Degeneration of Polylogarithms and Special Values of L-Functions for Totally Real Fields

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Abstract. The degeneration of the polylogarithm on the universal abelian scheme over a Hilbert modular variety at the boundary is described in terms of (critical) special values of the L-function of the totally real field defining the variety. This gives a relation between the polylogarithm on abelian schemes and special values of L-functions.

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Introduction

The polylogarithm is a very powerful tool in studying special values of L-functions and subject to many conjectures. Most notably, the Zagier conjecture claims that all values of L-functions of number fields can be described by polylogarithms. The interpretation of the polylogarithm functions in terms of periods of variations of Hodge structures has lead to a motivic theory of the polylog and to generalizations as the elliptic polylog by Beilinson and Levin. Building on this work, Wildeshaus has defined polylogarithms in a more general context and in particular for abelian schemes.
Not very much is known about the extension classes arising from these "abelian polylogarithms". In an earlier paper [K] we were able to show that the abelian

[1]Dedicated to John Coates with admiration and respect.
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polylogarithm, is indeed of motivic origin, i.e., is in the image of the regulator from K-theory.
It was Levin in [L], who started to investigate certain "polylogarithmic currents" on abelian schemes, which are related to the construction by Wildeshaus. In [B1], Blottière could show that these currents actually represent the polylogarithmic extension in the category of Hodge modules.
In this paper we will, following and extending ideas from the case of the elliptic polylog treated in [HK] (which is in turn inspired by [BL]), consider the problem of the degeneration of the abelian polylog on Hilbert modular varieties. The main result will describe this degeneration in terms of (critical) special values of L-functions of the totally real field, which defines the Hilbert modular variety. To describe the theorem more precisely, consider the specialization of the polylog, which gives Eisenstein classes (say in the category of mixed étale sheaves to fix ideas)

$$\mathrm{Eis}^k(\alpha) \in \mathrm{Ext}_S^{2g-1}(\mathbb{Q}_\ell, \mathrm{Sym}^k \mathcal{H}(g)),$$

where S is the Hilbert modular variety of dimension g and $\mathcal{H}$ is the locally constant sheaf of relative Tate-modules of the universal abelian scheme. Let $j : S \to \overline{S}$ be the Baily-Borel compactification of S and $i : \partial S := \overline{S} \setminus S \to \overline{S}$ the inclusion of the cusps. The degeneration or residue map (see 1.5.2 for the precise definition) is then

$$\mathrm{res} : \mathrm{Ext}_S^{2g-1}(\mathbb{Q}_\ell, \mathrm{Sym}^k \mathcal{H}(g)) \to \mathrm{Hom}_{\partial S}(\mathbb{Q}_\ell, \mathbb{Q}_\ell).$$

The target of this map is sitting inside a sum of copies of $\mathbb{Q}_\ell$ and the main result of this paper 1.7.1 describes $\mathrm{res}(\mathrm{Eis}^k(\alpha))$ in terms of special values of (partial) L-functions of the totally real field defining S.
The same result was also obtained by Blottière in [B2] with different methods. His computation uses the explicit description of the polylogarithm in terms of the currents constructed by Levin.
Our method of proof is inspired by [BL] 2.4. and [HK] and follows a different line. Instead of computing directly the degeneration on the base we work with the polylog, which lives on the universal abelian scheme, and use the fact that the universal abelian variety can be written in a neighborhood of the cusps as an extension of real tori. The idea is to view the problem as of topological nature and use the good functorial properties of the topological polylog to compute the degeneration. In fact, we avoid computations by reducing to the situation considered by Nori [N] and Sczech [Sc]. For the convenience of the reader we reproduce then their computations, which lead to the relation with the L-values.
There is a very interesting question raised by the results in this paper. In [HK] we were able to construct extension classes related to non-critical values of Dirichlet-L-functions, if the residue map was zero on the specialization of the polylog. Is there an analogous result here?
The paper is organized as follows: In the first section we review the definition of the Hilbert modular variety, define the residue or degeneration map and

formulate our main theorem. The second section reviews the theory of the polylog and the Eisenstein classes emphasing the topological situation, which is not extensively covered in the literature. In the third section we give the proof of the main theorem.

It is a pleasure to thank David Blottière for a series of interesting and stimulating discussions during his stay in Regensburg. Moreover, I like to thank Sascha Beilinson for making available some time ago his notes about his and A. Levin's interpretation of Nori's work.

1 Polylogarithms and degeneration

We review the definition of a Hilbert modular variety to fix notations and pose the problem of computing the degeneration of the specializations of the polylogarithm at the boundary. The main theorem describes this residue in terms of special values of L-functions.

1.1 Notation

As in [BL] we deal with three different types of sheaves simultaneously. Let X/k be a variety and L a coefficient ring for our sheaf theory, then we consider

i) $k = \mathbb{C}$ the usual topology on $X(\mathbb{C})$ and L any commutative ring

ii) $k = \mathbb{R}$ or $\mathbb{C}$ and $L = \mathbb{Q}$ or $\mathbb{R}$ and we work with the category of mixed Hodge modules

iii) $k = \mathbb{Q}$ and $L = \mathbb{Z}/l^r\mathbb{Z}, \mathbb{Z}_l$ or $\mathbb{Q}_l$ and we work with the category of étale sheaves

1.2 Hilbert modular varieties

We recall the definition of Hilbert modular varieties following Rapoport [R]. To avoid all technicalities, we will only consider the moduli scheme over $\mathbb{Q}$. The theory works over more general base schemes without any modification.

Let F be a totally real field, $g := [F : \mathbb{Q}]$, $\mathcal{O}$ the ring of integers, $\mathfrak{D}^{-1}$ the inverse different and d_F its discriminant. Fix an integer $n \geq 3$. We consider the functor, which associates to a scheme T over $\operatorname{Spec}\mathbb{Q}$ the isomorphism classes of triples (A, α, λ), where A/T is an abelian scheme of dimension g, with real multiplication by $\mathcal{O}$, $\alpha : \underline{\mathrm{Hom}}_{et,\mathcal{O},sym}(\mathcal{A}, \mathcal{A}^*) \to \mathfrak{D}^{-1}$ is a $\mathfrak{D}^{-1}$-polarization in the sense of [R] 1.19, i.e., an $\mathcal{O}$-module isomorphism respecting the positivity of the totally positive elements in $\mathfrak{D}^{-1} \subset F$, and $\lambda : A[n] \cong (\mathcal{O}/n\mathcal{O})^2$ is a level n structure satisfying the compatibility of [R] 1.21. For $n \geq 3$ this functor is represented by a smooth scheme $S := S_n^{\mathfrak{D}^{-1}}$ of finite type over $\operatorname{Spec}\mathbb{Q}$. Let

$$\mathcal{A} \xrightarrow{\pi} S$$

be the universal abelian scheme over S. In any of the three categories of sheaves i)-iii) from 1.1 we let

$$\mathcal{H} := \underline{\mathrm{Hom}}_S(R^1\pi_*L, L)$$

the first homology of $\mathcal{A}/S$. In the étale case and $L = \mathbb{Z}_\ell$, the fiber of $\mathcal{H}$ at a point is the Tate module of the abelian variety over that point.

1.3 Transcendental description

For the later computation we need a description in group theoretical terms of the complex points $S(\mathbb{C})$ and of $\mathcal{H}$.
Define a group scheme $G/\operatorname{Spec}\mathbb{Z}$ by the Cartesian diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d] & \mathrm{Res}_{\mathcal{O}/\mathbb{Z}}\,\mathrm{Gl}_2 \arrow[d, "\det"] \\ \mathbb{G}_m \arrow[r] & \mathrm{Res}_{\mathcal{O}/\mathbb{Z}}\,\mathbb{G}_m \end{tikzcd}$$

and let

$$\mathfrak{H}^g_\pm := \{\tau \in F\otimes\mathbb{C} | Im\,\tau \text{ totally positive or totally negative}\}.$$

Then $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathbb{R})$ acts on $\mathfrak{H}^g_\pm$ by the usual formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\tau = \frac{a\tau + b}{c\tau + d}$$

and the stabilizer of $1\otimes i \in \mathfrak{H}^g_\pm$ is

$$K_\infty := (F\otimes\mathbb{C})^* \cap G(\mathbb{R}),$$

so that

$$\mathfrak{H}^g_\pm \cong G(\mathbb{R})/K_\infty.$$

With this notation one has

$$S(\mathbb{C}) = G(\mathbb{Z})\backslash(\mathfrak{H}^g_\pm \times G(\mathbb{Z}/n\mathbb{Z})).$$

On $S(\mathbb{C})$ acts $G(\mathbb{Z}/n\mathbb{Z})$ by right multiplication. The determinant $\det : G \to \mathbb{G}_m$ induces

$$S(\mathbb{C}) \to \mathbb{G}_m(\mathbb{Z}/n\mathbb{Z})$$

and the fibers are the connected components. Define a subgroup $D \subset G$ isomorphic to $\mathbb{G}_m$ by $D := \{\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \in G : a \in \mathbb{G}_m\big)$. This gives a section of det. Then the action of $D(\mathbb{Z}/n\mathbb{Z})$ by right multiplication is transitive on the set of connected components.

The embedding $G(\mathbb{Z}) \subset \mathrm{Gl}_2(\mathcal{O})$ defines an action of $G(\mathbb{Z})$ on $\mathcal{O}^{\oplus 2}$ and in the topological realization the local system $\mathcal{H}$ is given by the quotient

$$G(\mathbb{Z})\backslash(\mathfrak{H}_{\pm}^{g} \times \mathcal{O}^{\oplus 2} \times G(\mathbb{Z}/n\mathbb{Z})).$$

In particular, as a family of real ($2g$-dimensional) tori, the complex points $\mathcal{A}(\mathbb{C})$ of the universal abelian scheme can be written as

$$G(\mathbb{Z})\backslash(\mathfrak{H}_{\pm}^{g} \times (F \otimes \mathbb{R}/\mathcal{O})^{\oplus 2} \times G(\mathbb{Z}/n\mathbb{Z}))$$

and the level n structure is given by the subgroup

$$(\frac{1}{n}\mathcal{O}/\mathcal{O})^{\oplus 2} \subset (F \otimes \mathbb{R}/\mathcal{O})^{\oplus 2}.$$

The $\mathcal{O}$-multiplication on $\mathcal{A}(\mathbb{C})$ is in this description given by the natural $\mathcal{O}$-module structure on $F \otimes \mathbb{R}$.

1.4 Transcendental description of the cusps

The following description of the boundary cohomology is inspired by [H]. For further details we refer to [H] 2.1. Let $B \subset G$ the subgroup of upper triangular matrices, $T \subset B$ its maximal torus and $N \subset B$ its unipotent radical. We have an exact sequence

$$1 \to N \to B \xrightarrow{q} T \to 1.$$

We denote by G^1, B^1 and T^1 the subgroups of determinant 1. Note that $G^1 = \mathrm{Res}_{\mathcal{O}/\mathbb{Z}} \mathrm{Sl}_2$. Let $K_\infty^B := B(\mathbb{R}) \cap K_\infty$, then the Cartan decomposition shows that $\mathfrak{H}_{\pm}^{g} = B(\mathbb{R})/K_\infty^B$. A pointed neighborhood of the set of all cusps is given by

$$\widetilde{S}_B := B(\mathbb{Z})\backslash\big(B(\mathbb{R})/K_\infty^B \times G(\mathbb{Z}/n\mathbb{Z})\big). \tag{1}$$

In particular, the set of cusps is (cf. also [R] p.305)

$$\partial S(\mathbb{C}) = B^1(\mathbb{Z})\backslash G(\mathbb{Z}/n\mathbb{Z}). \tag{2}$$

The fibres of the map $\partial S(\mathbb{C}) \to \mathbb{G}_m(\mathbb{Z}/n\mathbb{Z})$ induced by the determinant are

$$B^1(\mathbb{Z})\backslash G^1(\mathbb{Z}/n\mathbb{Z}) \cong \Gamma_G\backslash\mathbb{P}^1(\mathcal{O}), \tag{3}$$

where $\Gamma_G := \ker(G^1(\mathbb{Z}) \to G^1(\mathbb{Z}/n\mathbb{Z}))$. In particular, we can think of a cusp represented by $h \in G^1(\mathbb{Z}/n\mathbb{Z})$ as a rank 1 $\mathcal{O}$-module $\mathfrak{b}_h$, which is a quotient

$$\mathcal{O}^2 \xrightarrow{p_h} \mathfrak{b}_h, \tag{4}$$

together with a level structure, i.e., a basis $h \in G^1(\mathbb{Z}/n\mathbb{Z})$. Explicitly, the fractional ideal $\mathfrak{b}_h$ is generated by any representatives $u, v \in \mathcal{O}$ of the second row of h.

