Equivariant Iwasawa Theory: An Example

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Abstract. The equivariant main conjecture of Iwasawa theory is shown to hold for a Galois extension K/k of totally real number fields with Galois group an l-adic pro-l Lie group of dimension 1 containing an abelian subgroup of index l, provided that Iwasawa's μ-invariant $\mu(K/k)$ vanishes.

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This note justifies a remark made in the introduction of [6] according to which the "main conjecture" of equivariant Iwasawa theory, as formulated in [2, p.564], holds when $G = G(K/k)$ is a pro-l group with an abelian subgroup G' of index l.

We quickly repeat the general set-up and, in doing so, refer the reader to [5,§1] for facts and notation that is taken from our earlier papers on Iwasawa theory. Namely, l is a fixed odd prime number and K/k a Galois extension of totally real number fields, with $k/\mathbb{Q}$ and K/k_∞ finite, where k_∞ is the cyclotomic $\mathbb{Z}_l$-extension of k. Throughout it will be assumed that Iwasawa's μ-invariant $\mu(K/k)$ vanishes. We also fix a finite set S of primes of k containing all primes above ∞ and all those whose ramification index in K/k is divisible by l.

In this situation it is shown in [5] that the "main conjecture" of equivariant Iwasawa theory would follow from two kinds of hypothetical congruences between values of Iwasawa L-functions. One of these kinds, the so-called *torsion congruences* (see [5, Proposition 3.2]), stated as

$$\frac{\mathrm{ver}(\lambda_{K_{\mathrm{ab}}/k})}{\lambda_{K/k'}} \equiv 1 \mod \mathcal{T}'$$

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in the proof of the proposition in §2, has meanwhile been verified in [6].

The purpose of the present paper is to show that the torsion congruences already suffice to obtain the whole conjecture in the special case when G is a pro-l group with an abelian subgroup G' of index l. Before stating the precise theorem we need to recall some notation (compare [5,§1]).

> $\Lambda_\wedge G$ is the l-completion of the localization $\Lambda_\bullet G$ which is obtained from the Iwasawa algebra $\Lambda G = \mathbb{Z}_l[[G]]$ by inverting all central elements which are regular in $\Lambda G / l\Lambda G$; $\mathcal{Q}_\wedge G$ is the total ring of fractions of $\Lambda_\wedge G$;
>
> $T(\mathcal{Q}_\wedge G) = \mathcal{Q}_\wedge G/[\mathcal{Q}_\wedge G, \mathcal{Q}_\wedge G]$ is the quotient of $\mathcal{Q}_\wedge G$ by Lie commutators;
>
> if G is a pro-l group, then (see [5,§2] [2])
>
> $$\text{(LD)} \qquad \begin{array}{ccc} K_1(\Lambda_\wedge G) & \xrightarrow{\mathbb{L}} & T(\mathcal{Q}_\wedge G) \\ \mathrm{Det} \downarrow & & \overset{\mathrm{Tr}}{\simeq} \downarrow \\ \mathrm{HOM}(R_l G, (\Lambda_\wedge^c \Gamma_k)^\times) & \xrightarrow{\mathbf{L}} & \mathrm{Hom}^*(R_l G, \mathcal{Q}_\wedge^c \Gamma_k) \end{array}$$
>
> is the logarithmic diagram defining the *logarithmic pseudomeasure*
>
> $$t_{K/k} \in T(\mathcal{Q}_\wedge G) \quad \text{by} \quad \mathrm{Tr}(t_{K/k}) = \mathbf{L}(L_{K/k})$$
>
> where $L_{K/k} = L_{K/k,S} \in \mathrm{HOM}(R_l G, (\Lambda_\wedge^c \Gamma_k)^\times)$ is the Iwasawa L-function.

Theorem. *With K/k and S as at the beginning and $G = G(K/k)$ a pro-l group, $t_{K/k}$ is integral (i.e., $t_{K/k} \in T(\Lambda_\wedge G)$) whenever G has an abelian subgroup G' of index l.*

As a corollary, by [5, Proposition 3.2] and [6, Theorem], $L_{K/k} \in \mathrm{Det}\, K_1(\Lambda_\wedge G)$, which implies the conjecture (see [3, Theorem A]), up to its uniqueness assertion. However, $SK_1(\mathcal{Q}G) = 1$ because each simple component, after tensoring up with a suitable extension field of its centre, becomes isomorphic to a matrix ring of dimension a divisor of l^2 by the proof of [2, Proposition 6], as the character degrees $\chi(1)$ all divide l. Now apply [7, p.334, Corollary].

The proof of the theorem is carried out in §2; before, in a short §1, we introduce restriction maps

$$\mathrm{Res}_G^{G'} : T(\mathcal{Q}_\wedge G) \to T(\mathcal{Q}_\wedge G')$$

and

$$\mathrm{Res}_G^{G'} : \mathrm{Hom}^*(R_l G, \mathcal{Q}_\wedge^c \Gamma_k) \to \mathrm{Hom}^*(R_l G', \mathcal{Q}_\wedge^c \Gamma_{k'})$$

[2] $R_l G$ is the ring of all (virtual) $\mathbb{Q}_l^c$-characters of G with open kernel; $\Gamma_k = G(k_\infty/k)$; $\Lambda_\wedge^c \Gamma_k = \mathbb{Z}_l^c \otimes_{\mathbb{Z}_l} \Lambda_\wedge \Gamma_k$ with $\mathbb{Z}_l^c$ the ring of integers in a fixed algebraic closure $\mathbb{Q}_l^c$ of $\mathbb{Q}_l$

making the diagram

$$\begin{array}{ccccc} K_1(\Lambda_\wedge G) & \xrightarrow{\mathbb{L}} & T(\mathcal{Q}_\wedge G) & \xrightarrow{\mathrm{Tr}} & \mathrm{Hom}^*(R_l G, \mathcal{Q}^{\mathrm{c}}_\wedge \Gamma_k) \\ \mathrm{res}_G^{G'} \downarrow & & \mathrm{Res}_G^{G'} \downarrow & & \mathrm{Res}_G^{G'} \downarrow \\ K_1(\Lambda_\wedge G') & \xrightarrow{\mathbb{L}'} & T(\mathcal{Q}_\wedge G') & \xrightarrow{\mathrm{Tr}'} & \mathrm{Hom}^*(R_l G', \mathcal{Q}^{\mathrm{c}}_\wedge \Gamma_{k'}) \end{array}$$

commute [3] for any pair of pro-l groups $G = G(K/k)$ and $G' = G(K/k') \leq G$ such that $[G : G']$ is finite. We remark that replacing $\mathrm{Res}_G^{G'}$ by the "natural" restriction map,

$$(\mathrm{res}_G^{G'} f)(\chi') = f(\mathrm{ind}_{G'}^G \chi')\,,\ f \in \mathrm{Hom}^*(R_l G, \mathcal{Q}^{\mathrm{c}}_\wedge \Gamma_k)\,,\ \chi' \in R_l G'\,,$$

does not work, because induction and Adams operations do not commute.

