Rational Real Algebraic Models of Topological Surfaces

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Abstract. Comessatti proved that the set of all real points of a rational real algebraic surface is either a nonorientable surface, or diffeomorphic to the sphere or the torus. Conversely, it is well known that each of these surfaces admits at least one rational real algebraic model. We prove that they admit exactly one rational real algebraic model. This was known earlier only for the sphere, the torus, the real projective plane and the Klein bottle.

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1 Introduction

Let X be a rational nonsingular projective real algebraic surface. Then the set $X(\mathbb{R})$ of real points of X is a compact connected topological surface. Comessatti showed that $X(\mathbb{R})$ cannot be an orientable surface of genus bigger than 1. To put it otherwise, $X(\mathbb{R})$ is either nonorientable, or it is orientable and diffeomorphic to the sphere S^2 or the torus $S^1 \times S^1$ [Co2, p. 257].
Conversely, each of these topological surfaces admits a *rational real algebraic model*, or *rational model* for short. In other words, if S is a compact connected topological surface which is either nonorientable, or orientable and diffeomorphic to the sphere or the torus, then there is a nonsingular rational projective real algebraic surface X such that $X(\mathbb{R})$ is diffeomorphic to S. Indeed, this is clear for the sphere, the torus and the real projective plane: the real projective surface defined by the affine equation $x^2 + y^2 + z^2 = 1$ is a rational

model of the sphere S^2, the real algebraic surface $\mathbb{P}^1 \times \mathbb{P}^1$ is a rational model of the torus $S^1 \times S^1$, and the real projective plane $\mathbb{P}^2$ is a rational model of the topological real projective plane $\mathbb{P}^2(\mathbb{R})$. If S is any of the remaining topological surfaces, then S is diffeomorphic to the n-fold connected sum of the real projective plane, where $n \geq 2$. A rational model of such a topological surface is the real surface obtained from $\mathbb{P}^2$ by blowing up $n-1$ real points. Therefore, any compact connected topological surface which is either nonorientable, or orientable and diffeomorphic to the sphere or the torus, admits at least one rational model.

Now, if S is a compact connected topological surface admitting a rational model X, then one can construct many other rational models of S. To see this, let P and $\overline{P}$ be a pair of complex conjugate complex points on X. The blow-up $\widetilde{X}$ of X at P and $\overline{P}$ is again a rational model of S. Indeed, since P and $\overline{P}$ are nonreal points of X, there are open subsets U of X and V of $\widetilde{X}$ such that

- $X(\mathbb{R}) \subseteq U(\mathbb{R})$, $\widetilde{X}(\mathbb{R}) \subseteq V(\mathbb{R})$, and
- U and V are isomorphic.

In particular, $X(\mathbb{R})$ and $\widetilde{X}(\mathbb{R})$ are diffeomorphic. This means that $\widetilde{X}$ is a rational model of S if X is so. Iterating the process, one can construct many nonisomorphic rational models of S. We would like to consider all such models of S to be equivalent. Therefore, we introduce the following equivalence relation on the collection of all rational models of a topological surface S.

Definition 1.1. Let X and Y be two rational models of a topological surface S. We say that X and Y are isomorphic as rational models of S if there is a sequence

$$\begin{tikzcd}[column sep=small]
 & X_1 \arrow[dl] \arrow[dr] & & X_3 \arrow[dl] \arrow[dr] & & X_{2n-1} \arrow[dl] \arrow[dr] & \\
X = X_0 & & X_2 & & \cdots & & X_{2n} = Y
\end{tikzcd}$$

where each morphism is a blowing-up at a pair of nonreal complex conjugate points.

We note that the equivalence relation, in Definition 1.1, on the collection of all rational models of a given surface S is the smallest one for which the rational models X and $\widetilde{X}$ mentioned above are equivalent.

Let X and Y be rational models of a topological surface S. If X and Y are isomorphic models of S, then the above sequence of blowing-ups defines a rational map

$$f \colon X \dashrightarrow Y$$

having the following property. There are open subsets U of X and V of Y such that

- the restriction of f to U is an isomorphism of real algebraic varieties from U onto V, and
- $X(\mathbb{R}) \subseteq U(\mathbb{R})$ and $Y(\mathbb{R}) \subseteq V(\mathbb{R})$.

