Essential Spectrum of Multiparticle Brown–Ravenhall Operators in External Field

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Received: February 4, 2008
Revised: March 7, 2008

Communicated by Heinz Siedentop

Abstract. The essential spectrum of multiparticle Brown–Ravenhall operators is characterized in terms of two–cluster decompositions for a wide class of external fields and interparticle interactions and for the systems with prescribed symmetries.

2000 Mathematics Subject Classification: 81V55, 81Q10

1 Introduction

It is well known that the eigenvalues of the one–particle Dirac operator are in much better accordance with the spectroscopic data then the eigenvalues of the Schrödinger operator. However, due to the presence of the negative continuum of positronic states the multiparticle Coulomb–Dirac operator has no eigenvalues and its essential spectrum is the whole real line. Coupling with the quantized electromagnetic field does not correct this situation. However, there are ways to construct a *semibounded* operator which will take the relativistic effects into account. Such models, although nonlocal, find their applications in numerical studies of heavy elements and cosmology, where the relativistic effects cannot be ignored.
The most obvious choice of the kinetic energy (sometimes called Chandrasekhar or Herbst operator) given by $\sqrt{\mathbf{p}^2c^2 + m^2c^4}$, $\mathbf{p}$ and m being the momentum and mass of the particle, suffers from the lack of semiboundedness for nuclear charges exceeding 87, as shown in [9]. Most other operators considered in the literature are obtained by reducing the (multiparticle) Dirac operator onto some subspace on which it becomes semibounded. One of such models, extensively studied recently, is by Brown and Ravenhall [4], see also Bethe and Salpeter [3], Sucher [18, 19]. In this model every particle is confined the positive spectral subspace of the *free* Dirac operator. Since the multiplication by interaction

potentials does not leave this subspace invariant, the potential energy terms are projected back by the corresponding projector.

The mathematical study of the Brown–Ravenhall operator started from the one–particle case in the article of Evans, Perry, and Siedentop [7]. The authors have proved that the atomic Hamiltonian is semibounded from below for nuclear charges not exceeding 124. This makes the Brown–Ravenhall model applicable to all existing elements. It was also proved in [7] that the essential spectrum of the one–particle atomic Brown–Ravenhall operator is $[mc^2, \infty)$ with m being the mass of the particle, and that the singular continuous spectrum is empty.

Further studies of the Brown–Ravenhall operator include the improved lower bounds by Tix [21, 22] (see also Burenkov and Evans [5]) in the atomic case, the proof that the eigenvalues of Brown–Ravenhall operator are strictly bigger than those of the one–particle Dirac operator by Griesemer et al. [8], proofs of stability of one-electron molecule by Balinsky and Evans [2], the proof of stability of matter by Hoever and Siedentop [10], and the asymptotic result on the ground state energy for large atomic charges Z (with Z/c fixed) by Cassanas and Siedentop [6].

The essential spectrum of the multiparticle operator was characterized by Jakubaßa–Amundsen [12, 13], and in our joint work with S. Vugalter [16] in terms of two–cluster decompositions. This is usually referred to as HVZ theorem after the well known result for the multiparticle Schrödinger operator. In [11] an analogous result is proved in presence of the constant magnetic field. It is also proved in [16] that the neutral atoms or positively charged atomic ions have infinitely many bound states.

In all these previous studies the nuclei were considered as fixed sources of the external field, the particles were assumed to be identical, and the interaction potentials were purely Coulombic.

In this paper we generalize the HVZ theorem of [12, 13, 16] as follows: We allow any number of (massive) particles of the system to be identical. We allow quite general matrix interaction potentials. In particular, our result applies in the presence of the magnetic fields if the vector potential decays at infinity in some weak sense. Another problem we address is the reduction to any irreducible representations of the groups of rotation–reflection symmetry and permutations of identical particles. Note that such a reduction allows to analyze the eigenvalues of some irreducible representations even if they are embedded into the continuous spectrum of some other representations. For some particular models (including atoms and molecules in the Born–Oppenheimer approximation) the existence of such eigenvalues can be shown along the same lines as in [16].

From the technical point of view, the nonlocality of the model due to the presence of the spectral projections of the free Dirac operator is overcome with the same ideas as in [16]. One more complication should be stressed: for the Brown–Ravenhall operator the center of mass motion cannot be separated in the same way as it is usually done for Schrödinger operators, where the complete

Hamiltonian without external field can be represented in suitable coordinates as

$$\mathcal{H} = A \otimes I + I \otimes B,$$

where A describes the free motion of the center of mass and B is the internal Hamiltonian of the system (see [14]). Such a decomposition appears to be especially fruitful in the presence of rotation symmetries. Since it cannot be obtained for pseudorelativistic operators due to the form of kinetic energy, we have used a completely different approach based on the commutation of the Hamiltonian with the *absolute value* of the total momentum of the system.

Note that the proof of the HVZ theorem for a system of particles described by the Chandrasekhar operator was till now not known for operators reduced to irreducible representations of the rotation–reflection symmetry group (see the article of Lewis, Siedentop and Vugalter [15] for the case without such reductions). Such a proof can now be obtained as a simplified modification of the proof given in this paper.

In Section 2 we introduce the model and make the necessary assumptions. At the end of this section we formulate the main result in Theorem 6. The rest of the article contains the proof of this theorem.

2 Setup and Main Result

$[A, B] = AB - BA$ is the commutator of two operators. $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ stand for the inner product and the norm in $L_2(\mathbb{R}^{3d}, \mathbb{C}^{4^d})$, where d is the dimension of the underlying configuration space. Irrelevant constants are denoted by C. I_Ω is the indicator function of the set Ω. For a selfadjoint operator A we denote its spectrum and the corresponding sesquilinear form by $\sigma(A)$ and $\langle A\cdot, \cdot \rangle = \langle \cdot, A\cdot \rangle$, respectively. We use the conventional units $\hbar = c = 1$. Sometimes we denote the unitary Fourier transform by $\widehat{\cdot}$.

In the Hilbert space $L_2(\mathbb{R}^3, \mathbb{C}^4)$ the Dirac operator describing a particle of mass $m > 0$ is given by

$$D_m = -i\boldsymbol{\alpha} \cdot \nabla + \beta m,$$

where $\boldsymbol{\alpha} := (\alpha_1, \alpha_2, \alpha_3)$ and β are the 4×4 Dirac matrices [20]. The form domain of D_m is the Sobolev space $H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$ and the spectrum is $(-\infty, -m] \cup [m, +\infty)$. Let Λ_m be the orthogonal projector onto the positive spectral subspace of D_m:

$$\Lambda_m := \frac{1}{2} + \frac{-i\boldsymbol{\alpha} \cdot \nabla + \beta m}{2\sqrt{-\Delta + m^2}}. \tag{2.1}$$

We consider a finite system of N particles with positive masses m_n, $n = 1, \dots, N$. To simplify the notation we write D_n and Λ_n for D_{m_n} and Λ_{m_n}, respectively. Let $\mathfrak{H}_N := \bigotimes_{n=1}^{N} \Lambda_n L_2(\mathbb{R}^3, \mathbb{C}^4)$ be the Hilbert space with the inner product induced by those of $\bigotimes_{n=1}^{N} L_2(\mathbb{R}^3, \mathbb{C}^4) \cong L_2(\mathbb{R}^{3N}, \mathbb{C}^{4^N})$. In this space the

N–particle Brown–Ravenhall operator is formally given by

$$\mathcal{H}_N = \Lambda^N \bigg(\sum_{n=1}^{N} (D_n + V_n) + \sum_{n<j}^{N} U_{nj} \bigg) \Lambda^N, \tag{2.2}$$

with

$$\Lambda^N := \prod_{n=1}^{N} \Lambda_n = \mathop{\otimes}_{n=1}^{N} \Lambda_n. \tag{2.3}$$

Here and below the indices n and j indicate the particle, on whose coordinates the corresponding operator acts. In (2.2) V_n is the external field potential for the n^{th} particle, i.e., the operator of multiplication by a hermitian 4×4 matrix–function $V_n(\mathbf{x}_n)$, $n = 1, \dots, N$, and U_{nj} is the potential energy of the interaction between the n^{th} and j^{th} particles, given by the operator of multiplication by a hermitian 16×16 matrix–function $U_{nj}(\mathbf{x}_n - \mathbf{x}_j)$, $n < j = 1, \dots, N$. More explicitly, if we let $s_j \in \{1,2,3,4\}$ be the spinor index of the j^{th} particle, then

$$\begin{aligned}
&(V_n \psi)(\mathbf{x}_1, s_1; \dots; \mathbf{x}_n, s_n; \dots; \mathbf{x}_N, s_N) \\
&:= \sum_{\widetilde{s}_n} V_n^{s_n, \widetilde{s}_n}(\mathbf{x}_n) \psi(\mathbf{x}_1, s_1; \dots; \mathbf{x}_n, \widetilde{s}_n; \dots; \mathbf{x}_N, s_N),
\end{aligned}$$

and

$$\begin{aligned}
&(U_{nj} \psi)(\mathbf{x}_1, s_1; \dots; \mathbf{x}_n, s_n; \dots; \mathbf{x}_j, s_j; \dots; \mathbf{x}_N, s_N) \\
&:= \sum_{\widetilde{s}_n, \widetilde{s}_j} U_{nj}^{s_n s_j, \widetilde{s}_n \widetilde{s}_j}(\mathbf{x}_n - \mathbf{x}_j) \psi(\mathbf{x}_1, s_1; \dots; \mathbf{x}_n, \widetilde{s}_n; \dots; \mathbf{x}_j, \widetilde{s}_j; \dots; \mathbf{x}_N, s_N).
\end{aligned}$$

Before we make other assumptions on the interaction potentials, let us consider possible decompositions of the system into two clusters. Let $Z = (Z_1, Z_2)$ be a decomposition of the index set $I := \{1, \dots, N\}$ into two disjoint subsets:

$$I = Z_1 \cup Z_2, \quad Z_1 \cap Z_2 = \varnothing.$$

Let

$$N_j := \# Z_j, \quad j = 1, 2 \tag{2.4}$$

be the number of particles in each cluster. We will write $n \# j$ if n and j belong to different clusters. Let

$$\mathcal{H}_{Z,1} := \sum_{n \in Z_1} (D_n + V_n) + \sum_{\substack{n,j \in Z_1 \\ n<j}} U_{nj}, \tag{2.5}$$

$$\mathcal{H}_{Z,2} := \sum_{n \in Z_2} D_n + \sum_{\substack{n,j \in Z_2 \\ n<j}} U_{nj}. \tag{2.6}$$

We omit $\mathcal{H}_{Z,j}$ if $Z_j = \varnothing$, $j = 1, 2$. Let us introduce the operators corresponding to noninteracting clusters, with the second cluster transferred far away from the sources of the external field:

$$\widetilde{\mathcal{H}}_{Z,j} := \Lambda_{Z,j}\mathcal{H}_{Z,j}\Lambda_{Z,j}, \quad \text{in} \quad \mathfrak{H}_{Z,j} := \underset{n\in Z_j}{\otimes} \Lambda_n L_2(\mathbb{R}^3, \mathbb{C}^4), \quad j = 1, 2, \tag{2.7}$$

where

$$\Lambda_{Z,j} := \prod_{n\in Z_j} \Lambda_n.$$

We make the following assumptions:

Assumption 1 *There exists $C > 0$ such that for any Z and $j = 1, 2$*

$$\left|\langle \mathcal{H}_{Z,j}\varphi, \psi\rangle\right| \leqslant C\|\varphi\|_{H^{1/2}}\|\psi\|_{H^{1/2}}, \quad \textit{for any} \quad \varphi, \psi \in \underset{n\in Z_j}{\otimes} H^{1/2}(\mathbb{R}^3, \mathbb{C}^4). \tag{2.8}$$

For Coulomb interaction potentials (2.8) follows from Kato's inequality.