1. Res

Let $G = G(K/k)$ be a pro-l group and $G' = G(K/k') \leq G$ an open subgroup. Recall that $\Psi : \Lambda^{\mathrm{c}}_\wedge \Gamma_k \to \Lambda^{\mathrm{c}}_\wedge \Gamma_k$ is the map induced by $\Psi(\gamma) = \gamma^l$ for $\gamma \in \Gamma_k$ (compare [5,§1]) and that ψ_l is the l^{th} Adams operation on $R_l(-)$.

Definition. $\mathrm{Res}_G^{G'} : \mathrm{Hom}^*(R_l G, \mathcal{Q}^{\mathrm{c}}_\wedge \Gamma_k) \to \mathrm{Hom}^*(R_l G', \mathcal{Q}^{\mathrm{c}}_\wedge \Gamma_{k'})$ *sends* f *to*

$$\mathrm{Res}_G^{G'} f = \big[\, \chi' \mapsto f(\mathrm{ind}_{G'}^G \chi') + \sum_{r \geq 1} \frac{\Psi^r}{l^r} \left(f(\psi_l^{r-1} \chi)\right) \big]\,,$$

where $\chi = \psi_l(\mathrm{ind}_{G'}^G \chi') - \mathrm{ind}_{G'}^G(\psi_l \chi')\,.$

To justify the definition we must show that the sum $\sum_{r \geq 1}$ is actually a finite sum. For this, let $\{t\}$ be a set of coset representatives of G' in G, so $G = \dot{\cup}_t tG'$, and define

$$m(g) = \min\{r \geq 0 : g^{l^r} \in G'\} \quad \text{for} \quad g \in G\,.$$

Then

$$\mathrm{ind}_{G'}^G \chi'(g) = \sum_t \dot{\chi}'(g^t) = \sum_{\{t : m(g^t) = 0\}} \chi'(g^t)\,,$$

if, as usual, $\dot{\chi}'$ coincides with χ' on G' and vanishes on $G \setminus G'$. Hence,

$$\begin{aligned} \chi(g) &= (\mathrm{ind}_{G'}^G \chi')(g^l) - \mathrm{ind}_{G'}^G(\psi_l \chi')(g) \\ &= \textstyle\sum_{m(g^{lt}) = 0} \chi'(g^{lt}) - \sum_{m(g^t) = 0} \chi'(g^{lt}) = \sum_{m(g^t) = 1} \chi'(g^{lt})\,. \end{aligned}$$

If r_0 is such that $G^{l^{r_0}} \subset G'$, then $\psi_l^{r_0 - 1} \chi = 0$ and $\sum_{r \geq 1} = \sum_{r=1}^{r_0 - 2}$, because the sum $\sum_{m(g^t) = 1}$ is empty when $g \in G^{l^{r_0 - 1}}$.

[3] For finite G, this $\mathrm{Res}_G^{G'}$, making the left square commute, appears already in [1, p.14]. For us the properties (HD),(TD) of $\mathrm{Res}_G^{G'}$ shown in §1 are equally necessary for the proof of the Theorem above.

It remains to show that $\mathrm{Res}_G^{G'} f \in \mathrm{Hom}^*(R_l G', \mathcal{Q}^{\mathrm{c}}_{\wedge}\Gamma_{k'})$, i.e., $\mathrm{Res}_G^{G'} f$ is a Galois stable homomorphism, compatible with W-twists (see [5,§1]), and taking values in $\mathcal{Q}^{\mathrm{c}}_{\wedge}\Gamma_{k'}$. The first property is easily checked and the third follows from the second as in [2, proof of Lemma 9]. We turn to twisting.

Let ρ' be a type-W character of G', so ρ' is inflated from $\Gamma_{k'}$, and write $\rho' = \mathrm{res}_G^{G'} \rho$ with ρ inflated from Γ_k to G. Then

$$f(\mathrm{ind}_{G'}^{G}(\rho'\chi')) = f(\rho \cdot \mathrm{ind}_{G'}^{G}\chi') = \rho^{\sharp}(f(\mathrm{ind}_{G'}^{G}\chi')) = (\rho')^{\sharp}(f(\mathrm{ind}_{G'}^{G}\chi'))$$

as $f(\mathrm{ind}_{G'}^{G}\chi') \in \mathcal{Q}^{\mathrm{c}}_{\wedge}\Gamma_{k'}$. Moreover, since ψ_l is multiplicative,

$$\psi_l(\mathrm{ind}_{G'}^{G}(\rho'\chi')) - \mathrm{ind}_{G'}^{G}(\psi_l(\rho'\chi')) = \psi_l(\rho \cdot \mathrm{ind}_{G'}^{G}\chi') - \mathrm{ind}_{G'}^{G}((\rho')^l \cdot \psi_l\chi') = \rho^l \cdot \chi$$

and thus

$$\begin{aligned}&\tfrac{\Psi^r}{l^r}\,(f(\psi_l^{r-1}(\rho^l \cdot \chi)))) = \tfrac{\Psi^r}{l^r}\, f(\rho^{l^r} \cdot \psi_l^{r-1}\chi) = \\ &\tfrac{\Psi^r}{l^r}\,((\rho^{l^r})^{\sharp}(f(\psi_l^{r-1}\chi))) = \rho^{\sharp}(\tfrac{\Psi^r}{l^r}\, f(\psi_l^{r-1}\chi)) = (\rho')^{\sharp}(\tfrac{\Psi^r}{l^r}\, f(\psi_l^{r-1}\chi)).\end{aligned}$$

Lemma 1. *The diagram below commutes. In it,* $\mathbf{L}$ *and* $\mathbf{L}'$ *are the lower horizontal maps of the logarithmic diagram (LD) for* G *and* G'*, respectively.*

$$\text{(HD)}\qquad \begin{array}{ccc} \mathrm{HOM}(R_l G, (\Lambda^{\mathrm{c}}_{\wedge}\Gamma_k)^{\times}) & \xrightarrow{\mathbf{L}} & \mathrm{Hom}^*(R_l G, \mathcal{Q}^{\mathrm{c}}_{\wedge}\Gamma_k) \\ \mathrm{res}_G^{G'} \downarrow & & \mathrm{Res}_G^{G'} \downarrow \\ \mathrm{HOM}(R_l G', (\Lambda^{\mathrm{c}}_{\wedge}\Gamma_{k'})^{\times}) & \xrightarrow{\mathbf{L}'} & \mathrm{Hom}^*(R_l G', \mathcal{Q}^{\mathrm{c}}_{\wedge}\Gamma_{k'}). \end{array}$$