On $\widetilde{S}_B$ acts $G(\mathbb{Z}/n\mathbb{Z})$ by multiplication from the right. This action is transitive on the connected components of $\widetilde{S}_B$. Define

$$S_B := B(\mathbb{Z})\backslash\big(B(\mathbb{R})/K_\infty^B \times B(\mathbb{Z}/n\mathbb{Z})\big), \tag{5}$$

then $S_B \subset \widetilde{S}_B$ is a union of connected components of $\widetilde{S}_B$. Let K_∞^T (respectively $T(\mathbb{Z})$) be the image of K_∞^B (respectively $B(\mathbb{Z})$) under $q : B(\mathbb{R}) \to T(\mathbb{R})$. Define

$$S_T := T(\mathbb{Z})\backslash\big(T(\mathbb{R})/K_\infty^T \times T(\mathbb{Z}/n\mathbb{Z})\big), \tag{6}$$

then the map $q : B \to T$ induces a fibration

$$q : S_B \to S_T, \tag{7}$$

whose fibers are $N(\mathbb{Z})\backslash\big(N(\mathbb{R})\times N(\mathbb{Z}/n\mathbb{Z})\big)$ with $N(\mathbb{Z}) := B(\mathbb{Z})\cap N(\mathbb{R})$. Denote by

$$u : S_T \to pt \tag{8}$$

the structure map to a point. For the study of the degeneration, one considers the diagram

$$\begin{array}{ccccc} S_B & \xrightarrow{q} & S_T & \xrightarrow{u} & pt \\ \downarrow & & & & \\ S & & & & \end{array} \tag{9}$$

In fact we are interested in the cohomology of certain local systems on these topological spaces. For the computations it is convenient to replace S_B and S_T by homotopy equivalent spaces as follows.
Define $K_\infty^{T^1} := K_\infty^T \cap T^1(\mathbb{R})$ and note that this is the kernel of the determinant $K_\infty^T \to \mathbb{R}^*$. Then the inclusion induces an isomorphism

$$T^1(\mathbb{Z})\backslash\big(T^1(\mathbb{R})/K_\infty^{T^1} \times T(\mathbb{Z}/n\mathbb{Z})\big) \cong S_T.$$

The map $a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}\right)$ defines isomorphisms $(F\otimes\mathbb{R})^* \cong T^1(\mathbb{R})$ and $\mathcal{O}^* \cong T^1(\mathbb{Z})$. Note that $K_\infty^{T^1} \subset (F\otimes\mathbb{R})^*$ is identified with the two torsion subgroup in $(F\otimes\mathbb{R})^*$ and that $K_\infty^{T^1} \cong (\mathbb{Z}/2\mathbb{Z})^g$ permutes the set of connected components of $T^1(\mathbb{R})$.

Lemma 1.4.1. *Let $(F \otimes \mathbb{R})^1 \subset T^1(\mathbb{R}) = (F \otimes \mathbb{R})^*$ be the subgroup of elements of norm 1 and $\mathcal{O}^{*,1} = \mathcal{O}^* \cap (F \otimes \mathbb{R})^1$. Then*

$$S_T^1 := \mathcal{O}^{*,1}\backslash\big((F \otimes \mathbb{R})^1/K_\infty^{T^1} \cap (F \otimes \mathbb{R})^1 \times T(\mathbb{Z}/n\mathbb{Z})\big)$$

is homotopy equivalent to S_T. Moreover, the inclusion of the totally positive elements $(F \otimes \mathbb{R})^1_+$ into $(F \otimes \mathbb{R})^1$ provides an identification

$$(F \otimes \mathbb{R})^1_+ \cong (F \otimes \mathbb{R})^1/K_\infty^{T^1} \cap (F \otimes \mathbb{R})^1.$$

Proof. The exact sequence

$$0 \to (F \otimes \mathbb{R})^1 \to (F \otimes \mathbb{R})^* \to \mathbb{R}^* \to 0$$

together with the fact that $K_\infty^{T^1}$ is the two torsion in $(F \otimes \mathbb{R})^*$ allows to identify

$$T^1(\mathbb{R})/K_\infty^{T^1} \cong \big((F \otimes \mathbb{R})^1/K_\infty^{T^1} \cap (F \otimes \mathbb{R})^1\big) \times \mathbb{R}_{>0}.$$

The last identity is clear. □

We define S_B^1 to be the inverse image of S_T^1 under q, so that we have a Cartesian diagram

$$\begin{tikzcd} S_B^1 \arrow[r, "q"] \arrow[d] & S_T^1 \arrow[d] \\ S_B \arrow[r, "q"] & S_T. \end{tikzcd} \tag{10}$$

Over S_B the representation $\mathcal{O}^2$ has a filtration

$$0 \to \mathcal{O} \to \mathcal{O}^2 \xrightarrow{p} \mathcal{O} \to 0, \tag{11}$$

where the first map sends $a \in \mathcal{O}$ to the vector $\binom{a}{0}$ and the second map is $\binom{a}{b} \mapsto b$. This induces a filtration on the local system $\mathcal{H}$

$$0 \to \mathcal{N} \to \mathcal{H} \to \mathcal{M} \to 0, \tag{12}$$

where $\mathcal{N}$ and $\mathcal{M}$ are the associated local systems. In particular, over S_B^1 one has a filtration of topological tori

$$0 \to \mathcal{T}_\mathcal{N} \to \mathcal{A}(\mathbb{C}) \xrightarrow{p} \mathcal{T}_\mathcal{M} \to 0, \tag{13}$$

where $\mathcal{T}_\mathcal{N} := \mathcal{N} \otimes \mathbb{R}/\mathbb{Z}$ and $\mathcal{T}_\mathcal{M} := \mathcal{M} \otimes \mathbb{R}/\mathbb{Z}$. By definition of N the fibration in (13) and (10) are compatible, i.e., one has a commutative diagram

$$\begin{tikzcd} \mathcal{A}(\mathbb{C}) \arrow[r, "p"] \arrow[d, "\pi"'] & \mathcal{T}_\mathcal{M} \arrow[d, "\pi_\mathcal{M}"] \\ S_B^1 \arrow[r, "q"] & S_T^1. \end{tikzcd} \tag{14}$$

1.5 The degeneration map

In this section we explain the degeneration problem we want to consider.
The polylogarithm on $\pi : \mathcal{A} \to S$ defines for certain linear combinations α of torsion sections of $\mathcal{A}$ an extension class

$$\operatorname{Eis}^k(\alpha) \in \operatorname{Ext}^{2g-1}_{?,S}(L, \operatorname{Sym}^k \mathcal{H}(g)), \tag{15}$$

where ? can be MHM, et, top. The construction of this class will be given in section 2 definition 2.4.1.
Let $\overline{S}$ be the Baily-Borel compactification of S. Denote by $\partial S := \overline{S} \setminus S$ the set of cusps. We get

$$\partial S \xrightarrow{i} \overline{S} \xleftarrow{j} S.$$

The adjunction map together with the edge morphism in the Leray spectral sequence for Rj_* gives

$$\begin{tikzcd} \operatorname{Ext}^{2g-1}_S(L, \operatorname{Sym}^k \mathcal{H}(g)) \arrow[r] \arrow[dr] & \operatorname{Ext}^{2g-1}_{\partial S}(L, i^* Rj_* \operatorname{Sym}^k \mathcal{H}(g)) \arrow[d] \\ & \operatorname{Hom}_{\partial S}(L, i^* R^{2g-1} j_* \operatorname{Sym}^k \mathcal{H}(g)). \end{tikzcd} \tag{16}$$

There are several possibilities to compute $i^* R^{2g-1} j_* \operatorname{Sym}^k \mathcal{H}(g)$.

THEOREM 1.5.1. *Assume that $\mathbb{Q} \subset L$. Then, in any of the categories MHM, et, top, there is a canonical isomorphism*

$$i^* R^{2g-1} j_* \operatorname{Sym}^k \mathcal{H}(g) \cong L,$$

where L has the trivial Hodge structure (resp. the trivial Galois action).

REMARK: J. Wildeshaus has pointed out that the determination of the weight on the right hand side is not necessary for our main result, but follows from it. In fact, our main result gives non-zero classes in

$$\operatorname{Hom}_{\partial S}(L, i^* R^{2g-1} j_* \operatorname{Sym}^k \mathcal{H}(g)),$$

so that the rank one sheaf $i^* R^{2g-1} j_* \operatorname{Sym}^k \mathcal{H}(g)$ has to be of weight zero.

Using this identification we define the residue or degeneration map:

DEFINITION 1.5.2. *The map from (16) together with the identification of 1.5.1 define the* residue map

$$\operatorname{res} : \operatorname{Ext}^{2g-1}_S(L, \operatorname{Sym}^k \mathcal{H}(g)) \to \operatorname{Hom}_{\partial S}(L, L).$$

The residue map is equivariant for the $G(\mathbb{Z}/n\mathbb{Z})$ action on both sides.

Proof. (of theorem 1.5.1). In the case of Hodge modules we use theorem 2.9. in Burgos-Wildeshaus [BW] and in the étale case we use theorem 5.3.1 in [P2]. Roughly speaking, both results asserts that the higher direct image can be calculated using group cohomology and the "canonical construction", which associates to a representation of the group defining the Shimura variety a Hodge module resp. an étale sheaf.
More precisely, from a topological point of view, the monodromy at the cusps is exactly the cohomology of $\widetilde{S}_B$. One has

$$H^{2g-1}(\widetilde{S}_B, \operatorname{Sym}^k \mathcal{H}(g)) \cong \operatorname{Ind}^{G(\mathbb{Z}/n\mathbb{Z})}_{B(\mathbb{Z}/n\mathbb{Z})} H^{2g-1}(S_B, \operatorname{Sym}^k \mathcal{H}(g))$$

and

$$H^{2g-1}(S_B, \operatorname{Sym}^k \mathcal{H}(g)) \cong \bigoplus_{r+s=2g-1} H^r(S_T, R^s q_* \operatorname{Sym}^k \mathcal{H}(g))).$$

As the cohomological dimension of Γ_T is $g-1$ and that of Γ_N is g, one has in fact

$$H^{2g-1}(S_B, \operatorname{Sym}^k \mathcal{H}(g)) \cong H^{g-1}(S_T, R^g q_* \operatorname{Sym}^k \mathcal{H}(g))).$$

The exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}^2 \xrightarrow{p} \mathcal{O} \to 0$$

from (11) shows that $R^g q_* \operatorname{Sym}^k \mathcal{H}(g)$ can be identified via p with $\operatorname{Sym}^k \mathcal{O} \otimes L$ with the induced $T(\mathbb{Z})$ action, which maps $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ to d^k. To compute the coinvariants, extend the coefficients to $\mathbb{R}$, so that

$$\mathcal{O} \otimes \mathbb{R} \cong \bigoplus_{\tau: F \to \mathbb{R}} \mathbb{R}$$

and $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \in T(\mathbb{Z})$ acts via $\tau(d)$ on the component indexed by τ. Thus $\operatorname{Sym}^k \mathcal{O} \otimes L$ can only have a trivial quotient, if $k \equiv 0 \bmod g$ and on this one dimensional quotient the action is by the norm map $T(\mathbb{Z}) \to \pm 1$. One gets:

$$H^{g-1}(S_T, \operatorname{Sym}^k \mathcal{O} \otimes L) \cong \begin{cases} L & \text{if } k \equiv 0 \bmod g \\ 0 & \text{else} \end{cases}$$