Assumption 2 *There exist $C_1 > 0$ and $C_2 \in \mathbb{R}$ such that for any Z*

$$\begin{aligned} \langle \widetilde{\mathcal{H}}_{Z,j}\psi, \psi\rangle \geqslant C_1\langle \sum_{n\in Z_j} D_n\psi, \psi\rangle - C_2\|\psi\|^2, \\ \textit{for any} \quad \psi \in \underset{n\in Z_j}{\otimes} \Lambda_n H^{1/2}(\mathbb{R}^3, \mathbb{C}^4), \quad j = 1, 2. \end{aligned} \tag{2.9}$$

Remark 3 *Note that for $\psi \in \underset{n\in Z_j}{\otimes} \Lambda_n H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$ the metric*

$$\langle \sum_{n\in Z_j} D_n\psi, \psi\rangle^{1/2} = \Big\| \sum_{n\in Z_j} |D_n|^{1/2}\psi\Big\|$$

is equivalent to the norm of ψ in $\underset{n\in Z_j}{\otimes} H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$, since

$$\Lambda_n D_n \Lambda_n = \Lambda_n |D_n| \Lambda_n = \Lambda_n \sqrt{-\Delta + m_n^2}\,\Lambda_n. \tag{2.10}$$

An equivalent formulation of Assumption 2 is that the operator $\widetilde{\mathcal{H}}_{Z,j}$ is semi-bounded from below even if we multiply all the interaction potentials by $1 + \varepsilon$ with $\varepsilon > 0$ small enough. This is only slightly more restrictive than the semiboundedness of $\widetilde{\mathcal{H}}_{Z,j}$.

Assumption 4 *For any $R > 0$ there exists a finite constant $C_R \geqslant 0$ such that*

$$\sum_{n=1}^{N}\left(\int_{|\mathbf{x}|\leqslant R} \left|V_n(\mathbf{x})\right|^2 d\mathbf{x}\right)^{1/2} + \sum_{n<j}^{N}\left(\int_{|\mathbf{x}|\leqslant R} \left|U_{nj}(\mathbf{x})\right|^2 d\mathbf{x}\right)^{1/2} \leqslant C_R. \tag{2.11}$$

This means that the interaction potentials are locally square integrable.

Assumption 5 *For any $\varepsilon > 0$ there exists $R > 0$ big enough such that for all $n = 1, \dots, N$*

$$\|V_n I_{\{|\mathbf{x}_n|>R\}}\psi\| \leqslant \varepsilon \big\||D_n|^{1/2}\psi\big\|, \quad \textit{for all} \quad \psi \in H^{1/2}(\mathbb{R}^3, \mathbb{C}^4), \tag{2.12}$$

and for all $n < j = 1, \dots, N$

$$\begin{aligned}\|U_{nj} I_{\{|\mathbf{x}_n-\mathbf{x}_j|>R\}}\varphi\| \leqslant \varepsilon \min\Big\{\big\||D_n|^{1/2}\varphi\big\|, \big\||D_j|^{1/2}\varphi\big\|\Big\},\\ \textit{for all} \quad \varphi \in H^{1/2}(\mathbb{R}^6, \mathbb{C}^{16}).\end{aligned} \tag{2.13}$$

By Remark 3 this assumption is *weaker* then the decay of L_∞ norms of the interaction potentials at infinity.
It follows from (2.9) and Remark 3 that for any Z there exists a constant $C > 0$ such that for any $\psi \in \underset{n\in Z_j}{\otimes} \Lambda_n H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$

$$\|\psi\|^2_{H^{1/2}} \leqslant C\big(\langle \widetilde{\mathcal{H}}_{Z,j}\psi, \psi\rangle + \|\psi\|^2\big), \quad j = 1, 2. \tag{2.14}$$

Hence by Assumptions 1 and 2, the quadratic forms of operators (2.7) (and, in particular, $\mathcal{H}_N$) are semibounded from below and closed on $\underset{n\in Z_j}{\otimes} \Lambda_n H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$. Thus these operators are well–defined in the form sense.
Some particles of the system (say, k^{th} and l^{th}) can be identical (in which case $m_k = m_l$, $V_k = V_l$, and $U_{kj} = U_{lj}$ for all j). Then the operator $\mathcal{H}_N$ can be reduced to the subspace of functions which transform in a certain way under permutations of identical particles. The most physically motivated assumption is that any transposition of two identical particles should change the sign of the wave function $\psi \in \mathfrak{H}_N$ describing the system. This is the Pauli principle applied to the identical fermions (the model describes spin 1/2 particles, thus fermions).
Let Π be the subgroup of the symmetric group $\mathcal{S}_N$ generated by transpositions of identical particles. We denote the number of elements of Π by h_Π. Let E be some irreducible representation of Π with dimension d_E and character ξ_E. For $\psi \in \mathfrak{H}_N$ let

$$P^E\psi := \frac{d_E}{h_\Pi} \sum_{\pi\in\Pi} \overline{\xi_E(\pi)}\pi\psi, \tag{2.15}$$

where π is the operator of permutation:

$$(\pi\psi)(\mathbf{x}_1, s_1; \dots; \mathbf{x}_N, s_N) = \psi(\mathbf{x}_{\pi^{-1}(1)}, s_{\pi^{-1}(1)}; \dots; \mathbf{x}_{\pi^{-1}(N)}, s_{\pi^{-1}(N)}).$$

Here $s_1, \dots, s_N$ are the spinor coordinates of the particles. The operator P^E defined in (2.15) is the projector to the subspace of functions in $\mathfrak{H}_N$ which transform according to the representation E of Π. Since any $\pi \in \Pi$ commutes with $\mathcal{H}_N$, P^E reduces $\mathcal{H}_N$. Let $\mathcal{H}_N^E$ be the corresponding reduced selfadjoint operator in

$$\mathfrak{H}_N^E := P^E \mathfrak{H}_N.$$

For a decomposition $Z = (Z_1, Z_2)$ let Π_j^Z be the group generated by transpositions of identical particles inside Z_j, $j = 1, 2$. For any irreducible representation E_j of Π_j^Z with dimension d_{E_j} and character ξ_{E_j} the projection to the space of functions in $\mathfrak{H}_{Z,j}$ transforming according to E_j under the action of Π_j^Z is given by

$$P^{E_j}\psi := \frac{d_{E_j}}{h_{\Pi_j^Z}} \sum_{\pi \in \Pi_j^Z} \overline{\xi_{E_j}(\pi)} \pi\psi, \quad \psi \in \mathfrak{H}_{Z,j},$$

where $h_{\Pi_j^Z}$ is the cardinality of Π_j^Z. Projectors P^{E_j} reduce operators $\widetilde{\mathcal{H}}_{Z,j}$. We introduce the reduced operators $\widetilde{\mathcal{H}}_{Z,j}^{E_j}$ in

$$\mathfrak{H}_{Z,j}^{E_j} := P^{E_j} \mathfrak{H}_{Z,j}, \quad j = 1, 2.$$

Given an irreducible representation E of Π and a decomposition $Z = (Z_1, Z_2)$, we have

$$\mathfrak{H}_N^E \subset \bigoplus_{(E_1,E_2)} \left(\mathfrak{H}_{Z,1}^{E_1} \otimes \mathfrak{H}_{Z,2}^{E_2} \right), \tag{2.16}$$

where $E_{1,2}$ are some irreducible representations of $\Pi_{1,2}^Z$. We write $(E_1, E_2) \underset{Z}{\prec} E$ if the corresponding term cannot be omitted on the r. h. s. of (2.16) without violation of the inclusion.

Apart from permutations of identical particles the operator $\mathcal{H}_N^E$ can have some rotation–reflection symmetries. Let γ be an orthogonal transform in $\mathbb{R}^3$: the rotation around the axis directed along a unit vector $\mathbf{n}_\gamma$ through an angle φ_γ, possibly combined with the reflection $\mathbf{x} \mapsto -\mathbf{x}$. The corresponding unitary operator O_γ acts on the functions $\psi \in \mathfrak{H}^N$ as (see [20], Chapter 2)

$$(O_\gamma \psi)(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \prod_{n=1}^N e^{-i\varphi_\gamma \mathbf{n}_\gamma \cdot \mathbf{S}_n} \psi(\gamma^{-1}\mathbf{x}_1, \ldots, \gamma^{-1}\mathbf{x}_N).$$

Here $\mathbf{S}_n = -\frac{i}{4}\alpha_n \wedge \alpha_n$ is the spin operator acting on the spinor coordinates of the n^{th} particle. The compact group of orthogonal transformations γ such that O_γ commutes with V_n and U_{nj} for all $n, j = 1, \ldots, N$ (and thus with $\mathcal{H}_N^E$) we denote by Γ. Further, we decompose $\mathfrak{H}_N^E$ into the orthogonal sum

$$\mathfrak{H}_N^E = \bigoplus_{\alpha \in A} \mathfrak{H}_N^{D_\alpha, E}, \tag{2.17}$$

where $\mathfrak{H}_N^{D_\alpha,E}$ consists of functions which transform under O_γ according to some irreducible representation D_α of Γ, and A is the set indexing all such irreducible representations. The decomposition (2.17) reduces $\mathcal{H}_N^E$. We denote the selfadjoint restrictions of $\mathcal{H}_N^E$ to $\mathfrak{H}_N^{D_\alpha,E}$ by $\mathcal{H}_N^{D_\alpha,E}$. For any fixed irreducible representation D with dimension d_D and character ζ_D the orthogonal projector in $\mathfrak{H}_N$ onto the subspace of functions which transform according to D is

$$P^D := d_D \int_\Gamma \overline{\zeta_D(\gamma)} O_\gamma d\mu(\gamma),$$

where μ is the invariant probability measure on Γ.
For $j = 1, 2$ let D_j be some irreducible representations of Γ with dimensions d_{D_j} and characters ζ_{D_j}. The corresponding projectors in $\mathfrak{H}_{Z,j}$ are given by

$$P^{D_j} = d_{D_j} \int_\Gamma \overline{\zeta_{D_j}(\gamma)} O_{\gamma,j} d\mu(\gamma),$$

where $O_{\gamma,j}$ is the restriction of O_γ to $\mathfrak{H}_{Z,j}$:

$$(O_{\gamma,j}\psi)(\mathbf{x}_{n_1}, \dots, \mathbf{x}_{n_{N_j}}) = \prod_{n \in Z_j} e^{-i\varphi_\gamma \mathbf{n}_\gamma \cdot \mathbf{S}_n} \psi(\gamma^{-1}\mathbf{x}_{n_1}, \dots, \gamma^{-1}\mathbf{x}_{n_{N_j}}).$$

Given representations D_j and E_j, projector $P^{D_j} P^{E_j} = P^{E_j} P^{D_j}$ reduces $\widetilde{\mathcal{H}}_{Z,j}$. We denote the reduced operators in

$$\mathfrak{H}_{Z,j}^{D_j,E_j} := P^{D_j} P^{E_j} \mathfrak{H}_{Z,j}$$

by $\widetilde{\mathcal{H}}_{Z,j}^{D_j,E_j}$. Let

$$\varkappa_j(Z, D_j, E_j) := \inf \sigma(\widetilde{\mathcal{H}}_{Z,j}^{D_j,E_j}). \tag{2.18}$$

We write $(D_1, E_1; D_2, E_2) \underset{Z}{\prec} (D, E)$ if the corresponding term cannot be omitted on the r. h. s. of

$$\mathfrak{H}_N^{D,E} \subset \bigoplus_{\substack{(D_1,E_1) \\ (D_2,E_2)}} \left(\mathfrak{H}_{Z,1}^{D_1,E_1} \otimes \mathfrak{H}_{Z,2}^{D_2,E_2} \right)$$

without violation of the inclusion. For $Z_2 \neq \varnothing$ let

$$\varkappa(Z, D, E) := \begin{cases} \inf\left\{ \varkappa_1(Z, D_1, E_1) + \varkappa_2(Z, D_2, E_2) : (D_1, E_1; D_2, E_2) \underset{Z}{\prec} (D, E) \right\}, & Z_1 \neq \varnothing, \\ \varkappa_2(Z, D, E), & Z_1 = \varnothing. \end{cases} \tag{2.19}$$

The main result of the article is

THEOREM 6 *Suppose Assumptions 1, 2, 4, and 5 hold true. For $N \in \mathbb{N}$ let D be some irreducible representation of Γ, and E some irreducible representation of Π, such that $P^D P^E \neq 0$. Then*

$$\sigma_{\mathrm{ess}}(\mathcal{H}_N^{D,E}) = \left[\varkappa(D, E), \infty \right),$$

where

$$\varkappa(D, E) = \min\left\{ \varkappa(Z, D, E) : Z = (Z_1, Z_2),\ Z_2 \neq \varnothing \right\}. \tag{2.20}$$

Remark 7 *We only need Assumption 2 for the operators* $\widetilde{\mathcal{H}}_{Z,j}^{D_j,E_j}$ *which appear in* (2.18), (2.19).