Indeed, for $f \in \mathrm{HOM}(R_l G, (\Lambda^{\mathrm{c}}_{\wedge}\Gamma_k)^{\times})$ we get

$$\begin{aligned}&(\mathrm{Res}_G^{G'}\mathbf{L}f)(\chi') = (\mathbf{L}f)(\mathrm{ind}_{G'}^{G}\chi') + \textstyle\sum_{r\geq 1} \tfrac{\Psi^r}{l^r}\,[(\mathbf{L}f)(\psi_l^{r-1}\chi)] \\ &\doteq (\mathbf{L}f)(\mathrm{ind}_{G'}^{G}\chi') + \textstyle\sum_{r\geq 1} \tfrac{\Psi^r}{l^r}\,[\log(f(\psi_l^{r-1}\chi)) - \tfrac{\Psi}{l}\,\log(f(\psi_l^{r}\chi))] \\ &= (\mathbf{L}f)(\mathrm{ind}_{G'}^{G}\chi') + \textstyle\sum_{r\geq 1} \tfrac{\Psi^r}{l^r}\,\log(f(\psi_l^{r-1}\chi)) - \textstyle\sum_{r\geq 2} \tfrac{\Psi^r}{l^r}\,\log(f(\psi_l^{r-1}\chi)) \\ &= (\mathbf{L}f)(\mathrm{ind}_{G'}^{G}\chi') + \tfrac{\Psi}{l}\,\log(f(\chi)) = \tfrac{1}{l}\,\log\,\tfrac{f(\mathrm{ind}\,\chi')^l}{\Psi(f(\psi_l \mathrm{ind}\,\chi'))} + \tfrac{\Psi}{l}\,\log\,\tfrac{f(\psi_l \mathrm{ind}\,\chi')}{f(\mathrm{ind}\,\psi_l\chi')} \\ &= \tfrac{1}{l}\,\log\,\tfrac{f(\mathrm{ind}\,\chi')^l\cdot \Psi f(\psi_l \mathrm{ind}\,\chi')}{\Psi f(\psi_l \mathrm{ind}\,\chi')\cdot \Psi f(\mathrm{ind}\,\psi_l\chi')} = \tfrac{1}{l}\,\log\,\tfrac{f(\mathrm{ind}\,\chi')^l}{\Psi f(\mathrm{ind}\,\psi_l\chi')} \\ &= (\mathbf{L}'\mathrm{res}_G^{G'} f)(\chi').\end{aligned}$$

The dotted equality sign, $\doteq$, is due to the congruence $\frac{f(\chi)^l}{\Psi f(\psi_l\chi)} \equiv 1 \mod l\Lambda^{\mathrm{c}}_{\wedge}\Gamma_k$ (see [5,§1]) and to $\chi(1) = 0$, so $(\psi_l^{r-1}\chi)(1) = 0$ for every r. In fact, with $\tilde{\chi} \overset{\mathrm{def}}{=} \psi_l^{r-1}\chi$, we have

$$\begin{aligned}&f(\tilde{\chi})^l \equiv \Psi f(\psi_l\tilde{\chi}) \mod l\Lambda^{\mathrm{c}}_{\wedge}\Gamma_k \implies \\ &\qquad f(\tilde{\chi})^{l^s} \equiv \Psi^s f(\psi_l^s\tilde{\chi}) \equiv \Psi^s f(\tilde{\chi}(1)1) = 1 \mod l\Lambda^{\mathrm{c}}_{\wedge}\Gamma_k\end{aligned}$$

for big enough s. Thus $(\mathbf{L}f)(\tilde{\chi}) = \log(f(\tilde{\chi})) - \frac{\Psi}{l}\,\log(f(\psi_l\tilde{\chi}))$ as 'log' converges on an element a power of which is $\equiv 1 \mod l\Lambda^{\mathrm{c}}_{\wedge}\Gamma_k$.

The proof of Lemma 1 is complete.

By means of the trace isomorphism $\mathrm{Tr}: T(-) \to \mathrm{Hom}^*(-)$ we next transport $\mathrm{Res}_G^{G'}$ to $\mathrm{Res}_G^{G'}: T(\mathcal{Q}_\wedge G) \to T(\mathcal{Q}_\wedge G')$, i.e., the diagram

$$\text{(TD)} \qquad \begin{array}{ccc} T(\mathcal{Q}_\wedge G) & \xrightarrow{\mathrm{Tr}} & \mathrm{Hom}^*(R_l G, \mathcal{Q}_\wedge^c \Gamma_k) \\ \mathrm{Res}_G^{G'} \downarrow & & \mathrm{Res}_G^{G'} \downarrow \\ T(\mathcal{Q}_\wedge G') & \xrightarrow{\mathrm{Tr}'} & \mathrm{Hom}^*(R_l G', \mathcal{Q}_\wedge^c \Gamma_{k'}) \end{array}$$

commutes.

Lemma 2.
$$\begin{array}{ccc} K_1(\Lambda_\wedge G) & \xrightarrow{\mathbb{L}} & T(\mathcal{Q}_\wedge G) \\ \mathrm{res}_G^{G'} \downarrow & & \mathrm{Res}_G^{G'} \downarrow \\ K_1(\Lambda_\wedge G') & \xrightarrow{\mathbb{L}'} & T(\mathcal{Q}_\wedge G') \end{array} \qquad \textit{commutes and } \mathrm{Res}_G^{G'} t_{K/k} = t_{K/k'}\,.$$

The first claim follows from gluing together the diagrams (LD), (HD), (TD) and applying [2, Lemma 9]; the second claim follows from $\mathrm{res}_G^{G'} L_{K/k} = L_{K/k'}$ [2, Proposition 12].

The next lemma already concentrates on the case when G' is abelian and $[G : G'] = l$. We set $A = G/G' = \langle a \rangle$ and observe that a acts on G' by conjugation.