It follows, in particular, that the restriction of f to $X(\mathbb{R})$ is an *algebraic diffeomorphism* from $X(\mathbb{R})$ onto $Y(\mathbb{R})$, or in other words, it is a *biregular map* from $X(\mathbb{R})$ onto $Y(\mathbb{R})$ in the sense of [BCR].

Let us recall the notion of an algebraic diffeomorphism. Let X and Y be smooth projective real algebraic varieties. Then $X(\mathbb{R})$ and $Y(\mathbb{R})$ are compact manifolds, not necessarily connected or nonempty. Let

$$f : X(\mathbb{R}) \longrightarrow Y(\mathbb{R}) \tag{1}$$

be a map. Choose affine open subsets U of X and V of Y such that $X(\mathbb{R}) \subseteq U(\mathbb{R})$ and $Y(\mathbb{R}) \subseteq V(\mathbb{R})$. Since U and V are affine, we may assume that they are closed subvarieties of $\mathbb{A}^m$ and $\mathbb{A}^n$, respectively. Then $X(\mathbb{R})$ is a closed submanifold of $\mathbb{R}^m$, and $Y(\mathbb{R})$ is a closed submanifold of $\mathbb{R}^n$. The map f in (1) is *algebraic* or *regular* if there are real polynomials $p_1, \ldots, p_n, q_1, \ldots, q_n$ in the variables $x_1, \ldots, x_m$ such that none of the polynomials $q_1, \ldots, q_n$ vanishes on $X(\mathbb{R})$, and

$$f(x) = \left(\frac{p_1(x)}{q_1(x)}, \ldots, \frac{p_n(x)}{q_n(x)}\right)$$

for all $x \in X(\mathbb{R})$.

One can check that the algebraicity of f depends neither on the choice of the affine open subsets U and V nor on the choice of the embeddings of U and V in affine space. Note that the algebraicity of f immediately implies that f is a C^∞-map.

The map f in (1) is an *algebraic diffeomorphism* if f is algebraic, bijective, and f^{-1} is algebraic.

Again let X and Y be rational models of a topological surface S. As observed above, if X and Y are isomorphic models of S, then there is an algebraic diffeomorphism

$$f \colon X(\mathbb{R}) \longrightarrow Y(\mathbb{R})\,.$$

Conversely, if there is an algebraic diffeomorphism $f \colon X(\mathbb{R}) \longrightarrow Y(\mathbb{R})$, then X and Y are isomorphic models of S, as it follows from the well known Weak Factorization Theorem for birational maps between real algebraic surfaces(see [BPV, Theorem III.6.3] for the WFT over $\mathbb{C}$, from which the WFT over $\mathbb{R}$ follows).

Here we address the following question. Given a compact connected topological surface S, what is the number of nonisomorphic rational models of S?

By Comessatti's Theorem, an orientable surface of genus bigger than 1 does not have any rational model. It is known that the topological surfaces S^2, $S^1 \times S^1$ and $\mathbb{P}^2(\mathbb{R})$ have exactly one rational model, up to isomorphism (see also Remark 3.2). Mangolte has shown that the same holds for the Klein bottle [Ma, Theorem 1.3] (see again Remark 3.2).

Mangolte asked how large n should be so that the n-fold connected sum of the real projective plane admits more than one rational model, up to isomorphism; see the comments following Theorem 1.3 in [Ma]. The following theorem shows that there is no such integer n.