3 Commutator Estimates

3.1 One Particle Commutator Estimate

Lemma 8 *Let* $\chi \in C_B^2(\mathbb{R}^3)$ *(i. e. a bounded twice–differentiable function with bounded derivatives). Then for* $m_n > 0$ *the commutator* $[\chi, \Lambda_n]$ *is a bounded operator from* $L_2(\mathbb{R}^3, \mathbb{C}^4)$ *to* $H^1(\mathbb{R}^3, \mathbb{C}^4)$*. There exists* $C(m) > 0$ *such that*

$$\big\|[\chi, \Lambda_n]\big\|_{L_2(\mathbb{R}^3,\mathbb{C}^4)\to H^1(\mathbb{R}^3,\mathbb{C}^4)} \leqslant C(m_n)\big(\|\nabla\chi\|_{L_\infty} + \|\partial^2\chi\|_{L_\infty}\big). \tag{3.1}$$

Here $\|\partial^2\chi\|_{L_\infty} = \max\limits_{\substack{\mathbf{z}\in\mathbb{R}^3\\ k,l\in\{1,2,3\}}} \big|\partial^2_{kl}\chi(\mathbf{z})\big|$.

Proof. In the coordinate representation for $f \in C_0^1(\mathbb{R}^3, \mathbb{C}^4)$ the operator Λ_n acts as

$$(\Lambda_n f)(\mathbf{x}) = \frac{f(\mathbf{x})}{2} + \frac{im_n}{2\pi^2}\lim_{\varepsilon\to+0}\int_{|\mathbf{y}-\mathbf{x}|\geqslant\varepsilon} \frac{\boldsymbol{\alpha}\cdot(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3} K_1\big(m_n|\mathbf{x}-\mathbf{y}|\big) f(\mathbf{y}) d\mathbf{y}$$
$$+ \frac{m_n^2}{4\pi^2}\int_{\mathbb{R}^3}\left(\beta\frac{K_1\big(m_n|\mathbf{x}-\mathbf{y}|\big)}{|\mathbf{x}-\mathbf{y}|} + \frac{i\boldsymbol{\alpha}\cdot(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2}K_0\big(m_n|\mathbf{x}-\mathbf{y}|\big)\right) f(\mathbf{y}) d\mathbf{y},$$

where the limit on the r. h. s. is the limit in $L_2(\mathbb{R}^3, \mathbb{C}^4)$ (see Appendix B of [16], where this formula is derived in the case $m_n = 1$). The rest of the proof is an obvious modification of the proof of Lemma 1 of [16], where the case $m_n = 1$ is considered. •

Remark 9 *Since we only deal with a finite number of particles with positive masses, we will not trace the m-dependence of the constant in* (3.1) *any longer.*

3.2 Multiparticle Commutator Estimate

Lemma 10 *For any* $d, k \in \mathbb{N}$ *there exists* $C > 0$ *such that for any* $\chi \in C_B^1(\mathbb{R}^d)$ *and* $u \in H^{1/2}(\mathbb{R}^d, \mathbb{C}^k)$

$$\|\chi u\|_{H^{1/2}(\mathbb{R}^d,\mathbb{C}^k)} \leqslant C\big(\|\chi\|_{L_\infty(\mathbb{R}^d)} + \|\nabla\chi\|_{L_\infty(\mathbb{R}^d)}\big)\|u\|_{H^{1/2}(\mathbb{R}^d,\mathbb{C}^k)}. \tag{3.2}$$

Proof of Lemma 10. We can choose the norm in $H^{1/2}(\mathbb{R}^d, \mathbb{C}^k)$ as (see [1], Theorem 7.48)

$$\|u\|^2_{H^{1/2}(\mathbb{R}^d,\mathbb{C}^k)} := \|u\|^2_{L_2(\mathbb{R}^d,\mathbb{C}^k)} + \iint \frac{\big|u(\mathbf{x}) - u(\mathbf{y})\big|^2}{|\mathbf{x}-\mathbf{y}|^{d+1}} d\mathbf{x}d\mathbf{y}.$$

Then

$$\|\chi u\|^2_{H^{1/2}(\mathbb{R}^d,\mathbb{C}^k)} = \|\chi u\|^2_{L_2(\mathbb{R}^d,\mathbb{C}^k)} + \iint \frac{\big|\chi(\mathbf{x})u(\mathbf{x}) - \chi(\mathbf{y})u(\mathbf{y})\big|^2}{|\mathbf{x}-\mathbf{y}|^{d+1}} d\mathbf{x}d\mathbf{y}$$
$$\leqslant \|\chi\|^2_{L_\infty}\|u\|^2_{L_2} + \iint \Big(\frac{\big|\chi(\mathbf{x})\big|^2\big|u(\mathbf{x}) - u(\mathbf{y})\big|^2}{|\mathbf{x}-\mathbf{y}|^{d+1}} + \frac{\big|\chi(\mathbf{x}) - \chi(\mathbf{y})\big|^2\big|u(\mathbf{y})\big|^2}{|\mathbf{x}-\mathbf{y}|^{d+1}} \Big) d\mathbf{x}d\mathbf{y}$$
$$\leqslant \|\chi\|^2_{L_\infty}\|u\|^2_{H^{1/2}} + \sup_{\mathbf{y}\in\mathbb{R}^d} \int \frac{\big|\chi(\mathbf{x}) - \chi(\mathbf{y})\big|^2}{|\mathbf{x}-\mathbf{y}|^{d+1}} d\mathbf{x} \|u\|^2_{L_2}. \tag{3.3}$$

The supremum on the r. h. s. of (3.3) can be estimated as

$$\begin{aligned} \sup_{\mathbf{y}\in\mathbb{R}^d} \int \frac{\big|\chi(\mathbf{x}) - \chi(\mathbf{y})\big|^2}{|\mathbf{x}-\mathbf{y}|^{d+1}} d\mathbf{x} &\leqslant \sup_{\mathbf{y}\in\mathbb{R}^d} \int_{|\mathbf{x}-\mathbf{y}|\leqslant 1} \frac{\big|\chi(\mathbf{x}) - \chi(\mathbf{y})\big|^2}{|\mathbf{x}-\mathbf{y}|^{d+1}} d\mathbf{x} \\ + \sup_{\mathbf{y}\in\mathbb{R}^d} \int_{|\mathbf{x}-\mathbf{y}|>1} \frac{\big|\chi(\mathbf{x}) - \chi(\mathbf{y})\big|^2}{|\mathbf{x}-\mathbf{y}|^{d+1}} d\mathbf{x} &\leqslant |\mathbb{S}^{d-1}|\big(\|\nabla\chi\|^2_{L_\infty} + 4\|\chi\|^2_{L_\infty}\big), \end{aligned} \tag{3.4}$$

where $|\mathbb{S}^{d-1}|$ is the area of $(d-1)$–dimensional unit sphere. Substituting (3.4) into (3.3) we obtain (3.2). •

LEMMA 11 *For any $\chi \in C^2_B(\mathbb{R}^{3N})$ the operator $[\chi, \Lambda^N]$ is bounded in $H^{1/2}(\mathbb{R}^{3N}, \mathbb{C}^{4^N})$, and for any $\psi \in H^{1/2}(\mathbb{R}^{3N}, \mathbb{C}^{4^N})$ we have*

$$\big\|[\chi, \Lambda^N]\psi\big\|_{H^{1/2}} \leqslant C\big(\|\nabla\chi\|_{L_\infty} + \|\partial^2\chi\|_{L_\infty}\big)\big(\|\chi\|_{L_\infty} + \|\nabla\chi\|_{L_\infty}\big)\|\psi\|_{H^{1/2}} \tag{3.5}$$

with C depending only on N and the masses of the particles.

PROOF. Successively commuting χ with Λ_n, $n = 1, \ldots, N$ (see (2.3)) we obtain

$$[\chi, \Lambda^N] = \sum_{n=1}^{N} \prod_{k=1}^{n-1} \Lambda_k [\chi, \Lambda_n] \prod_{l=n+1}^{N} \Lambda_l, \tag{3.6}$$

where the empty products should be replaced by identity operators. By (2.1) the operators Λ_n are bounded in $H^{1/2}$ for any $n = 1, \ldots, N$. This, together with (3.6) and Lemmata 8 and 10, implies (3.5). •

4 LOWER BOUND OF THE ESSENTIAL SPECTRUM

In this section we prove that

$$\inf \sigma_{\mathrm{ess}}(\mathcal{H}_N^{D,E}) \geqslant \varkappa(D, E). \tag{4.1}$$

4.1 Partition of Unity

Lemma 12 *There exists a set of nonnegative functions $\{\chi_Z\}$ indexed by possible 2–cluster decompositions $Z = (Z_1, Z_2)$ satisfying*

1. $\chi_Z \in C^\infty(\mathbb{R}^{3N})$ *for all* Z;
2. $\chi_Z(\kappa\mathbf{X}) = \chi_Z(\mathbf{X})$ *for all* $|\mathbf{X}| = 1,\ \kappa > 1,\ Z_2 \neq \varnothing$;
3. $\sum_Z \chi_Z^2(\mathbf{X}) = 1,$ *for all* $\mathbf{X} \in \mathbb{R}^{3N}$; (4.2)
4. *There exists* $C > 0$ *such that for any* $\mathbf{X} \in \operatorname{supp}\chi_Z$ $\min\big\{|\mathbf{x}_j - \mathbf{x}_n| : \mathbf{x}_j \in Z_1,\ \mathbf{x}_n \in Z_2;\ |\mathbf{x}_n| : \mathbf{x}_n \in Z_2\big\} \geqslant C|\mathbf{X}|$; (4.3)
5. $\chi_Z(\gamma\mathbf{x}_1, \ldots, \gamma\mathbf{x}_N) = \chi_Z(\mathbf{x}_1, \ldots, \mathbf{x}_N)$ *for any orthogonal transformation* γ;
6. χ_Z *is invariant under permutations of variables preserving* $Z_{1,2}$.

Proof. The proof is essentially based on the modification of the argument given in [17], Lemma 2.4.

1. We first prove that for any $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_N) \in \mathbb{R}^{3N}$ with $|\mathbf{X}| = 1$ there exists a 2–cluster decomposition $Z = (Z_1, Z_2)$ such that

$$\min\big\{|\mathbf{x}_j - \mathbf{x}_n| : \mathbf{x}_j \in Z_1,\ \mathbf{x}_n \in Z_2;\ |\mathbf{x}_n| : \mathbf{x}_n \in Z_2\big\} > N^{-3/2}. \tag{4.4}$$

Indeed, let k be such that $|\mathbf{x}_k| \geqslant |\mathbf{x}_j|$ for all $j = 1, \ldots, N$. Then, since $|\mathbf{X}| = 1$,

$$|\mathbf{x}_k| \geqslant N^{-\frac{1}{2}}. \tag{4.5}$$

Choose Cartesian coordinates in $\mathbb{R}^3$ with the first axis passing through the origin and $\mathbf{x}_k$, so that $\mathbf{x}_k = \big(|\mathbf{x}_k|, 0, 0\big)$. Consider N regions

$$R_1 := \big\{\mathbf{x} \in \mathbb{R}^3 : x^1 \leqslant |\mathbf{x}_k|/N\big\},$$
$$R_l := \Big\{\mathbf{x} \in \mathbb{R}^3 : x^1 \in \big((l-1)|\mathbf{x}_k|/N,\ l|\mathbf{x}_k|/N\big]\Big\}, \quad l = 2, \ldots, N.$$

At least one of these regions does not contain $\mathbf{x}_j$ with $j \neq k$. Let l_0 be the maximal index of such regions. Let Z_2 be the set of indices n such that $\mathbf{x}_n \in \bigcup_{l>l_0} R_l$. Z_2 is nonempty since $\mathbf{x}_k \in Z_2$. Setting $Z_1 := I \setminus Z_2$ we observe that

$$\min\big\{|\mathbf{x}_j - \mathbf{x}_n| : \mathbf{x}_j \in Z_1,\ \mathbf{x}_n \in Z_2;\ |\mathbf{x}_n| : \mathbf{x}_n \in Z_2\big\} > |\mathbf{x}_k|/N,$$

which together with (4.5) implies (4.4).

