omega^n\mathscr{F}$, in all degrees n. Recall that $\gamma(x)|_{\Omega^{2n}\mathscr{F}} = \phi^n(x)$ and $\gamma(\natural x\mathbf{d}y)|_{\Omega^{2n+1}\mathscr{F}} = \phi^n(x\mathbf{d}y + b(x\phi(y)))$ for any $x, y \in \mathscr{F}$, where the map $\phi : \Omega^n\mathscr{F} \to \Omega^{n+2}\mathscr{F}$ is obtained from its restriction to zero-forms as

$$\phi(x_0\mathbf{d}x_1\ldots\mathbf{d}x_n) = \sum_{i=0}^n (-)^{ni}\phi(x_i)\mathbf{d}x_{i+1}\ldots\mathbf{d}x_n\mathbf{d}x_0\ldots\mathbf{d}x_{i-1}\ .$$

Note the following important properties of ϕ. Firstly, it is invariant under the Karoubi operator $\kappa : \Omega^n\mathscr{F} \to \Omega^n\mathscr{F}$ in the sense that $\phi\circ\kappa = \phi$, and vanishes on the image of the boudaries $d, B : \Omega^n\mathscr{F} \to \Omega^{n+1}\mathscr{F}$. Secondly, the relation $\phi b - b\phi = B$ holds on $\Omega^n\mathscr{F}$ whenever $n \geq 1$ (see [22] §4). Since we have to apply successive powers of ϕ on the filtration $G^k\Omega^n\mathscr{F}$, the computation will be greatly simplified by exploiting κ-invariance. Define the linear map $\widetilde{\phi}:\Omega^n\mathscr{F}\to\Omega^{n+2}\mathscr{F}$ by

$$\widetilde{\phi}(x_0\mathbf{d}x_1\ldots\mathbf{d}x_n)=\sum_{i=0}^n(-)^i\mathbf{d}x_0\ldots\mathbf{d}x_{i-1}\phi(x_i)\mathbf{d}x_{i+1}\ldots\mathbf{d}x_n\ .$$

Then $\widetilde{\phi}$ coincides with ϕ modulo the image of $1-\kappa$. In particular the relation $\phi^n=\phi\circ\widetilde{\phi}^{n-1}$ holds. The advantage of the map $\widetilde{\phi}$ stems from the fact that it does not involve cyclic permutations of the elements x_i, and verifies the following optimal compatibility with the G-filtration

$$\widetilde{\phi}(G^k\Omega^n\mathscr{F})\subset G^{k-1}\Omega^{n+2}\mathscr{F}\qquad\forall k,n\geq 0\ ,$$

whereas the map ϕ is only compatible with the (coarser) H-filtration:

$$\phi(H^k\Omega^n\mathscr{F})\subset H^{k-1}\Omega^{n+2}\mathscr{F}\qquad\forall k,n\geq 0\ .$$

We shall now evaluate the image of the filtration $F^k_{\mathscr{G}}X(\mathscr{F})$ under the map $\gamma:X(\mathscr{F})\to\widehat{\Omega}\mathscr{F}$. Firstly, one has $\gamma(\mathscr{G}^{k+1})\cap\Omega^{2n}\mathscr{F}=\phi^n(\mathscr{G}^{k+1})$. But $\mathscr{G}^{k+1}\subset G^{k+1}\Omega^0\mathscr{F}$ and $\phi^n=\phi\circ\widetilde{\phi}^{n-1}$, hence

$$\gamma(\mathscr{G}^{k+1})\cap\Omega^{2n}\mathscr{F}\subset\phi(G^{k-n+2}\Omega^{2n-2}\mathscr{F})\ . \tag{47}$$

Secondly, the image of $\natural\mathscr{G}^k\mathbf{d}\mathscr{F}$ in $\Omega^{2n+1}\mathscr{F}$ is given by $\phi^n(\mathscr{G}^k\mathbf{d}\mathscr{F}+b(\mathscr{G}^k\phi(\mathscr{F})))$. One has $b(\mathscr{G}^k\phi(\mathscr{F}))\subset[\mathscr{G}^k\mathbf{d}\mathscr{F},\mathscr{F}]\subset\mathscr{G}^k\mathbf{d}\mathscr{F}$, hence we only need to compute $\phi^n(\mathscr{G}^k\mathbf{d}\mathscr{F})\subset\phi^n(G^k\Omega^1\mathscr{F})$, and

$$\gamma(\natural\mathscr{G}^k\mathbf{d}\mathscr{F})\cap\Omega^{2n+1}\mathscr{F}\subset\phi(G^{k-n+1}\Omega^{2n-1}\mathscr{F})\ . \tag{48}$$

Thirdly, $[\mathscr{G}^k,\mathscr{F}]=\overline{b}\natural\mathscr{G}^k\mathbf{d}\mathscr{F}$ so that $\gamma([\mathscr{G}^k,\mathscr{F}])=(b+B)\gamma(\natural\mathscr{G}^k\mathbf{d}\mathscr{F})$ because γ is a chain map. Therefore, the image of $[\mathscr{G}^k,\mathscr{F}]$ restricted to $\Omega^{2n}\mathscr{F}$ is contained in $B\phi^{n-1}(\mathscr{G}^k\mathbf{d}\mathscr{F})+b\phi^n(\mathscr{G}^k\mathbf{d}\mathscr{F})$. We may estimate coarsly the first term as $B\phi^{n-1}(\mathscr{G}^k\mathbf{d}\mathscr{F})\subset B\phi^{n-1}(H^k\Omega^1\mathscr{F})\subset H^{k-n+1}\Omega^{2n}\mathscr{F}$, and the second term as $b\phi\widetilde{\phi}^{n-1}(\mathscr{G}^k\mathbf{d}\mathscr{F})\subset b\phi(G^{k-n+1}\Omega^{2n-1}\mathscr{F})$. Hence

$$\gamma([\mathscr{G}^k,\mathscr{F}])\cap\Omega^{2n}\mathscr{F}\subset H^{k-n+1}\Omega^{2n}\mathscr{F}+b\phi(G^{k-n+1}\Omega^{2n-1}\mathscr{F})\ . \tag{49}$$

Fourthly, the image of $\natural\mathscr{G}^k\mathbf{d}\mathscr{G}$ in $\Omega^{2n+1}\mathscr{F}$ is given by $\phi^n(\mathscr{G}^k\mathbf{d}\mathscr{G}+b(\mathscr{G}^k\phi(\mathscr{G})))$. We estimate coarsly $\phi^n(\mathscr{G}^k\mathbf{d}\mathscr{G})\subset\phi^n(H^{k+1}\Omega^1\mathscr{F})\subset H^{k-n+1}\Omega^{2n+1}\mathscr{F}$. Then, one has $\mathscr{G}^k\phi(\mathscr{G})\subset\mathscr{G}^k\mathbf{d}\mathscr{F}\mathbf{d}\mathscr{F}\subset G^k\Omega^2\mathscr{F}$, and using repeatedly the relations $\phi b-b\phi=B$, $\phi B=0$ gives $\phi^n b(\mathscr{G}^k\phi(\mathscr{G}))\subset b\phi^n(G^k\Omega^2\mathscr{F})+B\phi^{n-1}(G^k\Omega^2\mathscr{F})\subset b\phi(G^{k-n+1}\Omega^{2n}\mathscr{F})+BH^{k-n+1}\Omega^{2n}\mathscr{F}$. Thus

$$\gamma(\natural\mathscr{G}^k\mathbf{d}\mathscr{G})\cap\Omega^{2n+1}\mathscr{F}\subset H^{k-n+1}\Omega^{2n+1}\mathscr{F}+b\phi(G^{k-n+1}\Omega^{2n}\mathscr{F})\ . \tag{50}$$

Now everything is set to evaluate the adic behaviour of the composites $\widehat{\chi}^n\gamma$ and $\widehat{\eta}^n\gamma$. We shall deal only with even degrees, the odd case is similar.

Hence let us start with the map $\widehat{\chi}_0^{2n}\gamma : \mathscr{F} \to \mathscr{R}$. For any $k \in \mathbb{Z}$, Eq. (47) gives $\widehat{\chi}_0^{2n}\gamma(\mathscr{G}^{k+1}) \subset \widehat{\chi}_0^{2n} \circ \phi(G^{k-n+2}\Omega^{2n-2}\mathscr{F})$. But $\widehat{\chi}_0^{2n}$ is κ-invariant, hence $\widehat{\chi}_0^{2n} \circ \phi = \widehat{\chi}_0^{2n} \circ \widetilde{\phi}$. Therefore, $\widehat{\chi}_0^{2n}\gamma(\mathscr{G}^{k+1}) \subset \widehat{\chi}_0^{2n}(G^{k-n+1}\Omega^{2n}\mathscr{F}) \subset \mathscr{J}^{k-n+1}$ using $G^{k-n+1}\Omega \subset H^{k-n+1}\Omega$ and (42). Now we look at its companion $\widehat{\chi}_1^{2n}\gamma : \Omega^1\mathscr{F}_\natural \to \Omega^1\mathscr{R}_\natural$. From (48) one gets $\widehat{\chi}_1^{2n}(\natural\mathscr{G}^k\mathbf{d}\mathscr{F}) \subset \widehat{\chi}_1^{2n} \circ \phi(G^{k-n+1}\Omega^{2n-1}\mathscr{F})$. But $\widehat{\chi}_1^{2n}$ is also κ-invariant and $\widehat{\chi}_1^{2n} \circ \phi = \widehat{\chi}_1^{2n} \circ \widetilde{\phi}$, thus $\widehat{\chi}_1^{2n}\gamma(\natural\mathscr{G}^k\mathbf{d}\mathscr{F}) \subset \widehat{\chi}_1^{2n}(G^{k-n}\Omega^{2n+1}\mathscr{F}) \subset \natural\mathscr{J}^{k-n}\mathbf{d}\mathscr{R}$ by (46). This allows to estimate the image of $[\mathscr{G}^k, \mathscr{F}] = \overline{b}\natural\mathscr{G}^k\mathbf{d}\mathscr{F}$ under the chain map $\widehat{\chi}^{2n}\gamma$. Indeed $\widehat{\chi}_0^{2n}\gamma(\overline{b}\natural\mathscr{G}^k\mathbf{d}\mathscr{F}) = \overline{b}\widehat{\chi}_1^{2n}\gamma(\natural\mathscr{G}^k\mathbf{d}\mathscr{F}) \subset \overline{b}\natural\mathscr{J}^{k-n}\mathbf{d}\mathscr{R}$, so that $\widehat{\chi}_0^{2n}\gamma([\mathscr{G}^k, \mathscr{F}]) \subset [\mathscr{J}^{k-n}, \mathscr{R}]$. Collecting these results shows the effect of the map $\widehat{\chi}^{2n}\gamma$ on the adic filtration in even degree:

$$\left\{ \begin{array}{lll} \widehat{\chi}_0^{2n}\gamma : & \mathscr{G}^{k+1} + [\mathscr{G}^k, \mathscr{F}] & \longrightarrow & \mathscr{J}^{k-n+1} + [\mathscr{J}^{k-n}, \mathscr{R}] \\ \widehat{\chi}_1^{2n}\gamma : & \natural\mathscr{G}^k\mathbf{d}\mathscr{F} & \longrightarrow & \natural\mathscr{J}^{k-n}\mathbf{d}\mathscr{R} \end{array} \right.$$

hence $\widehat{\chi}^{2n}\gamma : F^{2k}_{\mathscr{G}}X(\mathscr{F}) \to F^{2k-2n}_{\mathscr{J}}X(\mathscr{R})$. To understand the effect on the filtration in odd degree, one has to evaluate $\widehat{\chi}_1^{2n}\gamma$ on $\natural\mathscr{G}^k\mathbf{d}\mathscr{G}$. From (50), one gets $\widehat{\chi}_1^{2n}\gamma(\natural\mathscr{G}^k\mathbf{d}\mathscr{G}) \subset \widehat{\chi}_1^{2n}(H^{k-n+1}\Omega^{2n+1}\mathscr{F} + b\phi(G^{k-n+1}\Omega^{2n}\mathscr{F}))$. But (35) shows $\widehat{\chi}_1^{2n} \circ b = 0$, and (43) implies $\widehat{\chi}_1^{2n}\gamma(\natural\mathscr{G}^k\mathbf{d}\mathscr{G}) \subset \natural(\mathscr{J}^{k-n+1}\mathbf{d}\mathscr{R} + \mathscr{J}^{k-n}\mathbf{d}\mathscr{J})$. One thus gets the adic behaviour of the chain map $\widehat{\chi}^{2n}\gamma$ on the filtration of odd degree:

$$\left\{ \begin{array}{lll} \widehat{\chi}_0^{2n}\gamma : & \mathscr{G}^{k+1} & \longrightarrow & \mathscr{J}^{k-n+1} \\ \widehat{\chi}_1^{2n}\gamma : & \natural(\mathscr{G}^{k+1}\mathbf{d}\mathscr{F} + \mathscr{G}^k\mathbf{d}\mathscr{G}) & \longrightarrow & \natural(\mathscr{J}^{k-n+1}\mathbf{d}\mathscr{R} + \mathscr{J}^{k-n}\mathbf{d}\mathscr{J}) \end{array} \right.$$

hence $\widehat{\chi}^{2n}\gamma : F^{2k+1}_{\mathscr{G}}X(\mathscr{F}) \to F^{2k-2n+1}_{\mathscr{J}}X(\mathscr{R})$ and $\widehat{\chi}^{2n}\gamma$ is a map of order $2n$. Using similar methods, one shows that $\widehat{\chi}^{2n+1}\gamma$ is of order $2n+1$.

Now we investigate the eta-cochain. Consider $\widehat{\eta}_0^{2n}\gamma : \mathscr{F} \to \mathscr{R}$. (47) gives $\widehat{\eta}_0^{2n}\gamma(\mathscr{G}^{k+1}) \subset \widehat{\eta}_0^{2n}\phi(G^{k-n+2}\Omega^{2n-2}\mathscr{F})$. However $\widehat{\eta}^{2n}$ is not κ-invariant, so that we cannot replace ϕ by $\widetilde{\phi}$. We are forced to consider $\phi(G^{k-n+2}\Omega^{2n-2}\mathscr{F}) \subset H^{k-n+1}\Omega^{2n}\mathscr{F}$ and consequently $\widehat{\eta}_0^{2n}\gamma(\mathscr{G}^{k+1}) \subset \mathscr{J}^{k-n+1}$ by (44). Similarly, (49) implies $\widehat{\eta}_0^{2n}\gamma([\mathscr{G}^k, \mathscr{F}]) \subset \widehat{\eta}_0^{2n}(H^{k-n+1}\Omega^{2n}\mathscr{F}) + \widehat{\eta}_0^{2n}b\phi(G^{k-n+1}\Omega^{2n-1}\mathscr{F}) \subset \widehat{\eta}_0^{2n}(H^{k-n}\Omega^{2n}\mathscr{F})$ hence $\widehat{\eta}_0^{2n}\gamma([\mathscr{G}^k, \mathscr{F}]) \subset \mathscr{J}^{k-n}$. Its companion $\widehat{\eta}_1^{2n}\gamma : \Omega^1\mathscr{F}_\natural \to \Omega^1\mathscr{R}_\natural$ evaluated on $\natural\mathscr{G}^k\mathbf{d}\mathscr{F}$ uses equation (48) again with $\phi(G^{k-n+1}\Omega^{2n-1}\mathscr{F}) \subset H^{k-n}\Omega^{2n+1}\mathscr{F}$, so that $\widehat{\eta}_1^{2n}\gamma(\natural\mathscr{G}^k\mathbf{d}\mathscr{F}) \subset \widehat{\eta}_1^{2n}(H^{k-n}\Omega^{2n+1}\mathscr{F}) \subset \natural(\mathscr{J}^{k-n}\mathbf{d}\mathscr{R} + \mathscr{J}^{k-n-1}\mathbf{d}\mathscr{J})$ by (45). This shows the effect of $\widehat{\eta}^{2n}\gamma$ on the filtration of even degree

$$\left\{ \begin{array}{lll} \widehat{\eta}_0^{2n}\gamma : & \mathscr{G}^{k+1} + [\mathscr{G}^k, \mathscr{F}] & \longrightarrow & \mathscr{J}^{k-n} \\ \widehat{\eta}_1^{2n}\gamma : & \natural\mathscr{G}^k\mathbf{d}\mathscr{F} & \longrightarrow & \natural(\mathscr{J}^{k-n}\mathbf{d}\mathscr{R} + \mathscr{J}^{k-n-1}\mathbf{d}\mathscr{J}) \end{array} \right.$$

hence $\widehat{\eta}^{2n}\gamma : F^{2k}_{\mathscr{G}}X(\mathscr{F}) \to F^{2k-2n-1}_{\mathscr{J}}X(\mathscr{R})$. For the odd filtration, let us compute from (50) $\widehat{\eta}_1^{2n}\gamma(\natural\mathscr{G}^k\mathbf{d}\mathscr{G}) \subset \widehat{\eta}_1^{2n}(H^{k-n+1}\Omega^{2n+1}\mathscr{F}) + \widehat{\eta}_1^{2n}b\phi(G^{k-n+1}\Omega^{2n}\mathscr{F})$. But the identities (37) show that $\widehat{\eta}_1^{2n}b = \widehat{\chi}_1^{2n+1}$, hence using κ-invariance one gets $\widehat{\eta}_1^{2n}b\phi(G^{k-n+1}\Omega^{2n}\mathscr{F}) \subset \widehat{\chi}_1^{2n+1}\widetilde{\phi}(G^{k-n+1}\Omega^{2n}\mathscr{F}) \subset \widehat{\chi}_1^{2n+1}(G^{k-n}\Omega^{2n+2}\mathscr{F})$.

Therefore, (45) and (46) imply $\widehat{\eta}_1^{2n}\gamma(\natural\mathscr{G}^k\mathbf{d}\mathscr{G}) \subset \natural\mathscr{J}^{k-n}\mathbf{d}\mathscr{R}$. These results give the adic behaviour of $\widehat{\eta}^{2n}\gamma$ with respect to the odd filtration

$$\left\{ \begin{array}{llll} \widehat{\eta}_0^{2n}\gamma : & \mathscr{G}^{k+1} & \longrightarrow & \mathscr{J}^{k-n+1} \\ \widehat{\eta}_1^{2n}\gamma : & \natural(\mathscr{G}^{k+1}\mathbf{d}\mathscr{F} + \mathscr{G}^k\mathbf{d}\mathscr{G}) & \longrightarrow & \natural\mathscr{J}^{k-n}\mathbf{d}\mathscr{R} \end{array} \right.$$

hence $\widehat{\eta}^{2n}\gamma : F^{2k+1}_{\mathscr{G}}X(\mathscr{F}) \to F^{2k-2n}_{\mathscr{J}}X(\mathscr{R})$ and $\widehat{\eta}^{2n}\gamma$ is a map of order $2n+1$. Similarly, one shows that $\widehat{\eta}^{2n+1}\gamma$ is of order $2n+2$. ∎

Note that the chain maps γ and $\widehat{\chi}^n$ extend to the adic completions of all the algebras involved, so that from now on we will consider the bivariant Chern character $\mathrm{ch}^n(\rho) \in \mathrm{Hom}^n(\widehat{X}(T\mathscr{A}, J\mathscr{A}), \widehat{X}(\mathscr{R}, \mathscr{J}))$ as a chain map of pro-complexes

$$\mathrm{ch}^n(\rho) : X(\widehat{T}\mathscr{A}) \xrightarrow{\gamma} \widehat{\Omega}\widehat{T}\mathscr{A} \xrightarrow{\rho_*} \widehat{\Omega}\widehat{\mathscr{M}}^s_+ \xrightarrow{\widehat{\chi}^n} X(\widehat{\mathscr{R}}) \ . \tag{51}$$

We would like to introduce some equivalence relations among quasihomomorphisms, and discuss the corresponding invariance properties of the Chern character. The first equivalence relation is (smooth) homotopy. It involves the algebra $C^\infty[0,1]$ of smooth functions $f : [0,1] \to \mathbb{C}$, such that all the derivatives of order ≥ 1 vanish at the enpoints 0 and 1, while the values of f itself remain arbitrary. We have already seen that $C^\infty[0,1]$ endowed with its usual Fréchet topology is an m-algebra. It is moreover nuclear [14], so that its projective tensor product $\mathscr{A}\hat{\otimes}C^\infty[0,1]$ with any m-algebra $\mathscr{A}$ is isomorphic to the algebra of smooth $\mathscr{A}$-valued functions over $[0,1]$, with all derivatives of order ≥ 1 vanishing at the endpoints. We will usually denote by $\mathscr{A}[0,1]$ this m-algebra. The second equivalence relation of interest among quasihomomorphisms is conjugation by an invertible element of the unitalized algebra $(\mathscr{E}^s_+)^+$.

Definition 3.11 *Let $\rho_0 : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s\hat{\otimes}\mathscr{B}$ and $\rho_1 : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s\hat{\otimes}\mathscr{B}$ be two quasihomomorphisms with same parity. They are called*

i) homotopic *if there exists a quasihomomorphism $\rho : \mathscr{A} \to \mathscr{E}[0,1]^s \triangleright \mathscr{I}^s\hat{\otimes}\mathscr{B}[0,1]$ such that evaluation at the endpoints gives ρ_0 and ρ_1;*

ii) conjugate *if there exists an invertible element in the unitalized algebra $U \in (\mathscr{E}^s_+)^+$ with $U - 1 \in \mathscr{E}^s_+$, such that $\rho_1 = U^{-1}\rho_0 U$ as a homomorphism $\mathscr{A} \to \mathscr{E}^s_+$.*

Remark that the commutators $[F, U]$ and $[F, U^{-1}]$ always lie in the ideal $\mathscr{I}^s\hat{\otimes}\mathscr{B} \subset \mathscr{E}^s$. When the algebra $\mathscr{I}$ is M_2-stable (i.e. $M_2(\mathscr{I}) \cong \mathscr{I}$), two conjugate quasihomomorphisms are also homotopic, but the converse is not true. Hence conjugation is strictly stronger than homotopy as an equivalence relation. The former is an analogue of "compact perturbation" of quasihomomorphisms in Kasparov's bivariant K-theory for C^*-algebras, see [2].
The proposition below describes the compatibility between these equivalence relations and the bivariant Chern character.

Proposition 3.12 *Let* $\rho_0 : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s \hat{\otimes} \mathscr{B}$ *and* $\rho_1 : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s \hat{\otimes} \mathscr{B}$ *be two* $(p+1)$*-summable quasihomomorphisms of parity* $p \bmod 2$*, with* $\mathscr{E}$ *admissible with respect to a quasi-free extension* $\mathscr{R}$ *of* $\mathscr{B}$*. Let* $n \geq p$ *be any integer of the same parity.*

i) If ρ_0 *and* ρ_1 *are homotopic, then* $S\mathrm{ch}^n(\rho_0) \equiv S\mathrm{ch}^n(\rho_1)$ *in* $HC^{n+2}(\mathscr{A}, \mathscr{B})$*. In particular* $\mathrm{ch}^n(\rho_0) \equiv \mathrm{ch}^n(\rho_1)$ *whenever* $n \geq p+2$*.*

ii) If ρ_0 *and* ρ_1 *are conjugate, then* $\mathrm{ch}^n(\rho_0) \equiv \mathrm{ch}^n(\rho_1)$ *in* $HC^n(\mathscr{A}, \mathscr{B})$ *for all* $n \geq p$*.*

Proof: First observe that if $\rho : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s \hat{\otimes} \mathscr{B}$ is a quasihomomorphism with $\mathscr{R}$-admissible algebra $\mathscr{E}$, the lifting homomorphism $\rho_* : T\mathscr{A} \to \mathscr{M}^s_+$ factors through the tensor algebra $T\mathscr{E}^s_+$ by virtue of the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & J\mathscr{A} \arrow[r] \arrow[d, "\varphi"'] & T\mathscr{A} \arrow[r] \arrow[d, "\varphi"'] & \mathscr{A} \arrow[r] \arrow[d, "\rho"'] & 0 \\
0 \arrow[r] & J\mathscr{E}^s_+ \arrow[r] \arrow[d] & T\mathscr{E}^s_+ \arrow[r] \arrow[d] & \mathscr{E}^s_+ \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathscr{N}^s_+ \arrow[r] & \mathscr{M}^s_+ \arrow[r] & \mathscr{E}^s_+ \arrow[r] \arrow[l, dotted, bend right, "\sigma"'] & 0
\end{tikzcd}$$

where the homomorphism $\varphi : T\mathscr{A} \to T\mathscr{E}^s_+$ is $\varphi(a_1 \otimes \ldots \otimes a_n) = \rho(a_1) \otimes \ldots \otimes \rho(a_n)$, and the arrow $T\mathscr{E}^s_+ \to \mathscr{M}^s_+$ maps a tensor product $e_1 \otimes \ldots \otimes e_n \in T\mathscr{E}^s_+$ to the product $\sigma(e_1) \ldots \sigma(e_n)$. By the naturality of the Goodwillie equivalences $\gamma_{\mathscr{A}} : X(T\mathscr{A}) \to \widehat{\Omega} T\mathscr{A}$ and $\gamma_{\mathscr{E}^s_+} : X(T\mathscr{E}^s_+) \to \widehat{\Omega} T\mathscr{E}^s_+$, one immediately sees that the bivariant Chern character coincides with the composition of chain maps

$$\mathrm{ch}^n(\rho) : X(T\mathscr{A}) \xrightarrow{X(\varphi)} X(T\mathscr{E}^s_+) \xrightarrow{\gamma_{\mathscr{E}^s_+}} \widehat{\Omega} T\mathscr{E}^s_+ \xrightarrow{\widehat{\chi}^n} X(\mathscr{R}) \ .$$

Hence, all the information about the homomorphism $\rho : \mathscr{A} \to \mathscr{E}^s_+$ is concentrated in the chain map $X(\varphi) : X(T\mathscr{A}) \to X(T\mathscr{E}^s_+)$. This will simplify the comparison of Chern characters associated to homotopic or conjugate quasihomomorphisms.

i) Homotopy: the cocycles $\mathrm{ch}^n(\rho_0)$ and $\mathrm{ch}^n(\rho_1)$ differ only by the chain maps $X(\varphi_i) : X(T\mathscr{A}) \to X(T\mathscr{E}^s_+)$, $i = 0, 1$. We view $\rho : \mathscr{A} \to \mathscr{E}^s_+[0,1]$ as a smooth family of homomorphisms $\rho_t : \mathscr{A} \to \mathscr{E}^s_+$ parametrized by $t \in [0,1]$, giving a homotopy between the two endpoints ρ_0 and ρ_1. Cuntz and Quillen prove in [12] a Cartan homotopy formula which provides a transgression between the chain maps $X(\varphi_i)$. At any point $t \in [0,1]$, denote by $\dot{\varphi} = \frac{d}{dt}\varphi : T\mathscr{A} \to T\mathscr{E}^s_+$ the derivative of the homomorphism φ_t with respect to t, and define a linear map $\iota : \Omega^m T\mathscr{A} \to \Omega^{m-1} T\mathscr{E}^s_+$ by

$$\iota(x_0 \mathbf{d}x_1 \ldots \mathbf{d}x_m) = (\varphi x_0)(\dot{\varphi} x_1)\mathbf{d}(\varphi x_2) \ldots \mathbf{d}(\varphi x_m) \ .$$

The tensor algebra $T\mathscr{A}$ is quasi-free, hence consider any $\phi : T\mathscr{A} \to \Omega^2 T\mathscr{A}$ verifying $\phi(xy) = \phi(x)y + x\phi(y) + \mathbf{d}x\mathbf{d}y$, and let $h : X(T\mathscr{A}) \to X(T\mathscr{E}^s_+)$ be the linear map of odd degree

$$h(x) = \natural\iota\phi(x) \ , \qquad h(\natural x\mathbf{d}y) = \iota(x\mathbf{d}y + b(x\phi(y)))$$

(the latter is well-defined on $\natural x\mathbf{d}y$). Then Cuntz and Quillen show the following adic properties of h for any $k \in \mathbb{Z}$,

$$\begin{aligned} h(F^k_{J\mathscr{A}} X(T\mathscr{A})) &\subset F^{k-1}_{J\mathscr{E}^s_+} X(T\mathscr{E}^s_+) \quad \text{if} \quad \dot\varphi(J\mathscr{A}) \subset J\mathscr{E}^s_+ \ , \\ h(F^k_{J\mathscr{A}} X(T\mathscr{A})) &\subset F^{k}_{J\mathscr{E}^s_+} X(T\mathscr{E}^s_+) \quad \text{if} \quad \dot\varphi(T\mathscr{A}) \subset J\mathscr{E}^s_+ \ , \end{aligned}$$

and moreover the transgression formula $\frac{d}{dt} X(\varphi) = [\partial, h]$ holds. Hence if we define by integration over $[0,1]$ the odd chain $H = \int_0^1 dt\, h$, one has

$$X(\varphi_1) - X(\varphi_0) = [\partial, H]$$

in the complex $\mathrm{Hom}^1(\widehat{X}(T\mathscr{A}, J\mathscr{A}), \widehat{X}(T\mathscr{E}^s_+, J\mathscr{E}^s_+))$ in case $\dot\varphi(J\mathscr{A}) \subset J\mathscr{E}^s_+$, or in the complex $\mathrm{Hom}^0(\widehat{X}(T\mathscr{A}, J\mathscr{A}), \widehat{X}(T\mathscr{E}^s_+, J\mathscr{E}^s_+))$ in case $\dot\varphi(T\mathscr{A}) \subset J\mathscr{E}^s_+$. For a general homotopy we are in the first case $\dot\varphi(J\mathscr{A}) \subset J\mathscr{E}^s_+$. After composition by the chain map $\widehat{\chi}^n \gamma_{\mathscr{E}^s_+} \in \mathrm{Hom}^n(\widehat{X}(T\mathscr{E}^s_+, J\mathscr{E}^s_+), \widehat{X}(\mathscr{R}, \mathscr{J}))$, this shows the transgression relation

$$\mathrm{ch}^n(\rho_1) - \mathrm{ch}^n(\rho_0) = (-)^n [\partial, \widehat{\chi}^n \gamma_{\mathscr{E}^s_+} H] \in \mathrm{Hom}^{n+1}(\widehat{X}(T\mathscr{A}, J\mathscr{A}), \widehat{X}(\mathscr{R}, \mathscr{J})) \ ,$$

whence $S\mathrm{ch}^n(\rho_1) \equiv S\mathrm{ch}^n(\rho_0)$ in $HC^{n+2}(\mathscr{A}, \mathscr{B})$. The sign $(-)^n$ comes from the parity of the chain map $\widehat{\chi}^n \gamma_{\mathscr{E}^s_+}$.

ii) Conjugation: now $\varphi_0, \varphi_1 : T\mathscr{A} \to T\mathscr{E}^s_+$ are the homomorphism lifts of ρ_0 and $\rho_1 = U^{-1}\rho_0 U$. Introduce the pro-algebra $\widehat{T}\mathscr{E}^s_+ = \varprojlim_k T\mathscr{E}^s_+/(J\mathscr{E}^s_+)^k \cong \prod_{k\geq 0} \Omega^{2k}\mathscr{E}^s_+$, and consider the invertible $U \in (\mathscr{E}^s_+)^+$ as an element $\widehat{U}$ of the unitalization $(\widehat{T}\mathscr{E}^s_+)^+$, via the linear inclusion of zero-forms $\mathscr{E}^s_+ \hookrightarrow \widehat{T}\mathscr{E}^s_+$. By proceeding as in [12], it turns out that $\widehat{U}$ is invertible, with inverse given by the series

$$\widehat{U}^{-1} = \sum_{k\geq 0} U^{-1}(dU\, dU^{-1})^k \ \in (\widehat{T}\mathscr{E}^s_+)^+ \ .$$

Of course the image of $\widehat{U}^{-1}$ under the multiplication map $(\widehat{T}\mathscr{E}^s_+)^+ \to (\mathscr{E}^s_+)^+$ is U^{-1}. We will show that φ_1, viewed as a homomorphism $T\mathscr{A} \to \widehat{T}\mathscr{E}^s_+$, is homotopic to the homomorphism $\widehat{U}^{-1}\varphi_0\widehat{U}$. For any $t \in [0,1]$ define a linear map $\sigma_t : \mathscr{A} \to \widehat{T}\mathscr{E}^s_+$ by

$$\sigma_t(a) = (1-t)\rho_1(a) + t\, \widehat{U}^{-1}\rho_0(a)\widehat{U} \ , \quad \forall a \in \mathscr{A} \ ,$$

where $\rho_0(a)$ and $\rho_1(a)$ are considered as elements of the subspace of zero-forms $\mathscr{E}^s_+ \hookrightarrow \widehat{T}\mathscr{E}^s_+$. Thus σ_t is a linear lifting of the constant homomorphism

$\rho_1 : \mathscr{A} \to \mathscr{E}^s_+$. Then use the universal property of the tensor algebra $T\mathscr{A}$ to build a smooth family of homomorphisms $\varphi(t) : T\mathscr{A} \to \widehat{T}\mathscr{E}^s_+$ by means of the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & J\mathscr{A} \arrow[r] \arrow[d, "\varphi(t)"'] & T\mathscr{A} \arrow[r] \arrow[d, "\varphi(t)"'] & \mathscr{A} \arrow[r] \arrow[d, "\rho_1"] \arrow[dl, dotted, "\sigma_t"'] & 0 \\ 0 \arrow[r] & \widehat{J}\mathscr{E}^s_+ \arrow[r] & \widehat{T}\mathscr{E}^s_+ \arrow[r] & \mathscr{E}^s_+ \arrow[r] & 0 \end{tikzcd}$$

By construction one has $\varphi(0) = \varphi_1$, $\varphi(1) = \widehat{U}^{-1}\varphi_0\widehat{U}$ and the derivative $\dot{\varphi}$ sends $T\mathscr{A}$ to the ideal $\widehat{J}\mathscr{E}^s_+$. Hence from the Cartan homotopy formula of part i) we deduce that the chain maps $X(\varphi_1)$ and $X(\widehat{U}^{-1}\varphi_0\widehat{U})$ are cohomologous in the complex $\mathrm{Hom}^0(\widehat{X}(T\mathscr{A}, J\mathscr{A}), \widehat{X}(T\mathscr{E}^s_+, J\mathscr{E}^s_+))$. Then we have to show that $X(\widehat{U}^{-1}\varphi_0\widehat{U})$ and $X(\varphi_0)$ are cohomologous. Consider the following linear map of odd degree $h : X(T\mathscr{A}) \to X(\widehat{T}\mathscr{E}^s_+) \cong \widehat{X}(T\mathscr{E}^s_+, J\mathscr{E}^s_+)$ defined by

$$h(x) = \natural(\widehat{U}^{-1}\varphi_0(x)\mathbf{d}\widehat{U}) \ , \qquad h(\natural x\mathbf{d}y) = 0 \ .$$

It is easy to see that h defines a cochain of order zero, i.e. lies in the complex $\mathrm{Hom}^0(\widehat{X}(T\mathscr{A}, J\mathscr{A}), \widehat{X}(T\mathscr{E}^s_+, J\mathscr{E}^s_+))$. Moreover, one has the transgression relation $X(\widehat{U}^{-1}\varphi_0\widehat{U}) - X(\varphi_0) = [\partial, h]$. Indeed (we replace $\varphi_0(x)$ by x for notational simplicity)

$$\begin{aligned} X(\widehat{U}^{-1}\varphi_0\widehat{U})(x) - X(\varphi_0)(x) &= \widehat{U}^{-1}x\widehat{U} - x \ = \ [\widehat{U}^{-1}x, \widehat{U}] \\ &= \overline{b}\natural(\widehat{U}^{-1}x\mathbf{d}\widehat{U}) \ = \ \overline{b}h(x) \ , \end{aligned}$$

$$\begin{aligned} X(\widehat{U}^{-1}\varphi_0\widehat{U})(\natural x\mathbf{d}y) - X(\varphi_0)(\natural x\mathbf{d}y) &= \natural\widehat{U}^{-1}x\widehat{U}\mathbf{d}(\widehat{U}^{-1}y\widehat{U}) - \natural x\mathbf{d}y \\ &= \natural(\widehat{U}^{-1}x\widehat{U}\mathbf{d}\widehat{U}^{-1}y\widehat{U} + \widehat{U}^{-1}x\mathbf{d}y\widehat{U} + \widehat{U}^{-1}xy\mathbf{d}\widehat{U} - x\mathbf{d}y) \\ &= \natural(-yx\mathbf{d}\widehat{U}\widehat{U}^{-1} + xy\mathbf{d}\widehat{U}\widehat{U}^{-1}) \ = \natural\widehat{U}^{-1}[x,y]\mathbf{d}\widehat{U} \\ &= h(\overline{b}\natural x\mathbf{d}y) \ , \end{aligned}$$

where in the second computation we use the identity $\mathbf{d}\widehat{U}^{-1} = -\widehat{U}^{-1}\mathbf{d}\widehat{U}\widehat{U}^{-1}$ deduced from $\mathbf{d}1 = 0$. This shows the equality of bivariant cyclic cohomology classes

$$X(\varphi_1) \equiv X(\varphi_0) \ \in HC^0(\mathscr{A}, \mathscr{E}^s_+) \ ,$$

so that after composition with $\widehat{\chi}^n\gamma_{\mathscr{E}^s_+} \in HC^n(\mathscr{E}^s_+, \mathscr{B})$, the equality $\mathrm{ch}^n(\rho_1) \equiv \mathrm{ch}^n(\rho_0)$ holds in $HC^n(\mathscr{A}, \mathscr{B})$. ∎

Part ii) of the above proof also shows the independence of the cohomology class $\mathrm{ch}^n(\rho) \in HC^n(\mathscr{A}, \mathscr{B})$ with respect to the choice of linear splitting $\sigma : \mathscr{E}^s_+ \to \mathscr{M}^s_+$ used to lift the homomorphism ρ, two such splittings being always homotopic. Then, from section 2 we know that any class in $HC^n(\mathscr{A}, \mathscr{B})$

induces linear maps of degree $-n$ between the cyclic homologies of $\mathscr{A}$ and $\mathscr{B}$, compatible with the SBI exact sequence. Hence, if the quasihomomorphism is $(p+1)$-summable and with parity p mod 2, the lowest degree representative of the Chern character $\mathrm{ch}^p(\rho) \in HC^p(\mathscr{A},\mathscr{B})$ carries the maximal information. We collect these results in a theorem:

Theorem 3.13 *Let $\rho : \mathscr{A} \to \mathscr{E}^s_+ \triangleright \mathscr{I}^s_+\hat{\otimes}\mathscr{B}$ be a $(p+1)$-summable quasihomomorphism of parity* p mod 2*, with $\mathscr{E}$ admissible with respect to a quasi-free extension of $\mathscr{B}$. The bivariant Chern character $\mathrm{ch}^p(\rho) \in HC^p(\mathscr{A},\mathscr{B})$ induces a graded-commutative diagram*

$$\begin{tikzcd} HP_{n+1}(\mathscr{A}) \arrow[r, "S"] \arrow[d, "\mathrm{ch}^p(\rho)"] & HC_{n-1}(\mathscr{A}) \arrow[r, "B"] \arrow[d, "\mathrm{ch}^p(\rho)"] & HN_n(\mathscr{A}) \arrow[r, "I"] \arrow[d, "\mathrm{ch}^p(\rho)"] & HP_n(\mathscr{A}) \arrow[d, "\mathrm{ch}^p(\rho)"] \\ HP_{n-p+1}(\mathscr{B}) \arrow[r, "S"] & HC_{n-p-1}(\mathscr{B}) \arrow[r, "B"] & HN_{n-p}(\mathscr{B}) \arrow[r, "I"] & HP_{n-p}(\mathscr{B}) \end{tikzcd}$$

invariant under conjugation of quasihomomorphisms. Moreover the arrow in periodic cyclic homology $HP_n(\mathscr{A}) \to HP_{n-p}(\mathscr{B})$ is invariant under homotopy of quasihomomorphisms.