The above mentioned theorems imply that this topological computation gives also the result in the categories MHM, et, top. The Hodge structure on $H^{g-1}(S_T, \operatorname{Sym}^k \mathcal{O} \otimes L)$ is the trivial one, as one sees from the explicit description of the action of T and the fact that the action of the Deligne torus $\mathbb{S}$, which defines the weight, is induced from the embedding $x \mapsto \left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)$., hence is trivial. The same remark and proposition 5.5.4. in [P2] show that the weight is also zero in the étale case. □

1.6 Partial zeta functions of totally real fields

Let $\mathfrak{b}, \mathfrak{f}$ be relatively prime integral ideals of $\mathcal{O}$, $\epsilon : (\mathbb{R} \otimes F)^* \to \{\pm 1\}$ a sign character. This is a product of characters $\epsilon_\tau : \mathbb{R}^* \to \{\pm 1\}$ for all embeddings $\tau : F \to \mathbb{R}$. Denote by $|\epsilon|$ the number of non-trivial ϵ_τ which occur in this product decomposition of ϵ. Moreover let $x \in \mathcal{O}$ such that $x \not\equiv 0 \bmod \mathfrak{b}^{-1}\mathfrak{f}$ and $\mathcal{O}_\mathfrak{f}^* := \{a \in \mathcal{O} | a \text{ totally positive and } a \equiv 1 \bmod \mathfrak{f}\}$. Define

$$F(\mathfrak{b}, \mathfrak{f}, \epsilon, x, s) := \sum_{\nu \in (x + \mathfrak{f}\mathfrak{b}^{-1})/\mathcal{O}_\mathfrak{f}^*} \frac{\epsilon(\nu)}{|N(\nu)|^s} \tag{17}$$

for $Re\ s > 1$. Here N is the norm. On the other hand let $\operatorname{Tr} : F \to \mathbb{Q}$ be the trace map and define

$$L(\mathfrak{b}, \mathfrak{f}, \epsilon, x, s) := \sum_{\lambda \in \mathfrak{b}(\mathfrak{f}\mathfrak{D})^{-1}/\mathcal{O}_\mathfrak{f}^*} \frac{\epsilon(\lambda) e^{2\pi i \operatorname{Tr}(x\lambda)}}{|N(\nu)|^s} \tag{18}$$

These two L-functions are related by a functional equation. To formulate it we introduce the Γ-factor

$$\Gamma_\epsilon(s) := \pi^{-\frac{1}{2}(sg+|\epsilon|)}\Gamma\left(\frac{s+1}{2}\right)^{|\epsilon|}\Gamma\left(\frac{s}{2}\right)^{g-|\epsilon|}.$$

The functional equation follows directly with Hecke's method for Grössencharacters and was first mentioned for these partial zeta functions by Siegel:

PROPOSITION 1.6.1 (cf.[Si] Formel (10)). *The functional equation reads:*

$$\Gamma_\epsilon(1-s)F(\mathfrak{b},\mathfrak{f},\epsilon,x,1-s) = i^{-|\epsilon|}|d_F|^{-\frac{1}{2}}N(\mathfrak{f}^{-1}\mathfrak{b})\Gamma_\epsilon(s)L(\mathfrak{b},\mathfrak{f},\epsilon,x,s),$$

where d_F is the discriminant of $F/\mathbb{Q}$.

The functional equation shows that $F(\mathfrak{b},\mathfrak{f},\epsilon,x,1-k)$ can be non-zero for $k = 1,2,\ldots$ only if $|\epsilon|$ is either g or 0. Let us introduce

$$\zeta(\mathfrak{b},\mathfrak{f},x,s) := \sum_{\nu\in(x+\mathfrak{f}\mathfrak{b}^{-1})/\mathcal{O}_\mathfrak{f}^*}\frac{1}{N(\nu)^s}.$$

We get:

COROLLARY 1.6.2. *The functional equation shows that $F(\mathfrak{b},\mathfrak{f},\epsilon,x,1-k)$ for $k = 1,2,\ldots$ is non-zero for $|\epsilon| = 0$ and k even or for $|\epsilon| = g$ and k odd. In these cases one has*

$$\zeta(\mathfrak{b},\mathfrak{f},x,1-k) = |d_F|^{-\frac{1}{2}}N(\mathfrak{f}^{-1}\mathfrak{b})\frac{((k-1)!)^g}{(2\pi i)^{kg}}L(\mathfrak{b},\mathfrak{f},\epsilon,x,k).$$

1.7 THE MAIN THEOREM

Here we formulate our main theorem. It computes the residue map from (1.5.2) in terms of the partial L-functions.

The transcendental description of the cusps gives

$$H^0(\partial S(\mathbb{C}),L) = \mathrm{Ind}_{B^1(\mathbb{Z})}^{G(\mathbb{Z}/n\mathbb{Z})} L$$

and $H^0(\partial S, L)$ is the subgroup of elements invariant under $D(\mathbb{Z}/n\mathbb{Z})$. Similarly, the n-torsion sections of $\mathcal{A}[n]$ over $S(\mathbb{C})$ can be identified with functions from $G(\mathbb{Z}/n\mathbb{Z})$ to $(\frac{1}{n}\mathcal{O}/\mathcal{O})^2$, which are equivariant with respect to the canonical $G^1(\mathbb{Z}) := \ker(G(\mathbb{Z}) \to \mathbb{Z}^*)$ action. The action of $G(\mathbb{Z}/n\mathbb{Z})$ on S induces via pull-back an action on $\mathcal{A}[n](S(\mathbb{C}))$ and we have:

$$\mathcal{A}[n](S(\mathbb{C})) = \mathrm{Ind}_{G^1(\mathbb{Z})}^{G(\mathbb{Z}/n\mathbb{Z})}(\frac{1}{n}\mathcal{O}/\mathcal{O})^2.$$

The group $\mathcal{A}[n](S)$ consists again of the elements invariant under $D(\mathbb{Z}/nZ)$. Let $D := \mathcal{A}[n](S)$ and consider the formal linear combinations

$$L[D]^0 := \{\sum_{\sigma\in D} l_\sigma(\sigma) : l_\sigma \in L \text{ and } \sum_{\sigma\in D} l_\sigma = 0\}.$$

The $G(\mathbb{Z}/n\mathbb{Z})$ action on D carries over to an action on $L[D]^0$. For $\alpha \in L[D]^0$ we construct in 2.4.1 a class

$$\mathrm{Eis}^k(\alpha) \in \mathrm{Ext}^{2g-1}_S(L, \mathrm{Sym}^k \mathcal{H}(g)),$$

which depends on α in a functorial way. Thus, the resulting map

$$L[D]^0 \xrightarrow{\mathrm{Eis}^k} \mathrm{Ext}^{2g-1}_S(L, \mathrm{Sym}^k \mathcal{H}(g)) \xrightarrow{\mathrm{res}} \mathrm{Ind}^{G(\mathbb{Z}/n\mathbb{Z})}_{B^1(\mathbb{Z})} L \tag{19}$$

is equivariant for the $G(\mathbb{Z}/n\mathbb{Z})$ action.

Theorem 1.7.1. *Let $L \supset \mathbb{Q}$ and $\alpha = \sum_{\sigma\in D} l_\sigma(\sigma)$. Then* $\mathrm{res}(\mathrm{Eis}^m(\alpha))$ *is non-zero only for $m \equiv 0(g)$ and for every $h \in G(\mathbb{Z}/n\mathbb{Z})$ and $k > 0$*

$$\mathrm{res}(\mathrm{Eis}^{gk}(\alpha))(h) = (-1)^{g-1} \sum_{\sigma\in D} l_\sigma \zeta(\mathcal{O}, \mathcal{O}, p(h\sigma), -k).$$

To use the basis given by the coinvariants in $\mathrm{Sym}^{gk} \mathcal{O} \otimes L$ as we did in the proof of theorem 1.5.1 is not natural. A better description is as follows: For each $h \in G(\mathbb{Z}/n\mathbb{Z})$ choose an element $d_h \in D(\mathbb{Z}/n\mathbb{Z})$ such that $\tilde{h} := hd_h^{-1} \in G^1(\mathbb{Z}/n\mathbb{Z})$. Then, as in (4) we have an ideal $\mathfrak{b}_{\tilde{h}}$ and a projection

$$\mathcal{O}^2 \xrightarrow{p_{\tilde{h}}} \mathfrak{b}_{\tilde{h}}.$$

Now use the identification $H^{g-1}(S_T, \mathrm{Sym}^{gk} \mathfrak{b}_{\tilde{h}} \otimes L) \cong L$ at the cusp h. With this basis the above result reads

Corollary 1.7.2. *In this basis*

$$\mathrm{res}(\mathrm{Eis}^{gk}(\alpha))(h) = (-1)^{g-1} N\mathfrak{b}_{\tilde{h}}^{-k-1} \sum_{\sigma\in D} l_\sigma \zeta(\mathfrak{b}_{\tilde{h}}, \mathcal{O}, p_{\tilde{h}}(\sigma), -k).$$

The theorem and the corollary will be proved in section 3.

2 Polylogarithms

In this section we review the theory of the polylogarithm on abelian schemes. Special emphasis is given the topological case, which will be important in the proof of the main theorem. The elliptic polylogarithm was introduced by Beilinson and Levin [BL] and the generalization to higher dimensional families of

abelian varieties is due to Wildeshaus [W]. The idea to interprete the construction by Nori in terms of the topological polylogarithm is due to Beilinson and Nori (unpublished).
The polylogarithm can be defined in any of the categories MHM, et, top for any abelian scheme $\pi : \mathcal{A} \to S$, with unit section $e : S \to \mathcal{A}$ of constant relative dimension g. If we work in top, it even suffices to assume that $\pi : \mathcal{A} \to S$ is a family of topological tori (i.e., fiberwise isomorphic to $(\mathbb{R}/\mathbb{Z})^g$). For more details in the case of abelian schemes, see [W] chapter III part I, or [L]. In the case of elliptic curves one can also consult [BL] or [HK].