2. Choose $\eta \in C^\infty\big(\mathbb{R}_+, [0,1]\big)$ so that

$$\eta(t) \equiv \begin{cases} 0, & t \in [0,1], \\ 1, & t \in [2,\infty). \end{cases}$$

Let

$$\zeta_Z(\mathbf{X}) := \begin{cases} \eta\big(2|\mathbf{X}|\big) \prod\limits_{n\in Z_2} \eta\Big(\dfrac{2|\mathbf{x}_n|}{|\mathbf{X}|N^{-3/2}}\Big) \prod\limits_{j\in Z_1} \eta\Big(\dfrac{2|\mathbf{x}_j-\mathbf{x}_n|}{|\mathbf{X}|N^{-3/2}}\Big), & Z_2 \neq \varnothing, \\ 1-\eta\big(2|\mathbf{X}|\big), & Z_2 = \varnothing. \end{cases} \tag{4.6}$$

Functions (4.6) satisfy conditions 1, 2, 4 (with $C = N^{-3/2}/2$), 5, and 6 of Lemma 12. Moreover, by the first part of the proof

$$\sum_Z \zeta_Z(\mathbf{X}) \geqslant 1, \quad \text{for all} \quad \mathbf{X} \in \mathbb{R}^{3N}.$$

Hence all the conditions are satisfied by the functions

$$\chi_Z := \zeta_Z^{1/2}\Big(\sum_Z \zeta_Z\Big)^{-1/2}.$$

•

Let

$$\chi_Z^R(\mathbf{X}) := \chi_Z(\mathbf{X}/R), \tag{4.7}$$

where the functions χ_Z are defined in Lemma 12. The derivatives of χ_Z^R decay as R tends to infinity:

$$\|\nabla\chi_Z^R\|_\infty \leqslant CR^{-1}, \quad \|\partial^2\chi_Z^R\|_\infty \leqslant CR^{-2}. \tag{4.8}$$

To simplify the notation we omit the superscript R further on.

4.2 Cluster Decomposition and Lower Bound

We now estimate from below the quadratic form of $\mathcal{H}_N^{D,E}$ on a function ψ from $\mathfrak{H}_N^{D,E} \cap \Lambda^N \bigotimes\limits_{n=1}^{N} H^{1/2}(\mathbb{R}^3,\mathbb{C}^4)$, which is the form domain of $\mathcal{H}_N^{D,E}$.

$$\begin{aligned} \langle\mathcal{H}_N^{D,E}\psi,\psi\rangle &= \langle\Big(\sum_{n=1}^N (D_n+V_n) + \sum_{n<j}^N U_{nj}\Big)\sum_Z \chi_Z^2\psi,\psi\rangle \\ &= \sum_Z \langle\Big(\sum_{n=1}^N (D_n+V_n) + \sum_{n<j}^N U_{nj}\Big)\chi_Z\psi,\chi_Z\psi\rangle. \end{aligned}$$

Here we have used (4.2) and the relation

$$\sum_Z \langle f, \sum_{n=1}^N \nabla_n(\chi_Z^2 g)\rangle = \sum_Z \langle\chi_Z f, \sum_{n=1}^N \nabla_n(\chi_Z g)\rangle + \sum_Z \langle f, \sum_{n=1}^N \nabla_n\Big(\frac{\chi_Z^2}{2}\Big)g\rangle \tag{4.9}$$

which holds for any $f, g \in \bigotimes_{n=1}^{N} H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$. The last term on the r. h. s. of (4.9) is equal to zero due to (4.2). Thus

$$\begin{aligned}\langle \mathcal{H}_N^{D,E}\psi, \psi\rangle = \sum_{Z=(Z_1,Z_2)} \Big(&\langle (\mathcal{H}_{Z,1} + \mathcal{H}_{Z,2})\Lambda^N \chi_Z\psi, \Lambda^N\chi_Z\psi\rangle \\ &+ \langle (\mathcal{H}_{Z,1} + \mathcal{H}_{Z,2})[\chi_Z, \Lambda^N]\psi, \Lambda^N\chi_Z\psi\rangle \\ &+ \langle (\mathcal{H}_{Z,1} + \mathcal{H}_{Z,2})\chi_Z\psi, [\chi_Z, \Lambda^N]\psi\rangle \\ &+ \langle \sum_{n\in Z_2} V_n\chi_Z^2\psi, \psi\rangle + \langle \sum_{\substack{n<j \\ n\# j}} U_{nj}\chi_Z^2\psi, \psi\rangle\Big).\end{aligned} \tag{4.10}$$

By (2.12), (2.13), (4.3), (4.7), and (2.14) the terms at the last line of (4.10) can be estimated as

$$\langle \sum_{n\in Z_2} V_n\chi_Z^2\psi, \psi\rangle + \langle \sum_{\substack{n<j \\ n\# j}} U_{nj}\chi_Z^2\psi, \psi\rangle \geqslant -\varepsilon_1(R)\big(\langle \mathcal{H}_N^{D,E}\psi, \psi\rangle + \|\psi\|^2\big) \tag{4.11}$$

with $\varepsilon_1(R) \to 0$ as $R \to \infty$. The terms at the second and third lines of (4.10) can also be estimated as

$$\begin{aligned}&\langle (\mathcal{H}_{Z,1} + \mathcal{H}_{Z,2})[\chi_Z, \Lambda^N]\psi, \Lambda^N\chi_Z\psi\rangle + \langle (\mathcal{H}_{Z,1} + \mathcal{H}_{Z,2})\chi_Z\psi, [\chi_Z, \Lambda^N]\psi\rangle \\ &\geqslant -\varepsilon_2(R)\big(\langle \mathcal{H}_N^{D,E}\psi, \psi\rangle + \|\psi\|^2\big), \quad \varepsilon_2(R) \underset{R\to\infty}{\longrightarrow} 0,\end{aligned}$$

according to (2.8), (3.2), (3.5), (4.8), and (2.14). For $Z_2 \neq \varnothing$ we estimate the terms at the first line of (4.10) in the following way (recall the definitions (2.18), (2.19) and (2.20)):

$$\begin{aligned}&\langle (\mathcal{H}_{Z,1} + \mathcal{H}_{Z,2})\Lambda^N\chi_Z\psi, \Lambda^N\chi_Z\psi\rangle \\ &= \sum_{(D_1,E_1;D_2,E_2)\underset{Z}{\prec}(D,E)} \langle (\mathcal{H}_{Z,1}P^{D_1}P^{E_1} + \mathcal{H}_{Z,2}P^{D_2}P^{E_2})\Lambda^N\chi_Z\psi, \Lambda^N\chi_Z\psi\rangle \\ &\geqslant \sum_{(D_1,E_1;D_2,E_2)\underset{Z}{\prec}(D,E)} \langle \big(\varkappa_1(Z, D_1, E_1)P^{D_1}P^{E_1} \\ &+ \varkappa_2(Z, D_2, E_2)P^{D_2}P^{E_2}\big)\Lambda^N\chi_Z\psi, \Lambda^N\chi_Z\psi\rangle \\ &\geqslant \varkappa(D, E)\langle \Lambda^N\chi_Z\psi, \Lambda^N\chi_Z\psi\rangle \\ &= \varkappa(D, E)\langle \chi_Z^2\psi, \psi\rangle + \varkappa(D, E)\langle [\Lambda^N, \chi_Z]\psi, \chi_Z\psi\rangle.\end{aligned} \tag{4.12}$$

By (3.2), (3.5), (4.8), and (2.14) the last term on the r. h. s. of (4.12) can be estimated as

$$\varkappa(D, E)\langle [\Lambda^N, \chi_Z]\psi, \chi_Z\psi\rangle \geqslant -\varepsilon_3(R)\big(\langle \mathcal{H}_N^{D,E}\psi, \psi\rangle + \|\psi\|^2\big), \quad \varepsilon_3(R) \underset{R\to\infty}{\longrightarrow} 0. \tag{4.13}$$

Substituting the estimates (4.11) — (4.13) into (4.10) we obtain

$$\langle \mathcal{H}_N^{D,E}\psi,\psi\rangle \geqslant \varkappa(D,E)\langle \sum_{\substack{Z=(Z_1,Z_2)\\ Z_2\neq\varnothing}} \chi_Z^2\psi,\psi\rangle + \langle \mathcal{H}_N^{D,E}\Lambda^N\chi_{(I,\varnothing)}\psi,\Lambda^N\chi_{(I,\varnothing)}\psi\rangle$$
$$-\varepsilon_4(R)\big(\langle \mathcal{H}_N^{D,E}\psi,\psi\rangle + \|\psi\|^2\big),\quad \varepsilon_4(R) \underset{R\to\infty}{\longrightarrow} 0. \tag{4.14}$$

4.3 Estimate Inside of the Compact Region

It remains to estimate from below the quadratic form of $\mathcal{H}_N^{D,E}$ on $\Lambda^N\chi_{(I,\varnothing)}\psi$. Note that according to Lemma 12 and (4.7) $\operatorname{supp}\chi_{(I,\varnothing)} \subset [-2R,2R]^{3N}$. To simplify the notation let

$$\chi_0 := \chi_{(I,\varnothing)}.$$

Lemma 13 *For $M>0$ let*

$$W_M := \big\{\mathbf{p}\in\mathbb{R}^{3N} : |p_i|\leqslant M, i=1,\dots,3N\big\},\ \widetilde{W}_M := \mathbb{R}^{3N}\setminus W_M.$$

There exists a finite set $Q_M\subset L_2(\mathbb{R}^{3N})$ such that for any $f\in L_2(\mathbb{R}^{3N})$ with $\operatorname{supp} f\subset[-2R,2R]^{3N}$, $f\perp Q_M$ *holds*

$$\|\hat f\|_{L_2(\widetilde{W}_M)} \geqslant \frac12\|\hat f\|_{L_2(\mathbb{R}^{3N})}.$$

The proof of Lemma 13 is analogous to the proof of Theorem 7 of [23] and is given in Appendix C of [16].

It follows from (2.9) that for any $M>0$

$$\langle \mathcal{H}_N^{D,E}\Lambda^N\chi_0\psi,\Lambda^N\chi_0\psi\rangle \geqslant C_1\langle\sum_{n=1}^N D_n I_{\widetilde{W}_M}\Lambda^N\chi_0\psi,\Lambda^N\chi_0\psi\rangle - C_2\|\chi_0\psi\|^2. \tag{4.15}$$

Here $I_{\widetilde{W}_M}$ is the operator of multiplication by the characteristic function of $\widetilde{W}_M$ in momentum space.
We choose

$$M := 8\big(\varkappa(D,E)+C_2\big)C_1^{-1} \tag{4.16}$$

and assume henceforth that $f:=\chi_0\psi$ is orthogonal to the set Q_M defined in Lemma 13. Since in momentum space the operator D_n acts on functions from $\Lambda_n L_2(\mathbb{R}^3,\mathbb{C}^4)$ as multiplication by $\sqrt{|\mathbf{p}|^2+m_n^2}$, by construction of $\widetilde{W}_M$ we have

$$\langle\sum_{n=1}^N D_n I_{\widetilde{W}_M}\Lambda^N\chi_0\psi,\Lambda^N\chi_0\psi\rangle \geqslant M\|I_{\widetilde{W}_M}\Lambda^N\chi_0\psi\|^2. \tag{4.17}$$

Inequalities (4.15) and (4.17) imply

$$
\begin{aligned}
&\langle \mathcal{H}_N^{D,E}\Lambda^N\chi_0\psi, \Lambda^N\chi_0\psi\rangle \geqslant C_1M\|I_{\widetilde{W}_M}\Lambda^N\chi_0\psi\|^2 - C_2\|\chi_0\psi\|^2\\
&\geqslant C_1M\Big(\|I_{\widetilde{W}_M}\chi_0\psi\| - \big\|I_{\widetilde{W}_M}[\Lambda^N,\chi_0]\psi\big\|\Big)^2 - C_2\|\chi_0\psi\|^2\\
&\geqslant C_1M\Big(\frac{1}{2}\|I_{\widetilde{W}_M}\chi_0\psi\|^2 - \big\|I_{\widetilde{W}_M}[\Lambda^N,\chi_0]\psi\big\|^2\Big) - C_2\|\chi_0\psi\|^2\\
&\geqslant 4\big(\varkappa(D,E)+C_2\big)\|I_{\widetilde{W}_M}\chi_0\psi\|^2\\
&-8\big(\varkappa(D,E)+C_2\big)\big\|[\Lambda^N,\chi_0]\psi\big\|^2 - C_2\|\chi_0\psi\|^2.
\end{aligned}
\tag{4.18}
$$

At the last step we have used (4.16). The second term on the r. h. s. of (4.18) can be estimated analogously to (4.13) as

$$
-8\big(\varkappa(D,E)+C_2\big)\big\|[\Lambda^N,\chi_0]\psi\big\|^2 \geqslant -\varepsilon_5(R)\big(\langle \mathcal{H}_N^{D,E}\psi,\psi\rangle + \|\psi\|^2\big),\ \varepsilon_5(R) \underset{R\to\infty}{\longrightarrow} 0.
$$

For the first term on the r. h. s. of (4.18) Lemma 13 implies

$$
4\|I_{\widetilde{W}_M}\chi_0\psi\|^2 \geqslant \|\chi_0\psi\|^2. \tag{4.19}
$$

As a consequence of (4.18) — (4.19), we have

$$
\begin{gathered}
\langle \mathcal{H}_N^{D,E}\Lambda^N\chi_0\psi, \Lambda^N\chi_0\psi\rangle \geqslant \varkappa(D,E)\|\chi_0\psi\|^2 - \varepsilon_5(R)\big(\langle \mathcal{H}_N^{D,E}\psi,\psi\rangle + \|\psi\|^2\big),\\
\varepsilon_5(R) \underset{R\to\infty}{\longrightarrow} 0.
\end{gathered}
\tag{4.20}
$$

4.4 Completion of the Proof

By (4.14), (4.20), and (4.2)

$$
\langle \mathcal{H}_N^{D,E}\psi,\psi\rangle \geqslant \varkappa(D,E)\|\psi\|^2 - \varepsilon_6(R)\big(\langle \mathcal{H}_N^{D,E}\psi,\psi\rangle + \|\psi\|^2\big), \quad \varepsilon_6(R) \underset{R\to\infty}{\longrightarrow} 0.
$$

for any ψ in the form domain of $\mathcal{H}_N^{D,E}$ orthogonal to the finite set of functions (cardinality of this set depends on R). This implies the discreteness of the spectrum of $\mathcal{H}_N^{D,E}$ below $\varkappa(D,E)$ and thus (4.1).