Theorem 1.2. *Let S be a compact connected real two-manifold.*

1. *If S is orientable of genus greater than* 1*, then S does not admit any rational model.*

2. *If S is either nonorientable, or it is diffeomorphic to one of S^2 and $S^1 \times S^1$, then there is exactly one rational model of S, up to isomorphism. In other words, any two rational models of S are isomorphic.*

Of course, statement 1 is nothing but Comessatti's Theorem referred to above. Our proof of statement 2 is based on the Minimal Model Program for real algebraic surfaces developed by János Kollár in [Ko1]. Using this Program, we show that a rational model X of a nonorientable topological surface S is obtained from $\mathbb{P}^2$ by blowing it up successively in a finite number of real points (Theorem 3.1). The next step of the proof of Theorem 1.2 involves showing that the model X is isomorphic to a model X' obtained from $\mathbb{P}^2$ by blowing up $\mathbb{P}^2$ at real points $P_1, \ldots, P_n$ of $\mathbb{P}^2$. At that point, the proof of Theorem 1.2 would have been finished if we were able to prove that the group $\mathrm{Diff}_{\mathrm{alg}}(\mathbb{P}^2(\mathbb{R}))$ of algebraic diffeomorphisms of $\mathbb{P}^2(\mathbb{R})$ acts n-transitively on $\mathbb{P}^2(\mathbb{R})$. However, we were unable to prove such a statement. Nevertheless, a statement we were able to prove is the following.

Theorem 1.3. *Let n be a natural integer. The group $\mathrm{Diff}_{\mathrm{alg}}(S^1 \times S^1)$ acts n-transitively on $S^1 \times S^1$.*

We conjecture, however, the following.

Conjecture 1.4. Let X be a smooth projective rational surface. Let n be a natural integer. Then the group $\mathrm{Diff}_{\mathrm{alg}}(X(\mathbb{R}))$ acts n-transitively on $X(\mathbb{R})$.[1]

The only true evidence we have for the above conjecture is that it holds for $X = \mathbb{P}^1 \times \mathbb{P}^1$ according to Theorem 1.3.

Now, coming back to the idea of the proof of Theorem 1.2, we know that any rational model of S is isomorphic to one obtained from $\mathbb{P}^2$ by blowing up $\mathbb{P}^2$ at real points $P_1, \ldots, P_n$. Since we have established n-transitivity of the group of algebraic diffeomorphisms of $S^1 \times S^1$, we need to realize X' as a blowing-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at a finite number of real points.

Let L be the real projective line in $\mathbb{P}^2$ containing P_1 and P_2. Applying a nontrivial algebraic diffeomorphism of $\mathbb{P}^2$ into itself, if necessary, we may assume that $P_i \notin L$ for $i \geq 3$. Then we can do the usual transformation of $\mathbb{P}^2$ into $\mathbb{P}^1 \times \mathbb{P}^1$ by first blowing-up P_1 and P_2, and then contracting the strict

[1] This conjecture is now proved [HM].

transform of L. This realizes X' as a surface obtained from $\mathbb{P}^1\times\mathbb{P}^1$ by blowing-up $\mathbb{P}^1\times\mathbb{P}^1$ at $n-1$ distinct real points. Theorem 1.2 then follows from the $(n-1)$-transitivity of $\mathrm{Diff}_{\mathrm{alg}}(S^1\times S^1)$.

We will also address the question of uniqueness of geometrically rational models of a topological surface. By yet another result of Comessatti, a geometrically rational real surface X is rational if $X(\mathbb{R})$ is nonempty and connected. Therefore, Theorem 1.2 also holds when one replaces "rational models" by "geometrically rational models". Since the set of real points of a geometrically rational surface is not neccesarily connected, it is natural to study geometrically rational models of not necessarily connected topological surfaces. We will show that such a surface has an infinite number of geometrically rational models, in general.