2.1 Construction of the polylog

For simplicity we assume $L \supset \mathbb{Q}$ in this section and discuss the necessary modifications for integral coefficients later. Define a lisse sheaf $\operatorname{Log}^{(1)}$ on $\mathcal{A}$, which is an extension

$$0 \to \mathcal{H} \to \operatorname{Log}^{(1)} \to L \to 0$$

together with a splitting $s : e^*L \to e^* \operatorname{Log}^{(1)}$ in any of the three categories MHM, et, top as follows: Consider the exact sequence

$$0 \to \operatorname{Ext}^1_S(L, \mathcal{H}) \xrightarrow{\pi^*} \operatorname{Ext}^1_{\mathcal{A}}(L, \pi^*\mathcal{H}) \to \operatorname{Hom}_S(L, R^1\pi_*\pi^*\mathcal{H}) \to 0,$$

which is split by e^*. Note that by the projection formula $R^1\pi_*\pi^*\mathcal{H} \cong R^1\pi_*L \otimes \mathcal{H}$ so that

$$\operatorname{Hom}_S(L, R^1\pi_*\pi^*\mathcal{H}) \cong \operatorname{Hom}_S(\mathcal{H}, \mathcal{H}).$$

Then $\operatorname{Log}^{(1)}$ is a sheaf representing the unique extension class in $\operatorname{Ext}^1_{\mathcal{A}}(L, \pi^*\mathcal{H})$, which splits when pulled back to S via e^* and which maps to $\mathrm{id} \in \operatorname{Hom}_S(\mathcal{H}, \mathcal{H})$. Define

$$\operatorname{Log}^{(k)} := \operatorname{Sym}^k \operatorname{Log}^{(1)}.$$

Definition 2.1.1. *The* logarithm sheaf *is the pro-sheaf*

$$\operatorname{Log} := \operatorname{Log}_{\mathcal{A}} := \varprojlim \operatorname{Log}^{(k)},$$

where the transition maps are induced by the map $\operatorname{Log}^{(1)} \to L$. *In particular, one has exact sequences*

$$0 \to \operatorname{Sym}^k \mathcal{H} \to \operatorname{Log}^{(k)} \to \operatorname{Log}^{(k-1)} \to 0$$

and a splitting induced by $s : e^*L \to e^* \operatorname{Log}^{(1)}$

$$e^* \operatorname{Log} \cong \prod_{k \geq 0} \operatorname{Sym}^k \mathcal{H}.$$

Any isogeny $\phi : \mathcal{A} \to \mathcal{A}$ of degree invertible in L induces an isomorphism $\mathrm{Log} \cong \phi^* \mathrm{Log}$, which is on the associated graded induced by $\mathrm{Sym}^k \phi : \mathrm{Sym}^k \mathcal{H} \to \mathrm{Sym}^k \mathcal{H}$. For every torsion point $x \in \mathcal{A}(S)_{\mathrm{tors}}$ one gets an isomorphism

$$x^* \mathrm{Log} \cong e^* \mathrm{Log} \cong \prod_{k \geq 0} \mathrm{Sym}^k \mathcal{H}. \tag{20}$$

The most important property of the sheaf Log is the vanishing of its higher direct images except in the highest degree.

Theorem 2.1.2 (Wildeshaus, [W], cor. 4.4., p. 70). *One has*

$$R^i \pi_* \mathrm{Log} = 0 \textit{ for } i \neq 2g$$

and the augmentation $\mathrm{Log} \to L$ *induces canonical isomorphisms*

$$R^{2g} \pi_* \mathrm{Log} \cong R^{2g} \pi_* L \cong L(-g).$$

For the construction of the polylogarithm one considers a non-empty disjoint union of torsion sections $i : D \subset \mathcal{A}$, whose orders are invertible in L (more generally, one can also consider D étale over S). Let

$$L[D] := \bigoplus_{\sigma \in D} L$$

and $L[D]^0 \subset L[D]$ the kernel of the augmentation map $L[D] \to L$. Elements $\alpha \in L[D]$ are written as formal linear combinations $\alpha = \sum_{\sigma \in D} l_\sigma(\sigma)$. Similarly, define

$$\mathrm{Log}[D] := \bigoplus_{\sigma \in D} \sigma^* \mathrm{Log}$$

and

$$\mathrm{Log}[D]^0 := \ker (\mathrm{Log}[D] \to L)$$

to be the kernel of the composition of the sum of the augmentation maps $\mathrm{Log}[D] \to L[D]$ and the augmentation $L[D] \to L$.

Corollary 2.1.3. *The localization sequence for* $U := \mathcal{A} \setminus D$ *induces an isomorphism*

$$\mathrm{Ext}^{2g-1}_U (L[D]^0, \mathrm{Log}(g)) \cong \mathrm{Hom}_S(L[D]^0, \mathrm{Log}[D]^0).$$

Proof. The vanishing result 2.1.2 implies that the localization sequence is of the form

$$0 \to \mathrm{Ext}^{2g-1}_U (L[D]^0, \mathrm{Log}(g)) \to \mathrm{Hom}_S(L[D]^0, i^* \mathrm{Log}) \to \mathrm{Hom}_S(L[D]^0, L) \to 0.$$

Inserting the definition of $\mathrm{Log}[D]^0$ gives the desired result. □

DEFINITION 2.1.4. *The* polylogarithm pol^D *is the extension class*

$$\mathrm{pol}^D \in \mathrm{Ext}_U^{2g-1}(L[D]^0, \mathrm{Log}(g)),$$

which maps to the canonical inclusion $L[D]^0 \to \mathrm{Log}[D]$ *under the isomorphism in 2.1.3. In particular, for every* $\alpha \in L[D]^0$ *we get by pull-back an extension class*

$$\mathrm{pol}_\alpha^D \in \mathrm{Ext}_U^{2g-1}(L, \mathrm{Log}(g)).$$

2.2 INTEGRAL VERSION OF THE POLYLOGARITHM, THE TOPOLOGICAL CASE

In the topological and the étale situation it is possible to define the polylogarithm with integral coefficients. In this section we treat the topological case and the étale case in the next section. The construction presented here is a reinterpretation by Beilinson and Levin (unpublished) of results of Nori and Sczech.

We start by defining the logarithm sheaf for any (commutative) coefficient ring L, in particular for $L = \mathbb{Z}$. In the topological situation, it is even possible to define more generally the polylogarithm for any smooth family of real tori of constant dimension g, which has a unit section.

Let

$$\pi : \mathcal{T} \to S$$

be such a family, $e : S \to \mathcal{T}$ the unit section and let $\mathcal{H}_L := \underline{\mathrm{Hom}}_S(R^1\pi_* L, L)$ be the local system of the homologies of the fibers with coefficients in L. Let $\tilde{\mathcal{H}}_\mathbb{R}$ be the associated vector bundle of $\mathcal{H}_\mathbb{R}$. Then $\mathcal{T} \cong \mathcal{H}_\mathbb{Z} \backslash \tilde{\mathcal{H}}_\mathbb{R}$ and we denote by

$$\tilde{\pi} : \tilde{\mathcal{H}}_\mathbb{R} \to \mathcal{T}$$

the associated map. Let

$$L[\mathcal{H}_\mathbb{Z}] := e^* \tilde{\pi}_! L$$

be the local system of group rings on S, which is stalk-wise the group ring of the stalk of the local system $\mathcal{H}_\mathbb{Z}$ with coefficients in L. The augmentation ideal of $L[\mathcal{H}_\mathbb{Z}] \to L$ is denoted by $\mathcal{I}$ and we define

$$L[[\mathcal{H}_\mathbb{Z}]] := \varprojlim_r L[\mathcal{H}_\mathbb{Z}]/\mathcal{I}^r$$

the completion along the augmentation ideal. Note that $\mathcal{I}^n/\mathcal{I}^{n+1} \cong \mathrm{Sym}^n \mathcal{H}_L$. If $L \supset \mathbb{Q}$, one has even a ring isomorphism

$$L[[\mathcal{H}_\mathbb{Z}]] \xrightarrow{\cong} \prod_{k \geq 0} \mathrm{Sym}^k \mathcal{H}_L, \tag{21}$$

induced by $h \mapsto \sum_{k \geq 0} h^{\otimes k}/k!$ for $h \in \mathcal{H}_\mathbb{Z}$. In the special case $L = \mathbb{Z}, \mathbb{Q}$ the canonical map of group rings $\mathbb{Z}[\mathcal{H}_\mathbb{Z}] \to \mathbb{Q}[\mathcal{H}_\mathbb{Z}]$ induces

$$\mathbb{Z}[[\mathcal{H}_\mathbb{Z}]] \to \mathbb{Q}[[\mathcal{H}_\mathbb{Z}]] \cong \prod_{k \geq 0} \mathrm{Sym}^k \mathcal{H}_\mathbb{Q}. \tag{22}$$

Definition 2.2.1. *The* logarithm sheaf Log *is the local system on $\mathcal{T}$ defined by*

$$\mathrm{Log} := \tilde{\pi}_! L \otimes_{L[\mathcal{H}_\mathbb{Z}]} L[[\mathcal{H}_\mathbb{Z}]].$$

As a local system of $L[[\mathcal{H}_\mathbb{Z}]]$-modules, Log *is of rank* 1.

Any isogeny $\phi : \mathcal{T} \to \mathcal{T}$ of order invertible in L induces an isomorphism $\mathrm{Log} \cong \phi^* \mathrm{Log}$, which is induced by $\phi : \mathcal{H}_\mathbb{Z} \to \mathcal{H}_\mathbb{Z}$. In particular, if the order of a torsion section $x : S \to \mathcal{T}$ is invertible in L, one has an isomorphism

$$x^* \mathrm{Log} \cong e^* \mathrm{Log} = L[[\mathcal{H}_\mathbb{Z}]].$$

To complete the definition of the polylogarithm, one has to compute the cohomology of Log. As $L[[\mathcal{H}_\mathbb{Z}]]$ is a flat $L[\mathcal{H}_\mathbb{Z}]$-module one gets

$$R^i \pi_* \mathrm{Log} \cong R^i \pi_* \tilde{\pi}_! L \otimes_{L[\mathcal{H}_\mathbb{Z}]} L[[\mathcal{H}_\mathbb{Z}]]$$

and because $\pi_* = \pi_!$ one has to consider $R^i(\pi \circ \tilde{\pi})_! L$. But the fibers of

$$\pi \circ \tilde{\pi} : \tilde{\mathcal{H}}_\mathbb{R} \to S$$

are just g-dimensional vector spaces and the cohomology with compact support lives only in degree g, where it is the dual of $\Lambda^{max} \mathcal{H}_L$. Hence, we have proved:

Lemma 2.2.2. *Denote by $\mu_\mathcal{T}^\vee$ the L-dual of $\mu_\mathcal{T} := \Lambda^{max} \mathcal{H}_L$. Then the higher direct images of* Log *are given by*

$$R^i \pi_* \mathrm{Log} \cong \begin{cases} \mu_\mathcal{T}^\vee & \text{if } i = g \\ 0 & \text{else.} \end{cases}$$

As in 2.1.3 one obtains

$$\mathrm{Ext}^g_U(L[D]^0, \mathrm{Log} \otimes \mu_\mathcal{T}) \cong \mathrm{Hom}_S(L[D]^0, \mathrm{Log}[D]^0)$$

and one defines the polylogarithm

$$\mathrm{pol}^D \in \mathrm{Ext}^{g-1}_U(L[D]^0, \mathrm{Log} \otimes \mu_\mathcal{T})$$

in the same way. For $\alpha \in L[D]^0$ one has again

$$\mathrm{pol}^D_\alpha \in \mathrm{Ext}^{g-1}_U(L, \mathrm{Log} \otimes \mu_\mathcal{T}) = H^{g-1}(U, \mathrm{Log} \otimes \mu_\mathcal{T}).$$

The relation to the polylog defined in 2.1.4 is as follows: If we denote the logarithm sheaf and the polylog from this section by $\mathrm{Log}_\mathbb{Z}$ and $\mathrm{pol}^D_\mathbb{Z}$ and similarly the ones from 2.1.4 by $\mathrm{Log}_\mathbb{Q}$ and $\mathrm{pol}^D_\mathbb{Q}$, we get from 22 a canonical map $\mathbb{Q} \otimes_\mathbb{Z} \mathrm{Log}_\mathbb{Z} \to \mathrm{Log}_\mathbb{Q}$ and hence a map

$$\mathrm{Ext}^{g-1}_U(\mathbb{Z}[D]^0, \mathrm{Log}_\mathbb{Z} \otimes \mu_\mathcal{T}) \to \mathrm{Ext}^{g-1}_U(\mathbb{Q}[D]^0, \mathrm{Log}_\mathbb{Q} \otimes \mu_\mathcal{T}),$$

which maps $\mathrm{pol}^D_\mathbb{Z}$ to
$pol^D_\mathbb{Q}$.