5 Spectrum of the Free Cluster

In this section we characterize the spectrum of the cluster Z_2 which does not interact with the external field.

Proposition 14 *For any irreducible representations D_2, E_2 of rotation–reflection and permutation groups the spectrum of $\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2}$ is*

$$
\sigma(\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2}) = \sigma_{\text{ess}}(\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2}) = \big[\varkappa_2(Z,D_2,E_2),\infty\big),
$$

with some $\varkappa_2(Z,D_2,E_2)\in\mathbb{R}$.

Proof. Let us introduce the new coordinates in the configuration space $\mathbb{R}^{3N_2}$ of the cluster $Z_2 = \{n_1, \ldots, n_{N_2}\}$, in the same manner as it is done in [15]. Let $M := \sum_{n\in Z_2} m_n$ be the total mass of the particles constituting the cluster. We introduce

$$\begin{aligned} \mathbf{y}_0 &:= \frac{1}{M}\sum_{n\in Z_2} m_n \mathbf{x}_n, \\ \mathbf{y}_k &:= \mathbf{x}_{n_{k+1}} - \mathbf{x}_{n_1}, \quad k = 1, \ldots, N_2 - 1. \end{aligned} \tag{5.1}$$

The Jacobian of this variable change is one. Here $\mathbf{y}_0$ is the coordinate of the center of mass, whereby $\mathbf{y}_k$, $k = 1, \ldots, N_2 - 1$ are the internal coordinates of the cluster. Accordingly,

$$\begin{aligned} \mathbf{x}_{n_1} &= \mathbf{y}_0 - \frac{1}{M}\sum_{k=1}^{N_2-1} m_{n_{k+1}} \mathbf{y}_k, \\ \mathbf{x}_{n_{l+1}} &= \mathbf{y}_0 + \mathbf{y}_l - \frac{1}{M}\sum_{k=1}^{N_2-1} m_{n_{k+1}} \mathbf{y}_k, \quad l = 1, \ldots, N_2 - 1. \end{aligned} \tag{5.2}$$

The momentum operators in the new coordinates are

$$\begin{aligned} \mathbf{p}_{n_1} &:= -i\nabla_{\mathbf{x}_{n_1}} = \frac{m_{n_1}}{M}\mathbf{P} - \sum_{k=1}^{N_2-1}(-i\nabla_{\mathbf{y}_k}), \\ \mathbf{p}_{n_k} &:= -i\nabla_{\mathbf{x}_{n_k}} = \frac{m_{n_k}}{M}\mathbf{P} + (-i\nabla_{\mathbf{y}_{k-1}}), \quad k = 2, \ldots, N_2, \end{aligned} \tag{5.3}$$

where $\mathbf{P}$ is the total momentum of the cluster:

$$\mathbf{P} := \sum_{n\in Z_2} -i\nabla_{\mathbf{x}_n} = -i\nabla_{\mathbf{y}_0}.$$

Let $\mathcal{F}_0$ be the partial Forurier transform on $\mathfrak{H}^{D_2,E_2}_{Z,2}$ defined by

$$(\mathcal{F}_0 f)(\mathbf{P}, \mathbf{y}_1, \ldots, \mathbf{y}_{N_2-1}) := \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} f(\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_{N_2-1}) e^{-i\mathbf{P}\mathbf{y}_0} d\mathbf{y}_0.$$

By (2.6) and (2.7) we have

$$\widetilde{\mathcal{H}}^{D_2,E_2}_{Z,2} = \mathcal{F}_0^{-1}\widehat{\Lambda}_{Z,2}\widehat{\mathcal{H}}^{D_2,E_2}_{Z,2}\widehat{\Lambda}_{Z,2}\mathcal{F}_0,$$

where in the new coordinates

$$\widehat{\mathcal{H}}^{D_2,E_2}_{Z,2} := \sum_{n\in Z_2}(\boldsymbol{\alpha}_n\cdot\mathbf{p}_n + \beta_n m_n) + \sum_{k=2}^{N_2-1} U_{n_1 n_k}(\mathbf{y}_k) + \sum_{1<k<l\leqslant N_2-1} U_{n_k n_l}(\mathbf{y}_k - \mathbf{y}_l), \tag{5.4}$$

$$\widehat{\Lambda}_{Z,2} := \prod_{n\in Z_2} \widehat{\Lambda}_n, \tag{5.5}$$

$$\widehat{\Lambda}_n := \frac{1}{2} + \frac{\boldsymbol{\alpha}_n \cdot \mathbf{p}_n + \beta_n m_n}{2\sqrt{\mathbf{p}_n^2 + m_n^2}},$$

operators $\mathbf{p}_n$ are given by (5.3), and $\mathbf{P}$ should now be interpreted as multiplication by the vector–function. The operators (5.4) and (5.5) obviously commute with $\mathfrak{P} := |\mathbf{P}|$. The operator $\mathcal{F}_0^{-1}\mathfrak{P}\mathcal{F}_0$ (unlike $\mathcal{F}_0^{-1}\mathbf{P}\mathcal{F}_0$) is well–defined in $\mathfrak{H}_{Z,2}^{D_2,E_2}$, since it commutes with P^{D_2} an P^{E_2} in $\mathfrak{H}_{Z,2}$. This implies that $\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2}$ commutes with $\mathcal{F}_0^{-1}\mathfrak{P}\mathcal{F}_0$.
Let $\omega := \mathbf{P}/\mathfrak{P} \in S^2$. We decompose the Hilbert space $\mathfrak{H}_{Z,2}^{D_2,E_2}$ into the direct integral

$$\mathfrak{H}_{Z,2}^{D_2,E_2} = \int_0^\infty \oplus \mathfrak{H}_{Z,2}^{D_2,E_2,\mathfrak{P}} \mathfrak{P}^2 d\mathfrak{P}. \tag{5.6}$$

The fibre space $\mathfrak{H}_{Z,2}^{D_2,E_2,\mathfrak{P}}$ can be considered as a subspace of $L_2(\mathbb{R}^{3N_2-3} \times S^2, \mathbb{C}^{4^{N_2}})$ with the inner product

$$\langle f, g\rangle_* := \int_{\mathbb{R}^{3(N_2-1)}\times S^2} \langle f, g\rangle_{\mathbb{C}^{4^{N_2}}} d\mathbf{y}_1 \cdots d\mathbf{y}_{N_2-1} d\omega.$$

For $f \in \mathfrak{H}_{Z,2}^{D_2,E_2}$ the corresponding element of $\mathfrak{H}_{Z,2}^{D_2,E_2,\mathfrak{P}}$ is given by

$$f_{\mathfrak{P}} := \mathcal{F}_0 f|_{|\mathbf{P}|=\mathfrak{P}}.$$

We have

$$\|f\|^2 = \int_0^\infty \|f_{\mathfrak{P}}\|_*^2 \mathfrak{P}^2 d\mathfrak{P} \tag{5.7}$$

in compliance with (5.6). The form domain of $\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2,\mathfrak{P}}$ is

$$\mathfrak{D}^{\mathfrak{P}} := \Lambda_{Z,2}^{\mathfrak{P}} P^{D_2} P^{E_2} H^{1/2}(\mathbb{R}^{3(N_2-1)} \times S^2, \mathbb{C}^{4^{N_2}}),$$

where $\Lambda_{Z,2}^{\mathfrak{P}}$ is given by (5.5) with the only difference that we should replace $\mathbf{P}$ by $\omega\mathfrak{P}$ in (5.3). The operators on fibres of the direct integral (5.6) are

$$\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2,\mathfrak{P}} := \Lambda_{Z,2}^{\mathfrak{P}} \mathcal{H}_{Z,2}^{D_2,E_2,\mathfrak{P}} \Lambda_{Z,2}^{\mathfrak{P}},$$

where $\mathcal{H}_{Z,2}^{D_2,E_2,\mathfrak{P}}$ is given by the r. h. s. of (5.4) with $\mathbf{P}$ replaced by $\omega\mathfrak{P}$ in (5.3). We thus have

$$\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2} = \int_0^\infty \oplus \widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2,\mathfrak{P}} \mathfrak{P}^2 d\mathfrak{P}. \tag{5.8}$$

The spectrum of $\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2}$ can be represented as

$$\sigma(\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2}) = \overline{\mathrm{ess} \bigcup_{\mathfrak{P}\in\mathbb{R}_+} \sigma(\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2,\mathfrak{P}})}, \tag{5.9}$$

where the essential union is taken with respect to the Lebesgue measure in $\mathbb{R}_+$. The bottom of the spectrum of $\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2,\mathfrak{P}}$ is given by

$$\mu(\mathfrak{P}) := \inf_{\psi\in\mathfrak{D}^{\mathfrak{P}}} \frac{\langle \widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2,\mathfrak{P}}\psi,\psi\rangle_*}{\|\psi\|_*^2}. \tag{5.10}$$

Lemma 15 *Function* (5.10) *is continuous on* $\mathbb{R}_+$.

Proof of Lemma 15. Let us fix $\mathfrak{P}\in\mathbb{R}_+$ and $\varepsilon>0$. We will prove that $\big|\mu(\mathfrak{P}+\mathfrak{p})-\mu(\mathfrak{P})\big|<\varepsilon$ if $|\mathfrak{p}|$ is small enough. Choose $\psi\in\mathfrak{D}^{\mathfrak{P}}$ such that

$$\left|\frac{\langle \widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2,\mathfrak{P}}\psi,\psi\rangle_*}{\|\psi\|_*^2}-\mu(\mathfrak{P})\right|\leqslant\frac{\varepsilon}{2}. \tag{5.11}$$

Let

$$\phi := \Lambda_{Z,2}^{\mathfrak{P}+\mathfrak{p}}\psi\in\mathfrak{D}^{\mathfrak{P}+\mathfrak{p}}.$$

We have

$$\phi-\psi=(\Lambda_{Z,2}^{\mathfrak{P}+\mathfrak{p}}-\Lambda_{Z,2}^{\mathfrak{P}})\psi=\sum_{k=1}^{N_2}\prod_{i<k}\Lambda_{n_i}^{\mathfrak{P}+\mathfrak{p}}(\Lambda_{n_k}^{\mathfrak{P}+\mathfrak{p}}-\Lambda_{n_k}^{\mathfrak{P}})\prod_{j>k}\Lambda_{n_j}^{\mathfrak{P}}\psi. \tag{5.12}$$

Let $\mathcal{F}$ be the unitary Fourier transform in $L_2(\mathbb{R}^{3(N_2-1)}\times S^2,\mathbb{C}^{4^{N_2}})$ defined by

$$\begin{aligned}&(\mathcal{F}\xi)(\omega,\mathbf{q}_1,\dots,\mathbf{q}_{N_2-1})\\ &:=(2\pi)^{3(1-N_2)/2}\int\limits_{\mathbb{R}^{3(N_2-1)}}\xi(\omega,\mathbf{y}_1,\dots,\mathbf{y}_{N_2-1})e^{-i\sum\limits_{k=1}^{N_2-1}\mathbf{q}_k\cdot\mathbf{y}_k}d\mathbf{y}_1\cdots d\mathbf{y}_{N_2-1}.\end{aligned}$$