The paper is organized as follows. In Section 2 we show that a real Hirzebruch surface is either isomorphic to the standard model $\mathbb{P}^1\times\mathbb{P}^1$ of the real torus $S^1\times S^1$, or isomorphic to the standard model of the Klein bottle. The standard model of the Klein bottle is the real algebraic surface $B_P(\mathbb{P}^2)$ obtained from the projective plane $\mathbb{P}^2$ by blowing up one real point P. In Section 3, we use the Minimal Model Program for real algebraic surfaces in order to prove that any rational model of any topological surface is obtained by blowing up one of the following three real algebraic surfaces: $\mathbb{P}^2$, $\mathbb{S}^2$ and $\mathbb{P}^1\times\mathbb{P}^1$ (Theorem 3.1). Here $\mathbb{S}^2$ is the real algebraic surface defined by the equation $x^2+y^2+z^2=1$. As a consequence, we get new proofs of the known facts that the sphere, the torus, the real projective plane and the Klein bottle admit exactly one rational model, up to isomorphism of course. In Section 4 we prove a lemma that will have two applications. Firstly, it allows us to conclude the uniqueness of a rational model for the "next" topological surface, the 3-fold connected sum of the real projective plane. Secondly, it also allows us to conclude that a rational model of a nonorientable topological surface is isomorphic to a model obtained from $\mathbb{P}^2$ by blowing up a finite number of distinct real points $P_1,\ldots,P_n$ of $\mathbb{P}^2$. In Section 5 we prove n-transitivity of the group of algebraic diffeomorphisms of the torus $S^1\times S^1$. In Section 6 we construct a nontrivial algebraic diffeomorphism f of $\mathbb{P}^2(\mathbb{R})$ such that the real points $f(P_i)$, for $i=3,\ldots,n$, are not on the real projective line through $f(P_1)$ and $f(P_2)$. In Section 7 we put all the pieces together and complete the proof of Theorem 1.2. In Section 8 we show by an example that the uniqueness does not hold for geometrically rational models of nonconnected topological surfaces.

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2 Real Hirzebruch surfaces

The set of real points of the rational real algebraic surface $\mathbb{P}^1\times\mathbb{P}^1$ is the torus $S^1\times S^1$. We call this model the *standard model* of the real torus. Fix a real point O of the projective plane $\mathbb{P}^2$. The rational real algebraic surface $B_O(\mathbb{P}^2)$

obtained from $\mathbb{P}^2$ by blowing up the real point O is a model of the Klein bottle K. We call this model the *standard model* of the Klein bottle.
Let d be a natural integer. Let $\mathbb{F}_d$ be the *real Hirzebruch surface* of degree d. Therefore, $\mathbb{F}_d$ is the compactification $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(d)\oplus\mathcal{O}_{\mathbb{P}^1})$ of the line bundle $\mathcal{O}_{\mathbb{P}^1}(d)$ over $\mathbb{P}^1$. Recall that the real algebraic surface $\mathbb{F}_d$ is isomorphic to $\mathbb{F}_e$ if and only if $d = e$. The restriction of the line bundle $\mathcal{O}_{\mathbb{P}^1}(d)$ to the set of real points $\mathbb{P}^1(\mathbb{R})$ of $\mathbb{P}^1$ is topologically trivial if and only if d is even. Consequently, $\mathbb{F}_d$ is a rational model of the torus $S^1 \times S^1$ if d is even, and it is a rational model of the Klein bottle K if d is odd (see [Si, Proposition VI.1.3] for a different proof).
The following statement is probably well known, and is an easy consequence of known techniques (compare the proof of Theorem 6.1 in [Ma]). We have chosen to include the statement and a proof for two reasons: the statement is used in the proof of Theorem 3.1, and the idea of the proof turns out also to be useful in Lemma 4.1.

Proposition 2.1. *Let d be a natural integer.*

1. *If d is even, then $\mathbb{F}_d$ is isomorphic to the standard model $\mathbb{P}^1 \times \mathbb{P}^1$ of $S^1 \times S^1$.*
2. *If d is odd, then $\mathbb{F}_d$ is isomorphic to the standard model $B_O(\mathbb{P}^2)$ of the Klein bottle K.*

(All isomorphisms are in the sense of Definition 1.1.)

Proof. Observe that

- the real algebraic surface $\mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to $\mathbb{F}_0$, and
- that the real algebraic surface $B_O(\mathbb{P}^2)$ is isomorphic to $\mathbb{F}_1$.