Proof: The fact that $S\mathrm{ch}^p(\rho) \in HC^{p+2}(\mathscr{A},\mathscr{B})$ is homotopy invariant shows its image in the periodic theory $HP^p(\mathscr{A},\mathscr{B})$ is homotopy invariant. ∎

Example 3.14 When $\mathscr{A}$ is arbitrary and $\mathscr{B} = \mathbb{C}$, we saw in Example 3.3 that a $(p+1)$-summable quasihomomorphism $\rho : \mathscr{A} \to \mathscr{L}^s \triangleright \mathscr{I}^s$, represents a K-homology class of $\mathscr{A}$. By hypothesis, the degree of the quasihomomorphism is p mod 2. The Chern character $\mathrm{ch}^p(\rho) \in HC^p(\mathscr{A},\mathbb{C}) \cong HC^p(\mathscr{A})$ is a cyclic cohomology class of degree p over $\mathscr{A}$, represented by a chain map $\widehat{X}(T\mathscr{A},J\mathscr{A}) \to \mathbb{C}$ vanishing on the subcomplex $F^p\widehat{X}(T\mathscr{A},J\mathscr{A})$. Using the pro-vector space isomorphism $\widehat{X}(T\mathscr{A},J\mathscr{A}) \cong \widehat{\Omega}\mathscr{A}$, one finds that $\mathrm{ch}^p(\rho)$ is non-zero only on the subspace of p-forms $\Omega^p\mathscr{A}$, explicitly

$$\mathrm{ch}^p(\rho)(a_0da_1\dots da_p) = \frac{c_p}{2}\,\mathrm{Tr}_s(F[F,a_0]\dots[F,a_p])\ ,$$

where $\mathrm{Tr}_s : (\mathscr{I}^s)^{p+1} \to \mathbb{C}$ is the supertrace of the $(p+1)$-summable algebra $\mathscr{I}$ and c_p is a constant depending on the degree. One has $c_p = (-)^n(n!)^2/p!$ when $p = 2n$ is even, and $c_p = \sqrt{2\pi i}\,(-)^n/2^p$ when $p = 2n+1$ is odd. This coincides with the Chern-Connes character [4, 5], up to a scaling factor accounting for the homotopy equivalence between the X-complex $\widehat{X}(T\mathscr{A},J\mathscr{A})$ and the $(b+B)$-complex $\widehat{\Omega}\mathscr{A}$.

Example 3.15 When $\mathscr{A} = \mathbb{C}$ and $\mathscr{B} = C^\infty(0,1)$, the Bott element (Example 3.5) represented by the odd 1-summable quasihomomorphism $\rho : \mathbb{C} \to \mathscr{E}^s \triangleright \mathscr{I}^s\hat{\otimes}\mathscr{B}$ with $\mathscr{B}$-admissible extension $\mathscr{E} = C^\infty[0,1]$, has a Chern character in

$HC^1(\mathbb{C},\mathscr{B}) \cong HP_1(\mathscr{B})$. The periodic cyclic homology of $\mathscr{B}$ is isomorphic to the de Rham cohomology of the open interval $(0,1)$, hence $HP_0(\mathscr{B}) = 0$ and $HP_1(\mathscr{B}) = \mathbb{C}$. Consequently, the Chern character $\mathrm{ch}^1(\rho)$ may be represented by a smooth one-form over $[0,1]$ vanishing at the endpoints. It involves a real-valued function $\xi \in \mathscr{E}$, with $\xi(0) = 0$ and $\xi(1) = \pi/2$, used in the construction of the homomorphism $\rho : \mathbb{C} \to \mathscr{E}^s_+$. One explicitly finds

$$\mathrm{ch}(\rho) = \sqrt{2\pi i}\,\mathbf{d}(\sin^2 \xi) \ ,$$

so that its integral over the interval $[0,1]$ is normalized to $\sqrt{2\pi i}$, and of course does no depend on the chosen function ξ. This is due to the fact that quasihomomorphisms associated to different choices of ξ are homotopic.

4 Topological K-theory

We review here the topological K-theory of Fréchet m-algebras following Phillips [27], and construct various Chern character maps with value in cyclic homology. Topological K-theory for Fréchet m-algebras is defined in analogy with Banach algebras and fulfills the same properties of homotopy invariance, Bott periodicity and excision [27]. For our purposes, only homotopy invariance and Bott periodicity are needed. A basic example of Fréchet m-algebra is provided by the algebra $\mathscr{K}$ of "smooth compact operators". $\mathscr{K}$ is the space of infinite matrices $(A_{ij})_{i,j\in\mathbb{N}}$ with entries in $\mathbb{C}$ and rapid decay, endowed with the family of submultiplicative norms

$$\|A\|_n = \sup_{(i,j)\in\mathbb{N}^2} (1+i+j)^n A_{ij} < \infty \qquad \forall n \in \mathbb{N} \ .$$

The multiplication of matrices makes $\mathscr{K}$ a Fréchet m-algebra. Moreover $\mathscr{K}$ is nuclear as a locally convex vector space [14]. If $\mathscr{A}$ is any Fréchet m-algebra, the completed tensor product $\mathscr{K}\hat{\otimes}\mathscr{A}$ is the *smooth stabilization* of $\mathscr{A}$. Other important examples are the algebras $C^\infty[0,1]$, resp. $C^\infty(0,1)$, of smooth $\mathbb{C}$-valued functions over the interval, with all derivatives of order ≥ 1, resp. ≥ 0, vanishing at the endpoints. As already mentioned in section 3, these are again nuclear Fréchet m-algebras and the completed tensor products $\mathscr{A}[0,1] = \mathscr{A}\hat{\otimes}C^\infty[0,1]$ and $\mathscr{A}(0,1) = \mathscr{A}\hat{\otimes}C^\infty(0,1)$ are isomorphic to the algebras of smooth $\mathscr{A}$-valued functions over the interval, with the appropriate vanishing boundary conditions. In particular $S\mathscr{A} := \mathscr{A}(0,1)$ is the *smooth suspension* of $\mathscr{A}$. We say that two idempotents e_0, e_1 of an algebra $\mathscr{A}$ are smoothly homotopic if there exists an id empotent $e \in \mathscr{A}[0,1]$ whose evaluation at the endpoints gives e_0 and e_1. Similarly for invertible elements.
The definition of topological K-theory involves idempotents and invertibles of the unitalized algebra $(\mathscr{K}\hat{\otimes}\mathscr{A})^+$. Choosing an isomorphism $M_2(\mathscr{K}) \cong \mathscr{K}$ makes $(\mathscr{K}\hat{\otimes}\mathscr{A})^+$ a semigroup for the direct sum $a \oplus b = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$. We denote by p_0 the idempotent $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ of the matrix algebra $M_2(\mathscr{K}\hat{\otimes}\mathscr{A})^+$.

Definition 4.1 (Phillips [27]) *Let $\mathscr{A}$ be a Fréchet m-algebra. The topological K-theory of $\mathscr{A}$ in degree zero and one is defined by*

$$\begin{aligned} K_0^{\mathrm{top}}(\mathscr{A}) &= \{\textit{set of smooth homotopy classes of idempotents } e \in M_2(\mathscr{K}\hat{\otimes}\mathscr{A})^+ \\ &\qquad \textit{such that } e - p_0 \in M_2(\mathscr{K}\hat{\otimes}\mathscr{A})\ \} \\ K_1^{\mathrm{top}}(\mathscr{A}) &= \{\textit{set of smooth homotopy classes of invertibles } g \in (\mathscr{K}\hat{\otimes}\mathscr{A})^+ \\ &\qquad \textit{such that } g - 1 \in \mathscr{K}\hat{\otimes}\mathscr{A}\ \} \end{aligned}$$

$K_0^{\mathrm{top}}(\mathscr{A})$ and $K_1^{\mathrm{top}}(\mathscr{A})$ are semigroups for the direct sum of idempotents and invertibles; in the case of idempotents, the direct sum $e \oplus e' \in M_4(\mathscr{K}\hat{\otimes}\mathscr{A})^+$ has to be conjugated by the invertible matrix

$$c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in M_4(\mathscr{K}\hat{\otimes}\mathscr{A})^+ \ , \quad c^{-1} = c \ , \tag{52}$$

in order to preserve the condition $c(e \oplus e')c - \tilde{p}_0 \in M_4(\mathscr{K}\hat{\otimes}\mathscr{A})$, with $\tilde{p}_0$ the diagonal matrix $\mathrm{diag}(1,1,0,0)$. The proof that $K_0^{\mathrm{top}}(\mathscr{A})$ and $K_1^{\mathrm{top}}(\mathscr{A})$ are actually abelian groups will be recalled in Lemma 5.2. The unit of $K_0^{\mathrm{top}}(\mathscr{A})$ is the class of the idempotent $p_0 \in M_2(\mathscr{K}\hat{\otimes}\mathscr{A})^+$, whereas the unit of $K_1^{\mathrm{top}}(\mathscr{A})$ is represented by $1 \in (\mathscr{K}\hat{\otimes}\mathscr{A})^+$.
The fundamental property of topological K-theory is Bott periodicity [27]. Let $S\mathscr{A} = \mathscr{A}(0,1)$ be the smooth suspension of $\mathscr{A}$. Define two additive maps

$$\alpha : K_1^{\mathrm{top}}(\mathscr{A}) \to K_0^{\mathrm{top}}(S\mathscr{A}) \ , \qquad \beta : K_0^{\mathrm{top}}(\mathscr{A}) \to K_1^{\mathrm{top}}(S\mathscr{A}) \tag{53}$$

as follows. First choose a real-valued function $\xi \in C^\infty[0,1]$ such that $\xi(0) = 0$ and $\xi(1) = \pi/2$ (we recall that all the derivatives of ξ vanish at the endpoints). Let $g \in (\mathscr{K}\hat{\otimes}\mathscr{A})^+$ represent an element of $K_1^{\mathrm{top}}(\mathscr{A})$. Then the idempotent

$$\alpha(g) = G^{-1} p_0 G \ , \qquad \alpha(g) - p_0 \in M_2(\mathscr{K}\hat{\otimes}S\mathscr{A})$$

defines an element of $K_0^{\mathrm{top}}(S\mathscr{A})$, where $G : [0,1] \to M_2(\mathscr{K}\hat{\otimes}\mathscr{A})^+$ is the matrix function over $[0,1]$

$$G = R^{-1} \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} R \quad \text{with} \quad R = \begin{pmatrix} \cos\xi & \sin\xi \\ -\sin\xi & \cos\xi \end{pmatrix} \ .$$

Now $z = \exp(4i\xi)$ is a complex-valued invertible function over $[0,1]$ with winding number 1. The functions $z - 1$ and $z^{-1} - 1$ lie in $C^\infty(0,1)$. Then for any idempotent $e \in M_2(\mathscr{K}\hat{\otimes}\mathscr{A})^+$ representing a class in $K_0^{\mathrm{top}}(\mathscr{A})$, we define the invertible element

$$\beta(e) = (1 + (z-1)e)(1 + (z-1)p_0)^{-1} \ .$$

One has $(1 + (z-1)p_0)^{-1} = (1 + (z^{-1} - 1)p_0)$, and the idempotent relations $e^2 = e$, $p_0^2 = p_0$ imply $\beta(e) = 1 + (z-1)e(e - p_0) + (z^{-1} - 1)(p_0 - e)p_0$, which shows that $\beta(e) - 1$ is an element of the algebra $M_2(\mathscr{K}\hat{\otimes}S\mathscr{A}) \cong \mathscr{K}\hat{\otimes}S\mathscr{A}$, hence $\beta(e)$ defines a class in $K_1^{\mathrm{top}}(S\mathscr{A})$.

Proposition 4.2 (Bott periodicity [27]) *The two maps defined above* $\alpha : K_1^{\mathrm{top}}(\mathscr{A}) \to K_0^{\mathrm{top}}(S\mathscr{A})$ *and* $\beta : K_0^{\mathrm{top}}(\mathscr{A}) \to K_1^{\mathrm{top}}(S\mathscr{A})$ *are isomophisms of abelian groups.* ∎

Hence Bott periodicity implies $K_i^{\mathrm{top}}(S^2\mathscr{A}) = K_i^{\mathrm{top}}(\mathscr{A})$ for $i = 0, 1$, so that we may define topological K-theory groups in any degree $n \in \mathbb{Z}$:

$$K_n^{\mathrm{top}}(\mathscr{A}) = \begin{cases} K_0^{\mathrm{top}}(\mathscr{A}) & n \text{ even} \\ K_1^{\mathrm{top}}(\mathscr{A}) & n \text{ odd} . \end{cases} \tag{54}$$

Following Cuntz and Quillen [12], we construct Chern characters with values in periodic cyclic homology $K_n^{\mathrm{top}}(\mathscr{A}) \to HP_n(\mathscr{A})$. Recall (section 2) that periodic cyclic homology is computed from any quasi-free extension $0 \to \mathscr{J} \to \mathscr{R} \to \mathscr{A} \to 0$ by the pro-complex

$$\widehat{X}(\mathscr{R}, \mathscr{J}) = X(\widehat{\mathscr{R}}) \ : \ \widehat{\mathscr{R}} \rightleftarrows \Omega^1\widehat{\mathscr{R}}_\natural \ ,$$

where the pro-algebra $\widehat{\mathscr{R}} = \varprojlim_n \mathscr{R}/\mathscr{J}^n$ is the $\mathscr{J}$-adic completion of the quasi-free algebra $\mathscr{R}$. In particular, the universal free extension $0 \to J\mathscr{A} \to T\mathscr{A} \to \mathscr{A} \to 0$ is quasi-free and the universal property of the tensor algebra leads to a classifying homomorphism $T\mathscr{A} \to \mathscr{R}$ compatible with the ideals $J\mathscr{A}$ and $\mathscr{J}$ by means of the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & J\mathscr{A} \arrow[r] \arrow[d] & T\mathscr{A} \arrow[r] \arrow[d] & \mathscr{A} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathscr{J} \arrow[r] & \mathscr{R} \arrow[r] & \mathscr{A} \arrow[r] \arrow[l, dotted, bend right, "\sigma"'] & 0 \end{tikzcd}$$

for any choice of continuous linear section $\sigma : \mathscr{A} \to \mathscr{R}$. The homomorphism $T\mathscr{A} \to \mathscr{R}$ thus extends to a homomorphism of pro-algebras $\widehat{T}\mathscr{A} \to \widehat{\mathscr{R}}$ and the induced morphism of complexes $X(\widehat{T}\mathscr{A}) \to X(\widehat{\mathscr{R}})$ is a homotopy equivalence. The Chern character on topological K-theory requires to lift idempotents and invertible elements from the algebra $\mathscr{K}\hat{\otimes}\mathscr{A}$ to the pro-algebra

$$\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}} = \varprojlim_n \mathscr{K}\hat{\otimes}(\mathscr{R}/\mathscr{J}^n) \ .$$

If $e \in M_2(\mathscr{K}\hat{\otimes}\mathscr{A})^+$ is an idempotent such that $e - p_0 \in M_2(\mathscr{K}\hat{\otimes}\mathscr{A})$, there always exists an idempotent lift $\hat{e} \in M_2(\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}})^+$ with $\hat{e} - p_0 \in M_2(\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}})$, and two such liftings are always conjugate [12]. A concrete way to construct an idempotent lift is to work first with the tensor algebra and then push forward by the homomorphism $\mathscr{K}\hat{\otimes}\widehat{T}\mathscr{A} \to \mathscr{K}\hat{\otimes}\widehat{\mathscr{R}}$. Using the isomorphism of pro-vector spaces $\widehat{T}\mathscr{A} \cong \widehat{\Omega}^+\mathscr{A}$, the following differential form of even degree defines an idempotent [12]

$$\hat{e} = e + \sum_{k\geq 1} \frac{(2k)!}{(k!)^2}(e - \frac{1}{2})(dede)^k \ \in M_2(\mathscr{K}\hat{\otimes}\widehat{T}\mathscr{A})^+ \ , \tag{55}$$

where concatenation products over $M_2(\mathscr{K})$ are taken. We will refer to (55) as the *canonical lift* of e, but it should be stressed that other choices are possible. Denoting also by $\hat{e}$ its image in $M_2(\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}})^+$, the Chern character of e is represented by the cycle of even degree

$$\mathrm{ch}_0(\hat{e}) = \mathrm{Tr}(\hat{e} - p_0) \ \in \widehat{\mathscr{R}}\ , \tag{56}$$

where the partial trace $\mathrm{Tr} : M_2(\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}}) \to \widehat{\mathscr{R}}$ comes from the usual trace of matrices with rapid decay. We will show below that the cyclic homology class of $\mathrm{ch}_0(\hat{e})$ is invariant under smooth homotopies of $\hat{e}$. Moreover, the invariance of the trace under similarity implies that ch_0 is additive. Next, if $g \in (\mathscr{K}\hat{\otimes}\mathscr{A})^+$ is an invertible element such that $g - 1 \in \mathscr{K}\hat{\otimes}\mathscr{A}$, we have again to choose an invertible lift $\hat{g} \in (\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}})^+$ with $\hat{g} - 1 \in \mathscr{K}\hat{\otimes}\widehat{\mathscr{R}}$. It turns out that any lifting of g is invertible, and two such liftings are always homotopic [12]. A concrete way to construct an invertible lift is to use the linear inclusion of zero-forms $\mathscr{K}\hat{\otimes}\mathscr{A} \hookrightarrow \mathscr{K}\hat{\otimes}\widehat{T}\mathscr{A} \cong \mathscr{K}\hat{\otimes}\widehat{\Omega}^+\mathscr{A}$ and consider g as an element $\hat{g} = g \in (\mathscr{K}\hat{\otimes}\widehat{T}\mathscr{A})^+$. A simple computation shows that it is invertible, with inverse

$$\hat{g}^{-1} = \sum_{k\geq 0} g^{-1}(dg\, dg^{-1})^k \ \in (\mathscr{K}\hat{\otimes}\widehat{T}\mathscr{A})^+\ . \tag{57}$$

Here again we shall refer to the above $\hat{g}$ as the *canonical lift* of g, but other choices are possible. Then denoting also by $\hat{g}$ its image in $(\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}})^+$, the Chern character of g is represented by the cycle of odd degree

$$\mathrm{ch}_1(\hat{g}) = \frac{1}{\sqrt{2\pi i}}\,\mathrm{Tr}\natural\hat{g}^{-1}\mathbf{d}\hat{g} \ \in \Omega^1\widehat{\mathscr{R}}_\natural\ , \tag{58}$$

with the trace map $\mathrm{Tr} : \Omega^1(\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}})_\natural \to \Omega^1\widehat{\mathscr{R}}_\natural$. In this case also we will show that the cyclic homology class of $\mathrm{ch}_1(\hat{g})$ is invariant under smooth homotopies of $\hat{g}$. Clearly ch_1 is additive. The factor $1/\sqrt{2\pi i}$ is chosen for consistency with the bivariant Chern character.

Note the following important property of idempotents and invertibles: two idempotents $\hat{e}_0, \hat{e}_1 \in M_2(\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}})^+$ are homotopic if and only if their projections $e_0, e_1 \in M_2(\mathscr{K}\hat{\otimes}\mathscr{A})^+$ are homotopic, and similarly with invertibles [12]. Since the cyclic homology classes of the Chern characters $\mathrm{ch}_0(\hat{e})$ and $\mathrm{ch}_1(\hat{g})$ are homotopy invariant with respect to $\hat{e}$ and $\hat{g}$, one gets well-defined additive maps $\mathrm{ch}_0 : K_0^{\mathrm{top}}(\mathscr{A}) \to HP_0(\mathscr{A})$ and $\mathrm{ch}_1 : K_1^{\mathrm{top}}(\mathscr{A}) \to HP_1(\mathscr{A})$ on the topological K-theory groups. They do not depend on the quasi-free extension $\mathscr{R}$ since we know that the classifying homomorphism $\widehat{T}\mathscr{A} \to \widehat{\mathscr{R}}$ induces a homotopy equivalence of pro-complexes $X(\widehat{T}\mathscr{A}) \xrightarrow{\sim} X(\widehat{\mathscr{R}})$.

To show the homotopy invariance of the Chern characters, we introduce the Cherns-Simons transgressions. Let $\widehat{\mathscr{R}}[0,1]$ be the tensor product $\widehat{\mathscr{R}}\hat{\otimes}C^\infty[0,1]$, and let $\hat{e}$ be any idempotent of $M_2(\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}}[0,1])^+$ with $\hat{e}-p_0 \in M_2(\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}}[0,1])$. Denote by $s : C^\infty[0,1] \to \Omega^1[0,1]$ the de Rham coboundary map with values in ordinary (commutative) one-forms over the interval. We then define the

Chern-Simons form associated to $\hat{e}$ as the chain of odd degree

$$\mathrm{cs}_1(\hat{e}) = \int_0^1 \mathrm{Tr}\natural(-2\hat{e}+1)\, s\hat{e}\,\mathbf{d}\hat{e} \ \in \Omega^1\widehat{\mathscr{R}}_\natural\ , \tag{59}$$

with obvious notations. Now let $\hat{g} \in (\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}}[0,1])^+$ be any invertible element such that $\hat{g}-1 \in \mathscr{K}\hat{\otimes}\widehat{\mathscr{R}}[0,1]$. The Chern-Simons form associated to $\hat{g}$ is the chain of even degree

$$\mathrm{cs}_0(\hat{g}) = \frac{1}{\sqrt{2\pi i}} \int_0^1 \mathrm{Tr}(\hat{g}^{-1}s\hat{g}) \ \in \widehat{\mathscr{R}}\ . \tag{60}$$

Lemma 4.3 *Let $\hat{e}$ be an idempotent of the algebra $M_2(\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}}[0,1])^+$ with $\hat{e}-p_0 \in M_2(\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}}[0,1])$. Denote by $\hat{e}_0$ and $\hat{e}_1$ the idempotents of $M_2(\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}})^+$ obtained by evaluation at 0 and 1. Then one has*

$$\overline{b}\mathrm{cs}_1(\hat{e}) = \mathrm{ch}_0(\hat{e}_1) - \mathrm{ch}_0(\hat{e}_0) \ \in \widehat{\mathscr{R}}\ . \tag{61}$$

Let $\hat{g} \in (\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}}[0,1])^+$ be an invertible element such that $\hat{g}-1 \in \mathscr{K}\hat{\otimes}\widehat{\mathscr{R}}[0,1]$. Denote by $\hat{g}_0$ and $\hat{g}_1$ the invertibles of $(\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}})^+$ obtained by evaluation at 0 and 1. Then one has

$$\natural\mathbf{d}\mathrm{cs}_0(\hat{g}) = \mathrm{ch}_1(\hat{g}_1) - \mathrm{ch}_1(\hat{g}_0) \ \in \Omega^1\widehat{\mathscr{R}}_\natural\ . \tag{62}$$

Proof: First notice that the current $\int_0^1$ is odd, so that

$$\overline{b}\mathrm{cs}_1(\hat{e}) = -\int_0^1 \overline{b}\mathrm{Tr}\natural(-2\hat{e}+1)\, s\hat{e}\,\mathbf{d}\hat{e}\ ,$$

and taking into account the fact that $s\hat{e}$ is also odd, one has

$$\overline{b}\mathrm{Tr}\natural(-2\hat{e}+1)\, s\hat{e}\,\mathbf{d}\hat{e} = -\mathrm{Tr}[(-2\hat{e}+1)\, s\hat{e}, \hat{e}] = \mathrm{Tr}((2\hat{e}-1)s\hat{e}\,\hat{e} - \hat{e}s\hat{e})\ ,$$

where we use the idempotent property $\hat{e}^2=\hat{e}$ for the last equality. Since s is a derivation, one has $s\hat{e} = s(\hat{e}^2) = s\hat{e}\,\hat{e} + \hat{e}s\hat{e}$ and $\hat{e}s\hat{e}\,\hat{e} = 0$, whence

$$\overline{b}\mathrm{cs}_1(e) = \int_0^1 \mathrm{Tr}(s\hat{e}\,\hat{e} + \hat{e}s\hat{e}) = \int_0^1 s\mathrm{Tr}\hat{e} = \mathrm{Tr}(\hat{e}_1 - \hat{e}_0) = \mathrm{ch}_0(\hat{e}_1) - \mathrm{ch}_0(\hat{e}_0)\ ,$$

because $\hat{e}_0$ and $\hat{e}_1$ are the evaluations of $\hat{e}$ respectively at 0 and 1. Let us proceed now with invertibles:

$$\natural\mathbf{d}\mathrm{cs}_0(\hat{g}) = \frac{-1}{\sqrt{2\pi i}} \int_0^1 \mathrm{Tr}\natural\mathbf{d}(\hat{g}^{-1}s\hat{g})\ ,$$

and because $\mathbf{d}$ is an odd derivation anticommuting with s, one has

$$\mathrm{Tr}\natural\mathbf{d}(\hat{g}^{-1}s\hat{g}) = \mathrm{Tr}\natural(\mathbf{d}\hat{g}^{-1}s\hat{g} - \hat{g}^{-1}s\mathbf{d}\hat{g})\ .$$

Then, $\mathbf{d}1 = 0 = s1$ implies $\mathbf{d}\hat{g}^{-1} = -\hat{g}^{-1}\mathbf{d}\hat{g}\hat{g}^{-1}$ and $s\hat{g}^{-1} = -\hat{g}^{-1}s\hat{g}\hat{g}^{-1}$. But $\mathrm{Tr}\natural$ is a supertrace, hence

$$\mathrm{Tr}\natural(-\hat{g}^{-1}\mathbf{d}\hat{g}\hat{g}^{-1}s\hat{g} - \hat{g}^{-1}s\mathbf{d}\hat{g}) = \mathrm{Tr}\natural(\hat{g}^{-1}s\hat{g}\hat{g}^{-1}\mathbf{d}\hat{g} - \hat{g}^{-1}s\mathbf{d}\hat{g}) = -s\mathrm{Tr}\natural(\hat{g}^{-1}\mathbf{d}\hat{g}) \ .$$

By integration over the interval $[0,1]$, one gets

$$\natural\mathbf{d}\mathrm{cs}_0(\hat{g}) = \frac{1}{\sqrt{2\pi i}}\mathrm{Tr}\natural(\hat{g}_1^{-1}\mathbf{d}\hat{g}_1 - \hat{g}_0^{-1}\mathbf{d}\hat{g}_0) = \mathrm{ch}_1(\hat{g}_1) - \mathrm{ch}_1(\hat{g}_0)$$

as wanted. ∎

Hence the cyclic homology classes of the Chern characters are homotopy invariant as claimed. There is another consequence of the above lemma. Observe that the suspensions $S\mathscr{A} = \mathscr{A}(0,1)$ and $S\widehat{\mathscr{R}} = \widehat{\mathscr{R}}(0,1)$ are subalgebras of $\mathscr{A}[0,1]$ and $\widehat{\mathscr{R}}[0,1]$. If $e \in M_2(\mathscr{K}\hat{\otimes}S\mathscr{A})^+$ is an idempotent representing a class in $K_0^{\mathrm{top}}(S\mathscr{A})$, and $g \in (\mathscr{K}\hat{\otimes}S\mathscr{A})^+$ an invertible representing a class in $K_1^{\mathrm{top}}(S\mathscr{A})$, we choose some lifts $\hat{e} \in M_2(\mathscr{K}\hat{\otimes}S\widehat{\mathscr{R}})^+$ and $\hat{g} \in (\mathscr{K}\hat{\otimes}S\widehat{\mathscr{R}})^+$. Then $\mathrm{cs}_1(\hat{e})$ and $\mathrm{cs}_0(\hat{g})$ are closed and define homology classes in $HP_1(\mathscr{A})$ and $HP_0(\mathscr{A})$ respectively. The following lemma shows the compatibility with Bott periodicity:

Lemma 4.4 *The Chern-Simons forms define additive maps* $\mathrm{cs}_1 : K_0^{\mathrm{top}}(S\mathscr{A}) \to HP_1(\mathscr{A})$ *and* $\mathrm{cs}_0 : K_1^{\mathrm{top}}(S\mathscr{A}) \to HP_0(\mathscr{A})$. *Moreover they are compatible with the Bott isomorphisms* $\alpha : K_1^{\mathrm{top}}(\mathscr{A}) \to K_0^{\mathrm{top}}(S\mathscr{A})$ *and* $\beta : K_0^{\mathrm{top}}(\mathscr{A}) \to K_1^{\mathrm{top}}(S\mathscr{A})$ *and Chern characters, up to multiplication by a factor* $\sqrt{2\pi i}$*:*

$$\begin{aligned} \mathrm{cs}_1 \circ \alpha \equiv \sqrt{2\pi i}\,\mathrm{ch}_1 \quad &: \quad K_1^{\mathrm{top}}(\mathscr{A}) \to HP_1(\mathscr{A}) \ , \\ \mathrm{cs}_0 \circ \beta \equiv \sqrt{2\pi i}\,\mathrm{ch}_0 \quad &: \quad K_0^{\mathrm{top}}(\mathscr{A}) \to HP_0(\mathscr{A}) \ . \end{aligned}$$

Proof: Let $\hat{e}$ be any idempotent of the pro-algebra $M_2(\mathscr{K}\hat{\otimes}S\widehat{\mathscr{R}})^+$. We have to prove the homotopy invariance of the cyclic homology class determined by the cycle

$$\mathrm{cs}_1(\hat{e}) = \int_0^1 \mathrm{Tr}\natural(-2\hat{e}+1)\, s\hat{e}\,\mathbf{d}\hat{e} \ .$$

To this end, consider a smooth family of idempotents $\hat{e}_t \in M_2(\mathscr{K}\hat{\otimes}S\widehat{\mathscr{R}})^+$ parametrized by $t \in \mathbb{R}$, such that $\hat{e}_t - p_0 \in M_2(\mathscr{K}\hat{\otimes}S\widehat{\mathscr{R}}), \forall t$. Denote by $\dot{\hat{e}}$ the derivative $\partial\hat{e}/\partial t$. The idempotent property of the family $\hat{e}$ implies the following identity:

$$\frac{\partial}{\partial t}\mathrm{Tr}\natural(-2\hat{e}+1)s\hat{e}\mathbf{d}\hat{e} = -\natural\mathbf{d}\mathrm{Tr}(\hat{e}(\dot{\hat{e}}s\hat{e} - s\hat{e}\dot{\hat{e}})) - s\mathrm{Tr}\natural\hat{e}(\dot{\hat{e}}\mathbf{d}\hat{e} - \mathbf{d}\hat{e}\dot{\hat{e}}) \ .$$

Since for any fixed t, the idempotent $\hat{e}_t$ equals p_0 at the boundaries of the suspended algebra $M_2(\mathscr{K}\hat{\otimes}S\widehat{\mathscr{R}})^+$, one gets $\int_0^1 s\mathrm{Tr}\natural\hat{e}(\dot{\hat{e}}\mathbf{d}\hat{e} - \mathbf{d}\hat{e}\dot{\hat{e}}) = 0$ and

$$\frac{\partial}{\partial t}\int_0^1 \mathrm{Tr}\natural(-2\hat{e}+1)\, s\hat{e}\,\mathbf{d}\hat{e} = \natural\mathbf{d}\int_0^1 \mathrm{Tr}(\hat{e}(\dot{\hat{e}}s\hat{e} - s\hat{e}\dot{\hat{e}})) \ .$$

This implies that the cyclic homology class of $\mathrm{cs}_1(\hat{e})$ is a homotopy invariant of $\hat{e}$. Hence if $\hat{e}$ lifts an idempotent $e \in M_2(\mathscr{K}\hat{\otimes}S\mathscr{A})^+$, the cyclic homology class of $\mathrm{cs}_1(\hat{e})$ only depends on the homotopy class of e, and the map $\mathrm{cs}_1 : K_0^{\mathrm{top}}(S\mathscr{A}) \to HP_1(\mathscr{A})$ is well-defined. We now have to show the compatibility with Bott periodicity. Thus let $g \in (\mathscr{K}\hat{\otimes}\mathscr{A})^+$ be an invertible such that $g-1 \in \mathscr{K}\hat{\otimes}\mathscr{A}$, and let $\alpha(g)$ be the idempotent $G^{-1}p_0G \in M_2(\mathscr{K}\hat{\otimes}S\mathscr{A})^+$ constructed by means of a rotation matrix

$$R = \begin{pmatrix} \cos\xi & \sin\xi \\ -\sin\xi & \cos\xi \end{pmatrix} , \qquad G = R^{-1}\begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} R ,$$

where $\xi \in C^\infty[0,1]$ is a real function with $\xi(0)=0$ and $\xi(1)=\pi/2$. Then, it is clear that the idempotent $\hat{e} = \widehat{G}^{-1}p_0\widehat{G} \in M_2(\mathscr{K}\hat{\otimes}S\widehat{\mathscr{R}})^+$ is a lifting of $\alpha(g)$, where the matrix $\widehat{G} = R^{-1}\left(\begin{smallmatrix} 1 & 0 \\ 0 & \hat{g} \end{smallmatrix}\right)R$ is built from any lifting $\hat{g} \in (\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}})^+$ of g. Hence, the cyclic homology class of $\mathrm{cs}_1(\widehat{\alpha(g)})$ is represented by $\mathrm{cs}_1(\hat{e})$. A direct computation shows the equality

$$\mathrm{Tr}\natural(-2\hat{e}+1)\, s\hat{e}\,\mathbf{d}\hat{e} = s(\cos\xi)\mathrm{Tr}\natural(-\hat{g}^{-1}\mathbf{d}\hat{g} + \frac{1}{2}\mathbf{d}\hat{g} - \frac{1}{2}\mathbf{d}\hat{g}^{-1}) ,$$

so that after integration over $[0,1]$ one gets, modulo boundaries $\natural\mathbf{d}(\cdot)$

$$\mathrm{cs}_1(\widehat{\alpha(g)}) \equiv \mathrm{Tr}\natural(\hat{g}^{-1}\mathbf{d}\hat{g}) \bmod \natural\mathbf{d} \equiv \sqrt{2\pi i}\,\mathrm{ch}_1(\hat{g}) \bmod \natural\mathbf{d} .$$

Next, we turn to the map cs_0. If $\hat{g} \in (\mathscr{K}\hat{\otimes}S\widehat{\mathscr{R}})^+$ is any invertible, one has

$$\mathrm{cs}_0(\hat{g}) = \frac{1}{\sqrt{2\pi i}}\int_0^1 \mathrm{Tr}(\hat{g}^{-1}s\hat{g}) .$$

We have to show that the cyclic homology class of $\mathrm{cs}_0(\hat{g})$ is a homotopy invariant of $\hat{g}$. To this end, consider a smooth one-parameter family of invertibles $\hat{g}_t \in (\mathscr{K}\hat{\otimes}S\widehat{\mathscr{R}})^+$. One has, with $\dot{\hat{g}} = \partial\hat{g}/\partial t$,

$$\frac{\partial}{\partial t}(\hat{g}^{-1}s\hat{g}) = -\hat{g}^{-1}\dot{\hat{g}}\hat{g}^{-1}s\hat{g} + \hat{g}^{-1}s\dot{\hat{g}} = [\hat{g}^{-1}s\hat{g}, \hat{g}^{-1}\dot{\hat{g}}] + s(\hat{g}^{-1}\dot{\hat{g}}) .$$

Since $\mathrm{Tr}[\hat{g}^{-1}s\hat{g}, \hat{g}^{-1}\dot{\hat{g}}] = -\overline{b}\mathrm{Tr}\natural\hat{g}^{-1}s\hat{g}\mathbf{d}(\hat{g}^{-1}\dot{\hat{g}})$, we get

$$\frac{\partial}{\partial t}\int_0^1 \mathrm{Tr}(\hat{g}^{-1}s\hat{g}) = \overline{b}\int_0^1 \mathrm{Tr}\natural\hat{g}^{-1}s\hat{g}\mathbf{d}(\hat{g}^{-1}\dot{\hat{g}}) .$$

Hence the cyclic homology class of $\mathrm{cs}_0(\hat{g})$ is homotopy invariant. In particular if $\hat{g}$ lifts an invertible $g \in (\mathscr{K}\hat{\otimes}S\mathscr{A})^+$, the cyclic homology class of $\mathrm{cs}_0(\hat{g})$ is a homotopy invariant of g and the map $\mathrm{cs}_0 : K_1^{\mathrm{top}}(S\mathscr{A}) \to HP_0(\mathscr{A})$ is well-defined. Now let $e \in M_2(\mathscr{K}\hat{\otimes}\mathscr{A})^+$ be an idempotent, with $e - p_0 \in M_2(\mathscr{K}\hat{\otimes}\mathscr{A})$. Its image under the Bott map β is the invertible element $\beta(e) \in (\mathscr{K}\hat{\otimes}S\mathscr{A})^+$ given by

$$\beta(e) = (1+(z-1)e)(1+(z-1)p_0)^{-1} ,$$

where $z = \exp(4i\xi)$. If $\hat e \in M_2(\mathscr{K}\hat\otimes\widehat{\mathscr{R}})^+$ is any idempotent lift of e, it is clear that the invertible

$$\hat g = (1+(z-1)\hat e)(1+(z-1)p_0)^{-1} \ \in (\mathscr{K}\hat\otimes S\widehat{\mathscr{R}})^+$$

is a lifting of $\beta(e)$. Hence the cyclic homology class of $\mathrm{cs}_0(\widehat{\beta(e)})$ is represented by $\mathrm{cs}_0(\hat g)$. Let us compute explicitly $\mathrm{Tr}(\hat g^{-1}s\hat g)$:

$$\begin{aligned}\mathrm{Tr}\big((1+(z-1)p_0)&(1+(z-1)\hat e)^{-1}s\big((1+(z-1)\hat e)(1+(z-1)p_0)^{-1}\big)\big)\\ &= \mathrm{Tr}\big((1+(z^{-1}-1)\hat e)sz\,\hat e-(1+(z^{-1}-1)p_0)sz\,p_0\big)\\ &= \mathrm{Tr}(\hat e-p_0)\,z^{-1}sz\ .\end{aligned}$$