We can rewrite (5.12) as

$$\phi-\psi=\mathcal{F}^{-1}\sum_{k=1}^{N_2}\prod_{i<k}\widehat{\Lambda}_{n_i}^{\mathfrak{P}+\mathfrak{p}}(\widehat{\Lambda}_{n_k}^{\mathfrak{P}+\mathfrak{p}}-\widehat{\Lambda}_{n_k}^{\mathfrak{P}})\prod_{j>k}\widehat{\Lambda}_{n_j}^{\mathfrak{P}}\mathcal{F}\psi, \tag{5.13}$$

where $\widehat{\Lambda}_n^{\mathfrak{P}}$, $n\in Z_2$ are the operators of multiplication by the symbols

$$\widehat{\Lambda}_n^{\mathfrak{P}} := \frac{1}{2}+\frac{\boldsymbol{\alpha}_n\cdot\widehat{\mathbf{p}}_n+\beta_n m_n}{2\sqrt{\widehat{\mathbf{p}}_n^2+m_n^2}}, \tag{5.14}$$

$$\begin{aligned}\widehat{\mathbf{p}}_{n_1} &:= \frac{m_{n_1}}{M}\omega\mathfrak{P}-\sum_{k=1}^{N_2-1}\mathbf{q}_k,\\ \widehat{\mathbf{p}}_{n_k} &:= \frac{m_{n_k}}{M}\omega\mathfrak{P}+\mathbf{q}_{k-1},\quad k=2,\dots,N_2.\end{aligned} \tag{5.15}$$

The matrix–functions (5.14) are uniformly continuous in $\mathfrak{P}$. Thus by (5.13)

$$\|\phi-\psi\|_{H^{1/2}(\mathbb{R}^{3(N_2-1)}\times S^2,\mathbb{C}^{4^{N_2}})} \leqslant C\sum_{k=1}^{N_2}\|\widehat{\Lambda}_{n_k}^{\mathfrak{P}+\mathfrak{p}}-\widehat{\Lambda}_{n_k}^{\mathfrak{P}}\|_{L_\infty}\|\psi\|_{H^{1/2}} \underset{|\mathfrak{p}|\to 0}{\longrightarrow} 0. \quad (5.16)$$

We write

$$\begin{aligned}\langle\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2,\mathfrak{P}+\mathfrak{p}}\phi,\phi\rangle_* &= \langle\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2,\mathfrak{P}}\psi,\psi\rangle_* + \langle\mathcal{H}_{Z,2}^{D_2,E_2,\mathfrak{P}}(\phi-\psi),\psi\rangle_* \\ &+ \langle\mathcal{H}_{Z,2}^{D_2,E_2,\mathfrak{P}}\phi,(\phi-\psi)\rangle_* + \langle(\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2,\mathfrak{P}+\mathfrak{p}} - \mathcal{H}_{Z,2}^{D_2,E_2,\mathfrak{P}})\phi,\phi\rangle_*.\end{aligned} \quad (5.17)$$

The second and third terms on the r. h. s. of (5.17) tend to zero as $|\mathfrak{p}| \to 0$ according to (5.16) and (2.8). The last term also tends to zero for small $|\mathfrak{p}|$, since the symbol of the difference is

$$\mathcal{F}(\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2,\mathfrak{P}+\mathfrak{p}} - \mathcal{H}_{Z,2}^{D_2,E_2,\mathfrak{P}})\mathcal{F}^{-1} = \sum_{n\in Z_2}\frac{m_n}{M}\boldsymbol{\alpha}_n\cdot\omega\mathfrak{p}.$$

From (5.16) and (5.17) follows that

$$\left|\frac{\langle\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2,\mathfrak{P}}\psi,\psi\rangle_*}{\|\psi\|_*^2} - \frac{\langle\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2,\mathfrak{P}+\mathfrak{p}}\phi,\phi\rangle_*}{\|\phi\|_*^2}\right| \leqslant \frac{\varepsilon}{2}, \quad (5.18)$$

if $|\mathfrak{p}|$ is small enough. Hence by (5.11) and (5.18) for any $\varepsilon > 0$

$$\left|\mu(\mathfrak{P}+\mathfrak{p}) - \mu(\mathfrak{P})\right| < \varepsilon$$

for $|\mathfrak{p}|$ small enough. •

Now we prove that μ is semibounded from below and tends to infinity as $|\mathfrak{P}| \to \infty$. This, together with (5.9) and Lemma 15, implies that the spectrum of $\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2}$ is purely essential and is concentrated on a semi–axis. Proposition 14 will be thus proved.

According to (2.9) for $j = 2$ and (2.10) we have

$$\begin{aligned}\langle\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2}\psi,\psi\rangle \geqslant C_1\langle\sum_{n\in Z_2}\sqrt{-\Delta_n+m_n^2}\psi,\psi\rangle - C_2\|\psi\|^2,\\ \text{for any}\quad \psi\in P^DP^E\bigotimes_{n\in Z_2}\Lambda_nH^{1/2}(\mathbb{R}^3,\mathbb{C}^4).\end{aligned} \quad (5.19)$$

Since all the operators corresponding to the quadratic forms involved in (5.19) commute with $\mathcal{F}_0^{-1}\mathfrak{P}\mathcal{F}_0$, it follows from (5.8) that for almost all $\mathfrak{P}$ the inequality

$$\langle\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2,\mathfrak{P}}\psi,\psi\rangle_* \geqslant C_1\langle\sum_{n\in Z_2}\sqrt{\widehat{\mathbf{p}}_n^2+m_n^2}\mathcal{F}\psi,\mathcal{F}\psi\rangle_* - C_2\|\psi\|_*^2 \quad (5.20)$$

holds for every $\psi\in\mathfrak{D}^{\mathfrak{P}}$, where $\widehat{\mathbf{p}}_n$ are defined in (5.15). Thus μ is semibounded from below. Since by (5.15)

$$\mathfrak{P} = \Big|\sum_{n\in Z_2}\widehat{\mathbf{p}}_n\Big|,$$

there exists $n \in Z_2$ such that

$$|\widehat{\mathbf{p}}_n| \geqslant \frac{\mathfrak{P}}{N_2}$$

and hence

$$\sum_{n \in Z_2} \sqrt{\widehat{\mathbf{p}}_n^2 + m_n^2} \geqslant \frac{\mathfrak{P}}{N_2}.$$

Thus the r. h. s. of (5.20) tends to infinity as $\mathfrak{P} \to \infty$. •

6 Absence of Gaps

We are now ready to finish the proof of Theorem 6 by proving that

$$\big[\varkappa(D,E), \infty\big) \subseteq \sigma(\mathcal{H}_N^{D,E}). \tag{6.1}$$

Let us first fix a decomposition Z on which the minimum is attained in (2.20). Following the general strategy of [14], we will prove that for any irreducible representations $(D_1, E_1; D_2, E_2) \underset{Z}{\prec} (D, E)$ any

$$\lambda \geqslant \varkappa_1(Z, D_1, E_1) + \varkappa_2(Z, D_2, E_2)$$

belongs to $\sigma(\mathcal{H}_N^{D,E})$. This will imply (6.1) according to the definition (2.19). Let

$$\lambda_1 := \lambda - \varkappa_1(Z, D_1, E_1) \geqslant \varkappa_2(Z, D_2, E_2). \tag{6.2}$$

We will use the notation and results of Section 5. The following lemma is a slight modification of Theorem 8.11 of [14] and is proved along the same lines:

Lemma 16 *Let A be a selfadjoint operator in a Hilbert space $\mathfrak{H}$ and $U(\gamma)$ be a continuous representation of a compact group Γ by unitary operators in $\mathfrak{H}$ such that $U(\gamma)\operatorname{Dom} A \subset \operatorname{Dom} A$ and $U(\gamma)A = AU(\gamma)$ for any $\gamma \in \Gamma$. Then for any irreducible (matrix) representation D of Γ the corresponding subspace $P^D\mathfrak{H}$ reduces A. For every $\lambda \in \sigma(A^D)$ where A^D is the reduced operator and every $\varepsilon > 0$ there exists a D–generating subspace G of* $\operatorname{Dom} A$ *such that*

$$\|Au - \lambda u\| \leqslant \varepsilon \|u\|, \textit{ for all } u \in G.$$

Remark 17 *Recall that a subspace G of $\mathfrak{H}$ is called D–*generating *if the operator $U(\gamma)|G$ is unitary in G for all $\gamma \in \Gamma$ and there exists an orthonormal base in G such that for every $\gamma \in \Gamma$ the operator $U(\gamma)|G$ is represented by the matrix $D(\gamma)$.*

Proof of Lemma 16. Let r be the dimension of the representation $D : \gamma \mapsto \big(D_{lk}(\gamma)\big)_{l,k=1}^r$. Let us introduce in $\mathfrak{H}$ the bounded operators P_{lk} by

$$P_{lk} := r \int_\Gamma \overline{D_{lk}(\gamma)} U(\gamma) d\mu(\gamma), \quad l, k = 1, \dots, r,$$

where μ is the invariant probability measure on Γ. It is shown in the proof of Theorem 8.11 of [14] that P_{ll} are orthogonal projections onto mutually orthogonal subspaces of $\mathfrak{H}$ and that

$$P^D = \sum_{l=1}^{r} P_{ll}. \tag{6.3}$$

In fact, P_{ll} is the projection on the subspace of function which belong to the l^{th} row of the representation D. Moreover, P_{lk} is a partial isometry between $P_{kk}\mathfrak{H}$ and $P_{ll}\mathfrak{H}$. Since $\lambda \in \sigma(A^D)$, there exists a vector $u_0 \in \operatorname{Dom} A^D$ such that

$$\|A^D u_0 - \lambda u_0\| \leqslant \varepsilon \|u_0\|.$$

It follows from (6.3) that there exists $l \in \{1, \dots, r\}$ such that $\|P_{ll}u_0\| \geqslant r^{-1}$. We can thus define $u_l := P_{ll}u_0/\|P_{ll}u_0\|$ and then $u_k := P_{kl}u_l$ for $k = 1, \dots, r$. The subspace G spanned by $\{u_k\}_{k=1}^r$ satisfies the statement of the lemma. •

Let

$$r_j := \dim(D_j \otimes E_j), \quad j = 1, 2. \tag{6.4}$$

Since $\varkappa_1(Z, D_1, E_1)$ belongs to the spectrum of $\widetilde{\mathcal{H}}_{Z,1}^{D_1,E_1}$ (see definition (2.18)), by Lemma 16 we can choose a sequence of $(D_1 \otimes E_1)$–generating subspaces $\{G_q\}_{q=1}^\infty$ of $\operatorname{Dom}(\widetilde{\mathcal{H}}_{Z,1}^{D_1,E_1})$ such that for all $q \in \mathbb{N}$

$$\left\|\widetilde{\mathcal{H}}_{Z,1}^{D_1,E_1}\phi_q - \varkappa_1(Z, D_1, E_1)\phi_q\right\|_{\mathfrak{H}_{Z,1}} \leqslant q^{-1}\|\phi_q\|_{\mathfrak{H}_{Z,1}}, \text{ for all } \phi_q \in G_q. \tag{6.5}$$

Analogously, for any $\mathfrak{P} \geqslant 0$ we can find a sequence $\{G_q^{\mathfrak{P}}\}_{q=1}^\infty$ of $(D_2 \otimes E_2)$–generating subspaces of $\operatorname{Dom}\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2,\mathfrak{P}}$ such that

$$\left\|\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2,\mathfrak{P}}\psi_q^{\mathfrak{P}} - \mu(\mathfrak{P})\psi_q^{\mathfrak{P}}\right\|_* \leqslant q^{-1}\left\|\psi_q^{\mathfrak{P}}\right\|_*, \text{ for all } \psi_q^{\mathfrak{P}} \in G_q^{\mathfrak{P}}. \tag{6.6}$$

Moreover, we can choose an orthonormal basis $\{\psi_{q,l}^{\mathfrak{P}}\}_{l=1}^{r_2}$ in $G_q^{\mathfrak{P}}$ in such a way that for every $q \in \mathbb{N}$ and $l = 1, \dots, r_2$ $\psi_{q,l}$ belongs to the l^{th} row of the representation $(D_2 \otimes E_2)$ and satisfies (6.6). By Proposition 14, Lemma 15, and (6.2) we can choose $\mathfrak{P}_0$ in such a way that

$$\mu(\mathfrak{P}_0) = \lambda_1. \tag{6.7}$$

We choose $R_q > q$ so that (2.12) and (2.13) hold true for all $n, j = 1, \dots, N$, $n < j$ with

$$\varepsilon := q^{-1}(N_1+1)^{-1}N_2^{-1/2}C_1^{1/2}\big(C_2 + |\lambda_1| + 2\big)^{-1/2}, \tag{6.8}$$

where $N_{1,2}$ are the numbers of elements in $Z_{1,2}$, and $C_{1,2}$ are the constants in (2.9) for $j = 2$, and so that for some orthonormal base $\{\phi_{q,k}\}_{k=1}^{r_1}$ of G_q

$$\left\|\Big(1 - \prod_{j \in Z_1} I_{\{|\mathbf{x}_j| < R_q\}}\Big)\phi_{q,k}\right\|_{L_2(\mathbb{R}^{3N_1}, \mathbb{C}^{4^{N_1}})} \leqslant \frac{\nu_0}{4d_E^2 r_1 r_2}, \tag{6.9}$$

where d_E is the dimension of E, $r_{1,2}$ are defined in (6.4), and the constant $\nu_0 > 0$ depending only on E, E_1, E_2 will be specified later in the proof of Lemma 21.