Therefore, the proposition follows from the following lemma. □

Lemma 2.2. *Let d and e be natural integers. Then the two models $\mathbb{F}_d$ and $\mathbb{F}_e$ are isomorphic if and only if $d \equiv e \pmod 2$.*

Proof. Since the torus is not diffeomorphic to the Klein bottle, the rational models $\mathbb{F}_d$ and $\mathbb{F}_e$ are not isomorphic if $d \not\equiv e \pmod 2$. Conversely, if $d \equiv e \pmod 2$, then $\mathbb{F}_d$ and $\mathbb{F}_e$ are isomorphic models, as follows from the following lemma using induction. □

Lemma 2.3. *Let d be a natural integer. The two rational models $\mathbb{F}_d$ and $\mathbb{F}_{d+2}$ are isomorphic.*

Proof. Let E be the section at infinity of $\mathbb{F}_d$. The self-intersection of E is equal to $-d$. Choose nonreal complex conjugate points P and $\overline{P}$ on E. Let F and $\overline{F}$ be the fibers of the fibration of $\mathbb{F}_d$ over $\mathbb{P}^1$ that contain P and $\overline{P}$, respectively. Let X be the real algebraic surface obtained from $\mathbb{F}_d$ by blowing up P and $\overline{P}$.

Denote again by E the strict transform of E in X. The self-intersection of E is equal to $-d-2$. The strict transforms of F and $\overline{F}$, again denoted by F and $\overline{F}$ respectively; they are disjoint smooth rational curves of self-intersection -1, and they do not intersect E. The real algebraic surface Y obtained from X by contracting F and $\overline{F}$ is a smooth $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. The image of E in Y has self-intersection $-d-2$. It follows that Y is isomorphic to $\mathbb{F}_{d+2}$ as a real algebraic surface. Therefore, we conclude that $\mathbb{F}_d$ and $\mathbb{F}_{d+2}$ are isomorphic models. □

3 RATIONAL MODELS

Let Y be a real algebraic surface. A real algebraic surface X is said to be *obtained from* Y *by blowing up* if there is a nonnegative integer n, and a sequence of morphisms

$$X = X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} X_0 = Y\,,$$

such that for each $i = 1, \ldots, n$, the morphism f_i is either the blow up of X_{i-1} at a real point, or it is the blow up of X_{i-1} at a pair of distinct complex conjugate points.

The surface X is said to be obtained from Y by blowing up *at real points only* if for each $i = 1, \ldots, n$, the morphism f_i is a blow up of X_{i-1} at a real point of X_{i-1}.

One defines, similarly, the notion of a real algebraic surface obtained from Y by blowing up *at nonreal points only.*

The real algebraic surface defined by the affine equation

$$x^2 + y^2 + z^2 = 1$$

will be denoted by $\mathbb{S}^2$. Its set of real points is the two-sphere S^2. The real Hirzebruch surface $\mathbb{F}_1$ will be simply denoted by $\mathbb{F}$. Its set of real points is the Klein bottle K.

Thanks to the Minimal Model Program for real algebraic surfaces due to János Kollár [Ko1, p. 206, Theorem 30], one has the following statement:

THEOREM 3.1. *Let S be a compact connected topological surface. Let X be a rational model of S.*

1. *If S is not orientable then X is isomorphic to a rational model of S obtained from $\mathbb{P}^2$ by blowing up at real points only.*
2. *If S is orientable then X is isomorphic to $\mathbb{S}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$, as a model.*

Proof. Apply the Minimal Model Program to X in order to obtain a sequence of blowing-ups as above, where Y is one of the following:

1. a minimal surface,

2. a conic bundle over a smooth real algebraic curve,
3. a Del Pezzo surface of degree 1 or 2, and
4. $\mathbb{P}^2$ or $\mathbb{S}^2$.

(See [Ko1, p. 206, Theorem 30].) The surface X being rational, we know that X is not birational to a minimal surface. This rules out the case of Y being a minimal surface. Since $X(\mathbb{R})$ is connected, it can be shown that X is not birational to a Del Pezzo surface of degree 1 or 2. Indeed, such Del Pezzo surfaces have disconnected sets of real points [Ko1, p. 207, Theorem 33(D)(c–d)]. This rules out the case of Y being a Del Pezzo surface of degree 1 or 2. It follows that

- either Y is a conic bundle, or
- Y is isomorphic to $\mathbb{P}^2$, or
- Y is isomorphic to $\mathbb{S}^2$.