Lemma 3. *Let $\tau : \Lambda_\wedge G \to T(\Lambda_\wedge G)$ denote the canonical map and $g \in G$. If G' is abelian* [4] *and of index l in G, then*

$$\mathrm{Res}_G^{G'}(\tau g) = \begin{cases} \sum_{i=0}^{l-1} g^{a^i} & \text{if } g \in G' \\ g^l & \text{if } g \notin G'\,. \end{cases}$$

The lemma is just a special case of

Proposition A. *Let H be an open subgroup of $G = G(K/k)$. For $g \in G$ set $m_G^H(g) = \min\{r \geq 0 : g^{l^r} \in H\}$* [5]*, and let t run through a set of left representatives of H in G, i.e., $G = \dot{\bigcup} tH$. Then*

1. $\mathrm{Res}_G^H \tau_G(g) = \sum_t \tau_H((t^{-1}gt)^{l^{m_G^H(t^{-1}gt)}})/l^{m_G^H(t^{-1}gt)}$,
2. Res *is transitive,*
3. Res_G^H *is integral, i.e.,* $\mathrm{Res}_G^H(T(\Lambda_\wedge G)) \subset T(\Lambda_\wedge H)$ *for $H \leq G$ of finite index.*

Proposition A will be shown in the Appendix.

For the purpose of this paper it is enough to know Lemma 3 which we quickly prove directly on applying Tr' to both sides and employing the formula $\mathrm{Tr}'(\tau' g)(\chi') = \chi'(g)\overline{g}$ with $\overline{g}$ denoting the image of $g \in G'$ in $\Gamma_{k'}$ (see [5,§1]):

[4] whence $\tau' : \Lambda_\wedge G' \to T(\Lambda_\wedge G')$ is the identity map

[5] so $m_G^H(g)$ is the $m(g)$ defined earlier with $H = G'$

1. $(\mathrm{Tr}'\mathrm{Res}_G^{G'}(\tau g))(\chi') = \mathrm{Res}_G^{G'}(\mathrm{Tr}(\tau g))(\chi') = \mathrm{Tr}(\tau g)(\mathrm{ind}\,_{G'}^{G}\chi') + \frac{\Psi}{l}\mathrm{Tr}(\tau g)(\chi)$ since $G^l \subset G'$. Now, if $g \in G'$, $\mathrm{Tr}(\tau g)(\mathrm{ind}\,_{G'}^{G}\chi') = \sum_{i=0}^{l-1}\chi'(g^{a^i})\overline{g}$ and $\chi(g) = 0$. On the other hand, if $g \notin G'$, $\mathrm{Tr}(\tau g)(\mathrm{ind}\,_{G'}^{G}\chi') = 0$ and $\frac{\Psi}{l}\mathrm{Tr}(\tau g)(\chi) = \frac{1}{l}\mathrm{ind}\,_{G'}^{G}\chi'(g^l)\overline{g}^l = \chi'(g^l)\overline{g}^l$ since we may choose $a = g \mod G'$.

2. $\mathrm{Tr}'(\sum_{i=0}^{l-1} g^{a^i})(\chi') = \sum_{i=0}^{l-1}\chi'(g^{a^i})\overline{g}$, since g^{a^i} and g have the same image in Γ_k and so in $\Gamma_{k'}$. On the other hand, $\mathrm{Tr}'(g^l)(\chi') = \chi'(g^l)\overline{g^l}$.

The lemma is established.

We note that if $\Gamma\,(\simeq \mathbb{Z}_l)$ is a central subgroup of G contained in the abelian subgoup G' of index l, then the elements of $T(\Lambda_\wedge G)$ can uniquely be written as $\sum_g \beta_g\tau(g)$ with $\beta_g \in \Lambda_\wedge\Gamma$ and g running through a set of preimages of conjugacy classes of G/Γ (see [3, Lemma 5]). For each summand we have

$$\mathrm{Res}_G^{G'}(\beta_g\tau(g)) = \begin{cases} \sum_{i=0}^{l-1}\beta_g g^{a^i} & \text{if } g \in G' \\ \Psi(\beta_g)g^l & \text{if } g \notin G'\,. \end{cases}$$

2. Proof of the theorem

In this section $G = G(K/k)$ is a pro-l group and $G' = G(K/k')$ an abelian subgroup of index l (K/k is as in the introduction). As before, $A = G/G' = \langle a\rangle$, and we set $\hat{A} = 1 + a + \cdots + a^{l-1}$.

If G itself is abelian, the theorem holds by [4,§5, Example 1], whence we assume that G is non-abelian.

Lemma 4. *Assume that there exists an element $x \in T(\Lambda_\wedge G)$ such that $\mathrm{defl}_G^{G^{\mathrm{ab}}}x = \mathrm{defl}_G^{G^{\mathrm{ab}}}t_{K/k}$ and $\mathrm{Res}_G^{G'}x = \mathrm{Res}_G^{G'}t_{K/k}$. Then $t_{K/k} \in T(\Lambda_\wedge G)$.*

Denoting by K_{ab} the fixed field of the finite group $[G,G]$, we first observe, because of [5, Lemma 2.1] and Lemma 2, that $\mathrm{defl}_G^{G^{\mathrm{ab}}}t_{K/k} = t_{K_{\mathrm{ab}}/k}$ and $\mathrm{Res}_G^{G'}t_{K/k} = t_{K/k'}$ are integral: indeed, a logarithmic pseudomeasure is integral whenever the group is abelian.

From [4, Proposition 9] we obtain a power l^n of l such that $l^n t_{K/k} \in T(\Lambda_\wedge G)$. Consider the element $\tilde{x} = l^n(x - t_{K/k}) \in T(\Lambda_\wedge G)$. It satisfies $\mathrm{defl}_G^{G^{\mathrm{ab}}}\tilde{x} = 0 = \mathrm{Res}_G^{G'}\tilde{x}$. We are going to prove $\tilde{x} = 0$ which implies $x = t_{K/k}$ because $\mathrm{Hom}^*(R_lG, \mathcal{Q}_\wedge^c\Gamma_k)$, and so $T(\mathcal{Q}_\wedge G)$, is torsionfree; whence the the lemma will be verified.

The proof of $\tilde{x} = 0$ employs the commutative diagram shown in the proof of [5, Proposition 2.2]:

$$\begin{array}{ccccc} 1 + \mathfrak{a}_\wedge & \rightarrowtail & (\Lambda_\wedge G)^\times & \overset{\mathrm{defl}_G^{G^{\mathrm{ab}}}}{\twoheadrightarrow} & (\Lambda_\wedge G^{\mathrm{ab}})^\times \\ \downarrow & & \mathbb{L}\downarrow & & \mathbb{L}^{\mathrm{ab}}\downarrow \\ \tau(\mathfrak{a}_\wedge) & \rightarrowtail & T(\Lambda_\wedge G) & \overset{\mathrm{defl}_G^{G^{\mathrm{ab}}}}{\twoheadrightarrow} & \Lambda_\wedge G^{\mathrm{ab}} \end{array} ,$$