2.3 Integral version of the polylogarithm, the étale case

This section will not be used in the rest of the paper and can be omitted by any reader not interested in the integral étale case.
To define an integral étale polylogarithm, one has to modify the definition of the logarithm sheaf as in the topological case. The situation we consider here is again an abelian scheme

$$\pi : \mathcal{A} \to S$$

of constant fiber dimension g and unit section $e : S \to \mathcal{A}$. Let ℓ be a prime number, $L = \mathbb{Z}/\ell^k\mathbb{Z}$ and assume that ℓ is invertible in $\mathcal{O}_S$. Then the ℓ^r-multiplication $[\ell^r] : \mathcal{A} \to \mathcal{A}$ is étale and the sheaves $[\ell^r]_! L$ form a projective system via the trace maps

$$[\ell^r]_! L \to [\ell^{r-1}]_! L.$$

Definition 2.3.1. *The* logarithm sheaf *is the inverse limit*

$$\mathrm{Log}_L := \varprojlim_r [\ell^r]_! L$$

with respect to the above trace maps. The logarithm sheaf with $\mathbb{Z}_\ell$-coefficients is defined by

$$\mathrm{Log}_{\mathbb{Z}_\ell} := \varprojlim_k \mathrm{Log}_{\mathbb{Z}/\ell^k\mathbb{Z}}.$$

Let $\mathcal{H}_\ell := \varprojlim_r \mathcal{A}[\ell^r]$ be the Tate-module of $\mathcal{A}/S$. As ℓ is nilpotent in L, we get that $e^* \mathrm{Log} = L[[\mathcal{H}_\ell]]$ is the Iwasawa algebra of $\mathcal{H}_\ell$ with coefficients in L. Any isogeny $\phi : \mathcal{A} \to \mathcal{A}$ of degree prime to ℓ induces an isomorphism $[\ell^r]_! L \to \phi^*[\ell^r]_! L$, which induces

$$\mathrm{Log} \cong \phi^* \mathrm{Log}.$$

Proposition 2.3.2. *Let $L = \mathbb{Z}/\ell^k\mathbb{Z}$ or $L = \mathbb{Z}_\ell$. The higher direct images of* Log *are given by*

$$R^i\pi_* \mathrm{Log} \cong \begin{cases} L(-g) & \text{if } i = 2g \\ 0 & \text{else.} \end{cases}$$

Proof. It suffices to consider the case $L = \mathbb{Z}/\ell^k\mathbb{Z}$. We will show that the transition maps $R^i\pi_*[\ell^r]_! L \to R^i\pi_*[\ell^s]_! L$ are zero for $i < 2g$ and every s, if r is sufficiently big. By Poincaré duality we may consider the maps

$$R^{2g-i}\pi_![\ell^s]_* L(g) \to R^{2g-i}\pi_![\ell^r]_* L(g). \tag{23}$$

By base change we may assume that S is the spectrum of an algebraically closed field. Denote by $\mathcal{A}_s$ the variety $\mathcal{A}$ considered as covering of $\mathcal{A}$ via $[\ell^s]$. Then

$$R^1\pi_![\ell^s]_* L(g) = H^1(\mathcal{A}, [\ell^s]_* L(g)) = \mathrm{Hom}(\pi_1(\mathcal{A}_s), L(g)).$$

With this description we see that for every $f \in \operatorname{Hom}(\pi_1(\mathcal{A}_s), L(g))$ there is an r, such that the restriction to $\pi_1(\mathcal{A}_r)$ is trivial. This shows that the map in (23) is zero, if r is sufficiently big and $i < 2g$ as the cohomology in degree i is the i-th exterior power of the first cohomology. That (23) is an isomorphism for $i = 2g$ is clear. □

2.4 Eisenstein classes

The Eisenstein classes are specializations of the polylogarithm. The situation is as follows. First let $\alpha \in L[\mathcal{A}[n]]^0$ and assume that $\mathbb{Q} \subset L$. Then one can pull-back the class $\operatorname{pol}_\alpha^{\mathcal{A}[n]} \in \operatorname{Ext}_U^{g-1}(L, \operatorname{Log}(g))$ along e and gets:

$$e^* \operatorname{pol}_\alpha^{\mathcal{A}[n]} \in \operatorname{Ext}_S^{2g-1}(L, e^* \operatorname{Log}(g)) = \prod_{k \geq 0} \operatorname{Ext}_S^{2g-1}(L, \operatorname{Sym}^k \mathcal{H}(g)).$$

Definition 2.4.1. *For any $\alpha \in L[\mathcal{A}[n]]^0$, define the k-th* Eisenstein class *associated to α,*

$$\operatorname{Eis}^k(\alpha) \in \operatorname{Ext}_S^{2g-1}(L, \operatorname{Sym}^k \mathcal{H}(g)),$$

to be the k-th component of $e^* \operatorname{pol}_\alpha^{\mathcal{A}[n]}$.

Note that by the functoriality of the polylogarithm the map

$$L[\mathcal{A}[n]]^0 \xrightarrow{\operatorname{Eis}^k} \operatorname{Ext}_S^{2g-1}(L, \operatorname{Sym}^k \mathcal{H}(g)) \tag{24}$$

is equivariant for the $G(\mathbb{Z}/n\mathbb{Z})$ action on both sides.
These Eisenstein classes should be considered as analogs of Harder's Eisenstein classes (but observe that we have only classes in cohomological degree $2g - 1$). The advantage of the above classes is that they are defined by a universal condition, which makes a lot of their properties easy to verify.

3 Proof of the main theorem

In this section we assume that $\mathbb{Q} \subset L$.
The proof of the main theorem will be in several steps. First we reduce to the case of local systems for the usual topology. The second step consists of a trick already used in [HK]: instead of working with the Eisenstein classes directly, we work with the polylogarithm itself. The reason is that the polylog is characterized by a universal property and has a very good functorial behavior. The third step reviews the computations of Nori in [N]. In the fourth step we compute the integral over S_T^1 and the fifth step gives the final result.

3.1 1. Step: Reduction to the classical topology

We distinguish the MHM and the étale case. In the MHM case, the target of the residue map from (1.5.2)

$$\operatorname{res} : \operatorname{Ext}_S^{2g-1}(L, \operatorname{Sym}^k \mathcal{H}(g)) \to \operatorname{Hom}_{\partial S}(L, L). \tag{25}$$

is purely topological and does not depend on the Hodge structure. More precisely, the canonical map "forget the Hodge structure" denoted by rat induces an isomorphism

$$\mathrm{rat} : \mathrm{Hom}_{MHM,\partial S}(L,L) \cong \mathrm{Hom}_{top,\partial S}(L,L).$$

By [Sa] thm. 2.1 we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^{2g-1}_{MHM,S}(L, \mathrm{Sym}^k \mathcal{H}(g)) & \xrightarrow{\mathrm{res}} & \mathrm{Hom}_{MHM,\partial S}(L,L) \\ \downarrow{\scriptstyle \mathrm{rat}} & & \cong \downarrow{\scriptstyle \mathrm{rat}} \\ \mathrm{Ext}^{2g-1}_{top,S}(L, \mathrm{Sym}^k \mathcal{H}(g)) & \xrightarrow{\mathrm{res}} & \mathrm{Hom}_{top,\partial S}(L,L). \end{array} \tag{26}$$

This reduces the computation of the residue map for MHM to the case of local systems in the classical topology.
In the étale case one has an injection

$$\mathrm{Hom}_{et,\partial S}(L,L) \hookrightarrow \mathrm{Hom}_{et,\partial S\times\bar{\mathbb{Q}}}(L,L) \cong \mathrm{Hom}_{top,\partial S(\mathbb{C})}(L,L).$$

and a commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^{2g-1}_{et,S}(L, \mathrm{Sym}^k \mathcal{H}(g)) & \xrightarrow{\mathrm{res}} & \mathrm{Hom}_{et,\partial S}(L,L) \\ \downarrow & & \downarrow \\ \mathrm{Ext}^{2g-1}_{top,S(\mathbb{C})}(L, \mathrm{Sym}^k \mathcal{H}(g)) & \xrightarrow{\mathrm{res}} & \mathrm{Hom}_{top,\partial S(\mathbb{C})}(L,L). \end{array} \tag{27}$$

Again, this reduces the residue computation to the classical topology.

3.2 2. Step: Topological degeneration

In this section we reduce the computation of $\mathrm{res}\circ\mathrm{Eis}^k$ to a computation of the polylog on $\mathcal{T}_{\mathcal{M}}$.
We are now in the topological situation and use again the notations ∂S and S instead of $\partial S(\mathbb{C})$ and $S(\mathbb{C})$.
Recall from (19) that $\mathrm{res}\circ\mathrm{Eis}^k$ is $G(\mathbb{Z}/n\mathbb{Z})$ equivariant. In particular,

$$\mathrm{res}(\mathrm{Eis}^k(\alpha))(h) = \mathrm{res}(\mathrm{Eis}^k(h\alpha))(\mathrm{id}),$$

where $h\alpha$ denotes the action of h on α. To compute the residue it suffices to consider the residue at id.
Recall from (14) that we have a commutative diagram of fibrations

$$\begin{array}{ccc} \mathcal{A}(\mathbb{C}) & \xrightarrow{p} & \mathcal{T}_{\mathcal{M}} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{M}}} \\ S^1_B & \xrightarrow{q} & S^1_T. \end{array} \tag{28}$$

The map $p : \mathcal{H} \to \mathcal{M}$ induces $\mathrm{Log}_{\mathcal{A}} \to p^* \mathrm{Log}_{\mathcal{M}}$. Let $D = \mathcal{A}[n]$ and $U := \mathcal{A} \setminus D$ be the complement. Let $p(D) = \mathcal{T}_{\mathcal{M}}[n]$ be the image of D in $\mathcal{T}_{\mathcal{M}}$ and $V := \mathcal{T}_{\mathcal{M}} \setminus p(D)$ be its complement in $\mathcal{T}_{\mathcal{M}}$. Then p induces a map

$$p : U \setminus p^{-1}(p(D)) \to V.$$

We define a trace map

$$p_* : \mathrm{Ext}_U^{2g-1}(L, \mathrm{Log}_{\mathcal{A}} \otimes \mu_{\mathcal{A}}) \to \mathrm{Ext}_V^{g-1}(L, \mathrm{Log}_{\mathcal{M}} \otimes \mu_{\mathcal{T}_{\mathcal{M}}}) \tag{29}$$

as the composition of the restriction to $U \setminus p^{-1}(p(D))$

$$\mathrm{Ext}_U^{2g-1}(L, \mathrm{Log}_{\mathcal{A}} \otimes \mu_{\mathcal{A}}) \to \mathrm{Ext}_{U \setminus p^{-1}(p(D))}^{2g-1}(L, \mathrm{Log}_{\mathcal{A}} \otimes \mu_{\mathcal{A}})$$

with the adjunction map

$$\mathrm{Ext}_{U \setminus p^{-1}(p(D))}^{2g-1}(L, \mathrm{Log}_{\mathcal{A}} \otimes \mu_{\mathcal{A}}) \to \mathrm{Ext}_V^{g-1}(L, R^g p_* p^* \mathrm{Log}_{\mathcal{M}} \otimes \mu_{\mathcal{A}}).$$

As $\mu_{\mathcal{A}} \cong \mu_{\mathcal{T}_{\mathcal{N}}} \otimes \mu_{\mathcal{T}_{\mathcal{M}}}$, the projection formula gives

$$R^g p_* p^* \mathrm{Log}_{\mathcal{M}} \cong \mathrm{Log}_{\mathcal{M}} \otimes \mu_{\mathcal{T}_{\mathcal{M}}}.$$

The composition of these maps gives the desired p_* in (29). The crucial fact is that the polylogarithm behaves well under this trace map.