Since the integration of $z^{-1}sz$ over the interval $[0,1]$ yields a factor $2\pi i$, one is left with equivalences modulo boundaries $\overline{b}(\cdot)$

$$\mathrm{cs}_0(\widehat{\beta(e)}) \equiv \sqrt{2\pi i}\,\mathrm{Tr}(\hat e-p_0) \bmod \overline{b} \equiv \sqrt{2\pi i}\,\mathrm{ch}_0(\hat e) \bmod \overline{b}$$

as wanted. ∎

Since our main motivation is index theory we will have to consider the stabilization of $\mathscr{A}$ by a p-summable Fréchet m-algebra $\mathscr{I}$, that is, $\mathscr{I}$ is provided with a continuous trace $\mathrm{Tr}: \mathscr{I}^p \to \mathbb{C}$ as in section 3. Hence it will be convenient to define a Chern character $K_n^{\mathrm{top}}(\mathscr{I}\hat\otimes\mathscr{A}) \to HP_n(\mathscr{A})$. The difficulty of course is that the trace is not defined on the algebra $\mathscr{K}\hat\otimes\mathscr{I}$ but only on its p-th power. To cope with this problem, we shall construct higher analogues of the Chern characters and Chern-Simons forms associated to idempotents and invertibles. Consider the following p-summable quasihomomorphism of even degree, from the algebra $\mathscr{I}\mathscr{A} := \mathscr{I}\hat\otimes\mathscr{A}$ to $\mathscr{A}$:

$$\rho : \mathscr{I}\mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s\mathscr{A}\ , \qquad \mathscr{E} = \mathscr{I}^+\hat\otimes\mathscr{A}\ , \tag{63}$$

where $\mathscr{I}^+$ is the unitalization of $\mathscr{I}$. Because ρ is of even degree, it is entirely specified by a pair of homomorphisms $(\rho_+,\rho_-) : \mathscr{I}\mathscr{A} \rightrightarrows \mathscr{E}$ which agree modulo the ideal $\mathscr{I}\mathscr{A} \subset \mathscr{E}$. Equivalently if we represent $\mathscr{E}^s$ in the $\mathbb{Z}_2$-graded matrix algebra $M_2(\mathscr{E})$ we can write $\rho = \left(\begin{smallmatrix}\rho_+ & 0\\ 0 & \rho_-\end{smallmatrix}\right)$. By definition we set

$$\rho_+ = \mathrm{Id} : \mathscr{I}\mathscr{A} \to \mathscr{I}\mathscr{A} \subset \mathscr{E}\ , \qquad \rho_- = 0\ .$$

Let $0 \to \mathscr{J} \to \mathscr{R} \to \mathscr{A} \to 0$ be any quasi-free extension of $\mathscr{A}$, with continuous linear splitting $\sigma : \mathscr{A} \to \mathscr{R}$. Then choosing $\mathscr{M} = \mathscr{I}^+\hat\otimes\mathscr{R}$ and $\mathscr{N} = \mathscr{I}^+\hat\otimes\mathscr{J}$, one gets a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{N} \arrow[r] & \mathscr{M} \arrow[r] & \mathscr{E} \arrow[r] & 0 \\ 0 \arrow[r] & \mathscr{I}\mathscr{J} \arrow[r] \arrow[u] & \mathscr{I}\mathscr{R} \arrow[r] \arrow[u] & \mathscr{I}\mathscr{A} \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

Since $(\mathscr{I}^+)^n = \mathscr{I}^+$ for any integer n, one sees that $\mathscr{N}^n = \mathscr{I}^+ \hat{\otimes} \mathscr{I}^n$ is a direct summand in $\mathscr{M}$. Moreover, there is an obvious chain map $\mathrm{Tr} : F^{2n+1}_{\mathscr{I}\mathscr{R}} X(\mathscr{M}) \to X(\mathscr{R})$ for any $n \geq p-1$, obtained by taking the trace $\mathscr{I}^{n+1} \to \mathbb{C}$. It follows that the algebra $\mathscr{E} \triangleright \mathscr{I}\mathscr{A}$ is $\mathscr{R}$-admissible (Definition 3.2), hence in all degrees $2n+1 \geq p$ the bivariant Chern characters $\mathrm{ch}^{2n}(\rho) \in HC^{2n}(\mathscr{I}\mathscr{A}, \mathscr{A})$ are defined and related by the S-operation in bivariant cyclic cohomology $\mathrm{ch}^{2n+2}(\rho) \equiv S\mathrm{ch}^{2n}(\rho)$. We recall briefly the construction of $\mathrm{ch}^{2n}(\rho)$. By the universal properties of the tensor algebra $T(\mathscr{I}\mathscr{A})$, the homomorphism $\rho : \mathscr{I}\mathscr{A} \to \left(\begin{smallmatrix} \mathscr{I}\mathscr{A} & 0 \\ 0 & 0 \end{smallmatrix}\right) \subset \mathscr{E}^s_+$ lifts to a classifying homomorphism ρ_* through the commutative diagram (31)

$$\begin{tikzcd}
0 \arrow[r] & J(\mathscr{I}\mathscr{A}) \arrow[r] \arrow[d, "\rho_*"'] & T(\mathscr{I}\mathscr{A}) \arrow[r] \arrow[d, "\rho_*"'] & \mathscr{I}\mathscr{A} \arrow[r] \arrow[d, "\rho"] & 0 \\
0 \arrow[r] & \mathscr{N}^s_+ \arrow[r] & \mathscr{M}^s_+ \arrow[r, shift left] & \mathscr{E}^s_+ \arrow[r] \arrow[l, dotted, shift left, "\mathrm{Id}\otimes\sigma"'] & 0
\end{tikzcd}$$

induced by the linear splitting. It extends to a homomorphism of pro-algebras $\rho_* : \widehat{T}(\mathscr{I}\mathscr{A}) \to \left(\begin{smallmatrix} \mathscr{I}\widehat{\mathscr{R}} & 0 \\ 0 & 0 \end{smallmatrix}\right) \subset \widehat{\mathscr{M}^s_+}$. The bivariant Chern character $\mathrm{ch}^{2n}(\rho)$ is the composite of the Goodwillie equivalence $\gamma : X(\widehat{T}(\mathscr{I}\mathscr{A})) \to \widehat{\Omega}\widehat{T}(\mathscr{I}\mathscr{A})$ with the chain maps $\rho_* : \widehat{\Omega}\widehat{T}(\mathscr{I}\mathscr{A}) \to \widehat{\Omega}\widehat{\mathscr{M}^s_+}$ and $\widehat{\chi}^{2n} : \widehat{\Omega}\widehat{\mathscr{M}^s_+} \to X(\widehat{\mathscr{R}})$. The two non-zero components of $\widehat{\chi}^{2n}$ are given by Eqs. (34) and defined on $2n$ and $(2n+1)$-forms respectively:

$$\widehat{\chi}^{2n}_0 : \Omega^{2n}\widehat{\mathscr{M}^s_+} \to \widehat{\mathscr{R}}\ , \qquad \widehat{\chi}^{2n}_1 : \Omega^{2n+1}\widehat{\mathscr{M}^s_+} \to \Omega^1\widehat{\mathscr{R}}_\natural\ .$$

The bivariant Chern character is designed to improve the summability degree and can be used to define the higher Chern characters of idempotents and invertibles via the composition

$$\mathrm{ch}^{2n}_i : K^{\mathrm{top}}_i(\mathscr{I}\hat{\otimes}\mathscr{A}) \to HP_i(\mathscr{I}\hat{\otimes}\mathscr{A}) \xrightarrow{\mathrm{ch}^{2n}_i(\rho)} HP_i(\mathscr{A})\ , \qquad i = 0, 1\ . \tag{64}$$

We shall now establish very explicit formulas for these higher characters. Let $\hat{e}$ be an idempotent of the algebra $M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$, such that $\hat{e} - p_0 \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})$. It is well-known (see for example [5]) that the differential forms

$$\begin{aligned}
\mathrm{ch}_{2n}(\hat{e}) &= (-)^n \frac{(2n)!}{n!} \mathrm{Tr}\big((\hat{e} - \tfrac{1}{2})(\mathbf{d}\hat{e}\mathbf{d}\hat{e})^n\big) \ \in \Omega^{2n}(\mathscr{I}\widehat{\mathscr{R}}) \quad \text{for } n \geq 1 \\
\mathrm{ch}_0(\hat{e}) &= \mathrm{Tr}(\hat{e} - p_0) \ \in \Omega^0(\mathscr{I}\widehat{\mathscr{R}})
\end{aligned} \tag{65}$$

are the components of a $(b+B)$-cycle of even degree over $\mathscr{I}\widehat{\mathscr{R}}$, i.e. fulfill the relations $B\mathrm{ch}_{2n}(\hat{e}) + b\mathrm{ch}_{2n+2}(\hat{e}) = 0$ for any n. Here Tr is the trace over $\mathscr{K}$. In the odd case, any invertible element $\hat{g} \in (\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$ such that $\hat{g} - 1 \in \mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}$ gives rise to a $(b+B)$-cycle of odd degree with components

$$\mathrm{ch}_{2n+1}(\hat{g}) = \frac{(-)^n}{\sqrt{2\pi i}} n!\, \mathrm{Tr}\big(\hat{g}^{-1}\mathbf{d}\hat{g}(\mathbf{d}\hat{g}^{-1}\mathbf{d}\hat{g})^n\big) \ \in \Omega^{2n+1}(\mathscr{I}\widehat{\mathscr{R}})\ . \tag{66}$$

The homology classes of these cycles are of course homotopy invariant. If $\hat e \in M_2(\mathscr{K}\hat\otimes\mathscr{I}\widehat{\mathscr{R}}[0,1])^+$ is a smooth path of idempotents, we define the components of the associated Chern-Simons form as

$$\mathrm{cs}_{2n+1}(\hat e) = (-)^n \frac{(2n)!}{n!}\int_0^1 \mathrm{Tr}\big((-2\hat e+1)\sum_{i=0}^{2n}(\mathbf{d}\hat e)^i s\hat e(\mathbf{d}\hat e)^{2n+1-i}\big) \tag{67}$$

in $\Omega^{2n+1}(\mathscr{I}\widehat{\mathscr{R}})$. Similarly if $\hat g \in (\mathscr{K}\hat\otimes\mathscr{I}\widehat{\mathscr{R}}[0,1])^+$ is a smooth path of invertibles, the components of the Chern-Simons form are for $n \geq 1$

$$\mathrm{cs}_{2n}(\hat g) = \frac{(-)^n}{\sqrt{2\pi i}}(n-1)!\int_0^1 \mathrm{Tr}\big(\hat g^{-1}\mathbf{d}\hat g\sum_{i=0}^{n-1}(\mathbf{d}\hat g^{-1}\mathbf{d}\hat g)^i\mathbf{d}\omega(\mathbf{d}\hat g^{-1}\mathbf{d}\hat g)^{n-1-i}\big) \tag{68}$$

in $\Omega^{2n}(\mathscr{I}\widehat{\mathscr{R}})$, where $\omega = \hat g^{-1}s\hat g$, and for $n = 0$ we set as before $\mathrm{cs}_0(\hat g) = \frac{1}{\sqrt{2\pi i}}\int_0^1 \mathrm{Tr}(\omega)$. Simple algebraic manipulations show that the $(b+B)$-boundaries of the Chern-Simons forms yield the difference of evaluations of the Chern characters at the endpoints:

$$\begin{aligned} B\mathrm{cs}_{2n-1}(\hat e) + b\mathrm{cs}_{2n+1}(\hat e) &= \mathrm{ch}_{2n}(\hat e_1) - \mathrm{ch}_{2n}(\hat e_0)\ , \\ B\mathrm{cs}_{2n}(\hat g) + b\mathrm{cs}_{2n+2}(\hat g) &= \mathrm{ch}_{2n+1}(\hat g_1) - \mathrm{ch}_{2n+1}(\hat g_0)\ . \end{aligned} \tag{69}$$

The higher Chern characters (64) and their associated Chern-Simons forms are obtained by evaluation of these $(b+B)$-chains on the inclusion homomorphism $\iota_* : \mathscr{I}\widehat{\mathscr{R}} \hookrightarrow \left(\begin{smallmatrix}\mathscr{I}\widehat{\mathscr{R}} & 0\\ 0 & 0\end{smallmatrix}\right) \subset \widehat{\mathscr{M}}^s_+$ followed by the chain map $\widehat\chi^{2n}$ whenever $2n+1 \geq p$.

Lemma 4.5 *Let $\mathscr{I}$ be p-summable and $2n+1 \geq p$. For any idempotent $\hat e \in M_2(\mathscr{K}\hat\otimes\mathscr{I}\widehat{\mathscr{R}})^+$ such that $\hat e - p_0 \in M_2(\mathscr{K}\hat\otimes\mathscr{I}\widehat{\mathscr{R}})$, and any invertible $\hat g \in (\mathscr{K}\hat\otimes\mathscr{I}\widehat{\mathscr{R}})^+$ such that $\hat g - 1 \in \mathscr{K}\hat\otimes\mathscr{I}\widehat{\mathscr{R}}$, we define the higher Chern characters by the explicit formulas*

$$\begin{aligned} \mathrm{ch}_0^{2n}(\hat e) &= \mathrm{Tr}\,(\hat e - p_0)^{2n+1}\ , \\ \mathrm{ch}_1^{2n}(\hat g) &= \frac{1}{\sqrt{2\pi i}}\frac{(n!)^2}{(2n)!}\mathrm{Tr}\natural\,\hat g^{-1}[(\hat g-1)(\hat g^{-1}-1)]^n\mathbf{d}\hat g\ , \end{aligned} \tag{70}$$

where we take concatenation products over $\mathscr{I}$ and Tr is the trace over the p-th power of $\mathscr{K}\hat\otimes\mathscr{I}$. Then one has $\mathrm{ch}_0^{2n}(\hat e) = \widehat\chi_0^{2n}\iota_*\mathrm{ch}_{2n}(\hat e)$ in $\widehat{\mathscr{R}}$ and $\mathrm{ch}_1^{2n}(\hat g) = \widehat\chi_1^{2n}\iota_*\mathrm{ch}_{2n+1}(\hat g)$ in $\Omega^1\widehat{\mathscr{R}}_\natural$.
Similarly, for any idempotent $\hat e \in M_2(\mathscr{K}\hat\otimes\mathscr{I}\widehat{\mathscr{R}}[0,1])^+$ and any invertible $\hat g \in (\mathscr{K}\hat\otimes\mathscr{I}\widehat{\mathscr{R}}[0,1])^+$, we define the higher Chern-Simons forms by the explicit formulas

$$\mathrm{cs}_1^{2n}(\hat e) = \int_0^1 \mathrm{Tr}\natural(-2\hat e+1)\sum_{i=0}^{n}(\hat e-p_0)^{2i}s\hat e(\hat e-p_0)^{2(n-i)}\mathbf{d}\hat e\ , \tag{71}$$

$$\mathrm{cs}_0^{2n}(\hat g) = \frac{1}{\sqrt{2\pi i}}\frac{(n!)^2}{(2n)!}\int_0^1 \mathrm{Tr}\,\hat g^{-1}[(\hat g-1)(\hat g^{-1}-1)]^n s\hat g$$

Then $\mathrm{cs}_1^{2n}(\hat e)=\widehat\chi_1^{2n}\iota_*\mathrm{cs}_{2n+1}(\hat e)$ *in* $\Omega^1\widehat{\mathscr R}_\natural$ *and* $\mathrm{cs}_0^{2n}(\hat g)\equiv\widehat\chi_0^{2n}\iota_*\mathrm{cs}_{2n}(\hat g)\mod\overline b$ *in* $\widehat{\mathscr R}$. *Moreover the transgression relations hold:*

$$\overline b\mathrm{cs}_1^{2n}(\hat e)=\mathrm{ch}_0^{2n}(\hat e_1)-\mathrm{ch}_0^{2n}(\hat e_0)\ ,\qquad \natural\mathbf{d}\mathrm{cs}_0^{2n}(\hat g)=\mathrm{ch}_1^{2n}(\hat g_1)-\mathrm{ch}_1^{2n}(\hat g_0)\ .$$

Proof: Let us briefly explain the computation of the cycles $\widehat\chi_0^{2n}\iota_*\mathrm{ch}_{2n}(\hat e)$ and $\widehat\chi_1^{2n}\iota_*\mathrm{ch}_{2n+1}(\hat g)$ associated to idempotents $\hat e\in M_2(\mathscr K\hat\otimes\mathscr I\widehat{\mathscr R})^+$ and invertibles $\hat g\in(\mathscr K\hat\otimes\mathscr I\widehat{\mathscr R})^+$. The upper left corner inclusion $\iota_*:\mathscr I\widehat{\mathscr R}\hookrightarrow\widehat{\mathscr M^s_+}$ canonically extends to a unital homomorphism $(\mathscr K\hat\otimes\mathscr I\widehat{\mathscr R})^+\to(\mathscr K\hat\otimes\widehat{\mathscr M^s_+})^+$, and in matrix form we can write

$$\iota_*\hat e=\begin{pmatrix}\hat e & 0\\ 0 & p_0\end{pmatrix}\ ,\qquad \iota_*\hat g=\begin{pmatrix}\hat g & 0\\ 0 & 1\end{pmatrix}\ .$$

Consequently the commutators with the odd multiplier $F=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ read

$$[F,\iota_*\hat e]=\begin{pmatrix}0 & p_0-\hat e\\ \hat e-p_0 & 0\end{pmatrix}\ ,\qquad [F,\iota_*\hat g]=\begin{pmatrix}0 & 1-\hat g\\ \hat g-1 & 0\end{pmatrix}\ .$$

It is therefore straightforward to evaluate the differential forms $\mathrm{ch}_{2n}(\hat e)$ and $\mathrm{ch}_{2n+1}(\hat g)$ on the chain map $\widehat\chi^{2n}$ given by (34). One finds $\widehat\chi_0^{2n}\iota_*\mathrm{ch}_{2n}(\hat e)=\mathrm{ch}_0^{2n}(\hat e)$ and $\widehat\chi_1^{2n}\iota_*\mathrm{ch}_{2n+1}(\hat g)=\mathrm{ch}_1^{2n}(\hat g)$. Similarly with the Chern-Simons forms one finds $\widehat\chi_1^{2n}\iota_*\mathrm{cs}_{2n+1}(\hat e)=\mathrm{cs}_1^{2n}(\hat e)$, whereas by setting $\omega=\hat g^{-1}s\hat g$

$$\begin{aligned}\widehat\chi_0^{2n}\iota_*\mathrm{cs}_{2n}(\hat g)=\frac{1}{\sqrt{2\pi i}}\frac{(n!)^2}{(2n+1)!}\int_0^1\mathrm{Tr}\big(\omega[(\hat g^{-1}-1)(\hat g-1)]^n+\\ (\hat g-1)\omega[(\hat g^{-1}-1)(\hat g-1)]^{n-1}(\hat g^{-1}-1)+\ldots+[(\hat g^{-1}-1)(\hat g-1)]^n\omega\big)\end{aligned}$$

coincides with $\mathrm{cs}_0^{2n}(\hat g)$ only modulo commutators.
Finally, the transgression relations are an immediate consequence of Eqs. (69) and the fact that $\widehat\chi^{2n}$ is a chain map from the $(b+B)$-complex over $\widehat{\mathscr M^s_+}$ to the complex $X(\widehat{\mathscr R})$. ∎

In any degree $2n+1\geq p$ the Chern characters $\mathrm{ch}_0^{2n}:K_0^{\mathrm{top}}(\mathscr I\hat\otimes\mathscr A)\to HP_0(\mathscr A)$ and $\mathrm{ch}_1^{2n}:K_1^{\mathrm{top}}(\mathscr I\hat\otimes\mathscr A)\to HP_1(\mathscr A)$ are thus obtained by first lifting idempotents $e\in M_2(\mathscr K\hat\otimes\mathscr I\mathscr A)^+$ and invertibles $g\in(\mathscr K\hat\otimes\mathscr I\mathscr A)^+$ to some $\hat e\in M_2(\mathscr K\hat\otimes\mathscr I\widehat{\mathscr R})^+$ and $\hat g\in(\mathscr K\hat\otimes\mathscr I\widehat{\mathscr R})^+$, and then taking the cyclic homology classes of $\mathrm{ch}_0^{2n}(\hat e)\in\widehat{\mathscr R}$ and $\mathrm{ch}_1^{2n}(\hat g)\in\Omega^1\widehat{\mathscr R}_\natural$. Although $\hat e$ and $\hat g$ are only defined up to homotopy, the above lemma shows these higher Chern characters are well-defined, and moreover independent of the degree $2n$ because the cocycles $\widehat\chi^{2n}$ are all related by the transgression relations

$$\widehat\chi^{2n}-\widehat\chi^{2n+2}=[\partial,\widehat\eta^{2n+1}]\ \in\mathrm{Hom}(\widehat\Omega\widehat{\mathscr M^s_+},X(\widehat{\mathscr R}))\ .$$

Passing to suspensions, lifting any idempotent $e\in M_2(\mathscr K\hat\otimes\mathscr I S\mathscr A)^+$ or invertible $g\in(\mathscr K\hat\otimes\mathscr I S\mathscr A)^+$ gives rise to an odd cycle $\mathrm{cs}_1^{2n}(\hat e)\in\Omega^1\widehat{\mathscr R}_\natural$ or an even cycle $\mathrm{cs}_0^{2n}(\hat g)\in\widehat{\mathscr R}$. As expected, this is well-defined at the K-theory level and compatible with Bott periodicity:

LEMMA 4.6 *Let $\mathscr{I}$ be p-summable. In any degre $2n+1 \geq p$, the Chern-Simons forms define additive maps $\mathrm{cs}_1^{2n} : K_0^{\mathrm{top}}(\mathscr{I}\hat{\otimes}S\mathscr{A}) \to HP_1(\mathscr{A})$ and cs_0^{2n} : $K_1^{\mathrm{top}}(\mathscr{I}\hat{\otimes}S\mathscr{A}) \to HP_0(\mathscr{A})$, independent of n, and compatible with the Bott isomorphisms:*

$$\begin{array}{rcl} \mathrm{cs}_1^{2n} \circ \alpha \equiv \sqrt{2\pi i}\, \mathrm{ch}_1^{2n} & : & K_1^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \to HP_1(\mathscr{A}) \ , \\ \mathrm{cs}_0^{2n} \circ \beta \equiv \sqrt{2\pi i}\, \mathrm{ch}_0^{2n} & : & K_0^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \to HP_0(\mathscr{A}) \ . \end{array}$$

Proof: Consider an idempotent $\hat{e} \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}S\widehat{\mathscr{R}})^+$. We have to show the homotopy invariance of the cyclic homology class $\mathrm{cs}_1^{2n}(\hat{e})$ with respect to $\hat{e}$. This can be shown by direct computation from Formulas (71). Define the matrix idempotent $\hat{f} = \left(\begin{smallmatrix} \hat{e} & 0 \\ 0 & p_0 \end{smallmatrix}\right)$. Then one has $s\hat{f} = \left(\begin{smallmatrix} s\hat{e} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\mathbf{d}\hat{f} = \left(\begin{smallmatrix} \mathbf{d}\hat{e} & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and $\mathrm{cs}_1^{2n}(\hat{e})$ can be rewritten by means of the operator $F = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and the supertrace τ:

$$\mathrm{cs}_1^{2n}(\hat{e}) = (-)^n \int_0^1 \tau\natural(-2\hat{f}+1) \sum_{i=0}^n [F,\hat{f}]^{2i} s\hat{f} [F,\hat{f}]^{2(n-i)} \mathbf{d}\hat{f} \ .$$

Now suppose that $\hat{e}$ depends smoothly on an additional parameter t. The above integrand may be expressed in terms of the odd differential $\delta = s + \mathbf{d} + dt\frac{\partial}{\partial t} + [F,\ ]$ as

$$\tau\natural(-2\hat{f}+1) \sum_{i=0}^n [F,\hat{f}]^{2i} s\hat{f} [F,\hat{f}]^{2(n-i)} \mathbf{d}\hat{f} = \frac{-1}{n+1}\, \tau\natural\, \hat{f}(\delta\hat{f})^{2n+1}|_{s,\mathbf{d}}$$

where $|_{s,\mathbf{d}}$ means that we select the terms containing only s, $\mathbf{d}$ and not dt. Because $\tau\natural$ is a supertrace, the cocycle property $(s+\mathbf{d}+dt\frac{\partial}{\partial t})\tau\natural\, \hat{f}(\delta\hat{f})^{2n+1} = \tau\natural\, (\delta\hat{f})^{2n+2} = 0$ holds, and projecting this relation on $s, \mathbf{d}, dt$ yields

$$s\,\tau\natural\, \hat{f}(\delta\hat{f})^{2n+1}|_{\mathbf{d},dt} + \tau\natural\, \mathbf{d}(\hat{f}(\delta\hat{f})^{2n+1})|_{s,dt} + dt\frac{\partial}{\partial t}\tau\natural\, \hat{f}(\delta\hat{f})^{2n+1}|_{s,\mathbf{d}} = 0 \ .$$

This may be rephrased as

$$\begin{aligned} &\frac{\partial}{\partial t}\big(\mathrm{Tr}\natural(-2\hat{e}+1) \sum_{i=0}^n (\hat{e}-p_0)^{2i} s\hat{e} (\hat{e}-p_0)^{2(n-i)} \mathbf{d}\hat{e}\big) \\ &\quad \equiv \quad s\big(\mathrm{Tr}\natural(-2\hat{e}+1) \sum_{i=0}^n (\hat{e}-p_0)^{2i} \frac{\partial\hat{e}}{\partial t} (\hat{e}-p_0)^{2(n-i)} \mathbf{d}\hat{e}\big) \ \mathrm{mod}\ \natural\mathbf{d} \ , \end{aligned}$$

and integration over the current $\int_0^1$ shows the homotopy invariance of the class $\mathrm{cs}_1^{2n}(\hat{e})$. Hence the map $\mathrm{cs}_1^{2n} : K_0^{\mathrm{top}}(\mathscr{I}\hat{\otimes}S\mathscr{A}) \to HP_1(\mathscr{A})$ is well-defined. Its compatibility with Bott periodicity can be established, without computation, as follows. Let $g \in (\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A})^+$ be an invertible element and

$e = \alpha(g) \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}S\mathscr{A})^+$ its idempotent image under the Bott isomorphism. Choose an invertible lift $\widetilde{g} \in (\mathscr{K}\hat{\otimes}S\widehat{T}(\mathscr{I}\mathscr{A}))^+$ of g and an idempotent lift $\widetilde{e} \in M_2(\mathscr{K}\hat{\otimes}S\widehat{T}(\mathscr{I}\mathscr{A}))^+$ of e. The differential forms $\mathrm{ch}_{2n+1}(\widetilde{g})$ and $\mathrm{cs}_{2n+1}(\widetilde{e})$ in $\Omega^{2n+1}\widehat{T}(\mathscr{I}\mathscr{A})$ defined by (66) and (67) are the components of two $(b+B)$-cycles $\mathrm{ch}_*(\widetilde{g})$ and $\mathrm{cs}_*(\widetilde{e})$, whose projections on the odd part of the complex $X(\widehat{T}(\mathscr{I}\mathscr{A}))$ are

$$\natural\mathrm{ch}_1(\widetilde{g}) = \frac{1}{\sqrt{2\pi i}}\mathrm{Tr}\natural\widetilde{g}^{-1}\mathbf{d}\widetilde{g}\ , \qquad \natural\mathrm{cs}_1(\widetilde{e}) = \int_0^1 \mathrm{Tr}\natural(-2\widetilde{e}+1)s\widetilde{e}\mathbf{d}\widetilde{e})\ .$$

By Lemma 4.4, the cycles $\sqrt{2\pi i}\natural\mathrm{ch}_1(\widetilde{g})$ and $\natural\mathrm{cs}_1(\widetilde{e})$ are homologous. But we know that the projection $\widehat{\Omega}\widehat{T}(\mathscr{I}\mathscr{A}) \to X(\widehat{T}(\mathscr{I}\mathscr{A}))$ is a homotopy equivalence, with inverse the Goodwillie map γ. Hence $\sqrt{2\pi i}\,\mathrm{ch}_*(\widetilde{g})$ and $\mathrm{cs}_*(\widetilde{e})$ are $(b+B)$-homologous in $\widehat{\Omega}\widehat{T}(\mathscr{I}\mathscr{A})$. Finally, it remains to observe that under the homomorphism $\rho_* : \widehat{T}\mathscr{A} \to \mathscr{M}^s_+$, the invertible $\rho_*\widetilde{g} = \left(\begin{smallmatrix} \hat{g} & 0 \\ 0 & 1\end{smallmatrix}\right)$ gives a choice of lifting $\hat{g} \in (\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$ and the idempotent $\rho_*(\widetilde{e}) = \left(\begin{smallmatrix} \hat{e} & 0 \\ 0 & p_0\end{smallmatrix}\right)$ gives a choice of lifting $\hat{e} \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$. Because the cycles $\sqrt{2\pi i}\,\widehat{\chi}^{2n}\rho_*\mathrm{ch}_*(\widetilde{g})$ and $\widehat{\chi}^{2n}\rho_*\mathrm{cs}_*(\widetilde{e})$ are homologous in $X(\widehat{\mathscr{R}})$, we have $\sqrt{2\pi i}\,\mathrm{ch}_1^{2n}(\hat{g}) \equiv \mathrm{cs}_1^{2n}(\hat{e})$ in $HP_1(\mathscr{A})$.

We proceed similarly with the map cs_0^{2n}. Let $\hat{g} \in (\mathscr{K}\hat{\otimes}\mathscr{I}S\widehat{\mathscr{R}})^+$ be an invertible. We have to show the homotopy invariance of the cyclic homology class $\mathrm{cs}_0^{2n}(\hat{g})$. Define the invertible matrix $\hat{u} = \left(\begin{smallmatrix} \hat{g} & 0 \\ 0 & 1\end{smallmatrix}\right)$. Then $s\hat{u} = \left(\begin{smallmatrix} s\hat{g} & 0 \\ 0 & 0\end{smallmatrix}\right)$ and one has

$$\mathrm{cs}_0^{2n}(\hat{g}) = \frac{(-)^n}{\sqrt{2\pi i}}\frac{(n!)^2}{(2n)!}\int_0^1 \tau(\hat{u}^{-1}([F,\hat{u}][F,\hat{u}^{-1}])^n s\hat{u})\ .$$

Now suppose that $\hat{g}$ is a smooth family of invertibles depending on an additional parameter t. The above integrand may be expressed in terms of the odd derivation $\delta = s + dt\frac{\partial}{\partial t} + [F,\ ]$ as

$$\tau\,\hat{u}^{-1}([F,\hat{u}][F,\hat{u}^{-1}])^n s\hat{u} \equiv \frac{(-)^n}{2n+1}\,\tau(\hat{u}^{-1}\delta\hat{u})^{2n+1}|_s \bmod \overline{b}\ .$$

One has the relation $(s + dt\frac{\partial}{\partial t})\tau(\hat{u}^{-1}\delta\hat{u})^{2n+1} = -\tau(\hat{u}^{-1}\delta\hat{u})^{2n+2} \equiv 0 \bmod \overline{b}$, hence by projection on s, dt

$$s\,\tau(\hat{u}^{-1}\delta\hat{u})^{2n+1}|_{dt} + dt\frac{\partial}{\partial t}\tau(\hat{u}^{-1}\delta\hat{u})^{2n+1}|_s \equiv 0 \bmod \overline{b}\ .$$

This may be rephrased as

$$\frac{\partial}{\partial t}\big(\mathrm{Tr}\,\hat{g}^{-1}[(\hat{g}-1)(\hat{g}^{-1}-1)]^n s\hat{g}\big) \equiv s\big(\mathrm{Tr}\,\hat{g}^{-1}[(\hat{g}-1)(\hat{g}^{-1}-1)]^n\frac{\partial\hat{g}}{\partial t}\big) \bmod \overline{b}\ ,$$

and integration over the current $\int_0^1$ shows the homotopy invariance of the class $\mathrm{cs}_0^{2n}(\hat{g})$. Hence the map $\mathrm{cs}_0^{2n} : K_1^{\mathrm{top}}(\mathscr{I}\hat{\otimes}S\mathscr{A}) \to HP_0(\mathscr{A})$ is well-defined. Its compatibility with Bott periodicity is be established as before, replacing invertibles by idempotents and conversely. ∎

5 MULTIPLICATIVE K-THEORY

Let $\mathscr{A}$ and $\mathscr{I}$ be Fréchet m-algebras, $\mathscr{I}$ being p-summable. We shall define the multiplicative K-theory groups $MK_n^{\mathscr{I}}(\mathscr{A})$ in any degree $n \in \mathbb{Z}$. They are intermediate between the topological K-theory $K_n^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A})$ and the non-periodic cyclic homology $HC_n(\mathscr{A})$. Recall from section 2 that if $0 \to \mathscr{J} \to \mathscr{R} \to \mathscr{A} \to 0$ is any quasi-free extension, $HC_n(\mathscr{A})$ is computed by the quotient complex $X_n(\mathscr{R},\mathscr{J}) = X(\mathscr{R})/F^n_{\mathscr{J}}X(\mathscr{R})$ induced by the $\mathscr{J}$-adic filtration:

$$HC_n(\mathscr{A}) = H_{n+2\mathbb{Z}}(X_n(\mathscr{R},\mathscr{J}))\ , \qquad \forall n \in \mathbb{Z}\ .$$

Of course $HC_n(\mathscr{A})$ vanishes whenever $n < 0$. Multiplicative K-theory classes are represented by idempotents or invertibles whose higher Chern characters (Lemma 4.5) can be transgressed up to a certain order. As before we adopt the notation $\mathscr{I}\mathscr{A} = \mathscr{I}\hat{\otimes}\mathscr{A}$ and $\mathscr{I}\widehat{\mathscr{R}} = \mathscr{I}\hat{\otimes}\widehat{\mathscr{R}}$, where $\widehat{\mathscr{R}}$ is the $\mathscr{I}$-adic completion of $\mathscr{R}$.

DEFINITION 5.1 *Let $0 \to \mathscr{J} \to \mathscr{R} \to \mathscr{A} \to 0$ be any quasi-free extension of Fréchet m-algebras, and let $\mathscr{I}$ be a p-summable Fréchet m-algebra. Choose an integer q such that $2q+1 \geq p$. We define the multiplicative K-theory $MK_n^{\mathscr{I}}(\mathscr{A})$, in any even degree $n = 2k \in \mathbb{Z}$, as the set of equivalence classes of pairs $(\hat{e},\theta)$ such that $\hat{e} \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$ is an idempotent and $\theta \in X_{n-1}(\mathscr{R},\mathscr{J})$ is a chain of odd degree related by the transgression formula*

$$\mathrm{ch}_0^{2q}(\hat{e}) = \overline{b}\theta \ \in X_{n-1}(\mathscr{R},\mathscr{J})\ .$$

Two pairs $(\hat{e}_0,\theta_0)$ and $(\hat{e}_1,\theta_1)$ are equivalent if and only if there exists an idempotent $\hat{e} \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1])^+$ whose evaluation yields $\hat{e}_0$ and $\hat{e}_1$ at the endpoints, and a chain $\lambda \in X_{n-1}(\mathscr{R},\mathscr{J})$ of even degree such that

$$\theta_1 - \theta_0 = \mathrm{cs}_1^{2q}(\hat{e}) + \natural\mathbf{d}\lambda \ \in X_{n-1}(\mathscr{R},\mathscr{J})\ .$$

In the same way, we define the multiplicative K-theory $MK_n^{\mathscr{I}}(\mathscr{A})$, in any odd degree $n = 2k+1 \in \mathbb{Z}$, as the set of equivalence classes of pairs $(\hat{g},\theta)$ such that $\hat{g} \in (\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$ is an invertible and $\theta \in X_{n-1}(\mathscr{R},\mathscr{J})$ is a chain of even degree related by the transgression formula

$$\mathrm{ch}_1^{2q}(\hat{g}) = \natural\mathbf{d}\theta \ \in X_{n-1}(\mathscr{R},\mathscr{J})\ .$$

Two pairs $(\hat{g}_0,\theta_0)$ and $(\hat{g}_1,\theta_1)$ are equivalent if and only if there exists an invertible $\hat{g} \in (\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1])^+$ whose evaluation yields $\hat{g}_0$ and $\hat{g}_1$ at the endpoints, and a chain $\lambda \in X_{n-1}(\mathscr{R},\mathscr{J})$ of odd degree such that

$$\theta_1 - \theta_0 = \mathrm{cs}_0^{2q}(\hat{g}) + \overline{b}\lambda \ \in X_{n-1}(\mathscr{R},\mathscr{J})\ .$$

We will prove in Proposition 5.4 that for any $n \in \mathbb{Z}$ the set $MK_n^{\mathscr{I}}(\mathscr{A})$ depends neither on the degree $2q+1 \geq p$ chosen to represent the Chern characters nor on the quasi-free extension $\mathscr{R}$.

Recall from section 2 that the homology $H_{n-1+2\mathbb{Z}}(X_n(\mathscr{R},\mathscr{J}))$ is the noncommutative de Rham homology $HD_{n-1}(\mathscr{A})$. Hence the transgression relation $\mathrm{ch}_0^{2q}(\hat{e})=\overline{b}\theta\in X_{n-1}(\mathscr{R},\mathscr{J})$ exactly means that the class of $\mathrm{ch}_0^{2q}(\hat{e})$ vanishes in $HD_{n-2}(\mathscr{A})$. Similarly in the odd case, $\mathrm{ch}_1^{2q}(\hat{g})=\natural\mathbf{d}\theta\in X_{n-1}(\mathscr{R},\mathscr{J})$ is equivalent to $\mathrm{ch}_1^{2q}(\hat{g})\equiv 0$ in $HD_{n-2}(\mathscr{A})$. Since the complexes $X_{n-1}(\mathscr{R},\mathscr{J})$ vanish for $n\le 0$, we immediately deduce that $MK_n^{\mathscr{I}}(\mathscr{A})$ coincides with the topological K-theory $K_n^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A})$ whenever $n\le 0$.
As in the case of topological K-theory, define an addition on $MK_n^{\mathscr{I}}(\mathscr{A})$ by direct sum of idempotents and invertibles as follows (c is the permutation matrix (52)):

$$\begin{aligned} \text{even case:}\qquad & (\hat{e},\theta)+(\hat{e}',\theta')=(c(\hat{e}\oplus\hat{e}')c,\theta+\theta')\ ,\\ \text{odd case:}\qquad & (\hat{g},\theta)+(\hat{g}',\theta')=(\hat{g}\oplus\hat{g}',\theta+\theta')\ .\end{aligned}$$

This turns $MK_n^{\mathscr{I}}(\mathscr{A})$ into a semigroup, the unit being represented by $(p_0,0)$ in the even case and $(1,0)$ in the odd case.