By Assumption 4 and Lemma 15, we can choose a sequence of positive numbers $\{\delta_q\}_{q=1}^{\infty}$ tending to zero in such a way that

$$\big|\mu(\mathfrak{P}) - \lambda_1\big| \leqslant q^{-1} \quad \text{for all} \quad \mathfrak{P} \in [\mathfrak{P}_0, \mathfrak{P}_0 + \delta_q], \tag{6.10}$$

$$\frac{1}{2\pi^2}(\mathfrak{P}_0 + \delta_q)^2 \delta_q C_{R_q} < q^{-2}, \tag{6.11}$$

where C_{R_q} is the constant in (2.11), and

$$\frac{1}{2\pi^2}(\mathfrak{P}_0 + \delta_q)^2 \delta_q \cdot \frac{4}{3}\pi R_q^3 < \frac{\nu_0^2}{16 d_E^4 r_1^2 r_2^2}. \tag{6.12}$$

Let us choose a function $f_q \in L_2(\mathbb{R}_+)$ with $\operatorname{supp} f_q \subset [\mathfrak{P}_0, \mathfrak{P}_0 + \delta_q]$ so that

$$\int_{\mathfrak{P}_0}^{\mathfrak{P}_0+\delta_q} \big|f_q(\mathfrak{P})\big|^2 \mathfrak{P}^2 d\mathfrak{P} = 1. \tag{6.13}$$

Let

$$\begin{aligned} &\psi_{q,l}(\mathbf{y}_0, \dots, \mathbf{y}_{N_2-1}) \\ &:= \frac{1}{(2\pi)^{\frac{3}{2}}} \int\limits_{\mathfrak{P}_0}^{\mathfrak{P}_0+\delta_q} \int\limits_{S^2} e^{i\mathfrak{P}\omega\mathbf{y}_0} f_q(\mathfrak{P}) \psi_{q,l}^{\mathfrak{P}}(\omega, \mathbf{y}_1, \dots, \mathbf{y}_{N_2-1}) \mathfrak{P}^2 d\omega d\mathfrak{P}, \end{aligned} \tag{6.14}$$

where $\{\mathbf{y}_0, \dots, \mathbf{y}_{N_2-1}\}$ and $\{\mathbf{x}_n\}_{n\in Z_2}$ are related by (5.1) and (5.2). It follows from (6.13) and the choice of $\psi_{q,l}^{\mathfrak{P}}$ that

$$\|\psi_{q,l}\|_{\mathfrak{H}_{Z,2}} = 1, \quad l = 1, \dots, N_2, \tag{6.15}$$

and that $\psi_{q,l}$ belongs to the l^{th} row of $(D_2 \otimes E_2)$. Clearly the linear subspace $\widetilde{G}_q$ spanned by $\{\psi_{q,l}\}_{l=1}^{r_2}$ is a $(D_2 \otimes E_2)$–generating subspace of $\operatorname{Dom} \widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2}$.

LEMMA 18 *For any $n \in Z_2$ and $\psi \in \widetilde{G}_q$ with $\|\psi\| = 1$ the one–particle density*

$$\rho_{\psi,n}(\mathbf{x}_n) := \int_{\mathbb{R}^{3N_2-3}} \big|\psi(\mathbf{x}_{n_1}, \dots, \mathbf{x}_{n_{N_2}})\big|^2 (d\mathbf{x}_{n_1} \cdots d\mathbf{x}_{n_{N_2}})/d\mathbf{x}_n$$

satisfies

$$\|\rho_{\psi,n}\|_{L_\infty(\mathbb{R}^3)} \leqslant \frac{1}{2\pi^2}(\mathfrak{P}_0 + \delta_q)^2 \delta_q.$$

Proof. By (6.14)

$$
\begin{aligned}
&\|\rho_{\psi,n}\|_{L_\infty(\mathbb{R}^3)} \leqslant (2\pi)^{-3/2}\|\widehat{\rho}_{\psi,n}\|_{L_1(\mathbb{R}^3)} \\
&= \frac{1}{(2\pi)^6}\int_{\mathbb{R}^3}\bigg|\int_{\mathbb{R}^{3N_2}}\int_{\mathfrak{P}_0}^{\mathfrak{P}_0+\delta_q}\int_{S^2}\int_{\mathfrak{P}_0}^{\mathfrak{P}_0+\delta_q}\int_{S^2} e^{-i\mathbf{p}(\mathbf{y}_0+\mathbf{r}_n)}e^{-i\mathfrak{P}\omega\mathbf{y}_0}\overline{f_q(\mathfrak{P})} \\
&\times \psi_q^{\mathfrak{P}*}(\omega,\mathbf{y}_1,\dots,\mathbf{y}_{N_2-1})e^{i\widetilde{\mathfrak{P}}\widetilde{\omega}\mathbf{y}_0}f_q(\widetilde{\mathfrak{P}})\psi_q^{\widetilde{\mathfrak{P}}}(\widetilde{\omega},\mathbf{y}_1,\dots,\mathbf{y}_{N_2-1})\mathfrak{P}^2\widetilde{\mathfrak{P}}^2 \\
&\times d\widetilde{\omega}\,d\widetilde{\mathfrak{P}}\,d\omega\,d\mathfrak{P}\,d\mathbf{y}_0\,d\mathbf{y}_1\cdots d\mathbf{y}_{N_2-1}\bigg|d\mathbf{p},
\end{aligned}
\tag{6.16}
$$

where $\mathbf{r}_n := \mathbf{x}_n - \mathbf{y}_0$, see (5.2). Integrating the r. h. s. of (6.16) in $\mathbf{y}_0$ we obtain $(2\pi)^3\delta(\mathbf{p}+\mathfrak{P}\omega-\widetilde{\mathfrak{P}}\widetilde{\omega})$ from all the factors involving $\mathbf{y}_0$. Estimating the absolute value of the integral by the integral of absolute value and taking into account that $\int\delta(\mathbf{p}+\dots)d\mathbf{p} = 1$ we get

$$
\begin{aligned}
&\|\rho_{\psi,n}\|_{L_\infty(\mathbb{R}^3)} \leqslant \frac{1}{(2\pi)^3}\int_{\mathbb{R}^{3N_2-3}}\int_{\mathfrak{P}_0}^{\mathfrak{P}_0+\delta_q}\int_{S^2}\int_{\mathfrak{P}_0}^{\mathfrak{P}_0+\delta_q}\int_{S^2}\big|f_q(\mathfrak{P})\big|\big|f_q(\widetilde{\mathfrak{P}})\big| \\
&\times \big|\psi_q^{\mathfrak{P}}(\omega,\mathbf{y}_1,\dots,\mathbf{y}_{N_2-1})\big|\big|\psi_q^{\widetilde{\mathfrak{P}}}(\widetilde{\omega},\mathbf{y}_1,\dots,\mathbf{y}_{N_2-1})\big|\mathfrak{P}^2\widetilde{\mathfrak{P}}^2 \\
&\times d\widetilde{\omega}\,d\widetilde{\mathfrak{P}}\,d\omega\,d\mathfrak{P}\,d\mathbf{y}_1\cdots d\mathbf{y}_{N_2-1} \leqslant \frac{1}{(2\pi)^3}4\pi(\mathfrak{P}_0+\delta_q)^2\delta_q,
\end{aligned}
\tag{6.17}
$$

where at the last step we have used Schwarz inequality and $\|\psi\| = 1$. The formal calculation (6.16) — (6.17) is justified by the fact that the integral over $\mathbb{R}^{3N_2}$ can be considered as a limit of integrals over expanding finite volumes, since $\psi \in L_2(\mathbb{R}^{3N_2})$. •

Corollary 19 *For any $W \in L_2(\mathbb{R}^3)$, $n \in Z_2$, and $\psi \in \widetilde{G}_q$ with $\|\psi\| = 1$ we have*

$$
\int_{\mathbb{R}^{3N_2}}\big|W(\mathbf{x}_n)\psi(\mathbf{x}_{n_1},\dots,\mathbf{x}_{n_{N_2}})\big|^2 d\mathbf{x}_{n_1}\cdots d\mathbf{x}_{n_{N_2}} \leqslant \frac{1}{2\pi^2}(\mathfrak{P}_0+\delta_q)^2\delta_q\|W\|^2.
$$

Let F_q be the subspace of $\mathfrak{H}_N$ spanned by the functions

$$
\begin{aligned}
\varphi_{q,k,l}(\mathbf{x}_1,\dots,\mathbf{x}_N) := \phi_{q,k}(\mathbf{x}_j : j \in Z_1)\otimes\psi_{q,l}(\mathbf{x}_n : n \in Z_2),& \\
k = 1,\dots,r_1, \quad l = 1,\dots,r_2,&
\end{aligned}
\tag{6.18}
$$

where $\{\phi_{q,k}\}_{k=1}^{r_1}$ and $\{\psi_{q,l}\}_{l=1}^{r_2}$ are orthonormal bases of G_q and $\widetilde{G}_q$, respectively. We obviously have $\|\varphi_{q,k,l}\|_{L_2(\mathbb{R}^{3N},\mathbb{C}^{4^N})} = 1$.

Lemma 20 *For any $q \in \mathbb{N}$ $F_q \subset \operatorname{Dom}\mathcal{H}_N$. For any $\varphi \in F_q$*

$$
\big\|(\mathcal{H}_N - \lambda)\varphi\big\| \leqslant 5q^{-1}r_1^{1/2}r_2^{1/2}\|\varphi\|.
$$

PROOF. It is enough to show that the functions (6.18) belong to $\operatorname{Dom}\mathcal{H}_N$ and satisfy

$$\big\|(\mathcal{H}_N - \lambda)\varphi_{q,k,l}\big\|_{L_2(\mathbb{R}^{3N},\mathbb{C}^{4^N})} \leqslant 5q^{-1}. \tag{6.19}$$

Indeed, by triangle and Cauchy inequalities for

$$\varphi = \sum_{k=1}^{r_1}\sum_{l=1}^{r_2} c_{kl}\varphi_{q,k,l} \tag{6.20}$$

we have

$$\begin{aligned}\big\|(\mathcal{H}_N - \lambda)\varphi\big\| &\leqslant \sum_{k=1}^{r_1}\sum_{l=1}^{r_2} |c_{kl}|\big\|(\mathcal{H}_N - \lambda)\varphi_{q,k,l}\big\| \\ &\leqslant \sup_{k,l}\big\|(\mathcal{H}_N - \lambda)\varphi_{q,k,l}\big\| r_1^{1/2} r_2^{1/2}\|\varphi\|.\end{aligned}$$

The operator domain of $\mathcal{H}_N$ can be characterized as the set of functions ξ from the form domain $\overset{N}{\underset{n=1}{\otimes}}\Lambda_n H^{1/2}(\mathbb{R}^3,\mathbb{C}^4)$ on which the sesquilinear form $\langle\mathcal{H}_N\xi,\cdot\rangle$ is a bounded linear functional in $\mathfrak{H}_N$. Functions (6.18) belong to $\overset{N}{\underset{n=1}{\otimes}}\Lambda_n H^{1/2}(\mathbb{R}^3,\mathbb{C}^4)$ by construction. By (2.2), (2.5), and (2.6) we have

$$\mathcal{H}_N = \mathcal{H}_{Z,1} + \mathcal{H}_{Z,2} + \Lambda^N\bigg(\sum_{n\in Z_2} V_n + \sum_{\substack{n<j\\ n\# j}} U_{nj}\bigg)\Lambda^N. \tag{6.21}$$