Proposition 3.2.1. *With the notations above, let $\alpha \in L[D]^0$ and $\mathrm{pol}_{\mathcal{A},\alpha}^D \in \mathrm{Ext}_U^{2g-1}(L, \mathrm{Log}_{\mathcal{A}})$ be the associated polylogarithm. Denote by $p(\alpha)$ the image of α under the map*

$$p : L[D]^0 \to L[p(D)]^0$$

induced by $p : \mathcal{A}(\mathbb{C}) \to \mathcal{T}_{\mathcal{M}}$. Then

$$p_* \mathrm{pol}_{\mathcal{A},\alpha}^D = \mathrm{pol}_{\mathcal{T}_{\mathcal{M}},p(\alpha)}^{p(D)}.$$

Proof. This is a quite formal consequence of the definition and the fact that the residue map commutes with the trace map. We use cohomological notation, then one has a commutative diagram

$$\begin{tikzcd}
H^{2g-1}(U, \mathrm{Log}_{\mathcal{A}} \otimes \mu_{\mathcal{A}}) \arrow[r] \arrow[d] & H_D^{2g}(\mathcal{A}, \mathrm{Log}_{\mathcal{A}} \otimes \mu_{\mathcal{A}}) \arrow[d] \\
H^{2g-1}(U \setminus p^{-1}(p(D)), \mathrm{Log}_{\mathcal{A}} \otimes \mu_{\mathcal{A}}) \arrow[r] \arrow[d, "p_*"] & H_{p^{-1}(p(D))}^{2g}(\mathcal{A}, \mathrm{Log}_{\mathcal{A}} \otimes \mu_{\mathcal{A}}) \arrow[d, "p_*"] \\
H^{g-1}(V, \mathrm{Log}_{\mathcal{M}} \otimes \mu_{\mathcal{T}_{\mathcal{M}}}) \arrow[r] & H_{p(D)}^{g}(\mathcal{T}_{\mathcal{M}}, \mathrm{Log}_{\mathcal{M}} \otimes \mu_{\mathcal{T}_{\mathcal{M}}}).
\end{tikzcd}$$

We can identify

$$H^{2g}_D(\mathcal{A}, \operatorname{Log}_{\mathcal{A}} \otimes \mu_{\mathcal{A}}) \cong \bigoplus_{\sigma \in D} \sigma^* \operatorname{Log}_{\mathcal{A}}$$

and

$$H^{g}_{p(D)}(\mathcal{T}_{\mathcal{M}}, \operatorname{Log}_{\mathcal{M}} \otimes \mu_{\mathcal{T}_{\mathcal{M}}}) \cong \bigoplus_{\sigma \in p(D)} \sigma^* \operatorname{Log}_{\mathcal{T}_{\mathcal{M}}}.$$

With this identification the composition of the vertical arrows on the right is induced by $\operatorname{Log}_{\mathcal{A}} \to p^* \operatorname{Log}_{\mathcal{T}_{\mathcal{M}}}$. The polylog $\operatorname{pol}^D_{\mathcal{A},\alpha}$ belongs to the section $\alpha \in L[D]^0 \subset \bigoplus_{\sigma \in D} \sigma^* \operatorname{Log}_{\mathcal{A}}$. This maps to $p(\alpha) \in L[p(D)]^0 \subset \bigoplus_{\sigma \in p(D)} \sigma^* \operatorname{Log}_{\mathcal{T}_{\mathcal{M}}}$. Thus $\operatorname{pol}^D_{\mathcal{A},\alpha}$ is mapped under p_* to $\operatorname{pol}^{p(D)}_{\mathcal{T}_{\mathcal{M}},p(\alpha)}$. $\square$

We want to prove the same sort of result for the Eisenstein classes themselves. To formulate it properly, we need:

Lemma 3.2.2. *Let $q : S^1_B \to S^1_T$ be the fibration from (28). Then*

$$R^g q_* \operatorname{Sym}^k \mathcal{H} \cong \operatorname{Sym}^k \mathcal{M} \otimes \mu^\vee_{\mathcal{T}_{\mathcal{N}}}.$$

Proof. Recall the exact sequence

$$0 \to \mathcal{N} \to \mathcal{H} \to \mathcal{M} \to 0$$

from (12). By definition of $N(\mathbb{Z})$, the coinvariants of $\operatorname{Sym}^k \mathcal{H}$ for $N(\mathbb{Z})$ are exactly $\operatorname{Sym}^k \mathcal{M}$. The lemma follows, as $R^g q_*$ corresponds by definition of the fibering exactly to the coinvariants under $N(\mathbb{Z})$. $\square$

Define a trace map

$$q_* : \operatorname{Ext}^{2g-1}_{S_B}(L, \operatorname{Sym}^k \mathcal{H} \otimes \mu_{\mathcal{A}}) \to \operatorname{Ext}^{g-1}_{S_T}(L, \operatorname{Sym}^k \mathcal{M} \otimes \mu_{\mathcal{T}_{\mathcal{M}}})$$

by adjunction for q, the isomorphism $R^g q_* \operatorname{Sym}^k \mathcal{H} \cong \operatorname{Sym}^k \mathcal{M} \otimes \mu^\vee_{\mathcal{T}_{\mathcal{N}}}$ from lemma 3.2.2 and the isomorphism $\mu_{\mathcal{A}} \cong \mu_{\mathcal{T}_{\mathcal{N}}} \otimes \mu_{\mathcal{T}_{\mathcal{M}}}$. The behaviour of $\operatorname{Eis}^k(\alpha)$ under q_* is given by:

Theorem 3.2.3. *Let $k > 0$ and $\alpha \in L[D]^0$. Then*

$$q_*(\operatorname{Eis}^k_{\mathcal{A}}(\alpha)) = \operatorname{Eis}^k_{\mathcal{T}_{\mathcal{M}}}(p(\alpha)),$$

where $p : L[D]^0 \to L[p(D)]^0$ is the map from 3.2.1.

Proof. Consider the following diagram in the derived category:

$$\begin{tikzcd}
Rp_* \operatorname{Log}_{\mathcal{A}} \otimes \mu_{\mathcal{A}} \arrow[r] \arrow[d] & Rp_* e_* e^* \operatorname{Log}_{\mathcal{A}} \otimes \mu_{\mathcal{A}} \arrow[d, equal] \\
Rp_* p^* \operatorname{Log}_{\mathcal{T}_{\mathcal{M}}} \otimes \mu_{\mathcal{A}} \arrow[d] & e'_* Rq_* e^* \operatorname{Log}_{\mathcal{A}} \otimes \mu_{\mathcal{A}} \arrow[d] \\
\operatorname{Log}_{\mathcal{T}_{\mathcal{M}}} \otimes \mu_{\mathcal{T}_{\mathcal{M}}}[-g] \arrow[r] & e'_* e'^* \operatorname{Log}_{\mathcal{T}_{\mathcal{M}}} \otimes \mu_{\mathcal{T}_{\mathcal{M}}}[-g]
\end{tikzcd} \tag{30}$$

We will show that this diagram is commutative and thereby explain all the maps. First consider the commutative diagram

$$\begin{array}{ccc} Rp_*\operatorname{Log}_{\mathcal{A}}\otimes\mu_{\mathcal{A}} & \longrightarrow & Rp_*e_*e^*\operatorname{Log}_{\mathcal{A}}\otimes\mu_{\mathcal{A}} \\ \downarrow & & \downarrow \\ Rp_*p^*\operatorname{Log}_{\mathcal{T_M}}\otimes\mu_{\mathcal{A}} & \longrightarrow & Rp_*e_*e^*p^*\operatorname{Log}_{\mathcal{T_M}}\otimes\mu_{\mathcal{A}}, \end{array}$$

where the horizontal arrows are induced from adjunction $\mathrm{id}\to e_*e^*$ and the vertical arrows from $\operatorname{Log}_{\mathcal{A}}\to p^*\operatorname{Log}_{\mathcal{T_M}}$. One has $p\circ e = e'\circ q$ and hence

$$Rp_*e_*e^*p^*\operatorname{Log}_{\mathcal{T_M}}\otimes\mu_{\mathcal{A}}\cong e'_*Rq_*q^*e'^*\operatorname{Log}_{\mathcal{T_M}}\otimes\mu_{\mathcal{A}}.$$

The projection formula gives

$$e'_*Rq_*q^*e'^*\operatorname{Log}_{\mathcal{T_M}}\otimes\mu_{\mathcal{A}}\cong e'_*e'^*\operatorname{Log}_{\mathcal{T_M}}\otimes\mu_{\mathcal{A}}\otimes Rq_*L.$$

Projection to the highest cohomology gives a commutative diagram

$$\begin{array}{ccc} Rp_*p^*\operatorname{Log}_{\mathcal{T_M}}\otimes\mu_{\mathcal{A}} & \longrightarrow & e'_*e'^*\operatorname{Log}_{\mathcal{T_M}}\otimes\mu_{\mathcal{A}}\otimes Rq_*L \\ \downarrow & & \downarrow \\ \operatorname{Log}_{\mathcal{T_M}}\otimes\mu_{\mathcal{A}}\otimes\mu^{\vee}_{\mathcal{T_N}} & \longrightarrow & e'_*e'^*\operatorname{Log}_{\mathcal{T_M}}\otimes\mu_{\mathcal{A}}\otimes\mu^{\vee}_{\mathcal{T_N}}, \end{array}$$

where the horizontal maps are adjunction maps $\mathrm{id}\to e'_*e'^*$. Finally we use $\mu_{\mathcal{A}}\otimes\mu^{\vee}_{\mathcal{T_N}}\cong\mu_{\mathcal{T_M}}$ to obtain the commutative diagram (30). Applying $\operatorname{Ext}^{2g-1}_V(L,-)$ to this diagram, where $V:=\mathcal{T_M}\setminus p(D)$ we get

$$\begin{array}{ccc} \operatorname{Ext}^{2g-1}_U(L,\operatorname{Log}_{\mathcal{A}}\otimes\mu_{\mathcal{A}}) & \longrightarrow & \operatorname{Ext}^{2g-1}_S(L,e^*\operatorname{Log}\otimes\mu_{\mathcal{A}}) \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle q_*} \\ \operatorname{Ext}^{g-1}_V(L,\operatorname{Log}_{\mathcal{T_M}}\otimes\mu_{\mathcal{T_M}}) & \longrightarrow & \operatorname{Ext}^{g-1}_S(L,e'^*\operatorname{Log}_{\mathcal{T_M}}\otimes\mu_{\mathcal{T_M}}). \end{array}$$

Now, as $k>0$, we may assume that $\alpha\in L[D\backslash e(S)]^0$ and $p(\alpha)\in L[p(D)\backslash e'(S)]^0$. The result follows then from proposition 3.2.1. $\square$

In a similar (but simpler) way one shows:

Theorem 3.2.4. *Let $\phi:\mathcal{T_M}\to\mathcal{T_{M'}}$ be an isogeny of tori, then ϕ induces a morphism $\phi_*:e^*\operatorname{Log}_{\mathcal{M}}\to e^*\operatorname{Log}_{\mathcal{M}'}$ and*

$$\phi_*\operatorname{Eis}^k_{\mathcal{T_M}}(\alpha)=\operatorname{Eis}^k_{\mathcal{T_{M'}}}(\phi(\alpha)).$$

3.3 3. Step: Explicit description of the polylog

In this section we follow Nori [N] to describe the polylog $\mathrm{pol}_\beta^{\mathcal{T}_\mathcal{M}[n]}$ for any $\beta \in L[\mathcal{T}_\mathcal{M}[n] \setminus 0]^0$ explicitly. The presentation is also influenced by unpublished notes of Beilinson and Levin.

In fact it is useful for the connection with L-functions to consider a more general situation and to allow arbitrary fractional ideals $\mathfrak{a}$ instead just $\mathcal{O}$.