Lemma 5.2 *$MK_n^{\mathscr{I}}(\mathscr{A})$ is an abelian group for any $n\in\mathbb{Z}$.*

Proof: We first need to recall the proof that $K_0^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A})$ is a group. Let $e\in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A})^+$ be an idempotent, with $e-p_0\in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A})$. The idempotent $1-e$ is orthogonal to e, as $e(1-e)=(1-e)e=0$. If $X\in M_2(\mathbb{C})$ is the permutation matrix $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, we claim that the idempotent

$$X(1-e)X\in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A})^+\ ,\quad\text{with}\quad X(1-e)X-p_0\in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A})\ ,$$

represents the inverse class of e. Indeed, we shall construct a homotopy between the direct sum $c(e\oplus X(1-e)X)c\in M_4(\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A})^+$ and the unit

$$\tilde{p}_0=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\in M_4(\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A})^+\ .$$

Choose a smooth real-valued function $\xi\in C^\infty[0,1]$ ranging from $\xi(0)=0$ to $\xi(1)=\pi/2$, and consider the paths of invertible matrices

$$R_{23}=\begin{pmatrix}1&0&0&0\\0&\cos\xi&\sin\xi&0\\0&\sin\xi&-\cos\xi&0\\0&0&0&1\end{pmatrix}\ ,\quad R_{14}=\begin{pmatrix}\cos\xi&0&0&\sin\xi\\0&1&0&0\\0&0&1&0\\\sin\xi&0&0&-\cos\xi\end{pmatrix}\ .$$

A direct computation shows that the idempotents $R_{23}(t)^{-1}\left(\begin{smallmatrix}e&0\\0&0\end{smallmatrix}\right)R_{23}(t)$ and $R_{14}(t)^{-1}\left(\begin{smallmatrix}1-e&0\\0&0\end{smallmatrix}\right)R_{14}(t)$ are orthogonal for any $t\in[0,1]$. Hence the sum

$$f=R_{23}^{-1}\begin{pmatrix}e&0\\0&0\end{pmatrix}R_{23}+R_{14}^{-1}\begin{pmatrix}1-e&0\\0&0\end{pmatrix}R_{14}\ \in M_4(\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A}[0,1])^+$$

is an idempotent path such that $f - \tilde{p}_0 \in M_4(\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A}[0,1])$, and interpolates $f_0 = \tilde{p}_0$ and $f_1 = c(e \oplus X(1-e)X)c$. This shows that $K_0^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A})$ is a group. It is abelian because a direct sum $e \oplus e'$ can be connected via a smooth path (by conjugation with respect to rotation matrices) to $e' \oplus e$.
Now fix an integer $2q+1 \geq p$ and let $(\hat{e}, \theta)$ represent an element of $MK_n^{\mathscr{I}}(\mathscr{A})$ of even degree $n = 2k$. Hence $\hat{e} \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$ is an idempotent such that $\mathrm{ch}_0^{2q}(\hat{e}) = \overline{b}\theta$ in the quotient complex $X_{n-1}(\mathscr{R}, \mathscr{J})$. Consider the smooth idempotent path $\hat{f} \in M_4(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1])^+$ constructed as above replacing $\mathscr{A}$ by its extension $\widehat{\mathscr{R}}$ and e by $\hat{e}$. It provides an interpolation between $\hat{f}_0 = \tilde{p}_0$ and $\hat{f}_1 = c(\hat{e} \oplus X(1-\hat{e})X)c$. We guess that the inverse of $(\hat{e}, \theta)$ is represented by the pair $(X(1-\hat{e})X, \mathrm{cs}_1^{2q}(\hat{f}) - \theta)$. Indeed, one has

$$\overline{b}\mathrm{cs}_1^{2q}(\hat{f}) = \mathrm{ch}_0^{2q}(\hat{f}_1) - \mathrm{ch}_0^{2q}(\hat{f}_0) = \mathrm{ch}_0^{2q}(\hat{e}) + \mathrm{ch}_0^{2q}(X(1-\hat{e})X) \ ,$$

so that $\mathrm{ch}_0^{2q}(X(1-\hat{e})X) = \overline{b}(\mathrm{cs}_1^{2q}(\hat{f}) - \theta)$ in the complex $X_{n-1}(\mathscr{R}, \mathscr{J})$ and the pair $(X(1-\hat{e})X, \mathrm{cs}_1^{2q}(\hat{f}) - \theta)$ represents a class in $MK_n^{\mathscr{I}}(\mathscr{A})$. Moreover, the sum

$$(\hat{e}, \theta) + (X(1-\hat{e})X, \mathrm{cs}_1^{2q}(\hat{f}) - \theta) = (c(\hat{e} \oplus X(1-\hat{e})X)c, \mathrm{cs}_1^{2q}(\hat{f}))$$

is equivalent to the unit $(\tilde{p}_0, 0)$ because $\hat{f}$ provides the interpolating idempotent. Hence $MK_n^{\mathscr{I}}(\mathscr{A})$, $n = 2k$ is a group. Abelianity is shown as for topological K-theory, by means of another interpolation between the idempotents $c(\hat{e}\oplus\hat{e}')c$ and $c(\hat{e}' \oplus \hat{e})c$ with the property that its Chern-Simons form cs_1 vanishes.
One proceeds similarly in the odd case. Let $(\hat{g}, \theta)$ represent an element of $MK_n^{\mathscr{I}}(\mathscr{A})$ of odd degree $n = 2k+1$. Hence $\mathrm{ch}_1^{2q}(\hat{g}) = \natural\mathbf{d}\theta$ in $X_{n-1}(\mathscr{R}, \mathscr{J})$. Define an invertible path $\hat{u} \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1])^+$ by means of the rotation matrix $R = \left(\begin{smallmatrix} \cos\xi & \sin\xi \\ -\sin\xi & \cos\xi \end{smallmatrix}\right)$:

$$\hat{u} = \begin{pmatrix} \hat{g} & 0 \\ 0 & 1 \end{pmatrix} R^{-1} \begin{pmatrix} \hat{g}^{-1} & 0 \\ 0 & 1 \end{pmatrix} R \ .$$

Then $\hat{u} - 1 \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1])$, and $\hat{u}$ provides a smooth homotopy between the invertibles $\hat{u}_0 = 1$ and $\hat{u}_1 = \left(\begin{smallmatrix} \hat{g} & 0 \\ 0 & \hat{g}^{-1} \end{smallmatrix}\right)$ (the same argument shows that $K_1^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A})$ is an abelian group). We guess that the inverse class of $(\hat{g}, \theta)$ is represented by the pair $(\hat{g}^{-1}, \mathrm{cs}_0^{2q}(\hat{u}) - \theta)$. Indeed, one has

$$\natural\mathbf{d}\mathrm{cs}_0^{2q}(\hat{u}) = \mathrm{ch}_1^{2q}(\hat{u}_1) - \mathrm{ch}_1^{2q}(\hat{u}_0) = \mathrm{ch}_1^{2q}(\hat{g}) + \mathrm{ch}_1^{2q}(\hat{g}^{-1})$$

so that $\mathrm{ch}_1^{2q}(\hat{g}^{-1}) = \natural\mathbf{d}(\mathrm{cs}_0^{2q}(\hat{u}) - \theta)$ in $X_{n-1}(\mathscr{R}, \mathscr{J})$ and $(\hat{g}^{-1}, \mathrm{cs}_0^{2q}(\hat{u}) - \theta)$ represents a class in $MK_n^{\mathscr{I}}(\mathscr{A})$. Moreover, the sum

$$(\hat{g}, \theta) + (\hat{g}^{-1}, \mathrm{cs}_0^{2q}(\hat{u}) - \theta) = (\hat{g} \oplus \hat{g}^{-1}, \mathrm{cs}_0^{2q}(\hat{u}))$$

is equivalent to the unit $(1, 0)$ through the interpolating invertible $\hat{u}$. Hence $MK_n^{\mathscr{I}}(\mathscr{A})$, $n = 2k+1$ is a group as claimed. Abelianity is shown once again by means of rotation matrices. ∎

Remark 5.3 We know that two different liftings of a given idempotent $e \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A})^+$ are always homotopic in $M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$. Hence choosing the universal free extension $\mathscr{R} = T\mathscr{A}$ allows to represent any multiplicative K-theory class of even degree by a pair $(\hat{e}, \theta)$ where $\hat{e} \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{T}\mathscr{A})^+$ is the *canonical lift* of some idempotent e. Moreover, the transgression formula established in the proof of Lemma 4.6 shows that two such pairs $(\hat{e}_0, \theta_0)$ and $(\hat{e}_1, \theta_1)$ are equivalent if and only if e_0 and e_1 can be joined by an idempotent $e \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A}[0,1])^+$ such that $\theta_1 - \theta_0 \equiv \mathrm{cs}_1^{2q}(\hat{e}) \bmod \natural\mathbf{d}$, where $\hat{e} \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{T}\mathscr{A}[0,1])^+$ is the canonical lift of e. The same is true with invertibles: any multiplicative K-theory class of odd degree may be represented by a pair $(\hat{g}, \theta)$ where $\hat{g} \in (\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{T}\mathscr{A})^+$ is the *canonical lift* of some invertible $g \in (\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A})^+$. Two such pairs $(\hat{g}_0, \theta_0)$ and $(\hat{g}_1, \theta_1)$ are equivalent if and only if g_0 and g_1 can be joined by an invertible $g \in (\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A}[0,1])^+$ such that $\theta_1 - \theta_0 \equiv \mathrm{cs}_0^{2q}(\hat{g}) \bmod \overline{b}$, where $\hat{g} \in (\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{T}\mathscr{A}[0,1])^+$ is the canonical lift of g.

The particular case $\mathscr{I} = \mathbb{C}$ is essentially equivalent Karoubi's definition of multiplicative K-theory [16, 17]. The groups $MK_n^{\mathscr{I}}(\mathscr{A})$ are designed to fit in a long exact sequence

$$K_{n+1}^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \to HC_{n-1}(\mathscr{A}) \xrightarrow{\delta} MK_n^{\mathscr{I}}(\mathscr{A}) \to K_n^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \to HC_{n-2}(\mathscr{A}) \tag{72}$$

The map $K_n^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \to HC_{n-2}(\mathscr{A})$ corresponds to the composition of the Chern character $K_n^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \to HP_n(\mathscr{A})$ with the natural map $HP_n(\mathscr{A}) \to HC_{n-2}(\mathscr{A})$ induced by the projection $\widehat{X}(\mathscr{R}, \mathscr{J}) \to X_{n-2}(\mathscr{R}, \mathscr{J})$. The map $MK_n^{\mathscr{I}}(\mathscr{A}) \to K_n^{\mathrm{top}}(\mathscr{A})$ is the forgetful map, which sends a pair $(\hat{e}, \theta)$ or $(\hat{g}, \theta)$ respectively on its image e or g under the projection homomorphism $\widehat{\mathscr{R}} \to \mathscr{A}$. The connecting map $\delta : HC_{n-1}(\mathscr{A}) \to MK_n^{\mathscr{I}}(\mathscr{A})$ sends a cycle $\theta \in X_{n-1}(\mathscr{R}, \mathscr{J})$ to

$$\delta(\theta) = \begin{cases} (p_0, \sqrt{2\pi i}\,\theta) & n \text{ even,} \\ (1, \sqrt{2\pi i}\,\theta) & n \text{ odd.} \end{cases} \tag{73}$$

There is also an additive Chern character map $\mathrm{ch}_n : MK_n^{\mathscr{I}}(\mathscr{A}) \to HN_n(\mathscr{A})$ defined in all degrees $n \in \mathbb{Z}$, with values in *negative* cyclic homology. Recall that the latter is the homology in degree n mod 2 of the subcomplex $F^{n-1}\widehat{X}(\mathscr{R}, \mathscr{J}) = \mathrm{Ker}(\widehat{X}(\mathscr{R}, \mathscr{J}) \to X_{n-1}(\mathscr{R}, \mathscr{J}))$:

$$HN_n(\mathscr{A}) = H_{n+2\mathbb{Z}}(F^{n-1}\widehat{X}(\mathscr{R}, \mathscr{J}))\ .$$

Hence in particular $HN_n(\mathscr{A}) = HP_n(\mathscr{A})$ whenever $n \leq 0$, and HP_*, HC_* HN_* are related by the SBI long exact sequence (section 2). To define the Chern character $\mathrm{ch}_n : MK_n^{\mathscr{I}}(\mathscr{A}) \to HN_n(\mathscr{A})$ in even degree $n = 2k$, we first have to choose an integer $2q + 1 \geq p$. Then, a multiplicative K-theory class of degree n is represented by a pair $(\hat{e}, \theta)$, such that the transgression formula $\mathrm{ch}_0^{2q}(\hat{e}) = \overline{b}\theta$ holds in $X_{n-1}(\mathscr{R}, \mathscr{J})$. Choose an arbitrary lifting $\tilde{\theta} \in \widehat{X}(\mathscr{R}, \mathscr{J})$

of θ, and define the negative Chern character as

$$\mathrm{ch}_n(\hat{e},\theta) = \mathrm{ch}_0^{2q}(\hat{e}) - \overline{b}\tilde{\theta} \ \in F^{n-1}\widehat{X}(\mathscr{R},\mathscr{J}) \ . \tag{74}$$

It is clearly closed, and its negative cyclic homology class does not depend on the choice of lifting $\tilde{\theta}$, since the difference of two such liftings lies in the subcomplex $F^{n-1}\widehat{X}(\mathscr{R},\mathscr{J})$. We will show in the proposition below that it does not depend on the representative $(\hat{e},\theta)$ of the K-theory class, nor on the integer $2q+1\geq p$. In odd degree $n=2k+1$, the Chern character $\mathrm{ch}_n : MK_n^{\mathscr{I}}(\mathscr{A}) \to HN_n(\mathscr{A})$ is defined exactly in the same way: take a representative $(\hat{g},\theta)$ of a multiplicative K-theory class, with $\mathrm{ch}_1^{2q}(\hat{g}) = \natural\mathbf{d}\theta$ in $X_{n-1}(\mathscr{R},\mathscr{J})$. Then if $\tilde{\theta}\in\widehat{X}(\mathscr{R},\mathscr{J})$ denotes an arbitrary lifting of θ, the cycle

$$\mathrm{ch}_n(\hat{g},\theta) = \mathrm{ch}_1^{2q}(\hat{g}) - \natural\mathbf{d}\tilde{\theta} \ \in F^{n-1}\widehat{X}(\mathscr{R},\mathscr{J}) \tag{75}$$

defines a negative cyclic homology class. The following proposition shows the compatibility between the K-theory exact sequence (72) and the SBI exact sequence (8), through the various Chern character maps.

Proposition 5.4 *Let $\mathscr{A}$ and $\mathscr{I}$ be Fréchet m-algebras, such that $\mathscr{I}$ is p-summable. Then one has a commutative diagram with long exact rows*

$$\begin{tikzcd} K^{\mathrm{top}}_{n+1}(\mathscr{I}\hat{\otimes}\mathscr{A}) \arrow[r] \arrow[d] & HC_{n-1}(\mathscr{A}) \arrow[r, "\delta"] \arrow[d, equal] & MK_n^{\mathscr{I}}(\mathscr{A}) \arrow[r] \arrow[d] & K^{\mathrm{top}}_n(\mathscr{I}\hat{\otimes}\mathscr{A}) \arrow[d] \\ HP_{n+1}(\mathscr{A}) \arrow[r, "S"] & HC_{n-1}(\mathscr{A}) \arrow[r, "\widetilde{B}"] & HN_n(\mathscr{A}) \arrow[r, "I"] & HP_n(\mathscr{A}) \end{tikzcd} \tag{76}$$

where $\widetilde{B}$ is the connecting map of the SBI sequence rescaled by a factor $-\sqrt{2\pi i}$.

Proof: We show the exactness of the sequence (72) in the case of even degree $n=2k$ (the odd case is completely similar):

$$K_1^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \xrightarrow{\mathrm{ch}_1} HC_{n-1}(\mathscr{A}) \xrightarrow{\delta} MK_n^{\mathscr{I}}(\mathscr{A}) \xrightarrow{\iota} K_0^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \xrightarrow{\mathrm{ch}_0} HC_{n-2}(\mathscr{A}) \ .$$

Fix once and for all an integer $2q+1\geq p$ to represent the Chern characters. We first have to check that the maps δ and ι are well-defined. Let θ_0 and $\theta_1 = \theta_0 + \natural\mathbf{d}\lambda$ be two homologous odd cycles in $X_{n-1}(\mathscr{R},\mathscr{J})$ representing the same cyclic homology class $[\theta]\in HC_{n-1}(\mathscr{A})$. Their images by δ are respectively $(p_0,\sqrt{2\pi i}\,\theta_0)$ and $(p_0,\sqrt{2\pi i}\,\theta_1)$, which obviously represent the same class in $MK_n^{\mathscr{I}}(\mathscr{A})$ by virtue of the equivalence relation $\theta_1-\theta_0 = \natural\mathbf{d}\lambda$ (take the constant idempotent $p_0\in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1])^+$ as interpolation, with $\mathrm{cs}_1^{2q}(p_0)=0$). Hence δ is well-defined.
Now take two equivalent pairs $(\hat{e}_0,\theta_0)$ and $(\hat{e}_1,\theta_1)$ representing the same element in $MK_n^{\mathscr{I}}(\mathscr{A})$. In particular, the idempotents $\hat{e}_0$ and $\hat{e}_1$ are smoothly homotopic and their projections $e_0,e_1\in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A})^+$ define the same

class in $K_0^{\rm top}(\mathscr{I}\hat{\otimes}\mathscr{A})$. Since $\iota(\hat{e}_0,\theta_0)=e_0$ and $\iota(\hat{e}_1,\theta_1)=e_1$, the map ι is well-defined.

Exactness at $HC_{n-1}(\mathscr{A})$: Let $[g]\in K_1^{\rm top}(\mathscr{I}\hat{\otimes}\mathscr{A})$ be a class represented by an invertible element $g\in(\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A})^+$, and consider its idempotent image $\alpha(g)\in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}S\mathscr{A})^+$ under the Bott isomorphism $\alpha: K_1^{\rm top}(\mathscr{I}\hat{\otimes}\mathscr{A})\to K_0^{\rm top}(\mathscr{I}\hat{\otimes}S\mathscr{A})$. Choose any invertible lift $\hat{g}\in(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$ and any idempotent lift $\widehat{\alpha(g)}\in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}S\widehat{\mathscr{R}})^+$. By Lemma 4.6, we have the equality of periodic cyclic homology classes

$$\mathrm{ch}_1^{2q}(\hat{g})\equiv\frac{1}{\sqrt{2\pi i}}\mathrm{cs}_1^{2q}(\widehat{\alpha(g)})\ \in HP_1(\mathscr{A})\ ,$$

hence this equality also holds in $HC_{n-1}(\mathscr{A})$. It follows that $\delta(\mathrm{ch}_1(g))$ is represented by

$$\delta\Big(\frac{1}{\sqrt{2\pi i}}\mathrm{cs}_1^{2q}(\widehat{\alpha(g)})\Big)=\big(p_0,\mathrm{cs}_1^{2q}(\widehat{\alpha(g)})\big)\ .$$

But the idempotent path $\widehat{\alpha(g)}$ evaluated at the endpoints is p_0, so that the pairs $(p_0,\mathrm{cs}_1^{2q}(\widehat{\alpha(g)}))$ and $(p_0,0)$ are equivalent in $MK_n^{\mathscr{I}}(\mathscr{A})$. Hence $\delta\circ\mathrm{ch}_1=0$. Now let a class $[\theta]\in HC_{n-1}(\mathscr{A})$ be in the kernel of δ. It means that the pair $\delta(\theta)=(p_0,\sqrt{2\pi i}\,\theta)$ is equivalent to $(p_0,0)$. Hence there exists an idempotent $\hat{e}\in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}S\widehat{\mathscr{R}})^+$ and a chain λ such that $\sqrt{2\pi i}\,\theta=\mathrm{cs}_1^{2q}(\hat{e})+\natural\mathbf{d}\lambda$ in $X_{n-1}(\mathscr{R},\mathscr{J})$. By Bott periodicity, there exists an element $[g]\in K_1^{\rm top}(\mathscr{I}\hat{\otimes}\mathscr{A})$ such that $\mathrm{cs}_1^{2q}(\hat{e})\equiv\sqrt{2\pi i}\,\mathrm{ch}_1^{2q}(\hat{g})$ in $HC_{n-1}(\mathscr{A})$, where $\hat{g}\in(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$ is any invertible lift of g. Whence the equality of cylic homology classes $[\theta]\equiv\mathrm{ch}_1^{2q}(\hat{g})$. It follows that $\mathrm{Ker}\,\delta=\mathrm{Im}\,\mathrm{ch}_1$.

Exactness at $MK_n^{\mathscr{I}}(\mathscr{A})$: Let $[\theta]\in HC_{n-1}(\mathscr{A})$ be any cyclic homology class. Then $\delta(\theta)=(p_0,\sqrt{2\pi i}\,\theta)$, and $\iota(\delta(\theta))=p_0$ is the zero-class in topological K-theory. Therefore $\iota\circ\delta=0$.
Now let $(\hat{e},\theta)\in MK_n^{\mathscr{I}}(\mathscr{A})$ be in the kernel of ι: it means that $\hat{e}$ is smoothly homotopic to p_0. Hence there exists an idempotent $\hat{f}\in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1])^+$, with evaluations $\hat{f}_0=\hat{e}$ and $\hat{f}_1=p_0$, and the pair $(\hat{e},\theta)$ is equivalent to $(p_0,\theta+\mathrm{cs}_1^{2q}(\hat{f}))$. Remark that the odd chain $\theta+\mathrm{cs}_1^{2q}(\hat{f})\in X_{n-1}(\mathscr{R},\mathscr{J})$ is closed (indeed, $\overline{b}\theta=\mathrm{ch}_0^{2q}(\hat{e})$ and $\overline{b}\mathrm{cs}_1^{2q}(\hat{f})=-\mathrm{ch}_0^{2q}(\hat{e})$), and we can write

$$(p_0,\theta+\mathrm{cs}_1^{2q}(\hat{f}))=\delta\Big(\frac{1}{\sqrt{2\pi i}}(\theta+\mathrm{cs}_1^{2q}(\hat{f}))\Big)\ .$$

It follows that $\mathrm{Ker}\,\iota=\mathrm{Im}\,\delta$.

Exactness at $K_0^{\rm top}(\mathscr{I}\hat{\otimes}\mathscr{A})$: Let $(\hat{e},\theta)\in MK_n^{\mathscr{I}}(\mathscr{A})$ represent any multiplicative K-theory class. Then $\mathrm{ch}_0^{2q}(\hat{e})\equiv 0$ in non-commutative de Rham homology $HD_{n-2}(\mathscr{A})$, and therefore also in $HC_{n-2}(\mathscr{A})$. Thus, the Chern character of $\iota(\hat{e},\theta)=e$ vanishes and $\mathrm{ch}_0\circ\iota=0$.

Now let $[e] \in K_0^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A})$ be in the kernel of ch_0. We know from section 2 that the natural map $HD_{n-2}(\mathscr{A}) \to HC_{n-2}(\mathscr{A})$ is injective, so that $\mathrm{ch}_0^{2q}(\hat{e}) \equiv 0$ in $HD_{n-2}(\mathscr{A})$ for any idempotent lift $\hat{e} \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$. Hence there exists an odd chain $\theta \in X_{n-1}(\mathscr{R}, \mathscr{J})$ such that $\mathrm{ch}_0^{2q}(\hat{e}) = \overline{b}\theta$, and $e = \iota(\hat{e}, \theta)$. This shows that $\mathrm{Ker}\,\mathrm{ch}_0 = \mathrm{Im}\,\iota$.

Let us now show the independence of multiplicative K-theory upon the choice of degree $2q+1 \geq p$. To this end, write $(\hat{e}, \theta)^q \in MK_n^{\mathscr{I}}(\mathscr{A})^q$ for a representative of a class obtained using the higher Chern character $\mathrm{ch}_0^{2q}(\hat{e}) = \overline{b}\theta$ of degree $2q$. We shall construct a map $MK_n^{\mathscr{I}}(\mathscr{A})^q \to MK_n^{\mathscr{I}}(\mathscr{A})^{q+1}$ which turns out to be an isomorphism. Let $\rho : \mathscr{I}\mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s\mathscr{A}$ be the canonical p-summable quasihomomorphism of even degree considered in section 4, for the construction of the higher Chern characters. Recall that $\mathscr{E} = \mathscr{I}^+\hat{\otimes}\mathscr{A}$ with extension $\mathscr{M} = \mathscr{I}^+\hat{\otimes}\mathscr{R}$. From Proposition 3.10, we know that the chain maps $\chi^{2q} : \widehat{\Omega}(\widehat{\mathscr{M}_+^s}) \to X(\widehat{\mathscr{R}})$ associated to ρ are related in successive degrees by the transgression formula involving the eta-cochain $\widehat{\chi}^{2q} - \widehat{\chi}^{2q+2} = [\partial, \widehat{\eta}^{2q+1}]$. More precisely:

$$\begin{aligned} \widehat{\chi}_0^{2q} - \widehat{\chi}_0^{2q+2} &= \overline{b}\widehat{\eta}_1^{2q+1} + \widehat{\eta}_0^{2q+1}(b+B) \ , \\ \widehat{\chi}_1^{2q} - \widehat{\chi}_1^{2q+2} &= \natural\mathbf{d}\widehat{\eta}_0^{2q+1} + \widehat{\eta}_1^{2q+1}(b+B) \ . \end{aligned}$$

The evaluation of the first equation on the $(b+B)$-cycle $\mathrm{ch}_*(\hat{e}) \in \widehat{\Omega}^+(\mathscr{I}\widehat{\mathscr{R}})$ yields (see section 4; we also omit reference to the inclusion homomorphism $\iota_* : \mathscr{I}\widehat{\mathscr{R}} \hookrightarrow \left(\begin{smallmatrix} \mathscr{I}\widehat{\mathscr{R}} & 0 \\ 0 & 0 \end{smallmatrix}\right) \subset \widehat{\mathscr{M}_+^s}$)

$$\mathrm{ch}_0^{2q}(\hat{e}) - \mathrm{ch}_0^{2q+2}(\hat{e}) = \overline{b}\big(\widehat{\eta}_1^{2q+1}\mathrm{ch}_{2q+2}(\hat{e})\big) \ ,$$

with $\mathrm{ch}_0^{2q}(\hat{e}) = \widehat{\chi}_0^{2q}\mathrm{ch}_{2q}(\hat{e})$ by Lemma 4.5. Therefore, we guess that the map $MK_n^{\mathscr{I}}(\mathscr{A})^q \to MK_n^{\mathscr{I}}(\mathscr{A})^{q+1}$ should send a pair $(\hat{e}, \theta)^q$ to the pair $(\hat{e}, \theta')^{q+1}$ with $\theta' = \theta - \widehat{\eta}_1^{2q+1}\mathrm{ch}_{2q+2}(\hat{e})$. Indeed, one has the correct transgression relation

$$\mathrm{ch}_0^{2q+2}(\hat{e}) = \mathrm{ch}_0^{2q}(\hat{e}) - \overline{b}(\widehat{\eta}_1^{2q+1}\mathrm{ch}_{2q+2}(\hat{e})) = \overline{b}\theta'$$

in the complex $X_{n-1}(\mathscr{R}, \mathscr{J})$. Moreover, this map is well-defined at the level of equivalence classes: let $(\hat{e}_0, \theta_0)^q$ and $(\hat{e}_1, \theta_1)^q$ be two equivalent pairs. Then there exists an interpolating idempotent $\hat{e} \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1])^+$ and a chain λ such that $\theta_1 - \theta_0 = \mathrm{cs}_1^{2q}(\hat{e}) + \natural\mathbf{d}\lambda$. Hence the respective images $(\hat{e}_0, \theta_0')^{q+1}$ and $(\hat{e}_1, \theta_1')^{q+1}$ verify

$$\theta_1' - \theta_0' = \theta_1 - \theta_0 - \widehat{\eta}_1^{2q+1}\left(\mathrm{ch}_{2q+2}(\hat{e}_1) - \mathrm{ch}_{2q+2}(\hat{e}_0)\right) \ .$$

But we know the transgression relation (69)

$$\mathrm{ch}_{2q+2}(\hat{e}_1) - \mathrm{ch}_{2q+2}(\hat{e}_0) = B\mathrm{cs}_{2q+1}(\hat{e}) + b\mathrm{cs}_{2q+3}(\hat{e}) \ ,$$

so that, using the identities $\widehat{\chi}_1^{2q}\mathrm{cs}_{2q+1}(\hat{e}) = (\natural\mathbf{d}\widehat{\eta}_0^{2q+1} + \widehat{\eta}_1^{2q+1}B)\mathrm{cs}_{2q+1}(\hat{e})$ and $-\widehat{\chi}_1^{2q+2}\mathrm{cs}_{2q+3}(\hat{e}) = \widehat{\eta}_1^{2q+1}b\mathrm{cs}_{2q+3}(\hat{e})$ one gets (recall $\mathrm{cs}_1^{2q}(\hat{e}) = \widehat{\chi}_1^{2q}\mathrm{cs}_{2q+1}(\hat{e})$)

$$\begin{aligned}\theta_1' - \theta_0' &= \mathrm{cs}_1^{2q}(\hat{e}) + \natural\mathbf{d}\lambda \\ &\quad -\widehat{\chi}_1^{2q}\mathrm{cs}_{2q+1}(\hat{e}) + \widehat{\chi}_1^{2q+2}\mathrm{cs}_{2q+3}(\hat{e}) + \natural\mathbf{d}\big(\widehat{\eta}_0^{2q+1}\mathrm{cs}_{2q+1}(\hat{e})\big) \\ &= \mathrm{cs}_1^{2q+2}(\hat{e}) + \natural\mathbf{d}\big(\lambda + \widehat{\eta}_0^{2q+1}\mathrm{cs}_{2q+1}(\hat{e})\big)\ .\end{aligned}$$

Hence $(\hat{e}_0, \theta_0')^{q+1}$ and $(\hat{e}_1, \theta_1')^{q+1}$ are equivalent in $MK_n^{\mathscr{I}}(\mathscr{A})^{q+1}$. It remains to show that the map $MK_n^{\mathscr{I}}(\mathscr{A})^q \to MK_n^{\mathscr{I}}(\mathscr{A})^{q+1}$ is an isomorphism. Consider the following diagram:

$$\begin{tikzcd} HC_{n-1}(\mathscr{A}) \arrow[r, "\delta"] \arrow[d, equal] & MK_n^{\mathscr{I}}(\mathscr{A})^q \arrow[r] \arrow[d] & K_0^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \arrow[d, equal] \\ HC_{n-1}(\mathscr{A}) \arrow[r, "\delta"] & MK_n^{\mathscr{I}}(\mathscr{A})^{q+1} \arrow[r] & K_0^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \end{tikzcd}$$

For any cyclic homology class $[\theta] \in HC_{n-1}(\mathscr{A})$ represented by a closed chain θ, one has $\delta(\theta) = (p_0, \sqrt{2\pi i}\,\theta)^q$ in $MK_n^{\mathscr{I}}(\mathscr{A})^q$. But observe that $\widehat{\eta}_1^{2q+1}\mathrm{ch}_{2q+2}(p_0) = 0$, so that $(p_0, \sqrt{2\pi i}\,\theta)^q$ is mapped to $(p_0, \sqrt{2\pi i}\,\theta)^{q+1}$ in $MK_n^{\mathscr{I}}(\mathscr{A})^{q+1}$. Hence the left square is commutative. Moreover the right square is obviously commutative. The isomorphism $MK_n^{\mathscr{I}}(\mathscr{A})^q \cong MK_n^{\mathscr{I}}(\mathscr{A})^{q+1}$ then follows from the five-lemma.

The negative Chern character $\mathrm{ch}_n : MK_n^{\mathscr{I}}(\mathscr{A})^q \to HN_n(\mathscr{A})$ is also independent of q, and compatible with the SBI exact sequence. Indeed if $(\hat{e}, \theta)^q$ is a representative of a class in $MK_n^{\mathscr{I}}(\mathscr{A})^q$, one has by definition

$$\mathrm{ch}_n(\hat{e}, \theta)^q = \mathrm{ch}_0^{2q}(\hat{e}) - \overline{b}\tilde{\theta}\ ,$$

where $\tilde{\theta} \in \widehat{X}(\mathscr{R}, \mathscr{J})$ is an arbitrary lifting of θ. First remark that ch_n is well-defined at the level of equivalence classes: if $(\hat{e}_0, \theta_0)^q$ and $(\hat{e}_1, \theta_1)^q$ are equivalent, there exists an interpolation $\hat{e} \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1])^+$ and a chain λ such that $\theta_1 - \theta_0 = \mathrm{cs}_1^{2q}(\hat{e}) + \natural\mathbf{d}\lambda$ in $X_{n-1}(\mathscr{R}, \mathscr{J})$. Let $\tilde{\theta}_0$, $\tilde{\theta}_1$ and $\tilde{\lambda}$ be arbitrary liftings; then there exists a chain $\mu \in F^{n-1}\widehat{X}(\mathscr{R}, \mathscr{J})$ such that $\tilde{\theta}_1 - \tilde{\theta}_0 = \mathrm{cs}_1^{2q}(\hat{e}) + \natural\mathbf{d}\tilde{\lambda} + \mu$ in $\widehat{X}(\mathscr{R}, \mathscr{J})$. Hence the difference

$$\begin{aligned}\mathrm{ch}_n(\hat{e}_1, \theta_1)^q - \mathrm{ch}_n(\hat{e}_0, \theta_0)^q &= \mathrm{ch}_0^{2q}(\hat{e}_1) - \mathrm{ch}_0^{2q}(\hat{e}_0) - \overline{b}(\tilde{\theta}_1 - \tilde{\theta}_0) \\ &= \overline{b}\mathrm{cs}_1^{2q}(\hat{e}) - \overline{b}(\mathrm{cs}_1^{2q}(\hat{e}) + \natural\mathbf{d}\tilde{\lambda} + \mu) = -\overline{b}\mu\end{aligned}$$

is a coboundary of the subcomplex $F^{n-1}\widehat{X}(\mathscr{R}, \mathscr{J})$, and the Chern character $\mathrm{ch}_n : MK_n^{\mathscr{I}}(\mathscr{A})^q \to HN_n(\mathscr{A})$ is well-defined. Now if $(\hat{e}, \theta)^q \in MK_n^{\mathscr{I}}(\mathscr{A})^q$ is any class, its image in $MK_n^{\mathscr{I}}(\mathscr{A})^{q+1}$ is represented by $(\hat{e}, \theta')^{q+1}$ with $\theta' =$

$\theta - \widehat{\eta}_1^{2q+1}\mathrm{ch}_{2q+2}(\hat{e})$. One has

$$\begin{aligned}\mathrm{ch}_n(\hat{e},\theta')^{q+1} &= \mathrm{ch}_0^{2q+2}(\hat{e}) - \overline{b}(\tilde{\theta} - \widehat{\eta}_1^{2q+1}\mathrm{ch}_{2q+2}(\hat{e}))\\ &= \mathrm{ch}_0^{2q+2}(\hat{e}) - \overline{b}(\tilde{\theta}) + \widehat{\chi}_0^{2q}\mathrm{ch}_{2q}(\hat{e}) - \widehat{\chi}_0^{2q+2}\mathrm{ch}_{2q+2}(\hat{e})\\ &= \mathrm{ch}_0^{2q}(\hat{e}) - \overline{b}(\tilde{\theta}) \ = \ \mathrm{ch}_n(\hat{e},\theta)^q \ ,\end{aligned}$$

and the negative Chern character does not depend on the degree q. Finally, for any cyclic homology class $[\theta] \in HC_{n-1}(\mathscr{A})$, one has

$$\mathrm{ch}_n(\delta(\theta)) = \mathrm{ch}_n(p_0, \sqrt{2\pi i}\,\theta) = -\sqrt{2\pi i}\,\overline{b}(\tilde{\theta}) \ ,$$

which shows the commutativity of the middle square (76). The compatibility between the negative Chern character and the periodic Chern character on topological K-theory is obvious, whence the commutativity of the right square (76).