in which $\mathbb{L}$ is extended to $(\Lambda_\wedge G)^\times$ by means of the canonical surjection $(\Lambda_\wedge G)^\times \twoheadrightarrow K_1(\Lambda_\wedge G)$ and $\mathfrak{a}_\wedge = \ker(\Lambda_\wedge G \to \Lambda_\wedge G^{\mathrm{ab}})$. The diagram yields a $v \in (\Lambda_\wedge G)^\times$ with $\mathbb{L}(v) = \tilde{x}$, simply because $\mathrm{defl}_G^{G^{\mathrm{ab}}} \tilde{x} = 0$. Combining diagrams (HD), (LD) and (TD), we arrive at

$$\mathbf{L}'(\mathrm{res}_G^{G'}(\mathrm{Det}\, v)) = \mathrm{Res}_G^{G'}(\mathbf{L}(\mathrm{Det}\, v)) = \mathrm{Res}_G^{G'}(\mathrm{Tr}\,\mathbb{L}(v)) = \mathrm{Tr}'(\mathrm{Res}_G^{G'}\tilde{x}) = 0$$

and, with $\mathrm{res}_G^{G'}$ replaced by $\mathrm{defl}_G^{G^{\mathrm{ab}}}$, at

$$\begin{aligned}\mathbf{L}^{\mathrm{ab}}(\mathrm{defl}_G^{G^{\mathrm{ab}}}(\mathrm{Det}\, v)) &= \mathrm{defl}_G^{G^{\mathrm{ab}}}(\mathbf{L}(\mathrm{Det}\, v)) = \\ &= \mathrm{defl}_G^{G^{\mathrm{ab}}}(\mathrm{Tr}\,\mathbb{L}(v)) = \mathrm{Tr}^{\mathrm{ab}}(\mathrm{defl}_G^{G^{\mathrm{ab}}}\tilde{x}) = 0\,,\end{aligned}$$

since $\mathbf{L}$ and Tr commute with deflation.

The first displayed formula in [3, p.46] now implies that $\mathrm{res}_G^{G'}(\mathrm{Det}\, v)$ and $\mathrm{defl}_G^{G^{\mathrm{ab}}}(\mathrm{Det}\, v)$ are torsion elements in $\mathrm{HOM}(R_l G', (\Lambda_\wedge^{\mathrm{c}}\Gamma_{k'})^\times)$ and $\mathrm{HOM}(R_l(G^{\mathrm{ab}}), (\Lambda_\wedge^{\mathrm{c}}\Gamma_k)^\times)$, respectively. Moreover, the first paragraph of the proof of [5, Proposition 3.2] therefore shows that $\mathrm{Det}\, v$ itself is a torsion element in $\mathrm{HOM}(R_l G, (\Lambda_\wedge^{\mathrm{c}}\Gamma_k)^\times)$. Consequently, for some natural number m, $(\mathrm{Det}\, v)^{l^m} = 1$, so $l^m\mathbf{L}(\mathrm{Det}\, v) = 0 = l^m\mathrm{Tr}(\mathbb{L}v) = \mathrm{Tr}(l^m\tilde{x})$, and $\tilde{x} = 0$ follows, as has been claimed.

We now introduce the commutative diagram

$$\begin{array}{ccccc}
\tau(\mathfrak{a}_\wedge) & \rightarrowtail & T(\Lambda_\wedge G) & \twoheadrightarrow & \Lambda_\wedge G^{\mathrm{ab}} = T(\Lambda_\wedge G^{\mathrm{ab}}) \\
\mathrm{Res}\downarrow & & \mathrm{Res}_G^{G'}\downarrow & & \mathrm{Res}\downarrow \\
\mathfrak{b}'_\wedge = \tau'(\mathfrak{b}'_\wedge) & \rightarrowtail & \Lambda_\wedge G' = T(\Lambda_\wedge G') & \twoheadrightarrow & \Lambda_\wedge(G'/[G,G]) = T(\Lambda_\wedge(G'/[G,G]))
\end{array}$$

with exact rows (of which the upper one has already appeared in the diagram shown in the proof of the preceding lemma). The images of all vertical maps are fixed elementwise by A because of Lemma 3. Thus we can turn the diagram into

$$\text{(D)}\qquad \begin{array}{ccccc}
\tau(\mathfrak{a}_\wedge) & \rightarrowtail & T(\Lambda_\wedge G) & \twoheadrightarrow & \Lambda_\wedge G^{\mathrm{ab}} \\
\mathrm{Res}\downarrow & & \mathrm{Res}_G^{G'}\downarrow & & \mathrm{Res}\downarrow \\
\mathfrak{b}'^{\,A}_\wedge & \rightarrowtail & (\Lambda_\wedge G')^A & \to & (\Lambda_\wedge(G'/[G,G]))^A \\
\downarrow & & \downarrow & & \downarrow \\
\hat{H}^0(A, \mathfrak{b}'_\wedge) & \to & \hat{H}^0(A, \Lambda_\wedge G') & \to & \hat{H}^0(A, \Lambda_\wedge(G'/[G,G]))
\end{array}$$

with exact rows and canonical lower vertical maps.

Lemma 5. *In (D), the left vertical column is exact and the left bottom horizontal map is injective.*

Proof. The ideal $\mathfrak{a}_\wedge$ is (additively) generated by the elements $g(c-1)$ with $g \in G$, $c \in [G,G]$; those with $g \in G'$ generate $\mathfrak{b}'_\wedge$. We compute $\mathrm{Res}_G^{G'}\tau(g(c-1))$, using Lemma 3:

1. if $g \in G'$, $\mathrm{Res}_G^{G'}\tau(g(c-1)) = \sum_{i=0}^{l-1}((gc)^{a^i} - g^{a^i}) = \sum_{i=0}^{l-1}\Big(g(c-1)\Big)^{a^i} \in \mathrm{tr}_A\mathfrak{b}'_\wedge$,
2. if $g \notin G'$, $\mathrm{Res}_G^{G'}\tau(g(c-1)) = \mathrm{Res}_G^{G'}(\tau(gc)-\tau(g)) = (gc)^l - g^l = g^l c^{\hat{A}} - g^l = 0$, since

   $$(\star) \qquad [G,G]^{\hat{A}} = 1$$

   by $[G,G] \doteq [G,G']$ and $[G,G']^{\hat{A}} = ((G')^{a-1})^{\hat{A}} = 1$ as $(a-1)\hat{A} = 0$. Here, the dotted equality sign, $\doteq$, results from the equation

   $$[bg'_1, b^i g'_2] = (g'_1)^{-1}b^{-1}(g'_2)^{-1}b^{-i}bg'_1 b^i g'_2 = (g'_1)^{-1}({g'_2}^{-1})^b (g'_1)^{b^i} g'_2 = \Big((g'_1)^{-1}(g'_1)^{b^i}\Big)\Big(({g'_2}^{-1})^b g'_2\Big) \in [G',G]\cdot[G,G'] \le [G,G']$$

   for $g'_1, g'_2 \in G'$ and $b \in G \setminus G'$, because G' is abelian and normal in G.