We assume $L = \mathbb{C}$. The geometric situation is this: Recall that $T^1(\mathbb{Z}) = \mathcal{O}^*$ and let $\mathfrak{a} \subset F$ be a fractional ideal with the usual $T^1(\mathbb{Z})$-action. We can form as usual the semi direct product

$$\mathfrak{a} \rtimes T^1(\mathbb{Z}),$$

where the multiplication is given by the formula $(v,t)(v',t') = (v + tv', tt')$. Similarly, we can form $\mathfrak{a} \otimes \mathbb{R} \rtimes T^1(\mathbb{R})$ and we define

$$\mathcal{T}_\mathfrak{a} := \mathfrak{a} \rtimes T(\mathbb{Z}) \backslash \left(\mathfrak{a} \otimes \mathbb{R} \rtimes T^1(\mathbb{R}) \right) / K_\infty^T.$$

We have

$$\pi_\mathfrak{a} : \mathcal{T}_\mathfrak{a} \to S_T^1$$

and we consider the polylog for this real torus bundle of relative dimension g. The case $\mathcal{T}_\mathcal{M}$ is the one where $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \in T^1(\mathbb{Z})$ acts via $d \in \mathcal{O}^*$ on $\mathcal{O}$. Let us describe the logarithm sheaf $\mathrm{Log}_{\mathcal{T}_\mathfrak{a}}$ in this setting. As the coefficients are $L = \mathbb{C}$, we can use the isomorphism from (21)

$$\mathbb{C}[[\mathfrak{a}]] \xrightarrow{\cong} \prod_{k \geq 0} \mathrm{Sym}^k \mathfrak{a}_\mathbb{C} =: \hat{\mathcal{U}}(\mathfrak{a}) \tag{31}$$

$$v \mapsto \exp(v) := \sum_{k=0}^{\infty} \frac{v^{\otimes k}}{k!}$$

The action of $(0,t) \in \mathfrak{a} \rtimes T^1(\mathbb{Z})$ on $\hat{\mathcal{U}}(\mathfrak{a})$ is induced by the action of $T^1(\mathbb{Z})$ on $\mathfrak{a}$. The action of

$$(v, \mathrm{id}) \in \mathfrak{a} \rtimes T^1(\mathbb{Z})$$

on $\hat{\mathcal{U}}(\mathfrak{a})$ is given by multiplication with $\exp(v)$. The logarithm sheaf $\mathrm{Log}_{\mathcal{T}_\mathfrak{a}}$ is just the local system defined by the quotient

$$\mathfrak{a} \rtimes T^1(\mathbb{Z}) \backslash \left(\mathfrak{a} \otimes \mathbb{R} \rtimes T^1(\mathbb{R}) \times \hat{\mathcal{U}}(\mathfrak{a}) \right) / K_\infty^T.$$

A $\mathcal{C}^\infty$-section f of $\mathrm{Log}_{\mathcal{T}_\mathfrak{a}}$ is a function $f : \mathfrak{a} \otimes \mathbb{R} \rtimes T^1(\mathbb{R}) \to \hat{\mathcal{U}}(\mathfrak{a})$, which has the equivariance property

$$f((v,t)(v',t')) = (v,t)^{-1} f(v',t').$$

In a similar way, we can describe $\mathrm{Log}_{\mathcal{T}_\mathfrak{a}}$-valued currents. The global $\mathcal{C}^\infty$-section

$$\exp(-v) : (v,t) \mapsto \sum_{k=0}^{\infty} \frac{(-v)^{\otimes k}}{k!},$$

with $(v,t) \in \mathfrak{a}\otimes\mathbb{R} \rtimes T^1(\mathbb{R})$ defines a trivialization of $\mathrm{Log}_{\mathcal{T}_\mathfrak{a}}$ as $\mathcal{C}^\infty$-bundle. Every current $\mu(v,t)$ with values in $\mathrm{Log}_{\mathcal{T}_\mathfrak{a}}$ can then be written in the form

$$\mu(v,t) = \nu(v,t)\exp(-v),$$

where $\nu(v,t)$ is now a current with values in the constant bundle $\hat{\mathcal{U}}(\mathfrak{a})$. In particular, $\nu(v,t)$ is invariant under the action of $\mathfrak{a} \subset \mathfrak{a} \rtimes T^1(\mathbb{Z})$.

Lemma 3.3.1. *Let $\mathbf{v} : \mathfrak{a}\otimes\mathbb{R} \to \hat{\mathcal{U}}(\mathfrak{a})$ be the canonical inclusion given by $\mathfrak{a}\otimes\mathbb{R} \subset \mathrm{Sym}^1 \mathfrak{a}\otimes\mathbb{C}$, then the canonical connection ∇ on $\mathrm{Log}_{\mathcal{T}_\mathfrak{a}}$ acts on ν by*

$$\nabla\nu = (d - d\mathbf{v})\nu.$$

Proof. Straightforward computation. □

Following Nori [N] we describe the polylog as a $\mathrm{Log}_{\mathcal{T}_\mathfrak{a}}$-valued current $\mu(v,t)$ on $\mathcal{T}_\mathfrak{a}$, such that

$$\nabla\mu(v,t) = \delta_\beta, \tag{32}$$

where

$$\delta_\beta := \sum_{\sigma\in D} l_\sigma \delta_\sigma$$

and δ_σ are the currents defined by integration over the cycles on $\mathcal{T}_\mathfrak{a}$ given by the section σ. If we write as above

$$\mu(v,t) = \nu(v,t)\exp(-v)$$

we get the equivalent condition

$$(d - d\mathbf{v})\nu(v,t) = \delta_\beta. \tag{33}$$

As $\nu(v,t)$ is invariant under the $\mathfrak{a}$-action, we can develop $\nu(v,t)$ into a Fourier series

$$\nu(v,t) = \sum_{\rho\in\mathfrak{a}^\vee} \nu_\rho(t) e^{2\pi i\rho(v)}. \tag{34}$$

The property (33) reads for the Fourier coefficients $\nu_\rho(t)$:

$$(d + 2\pi i d\rho - d\underline{v})\nu_\rho(t) = (e^{-2\pi i\rho(\beta)})\,\mathrm{vol}, \tag{35}$$

where vol is the unique constant coefficient g-form on $\mathfrak{a}\otimes\mathbb{R}$, such that the integral $\int_{\mathfrak{a}\otimes\mathbb{R}/\mathfrak{a}} \mathrm{vol} = 1$ and

$$e^{-2\pi i\rho(\beta)} := \sum_\sigma l_\sigma e^{-2\pi i\rho(\sigma)}.$$

We do not explain in detail the method of Nori to solve this equation, we just give the result. This suffices, because the cohomology class of the polylogarithm is uniquely determined by the equation (32) and we just need to give a solution for it.

Fix a positive definite quadratic form q on $\mathfrak{a}\otimes\mathbb{R}$, viewed as an isomorphism

$$q : (\mathfrak{a}\otimes\mathbb{R})^\vee \cong \mathfrak{a}\otimes\mathbb{R}.$$

Define a left action of $t\in T^1(\mathbb{R})$ by $q_t(v,w) := q(t^{-1}v, t^{-1}w)$. Consider ρ as element in $(\mathfrak{a}\otimes\mathbb{R})^\vee$. Then $q_t(\rho)$ can be considered as a vector field and we denote by ι_ρ the contraction with this vector field $q_t(\rho)$. We may also consider $q_t(\rho)$ as element in $\hat{\mathcal{U}}(\mathfrak{a})$ and denote this by $\mathbf{q}_t(\rho)$.

THEOREM 3.3.2 (Nori). *With the notations above, one has for $0\neq\rho$*

$$\nu_\rho(t) = \sum_{m=0}^{g-1} \frac{(-1)^m(e^{-2\pi i\rho(\beta)})}{(2\pi i\rho(q_t(\rho)) - \mathbf{q}_t(\rho))^{m+1}} \iota_\rho(d\circ\iota_\rho)^m \operatorname{vol}$$

and

$$\nu_0(t) = 0$$

Proof. Write Φ_ρ for the operator multiplication by $2\pi i d\rho - d\mathbf{v}$ and $\Psi_\rho := d+\Phi_\rho$. One checks that $\Psi_\rho\circ\Psi_\rho = 0 = \iota_\rho\circ\iota_\rho$ and that $\Psi_\rho\circ\iota_\rho + \iota_\rho\circ\Psi_\rho$ is an isomorphism. Indeed $\Phi_\rho\circ\iota_\rho + \iota_\rho\circ\Phi_\rho$ is multiplication by $2\pi i\rho(q_t(\rho)) - \mathbf{q}_t(\rho)$ and $\mathcal{L}_\rho := d\circ\iota_\rho + \iota_\rho\circ$ is the Lie derivative with respect to the vector field $q_t(\rho)$. The formula in the theorem is just

$$\iota_\rho\circ(\Psi_\rho\circ\iota_\rho + \iota_\rho\circ\Psi_\rho)^{-1}(e^{-2\pi i\rho(\beta)}) \operatorname{vol}$$

and to check that

$$\Psi_\rho\circ\iota_\rho\circ(\Psi_\rho\circ\iota_\rho + \iota_\rho\circ\Psi_\rho)^{-1} = \mathrm{id}$$

note that $\iota_\rho\circ\Psi_\rho$ commutes with $(\Psi_\rho\circ\iota_\rho+\iota_\rho\circ\Psi_\rho)^{-1}$ and $\iota_\rho\circ\Psi_\rho(e^{-2\pi i\rho(\beta)}) \operatorname{vol} = 0$. ☐

COROLLARY 3.3.3. *The polylogarithm $\mathrm{pol}_\beta^{\mathcal{T}_\mathfrak{a}[n]}$ is given in the topological realization by the current*

$$\mu(v,t) = \nu(v,t)\exp(-v)$$

where $\nu(v,t)$ is the current given by

$$\sum_{m=0}^{g-1}\sum_{k=0}^{\infty}\binom{k+m}{k}\sum_{\rho\in\mathfrak{a}^\vee\setminus 0}\frac{(-1)^m e^{2\pi i\rho(v-\beta)}}{(2\pi i\rho(q_t(\rho)))^{k+m+1}}\mathbf{q}_t(\rho)^{\otimes k}\iota_\rho(d\circ\iota_\rho)^m \operatorname{vol}.$$

Proof. This follows from the formula $\frac{1}{(A-B)^{m+1}} = \sum_{k=0}^\infty \frac{B^k}{A^{k+m+1}}\binom{k+m}{k}$. ☐

The Eisenstein classes are obtained by pull-back of this current along the zero section e. As for $k > 0$ the series over the ρ converges absolutely, this is defined and only terms with $m = g-1$ survive. We get the following formula for the Eisenstein classes.

Corollary 3.3.4. *Let $\beta \in \mathbb{C}[\mathcal{T}_{\mathfrak{a}}[n]]^0$ and $k > 0$, then the topological Eisenstein class is given by*

$$\mathrm{Eis}^k(\beta) = \frac{(k+g-1)!}{k!} \sum_{\rho \in \mathfrak{a}^\vee \setminus 0} \frac{(-1)^{g-1} e^{-2\pi i \rho(\beta)}}{(2\pi i \rho(q_t(\rho)))^{k+g}} \mathbf{q}_t(\rho)^{\otimes k} q_t(\rho)^* \iota_{\mathcal{E}} \,\mathrm{vol}\,.$$

Here, we have written $\mathcal{E}$ for the Euler vector field and $q_t(\rho)$ is considered as a function $q_t(\rho) : S_T \to \mathfrak{a} \otimes \mathbb{R}$, which maps t to the vector $q_t(\rho)$.