Concerning the independence of $MK_n^{\mathscr{I}}(\mathscr{A})$ with respect to the choice of quasi-free extension $\mathscr{R}$, it suffices to consider the universal extension $0 \to J\mathscr{A} \to T\mathscr{A} \to \mathscr{A} \to 0$ together with the classifying homomorphisms $T\mathscr{A} \to \mathscr{R}$ and $J\mathscr{A} \to \mathscr{J}$. The various Chern characters and Chern-Simons forms constructed in $\widehat{X}(\mathscr{R},\mathscr{J})$ are obtained from the universal ones in $\widehat{X}(T\mathscr{A},J\mathscr{A})$ by applying the chain map $\widehat{X}(T\mathscr{A},J\mathscr{A}) \to \widehat{X}(\mathscr{R},\mathscr{J})$, which we know is a homotopy equivalence compatible with the adic filtrations. Once again the conclusion follows from the five-lemma.
The case of odd degree $n = 2k+1$ is established along the same lines, replacing idempotents by invertibles. ∎

Before ending this section we need to establish the invariance of topological and multiplicative K-theory with respect to adjoint actions of multipliers on the p-summable Fréchet m-algebra $\mathscr{I}$. We say that U is a multiplier if it defines continuous linear maps (left and right multiplications) $x \mapsto Ux$ and $x \mapsto xU$ on $\mathscr{I}$, which commute and fulfill

$$\begin{array}{ll} i) & U(xy) = (Ux)y \ , \ (xU)y = x(Uy) \ , \ (xy)U = x(yU) \qquad \forall x,y \in \mathscr{I} \ ,\\ ii) & \mathrm{Tr}([U,\mathscr{I}^p]) = 0 \ .\end{array}$$

U is invertible if there exists a multiplier U^{-1} such that the compositions UU^{-1} and $U^{-1}U$ induce the identity on $\mathscr{I}$, while left and right multiplications by U and U^{-1} commute. In this case the adjoint action of U defined by $\mathrm{Ad}\,U(x) = U^{-1}xU$ is a continuous automorphism of $\mathscr{I}$ preserving the trace on $\mathscr{I}^p$. If $\mathscr{A}$ is any Fréchet m-algebra, the adjoint action of U extends to the tensor product $\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A}$ by acting trivially on the factors $\mathscr{K}$ and $\mathscr{A}$, thus defines an automorphism of $K_n^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A})$. Similarly if $0 \to \mathscr{J} \to \mathscr{R} \to \mathscr{A} \to 0$ is a quasi-free extension, and $(\hat{e},\theta)$ (resp. $(\hat{g},\theta)$) represents a multiplicative

K-theory class of even (resp. odd) degree, the adjoint action of U extends to an automorphism of the pro-algebra $\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}$ and define maps

$$\mathrm{Ad}\, U: \quad (\hat{e},\theta)\mapsto (U^{-1}\hat{e}U,\theta)\ , \quad (\hat{g},\theta)\mapsto (U^{-1}\hat{g}U,\theta)\ . \tag{77}$$

The images represent multiplicative K-theory classes because the invariance of the trace implies $\mathrm{ch}_0^{2q}(U^{-1}\hat{e}U)=\mathrm{ch}_0^{2q}(\hat{e})=\overline{b}\theta$ and $\mathrm{ch}_0^{2q}(U^{-1}\hat{g}U)=\mathrm{ch}_0^{2q}(\hat{g})=\natural\mathbf{d}\theta$. The adjoint action is actually well-defined at the level of K-theory:

Lemma 5.5 *Let U be an invertible multiplier of $\mathscr{I}$. Then the adjoint action $\mathrm{Ad}\, U$ induces the identity on $K_n^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A})$ and $MK_n^{\mathscr{I}}(\mathscr{A})$.*

Proof: First we show that an idempotent $e\in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A})^+$, with $e-p_0\in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A})$, is smoothly homotopic to its adjoint $U^{-1}eU$. Introduce the idempotent $f_0=\left(\begin{smallmatrix} e & 0\\ 0 & p_0\end{smallmatrix}\right)\in M_4(\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A})^+$, and choose a smooth real-valued function $\xi\in C^\infty[0,1]$ such that $\xi(0)=0$ and $\xi(1)=\pi/2$. We define a path W of invertible multipliers of $M_4(\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A})$ by means of the formula

$$W=R^{-1}\begin{pmatrix}1 & 0\\ 0 & U\end{pmatrix}R\ ,\qquad R=\begin{pmatrix}\cos\xi & \sin\xi\\ -\sin\xi & \cos\xi\end{pmatrix}\ ,$$

where each entry should be viewed as a 2×2 block matrix. Hence, W commutes with the matrix $\tilde{p}_0=\left(\begin{smallmatrix} p_0 & 0\\ 0 & p_0\end{smallmatrix}\right)$. The smooth path of idempotents $f=W^{-1}f_0W$ thus provides an interpolation between f_0 and $f_1=\left(\begin{smallmatrix} U^{-1}eU & 0\\ 0 & p_0\end{smallmatrix}\right)$. Put in another way, cfc interpolates the K-theoretic sums $e+p_0$ and $U^{-1}eU+p_0$. This shows that e and $U^{-1}eU$ define the same topological K-theory class.
Now suppose that $(\hat{e},\theta)\in MK_n^{\mathscr{I}}(\mathscr{A})$ represents a multiplicative K-theory class, with $\hat{e}\in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$, $\theta\in X_{n-1}(\mathscr{R},\mathscr{J})$ and $\mathrm{ch}_0^{2q}(\hat{e})=\overline{b}\theta$. We define as before $\hat{f}_0=\left(\begin{smallmatrix} \hat{e} & 0\\ 0 & p_0\end{smallmatrix}\right)$, and $\hat{f}=W^{-1}\hat{f}_0W$ provides an interpolation between $\hat{f}_0$ and $\hat{f}_1=\left(\begin{smallmatrix} U^{-1}\hat{e}U & 0\\ 0 & p_0\end{smallmatrix}\right)$. If $s: C^\infty[0,1]\to\Omega^1[0,1]$ denotes the differential over $[0,1]$ and $\mathbf{d}:\widehat{\mathscr{R}}\to\Omega^1\widehat{\mathscr{R}}$ the noncommutative differential, the Chern-Simons form (71) associated to $c\hat{f}c$ reads

$$\mathrm{cs}_1^{2q}(c\hat{f}c)=\int_0^1 \mathrm{Tr}\natural(-2\hat{f}+1)\sum_{i=0}^{q}(\hat{f}-\tilde{p}_0)^{2i}s\hat{f}(\hat{f}-\tilde{p}_0)^{2(q-i)}\mathbf{d}\hat{f}\ .$$

One has $\mathbf{d}\hat{f}=W^{-1}\mathbf{d}\hat{f}_0W$ and $s\hat{f}=W^{-1}(-sWW^{-1}\hat{f}_0+\hat{f}_0sWW^{-1})W$, hence

$$\begin{aligned}
&\mathrm{Tr}\natural(-2\hat{f}+1)\sum_{i=0}^{q}(\hat{f}-\tilde{p}_0)^{2i}s\hat{f}(\hat{f}-\tilde{p}_0)^{2(q-i)}\mathbf{d}\hat{f}\\
&= \quad -\mathrm{Tr}\natural(-2\hat{f}_0+1)\sum_{i=0}^{q}(\hat{f}_0-\tilde{p}_0)^{2i}sWW^{-1}\hat{f}_0(\hat{f}_0-\tilde{p}_0)^{2(q-i)}\mathbf{d}\hat{f}_0\\
&\qquad +\mathrm{Tr}\natural(-2\hat{f}_0+1)\sum_{i=0}^{q}(\hat{f}_0-\tilde{p}_0)^{2i}\hat{f}_0sWW^{-1}(\hat{f}_0-\tilde{p}_0)^{2(q-i)}\mathbf{d}\hat{f}_0
\end{aligned}$$

Observe that $\mathrm{Tr}\natural$ is a trace. In the first term of the r.h.s., we can use the identity $\hat f_0(\hat f_0-\tilde p_0)^{2(q-i)}=(\hat f_0-\tilde p_0)^{2(q-i)}\hat f_0$ which holds for any two idempotents $\hat f_0$ and $\tilde p_0$, and then $\hat f_0\mathbf{d}\hat f_0(-2\hat f_0+1)=\hat f_0\mathbf{d}\hat f_0$. In the second term of the r.h.s., we simply write $(-2\hat f_0+1)(\hat f_0-\tilde p_0)^{2i}\hat f_0=(-2\hat f_0+1)\hat f_0(\hat f_0-\tilde p_0)^{2i}=-\hat f_0(\hat f_0-\tilde p_0)^{2i}$. Hence we arrive at

$$\begin{aligned}
&\mathrm{Tr}\natural(-2\hat f+1)\sum_{i=0}^{q}(\hat f-\tilde p_0)^{2i}s\hat f(\hat f-\tilde p_0)^{2(q-i)}\mathbf{d}\hat f\\
&= \ -\sum_{i=0}^{q}\mathrm{Tr}\natural\,(\hat f_0-\tilde p_0)^{2i}sWW^{-1}(\hat f_0-\tilde p_0)^{2(q-i)}\hat f_0\mathbf{d}\hat f_0\\
&\qquad -\sum_{i=0}^{q}\mathrm{Tr}\natural\,(\hat f_0-\tilde p_0)^{2i}sWW^{-1}(\hat f_0-\tilde p_0)^{2(q-i)}\mathbf{d}\hat f_0\,\hat f_0\\
&= \ -\sum_{i=0}^{q}\mathrm{Tr}\natural\,sWW^{-1}(\hat f_0-\tilde p_0)^{2i}\mathbf{d}\hat f_0(\hat f_0-\tilde p_0)^{2(q-i)}
\end{aligned}$$

by the idempotent property $\hat f_0\mathbf{d}\hat f_0+\mathbf{d}\hat f_0\,\hat f_0=\mathbf{d}\hat f_0$. It remains to show that the latter sum is a boundary:

$$-\sum_{i=0}^{q}\mathrm{Tr}\natural\,sWW^{-1}(\hat f_0-\tilde p_0)^{2i}\mathbf{d}\hat f_0(\hat f_0-\tilde p_0)^{2(q-i)}=\natural\mathbf{d}\big(\mathrm{Tr}\,sWW^{-1}(\hat f_0-\tilde p_0)^{2q+1}\big)\ .$$

Indeed $\mathbf{d}$ anticommutes with sWW^{-1}, and $\mathbf{d}(\hat f_0-\tilde p_0)=\mathbf{d}\hat f_0$. Hence if we can show that the terms $\mathrm{Tr}\natural\,sWW^{-1}(\hat f_0-\tilde p_0)^{2i+1}\mathbf{d}\hat f_0(\hat f_0-\tilde p_0)^{2j+1}$ vanish, the conclusion follows. Since $\hat f_0\mathbf{d}\hat f_0\hat f_0=0$, we can write

$$\begin{aligned}
&\mathrm{Tr}\natural\,sWW^{-1}(\hat f_0-\tilde p_0)^{2i+1}\mathbf{d}\hat f_0(\hat f_0-\tilde p_0)^{2j+1}\\
&\quad= \ \mathrm{Tr}\natural\,sWW^{-1}(\hat f_0-\tilde p_0)^{2i}(-\hat f_0\mathbf{d}\hat f_0\tilde p_0-\tilde p_0\mathbf{d}\hat f_0\hat f_0+\tilde p_0\mathbf{d}\hat f_0\tilde p_0)(\hat f_0-\tilde p_0)^{2j}\\
&\quad= \ \mathrm{Tr}\natural\,sWW^{-1}(\hat f_0-\tilde p_0)^{2i}(-\hat f_0\mathbf{d}\hat f_0\tilde p_0-\mathbf{d}\hat f_0\hat f_0\tilde p_0+\mathbf{d}\hat f_0\tilde p_0)(\hat f_0-\tilde p_0)^{2j}\\
&\quad= \ 0\ ,
\end{aligned}$$

where we used the fact that $\tilde p_0$ commutes with sWW^{-1} and the even powers of $\hat f_0-\tilde p_0$. Hence $\mathrm{cs}_1^{2q}(c\hat f c)\equiv 0 \bmod \natural\mathbf{d}$, which shows that the pairs $(\hat e,\theta)$ and $(\widehat{U}^{-1}\hat e U,\theta)$ are equivalent. The adjoint action of U on multiplicative K-theory groups in even degrees is thus the identity.

One proceeds in the same fashion with odd groups. Let $g\in(\mathscr{K}\hat\otimes\mathscr{I}\mathscr{A})^+$ be an invertible such that $g-1\in\mathscr{K}\hat\otimes\mathscr{I}\mathscr{A}$. Introduce $u_0=\left(\begin{smallmatrix} g&0\\0&1\end{smallmatrix}\right)$ and the invertible path $u=W^{-1}u_0W$, where $W=R^{-1}\left(\begin{smallmatrix}1&0\\0&U\end{smallmatrix}\right)R$ is now viewed as a path of invertible multipliers of $M_2(\mathscr{K}\hat\otimes\mathscr{I}\mathscr{A})$. Hence u interpolates between u_0 and $u_1=\left(\begin{smallmatrix}U^{-1}gU&0\\0&1\end{smallmatrix}\right)$. This shows that g and $U^{-1}gU$ define the same topological K-theory class.

Now suppose that $(\hat g,\theta)\in MK_n^{\mathscr{I}}(\mathscr{A})$ represents a multiplicative K-theory class, with $\hat{g} \in (\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$, $\theta \in X_{n-1}(\mathscr{R},\mathscr{J})$ and $\mathrm{ch}_1^{2q}(\hat{g}) = \natural \mathbf{d}\theta$. We define $\hat{u}_0 = \left(\begin{smallmatrix} \hat{g} & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and $\hat{u} = W^{-1}\hat{u}_0 W$ provides an interpolation between $\hat{u}_0$ and $\hat{u}_1 = \left(\begin{smallmatrix} U^{-1}\hat{g}U & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The Chern-Simons form (71) associated to $\hat{u}$ reads

$$\mathrm{cs}_0^{2q}(\hat{u}) = \frac{1}{\sqrt{2\pi i}}\frac{(q!)^2}{(2q)!}\int_0^1 \mathrm{Tr}\,\hat{u}^{-1}[(\hat{u}-1)(\hat{u}^{-1}-1)]^q s\hat{u}\ .$$

Using $s\hat{u} = W^{-1}(-sWW^{-1}\hat{u}_0 + \hat{u}_0 sWW^{-1})W$, one gets

$$\begin{aligned} \mathrm{Tr}\,\hat{u}^{-1}[(\hat{u}-1)(\hat{u}^{-1}-1)]^q s\hat{u} &= -\mathrm{Tr}\,\hat{u}_0^{-1}[(\hat{u}_0-1)(\hat{u}_0^{-1}-1)]^q sWW^{-1}\hat{u}_0 \\ &\quad +\mathrm{Tr}\,\hat{u}_0^{-1}[(\hat{u}_0-1)(\hat{u}_0^{-1}-1)]^q \hat{u}_0 sWW^{-1} \\ &\equiv 0 \bmod \overline{b} \end{aligned}$$

Hence $\mathrm{cs}_0^{2q}(\hat{u}) \equiv 0 \bmod \overline{b}$ and the pair $(\widehat{U}^{-1}\hat{g}U, \theta)$ is equivalent to $(\hat{g},\theta)$. The adjoint action of U on the odd multiplicative K-theory groups is therefore the identity. ∎

Example 5.6 Take $\mathscr{A} = \mathbb{C}$ and $\mathscr{I} = \mathscr{L}^p(H)$ a Schatten ideal. It is known that $K_0^{\mathrm{top}}(\mathscr{I}) = \mathbb{Z}$ and $K_1^{\mathrm{top}}(\mathscr{I}) = 0$. Furthermore $HC_n(\mathbb{C}) = \mathbb{C}$ for $n = 2k \geq 0$ and vanishes otherwise. Hence the exact sequence yields

$$MK_n^{\mathscr{I}}(\mathbb{C}) = \begin{cases} \mathbb{Z} & n \leq 0 \text{ even} \\ \mathbb{C}^\times & n > 0 \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

The multiplicative K-theory of $\mathbb{C}$ is the natural target for index maps in even degree, and for regulator maps in odd degree (see [6] and Example 6.4).

Multiplicative K-theory has close connections with higher algebraic K-theory [16, 29]. In fact there exists a morphism $K_n^{\mathrm{alg}}(\mathscr{A}) \to MK_n^{\mathscr{I}}(\mathscr{A})$ in all degrees, and composition with the negative Chern character coincides with the Jones-Goodwillie map $K_n^{\mathrm{alg}}(\mathscr{A}) \to HN_n(\mathscr{A})$ [15]. See [7] for an exact sequence relating topological and algebraic K-theories of locally convex algebras stabilized by operator ideals.

6 Riemann-Roch-Grothendieck theorem

In this section we construct direct images of topological and multiplicative K-theory under quasihomomorphisms and show their compatibility with the K-theory and cyclic homology exact sequences. This provides a noncommutative version of the Riemann-Roch-Grothendieck theorem. If $\mathscr{I}$ is a p-summable Fréchet m-algebra, with trace $\mathrm{Tr} : \mathscr{I}^p \to \mathbb{C}$, the tensor product $\mathscr{I}\hat{\otimes}\mathscr{I}$ is in a natural way a p-sumable algebra with trace $\mathrm{Tr}\hat{\otimes}\mathrm{Tr}$. We demand that $\mathscr{I}$ is provided with an external product as follows.

Definition 6.1 *A p-summable Fréchet m-algebra $\mathscr{I}$ is* multiplicative *if there exists a continuous algebra homomorphism (external product)*

$$\boxtimes : \mathscr{I}\hat{\otimes}\mathscr{I} \to \mathscr{I}$$

such that the composition $\mathrm{Tr}\circ\boxtimes$ *coincides with the trace* $\mathrm{Tr}\hat{\otimes}\mathrm{Tr}$ *on* $(\mathscr{I}\hat{\otimes}\mathscr{I})^p$. *Two external products* $\boxtimes$ *and* $\boxtimes'$ *are* equivalent *if there exists an invertible multiplier U of $\mathscr{I}$ such that* $\boxtimes' = \mathrm{Ad}\,U\circ\boxtimes$ *on* $\mathscr{I}\hat{\otimes}\mathscr{I}$.

Hence if $\mathscr{I}$ is multiplicative the homomorphism $\boxtimes$ induces additive maps $K_n^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{A}) \to K_n^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A})$ and $MK_n^{\mathscr{I}\hat{\otimes}\mathscr{I}}(\mathscr{A}) \to MK_n^{\mathscr{I}}(\mathscr{A})$, clearly compatible with the commutative diagram of Proposition 5.4. Moreover two equivalent products induce the same maps, the adjoint action $\mathrm{Ad}\,U$ being trivial on K-theory by Lemma 5.5. In practice the algebra $\mathscr{I}$ often arises with external products defined only modulo equivalence:

Example 6.2 Let $\mathscr{I} = \mathscr{L}^p(H)$ be the Schatten ideal of p-summable operators on a separable infinite-dimensional Hilbert space H, provided with the operator trace. The tensor product $\mathscr{L}^p(H)\hat{\otimes}\mathscr{L}^p(H)$ is naturally mapped to the algebra $\mathscr{L}^p(H\otimes H)$, and choosing an isomorphism of Hilbert spaces $H\otimes H \cong H$ allows to identify $\mathscr{L}^p(H\otimes H)$ with $\mathscr{L}^p(H)$ modulo the adjoint action of unitary operators $U \in \mathscr{L}(H)$. The product $\boxtimes : \mathscr{L}^p(H)\hat{\otimes}\mathscr{L}^p(H) \to \mathscr{L}^p(H)$ is thus compatible with the traces, and canonically defined modulo the adjoint action of unitary operators.

Let $\mathscr{A}$ and $\mathscr{B}$ be any Fréchet m-algebras. Let $\rho : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s\hat{\otimes}\mathscr{B}$ be a quasi-homomorphism of parity p mod 2, and suppose that $\mathscr{I}$ is finitely summable (the exact summability degree is irrelevant for the moment). We want to show that ρ induces an additive map

$$\rho_! : K_n^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \to K_{n-p}^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{B}) \qquad \forall n\in\mathbb{Z} \tag{78}$$

provided $\mathscr{I}$ is multiplicative. This is has nothing to do with cyclic homology and we don't need to assume $\mathscr{E}$ admissible. Thanks to Bott periodicity, it is sufficient to define $\rho_!$ on $K_1^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A})$, where it is given by very explicit formulas. Suppose first that p is even. Then ρ is described by a pair of homomorphisms $(\rho_+,\rho_-) : \mathscr{A} \rightrightarrows \mathscr{E}$ which coincide modulo $\mathscr{I}\hat{\otimes}\mathscr{B}$. For any invertible element $g \in (\mathscr{K}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{A})^+$ with $g-1 \in \mathscr{K}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{A}$ representing a K-theory class in $K_1^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A})$, one has $\rho_\pm(g) \in (\mathscr{K}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{E})^+$ and $\rho_+(g)-\rho_-(g) \in \mathscr{K}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{B}$, where the homomorphisms ρ_+ and ρ_- are extended to the unitalized algebra $(\mathscr{K}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{A})^+$ by acting trivially on the factor $\mathscr{K}\hat{\otimes}\mathscr{I}$ and preserving the unit. set

$$\rho_!(g) = \rho_+(g)\rho_-(g)^{-1} \ \in (\mathscr{K}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{B})^+ \ . \tag{79}$$

Using the homomorphism $\boxtimes : \mathscr{I}\hat{\otimes}\mathscr{I} \to \mathscr{I}$, we may therefore consider $\rho_!(g)$ as an invertible element of $(\mathscr{K}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{B})^+$ such that $\rho_!(g)-1 \in \mathscr{K}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{B}$. It is

clear that the homotopy class of $\rho_!(g)$ only depends on the homotopy class of g, hence the map $\rho_! : K_1^{\rm top}(\mathscr{I}\hat{\otimes}\mathscr{A}) \to K_1^{\rm top}(\mathscr{I}\hat{\otimes}\mathscr{B})$ is well-defined.
When p is odd, ρ is a homomorphism $\mathscr{A} \to M_2(\mathscr{E})$ such that the off-diagonal terms lie in $\mathscr{I}\hat{\otimes}\mathscr{B}$. For any invertible element $g \in (\mathscr{K}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{A})^+$ as above, one has $\rho(g) \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{E})^+$ with off-diagonal elements in $\mathscr{K}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{B}$. Set

$$\rho_!(g) = \rho(g)^{-1} p_0 \rho(g) \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{B})^+ \ , \tag{80}$$

where $p_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is the trivial matrix idempotent. Again applying the external product $\boxtimes$ we may consider $\rho_!(g)$ as an idempotent of $M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{B})^+$ such that $\rho_!(g) - p_0 \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{B})$. The homotopy class only depends on the homotopy class of g and we thus obtain $\rho_! : K_1^{\rm top}(\mathscr{I}\hat{\otimes}\mathscr{A}) \to K_0^{\rm top}(\mathscr{I}\hat{\otimes}\mathscr{B})$.
To define $\rho_!$ on $K_0^{\rm top}(\mathscr{I}\hat{\otimes}\mathscr{A})$ it suffices to pass to the suspensions $S\mathscr{A} = \mathscr{A}(0,1)$ and $S\mathscr{B} = \mathscr{B}(0,1)$, then apply the pushforward map constructed above $\rho_! : K_1^{\rm top}(\mathscr{I}\hat{\otimes}S\mathscr{A}) \to K_{1-p}^{\rm top}(\mathscr{I}\hat{\otimes}S\mathscr{B})$ with trivial action on the factor $C^\infty(0,1)$. The Bott isomorphisms $K_n^{\rm top}(\mathscr{I}\hat{\otimes}\cdot) \cong K_{n+1}^{\rm top}(\mathscr{I}\hat{\otimes}S\cdot)$ allow to define $\rho_!$ for the original algebras through a *graded*-commutative diagram

$$\begin{tikzcd} K_0^{\rm top}(\mathscr{I}\hat{\otimes}\mathscr{A}) \arrow[r, "\sim"] \arrow[d, "\rho_!"'] & K_1^{\rm top}(\mathscr{I}\hat{\otimes}S\mathscr{A}) \arrow[d, "\rho_!"] \\ K_{-p}^{\rm top}(\mathscr{I}\hat{\otimes}\mathscr{B}) \arrow[r, "\sim"] & K_{1-p}^{\rm top}(\mathscr{I}\hat{\otimes}S\mathscr{B}) \end{tikzcd} \tag{81}$$

Note the following subtlety concerning graduations: since $K_n^{\rm top}$ has parity n mod 2 by definition, the Bott isomorphisms are *odd*. As a consequence, when p is also odd, the above diagram must be *anti*-commutative. These conventions are necessary if we want to avoid sign problems with the theorem below.

Now choose a quasi-free extension $0 \to \mathscr{J} \to \mathscr{R} \to \mathscr{B} \to 0$ for $\mathscr{B}$, and suppose that the algebra $\mathscr{E} \triangleright \mathscr{I}\hat{\otimes}\mathscr{B}$ is $\mathscr{R}$-admissible. We impose the following compatibility between the parity of the quasihomomorphism ρ and the summability degree of $\mathscr{I}$: in the even case $\mathscr{I}$ is $(p+1)$-summable with p even, and in the odd case $\mathscr{I}$ is p-summable with p odd (this complicated choice is dictated by the theorem below). In both cases the bivariant Chern character $\mathrm{ch}^p(\rho) \in HC^p(\mathscr{A},\mathscr{B})$ constructed in section 3 induces a map

$$\mathrm{ch}^p(\rho) : HC_n(\mathscr{A}) \to HC_{n-p}(\mathscr{B}) \qquad \forall n \in \mathbb{Z} \ . \tag{82}$$

Combining $\rho_!$ with the bivariant Chern character yields a transformation in multiplicative K-theory, compatible with the diagram (76) of Proposition 5.4. This will be detailed in the proof of the following noncommutative version of the Riemann-Roch-Grothendieck theorem:

Theorem 6.3 *Let $\mathscr{A}$, $\mathscr{B}$ be Fréchet m-algebras, and choose a quasi-free extension $0 \to \mathscr{J} \to \mathscr{R} \to \mathscr{B} \to 0$. Let $\rho : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s\hat{\otimes}\mathscr{B}$ be a quasihomomorphism of parity p mod 2, where $\mathscr{I}$ is multiplicative and $(p+1)$-summable* *in the even case, p-summable in the odd case. Suppose that $\mathscr{E} \triangleright \mathscr{I}\hat{\otimes}\mathscr{B}$ is $\mathscr{R}$-admissible. Then ρ defines a transformation in multiplicative K-theory $\rho_! : MK_n^{\mathscr{I}}(\mathscr{A}) \to MK_{n-p}^{\mathscr{I}}(\mathscr{B})$ compatible with the K-theory exact sequences for $\mathscr{A}$ and $\mathscr{B}$, whence a graded-commutative diagram*

$$\begin{tikzcd} K_{n+1}^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \arrow[r] \arrow[d, "\rho_!"] & HC_{n-1}(\mathscr{A}) \arrow[r] \arrow[d, "\mathrm{ch}^p(\rho)"] & MK_n^{\mathscr{I}}(\mathscr{A}) \arrow[r] \arrow[d, "\rho_!"] & K_n^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \arrow[d, "\rho_!"] \\ K_{n+1-p}^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{B}) \arrow[r] & HC_{n-1-p}(\mathscr{B}) \arrow[r] & MK_{n-p}^{\mathscr{I}}(\mathscr{B}) \arrow[r] & K_{n-p}^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{B}) \end{tikzcd} \tag{83}$$

The vertical arrows are invariant under conjugation of quasihomomorphisms; the arrow in topological K-theory $K_n^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \to K_{n-p}^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{B})$ is also invariant under homotopy of quasihomomorphisms. Moreover (83) is compatible with the commutative diagram of Theorem 3.13 (with connecting map B rescaled by a factor $-\sqrt{2\pi i}$) after taking the Chern characters $MK_n^{\mathscr{I}} \to HN_n$ and $K_n^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\ .\ ) \to HP_n$.

Proof: As a general rule, the bivariant cyclic cohomomology $HC^p(\mathscr{A},\mathscr{B})$ is described as the cohomology of the complex $\mathrm{Hom}^p(\widehat{X}(T\mathscr{A}, J\mathscr{A}), \widehat{X}(\mathscr{R},\mathscr{J}))$ of linear maps of order $\leq p$, where we choose the universal free extension $0 \to J\mathscr{A} \to T\mathscr{A} \to \mathscr{A} \to 0$ for $\mathscr{A}$ and the quasi-free extension $0 \to \mathscr{J} \to \mathscr{R} \to \mathscr{B} \to 0$ for $\mathscr{B}$. By hypothesis, the algebra $\mathscr{E} \triangleright \mathscr{I}\hat{\otimes}\mathscr{B}$ is $\mathscr{R}$-admissible (Definition 3.2), i.e. one has a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{N} \arrow[r] & \mathscr{M} \arrow[r] & \mathscr{E} \arrow[r] & 0 \\ 0 \arrow[r] & \mathscr{I}\hat{\otimes}\mathscr{J} \arrow[r] \arrow[u] & \mathscr{I}\hat{\otimes}\mathscr{R} \arrow[r] \arrow[u] & \mathscr{I}\hat{\otimes}\mathscr{B} \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

verifying adequate properties with respect to the trace over $\mathscr{I}$. The detailed construction of the pushforward map in multiplicative K-theory $\rho_! : MK_n^{\mathscr{I}}(\mathscr{A}) \to MK_{n-p}^{\mathscr{I}}(\mathscr{B})$ depends on the respective parities of n and p.

I) $n = 2k+1$ IS ODD AND $p = 2q$ IS EVEN. Our first task is to understand the composition of the topological Chern character $\mathrm{ch}_1^p : K_1^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \to HP_1(\mathscr{A})$ with the bivariant Chern character $\mathrm{ch}^p(\rho) \in HC^p(\mathscr{A},\mathscr{B})$. For notational simplicity, we write as usual $\mathscr{I}\mathscr{A}$ for the tensor product $\mathscr{I}\hat{\otimes}\mathscr{A}$. Without loss of generality, we may suppose that an element of $K_1^{\mathrm{top}}(\mathscr{I}\mathscr{A})$ is represented by an invertible $g \in (\mathscr{I}\mathscr{A})^+$ such that $g - 1 \in \mathscr{I}\mathscr{A}$ (indeed the algebra $\mathscr{K}$ of smooth compact operators plays a trivial role in what follows). Since the universal free extension $T\mathscr{A}$ is chosen, we can take the *canonical lift* $\hat{g} \in (\mathscr{I}\widehat{T}\mathscr{A})^+$ which corresponds to the image of g under the canonical linear inclusion $\mathscr{A} \hookrightarrow \widehat{\Omega}^+\mathscr{A} \cong \widehat{T}\mathscr{A}$ as the the subspace of zero-forms. Its inverse is given by the series (57), with $\mathscr{K}$ replaced by $\mathscr{I}$. The p-th higher Chern character of $\hat g$ is then represented by the cycle (70)

$$\mathrm{ch}^p_1(\hat g)=\frac{1}{\sqrt{2\pi i}}\,\frac{(q!)^2}{p!}\,\mathrm{Tr}\natural\hat g^{-1}[(\hat g-1)(\hat g^{-1}-1)]^q\mathbf{d}\hat g\ \in\Omega^1\widehat T\mathscr{A}_\natural\ ,$$

Observe that $(p+1)$ powers of $\mathscr{I}$ appear in the products of $(\hat g-1)$, $(\hat g^{-1}-1)$ and $\mathbf{d}\hat g$ hence the trace $\mathrm{Tr}:\mathscr{I}^{p+1}\to\mathbb{C}$ is well-defined. On the other hand, the bivariant Chern character of the quasihomomorphism ρ (section 3) is represented by the composition of chain maps $\mathrm{ch}^p(\rho)=\widehat\chi^p\rho_*\gamma: X(\widehat T\mathscr{A})\to\widehat\Omega\widehat T\mathscr{A}\to\widehat\Omega\widehat{\mathscr{M}^s_+}\to X(\widehat{\mathscr{R}})$, hence the composite $\mathrm{ch}^p(\rho)\cdot\mathrm{ch}^p_1(\hat g)$ requires to compute first the image of $\mathrm{ch}^p_1(\hat g)$ under the Goodwillie equivalence $\gamma: X(\widehat T\mathscr{A})\to\widehat\Omega\widehat T\mathscr{A}$. This tricky computation can be simplified as follows. We use the isomorphism $\mathscr{I}\mathscr{A}\cong\mathbb{C}\hat\otimes\mathscr{I}\mathscr{A}$ to identify $\hat g$ with the invertible element

$$\hat u=1+e\otimes(\hat g-1)\ \in(\mathbb{C}\hat\otimes\mathscr{I}\mathscr{A})^+\hookrightarrow(\widehat T\mathbb{C}\hat\otimes\mathscr{I}\widehat T\mathscr{A})^+\ ,$$

where e is the unit of $\mathbb{C}$. As usual we regard $\mathbb{C}\hat\otimes\mathscr{I}\mathscr{A}$ as the subspace of zero-forms of the algebra $\widehat T\mathbb{C}\hat\otimes\mathscr{I}\widehat T\mathscr{A}\cong\widehat\Omega^+\mathbb{C}\hat\otimes\mathscr{I}\widehat\Omega^+\mathscr{A}$. It is not hard to calculate that the inverse of $\hat u$ is given by the series

$$\hat u^{-1}=\sum_{i\geq 0}\big((dede)^i\otimes[(\hat g^{-1}-1)(\hat g-1)]^i+e(dede)^i\otimes[(\hat g^{-1}-1)(\hat g-1)]^i(\hat g^{-1}-1)\big)$$

with the convention $(dede)^0=1$. Observe that the power of $\mathscr{I}$ is equal to the power of e in each term of this series. Also, recall that the canonical lift of e is the idempotent

$$\hat e=e+\sum_{i\geq 1}\frac{(2i)!}{(i!)^2}(e-\frac12)(dede)^i\ \in\widehat T\mathbb{C}\ .$$

We define the *fundamental class* of degree $p=2q$ as the trace $[2q]:\widehat T\mathbb{C}\to\mathbb{C}$ vanishing on all the differential forms $e(dede)^i$ and $(dede)^i$ except $e(dede)^q$, and normalized so that $[2q]\,\hat e=1$. One thus have

$$[2q]\,e(dede)^q=\frac{(q!)^2}{p!}\ ,\qquad [2q]\,(\text{anything else})=0\ .$$

Of course, $[2q]$ is the generator in degree p of the cyclic cohomology of $\mathbb{C}$. The fact that it is a trace over $\widehat T\mathbb{C}$ is crucial. Indeed, one finds the identity

$$\mathrm{Tr}\natural[2q]\,\hat u^{-1}\mathbf{d}\hat u=\frac{(q!)^2}{p!}\,\mathrm{Tr}\natural\hat g^{-1}[(\hat g-1)(\hat g^{-1}-1)]^q\mathbf{d}\hat g\ \in\Omega^1\widehat T\mathscr{A}_\natural\ ,$$

so that the Chern character $\mathrm{ch}^p_1(\hat g)$ is exactly the cycle $\frac{1}{\sqrt{2\pi i}}\mathrm{Tr}\natural[2q]\,\hat u^{-1}\mathbf{d}\hat u$. This simplifies drastically the computation of $\gamma\mathrm{ch}^p_1(\hat g)$. The Goodwillie equivalence γ is explicitly constructed in section 2; it is based on the linear map $\phi:\widehat T\mathscr{A}\to$

$\Omega^2\widehat{T}\mathscr{A}$ verifying the properties $\phi(xy) = \phi(x)y + x\phi(y) + \mathbf{d}x\mathbf{d}y$ for all $x, y \in \widehat{T}\mathscr{A}$, and $\phi(a) = 0$ whenever $a \in \mathscr{A}$. We extend ϕ to a linear map

$$\phi : (\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{T}\mathscr{A})^+ \to \widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\Omega^2\widehat{T}\mathscr{A}$$

acting by the identity on the factor $\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}$ and setting $\phi(1) = 0$. This implies $\phi(\hat{u}\hat{u}^{-1}) = 0 = \phi(\hat{u})\hat{u}^{-1} + \hat{u}\phi(\hat{u}^{-1}) + \mathbf{d}\hat{u}\mathbf{d}\hat{u}^{-1}$. Moreover $\hat{u}$ lies in $(\mathbb{C}\hat{\otimes}\mathscr{I}\mathscr{A})^+$ so that $\phi(\hat{u}) = 0$, and one gets

$$\phi(\hat{u}^{-1}) = -\hat{u}^{-1}\mathbf{d}\hat{u}\mathbf{d}\hat{u}^{-1} \ .$$

Now, extending ϕ in all degrees as in section 2 one gets a linear map $\phi : \widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\Omega^i\widehat{T}\mathscr{A} \to \widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\Omega^{i+2}\widehat{T}\mathscr{A}$ for any $i \in \mathbb{N}$. The following computation is then straightforward (remark that $\mathrm{Tr}[2q]$ is a trace hence cyclic permutations are allowed; moreover the fundamental class $[2q]$ selects $(p+1)$ powers of e, hence of $\mathscr{I}$, so that Tr is well-defined):

$$\gamma(\mathrm{Tr}\natural[2q]\,\hat{u}^{-1}\mathbf{d}\hat{u}) = \sum_{i\geq 0}\mathrm{Tr}[2q]\,\phi^i(\hat{u}^{-1}\mathbf{d}\hat{u}) = \sum_{i\geq 0}(-)^i i!\,\mathrm{Tr}[2q]\,\hat{u}^{-1}\mathbf{d}\hat{u}(\mathbf{d}\hat{u}^{-1}\mathbf{d}\hat{u})^i \ .$$