Thus, $\mathrm{Res}_G^{G'}\tau(\mathfrak{a}_\wedge) = \mathrm{tr}_A\mathfrak{b}'_\wedge$, which proves the first claim of the lemma.
The second claim follows from $\hat{H}^{-1}(A, \Lambda_\wedge(G'/[G,G])) = 0$ and this in turn from the trivial action of A on $G'/[G,G]$ and the torsion freeness of $\Lambda_\wedge(G'/[G,G])$.
Lemma 5 is established.

As seen in diagram (D), there is an element $x_1 \in T(\Lambda_\wedge G)$ with $\mathrm{defl}_G^{G^{\mathrm{ab}}} x_1 = t_{K_{\mathrm{ab}}/k}$. We define $x'_1 \in \Lambda_\wedge G'$ by $\mathrm{Res}_G^{G'} x_1 = t_{K/k'} + x'_1$. Because of [5, Lemma 3.1], x'_1 is fixed by A. We want to change x_1 modulo $\tau(\mathfrak{a}_\wedge)$ so that the new x'_1 becomes zero: then we have arrived at an $x \in T(\Lambda_\wedge G)$ as assumed in Lemma 4 and the theorem will have been confirmed.
The above change is possible if, and only if, $x'_1 \in \mathrm{Res}_G^{G'}(\tau(\mathfrak{a}_\wedge))$ and so, because of Lemma 5, if x'_1 is in $\mathcal{T}' \stackrel{\mathrm{def}}{=} \mathrm{tr}_A(\Lambda_\wedge G')$, the A-trace ideal of the A-action on $\Lambda_\wedge G'$.

Proposition. *There exists an element x_1 of $T(\Lambda_\wedge G)$ with $\mathrm{defl}_G^{G^{\mathrm{ab}}} x_1 = t_{K_{\mathrm{ab}}/k}$ and $x'_1 \in \mathcal{T}'$.*

This is seen as follows. From [5,§1] we recall the existence of *pseudomeasures* $\lambda_{K_{\mathrm{ab}}/k}, \lambda_{K/k'}$ in $K_1(\Lambda_\wedge G^{\mathrm{ab}})$ and $K_1(\Lambda_\wedge G')$, respectively, satisfying $\mathrm{Det}\,\lambda_{K_{\mathrm{ab}}/k} = L_{K_{\mathrm{ab}}/k}$, $\mathrm{Det}\,\lambda_{K/k'} = L_{K/k'}$ (so $\mathbb{L}^{\mathrm{ab}}(\lambda_{K_{\mathrm{ab}}/k}) = t_{K_{\mathrm{ab}}/k}$, , $\mathbb{L}'(\lambda_{K/k'}) = t_{K/k'}$). From [5, 2. of Proposition 3.2] and [6, Theorem] we know that

$$\frac{\mathrm{ver}(\lambda_{K_{\mathrm{ab}}/k})}{\lambda_{K/k'}} \equiv 1 \mod \mathcal{T}'$$

where 'ver' is the map induced from the transfer homomorphism $G^{\mathrm{ab}} \to G'$.
Let $y \in (\Lambda_\wedge G)^\times$ have $\mathrm{defl}_G^{G^{\mathrm{ab}}} y = \lambda_{K_{\mathrm{ab}}/k}$ and set $\mathrm{res}_G^{G'} y = \lambda_{K/k'} \cdot y'$. Then

$$y' = \frac{\mathrm{res}_G^{G'} y}{\lambda_{K/k'}} \equiv \frac{\mathrm{ver}(\lambda_{K_{\mathrm{ab}}/k})}{\lambda_{K/k'}} \equiv 1 \mod \mathcal{T}'$$

(see the proof of [5, Proposition 3.2]). Moreover, $y' \in 1+\mathfrak{b}'_\wedge$. Now, $x_1 \stackrel{\text{def}}{=} \mathbb{L}(y)$ has $\operatorname{Res}^{G'}_G x_1 = \operatorname{Res}^{G'}_G \mathbb{L}(y) = t_{K/k'} + x'_1$ with $x'_1 \stackrel{\text{def}}{=} \mathbb{L}'(y')$, and $x'_1 \in \mathfrak{b}'_\wedge$ because of the commutativity of

$$\begin{array}{ccccc} 1+\mathfrak{b}'_\wedge & \rightarrowtail & (\Lambda_\wedge G')^\times & \twoheadrightarrow & (\Lambda_\wedge(G'/[G,G]))^\times \\ \mathbb{L}'\downarrow & & \mathbb{L}'\downarrow & & \overline{\mathbb{L}'}\downarrow \\ \mathfrak{b}'_\wedge & \rightarrowtail & \Lambda_\wedge G' & \twoheadrightarrow & \Lambda_\wedge(G'/[G,G]) \end{array}.$$

Hence, the proposition (and therefore the theorem) will be proved, if

$$x'_1 = \mathbb{L}'(y') \in \mathcal{T}'\,.$$

However, Lemma 5 gives

$$y' \in (1+{\mathfrak{b}'_\wedge}^A) \cap (1+\mathcal{T}') = 1 + ({\mathfrak{b}'_\wedge}^A \cap \mathcal{T}') = 1 + \mathrm{tr}_A \mathfrak{b}'_\wedge$$

and as $\mathbb{L}'(y') = \frac{1}{l}\log\frac{y'^l}{\Psi(y')}$ (compare [3, p.39]), we see that

$$\text{(1)} \qquad \mathbb{L}'(y') \in \mathcal{T}' \quad \text{if} \quad \frac{y'^l}{\Psi(y')} \equiv 1 \mod l\mathcal{T}'\,.$$

So it suffices to show this last congruence.