The sesquilinear forms $\langle(\mathcal{H}_{Z,1} + \mathcal{H}_{Z,2})\varphi_{q,k,l},\cdot\rangle$ are bounded linear functionals over $L_2(\mathbb{R}^{3N},\mathbb{C}^{4^N})$, since $\phi_{q,k} \in \operatorname{Dom}(\widetilde{\mathcal{H}}_{Z,1}^{D_1,E_1})$ and $\psi_{q,l} \in \operatorname{Dom}\widetilde{H}_{Z,2}^{D_2,E_2}$. Moreover, by (6.5)

$$\Big\|\big(\mathcal{H}_{Z,1} - \varkappa_1(Z,D_1,E_1)\big)\varphi_{q,k,l}\Big\| = \Big\|\big(\widetilde{\mathcal{H}}_{Z,1}^{D_1,E_1} - \varkappa_1(Z,D_1,E_1)\big)\phi_{q,k}\Big\| \leqslant q^{-1},$$

and by (6.6), (6.7), (6.10), (6.14), and (6.15)

$$\big\|(\mathcal{H}_{Z,2} - \lambda_1)\varphi_{q,k,l}\big\| = \big\|(\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2} - \lambda_1)\psi_{q,l}\big\| \leqslant 2q^{-1}. \tag{6.22}$$

In view of (6.21)—(6.22) and (6.2), to prove that $\varphi_{q,k,l} \in \operatorname{Dom}\mathcal{H}_N$ and that (6.19) holds true it is enough to obtain that

$$\bigg\|\bigg(\sum_{n\in Z_2} V_n + \sum_{\substack{n<j\\ n\# j}} U_{nj}\bigg)\varphi_{q,k,l}\bigg\| \leqslant 2q^{-1}. \tag{6.23}$$

To do this, we first note that by (2.12), (2.13), and Cauchy inequality

$$\Big\| \Big(\sum_{n \in Z_2} V_n I_{\{|\mathbf{x}_n| > R_q\}} + \sum_{\substack{n<j \\ n\#j}} U_{nj} I_{\{|\mathbf{x}_n - \mathbf{x}_j| > R_q\}} \Big) \varphi_{q,k,l} \Big\|$$
$$\leqslant \varepsilon(N_1+1) \sum_{n \in Z_2} \big\| |D_n|^{\frac12} \psi_{q,l} \big\| \leqslant \varepsilon(N_1+1) N_2^{\frac12} \Big(\sum_{n \in Z_2} \big\| |D_n|^{\frac12} \psi_{q,l} \big\|^2 \Big)^{\frac12}. \tag{6.24}$$

By (2.9), (6.15), and (6.22),

$$\begin{aligned} \sum_{n \in Z_2} \|D_n^{1/2} \psi_{q,l}\|^2 &\leqslant C_1^{-1} \Big(\big\| (\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2} - \lambda_1) \psi_{q,l} \big\| + C_2 + |\lambda_1| \Big) \\ &\leqslant C_1^{-1} \big(C_2 + |\lambda_1| + 2q^{-1} \big). \end{aligned} \tag{6.25}$$

Thus by (6.24), (6.25) and (6.8) for $q \geqslant 1$ we obtain

$$\Big\| \Big(\sum_{n \in Z_2} V_n I_{\{|\mathbf{x}_n| > R_q\}} + \sum_{\substack{n<j \\ n\#j}} U_{nj} I_{\{|\mathbf{x}_n - \mathbf{x}_j| > R_q\}} \Big) \varphi_{q,k,l} \Big\| \leqslant q^{-1}. \tag{6.26}$$

Now the scalar functions

$$V_{n,q}(\mathbf{x}) := \big|V_n(\mathbf{x})\big| I_{\{|\mathbf{x}| \leqslant R_q\}}(\mathbf{x}) \quad \text{and} \quad U_{nj,q}(\mathbf{x}) := \big|U_{nj}(\mathbf{x})\big| I_{\{|\mathbf{x}| \leqslant R_q\}}(\mathbf{x}) \tag{6.27}$$

are square integrable by (2.11). By Corollary 19, for $n \in Z_2$

$$\|V_{n,q} \varphi_{q,k,l}\|^2 = \|V_{n,q} \psi_{q,l}\|^2 \leqslant \frac{1}{2\pi^2} \delta_q (\mathfrak{P}_0 + \delta_q)^2 \|V_{n,q}\|^2_{L_2(\mathbb{R}^3)} \tag{6.28}$$

and for $n < j$, $n\#j$

$$\|U_{nj,q} \varphi_{q,k,l}\|^2 \leqslant \sup_{\mathbf{z} \in \mathbb{R}^3} \big\| U_{nj,q}(\cdot - \mathbf{z}) \psi_{q,l} \big\|^2 \leqslant \frac{1}{2\pi^2} \delta_q (\mathfrak{P}_0 + \delta_q)^2 \|U_{nj,q}\|^2_{L_2(\mathbb{R}^3)}. \tag{6.29}$$

Hence by (6.27), (6.28), (6.29), (2.11), and (6.11)

$$\Big\| \Big(\sum_{n \in Z_2} V_n I_{\{|\mathbf{x}_n| \leqslant R_q\}} + \sum_{\substack{n<j \\ n\#j}} U_{nj} I_{\{|\mathbf{x}_n - \mathbf{x}_j| \leqslant R_q\}} \Big) \varphi_{q,k,l} \Big\| \leqslant q^{-1}. \tag{6.30}$$

It remains to add (6.26) and (6.30) to obtain (6.23), finishing the proof of the lemma. •

The subspace F_q spanned by the functions (6.18) is $D_1 \otimes E_1 \otimes D_2 \otimes E_2$–generating. Since $(D_1, E_1; D_2, E_2) \underset{Z}{\prec} (D, E)$, F_q contains some nontrivial D–generating subspace. Hence the subspace $K_q := P^D F_q$ is not equal to $\{0\}$ and is contained in F_q.

Lemma 21 *There exists a constant $C_E > 0$ such that for every $q \in \mathbb{N}$*

$$\|P^E\varphi\| \geqslant C_E\|\varphi\|, \quad \textit{for all} \quad \varphi \in F_q. \tag{6.31}$$

Proof. Projector (2.15) can be written as

$$P^E = \frac{d_E}{h_\Pi} \sum_{\pi\in\Pi_1^Z\times\Pi_2^Z} \overline{\xi_E(\pi)}\pi + \frac{d_E}{h_\Pi} \sum_{\pi\in\Pi\setminus(\Pi_1^Z\times\Pi_2^Z)} \overline{\xi_E(\pi)}\pi. \tag{6.32}$$

We will denote the first term in (6.32) by Q^E, and the second by R^E. Then

$$\|P^E\varphi\|^2 = \langle\varphi, P^E\varphi\rangle = \langle\varphi, Q^E\varphi\rangle + \langle\varphi, R^E\varphi\rangle. \tag{6.33}$$

Relation $(D_1, E_1; D_2, E_2) \underset{Z}{\prec} (D, E)$ implies that the representation $E|\Pi_1^Z \times \Pi_2^Z$ is unitarily equivalent to a sum $\overset{k}{\underset{i=0}{\oplus}} n_i E^{(i)}$, where $n_i > 0$ are multiplicities of the irreducible representations $E^{(i)}$ of the group $\Pi_1^Z \times \Pi_2^Z$ with $E^{(0)} = E_1 \otimes E_2$. For the corresponding characters this gives

$$\xi_E(\pi) = \sum_{i=0}^{k} n_i \xi^{(i)}(\pi), \quad \text{for all} \quad \pi \in \Pi_1^Z \times \Pi_2^Z.$$

Hence

$$Q^E = \sum_{i=0}^{k} \nu_i P_i,$$

where $\nu_i > 0$ and P_i is the projector corresponding to the representation $E^{(i)}$. By construction, $P_0\varphi = \varphi$ for any $\varphi \in F_q$, hence $P_i\varphi = 0$ for $i = 1, \ldots, k$. Thus for any $\varphi \in F_q$

$$\langle\varphi, Q^E\varphi\rangle = \nu_0\|\varphi\|^2, \quad \nu_0 > 0. \tag{6.34}$$

We will now estimate the second term on the r. h. s. of (6.33). For any $n \in Z_2$ and any $\psi \in \widetilde{G}_q$ with $\|\psi\| = 1$ by Corollary 19 and (6.12) we have

$$\|I_{\{|\mathbf{x}_j|<R_q\}}\psi\|^2 \leqslant \frac{\nu_0^2}{16 d_E^4 r_1^2 r_2^2}. \tag{6.35}$$

For any functions (6.18) and any $\pi \in \Pi$ inequality (6.9) implies that

$$\big|\langle\varphi_{q,k,l}, \pi\varphi_{q,\widetilde{k},\widetilde{l}}\rangle\big| \leqslant \langle \prod_{j\in Z_1} I_{\{|\mathbf{x}_j|<R_q\}}|\varphi_{q,k,l}|, \pi|\varphi_{q,\widetilde{k},\widetilde{l}}|\rangle_{L_2(\mathbb{R}^{3N})} + \frac{\nu_0}{4 d_E^2 r_1 r_2}.$$

Now if $\pi \in \Pi \setminus (\Pi_1^Z \times \Pi_2^Z)$, then there exists $j_0 \in Z_1$ such that $\pi j_0 \in Z_2$. Hence by (6.35)

$$\langle \prod_{j\in Z_1} I_{\{|\mathbf{x}_j|<R_q\}}|\varphi_{q,k,l}|, \pi|\varphi_{q,\widetilde{k},\widetilde{l}}|\rangle \leqslant \langle|\varphi_{q,k,l}|, I_{\{|\mathbf{x}_{j_0}|<R_q\}}\pi|\varphi_{q,\widetilde{k},\widetilde{l}}|\rangle \leqslant \frac{\nu_0}{4 d_E^2 r_1 r_2}.$$

Thus

$$\left|\langle\varphi_{q,k,l}, \pi\varphi_{q,\widetilde{k},\widetilde{l}}\rangle\right| \leqslant \frac{\nu_0}{2d_E^2 r_1 r_2}, \quad \pi \in \Pi \setminus (\Pi_1^Z \times \Pi_2^Z). \tag{6.36}$$

Any $\varphi \in F_q$ can be written as (6.20). By (6.36) and Cauchy inequality for any $\pi \in \Pi \setminus (\Pi_1^Z \times \Pi_2^Z)$

$$\left|\langle\varphi, \pi\varphi\rangle\right| \leqslant \sum_{k,l,\widetilde{k},\widetilde{l}} |c_{kl}||c_{\widetilde{k}\widetilde{l}}|\left|\langle\varphi_{q,k,l}, \pi\varphi_{q,\widetilde{k},\widetilde{l}}\rangle\right| \leqslant \frac{\nu_0}{2d_E^2}\|\varphi\|^2. \tag{6.37}$$

Since the number of elements of $\Pi \setminus (\Pi_1^Z \times \Pi_2^Z)$ does not exceed d_Π and for any π $\left|\xi_E(\pi)\right| \leqslant d_E$ as a trace of unitary matrix of dimension d_E, (6.37) implies that

$$\left|\langle\varphi, R^E\varphi\rangle\right| \leqslant \nu_0\|\varphi\|^2/2.$$

By (6.33) and (6.34) we conclude that (6.31) holds with $C_E = \sqrt{\nu_0/2}$. •
Lemmata 20 and 21 imply that $L_q := P^E K_q$ is a nontrivial subspace of $\operatorname{Dom}\mathcal{H}_N^{D,E}$ and for every $f = P^E\varphi \in L_q$

$$\left\|(\mathcal{H}_N^{D,E} - \lambda)f\right\| \leqslant \left\|(\mathcal{H}_N - \lambda)\varphi\right\| \leqslant 5q^{-1}r_1^{\frac{1}{2}}r_2^{\frac{1}{2}}\|\varphi\| \leqslant 5q^{-1}r_1^{\frac{1}{2}}r_2^{\frac{1}{2}}C_E^{-1}\|f\|, \quad q \in \mathbb{N}.$$

This implies that $\lambda \in \sigma(\mathcal{H}_N^{D,E})$, and thus finishes the proof of Theorem 6.

Acknowledgement. The author was supported by the DFG grant SI 348/12–2. Part of this work was done during the stay at the Erwin Schrödinger Institute, Vienna.

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