Hence $\gamma\mathrm{ch}^p_1(\hat{g})$ is equal to this $(b+B)$-cycle over $\widehat{T}\mathscr{A}$, divided by a factor $\sqrt{2\pi i}$. It remains to apply the chain map $\chi^p\rho_* : \widehat{\Omega}\widehat{T}\mathscr{A} \to X(\widehat{\mathscr{R}})$ associated to the quasihomomorphism $\rho : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s\mathscr{B}$. In 2×2 matrix notation, the image of any $x \in \widehat{T}\mathscr{A}$ under the lifted quasihomomorphism $\rho_* : \widehat{T}\mathscr{A} \to \widehat{\mathscr{M}^s} \triangleright \mathscr{I}^s\widehat{\mathscr{R}}$ and the odd multiplier F read

$$\rho_* x = \begin{pmatrix} x_+ & 0 \\ 0 & x_- \end{pmatrix} \in \widehat{\mathscr{M}^s_+} \ , \qquad F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ ,$$

and the difference $x_+ - x_-$ is therefore an element of the pro-algebra $\mathscr{I}\widehat{\mathscr{R}}$. On the other hand, the odd component of the chain map $\widehat{\chi}^p\rho_*$ evaluated on a $(p+1)$-form $x_0\mathbf{d}x_1\ldots\mathbf{d}x_{p+1}$ is given by Eqs. (34):

$$\frac{q!}{(p+1)!}\sum_{i=1}^{p+1}\tau'\natural(\rho_*x_0[F,\rho_*x_1]\ldots\mathbf{d}(\rho_*x_i)\ldots[F,\rho_*x_{p+1}])$$

where $\tau' = \frac{1}{2}\tau(F[F,\ ])$ is the modified supertrace of even degree. Then we extend canonically ρ_* to a unital homomorphism $(\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{T}\mathscr{A})^+ \to (\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{\mathscr{M}^s_+})^+$ by taking the identity on the factor $\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}$. One thus has $\rho_*\hat{u} = \left(\begin{smallmatrix} \hat{u}_+ & 0 \\ 0 & \hat{u}_- \end{smallmatrix}\right)$ with $\hat{u}_+ - \hat{u}_- \in \widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\mathscr{I}\widehat{\mathscr{R}}$. A direct computation gives

$$\begin{aligned}\mathrm{ch}^p(\rho)\,(\mathrm{Tr}\natural[2q]\,\hat{u}^{-1}\mathbf{d}\hat{u}) &= (-)^q q!\,\widehat{\chi}^p_1\rho_*\,\mathrm{Tr}[2q](\hat{u}^{-1}\mathbf{d}\hat{u}(\mathbf{d}\hat{u}^{-1}\mathbf{d}\hat{u})^q) \\ &= \frac{(q!)^2}{p!}\,\mathrm{Tr}\natural[2q]\,\widetilde{u}^{-1}[(\widetilde{u}-1)(\widetilde{u}^{-1}-1)]^q\mathbf{d}\widetilde{u} \ ,\end{aligned}$$

where $\widetilde{u} = \hat{u}_+\hat{u}_-^{-1} \in (\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\mathscr{I}\widehat{\mathscr{R}})^+$ may be considered as an invertible element of the pro-algebra $(\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$ after applying the homomorphism $\boxtimes : \mathscr{I}\hat{\otimes}\mathscr{I} \to \mathscr{I}$. Dividing by a factor $\sqrt{2\pi i}$, the right-hand-side should be defined as the Chern character $\mathrm{ch}_1^p(\widetilde{u})$, cf. (70). One thus gets the identity

$$\mathrm{ch}^p(\rho)\cdot \mathrm{ch}_1^p(\hat{g}) = \mathrm{ch}_1^p(\widetilde{u})$$

at the level of cycles in $X(\widehat{\mathscr{R}})$. Now, observe that the projection of $\widetilde{u}$ onto the quotient algebra $(\mathbb{C}\hat{\otimes}\mathscr{I}\mathscr{B})^+$ is $u_+u_-^{-1} = 1 + e\hat{\otimes}(\rho_+(g)\rho_-(g)^{-1} - 1)$. It corresponds to the direct image $\rho_+(g)\rho_-(g)^{-1} = \rho_!(g)$ by virtue of the isomorphism $(\mathbb{C}\hat{\otimes}\mathscr{I}\mathscr{B})^+ \cong (\mathscr{I}\mathscr{B})^+$. Hence we expect that $\mathrm{ch}_1^p(\widetilde{u})$ is homologous to the Chern character of any invertible lift $\widehat{\rho_!(g)} \in (\mathscr{I}\widehat{\mathscr{R}})^+$. To see this, consider an invertible path $\hat{v} \in (\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1])^+$ connecting homotopically $\hat{v}_0 = \widetilde{u}$ to $\hat{v}_1 = 1+\hat{e}\otimes(\widehat{\rho_!(g)}-1)$, and such that its projection onto $(\mathbb{C}\hat{\otimes}\mathscr{I}\mathscr{B}[0,1])^+$ is the constant invertible function $1 + e\otimes(\rho_!(g) - 1)$ over $[0, 1]$. Such a path always exists, for example the linear interpolation

$$\hat{v}_t = t\big(1 + \hat{e}\otimes(\widehat{\rho_!(g)} - 1)\big) + (1-t)\widetilde{u}\ , \qquad t\in[0,1] \tag{84}$$

works. Since the evaluation of the fundamental class $[2q]$ on the canonical idempotent lift $\hat{e}$ is the unit, a little computation shows the equality

$$\mathrm{ch}_1^p(\hat{v}_1) = \mathrm{ch}_1^p\big(\widehat{\rho_!(g)}\big) \ \in \Omega^1\widehat{\mathscr{R}}_\natural$$

at the level of cycles. Moreover, the Chern-Simons form associated to $\hat{v}$, defined in analogy with formulas (71)

$$\mathrm{cs}_0^p(\hat{v}) = \frac{1}{\sqrt{2\pi i}}\frac{(q!)^2}{p!}\int_0^1 dt\, \mathrm{Tr}[2q]\, \hat{v}^{-1}[(\hat{v}-1)(\hat{v}^{-1}-1)]^q \frac{\partial \hat{v}}{\partial t}$$

fulfills the transgression relation in the complex $X(\widehat{\mathscr{R}})$

$$\natural\mathbf{d}\mathrm{cs}_0^p(\hat{v}) = \mathrm{ch}_1^p(\hat{v}_1) - \mathrm{ch}_1^p(\hat{v}_0) = \mathrm{ch}_1^p(\widehat{\rho_!(g)}) - \mathrm{ch}_1^p(\widetilde{u})$$

as wanted. We are now in a position to define the map $\rho_!$ on multiplicative K-theory. Let a pair $(\hat{g},\theta)$ represent a class in $MK_n^{\mathscr{I}}(\mathscr{A})$ of odd degree $n = 2k+1$. From Remark 5.3, we know that $\hat{g} \in (\mathscr{I}\widehat{T}\mathscr{A})^+$ can be taken as the *canonical lift* of some invertible element $g \in (\mathscr{I}\mathscr{A})^+$. Then, the transgression θ is a chain of even degree in the quotient complex $X_{n-1}(T\mathscr{A}, J\mathscr{A})$, and the relation $\mathrm{ch}_1^p(\hat{g}) = \natural\mathbf{d}\theta$ holds in $X_{n-1}(T\mathscr{A}, J\mathscr{A})$. We set

$$\boxed{\rho_!(\hat{g},\theta) = \big(\widehat{\rho_!(g)}\ ,\ \mathrm{ch}^p(\rho)\cdot\theta + \mathrm{cs}_0^p(\hat{v})\big) \ \in MK_{n-p}^{\mathscr{I}}(\mathscr{B})} \tag{85}$$

where $\widehat{\rho_!(g)} \in (\mathscr{I}\widehat{\mathscr{R}})^+$ is any invertible lift of $\rho_!(g)$ and $\hat{v}$ is an invertible path constructed as above. Let us explain why this defines a multiplicative K-theory class. First, the bivariant Chern character $\mathrm{ch}^p(\rho) \in HC^p(\mathscr{A},\mathscr{B})$ induces a morphism of quotient complexes $X_{n-1}(T\mathscr{A},J\mathscr{A}) \to X_{n-p-1}(\mathscr{R},\mathscr{J})$, hence $\mathrm{ch}^p(\rho)\cdot\theta$ is a well-defined chain of even degree in $X_{n-p-1}(\mathscr{R},\mathscr{J})$. Regarding also $\mathrm{cs}_0^p(\hat{v})$ as an element of $X_{n-p-1}(\mathscr{R},\mathscr{J})$, we see that the relation

$$\natural\mathbf{d}(\mathrm{ch}^p(\rho)\cdot\theta + \mathrm{cs}_0^p(\hat{v})) = \mathrm{ch}^p(\rho)\cdot\mathrm{ch}_1^p(\hat{g}) + \mathrm{ch}_1^p\big(\widehat{\rho_!(g)}\big) - \mathrm{ch}_1^p(\widetilde{u}) = \mathrm{ch}_1^p\big(\widehat{\rho_!(g)}\big)$$

holds in this quotient complex, hence $\rho_!(\hat{g},\theta)$ represents a class in $MK^{\mathscr{J}}_{n-p}(\mathscr{B})$. In fact, the latter does not depend on the choice of lifting $\widehat{\rho_!(g)}$, nor on the invertible path $\hat{v}$. This can be proved simultaneously with the fact that the equivalence class of $\rho_!(\hat{g},\theta)$ depends only on the equivalence class of $(\hat{g},\theta)$. To show this, consider two equivalent pairs $(\hat{g}_0,\theta_0)$ and $(\hat{g}_1,\theta_1)$ representing the same element of $MK^{\mathscr{I}}_n(\mathscr{A})$. It means there exists a homotopy $\hat{g}\in(\mathscr{I}\widehat{T}\mathscr{A}[0,1]_x)^+$ between $\hat{g}_0$ and $\hat{g}_1$ (we denote by x the variable of this homotopy, which should not be confused with the variable t used in the definition of the interpolation (84)), and a chain $\lambda\in X_{n-1}(T\mathscr{A},J\mathscr{A})$ such that $\theta_1-\theta_0=\mathrm{cs}_0^p(\hat{g})+\overline{b}\lambda$. From Remark 5.3, we may suppose that $\hat{g}_0$, $\hat{g}_1$ and $\hat{g}$ are respectively the canonical lifts of invertibles $g_0,g_1\in(\mathscr{I}\mathscr{A})^+$ and $g\in(\mathscr{I}\mathscr{A}[0,1]_x)^+$. By definition one has

$$\rho_!(\hat{g}_i,\theta_i) = \big(\widehat{\rho_!(g_i)}\ ,\ \mathrm{ch}^p(\rho)\cdot\theta_i + \mathrm{cs}_0^p(\hat{v}(g_i))\big)\ ,\quad i=0,1\ ,$$

where $\hat{v}(g_i)\in(\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1]_t)^+$ is a choice of invertible path associated to g_i, for example by Eq. (84). Choose an invertible path $\widehat{\rho_!(g)}\in(\mathscr{I}\widehat{\mathscr{R}}[0,1]_x)^+$ interpolating $\widehat{\rho_!(g_0)}$ and $\widehat{\rho_!(g_1)}$: it can be chosen as a lift of the path $\rho_!(g)=\rho_+(g)\rho_-(g)^{-1}$. Our goal is to show that the relation

$$\mathrm{ch}^p(\rho)\cdot(\theta_1-\theta_0)+\mathrm{cs}_0^p(\hat{v}(g_1))-\mathrm{cs}_0^p(\hat{v}(g_0)) \equiv \mathrm{cs}_0^p(\widehat{\rho_!(g)}) \bmod \overline{b}$$

holds in $X_{n-p-1}(\mathscr{R},\mathscr{J})$. As before we identify the canonical lift $\hat{g}$ of g with the invertible element

$$\hat{u} = 1+e\otimes(\hat{g}-1)\ \in(\mathbb{C}\hat{\otimes}\mathscr{I}\mathscr{A}[0,1]_x)^+ \hookrightarrow (\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{T}\mathscr{A}[0,1]_x)^+\ ,$$

and we remark that the higher Chern-Simons form $\mathrm{cs}_0^p(\hat{g})$ given by Lemma 4.5, Eqs. (71), can be written as

$$\mathrm{cs}_0^p(\hat{g}) = \frac{1}{\sqrt{2\pi i}}\int_0^1 \mathrm{Tr}[2q]\,\hat{u}^{-1}s\hat{u}$$

where $s=dx\frac{\partial}{\partial x}$ is the de Rham coboundary acting on the space of differential forms $\Omega[0,1]_x$. It follows that the computation of $\mathrm{ch}^p(\rho)\cdot\mathrm{cs}_0^p(\hat{g})$ requires first to evaluate the Goodwillie equivalence γ on the one-form $\hat{\omega}=\hat{u}^{-1}s\hat{u}$. To this end, we extend ϕ to a linear map

$$\phi:\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\Omega^i\widehat{T}\mathscr{A}\hat{\otimes}\Omega[0,1]_x\to\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\Omega^{i+2}\widehat{T}\mathscr{A}\hat{\otimes}\Omega[0,1]_x$$

acting by the identity on the factors $\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}$ and $\Omega[0,1]_x$. The algebraic property of ϕ implies $\phi(\hat{u}^{-1}s\hat{u}) = \phi(\hat{u}^{-1})s\hat{u} + \hat{u}^{-1}s\phi(\hat{u}) + \mathbf{d}\hat{u}^{-1}\mathbf{d}(s\hat{u})$. From $\phi(\hat{u}) = 0$, $\phi(\hat{u}^{-1}) = -\hat{u}^{-1}\mathbf{d}\hat{u}\mathbf{d}\hat{u}^{-1}$ and $\mathbf{d}\hat{u}^{-1} = -\hat{u}^{-1}\mathbf{d}\hat{u}\hat{u}^{-1}$ we deduce

$$\phi(\hat{\omega}) = -\hat{u}^{-1}\mathbf{d}\hat{u}\mathbf{d}\hat{\omega}\ .$$

Then the image of $[2q]\,\hat{\omega}$ under the Goodwillie equivalence is a straightforward computation, taking into account the tracial property of the fundamental class $\mathrm{Tr}[2q]$ and the fact that $\Omega[0,1]_x$ is a commutative algebra:

$$\gamma(\mathrm{Tr}[2q]\,\hat{\omega}) = \sum_{i\geq 0}\mathrm{Tr}[2q]\phi^i(\hat{\omega}) =$$

$$\mathrm{Tr}[2q]\,\hat{\omega} + \sum_{i\geq 1}(-)^i(i-1)!\sum_{j=0}^{i-1}\mathrm{Tr}[2q]\,\hat{u}^{-1}\mathbf{d}\hat{u}(\mathbf{d}\hat{u}^{-1}\mathbf{d}\hat{u})^j\mathbf{d}\hat{\omega}(\mathbf{d}\hat{u}^{-1}\mathbf{d}\hat{u})^{i-j-1}\ .$$

This is a chain in the bicomplex $\widehat{\Omega}\widehat{T}\mathscr{A}\hat{\otimes}\Omega[0,1]_x$ endowed with the boundary maps $(b+B)$ and s. Its is related to the $(b+B)$-cocycle $\gamma(\mathrm{Tr}\natural[2q]\hat{u}^{-1}\mathbf{d}\hat{u})$ via the descent equation

$$(b+B)\gamma(\mathrm{Tr}[2q]\hat{\omega}) + s\gamma(\mathrm{Tr}\natural[2q]\hat{u}^{-1}\mathbf{d}\hat{u}) = 0\ ,$$

which can be shown either by direct computation, or simply by observing that $\natural\mathbf{d}(\mathrm{Tr}[2q]\omega) + s(\mathrm{Tr}\natural[2q]\hat{u}^{-1}\mathbf{d}\hat{u}) = 0$ and $\gamma\natural\mathbf{d} = (b+B)\gamma$, $\gamma s = s\gamma$. Next we have to evaluate the image of $\gamma(\mathrm{Tr}[2q]\,\hat{\omega})$ by the chain map $\widehat{\chi}^p\rho_* : \widehat{\Omega}\widehat{T}\mathscr{A} \to X(\widehat{\mathscr{R}})$, whose even component evaluated on a p-form $x_0\mathbf{d}x_1\ldots\mathbf{d}x_p$ over $\widehat{T}\mathscr{A}$ reads

$$\frac{q!}{(p+1)!}\sum_{\lambda\in S_{p+1}}\varepsilon(\lambda)\tau'(\rho_*x_{\lambda(0)}[F,\rho_*x_{\lambda(1)}]\ldots[F,\rho_*x_{\lambda(p)}])\ .$$

Denote as before $\widetilde{u} = \hat{u}_+\hat{u}_-^{-1} \in (\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1]_x)^+$, and $\widetilde{\omega} = \widetilde{u}^{-1}s\widetilde{u}$. One finds:

$$\mathrm{ch}^p(\rho)\,(\mathrm{Tr}[2q]\,\hat{\omega}) = \frac{(q!)^2}{(p+1)!}\,\mathrm{Tr}[2q]\,\Big(\hat{u}_-^{-1}\widetilde{\omega}[(\widetilde{u}^{-1}-1)(\widetilde{u}-1)]^q\hat{u}_- +$$
$$(\widetilde{u}-1)\widetilde{\omega}[(\widetilde{u}^{-1}-1)(\widetilde{u}-1)]^{q-1}(\widetilde{u}^{-1}-1) + \ldots + \hat{u}_-^{-1}[(\widetilde{u}^{-1}-1)(\widetilde{u}-1)]^q\widetilde{\omega}\hat{u}_-\Big)$$

After evaluation on the current $\frac{1}{\sqrt{2\pi i}}\int_{x=0}^1$, the right-hand-side may be identified, modulo commutators, with the Chern-Simons form $\mathrm{cs}_0^p(\widetilde{u})$ defined in analogy with (71). One thus gets

$$\mathrm{ch}^p(\rho)\cdot\mathrm{cs}_0^p(\hat{g}) \equiv \mathrm{cs}_0^p(\widetilde{u}) \bmod \overline{b}\ .$$

Now, introduce a parameter $t \in [0,1]$ and choose an invertible interpolation $\hat{v} \in (\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1]_x[0,1]_t)^+$ between $\hat{v}_{t=0} = \widetilde{u}$ and $\hat{v}_{t=1} = 1 + \hat{e}\otimes(\widehat{\rho_!(g)}-1)$, with the property that it restricts to $\hat{v}(g_0)$ for $x=0$ and to $\hat{v}(g_1)$ for $x=1$. The projection of $\hat{v}$ on the algebra $(\mathbb{C}\hat{\otimes}\mathscr{I}\mathscr{B}[0,1]_x[0,1]_t)^+$ may be chosen constant with respect to t. In the proof of Lemma 4.6 we established at any point $(x,t)\in[0,1]^2$ the identity

$$\frac{\partial}{\partial t}\big(\mathrm{Tr}[2q]\,\hat{v}^{-1}[(\hat{v}-1)(\hat{v}^{-1}-1)]^q s\hat{v}\big)\equiv s\big(\mathrm{Tr}[2q]\,\hat{v}^{-1}[(\hat{v}-1)(\hat{v}^{-1}-1)]^q\frac{\partial\hat{v}}{\partial t}\big)\bmod \overline{b}\,,$$

and integrating over $[0,1]^2$ this implies

$$\mathrm{cs}_0^p(\hat{v}_{t=1})-\mathrm{cs}_0^p(\hat{v}_{t=0})\equiv\mathrm{cs}_0^p(\hat{v}_{x=1})-\mathrm{cs}_0^p(\hat{v}_{x=0})\bmod\overline{b}\ .$$

Taking into account that $\hat{v}_{x=0}=\hat{v}(g_0)$ and $\hat{v}_{x=1}=\hat{v}(g_1)$, we calculate mod $\overline{b}$ in the complex $X_{n-p-1}(\mathscr{R},\mathscr{J})$

$$\begin{aligned}\mathrm{ch}^p(\rho)\cdot(\theta_1-\theta_0)&+\mathrm{cs}_0^p(\hat{v}(g_1))-\mathrm{cs}_0^p(\hat{v}(g_0))\\ &\equiv\mathrm{ch}^p(\rho)\cdot\mathrm{cs}_0^p(\hat{g})+\mathrm{cs}_0^p(\hat{v}_{t=1})-\mathrm{cs}_0^p(\hat{v}_{t=0})\bmod\overline{b}\\ &\equiv\mathrm{cs}_0^p(\widetilde{u})+\mathrm{cs}_0^p(1+\hat{e}\otimes(\widehat{\rho_!(g)}-1))-\mathrm{cs}_0^p(\widetilde{u})\bmod\overline{b}\\ &\equiv\mathrm{cs}_0^p(\widehat{\rho_!(g)})\bmod\overline{b}\ .\end{aligned}$$

Hence the direct images $\rho_!(\hat{g}_0,\theta_0)$ and $\rho_!(\hat{g}_1,\theta_1)$ are equivalent and the map $\rho_!:MK_n^{\mathscr{I}}(\mathscr{A})\to MK_{n-p}^{\mathscr{I}}(\mathscr{B})$ for $n=2k+1$ and $p=2q$ is well-defined. It is obviously compatible with the push-forward map in topological K-theory $\rho_!:K_1^{\mathrm{top}}(\mathscr{I}\mathscr{A})\to K_1^{\mathrm{top}}(\mathscr{I}\mathscr{B})$. The compatibility with the push-forward map in cyclic homology $\mathrm{ch}^p(\rho):HC_{n-1}(\mathscr{A})\to HC_{n-p-1}(\mathscr{B})$ is clear once we remark that the Chern-Simons form $\mathrm{cs}_0^{2q}(\hat{v})$ vanish whenever $\hat{v}=1$. Hence the diagram (83) is commutative.

We have to check the invariance of $\rho_!$ with respect to conjugation of quasihomomorphisms. Let ρ_0 and ρ_1 be two conjugate quasihomomorphisms $\mathscr{A}\to\mathscr{E}^s\triangleright\mathscr{I}^s\mathscr{B}$. Hence there exists an invertible element $U\in(\mathscr{E}^s_+)^+$ such that $\rho_1=U^{-1}\rho_0U$. We follow the proof of Proposition 3.12 and remark that the lifting homomorphisms $\rho_{0*},\rho_{1*}:\widehat{T}\mathscr{A}\to\mathscr{M}^s_+$ factor through homomorphisms $\varphi_0,\varphi_1:\widehat{T}\mathscr{A}\to\widehat{T}\mathscr{E}^s_+$. The maps $\rho_{0!},\rho_{1!}:MK_n^{\mathscr{I}}(\mathscr{A})\to MK_{n-p}^{\mathscr{I}}(\mathscr{B})$ are obtained by composition of the pushforward maps $\varphi_{0!},\varphi_{1!}:MK_n^{\mathscr{I}}(\mathscr{A})\to MK_n^{\mathscr{I}}(\mathscr{E}^s_+)$ induced by the homomorphisms

$$\varphi_{i!}(\hat{g},\theta)=(\varphi_i(\hat{g}),\varphi_i(\theta))$$

with the map $MK_n^{\mathscr{I}}(\mathscr{E}^s_+)\to MK_{n-p}^{\mathscr{I}}(\mathscr{B})$ associated with the natural $(p+1)$-summable quasihomomorphism of even degree $\mathscr{E}^s_+\to\mathscr{E}^s\triangleright\mathscr{I}^s\mathscr{B}$. Hence it is sufficient to check that the maps $\varphi_{0!}$ and $\varphi_{1!}:MK_n^{\mathscr{I}}(\mathscr{A})\to MK_n^{\mathscr{I}}(\mathscr{E}^s_+)$ coincide. From the proof of 3.12 we know that φ_1 is smoothly homotopic to $\widehat{U}^{-1}\varphi_0\widehat{U}$, where $\widehat{U}\in(\widehat{T}\mathscr{E}^s_+)^+$ is a lifting of U, and the interpolating homomorphism $\varphi:\widehat{T}\mathscr{A}\to\widehat{T}\mathscr{E}^s_+[0,1]$ is constant modulo the ideal $\widehat{J}\mathscr{E}^s_+$. Consequently the morphisms $X(\varphi_1)$ and $X(\widehat{U}^{-1}\varphi_0\widehat{U}):X(\widehat{T}\mathscr{A})\to X(\widehat{T}\mathscr{E}^s_+)$ are homotopic,

$$X(\varphi_1)-X(\widehat{U}^{-1}\varphi_0\widehat{U})=[\partial,H]$$

with $H \in \mathrm{Hom}^0(X(T\mathscr{A}, J\mathscr{A}), X(T\mathscr{E}^s_+, J\mathscr{E}^s_+))$ a cochain *of order zero*. Let us now compare the images of $(\hat{g}, \theta) \in MK^{\mathscr{I}}_n(\mathscr{A})$ under the pushworwards $\varphi_{1!}$ and $(\widehat{U}^{-1}\varphi_0\widehat{U})_!$. The images $\varphi_1(\hat{g})$ and $\widehat{U}^{-1}\varphi_0(\hat{g})\widehat{U}$ are smoothly homotopic, with interpolation $\varphi(\hat{g}) \in (\mathscr{I}\widehat{T}\mathscr{E}^s_+[0,1])^+$. If moreover $\hat{g}$ is the canonical lift of an invertible element $g \in (\mathscr{I}\mathscr{A})^+$, Chern-Simons form associated to $\varphi(\hat{g})$ can be written as

$$\begin{aligned}
\mathrm{cs}^p_0(\varphi(\hat{g})) &= \frac{1}{\sqrt{2\pi i}}\frac{(q!)^2}{p!}\int_0^1 \mathrm{Tr}\,\varphi(\hat{g}^{-1})\varphi[(\hat{g}-1)(\hat{g}^{-1}-1)]^q s\varphi(\hat{g}) \\
&= \frac{1}{\sqrt{2\pi i}}\int_0^1 \mathrm{Tr}[2q]\,\varphi(\hat{u}^{-1})s\varphi(\hat{u})\ ,
\end{aligned}$$

where as usual $\hat{u} = 1 + e\otimes(\hat{g}-1)$ is the invertible associated to $\hat{g}$, with $\varphi(\hat{u}) = 1 + e\otimes(\varphi(\hat{g})-1) \in (\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{T}\mathscr{E}^s_+[0,1])^+$. By construction (Proposition 3.12 i)), the r.h.s. coincides with the evaluation of H on the Chern character $\mathrm{ch}^p_1(\hat{g}) = \frac{1}{\sqrt{2\pi i}}\mathrm{Tr}\natural[2q]\,\hat{u}^{-1}\mathbf{d}\hat{u}$, and because H is of order zero one has

$$\mathrm{cs}^p_0(\varphi(\hat{g})) = H\mathrm{ch}^p_1(\hat{g}) = H(\natural\mathbf{d}\theta) \equiv \varphi_1(\theta) - (\widehat{U}^{-1}\varphi_0\widehat{U})(\theta) \bmod \overline{b}$$

in the complex $X_{n-1}(T\mathscr{E}^s_+, J\mathscr{E}^s_+)$. This proves that $\varphi_{1!}(\hat{g},\theta)$ is equivalent to $(\widehat{U}^{-1}\varphi_0\widehat{U})_!(\hat{g},\theta)$. Now it remains to show that the image $(\widehat{U}^{-1}\varphi_0\widehat{U})_!(\hat{g},\theta) = (\widehat{U}^{-1}\varphi_0(\hat{g})\widehat{U}, \widehat{U}^{-1}\varphi_0(\theta)\widehat{U})$ is equivalent to $\varphi_{0!}(\hat{g},\theta) = (\varphi_0(\hat{g}), \varphi_0(\theta))$. Here we mimic the proof of Lemma 5.5 and construct a homotopy between the invertible matrices $\left(\begin{smallmatrix}\varphi_0(\hat{g}) & 0\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}\widehat{U}^{-1}\varphi_0(\hat{g})\widehat{U} & 0\\ 0 & 1\end{smallmatrix}\right)$, whose associated Chern-Simons form is a $\overline{b}$-boundary. Since $\widehat{U}^{-1}\varphi_0(\theta)\widehat{U} \equiv \varphi_0(\theta) \bmod \overline{b}$, we conclude that $\rho_{0!}$ and $\rho_{1!}$ agree on topological and multiplicative K-theories.
Finally we have to check the compatibility with the negative Chern character $MK^{\mathscr{I}}_* \to HN_*$. For any pair $(\hat{g},\theta) \in MK^{\mathscr{I}}_n(\mathscr{A})$, one has

$$\mathrm{ch}_n(\hat{g},\theta) = \mathrm{ch}^p_1(\hat{g}) - \natural\mathbf{d}\tilde{\theta}\ \in F^{n-1}\widehat{X}(T\mathscr{A}, J\mathscr{A}),$$

where $\tilde{\theta}$ is any lift of θ in $\widehat{X}(T\mathscr{A}, J\mathscr{A})$. On the other hand, if $\hat{g}$ is the canonical lift of some $g \in (\mathscr{I}\mathscr{A})^+$, its image $\rho_!(\hat{g},\theta) \in MK^{\mathscr{J}}_{n-p}(\mathscr{B})$ is represented by the pair $(\widehat{\rho_!(g)}, \mathrm{ch}^p(\rho)\cdot\theta + \mathrm{cs}^p_0(\hat{v}))$ constructed above, so that

$$\mathrm{ch}_{n-p}\big(\rho_!(\hat{g},\theta)\big) = \mathrm{ch}^p_1\big(\widehat{\rho_!(g)}\big) - \natural\mathbf{d}(\mathrm{ch}^p(\rho)\cdot\tilde{\theta} + \mathrm{cs}^p_0(\hat{v}))\ \in F^{n-p-1}\widehat{X}(\mathscr{R}, \mathscr{J})\ .$$

But we know that the relation $\mathrm{ch}^p_1\big(\widehat{\rho_!(g)}\big) - \natural\mathbf{d}\mathrm{cs}^p_0(\hat{v}) = \mathrm{ch}^p(\rho)\cdot\mathrm{ch}^p_1(\hat{g})$ actually holds in the complex $\widehat{X}(\mathscr{R}, \mathscr{J}) = X(\widehat{\mathscr{R}})$. Therefore

$$\mathrm{ch}_{n-p}\big(\rho_!(\hat{g},\theta)\big) = \mathrm{ch}^p(\rho)\cdot(\mathrm{ch}^p_1(\hat{g}) - \natural\mathbf{d}\tilde{\theta}) = \mathrm{ch}^p(\rho)\cdot\mathrm{ch}_n(\hat{g},\theta)\ ,$$

and (83) is compatible with the diagram of Theorem 3.13.

ii) $n=2k$ is even and $p=2q$ is even. As in the case of topological K-theory we pass to the suspensions of $\mathscr{A}$ and $\mathscr{B}$. We shall only sketch the procedure. The multiplicative K-theory group of even degree $MK_n^{\mathscr{I}}(\mathscr{A})$ has an alternative description in terms of the set $MK_n'^{\mathscr{I}}(\mathscr{A})$ of equivalence classes of pairs $(\hat g,\theta)$, where $\hat g\in(\mathscr{K}\hat\otimes\mathscr{I}S\widehat{T}\mathscr{A})^+$ is an invertible and $\theta\in X_{n-1}(T\mathscr{A},J\mathscr{A})$ is a chain of odd degree such that $\mathrm{cs}_0^p(\hat g)=\overline{b}\theta$. The equivalence relation is based on a higher transgression of the Chern-Simons form: $(\hat g_0,\theta_0)$ is equivalent to $(\hat g_1,\theta_1)$ iff there exists an invertible interpolation $\hat g\in(\mathscr{K}\hat\otimes\mathscr{I}S\widehat{T}\mathscr{A}[0,1])^+$ and a chain of even degree λ such that

$$\theta_1-\theta_0=\mathrm{cs}_1'^p(\hat g)+\natural\mathbf{d}\lambda\ \in X_{n-1}(T\mathscr{A},J\mathscr{A})\ ,$$

where the odd chain $\mathrm{cs}_1'^p(\hat g)\in X(\widehat{T}\mathscr{A})$ is defined modulo $\natural\mathbf{d}$ by the higher transgression formula (see the proof of Lemma 4.6)

$$\overline{b}\mathrm{cs}_1'^p(\hat g)=\mathrm{cs}_0^p(\hat g_1)-\mathrm{cs}_0^p(\hat g_0)\ .$$

Like MK, one can show that $MK_n'^{\mathscr{I}}(\mathscr{A})$ is an abelian group inserted between $HC_*(\mathscr{A})$ and $K_*^{\mathrm{top}}(\mathscr{I}S\mathscr{A})$ in an exact sequence. More precisely there is a commutative diagram with exact rows:

$$\begin{tikzcd}
K_1^{\mathrm{top}}(\mathscr{I}\mathscr{A}) \arrow[r, "\mathrm{ch}_1"] \arrow[d, "\alpha"] & HC_{n-1}(\mathscr{A}) \arrow[r, "\delta"] \arrow[d, "\times\sqrt{2\pi i}"] & MK_n^{\mathscr{I}}(\mathscr{A}) \arrow[r] \arrow[d] & K_0^{\mathrm{top}}(\mathscr{I}\mathscr{A}) \arrow[r, "\mathrm{ch}_0"] \arrow[d, "\beta"'] & HC_{n-2}(\mathscr{A}) \arrow[d, "\times\sqrt{2\pi i}"'] \\
K_0^{\mathrm{top}}(\mathscr{I}S\mathscr{A}) \arrow[r, "\mathrm{cs}_1"] & HC_{n-1}(\mathscr{A}) \arrow[r, "\delta"] & MK_n'^{\mathscr{I}}(\mathscr{A}) \arrow[r] & K_1^{\mathrm{top}}(\mathscr{I}S\mathscr{A}) \arrow[r, "\mathrm{cs}_0"] & HC_{n-2}(\mathscr{A})
\end{tikzcd}$$

Because for even n the group MK_n' is constructed from invertibles, it has *odd* parity by convention. The (odd) map $MK_n^{\mathscr{I}}(\mathscr{A})\to MK_n'^{\mathscr{I}}(\mathscr{A})$ sends a pair $(\hat e,\theta)$ to $(\beta(\hat e),\sqrt{2\pi i}\theta+l_1^p(\hat e))$, where $\beta(\hat e)=(1+(z-1)\hat e)(1+(z-1)p_0)^{-1}$ is the invertible image of $\hat e$ under the Bott map, and $l_1^p(\hat e)$ is the transgressed cochain defined modulo $\natural\mathbf{d}$ by $\overline{b}(l_1^p(\hat e))=\mathrm{cs}_0^p(\beta(\hat e))-\sqrt{2\pi i}\,\mathrm{ch}_0^p(\hat e)$. The map $MK_n'^{\mathscr{I}}(\mathscr{A})\to K_1^{\mathrm{top}}(\mathscr{I}S\mathscr{A})$ is the forgetful map, and $HC_{n-1}(\mathscr{A})\to MK_n'^{\mathscr{I}}(\mathscr{A})$ sends a cycle θ to $(1,\sqrt{2\pi i}\,\theta)$. By the five lemma, $MK_n'^{\mathscr{I}}(\mathscr{A})$ is thus isomorphic to $MK_n^{\mathscr{I}}(\mathscr{A})$. One easily checks that the negative Chern character $\mathrm{ch}_n: MK_n'^{\mathscr{I}}(\mathscr{A})\to HN_n(\mathscr{A})$ given by $\mathrm{ch}_n(\hat g,\theta)=\mathrm{cs}_0^p(\hat g)-\overline{b}\tilde\theta$ coincides with the negative Chern character on $MK_n^{\mathscr{I}}(\mathscr{A})$ up to a factor $\sqrt{2\pi i}$. Hence it suffices to constru ct the pushforward morphism $\rho_!$ for the groups MK_n', whose elements are represented by *invertibles* of the suspended algebras:

$$\begin{tikzcd}
K_0^{\mathrm{top}}(\mathscr{I}S\mathscr{A}) \arrow[r] \arrow[d, "\rho_!"] & HC_{n-1}(\mathscr{A}) \arrow[r] \arrow[d, "\mathrm{ch}^p(\rho)"] & MK_n'^{\mathscr{I}}(\mathscr{A}) \arrow[r] \arrow[d, "\rho_!"] & K_1^{\mathrm{top}}(\mathscr{I}S\mathscr{A}) \arrow[d, "\rho_!"] \\
K_{-p}^{\mathrm{top}}(\mathscr{I}S\mathscr{B}) \arrow[r] & HC_{n-1-p}(\mathscr{B}) \arrow[r] & MK_{n-p}'^{\mathscr{I}}(\mathscr{B}) \arrow[r] & K_{1-p}^{\mathrm{top}}(\mathscr{I}S\mathscr{B})
\end{tikzcd}$$

This can be done explicitly as in case i), with the only difference that the Chern character ch_1^p and Chern-Simons transgression cs_0^p are now replaced respectively by cs_0^p and the higher transgression $\mathrm{cs}_1'^p$. The needed formulas were already established in i): let $g \in (\mathscr{I}S\mathscr{A})^+$ be any invertible with canonical lift $\hat{g} \in (\mathscr{I}S\widehat{T}\mathscr{A})^+$. One can write

$$\mathrm{ch}^p(\rho)\cdot \mathrm{cs}_0^p(\hat{g}) = \mathrm{cs}_0^p(\widetilde{u}) - \overline{b}k_1^p(\hat{u})$$

with the invertibles $\hat{u} = 1 + e\otimes(\hat{g}-1) \in (\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}S\widehat{T}\mathscr{A})^+$ and $\widetilde{u} = \hat{u}_+\hat{u}_-^{-1} \in (\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}S\widehat{\mathscr{R}})^+$, and $k_1^p(\hat{u})$ is a chain defined mod $\natural\mathbf{d}$. Let $\widehat{\rho_!(g)} \in (\mathscr{I}S\widehat{\mathscr{R}})^+$ be any invertible lift of $\rho_!(g) = \rho_+(g)\rho_-(g)^{-1}$, and $\hat{v} \in (\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}S\widehat{\mathscr{R}}[0,1])^+$ be an invertible interpolation between $\hat{v}_0 = \widetilde{u}$ and $\hat{v}_1 = 1+\hat{e}\otimes(\widehat{\rho_!(g)}-1)$. Then one has

$$\overline{b}\mathrm{cs}_1'^p(\hat{v}) = \mathrm{cs}_0^p(\widehat{\rho_!(g)}) - \mathrm{cs}_0^p(\widetilde{u})\ .$$

Therefore if $(\hat{g},\theta)$ represents a class in $MK_n'^{\mathscr{I}}(\mathscr{A})$ we define its pushforward as the multiplicative K-theory class over $\mathscr{B}$

$$\boxed{\rho_!(\hat{g},\theta) = \big(\widehat{\rho_!(g)}\ ,\ \mathrm{ch}^p(\rho)\cdot\theta + k_1^p(\hat{u}) + \mathrm{cs}_1'^p(\hat{v})\big)\ \in MK_{n-p}'^{\mathscr{I}}(\mathscr{B})} \tag{86}$$

with the odd chain $\mathrm{ch}^p(\rho)\cdot\theta + k_1^p(\hat{u}) + \mathrm{cs}_1'^p(\hat{v})$ sitting in $X_{n-p-1}(\mathscr{R},\mathscr{J})$. One shows the consistency of $\rho_!$ with the various equivalence relations using the properties of higher Chern-Simons transgressions. Details are left to the reader.

iii) $n = 2k+1$ is odd and $p = 2q+1$ is odd. We first establish an explicit formula for the composition of the topological Chern character $\mathrm{ch}_1^{2q} : K_1^{\mathrm{top}}(\mathscr{I}\mathscr{A}) \to HP_1(\mathscr{A})$ with the bivariant Chern character $\mathrm{ch}^p(\rho) \in HC^p(\mathscr{A},\mathscr{B})$. Remark that $\mathscr{I}$ is $(2q+1)$-summable by hypothesis hence ch_1^{2q} is well-defined. As in case i) let $g \in (\mathscr{I}\mathscr{A})^+$ be an invertible, $\hat{g} \in (\mathscr{I}\widehat{T}\mathscr{A})^+$ its canonical lift and $\hat{u} = 1+e\otimes(\hat{g}-1) \in (\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{T}\mathscr{A})^+$ the associated invertible. Recall that

$$\mathrm{ch}_1^{2q}(\hat{g}) = \frac{1}{\sqrt{2\pi i}}\,\mathrm{Tr}\natural[2q]\,\hat{u}^{-1}\mathbf{d}\hat{u}\ \in \Omega^1\widehat{T}\mathscr{A}_\natural\ ,$$

and the image of $\mathrm{Tr}\natural[2q]\,\hat{u}^{-1}\mathbf{d}\hat{u}$ under the Goodwillie equivalence is the $(b+B)$-cycle over $\widehat{T}\mathscr{A}$

$$\gamma(\mathrm{Tr}\natural[2q]\,\hat{u}^{-1}\mathbf{d}\hat{u}) = \sum_{i\geq 0}(-)^i i!\,\mathrm{Tr}[2q]\,\hat{u}^{-1}\mathbf{d}\hat{u}(\mathbf{d}\hat{u}^{-1}\mathbf{d}\hat{u})^i\ .$$