Proof. From 3.3.3 we have to compute

$$e^* \iota_\rho (d \circ \iota_\rho)^m \,\mathrm{vol}\,.$$

For this remark that the Lie derivative $\mathcal{L}_\rho = d \circ \iota_\rho + \iota_\rho \circ d$ with respect to the vector field $q_t(\rho)$ acts in the same way on vol as $d \circ \iota_\rho$. One sees immediately that $e^* \iota_\rho (d \circ \iota_\rho)^m \,\mathrm{vol} = 0$, if $m < g-1$ and a direct computation in coordinates gives that $\iota_\rho (\mathcal{L}_\rho)^{g-1} \,\mathrm{vol} = (g-1)! q_t(\rho)^* \iota_{\mathcal{E}} \,\mathrm{vol}$. □

3.4 4. Step: Computation of the integral

To finish the proof of theorem 1.7.1 we have to compute $u_* \,\mathrm{Eis}^k(\beta)$, where $u : S_T^1 \to pt$ is the structure map. As we need only to compute the corresponding integral for the component of S_T^1 corresponding to id, we let $\Gamma_T \subset T^1(\mathbb{Z})$ be the stabilizer of $\mathrm{id} \in T(\mathbb{Z}/n\mathbb{Z})$ and consider

$$u_{\mathrm{id}} : \Gamma_T \backslash \big(T^1(\mathbb{R}) / K_\infty^{T^1} \big) \to pt.$$

To compute the integral, we introduce coordinates on $T^1(\mathbb{R}) \cong (F \otimes \mathbb{R})^*$ and on the torus $\mathcal{T}_{\mathfrak{a}}$. We identify $F \otimes \mathbb{R} \cong \prod_{\tau : F \to \mathbb{R}} \mathbb{R}$ and denote by $e_1, \ldots, e_g$ the standard basis on the right hand side and by $x_1, \ldots, x_g$ the dual basis. For any element $u = \sum u_i e_i$ or $u = \sum u_i x_i$ we write $Nu := u_1 \cdots u_g$. Let q be the quadratic form given by $\sum x_i^2$. We identify the orbit of q under $T^1(\mathbb{R})$ with $(F \otimes \mathbb{R})^*_+$ by mapping

$$\begin{aligned} (F \otimes \mathbb{R})^* &\to T^1(\mathbb{R}) q \\ t &\mapsto q_t. \end{aligned} \tag{36}$$

This map factors over $(F \otimes \mathbb{R})^*_+$ and the map is compatible with the $T^1(\mathbb{Z})$ action on both sides. We let $t_1, \ldots, t_g$ be coordinates on $(F \otimes \mathbb{R})^1$ so that $t_1^2, \ldots, t_g^2$ are coordinates on $(F \otimes \mathbb{R})^1_+$. If we write $\rho = \sum \rho_i x_i$ and $t_i := x_i(t)$, then

$$\rho(q_t(\rho)) = \sum t_i^2 \rho_i^2$$

and $\mathbf{q}_t(\rho)$ has coordinates $t_i^2\rho_i$. More precisely, if we let $\mathbf{e}_1,\dots,\mathbf{e}_g$ be the basis $e_1,\dots,e_g$ considered as elements of $\hat{\mathcal{U}}(\mathfrak{a})$, which identifies $\hat{\mathcal{U}}(\mathfrak{a})$ with the power series ring $\mathbb{C}[[\mathbf{e}_1,\dots,\mathbf{e}_g]]$, then $\mathbf{q}_t(\rho)=\sum t_i^2\rho_i\mathbf{e}_i$. The volume form is given by

$$\mathrm{vol}=|d_F|^{-1/2}N\mathfrak{a}^{-1}dx_1\wedge\ldots\wedge dx_g$$

and we can write the Euler vector field as $\mathcal{E}=\sum x_i\partial_{x_i}$. One gets (observe that $Nt=1$)

$$q_t(\rho)^*\iota_{\mathcal{E}}\,\mathrm{vol}=|d_F|^{-1/2}2^{g-1}N(\rho)N\mathfrak{a}^{-1}\sum_{k=1}^{g}(-1)^{k-1}t_kdt_1\wedge\ldots\widehat{dt_k}\ldots\wedge dt_g.$$

Explicitly, the Eisenstein class is given as a current on $T^1(\mathbb{R})$ by

$$\mathrm{Eis}^k(\beta)(t)=\frac{(k+g-1)!}{k!}\sum_{\rho\in\mathfrak{a}^\vee\setminus 0}\frac{(-1)^{g-1}e^{-2\pi i\rho(\beta)}(\sum t_i^2\rho_i\mathbf{e}_i)^{\otimes k}}{(2\pi i\sum\rho_i^2t_i^2)^{k+g}}q_t(\rho)^*\iota_{\mathcal{E}}\,\mathrm{vol}\tag{37}$$

Define an isomorphism $(\mathbb{R}\otimes F)^1\times\mathbb{R}^*\cong(\mathbb{R}\otimes F)^*$ by mapping $(t,r)\mapsto y:=rt$. Then we get:

$$\frac{dy_1}{y_1}\wedge\ldots\wedge\frac{dy_g}{y_g}=\frac{dr}{r}\wedge\sum_{k=1}^{g}(-1)^{k-1}t_kdt_1\wedge\ldots\widehat{dt_k}\ldots\wedge dt_g.\tag{38}$$

We use this decomposition to write $\mathrm{Eis}^k(\beta)(t)$ as a Mellin transform:

$$\mathrm{Eis}^k(\beta)(t)=\sum_{\rho\in\mathfrak{a}^\vee\setminus 0}(-1)^{g-1}e^{-2\pi i\rho(\beta)}\int_{\mathbb{R}_{>0}}e^{-u(2\pi i\sum\rho_i^2t_i^2)}\frac{(\sum t_i^2\rho_i\mathbf{e}_i)^{\otimes k}}{k!}u^{k+g}\frac{du}{u}\wedge q_t(\rho)^*\iota_{\mathcal{E}}\,\mathrm{vol}.\tag{39}$$

Substitute $u=r^2=N(y)^{2/g}$ and use (38) to get

$$\mathrm{Eis}^k(\beta)(t)=\sum_{\rho\in\mathfrak{a}^\vee\setminus 0}\frac{(-1)^{g-1}2^ge^{-2\pi i\rho(\beta)}N(\rho)}{|d_F|^{1/2}N\mathfrak{a}}\int_{\mathbb{R}_{>0}}e^{-2\pi i\sum\rho_i^2y_i^2}\frac{(\sum y_i^2\rho_i\mathbf{e}_i)^{\otimes k}}{k!}N(y)dy_1\wedge\ldots\wedge dy_g.\tag{40}$$

The application of $u_{\mathrm{id},*}$ amounts to integration over

$$\Gamma_T\backslash\big(T^1(\mathbb{R})/K_\infty^{T^1}\big)\cong\mathcal{O}_{(n)}^*\backslash(F\otimes\mathbb{R})^1_+,$$

where $\mathcal{O}^*_{(n)}$ are the totally positive units, which are congruent to 1 modulo the ideal generated by (n). This gives with the usual trick

$$(41)\quad u_{\mathrm{id},*}\operatorname{Eis}^k(\beta) = \sum_{\rho\in\mathcal{O}^*_{(n)}\backslash\left(\mathfrak{a}^\vee\backslash 0\right)} \frac{(-1)^{g-1}2^g e^{-2\pi i\rho(\beta)}N(\rho)}{|d_F|^{1/2}N\mathfrak{a}} \times \int_{(F\otimes\mathbb{R})^*_+} e^{-2\pi i\sum\rho_i^2y_i^2}\frac{(\sum y_i^2\rho_i\mathbf{e}_i)^{\otimes k}}{k!}N(y)dy_1\wedge\ldots\wedge dy_g.$$

The integral is a product of integrals for $j=1,\ldots,g$:

$$\int_{\mathbb{R}_{>0}} e^{-2\pi i\rho_j^2y_j^2}\rho_j^k\frac{\mathbf{e}_j^{\otimes k}}{k!}y_j^{2k+2}\frac{dy_j}{y_j} = \frac{\mathbf{e}_j^{\otimes k}}{2\rho_j(2\pi i\rho_j)^{k+1}}.$$

We now consider $\operatorname{Eis}^{gk}(\beta)$ instead of $\operatorname{Eis}^k(\beta)$. If we consider $e^*\operatorname{pol}^D_\beta$ as a power series in the $\mathbf{e}_i$ we are interested in the coefficient of $\frac{N\mathbf{e}^{\otimes k}}{k!^g}$. In fact, the integrallity properties of $\operatorname{Eis}^{gk}(\beta)$ are better reflected if we write it in terms of a basis $a_1,\ldots,a_g$ of $\mathfrak{a}$. Then $N\mathbf{e}^{\otimes k} = N\mathfrak{a}^{-k}N\mathbf{a}^{\otimes k}$, where $\mathbf{a}_1,\ldots,\mathbf{a}_g$ denote again the images of $a_1,\ldots,a_g$ in $\hat{\mathcal{U}}(\mathfrak{a})$. We get:

Corollary 3.4.1. *With the above basis $\mathbf{a}_1,\ldots,\mathbf{a}_g$, The integral over the Eisenstein class is given by*

$$u_{\mathrm{id},*}\operatorname{Eis}^{gk}(\beta) = \frac{(-1)^{g-1}(k!)^g}{(2\pi i)^{g(k+1)}|d_F|^{1/2}N\mathfrak{a}^{k+1}}\sum_{\rho\in\mathcal{O}^*_{(n)}\backslash\left(\mathfrak{a}^\vee\backslash 0\right)}\frac{e^{-2\pi i\rho(\beta)}}{N(\rho)^{k+1}}\frac{N\mathbf{a}^{\otimes k}}{k!^g}.$$

3.5 5. Step: End of the proof

To finish the proof of the theorem 1.7.1, let $\alpha\in L[\mathcal{A}[n]]^0$ and suppose we want to compute $\operatorname{res}(\operatorname{Eis}^k(\alpha))(h)$. Using the equivariance of $\operatorname{res}\circ\operatorname{Eis}^k$ from (19), this amounts to compute $\operatorname{res}(\operatorname{Eis}^k(h\alpha))(\mathrm{id})$. Theorem 1.5.1 shows that

$$\operatorname{res}(\operatorname{Eis}^k(h\alpha))(\mathrm{id}) = u_{\mathrm{id},*}q_*\operatorname{Eis}^k(h\alpha),$$

where $q: S^1_B\to S^1_T$ and $u_{\mathrm{id}}:\Gamma_T\backslash\left(T^1(\mathbb{R})/K^{T^1}_\infty\right)\to pt$ is the structure map of the component corresponding to $\mathrm{id}\in T^1(\mathbb{Z}/n\mathbb{Z})$. From theorem 3.2.3 we get

$$q_*\operatorname{Eis}^k(h\alpha) = \operatorname{Eis}^k(p(h\alpha)).$$

Using corollary 3.4.1 for $\mathfrak{a}=\mathcal{O}$ and the formula 1.6.2 for $\mathfrak{b}=\mathfrak{f}=\mathcal{O}$ we get

$$(42)\quad \frac{(-1)^{g-1}(k!)^g}{(2\pi i)^{gk+g}|d_F|^{1/2}}\sum_{\rho\in\mathcal{O}^*_{(n)}\backslash\left(\mathcal{O}^\vee\backslash 0\right)}\frac{e^{-2\pi i\rho(p(h\alpha))}}{N(\rho)^{k+1}} = (-1)^{g-1}\sum_{\sigma\in D} l_\sigma\zeta(\mathcal{O},\mathcal{O},p(h\sigma),-k),$$

which is the formula in the main theorem 1.7.1. To prove the corollary, we use that the map of real tori

$$\mathcal{A}(\mathbb{C}) \xrightarrow{p} \mathcal{T}_{\mathcal{M}}$$

factors through $\phi : \mathcal{T}_{\mathfrak{b}_{\widetilde{h}}} \to \mathcal{T}_{\mathcal{M}}$, where ϕ is induced by the inclusion $\mathfrak{b}_{\widetilde{h}} \subset \mathcal{O}$. Using corollary 3.4.1 for $\mathfrak{a} = \mathfrak{b}_{\widetilde{h}}$, we get the desired formula

$$\operatorname{res}(\mathrm{Eis}^{gk}(\alpha))(h) = (-1)^{g-1} N\mathfrak{b}_{\widetilde{h}}^{-k-1} \sum_{\sigma \in D} l_\sigma \zeta(\mathfrak{b}_{\widetilde{h}}, \mathcal{O}, p_{\widetilde{h}}(\sigma), -k),$$

which ends the proof.

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