Write $y' = 1 + \mathrm{tr}_A\beta'$ with $\beta' \in \mathfrak{b}'_\wedge$. Since $(1+\mathrm{tr}_A\beta')^l \equiv 1 + (\mathrm{tr}_A\beta')^l \mod l\mathcal{T}'$, the congruence in (1) is equivalent to

$$\text{(2)} \qquad (\mathrm{tr}_A\beta')^l \equiv \Psi(\mathrm{tr}_A\beta') \mod l\mathcal{T}'.$$

On picking a central subgroup $\Gamma\,(\simeq \mathbb{Z}_l)$ of G and writing $\beta' = \sum_{g',c}\beta_{g',c}\,g'(c-1)$ with elements $\beta_{g',c} \in \Lambda_\wedge\Gamma$, $g' \in G'$, $c \in [G,G]$, we obtain

$$\text{(a)} \qquad \begin{aligned} (\mathrm{tr}_A\beta')^l &= \Big(\sum\nolimits_{g',c}\beta_{g',c}\mathrm{tr}_A(g'(c-1))\Big)^l \\ &\equiv \sum\nolimits_{g',c}(\beta_{g',c})^l\Big(\mathrm{tr}_A(g'(c-1))\Big)^l \\ &\equiv \sum\nolimits_{g',c}\Psi(\beta_{g',c})\Big((\mathrm{tr}_A(g'c))^l - (\mathrm{tr}_A g')^l\Big) \mod l\mathcal{T}' \end{aligned}$$

and

$$\text{(b)} \qquad \Psi(\mathrm{tr}_A\beta') = \sum\nolimits_{g',c}\Psi(\beta_{g',c})\Big(\mathrm{tr}_A((g'c)^l) - \mathrm{tr}_A({g'}^l)\Big)$$

as Ψ and tr_A commute. Thus congruence (2) will result from Lemma 6 below, since then subtracting (b) from (a) yields the sum

$$\begin{aligned} &\sum\nolimits_{g',c}\Psi(\beta_{g',c})\Big((\mathrm{tr}_A(g'c))^l - \mathrm{tr}_A((g'c)^l) - (\mathrm{tr}_A g')^l + \mathrm{tr}_A({g'}^l)\Big) \equiv \\ &\sum\nolimits_{g',c}\Psi(\beta_{g',c})\Big(-l(g'c)^{\hat{A}} + l{g'}^{\hat{A}}\Big) \equiv \\ &\sum\nolimits_{g',c}(-l)\Psi(\beta_{g',c}){g'}^{\hat{A}}(c^{\hat{A}}-1) \equiv 0 \mod l\mathcal{T}'\,, \end{aligned}$$

by $(\star)$ of the proof of Lemma 5.

Lemma 6. $(\mathrm{tr}_A g')^l - \mathrm{tr}_A({g'}^l) \equiv -l{g'}^{\hat{A}} \mod l\mathcal{T}'$ *for* $g' \in G'$.

Proof. Set $\tilde{A} = \mathbb{Z}/l \times A$ and make $M = \mathrm{Maps}(\mathbb{Z}/l, A)$ into an $\tilde{A}$-set by defining $m^{(z,a^i)}(x) = m(x-z)\cdot a^i$. Then

$$(\mathrm{tr}_A g')^l = (\sum_{i=0}^{l-1} {g'}^{a^i})^l = \sum_{m\in M}\prod_{z\in\mathbb{Z}/l} {g'}^{m(z)} = \sum_{m\in M} {g'}^{\sum_{z\in\mathbb{Z}/l} m(z)}$$

with $\sum_z m(z)$ read in $\mathbb{Z}[A]$.

We compute the subsums of $\sum_m$ in which m is constrained to an $\tilde{A}$-orbit.

If $m \in M$ has stabilizer $\{(0,1)\}$ in $\tilde{A}$, then the $\tilde{A}$-orbit sum is

$$\begin{aligned}&\textstyle\sum_{(z,a^i)\in\tilde{A}} {g'}^{\sum_{v\in\mathbb{Z}/l} m^{(z,a^i)}(v)} = \sum_{(z,a^i)\in\tilde{A}} {g'}^{\sum_{v\in\mathbb{Z}/l} m(v-z)a^i} = \\ &\textstyle\sum_{(z,a^i)} {g'}^{\sum_v m(v)a^i} = l\sum_i ({g'}^{\sum_v m(v)})^{a^i} = l\cdot \mathrm{tr}_A({g'}^{\sum_v m(v)}) \in l\mathcal{T}' .\end{aligned}$$

Note that no $m \in M$ is stabilized by $(0,a^i)$ with $a^i \neq 1$: for $m(z) = m^{(0,a^i)}(z) = m(z)a^i$ implies $a^i = 1$. It follows that the stabilizers of the elements with stabilizer different from $\{(0,1)\}$ must be cyclic of order l and different from $\{(0,a^i) : 0 \le i \le l-1\}$ and therefore $= \langle(1,a^j)\rangle$ for a unique $j \mod l$.

One now checks that for each j there is exactly one $\tilde{A}$-orbit with stabilizer $\langle(1,a^j)\rangle$ and that it is represented by m_j, $m_j(z) = a^{jz}$. Moreover, $\{(0,a^i) : 0 \le i \le l-1\}$ is a transversal of the stabilizer of m_j in $\tilde{A}$.

For each j, the sum of ${g'}^{\sum_z m(z)}$ over the $\tilde{A}$-orbit of m_j is $\sum_i {g'}^{\sum_z m_j^{(0,a^i)}(z)} = \sum_i {g'}^{\sum_z a^{jz}a^i}$. If $j = 0$, this is $\sum_i {g'}^{la^i} = \mathrm{tr}_A({g'}^l)$, accounting for that term in the claim. If $j \neq 0$, it is $\sum_i {g'}^{\hat{A}\cdot a^i} = l{g'}^{\hat{A}}$, and summing over $j \neq 0$ gives $(l-1)l\cdot {g'}^{\hat{A}} \equiv -l\cdot {g'}^{\hat{A}} \mod l\mathcal{T}'$ because $l^2\cdot {g'}^{\hat{A}} = l\cdot\mathrm{tr}_A({g'}^{\hat{A}}) \in l\mathcal{T}'$.

This finishes the proof of Lemma 6.

3. Appendix

Proof of Proposition A:

We start from the formula $\chi(g) = \sum_{m_G^H(g^t)=1} \chi'(g^{lt})$ which appeared in the justification of the definition in §1 (now with $H = G'$). It implies $\chi(g^{l^{r-1}}) = \sum_{m_G^H(t^{-1}gt)=r} \chi'((t^{-1}gt)^{l^r})$ for $r \ge 1$. Exploiting [3, Lemma 6] for the equality