Now the quasihomomorphism $\rho : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s\mathscr{B}$ is of odd degree. Hence, the image of an element $x \in \widehat{T}\mathscr{A}$ under the lifted quasihomomorphism $\rho_* : \widehat{T}\mathscr{A} \to \widehat{\mathscr{M}^s} \triangleright \mathscr{I}^s\widehat{\mathscr{R}}$ is a 2×2 matrix over $\widehat{\mathscr{M}}$ whose off-diagonal entries lie in $\mathscr{I}\widehat{\mathscr{R}}$. Moreover the multiplier F is given by the matrix

$$F = \varepsilon\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \varepsilon(2p_0-1)$$

where ε is the odd generator of the Clifford algebra C_1. Thus the commutator $[p_0, \rho_* x]$ lies in the matrix algebra $M_2(\mathscr{I}\widehat{\mathscr{R}})$ for any $x \in \widehat{T}\mathscr{A}$. On the other hand, the component of the chain map $\widehat{\chi}^p \rho_* : \widehat{\Omega}\widehat{T}\mathscr{A} \to X(\widehat{\mathscr{R}})$ evaluated on a p-form $x_0 \mathbf{d}x_1 \dots \mathbf{d}x_p$ reads

$$-\frac{\Gamma(q+\frac{3}{2})}{(p+1)!} \sum_{\lambda \in S_{p+1}} \varepsilon(\lambda) \tau(\rho_* x_{\lambda(0)} [F, \rho_* x_{\lambda(1)}] \dots [F, \rho_* x_{\lambda(p)}]) ,$$

where $\tau(\varepsilon \cdot) = -\sqrt{2i}\,\mathrm{Tr}(\cdot)$ is the odd supertrace (see section 3). As in case i), let us extend ρ_* to a unital homomorphism $(\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{T}\mathscr{A})^+ \to (\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{\mathscr{M}^s_+})^+$. Using $\Gamma(q+\frac{3}{2}) = \sqrt{\pi}\, p!/(2^p q!)$ with $p = 2q+1$, one gets by direct computation

$$\begin{aligned} \mathrm{ch}^p(\rho) \cdot \mathrm{ch}_1^{2q}(\hat{g}) &= \frac{1}{2}\mathrm{Tr}[2q]\, (\widetilde{u}^{-1}[p_0, \widetilde{u}])^p + \frac{1}{2}\mathrm{Tr}[2q]\, ([p_0, \widetilde{u}]\widetilde{u}^{-1})^p \\ &= \mathrm{Tr}[2q]\, (\widetilde{u}^{-1}[p_0, \widetilde{u}])^p - \overline{b}\frac{1}{2}\mathrm{Tr}\natural[2q]\, (\widetilde{u}^{-1}[p_0, \widetilde{u}])^p \widetilde{u}^{-1}\mathbf{d}\widetilde{u} \end{aligned}$$

where $\widetilde{u} = \rho_* \hat{u}$ is an invertible element of the algebra $(\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{\mathscr{M}^s_+})^+$, and the commutator $[p_0, \widetilde{u}] \in M_2(\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\mathscr{I}\widehat{\mathscr{R}})^+$ may be considered as an element of $M_2(\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$ after applying the homomorphism $\boxtimes : \mathscr{I}\hat{\otimes}\mathscr{I} \to \mathscr{I}$. The first term of the r.h.s. is recognized as the higher Chern character $\mathrm{ch}_0^{2q}(\widetilde{f}) = \mathrm{Tr}[2q](\widetilde{f} - p_0)^p$ given by (70) for the idempotent $\widetilde{f} = \widetilde{u}^{-1} p_0 \widetilde{u} \in M_2(\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\mathscr{I}\widehat{\mathscr{R}})^+$ (or $M_2(\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$), whence the equality

$$\mathrm{ch}^p(\rho) \cdot \mathrm{ch}_1^{2q}(\hat{g}) = \mathrm{ch}_0^{2q}(\widetilde{f}) - \overline{b}\,\frac{1}{2}\mathrm{Tr}\natural[2q]\, (\widetilde{f} - p_0)^p \widetilde{u}^{-1}\mathbf{d}\widetilde{u}$$

of cycles in $X(\widehat{\mathscr{R}})$. Then, observe that the projection of $\widetilde{f}$ to the algebra $M_2(\mathbb{C}\hat{\otimes}\mathscr{I}\mathscr{B})^+$ is the idempotent $p_0 + e \otimes (\rho(g)^{-1} p_0 \rho(g) - p_0)$. Using the isomorphism $(\mathbb{C}\hat{\otimes}\mathscr{I}\mathscr{B})^+ \cong (\mathscr{I}\mathscr{B})^+$, this idempotent may be identified with the direct image $\rho_!(g) = \rho(g)^{-1} p_0 \rho(g)$. Hence, it is possible to relate $\mathrm{ch}_0^{2q}(\widetilde{f})$ with the Chern character of a given idempotent lift $\widehat{\rho_!(g)} \in M_2(\mathscr{I}\widehat{\mathscr{R}})^+$, via a homotopy with parameter $t \in [0,1]$. Let $\hat{f} \in M_2(\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1])^+$ be an idempotent path lifting the constant family $p_0 + e \otimes (\rho_!(g) - p_0)$ and connecting the two endpoints

$$\hat{f}_0 = \widetilde{f} \ , \qquad \hat{f}_1 = p_0 + \hat{e} \otimes (\widehat{\rho_!(g)} - p_0) \ .$$

$\hat{e} \in \widehat{T}\mathbb{C}$ is the canonical idempotent lift of the unit $e \in \mathbb{C}$ as in case i). The lifting $\hat{f}$ is thus defined up to homotopy (at least after stabilization by the matrix algebra $\mathscr{K}$). The property $[2q]\,\hat{e} = 1$ implies the equality

$$\mathrm{ch}_0^{2q}(\hat{f}_1) = \mathrm{ch}_0^{2q}\big(\widehat{\rho_!(g)}\big) \in \widehat{\mathscr{R}}$$

at the level of cycles. Furthermore, in analogy with Eqs. (71) the Chern-Simons form associated to the idempotent $\hat{f}$ is defined by

$$\mathrm{cs}_1^{2q}(\hat{f}) = \int_0^1 dt\, \mathrm{Tr}\natural[2q]\, (-2\hat{f}+1) \sum_{i=0}^{q} (\hat{f} - p_0)^{2i} \frac{\partial \hat{f}}{\partial t} (\hat{f} - p_0)^{2(q-i)} \mathbf{d}\hat{f} \ ,$$

and fulfills the transgression relation in $X(\widehat{\mathscr{R}})$

$$\overline{b}\mathrm{cs}_1^{2q}(\hat{f}) = \mathrm{ch}_0^{2q}(\hat{f}_1) - \mathrm{ch}_0^{2q}(\hat{f}_0) = \mathrm{ch}_0^{2q}(\widehat{\rho_!(g)}) - \mathrm{ch}_0^{2q}(\widetilde{f}) \ .$$

This leads to the definition of the map $\rho_!$ on multiplicative K-theory. Let $(\hat{g},\theta)$ represent a class in $MK_n^{\mathscr{I}}(\mathscr{A})$ of odd degree $n = 2k+1$. By Remark 5.3 we may suppose that $\hat{g}$ is the canonical lift of some invertible $g \in (\mathscr{I}\mathscr{A})^+$, and $\theta \in X_{n-1}(T\mathscr{A}, J\mathscr{A})$ is a transgression of the Chern character $\mathrm{ch}_1^{2q}(\hat{g}) = \natural\mathbf{d}\theta$. We set

$$\boxed{\rho_!(\hat{g},\theta) = \big(\widehat{\rho_!(g)}\ ,\ -\mathrm{ch}^p(\rho)\cdot\theta + h_1^{2q}(\widetilde{u}) + \mathrm{cs}_1^{2q}(\hat{f})\big)\ \in MK_{n-p}^{\mathscr{I}}(\mathscr{B})} \qquad (87)$$

where $\widehat{\rho_!(g)} \in M_2(\mathscr{I}\widehat{\mathscr{R}})^+$ is any idempotent lift of $\rho_!(g)$, $h_1^{2q}(\widetilde{u})$ is the chain $\frac{1}{2}\mathrm{Tr}\natural[2q]\,(\widetilde{f}-p_0)^p\widetilde{u}^{-1}\mathbf{d}\widetilde{u}$, and $\hat{f}$ is an idempotent path constructed as above. The minus sign in front of $\mathrm{ch}^p(\rho)\cdot\theta$ is necessary because the bivariant Chern character $\mathrm{ch}^p(\rho)$ is of odd degree $p = 2q+1$. This ensures the correct transgression relation

$$\begin{aligned}
&\overline{b}(-\mathrm{ch}^p(\rho)\cdot\theta + h_1^{2q}(\widetilde{u}) + \mathrm{cs}_1^{2q}(\hat{f}))\\
&= \mathrm{ch}^p(\rho)\cdot\mathrm{ch}_1^{2q}(\hat{g}) + \overline{b}\,\frac{1}{2}\mathrm{Tr}\natural[2q]\,(\widetilde{f}-p_0)^p\widetilde{u}^{-1}\mathbf{d}\widetilde{u} + \mathrm{ch}_0^{2q}(\widehat{\rho_!(g)}) - \mathrm{ch}_0^{2q}(\widetilde{f})\\
&= \mathrm{ch}_0^{2q}(\widehat{\rho_!(g)})
\end{aligned}$$

in the quotient complex $X_{n-p-1}(\mathscr{R},\mathscr{J})$, which shows that $\rho_!(\hat{g},\theta)$ indeed defines an element of $MK_{n-p}^{\mathscr{I}}(\mathscr{B})$. Its class does not dependent on the chosen idempotent lift $\widehat{\rho_!(g)}$ nor on the path $\hat{f}$, and moreover $\rho_!$ is compatible with the equivalence relation on multiplicative K-theory. We proceed as in case i) and let $(\hat{g}_0,\theta_0)$ and $(\hat{g}_1,\theta_1)$ be two equivalent representatives of a class in $MK_n^{\mathscr{I}}(\mathscr{A})$, provided with an interpolation $\hat{g} \in (\mathscr{I}\widehat{T}\mathscr{A}[0,1]_x)^+$ and a chain $\lambda \in X_{n-1}(T\mathscr{A}, J\mathscr{A})$ such that $\theta_1 - \theta_0 = \mathrm{cs}_0^{2q}(\hat{g}) + \overline{b}\lambda$. From Remark 5.3 the elements $\hat{g}_0$, $\hat{g}_1$ and $\hat{g}$ can be taken as the canonical lifts of $g_0, g_1 \in (\mathscr{I}\mathscr{A})^+$ and $g \in (\mathscr{I}\mathscr{A}[0,1]_x)^+$. Denoting by $\rho_*\hat{u}(g_i) = \widetilde{u}(g_i) \in (\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{\mathscr{M}}_+^s)^+$ the invertible and $\hat{f}(g_i) \in M_2(\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1]_t)^+$ the idempotent path associated to g_i, we have to establish the relation

$$\begin{aligned}
-\mathrm{ch}^p(\rho)\cdot(\theta_1-\theta_0) + h_1^{2q}(\widetilde{u}(g_1)) - h_1^{2q}(\widetilde{u}(g_0)) + \mathrm{cs}_1^{2q}(\hat{f}(g_1)) - \mathrm{cs}_1^{2q}(\hat{f}(g_0))\\
\equiv \mathrm{cs}_1^{2q}\big(\widehat{\rho_!(g)}\big) \bmod \natural\mathbf{d}
\end{aligned}$$

in the complex $X_{n-p-1}(\mathscr{R},\mathscr{J})$, where $\widehat{\rho_!(g)} \in M_2(\mathscr{I}\widehat{\mathscr{R}}[0,1]_x)^+$ is a choice of idempotent interpolation between the liftings $\widehat{\rho_!(g_i)}$'s. As usual, let $\hat{u} = 1 + e\otimes(\hat{g}-1) \in (\mathbb{C}\hat{\otimes}\mathscr{I}\mathscr{A}[0,1]_x)^+ \hookrightarrow (\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{T}\mathscr{A}[0,1]_x)^+$ be the invertible identification with $\hat{g}$. We know the equality

$$\mathrm{cs}_0^{2q}(\hat{g}) = \frac{1}{\sqrt{2\pi i}}\int_0^1 \mathrm{Tr}[2q]\,\hat{u}^{-1}s\hat{u}\ ,$$

where s is the de Rham differential on $\Omega[0,1]_x$. Set $\hat\omega = \hat u^{-1} s\hat u$. The computation of $\mathrm{ch}^p(\rho)\cdot \mathrm{cs}_0^{2q}(\hat g)$ involves the formula

$$\gamma(\mathrm{Tr}[2q]\,\hat\omega) = \\ \mathrm{Tr}[2q]\,\hat\omega + \sum_{i\ge 1}(-)^i(i-1)!\sum_{j=0}^{i-1}\mathrm{Tr}[2q]\,\hat u^{-1}\mathbf{d}\hat u(\mathbf{d}\hat u^{-1}\mathbf{d}\hat u)^j\mathbf{d}\hat\omega(\mathbf{d}\hat u^{-1}\mathbf{d}\hat u)^{i-j-1}\ ,$$

as well as the component of the chain map $\widehat\chi^p\rho_*$ evaluated on a $(p+1)$-form $x_0\mathbf{d}x_1\ldots\mathbf{d}x_{p+1}$ over $\widehat T\mathscr{A}$:

$$-\frac{\Gamma(q+\frac32)}{(p+1)!}\sum_{i=1}^{p+1}\tau\natural(\rho_*x_0[F,\rho_*x_1]\ldots\mathbf{d}(\rho_*x_i)\ldots[F,\rho_*x_{p+1}])$$

Denote as before $\widetilde u = \rho_*\hat u$ the invertible image in $(\widehat T\mathbb{C}\hat\otimes\mathscr{I}\widehat{\mathscr{M}^s_+}[0,1]_x)^+$, the associated idempotent $\widetilde f = \widetilde u^{-1}p_0\widetilde u \in M_2(\widehat T\mathbb{C}\hat\otimes\mathscr{I}\widehat{\mathscr{R}}[0,1]_x)^+$, and the Maurer-Cartan form $\widetilde\omega = \widetilde u^{-1}s\widetilde u$. One gets

$$\mathrm{ch}^p(\rho)\,(\mathrm{Tr}[2q]\,\hat\omega) = \\ -\frac{\sqrt{2\pi i}}{2}\mathrm{Tr}\natural[2q]\,\Big(\sum_{i=0}^{2q}(\widetilde f-p_0)^i[p_0,\widetilde\omega](\widetilde f-p_0)^{2q-i}\widetilde u^{-1}\mathbf{d}\widetilde u + (\widetilde f-p_0)^p\mathbf{d}\widetilde\omega\Big)\ .$$

Now observe that $\widetilde u_{x=0} = \widetilde u(g_0)$ and $\widetilde u_{x=1} = \widetilde u(g_1)$, so that after integration over the current $\frac{1}{\sqrt{2\pi i}}\int_{x=0}^1$ we get the identity (recall $\mathrm{ch}^p(\rho)$ is odd)

$$\begin{aligned} &-\mathrm{ch}^p(\rho)\cdot\mathrm{cs}_0^{2q}(\hat g) + h_1^{2q}(\widetilde u(g_1)) - h_1^{2q}(\widetilde u(g_0)) \\ &= \frac{1}{\sqrt{2\pi i}}\int_0^1 \mathrm{ch}^p(\rho)\,(\mathrm{Tr}[2q]\,\hat\omega) + \frac12\int_0^1 s\,\mathrm{Tr}\natural[2q]\,(\widetilde f-p_0)^p\widetilde u^{-1}\mathbf{d}\widetilde u \\ &= \int_0^1 \mathrm{Tr}\natural[2q]\,\Big(\sum_{i=1}^{q}(\widetilde f-p_0)^{2i-1}[p_0,\widetilde\omega](\widetilde f-p_0)^{2(q-i)+1}\widetilde u^{-1}\mathbf{d}\widetilde u - (\widetilde f-p_0)^p\mathbf{d}\widetilde\omega\Big) \end{aligned}$$

On the other hand, let us calculate the Chern-Simons form associated to the idempotent $\widetilde f$,

$$\mathrm{cs}_1^{2q}(\widetilde f) = \int_0^1 \mathrm{Tr}\natural[2q]\,(-2\widetilde f+1)\sum_{i=0}^{q}(\widetilde f-p_0)^{2i}s\widetilde f(\widetilde f-p_0)^{2(q-i)}\mathbf{d}\widetilde f\ ,$$

in terms of $\widetilde\omega$. Since by definition $\widetilde f = \widetilde u^{-1}p_0\widetilde u$, the structure equation $s\widetilde f = [\widetilde f,\widetilde\omega]$ follows and one finds

$$\begin{aligned} \mathrm{cs}_1^{2q}(\widetilde f) &= -\natural\mathbf{d}\int_0^1 \mathrm{Tr}[2q]\,(\widetilde f-p_0)^p\widetilde\omega \\ &+ \int_0^1 \mathrm{Tr}\natural[2q]\,\Big(\sum_{i=1}^{q}(\widetilde f-p_0)^{2i-1}[p_0,\widetilde\omega](\widetilde f-p_0)^{2(q-i)+1}\widetilde u^{-1}\mathbf{d}\widetilde u - (\widetilde f-p_0)^p\mathbf{d}\widetilde\omega\Big) \end{aligned}$$

Thus holds the fundamental relation

$$-\mathrm{ch}^p(\rho)\cdot \mathrm{cs}_0^{2q}(\hat{g}) + h_1^{2q}(\widetilde{u}(g_1)) - h_1^{2q}(\widetilde{u}(g_0)) \equiv \mathrm{cs}_1^{2q}(\widetilde{f}) \bmod \natural\mathbf{d}\ .$$

Now let $\hat{f} \in M_2(\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1]_x[0,1]_t)^+$ be an idempotent interpolation between $\hat{f}_{t=0} = \widetilde{f}$ and $\hat{f}_{t=1} = p_0 + \hat{e}\otimes(\widehat{\rho_!(g)} - p_0)$, with the property that it restricts to $\hat{f}(g_0)$ for $x=0$ and to $\hat{f}(g_1)$ for $x=1$. The projection of $\hat{f}$ to the algebra $M_2(\mathbb{C}\hat{\otimes}\mathscr{I}\mathscr{B}[0,1]_x[0,1]_t)^+$ may be chosen constant with respect to t. In the proof of Lemma 4.6 we established the following identity at any point $(x,t)\in[0,1]^2$:

$$\begin{aligned}
&\frac{\partial}{\partial t}\big(\mathrm{Tr}\natural[2q]\,(-2\hat{f}+1)\sum_{i=0}^{q}(\hat{f}-p_0)^{2i}s\hat{f}(\hat{f}-p_0)^{2(q-i)}\mathbf{d}\hat{f}\big)\\
&\quad\equiv\ s\big(\mathrm{Tr}\natural[2q]\,(-2\hat{f}+1)\sum_{i=0}^{q}(\hat{f}-p_0)^{2i}\frac{\partial\hat{f}}{\partial t}(\hat{f}-p_0)^{2(q-i)}\mathbf{d}\hat{f}\big) \bmod \natural\mathbf{d}\ ,
\end{aligned}$$

and integration over the square $[0,1]^2$ yields

$$\mathrm{cs}_1^{2q}(\hat{f}_{t=1}) - \mathrm{cs}_1^{2q}(\hat{f}_{t=0}) \equiv \mathrm{cs}_1^{2q}(\hat{f}_{x=1}) - \mathrm{cs}_1^{2q}(\hat{f}_{x=0}) \bmod \natural\mathbf{d}\ .$$

Since $\hat{f}_{x=0} = \hat{f}(g_0)$ and $\hat{f}_{x=1} = \hat{f}(g_1)$ we calculate, modulo $\natural\mathbf{d}$ in the complex $X_{n-p-1}(\mathscr{R},\mathscr{J})$

$$\begin{aligned}
&-\mathrm{ch}^p(\rho)\cdot(\theta_1-\theta_0) + h_1^{2q}(\widetilde{u}(g_1)) - h_1^{2q}(\widetilde{u}(g_0)) + \mathrm{cs}_1^{2q}(\hat{f}(g_1)) - \mathrm{cs}_1^{2q}(\hat{f}(g_0))\\
&\quad\equiv\ -\mathrm{ch}^p(\rho)\cdot\mathrm{cs}_0^{2q}(\hat{g}) + h_1^{2q}(\widetilde{u}(g_1)) - h_1^{2q}(\widetilde{u}(g_0)) + \mathrm{cs}_1^{2q}(\hat{f}_{t=1}) - \mathrm{cs}_1^{2q}(\hat{f}_{t=0})\\
&\quad\equiv\ \mathrm{cs}_1^{2q}(\widetilde{f}) + \mathrm{cs}_1^{2q}(p_0 + \hat{e}\otimes(\widehat{\rho_!(g)} - p_0)) - \mathrm{cs}_1^{2q}(\widetilde{f})\\
&\quad\equiv\ \mathrm{cs}_1^{2q}(\widehat{\rho_!(g)}) \bmod \natural\mathbf{d}
\end{aligned}$$

as wanted. Hence $\rho_!(\hat{g}_0,\theta_0)$ and $\rho_!(\hat{g}_1,\theta_1)$ are equivalent and the map $\rho_! : MK_n^{\mathscr{I}}(\mathscr{A}) \to MK_{n-p}^{\mathscr{I}}(\mathscr{B})$ for $n=2k+1$ and $p=2q+1$ is well-defined. Its compatibility with the map $\rho_!$ on topological K-theory is obvious. Concerning its compatibility with the map $\mathrm{ch}^p(\rho) : HC_{n-1}(\mathscr{A}) \to HC_{n-p-1}(\mathscr{B})$, we should take care of a minus sign which shows that the middle square of (83) is actually anticommutative; this has to be so because all the maps involved in this square are of odd degree. Hence the diagram (83) is graded commutative. The invariance of $\rho_!$ with respect to conjugation of quasihomomorphisms is proved exactly as in case i), by decomposing $\rho_!$ as the pushforward map $\varphi_! : MK_n^{\mathscr{I}}(\mathscr{A}) \to MK_n^{\mathscr{I}}(\mathscr{E}_+^s)$ induced by the homomorphism $\varphi : \widehat{T}\mathscr{A} \to \widehat{T}\mathscr{E}_+^s$, followed by the map $MK_n^{\mathscr{I}}(\mathscr{E}_+^s) \to MK_{n-p}^{\mathscr{I}}(\mathscr{B})$ associated with the natural p-summable quasihomomorphism of odd degree $\mathscr{E}_+^s \to \mathscr{E}^s \triangleright \mathscr{I}^s\mathscr{B}$. Also the compatibility with the negative Chern character is easily established.

iv) $n=2k$ is even and $p=2q+1$ is odd. As in case ii) we pass to the suspensions of $\mathscr{A}$ and $\mathscr{B}$ and work with the group $MK_n'^{\mathscr{I}}(\mathscr{A})$. Hence a multiplicative K-theory class of degree n over $\mathscr{A}$ is represented by a pair $(\hat{g}, \theta)$ of an invertible $\hat{g} \in (\mathscr{K}\hat{\otimes}\mathscr{I}S\widehat{T}\mathscr{A})^+$ and an odd chain $\theta \in X_{n-1}(T\mathscr{A}, J\mathscr{A})$ such that $\mathrm{cs}_0^{2q}(\hat{g}) = \overline{b}\theta$. We are thus led to build a morphism

$$\rho_! : MK_n'^{\mathscr{I}}(\mathscr{A}) \to MK_{n-p}'^{\mathscr{I}}(\mathscr{B}) \ ,$$

where the group $MK_{n-p}'^{\mathscr{I}}(\mathscr{B})$, for $n-p$ odd, is represented by pairs $(\hat{e}, \theta)$ of idempotent $\hat{e} \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}S\widehat{\mathscr{R}})^+$ and chain of even degree $\theta \in X_{n-p-1}(\mathscr{R}, \mathscr{J})$ such that $\mathrm{cs}_1^{2q}(\hat{e}) = \natural\mathbf{d}\theta$. Note that the parity of $MK_{n-p}'^{\mathscr{I}}(\mathscr{B})$ is *even*. We already established the needed formulas in case iii): let $g \in (\mathscr{I}S\mathscr{A})^+$ be any invertible with canonical lift $\hat{g} \in (\mathscr{I}S\widehat{T}\mathscr{A})^+$. One can write

$$-\mathrm{ch}^p(\rho) \cdot \mathrm{cs}_0^{2q}(\hat{g}) = \mathrm{cs}_1^{2q}(\widetilde{f}) - \natural\mathbf{d}k_0^{2q}(\hat{u})$$

with the invertible $\hat{u} = 1 + e \otimes (\hat{g} - 1) \in (\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}S\widehat{T}\mathscr{A})^+$, the idempotent $\widetilde{f} = \widetilde{u}^{-1}p_0\widetilde{u} \in M_2(\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}S\widehat{\mathscr{R}})^+$ where $\widetilde{u} = \rho_*\hat{u}$, and the chain $k_0^{2q}(\hat{u}) = -\int_0^1 \mathrm{Tr}[2q]\, (\widetilde{f} - p_0)^p\widetilde{\omega}$ where $\widetilde{\omega} = \widetilde{u}^{-1}s\widetilde{u}$. Let $\widehat{\rho_!(g)} \in M_2(\mathscr{I}S\widehat{\mathscr{R}})^+$ be any idempotent lift of $\rho_!(g) = \rho(g)^{-1}p_0\rho(g)$, and $\hat{f} \in M_2(\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}S\widehat{\mathscr{R}}[0,1])^+$ be an idempotent interpolation between $\hat{f}_0 = \widetilde{f}$ and $\hat{f}_1 = p_0 + \hat{e} \otimes (\widehat{\rho_!(g)} - p_0)$. Then one has by means of the higher transgressions (see the proof of Lemma 4.6)

$$\natural\mathbf{d}\mathrm{cs}_0'^{2q}(\hat{f}) = \mathrm{cs}_1^{2q}(\widehat{\rho_!(g)}) - \mathrm{cs}_1^{2q}(\widetilde{f}) \ ,$$

with $\mathrm{cs}_0'^{2q}(\hat{f})$ defined modulo $\overline{b}$. Therefore if $(\hat{g}, \theta)$ represents a class in $MK_n'^{\mathscr{I}}(\mathscr{A})$ we define its pushforward as the multiplicative K-theory class over $\mathscr{B}$

$$\boxed{\rho_!(\hat{g}, \theta) = \big(\widehat{\rho_!(g)}\ ,\ \mathrm{ch}^p(\rho) \cdot \theta + k_0^{2q}(\hat{u}) + \mathrm{cs}_0'^{2q}(\hat{f})\big) \ \in MK_{n-p}'^{\mathscr{I}}(\mathscr{B})} \tag{88}$$

with the chain $\mathrm{ch}^p(\rho) \cdot \theta + k_0^{2q}(\hat{u}) + \mathrm{cs}_0'^{2q}(\hat{f})$ of even degree sitting in the quotient complex $X_{n-p-1}(\mathscr{R}, \mathscr{J})$. ∎

Example 6.4 For $\mathscr{B} = \mathbb{C}$ a quasihomomorphism $\mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s$ induces a map $MK_n^{\mathscr{I}}(\mathscr{A}) \to MK_{n-p}^{\mathscr{I}}(\mathbb{C})$. Thus if $\mathscr{I}$ is a Schatten ideal on a Hilbert space, Example 5.6 yields index maps or regulators, depending on the degrees:

$$\begin{array}{rcll} MK_n^{\mathscr{I}}(\mathscr{A}) & \to & \mathbb{Z} & \text{if } n \le p \ ,\ n \equiv p \bmod 2 \ , \\ MK_n^{\mathscr{I}}(\mathscr{A}) & \to & \mathbb{C}^\times & \text{if } n > p \ ,\ n \equiv p + 1 \bmod 2 \ . \end{array}$$

7 Assembly maps and crossed products

In this section we illustrate the general theory of secondary characteristic classes with the specific example of crossed product algebras, and build an "assembly map" for multiplicative K-theory modelled on the Baum-Connes construction [1].

Let $\mathscr{A}$ be a unital Fréchet m-algebra and Γ a countable discrete group acting on $\mathscr{A}$ from the right by automorphisms. The action of an element $\gamma \in \Gamma$ on $a \in \mathscr{A}$ reads a^γ. We impose the action to be almost isometric in the following sense: for each submultiplicative seminorm $\|\cdot\|_\alpha$ on $\mathscr{A}$ there exists a constant C_α such that

$$\|a^\gamma\|_\alpha \leq C_\alpha \|a\|_\alpha \qquad \forall\ a \in \mathscr{A}\ ,\ \gamma \in \Gamma\ . \tag{89}$$

The algebraic tensor product $\mathscr{A} \otimes \mathbb{C}\Gamma$ is identified with the space of $\mathscr{A}$-valued functions with finite support over Γ. Thus any element of $\mathscr{A} \otimes \mathbb{C}\Gamma$ is a finite linear combination of symbols $a\gamma^*$ with $a_\gamma \in \mathscr{A}$ and $\gamma \in \Gamma$. The star refers to a contravariant notation. We endow $\mathscr{A} \otimes \mathbb{C}\Gamma$ with the crossed product defined in terms of symbols by

$$(a_1\gamma_1^*)(a_2\gamma_2^*) = a_1(a_2)^{\gamma_1}(\gamma_2\gamma_1)^* \qquad \forall\ a_i \in \mathscr{A}\ ,\ \gamma_i \in \Gamma\ .$$

The crossed product algebra $\mathscr{A} \rtimes \Gamma$ is an adequate completion of the above space consisting of $\mathscr{A}$-valued functions with "rapid decay" over Γ. This requires to fix once and for all a right-invariant distance function $d : \Gamma \times \Gamma \to \mathbb{R}_+$. Endow the space $\mathscr{A} \otimes \mathbb{C}\Gamma$ with the seminorms

$$\|b\|_{\alpha,\beta} = C_\alpha \sum_{\gamma\in\Gamma} \sigma_\beta(\gamma)\|b(\gamma)\|_\alpha \qquad \forall\ b \in \mathscr{A} \otimes \mathbb{C}\Gamma\ ,$$

where the $\mathbb{R}_+$-valued function $\sigma_\beta(\gamma) := (1 + d(\gamma, 1))^\beta$, for $\beta \geq 0$, fulfills the property $\sigma_\beta(\gamma_1\gamma_2) \leq \sigma_\beta(\gamma_1)\sigma_\beta(\gamma_2)$. One easily checks that $\|\cdot\|_{\alpha,\beta}$ is submultiplicative with respect to the crossed product, hence the completion of $\mathscr{A} \otimes \mathbb{C}\Gamma$ for the family of seminorms $(\|\cdot\|_{\alpha,\beta})$ yields a Fréchet m-algebra $\mathscr{A} \rtimes \Gamma$.
Multiplicative K-theory classes of $\mathscr{A} \rtimes \Gamma$ may be obtained by adapting the assembly map construction of [1]. The idea is to replace the noncommutative space $\mathscr{A} \rtimes \Gamma$ by a more classical space, for which the secondary invariants are presumably easier to describe. Consider a compact Riemannian manifold M without boundary, and let $P \xrightarrow{\Gamma} M$ be a Γ-covering. Γ acts on P from the left by deck transformations. Denote by

$$\mathscr{A}_P := C^\infty(P; \mathscr{A})^\Gamma$$

the algebra of Γ-invariant smooth $\mathscr{A}$-valued functions over P: any function $a \in \mathscr{A}_P$ verifies $a(\gamma^{-1} \cdot x) = (a(x))^\gamma$, $\forall x \in P, \gamma \in \Gamma$. Thus $\mathscr{A}_P$ is the algebra of smooth sections of a non-trivial bundle with fibre $\mathscr{A}$ over M. It can be represented as a subalgebra of matrices over $C^\infty(M)\hat{\otimes}(\mathscr{A} \rtimes \Gamma) = C^\infty(M; \mathscr{A} \rtimes \Gamma)$ as follows. Let (U_i), $i = 1, \dots, m$ be a finite open covering of M trivializing the bundle P, via a set of sections $s_i : U_i \to P$ and locally constant transition functions $\gamma_{ij} : U_i \cap U_j \to \Gamma$:

$$\gamma_{ij}\gamma_{jk} = \gamma_{ik} \text{ over } U_i \cap U_j \cap U_k\ , \quad s_i(x) = \gamma_{ij} \cdot s_j(x) \quad \forall x \in U_i \cap U_j\ .$$

Choose a partition of unity $c_i \in C^\infty(M)$ relative to this covering: $\mathrm{supp}\, c_i \subset U_i$ and $\sum_{i=1}^m c_i(x)^2 = 1$. From these data consider the homomorphism $\rho : \mathscr{A}_P \to M_m(C^\infty(M)\hat{\otimes}(\mathscr{A} \rtimes \Gamma))$ sending an element $a \in \mathscr{A}_P$ to the $m \times m$ matrix $\rho(a)$ whose components, as $(\mathscr{A} \rtimes \Gamma)$-valued functions over M, read

$$\rho(a)_{ij}(x) := c_i(x)c_j(x)a(s_i(x))\gamma^*_{ji} \quad \forall\, i,j = 1,\dots,m\ ,\ \forall x \in M\ .$$

Of course ρ depends on the choice of trivialization (U_i, s_i) and partition of unity (c_i), but different choices are related by conjugation in a suitably large matrix algebra. Indeed, if (U'_α, s'_α), $\alpha = 1,\dots,\mu$ denotes another trivialization with transition functions $\gamma'_{\alpha\beta}$ and partition of unity (c'_α), we get a corresponding homomorphism $\rho' : \mathscr{A}_P \to M_\mu(C^\infty(M)\hat{\otimes}(\mathscr{A} \rtimes \Gamma))$. Introduce the rectangular matrices u, v over $C^\infty(M)\hat{\otimes}(\mathscr{A} \rtimes \Gamma)$ with components

$$u_{i\alpha}(x) = c_i(x)c'_\alpha(x)\gamma^*_{\alpha i}\ , \qquad v_{\alpha i}(x) = c'_\alpha(x)c_i(x)\gamma^*_{i\alpha}\ ,$$

(recall $\mathscr{A}$ is unital by hypothesis hence $\mathbb{C}\Gamma \subset \mathscr{A} \rtimes \Gamma$), where the mixed transition functions $\gamma_{i\alpha}$, $\gamma_{\alpha i}$ are defined by $s_i(x) = \gamma_{i\alpha} \cdot s'_\alpha(x)$ and $s'_\alpha(x) = \gamma_{\alpha i} \cdot s_i(x)$ for any $x \in U_i \cap U'_\alpha$. Then uv and vu are idempotent square matrices, and for any element $a \in \mathscr{A}_P$ one has $\rho(a) = u\rho'(a)v$ and $\rho'(a) = v\rho(a)u$. Moreover, the invertible square matrix of size $m + \mu$

$$W = \begin{pmatrix} 1-uv & -u \\ v & 1-vu \end{pmatrix}\ , \qquad W^{-1} = \begin{pmatrix} 1-uv & u \\ -v & 1-vu \end{pmatrix}$$

verifies $W^{-1}\left(\begin{smallmatrix} \rho(a) & 0 \\ 0 & 0 \end{smallmatrix}\right)W = \left(\begin{smallmatrix} 0 & 0 \\ 0 & \rho'(a) \end{smallmatrix}\right)$, which shows that the homomorphisms ρ and ρ' are stably conjugate.
In order to get a quasihomomorphism from $\mathscr{A}_P$ to $\mathscr{A} \rtimes \Gamma$, we need a K-cycle for the Fréchet algebra $C^\infty(M)$ (see Example 3.3). By a standard procedure [4, 5], such a K-cycle D may be constructed from an elliptic pseudodifferential operator or a Toeplitz operator over M. We shall suppose that D is of parity p mod 2, and of summability degree $p+1$ (even case) or p (odd case). Hence (see Example 3.3) in the even case one has an infinite-dimensional separable Hilbert space H with two continuous representations $C^\infty(M) \rightrightarrows \mathscr{L} = \mathscr{L}(H)$ which agree modulo the Schatten ideal $\mathscr{I} = \mathscr{L}^{p+1}(H)$, whereas in the odd case the algebra $C^\infty(M)$ is represented in the matrix algebra $\left(\begin{smallmatrix} \mathscr{L} & \mathscr{I} \\ \mathscr{I} & \mathscr{L} \end{smallmatrix}\right)$ with $\mathscr{I} = \mathscr{L}^p(H)$. Therefore, upon choosing an isomorphism $H \cong H\hat{\otimes}\mathbb{C}^m$ the composition of $\rho : \mathscr{A}_P \to M_m(C^\infty(M)\hat{\otimes}(\mathscr{A} \rtimes \Gamma))$ with the Hilbert space representation induced by the K-cycle D leads to a quasihomomorphism of parity p mod 2

$$\rho_D : \mathscr{A}_P \to \mathscr{E}^s \triangleright \mathscr{I}^s\hat{\otimes}(\mathscr{A} \rtimes \Gamma)\ ,$$

with intermediate algebra $\mathscr{E} = \mathscr{L}\hat{\otimes}(\mathscr{A} \rtimes \Gamma)$ (or $(\mathscr{L} \ltimes \mathscr{I})\hat{\otimes}(\mathscr{A} \rtimes \Gamma)$, see Example 3.4). Note that $\mathscr{L}$ and $\mathscr{I}$ may be replaced by other suitable operator algebras, if needed. From the discussion above we see that ρ_D depends only on D up to conjugation by an invertible element $W \in \mathscr{E}^s_+$. Taking into account the Chern characters in negative and periodic cyclic homology, the Riemann-Roch-Grothendieck Theorem 6.3 thus yields cube diagrams of the following kind:

Corollary 7.1 *A K-cycle D over $C^\infty(M)$ as above yields for any integer $n \in \mathbb{Z}$ a commutative diagram*

$$\begin{tikzcd}
 & K_n^{\mathrm{top}}(\mathscr{I}\mathscr{A}_P) \arrow[rr] \arrow[dd] & & K_{n-p}^{\mathrm{top}}(\mathscr{I}(\mathscr{A} \rtimes \Gamma)) \arrow[dd] \\
MK_n^{\mathscr{I}}(\mathscr{A}_P) \arrow[ur] \arrow[rr] \arrow[dd] & & MK_{n-p}^{\mathscr{I}}(\mathscr{A} \rtimes \Gamma) \arrow[ur] \arrow[dd] & \\
 & HP_n(\mathscr{A}_P) \arrow[rr] & & HP_{n-p}(\mathscr{A} \rtimes \Gamma) \\
HN_n(\mathscr{A}_P) \arrow[ur] \arrow[rr] & & HN_{n-p}(\mathscr{A} \rtimes \Gamma) \arrow[ur] &
\end{tikzcd}$$

where the horizontal arrows are induced by the quasihomomorphism $\rho_D : \mathscr{A}_P \to \mathscr{E}^s \triangleright \mathscr{I}^s \hat{\otimes}(\mathscr{A} \rtimes \Gamma)$.

The background square describes the topological side of the Riemann-Roch-Grothendieck theorem, namely the compatibility between the push-forward in topological K-theory and the bivariant Chern character in periodic cyclic homology. One may choose D as a representative of the fundamental class in the K-homology of M. If moreover M is a model for the classifying space $B\Gamma$, one may choose P as the universal bundle $E\Gamma$. For torsion-free groups Γ the morphism $K_n^{\mathrm{top}}(\mathscr{I}\mathscr{A}_P) \to K_{n-p}^{\mathrm{top}}(\mathscr{I}(\mathscr{A} \rtimes \Gamma))$ thus obtained is related to the Baum-Connes assembly map [1] and exhausts many (in some cases, all the) interesting topological K-theory classes of $\mathscr{A} \rtimes \Gamma$.
The foreground square provides a lifting of the topological situation at the level of multiplicative K-theory and negative cyclic homology, i.e. secondary characteristic classes. Hence a part of $MK_*^{\mathscr{I}}(\mathscr{A} \rtimes \Gamma)$ may be obtained by direct images of multiplicative K-theory over $\mathscr{A}_P$. Note that in contrast to the topological situation, the push-forward map in multiplicative K-theory does *not* exhaust all the interesting classes over $\mathscr{A} \rtimes \Gamma$.

Let us now deal with the case $\mathscr{A} = C^\infty(N)$, for a compact smooth Riemannian manifold N, endowed with a left action of Γ by diffeomorphisms. We provide $\mathscr{A}$ with its usual Fréchet topology, and condition (89) forces the Γ-action be "almost isometric" on N. The crossed product $\mathscr{A} \rtimes \Gamma$ is then isomorphic to a certain convolution algebra of functions over the smooth étale groupoid $\Gamma \ltimes N$, describing a highly noncommutative space when the action of Γ is not proper. The commutative algebra $\mathscr{A}_P$ is the subalgebra of smooth Γ-invariant functions $a \in C^\infty(P \times N)$, $a(\gamma \cdot x, \gamma \cdot y) = a(x,y)$ for any $(x,y) \in P \times N$, and is thus isomorphic to the algebra of smooth functions over the (compact) quotient manifold $Q = \Gamma \backslash (P \times N)$.

The problem is therefore reduced to the computation of secondary invariants for the classical space Q. The cyclic homology of $\mathscr{A}_P = C^\infty(Q)$ has been determined by Connes [4] and is computable from the de Rham complex of differential forms over Q. We will see that the multiplicative K-theory $MK_*^{\mathscr{I}}(\mathscr{A}_P)$ is closely related (though not isomorphic) to Deligne cohomology. We first recall some definitions. Let $\Omega^n(Q)$ denote the space of complex, smooth differential n-forms over Q, d the de Rham coboundary, $Z^n_{\mathrm{dR}}(Q) = \mathrm{Ker}(d : \Omega^n \to \Omega^{n+1})$ the space of closed n-forms and $B^n_{\mathrm{dR}}(Q) = \mathrm{Im}(d : \Omega^{n-1} \to \Omega^n)$ the space of exact n-forms. By de Rham's theorem, the de Rham cohomology $H^n_{\mathrm{dR}}(Q) = Z^n_{\mathrm{dR}}(Q)/B^n_{\mathrm{dR}}(Q)$ is isomorphic to the Čech cohomology of Q with complex coefficients $\check{H}^n(Q;\mathbb{C})$. For any half-integer q we define the additive group $\mathbb{Z}(q) := (2\pi i)^q\mathbb{Z} \subset \mathbb{C}$ (the square root of $2\pi i$ must be chosen consistentl y with the Chern character on K_1^{top}). Let $\underline{\Omega}^k$ denote the sheaf of differential k-forms over Q and consider for any $n \in \mathbb{N}$ the complex of sheaves

$$0 \longrightarrow \underline{\mathbb{Z}}(n/2) \longrightarrow \underline{\Omega}^0 \xrightarrow{d} \underline{\Omega}^1 \xrightarrow{d} \ldots \xrightarrow{d} \underline{\Omega}^{n-1} \longrightarrow 0 \tag{90}$$

where the constant sheaf $\underline{\mathbb{Z}}(n/2)$ sits in degree 0 and $\underline{\Omega}^k$ in degree $k+1$. The map $\underline{\mathbb{Z}}(n/2) \to \underline{\Omega}^0$ is induced by the natural inclusion of constant functions into complex-valued functions. By definition the (smooth) Deligne cohomology $H^n_{\mathscr{D}}(Q;\mathbb{Z}(n/2))$ is the hyperhomology of (90) in degree n. The natural projection onto the constant sheaf $\underline{\mathbb{Z}}(n/2)$ yields a well-defined map from $H^n_{\mathscr{D}}(Q;\mathbb{Z}(n/2))$ to the Čech cohomology with integral coefficients $\check{H}^n(Q;\mathbb{Z}(n/2))$. On the other extreme, the de Rham coboundary $d : \underline{\Omega}^{n-1} \to \underline{\Omega}^n$ sends a Deligne n-cocycle to a globally defined closed n-form over Q, called its *curvature*, which only depends on the Deligne cohomology class. It follows from the definitions that the image of the curvature in de Rham cohomology coincides with the complexification of the Čech cohomology class of the Deligne cocycle. One thus gets a commutative diagram in any degree n

$$\begin{tikzcd} H^n_{\mathscr{D}}(Q;\mathbb{Z}(n/2)) \arrow[r] \arrow[d, "d"'] & \check{H}^n(Q;\mathbb{Z}(n/2)) \arrow[d, "\otimes\mathbb{C}"] \\ Z^n_{\mathrm{dR}}(Q) \arrow[r] & H^n_{\mathrm{dR}}(Q) \end{tikzcd} \tag{91}$$

This has to be compared with the commutative square involving the multiplicative and topological K-theories of the algebra $\mathscr{A}_P = C^\infty(Q)$, with their Chern characters:

$$\begin{tikzcd} MK^{\mathscr{I}}_n(\mathscr{A}_P) \arrow[r] \arrow[d] & K^{\mathrm{top}}_n(\mathscr{I}\mathscr{A}_P) \arrow[d] \\ HN_n(\mathscr{A}_P) \arrow[r] & HP_n(\mathscr{A}_P) \end{tikzcd} \tag{92}$$

In fact one can construct, at least in low degrees n, an explicit transformation from Deligne cohomology to multiplicative K-theory, and the curvature morphism captures the lowest degree part of the negative Chern character. Let us explain this with some details. Firstly, it is well-known that the bottom line of (91) is included as a direct summand in the bottom line of (92). Since we deal essentially with the X-complex description of cyclic homology (section 2), we recall how the latter is related to the de Rham cohomology of Q. Choose the universal free extension $0 \to J\mathscr{A}_P \to T\mathscr{A}_P \to \mathscr{A}_P \to 0$. The cyclic homology of $\mathscr{A}_P$ is computed by the X-complex $X(\widehat{T}\mathscr{A}_P)$ of the pro-algebra $\widehat{T}\mathscr{A}_P = \varprojlim_n T\mathscr{A}_P/(J\mathscr{A}_P)^n$, together with its filtration by the subcomplexes $F^n\widehat{X}(T\mathscr{A}_P, J\mathscr{A}_P)$. As a pro-vector space, $X(\widehat{T}\mathscr{A}_P)$ is isomorphic to the completed space of noncommutative differential forms $\widehat{\Omega}\mathscr{A}_P$, and the $J\mathscr{A}_P$-adic filtration coincides with the Hodge filtration $F^n\widehat{\Omega}\mathscr{A}_P$. A canonical chain map $X(\widehat{T}\mathscr{A}_P) \to \Omega^*(Q)$ is given by projecting noncommutative differential forms to commutative ones, up to a rescaling:

$$\Omega^n\mathscr{A}_P \ni a_0da_1\dots da_n \to \lambda_n a_0da_1\dots da_n \in \Omega^n(Q)$$

with $\lambda_n = (-)^k\frac{k!}{(2k)!}$ if $n = 2k$ and $\lambda_n = (-)^k\frac{k!}{(2k+1)!}$ if $n = 2k+1$. These factors are fixed in order to get exactly a chain map. Clearly it sends the Hodge filtration of $\widehat{\Omega}\mathscr{A}_P$ onto the natural filtration by degree on $\Omega^*(Q)$. The following proposition is a reformulation of Connes' version of the Hochschild-Kostant-Rosenberg theorem [4].

Proposition 7.2 *The chain map $X(\widehat{T}\mathscr{A}_P) \to \Omega^*(Q)$ is a homotopy equivalence compatible with the filtrations. Hence follow the isomorphisms*

$$\begin{aligned} HP_n(\mathscr{A}_P) &= \bigoplus_{k\in\mathbb{Z}} H^{n+2k}_{\mathrm{dR}}(Q)\ , \\ HC_n(\mathscr{A}_P) &= \frac{\Omega^n(Q)}{B^n_{\mathrm{dR}}(Q)} \oplus \bigoplus_{k<0} H^{n+2k}_{\mathrm{dR}}(Q)\ , \\ HN_n(\mathscr{A}_P) &= Z^n_{\mathrm{dR}}(Q) \oplus \bigoplus_{k>0} H^{n+2k}_{\mathrm{dR}}(Q)\ . \end{aligned} \tag{93}$$

Of course the injections $Z^n_{\mathrm{dR}}(Q) \to HN_n(\mathscr{A}_P)$ and $H^n_{\mathrm{dR}}(Q) \to HP_n(\mathscr{A}_P)$ are compatible with the forgetful maps $Z^n_{\mathrm{dR}}(Q) \to H^n_{\mathrm{dR}}(Q)$ and $HN_n(\mathscr{A}_P) \to HP_n(\mathscr{A}_P)$. It is useful to calculate the image of the Chern character of idempotents and invertibles under the chain map $X(\widehat{T}\mathscr{A}_P) \to \Omega^*(Q)$. Let $e \in M_2(\mathscr{K}\hat{\otimes}\mathscr{A}_P)^+$ be an idempotent such that $e - p_0 \in M_2(\mathscr{K}\hat{\otimes}\mathscr{A}_P)$, and let $\hat{e} \in M_2(\mathscr{K}\hat{\otimes}\widehat{T}\mathscr{A}_P)^+$ be its canonical lift. The image of the Chern character $\mathrm{ch}_0(\hat{e}) = \mathrm{Tr}(\hat{e} - p_0) \in X(\widehat{T}\mathscr{A}_P)$ is the differential form of even degree

$$\mathrm{ch}_{\mathrm{dR}}(e) = \mathrm{Tr}(e-p_0) + \sum_{k\geq 1} \frac{(-)^k}{k!}\,\mathrm{Tr}((e-\frac{1}{2})(dede)^k)\ \in \Omega^+(Q)\ .$$

Let $g \in (\mathscr{K}\hat{\otimes}\mathscr{A}_P)^+$ be an invertible such that $g - 1 \in \mathscr{K}\hat{\otimes}\mathscr{A}_P$, and let $\hat{g} \in (\mathscr{K}\hat{\otimes}\widehat{T}\mathscr{A}_P)^+$ be its canonical lift. The image of the Chern character $\mathrm{ch}_1(\hat{g}) =$

$\frac{1}{\sqrt{2\pi i}}\natural\hat{g}^{-1}\mathbf{d}\hat{g} \in X(\widehat{T}\mathscr{A}_P)$ is the differential form of odd degree

$$\mathrm{ch}_{\mathrm{dR}}(g) = \frac{1}{\sqrt{2\pi i}} \sum_{k\geq 0} (-)^k \frac{k!}{(2k+1)!} \operatorname{Tr}(g^{-1}dg(dg^{-1}dg)^k) \ \in \Omega^-(Q) \ .$$

We now construct explicit morphisms $H^n_{\mathscr{D}}(Q;\mathbb{Z}(n/2)) \to MK^{\mathscr{I}}_n(\mathscr{A}_P)$ in degrees $n = 0,1,2$. In fact the ideal $\mathscr{I}$ is irrelevant and the previous morphisms factor through the multiplicative group $MK_n(\mathscr{A}_P) := MK^{\mathbb{C}}_n(\mathscr{A}_P)$ associated to the 1-summable algebra $\mathbb{C}$. Then choosing any rank one injection $\mathbb{C} \to \mathscr{I}$ induces a unique map $MK_n(\mathscr{A}_P) \to MK^{\mathscr{I}}_n(\mathscr{A}_P)$ by virtue of Lemma 5.5.

$n = 0$: Then $\mathbb{Z}(0) = \mathbb{Z}$ and the complex $0 \to \underline{\mathbb{Z}}(0) \to 0$ calculates the Čech cohomology of Q with coefficients in $\mathbb{Z}$. Hence $H^0_{\mathscr{D}}(Q;\mathbb{Z}(0)) = \check{H}^0(Q;\mathbb{Z})$ is the additive group of $\mathbb{Z}$-valued locally constant functions over Q. The map

$$H^0_{\mathscr{D}}(Q;\mathbb{Z}(0)) \to MK_0(\mathscr{A}_P) \cong K^{\mathrm{top}}_0(\mathscr{A}_P) \tag{94}$$

associates to such a function f the K-theory class of the trivial complex vector bundle of rank f over Q.

$n = 1$: Then $\mathbb{Z}(1/2) = \sqrt{2\pi i}\,\mathbb{Z}$ and $H^1_{\mathscr{D}}(Q;\mathbb{Z}(1/2))$ is the hyperhomology in degree 1 of the complex $0 \to \underline{\mathbb{Z}}(1/2) \to \underline{\Omega}^0 \to 0$. Choose a good covering (U_i) of Q. A Deligne 1-cocycle is given by a collection (f_i, n_{ij}) of smooth functions $f_i : U_i \to \mathbb{C}$ and locally constant functions $n_{ij} : U_i \cap U_j \to \mathbb{Z}$ related by the descent equations

$$f_i - f_j = \sqrt{2\pi i}\, n_{ij} \text{ over } U_i \cap U_j \ , \quad n_{jk} - n_{ik} + n_{ij} = 0 \text{ over } U_i \cap U_j \cap U_k \ .$$

The cocycle is trivial if the f_i's are $\mathbb{Z}(1/2)$-valued. Taking the exponentials $g_i = \exp(\sqrt{2\pi i} f_i)$ one gets invertible smooth functions which agree on the overlaps $U_i \cap U_j$, hence define a global invertible function g over Q. The latter is equal to 1 exactly when the cocycle (f_i, n_{ij}) is trivial. Hence $H^1_{\mathscr{D}}(Q;\mathbb{Z}(1/2))$ is the multiplicative group $C^\infty(Q)^\times$ of complex-valued invertible functions over Q. On the other hand, the elements of $MK_1(\mathscr{A}_P)$ are represented by pairs $(\hat{g}, \theta)$ of an invertible $\hat{g} \in (\mathscr{K}\hat{\otimes}\widehat{T}\mathscr{A}_P)^+$ and a cochain $\theta \in X_0(T\mathscr{A}_P, J\mathscr{A}_P) \cong C^\infty(Q)$. We get a map

$$H^1_{\mathscr{D}}(Q;\mathbb{Z}(1/2)) \cong C^\infty(Q)^\times \to MK_1(\mathscr{A}_P) \tag{95}$$

by sending an invertible $g \in C^\infty(Q)^\times$ to the multiplicative K-theory class of $(\hat{g}, 0)$, with $\hat{g}$ the canonical lift of g (to be precise one should replace g by $1 + (g - 1_{\mathscr{A}_P}) \in (\mathscr{A}_P)^+$). This map identifies the curvature morphism $H^1_{\mathscr{D}}(Q;\mathbb{Z}(1/2)) \to Z^1_{\mathrm{dR}}(Q)$ with the lowest degree part of the negative Chern character $\mathrm{ch}_1 : MK_1(\mathscr{A}_P) \to HN_1(\mathscr{A}_P)$. Indeed the curvature of an element $g \in C^\infty(Q)^\times$ is by definition the closed one-form

$$df_i = \frac{1}{\sqrt{2\pi i}}\, g_i^{-1} dg_i = \frac{1}{\sqrt{2\pi i}}\, g^{-1} dg \qquad \forall i \ ,$$

globally defined over Q. But this coincides with the $Z^1_{\rm dR}(Q)$-component of the negative Chern character $\mathrm{ch}_1(\hat{g},0)$.

$n=2$: Then $\mathbb{Z}(1)=2\pi i\,\mathbb{Z}$ and $H^2_{\mathscr{D}}(Q;\mathbb{Z}(1))$ is the hyperhomology in degree 2 of the complex $0\to\underline{\mathbb{Z}}(1)\to\underline{\Omega}^0\to\underline{\Omega}^1\to 0$. A Deligne cocycle relative to the finite good covering (U_i), $i=1,\ldots,m$ is a collection (A_i,f_{ij},n_{ijk}) of one-forms A_i over U_i, smooth functions $f_{ij}:U_i\cap U_j\to\mathbb{C}$ and locally constant functions $n_{ijk}:U_i\cap U_j\cap U_k\to\mathbb{Z}$, subject to the descent equations

$$A_i-A_j=df_{ij}\ ,\quad -f_{jk}+f_{ik}-f_{ij}=2\pi i\,n_{ijk}\ ,\quad n_{jkl}-n_{ikl}+n_{ijl}-n_{ijk}=0\ .$$

Equivalently, passing to the exponentials $g_{ij}=\exp f_{ij}$ a cocycle is a collection (A_i,g_{ij}) such that $A_i-A_j=g_{ij}^{-1}dg_{ij}$ and $g_{ij}g_{jk}=g_{ik}$. Two cocycles (A_i,g_{ij}) and (A'_i,g'_{ij}) are cohomologous iff there exists a collection of smooth invertible functions (gauge transformations) $\alpha_i:U_i\to\mathbb{C}^\times$ such that

$$A'_i=A_i+\alpha_i^{-1}d\alpha_i\ ,\qquad g'_{ij}=\alpha_i g_{ij}\alpha_j^{-1}\ .$$

One sees that a Deligne cohomology class is nothing else but a complex line bundle over Q, described by the smooth transition functions $g_{ij}:U_i\cap U_j\to\mathbb{C}^\times$, together with a connection given locally by the one-forms A_i, up to gauge transformation. Hence

$$H^2_{\mathscr{D}}(Q;\mathbb{Z}(1))=\{\text{isomorphism classes of complex line bundles with connection}\}$$

The group law is the tensor product of line bundles with connections. The curvature morphism $H^2_{\mathscr{D}}(Q;\mathbb{Z}(1))\to Z^2_{\rm dR}(Q)$ maps a cocycle (A_i,g_{ij}) to the globally defined closed two-form dA_i $\forall i$, i.e. the curvature of the connection of the corresponding line bundle. The construction of the morphism from Deligne cohomology to multiplicative K-theory requires to fix a partition of unity (c_i) relative to the finite covering: $\mathrm{supp}\,c_i\subset U_i$ and $\sum_i c_i(x)^2=1$ $\forall x\in Q$. Given a Deligne cocycle (A_i,g_{ij}), we construct the idempotent $e_+\in M_m(C^\infty(Q))$ of rank 1 whose matrix elements are the functions

$$(e_+)_{ij}=c_ig_{ij}c_j\ \in C^\infty_c(U_i\cap U_j)\ ,$$

and let $e_-=1_{\mathscr{A}_P}$ be the unit of $C^\infty(Q)$ (the constant function 1 over Q). Define $e\in M_2(\mathscr{A}_P)^+$ as the idempotent matrix $\left(\begin{smallmatrix}1-e_- & 0\\ 0 & e_+\end{smallmatrix}\right)$. The 0th degree of the Chern character $\mathrm{ch}_{\rm dR}(e)$ is $\mathrm{Tr}(e-p_0)=\mathrm{Tr}(e_+)-1_{\mathscr{A}_P}=0$, so that the class of e in $K_0^{\rm top}(\mathscr{A}_P)$ is a virtual bundle of rank 0. To get a multiplicative K-theory class $(\hat{e},\theta)\in MK_2(\mathscr{A}_P)$, we must adjoin to the canonical idempotent lift $\hat{e}\in M_2(\widehat{T}\mathscr{A}_P)^+$ an odd cochain $\theta\in X_1(T\mathscr{A}_P,J\mathscr{A}_P)\cong X(\mathscr{A}_P)$. Since $\mathscr{A}_P$ is the commutative algebra of smooth functions over a compact manifold, its X-complex reduces to the de Rham complex of Q truncated in degrees ≥ 2, $X(\mathscr{A}_P):C^\infty(Q)\xrightarrow{d}\Omega^1(Q)$. The fact that e is of virtual rank zero insures that any choice of one-form $\theta \in \Omega^1(Q)$ satisfies the correct transgression relation $\mathrm{ch}_0(\hat{e}) = \overline{b}\theta = 0$ in the complex $X(\mathscr{A}_P)$. We s et

$$\theta = -\sum_i c_i^2 A_i \ \in \Omega^1(Q) \ .$$

LEMMA 7.3 *The assignement $(A_i, g_{ij}) \mapsto (\hat{e}, \theta)$ yields a well-defined morphism*

$$H^2_{\mathscr{D}}(Q; \mathbb{Z}(1)) \to MK_2(\mathscr{A}_P) \ , \tag{96}$$

and the curvature of (A_i, g_{ij}) corresponds to the $Z^2_{\mathrm{dR}}(Q)$ component of the negative Chern character $\mathrm{ch}_2(\hat{e}, \theta) \in HN_2(\mathscr{A}_P)$.

Proof: We have to show that the multiplicative K-theory class of $(\hat{e}, \theta)$ only depends on the Deligne cohomology class of (A_i, g_{ij}). Thus let (A'_i, g'_{ij}) be another representative, with $A'_i = A_i + \alpha_i^{-1} d\alpha_i$ and $g'_{ij} = \alpha_i g_{ij} \alpha_j^{-1}$. This yields a new pair $(\hat{e}', \theta')$ with $(e'_+)_{ij} = c_i g'_{ij} c_j$ and $\theta' = -\sum_i c_i^2 A'_i$. We show that $(\hat{e}, \theta)$ and $(\hat{e}', \theta')$ represent the same class in $MK_2(\mathscr{A}_P)$ by using the following general fact: if e and $e' = W^{-1} e W$ are conjugate by an invertible W, then $(\hat{e}, \theta)$ is equivalent to $(\hat{e}', \theta + \mathrm{cs}_1(f))$, where f is the idempotent interpolation between the matrices $\left(\begin{smallmatrix} e & 0 \\ 0 & p_0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} W^{-1} e W & 0 \\ 0 & p_0 \end{smallmatrix}\right)$ constructed as in Lemma 5.5. One has

$$\mathrm{cs}_1(f) \equiv \mathrm{Tr}(W^{-1}(e - p_0) dW) \bmod d \quad \text{in } \Omega^1(Q) \ ,$$

so that finally $(\hat{e}, \theta)$ is equivalent to $(\hat{e}', \theta + \mathrm{Tr}(W^{-1}(e - p_0) dW))$. In the present situation $g'_{ij} = \alpha_i g_{ij} \alpha_j^{-1}$, hence the idempotent e'_+ is conjugate to e_+ via the diagonal matrix $W_+ = \mathrm{diag}\ (\alpha_1^{-1} \ldots, \alpha_m^{-1})$, and of course $e'_- = e_- = 1_{\mathscr{A}_P}$ so one can choose $W_- = 1$. One calculates

$$\begin{aligned} \mathrm{Tr}(W^{-1}(e - p_0) dW) &= \mathrm{Tr}(W_+^{-1} e_+ dW_+) \ = \ \sum_i \alpha_i c_i^2 d(\alpha_i^{-1}) \\ &= -\sum_i c_i^2 (A'_i - A_i) \ = \ \theta' - \theta \ , \end{aligned}$$

hence $(\hat{e}, \theta)$ and $(\hat{e}', \theta')$ represent the same multiplicative K-theory class.
Now we leave the cocycle (A_i, g_{ij}) fixed and change the partition of unity (c_i) to (c'_i), $\sum_i (c'_i)^2 = 1$, whence a new idempotent $(e'_+)_{ij} = c'_i g_{ij} c'_j$ and a new cochain $\theta' = -\sum_i (c'_i)^2 A_i$. Introduce the matrices $u_{ij} = c_i g_{ij} c'_j$ and $v_{ij} = c'_i g_{ij} c_j$. Then one has $e_+ = uv$, $e'_+ = vu$, and the invertible matrix $W_+ = \left(\begin{smallmatrix} 1-uv & -u \\ v & 1-vu \end{smallmatrix}\right)$ stably conjugates e_+ and e'_+ in the sense that $W_+^{-1} \left(\begin{smallmatrix} e_+ & 0 \\ 0 & 0 \end{smallmatrix}\right) W_+ = \left(\begin{smallmatrix} 0 & 0 \\ 0 & e'_+ \end{smallmatrix}\right)$. A direct computation yields

$$\mathrm{Tr}(W_+^{-1} \left(\begin{smallmatrix} e_+ & 0 \\ 0 & 0 \end{smallmatrix}\right) dW_+) = \sum_{i,j} (c'_i)^2 c_j^2 g_{ij} dg_{ji} = \sum_{i,j} (c'_i)^2 c_j^2 (A_j - A_i) = \theta' - \theta \ ,$$

hence the multiplicative K-theory class of $(\hat{e}, \theta)$ does not depend on the choice of partition of unity. A similar argument shows that it does not depend on the

good covering (U_i).
It remains to calulate the lowest component of the negative Chern character. By definition $\mathrm{ch}_2(\hat{e},\theta)$ is the cycle of even degree in $X(\widehat{T}\mathscr{A}_P)$ given by $\mathrm{Tr}(\hat{e}-p_0)-\overline{b}\tilde{\theta}$, where $\tilde{\theta}$ is an arbitrary lifting of $\theta \in X(\mathscr{A}_P)$. Thus, the image of $\mathrm{ch}_2(\hat{e},\theta)$ under the chain map $X(\widehat{T}\mathscr{A}_P) \to \Omega^*(Q)$ has a component of degree two given by

$$-\mathrm{Tr}((e-\frac{1}{2})dede) - d\theta = -\sum_i d(c_i^2)A_i + d\sum_i c_i^2 A_i = \sum_i c_i^2 dA_i = dA_i \ ,$$

and coincides with the curvature of the line bundle (A_i, g_{ij}). ∎

If one forgets the connection A_i, the morphism $H^2_{\mathscr{D}}(Q;\mathbb{Z}(1)) \to MK_2(\mathscr{A}_P)$ just reduces to the elementary map $\check{H}^2(Q;\mathbb{Z}(1)) \to K_2^{\mathrm{top}}(\mathscr{A}_P)$ which associates to an isomorphism class of line bundles over Q its topological K-theory class.

Example 7.4 The simplest non-trivial example is provided by the celebrated non-commutative torus [3]. Here $\mathscr{A}$ is the algebra of smooth functions over the circle $N = S^1 = \mathbb{Z}\backslash\mathbb{R}$. Conventionnally we parametrize the points of N by the variable y. The group $\Gamma = \mathbb{Z}$ acts on N by rotations of angle $\alpha \in \mathbb{R}$:

$$y \mapsto y + n\alpha \qquad \forall\, y \in N \ ,\ n \in \mathbb{Z} \ .$$

When $\mathbb{Z}$ is provided with its natural distance, the crossed product $\mathscr{A} \rtimes \mathbb{Z}$ is isomorphic to the algebra $\mathscr{A}_\alpha$ of the non-commutative torus, presented for example in [3] by generators and relations: let $V_1 \in \mathscr{A}$ be the function $V_1(y) = e^{2\pi i y}$ over N and $V_2 = 1^* \in \mathbb{C}\mathbb{Z}$ be the element corresponding to the generator $1 \in \mathbb{Z}$. Then V_1 and V_2 are invertible elements of $\mathscr{A}_\alpha$ and fulfill the noncommutativity relation

$$V_2 V_1 = e^{2\pi i\alpha} V_1 V_2 \ . \tag{97}$$

Moreover any element of $\mathscr{A}_\alpha$ is a power series $\sum_{(n_1,n_2)\in\mathbb{Z}^2} a_{n_1 n_2} V_1^{n_1} V_2^{n_2}$ with coefficients $a_{n_1 n_2} \in \mathbb{C}$ of rapid decay. For $\alpha \in \mathbb{Q}$ this algebra is Morita equivalent to the smooth functions over an ordinary (commutative) 2-torus, and its multiplicative K-theory in any degree turns out to be completely determined by Deligne cohomology in this case. The situation is more interesting for $\alpha \notin \mathbb{Q}$. Following the discussion above we introduce the universal principal $\mathbb{Z}$-bundle $P = E\mathbb{Z} = \mathbb{R}$ over the classifying space $M = B\mathbb{Z} = \mathbb{Z}\backslash\mathbb{R}$. Conventionnally we parametrize the points of P by the variable x. Thus, $\mathscr{A}_P = C^\infty(P;\mathscr{A})^{\mathbb{Z}}$ is the mapping torus algebra

$$\mathscr{A}_P = \{a \in C^\infty(P \times N) \mid a(x+1, y+\alpha) = a(x,y) \ ,\ \forall\, x \in P \ ,\ y \in N\} \ .$$

Equivalently it is the algebra of smooth functions over the commutative 2-torus $Q = \mathbb{Z}\backslash(P \times N)$, quotient of $\mathbb{R}^2$ by the lattice generated by the vectors $(1,\alpha)$ and $(0,1)$. Now to get a quasihomomorphism from $\mathscr{A}_P$ to $\mathscr{A}_\alpha$ we need a K-cycle

D for the circle manifold M. Let $H = L^2(M)$ be the Hilbert space of square-integrable complex-valued functions. The algebra $C^\infty(M)$ is represented in the algebra of bounded operators $\mathscr{L}(H)$ by pointwise multiplication. D will be represented by the Toeplitz operator in $\mathscr{L}(H)$ which projects H onto the Hardy space $H_+ \subset H$:

$$D(e^{2\pi i\, nx}) = \begin{cases} e^{2\pi i\, nx} & \text{if} \quad n \geq 0 \\ 0 & \text{if} \quad n < 0 \end{cases}$$

One thus obtains a polarization of the Hilbert space $H = H_+ \oplus H_-$, with H_- the kernel of D. In the Fourier basis $e^{2\pi i\, nx}$ of H, the representation $C^\infty(M) \to \mathscr{L}(H)$ is easily seen to factor through the matrix subalgebra

$$\begin{pmatrix} \mathscr{T} & \mathscr{K} \\ \mathscr{K} & \mathscr{T} \end{pmatrix} \subset \begin{pmatrix} \mathscr{L}(H_+) & \mathscr{L}(H_-, H_+) \\ \mathscr{L}(H_+, H_-) & \mathscr{L}(H_-) \end{pmatrix} = \mathscr{L}(H) \ ,$$

where $\mathscr{T}$ is the *smooth Toeplitz algebra* (the elementary non-trivial extension of $C^\infty(S^1)$ by th ealgebra $\mathscr{K}$ of smooth compact operators, see [9]). The induced quasihomomorphism $\rho_D : \mathscr{A}_P \to \mathscr{E}^s \triangleright \mathscr{K}^s \hat{\otimes} \mathscr{A}_\alpha$, with $\mathscr{E} = \mathscr{T} \hat{\otimes} \mathscr{A}_\alpha$, is therefore 1-summable and of odd parity. Theorem 6.3 yields a graded-commutative diagram (remark that $MK_n^{\mathscr{K}} \cong MK_n$)

$$\begin{tikzcd} K_{n+1}^{\rm top}(\mathscr{A}_P) \arrow[r] \arrow[d] & HC_{n-1}(\mathscr{A}_P) \arrow[r] \arrow[d, "\mathrm{ch}^1(\rho_D)"] & MK_n(\mathscr{A}_P) \arrow[r] \arrow[d] & K_n^{\rm top}(\mathscr{A}_P) \arrow[d] \\ K_n^{\rm top}(\mathscr{A}_\alpha) \arrow[r] & HC_{n-2}(\mathscr{A}_\alpha) \arrow[r] & MK_{n-1}(\mathscr{A}_\alpha) \arrow[r] & K_{n-1}^{\rm top}(\mathscr{A}_\alpha) \end{tikzcd} \qquad (98)$$

in any degree $n \in \mathbb{Z}$. The group $K_n^{\rm top}(\mathscr{A}_P)$ is isomorphic to the topological K-theory of the 2-torus Q. Hence in even degree, $K_0^{\rm top}(\mathscr{A}_P) = \mathbb{Z} \oplus \mathbb{Z}$ is generated by the trivial line bundle over Q together with the Bott class, whereas in odd degree $K_1^{\rm top}(\mathscr{A}_P) = \mathbb{Z} \oplus \mathbb{Z}$ is generated by the invertible functions $g_1(x,y) = e^{2\pi i\, x}$ and $g_2(x,y) = e^{2\pi i(\alpha x - y)}$. The pushforward map $K_n^{\rm top}(\mathscr{A}_P) \to K_{n-1}^{\rm top}(\mathscr{A}_\alpha)$ is known to be an isomorphism (Baum-Connes). In particular the Bott class and the trivial line bundle over Q are mapped respectively to the classes of the invertible elements V_1 and V_2 in $K_1^{\rm top}(\mathscr{A}_\alpha)$. For multiplicative K-theory the situation is more involved. In degree $n = 1$ the map

$$MK_1(\mathscr{A}_P) \cong H^1_{\mathscr{D}}(Q; \mathbb{Z}(1/2)) \cong C^\infty(Q)^\times \to MK_0(\mathscr{A}_\alpha) \cong K_0^{\rm top}(\mathscr{A}_\alpha)$$

simply factors through the topological K-theory group $K_1^{\rm top}(\mathscr{A}_P)$. In degree $n = 2$ one still has an isomorphism $MK_2(\mathscr{A}_P) \cong H^2_{\mathscr{D}}(Q; \mathbb{Z}(1))$, and (98) amounts to

$$\begin{tikzcd} K_1^{\rm top}(\mathscr{A}_P) \arrow[r, "\mathrm{ch}_1"] \arrow[d] & \frac{\Omega^1(Q)}{d\Omega^0(Q)} \arrow[r] \arrow[d, "\mathrm{ch}^1(\rho_D)"] & H^2_{\mathscr{D}}(Q; \mathbb{Z}(1)) \arrow[r] \arrow[d] & K_0^{\rm top}(\mathscr{A}_P) \arrow[d] & \\ K_0^{\rm top}(\mathscr{A}_\alpha) \arrow[r] & HC_0(\mathscr{A}_\alpha) \arrow[r] & MK_1(\mathscr{A}_\alpha) \arrow[r] & K_1^{\rm top}(\mathscr{A}_\alpha) \arrow[r] & 0 \end{tikzcd} \qquad (99)$$

The map $\Omega^1/d\Omega^0 \to H^2_{\mathscr{D}}(Q;\mathbb{Z}(1))$ associates to a one-form $\frac{1}{\sqrt{2\pi i}}A$ over Q the isomorphism class of the trivial line bundle with connection $-A$, while the range of $H^2_{\mathscr{D}}(Q;\mathbb{Z}(1)) \to K_0^{\mathrm{top}}(\mathscr{A}_P)$ is generated by the Bott class. For generic values of $\alpha \notin \mathbb{Q}$ the commutator subspace $[\mathscr{A}_\alpha, \mathscr{A}_\alpha]$ may not be closed in $\mathscr{A}_\alpha$, therefore the quotient $HC_0(\mathscr{A}_\alpha) = HH_0(\mathscr{A}_\alpha) = \mathscr{A}_\alpha/[\mathscr{A}_\alpha, \mathscr{A}_\alpha]$ may not be separated. However the quotient $\overline{HC}_0(\mathscr{A}_\alpha)$ by the *closure* of the commutator subspace turns out to be isomorphic to $\mathbb{C}$, via the canonical trace

$$\mathscr{A}_\alpha \to \mathbb{C}\ , \qquad V_1^{n_1}V_2^{n_2} \mapsto \begin{cases} 1 & \text{if } n_1 = n_2 = 0\ , \\ 0 & \text{otherwise}\ . \end{cases}$$

With these identifications the evaluation of the Chern character $\mathrm{ch}^1(\rho_D)$: $\Omega^1/d\Omega^0 \to \overline{HC}_0(\mathscr{A}_\alpha) \cong \mathbb{C}$ on a one-form $A = A_x dx + A_y dy$ is easily performed and one finds

$$\mathrm{ch}^1(\rho_D)\left(\frac{A}{\sqrt{2\pi i}}\right) = \frac{1}{2\pi i}\int_0^1 dy \int_0^1 dx\, A_x(x,y)\ .$$

In particular $\mathrm{ch}^1(\rho_D)\cdot \mathrm{ch}_1(g_1) = 1$ and $\mathrm{ch}^1(\rho_D)\cdot\mathrm{ch}_1(g_2) = \alpha$, and one recovers the well-known fact ([4]) that the image of $K_0^{\mathrm{top}}(\mathscr{A}_\alpha)$ in $\overline{HC}_0(\mathscr{A}_\alpha)$ is the subgroup $\mathbb{Z} + \alpha\mathbb{Z} \subset \mathbb{C}$.
We may analogously define a new multiplicative K-theory group $\overline{MK}_1(\mathscr{A}_\alpha)$ whose elements are represented by pairs $(\hat{g}, \theta)$ with $\theta \in \mathbb{C}$ instead of $\theta \in \mathscr{A}_\alpha/[\mathscr{A}_\alpha, \mathscr{A}_\alpha]$. Because $K_1^{\mathrm{top}}(\mathscr{A}_\alpha)$ is generated by the invertibles V_1 and V_2 any class in $\overline{MK}_1(\mathscr{A}_\theta)$ is represented by a pair $(V_1^{n_1}V_2^{n_2}, \theta)$ for some integers n_1, n_2 and a complex number θ. Using a homotopy one shows that this pair is equivalent to $(e^{-\sqrt{2\pi i}\theta}V_1^{n_1}V_2^{n_2}, 0)$, and by exactness $\overline{MK}_1(\mathscr{A}_\alpha)$ is the quotient of the multiplicative group $\mathbb{C}^\times\langle V_1\rangle\langle V_2\rangle \subset \mathrm{GL}_1(\mathscr{A}_\alpha)$ by its commutator subgroup $\langle e^{2\pi i\alpha}\rangle \subset \mathbb{C}^\times$, or equivalently the abelianization

$$\overline{MK}_1(\mathscr{A}_\alpha) = (\mathbb{C}^\times\langle V_1\rangle\langle V_2\rangle)_{\mathrm{ab}}\ . \tag{100}$$

Since the Bott class of $K_0(\mathscr{A}_P)$ is sent to $[V_1] \in K_1^{\mathrm{top}}(\mathscr{A}_\alpha)$, one sees that the range of $H^2_{\mathscr{D}}(Q;\mathbb{Z}(1)) \to \overline{MK}_1(\mathscr{A}_\alpha)$ coincides with the subgroup $\mathbb{C}^\times\langle V_1\rangle$.

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