$\doteq$ and [3, Proposition 3] for $\ddot{=}$ we obtain

$$\begin{aligned}
&(\mathrm{Res}_G^H \mathrm{Tr}_G \tau_G(g))(\chi') \\
&= (\mathrm{Tr}_G \tau_G(g))(\mathrm{ind}\,{}_H^G \chi') + \textstyle\sum_{r\geq 1} \frac{\Psi^r}{l^r}\,[(\mathrm{Tr}_G \tau_G(g))(\psi_l^{r-1}\chi)] \\
&\doteq \mathrm{trace}(g \mid \mathfrak{V}_{\mathrm{ind}\,{}_H^G\chi'}) + \textstyle\sum_{r\geq 1} \frac{\Psi^r}{l^r}\,[\mathrm{trace}(g \mid \mathfrak{V}_{\psi_l^{r-1}\chi})] \\
&\ddot{=} (\mathrm{ind}\,{}_H^G \chi')(g)\overline{g} + \textstyle\sum_{r\geq 1} \frac{\Psi^r}{l^r}\,[(\psi_l^{r-1}\chi)(g)\overline{g}] \\
&= (\mathrm{ind}\,{}_H^G \chi')(g)\overline{g} + \textstyle\sum_{r\geq 1} \chi(g^{l^{r-1}})\,\frac{\overline{g}^{l^r}}{l^r} \\
&= \textstyle\sum_{m_G^H(t^{-1}gt)=0} \chi'(t^{-1}gt)\overline{g} + \sum_{r\geq 1} \sum_{m_G^H(t^{-1}gt)=r} \chi'((t^{-1}gt)^{l^r})\,\frac{\overline{g}^{l^r}}{l^r} \\
&= \textstyle\sum_t \chi'((t^{-1}gt)^{l^{m_G^H(t^{-1}gt)}})\,\frac{\overline{g}^{l^{m_G^H(t^{-1}gt)}}}{l^{m_G^H(t^{-1}gt)}}\ .
\end{aligned}$$

On the other hand, if $h \in H$, the same calculation on the H-level gives

$$(\mathrm{Tr}_H \tau_H(h))(\chi') = \mathrm{trace}(h \mid \mathfrak{V}_{\chi'}) = \chi'(h)\overline{h}\,,$$

and therefore, with $\tau_H(\sum_t \frac{(t^{-1}gt)^{l^{m_G^H(t^{-1}gt)}}}{l^{m_G^H(t^{-1}gt)}}) \in T(\mathcal{Q}(H))$, we get

$$\begin{aligned}
&(\mathrm{Tr}_H \tau_H \textstyle\sum_t \frac{(t^{-1}gt)^{l^{m_G^H(t^{-1}gt)}}}{l^{m_G^H(t^{-1}gt)}})(\chi') \\
&= \textstyle\sum_t \frac{1}{l^{m_G^H(t^{-1}gt)}}\,\chi'((t^{-1}gt)^{l^{m_G^H(t^{-1}gt)}})\overline{(t^{-1}gt)^{l^{m_G^H(t^{-1}gt)}}} \\
&= \textstyle\sum_t \chi'((t^{-1}gt)^{l^{m_G^H(t^{-1}gt)}})\,\frac{\overline{g}^{l^{m_G^H(t^{-1}gt)}}}{l^{m_G^H(t^{-1}gt)}}
\end{aligned}$$

via $\mathcal{Q}^{\mathrm{c}}(\Gamma_{k'}) \hookrightarrow \mathcal{Q}^{\mathrm{c}}(\Gamma_k)$ (where k' is the fixed field of H).

Since these formulae agree for all characters χ' of H (with open kernel) it follows that $\mathrm{Res}_G^H \tau_G(g) = \tau_H(\sum_t \frac{(t^{-1}gt)^{l^{m_G^H(t^{-1}gt)}}}{l^{m_G^H(t^{-1}gt)}})$ by the uniqueness of this element, proving 1. of Proposition A.

For 2. we first consider the situation $H \leq G' \leq G$ with $[G:G'] = l$ and show $\mathrm{Res}_G^H = \mathrm{Res}_{G'}^H \circ \mathrm{Res}_G^{G'}$:

Write $G = \dot{\bigcup}_x xG'$, $G' = \dot{\bigcup}_y yH$, hence $G = \dot{\bigcup}_{x,y} xyH$, and recall [6] from 1. that

$$\mathrm{Res}_G^{G'} g = \begin{cases} \tau_{G'}(\sum_x x^{-1}gx) & ,\ g \in G' \\ \tau_{G'}(g^l) & ,\ g \notin G' \end{cases}\ .$$

[6] if $g \notin G'$, then we may use $\{t\} = \{g^i : 0 \leq i \leq l-1\}$

Then, accordingly,

$$\begin{aligned}
&\mathrm{Res}^H_{G'}\mathrm{Res}^{G'}_G\tau_G(g)\\
&\quad = \begin{cases}\sum_x \mathrm{Res}^H_{G'}\tau_{G'}(x^{-1}gx)\\ \mathrm{Res}^H_{G'}\tau_{G'}(g^l)\end{cases}\\
&\quad = \begin{cases}\sum_x\sum_y \tau_H((y^{-1}x^{-1}gxy)^{l^{m^H_{G'}(y^{-1}x^{-1}gxy)}})/l^{m^H_{G'}(y^{-1}x^{-1}gxy)}\\ \sum_y \tau_H((y^{-1}g^l y)^{l^{m^H_{G'}(y^{-1}g^l y)}})/l^{m^H_{G'}(y^{-1}g^l y)}\end{cases}\\
&\quad = \begin{cases}\sum_t \tau_H((t^{-1}gt)^{l^{m^H_G(t^{-1}gt)}})/l^{m^H_G(t^{-1}gt)}\\ \sum_t \tau_H((t^{-1}g^l t)^{l^{m^H_{G'}(t^{-1}g^l t)}})/l\cdot l^{m^H_{G'}(t^{-1}g^l t)}\end{cases}\\
&\quad = \sum_t \tau_H((t^{-1}gt)^{l^{m^H_G(t^{-1}gt)}})/l^{m^H_G(t^{-1}gt)} \;=\; \mathrm{Res}^H_G\tau_G(g)\,,
\end{aligned}$$

using $G' \lhd G$ and $m^H_{G'}(x^l)+1 = m^H_G(x)$ for $x \notin G'$.

Induction on $[G:H]$ now proves 2., $\mathrm{Res}^H_G = \mathrm{Res}^H_{G''}\circ\mathrm{Res}^{G''}_G$ for $H \le G'' \le G$. Indeed, if $G'' \neq G$, find $G' \geq G''$ with $[G:G'] = l$ and use

$$\begin{aligned}
\mathrm{Res}^H_{G''}\mathrm{Res}^{G''}_G &= \mathrm{Res}^H_{G''}(\mathrm{Res}^{G''}_{G'}\mathrm{Res}^{G'}_G) =\\
&= (\mathrm{Res}^H_{G''}\mathrm{Res}^{G''}_{G'})\mathrm{Res}^{G'}_G = \mathrm{Res}^H_{G'}\mathrm{Res}^{G'}_G = \mathrm{Res}^H_G\,.
\end{aligned}$$

Finally, for 3., proceed by induction on $[G:H]$ and use 1. if H has index l in G and 2. when the index is bigger, in which case there is a subgroup G' of G with $H \le G$, $[G:G'] = l$.

The proof of Proposition A is complete.

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