We will show that the statement of the theorem holds in all these three cases. If Y is isomorphic to $\mathbb{P}^2$, then $Y(\mathbb{R})$ is not orientable. Since X is obtained from Y by blowing up, it follows that $X(\mathbb{R})$ is not orientable either. Therefore, the surface S is not orientable, and also X is isomorphic to a rational model of S obtained from $\mathbb{P}^2$ by blowing up. Moreover, it is easy to see that X is then isomorphic to a rational model of S obtained from $\mathbb{P}^2$ by blowing up at real points only. This settles the case when Y is isomorphic to $\mathbb{P}^2$.
If Y is isomorphic to $\mathbb{S}^2$, then there are two cases to consider: (1) the case of S being orientable, (2) and the case of S being nonorientable. If S is orientable, then $X(\mathbb{R})$ is orientable too, and X is obtained from Y by blowing up at nonreal points only. It follows that X is isomorphic to $\mathbb{S}^2$ as a model.
If S is nonorientable, then $X(\mathbb{R})$ is nonorientable too, and X is obtained from $\mathbb{S}^2$ by blowing up a nonempty set of real points. Therefore, the map $X \longrightarrow Y$ factors through a blow up $\widetilde{\mathbb{S}}^2$ of $\mathbb{S}^2$ at a real point. Now, $\widetilde{\mathbb{S}}^2$ contains two smooth disjoint complex conjugated rational curves of self-intersection -1. When we contract them, we obtain a real algebraic surface isomorphic to $\mathbb{P}^2$. Therefore, X is obtained from $\mathbb{P}^2$ by blowing up. It follows again that X is isomorphic to a rational model of S obtained from $\mathbb{P}^2$ by blowing up at real points only. This settles the case when Y is isomorphic to $\mathbb{S}^2$.
The final case to consider is the one where Y is a conic bundle over a smooth real algebraic curve B. Since X is rational, B is rational. Moreover, B has real points because X has real points. Hence, the curve B is isomorphic to $\mathbb{P}^1$.
The singular fibers of the the conic bundle Y over B are real, and moreover, the number of singular fibers is even. Since $X(\mathbb{R})$ is connected, we conclude that $Y(\mathbb{R})$ is connected too. it follows that the conic bundle Y over B has either no singular fibers or exactly 2 singular fibers. If it has exactly 2 singular fibers, then Y is isomorphic to $\mathbb{S}^2$ [Ko2, Lemma 3.2.4], a case we have already dealt with.

Therefore, we may assume that Y is a smooth $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. Therefore, Y is a real Hirzebruch surface. By Proposition 2.1, we may suppose that $Y = \mathbb{P}^1 \times \mathbb{P}^1$, or that $Y = \mathbb{F}$. Since $\mathbb{F}$ is obtained from $\mathbb{P}^2$ by blowing up one real point, the case $Y = \mathbb{F}$ follows from the case of $Y = \mathbb{P}^2$ which we have already dealt with above.

Therefore, we may assume that $Y = \mathbb{P}^1 \times \mathbb{P}^1$. Again, two cases are to be considered: (1) the case of S being orientable, and (2) the case of S being nonorientable. If S is orientable, $X(\mathbb{R})$ is orientable, and X is obtained from Y by blowing up at non real points only. It follows that X is isomorphic as a model to $\mathbb{P}^1 \times \mathbb{P}^1$. If S is not orientable, X is obtained from Y by blowing up, at least, one real point. Since $Y = \mathbb{P}^1 \times \mathbb{P}^1$, a blow-up of Y at one real point is isomorphic to a blow-up of $\mathbb{P}^2$ at two real points. We conclude again by the case of $Y = \mathbb{P}^2$ dealt with above. □

Note that Theorem 3.1 implies Comessatti's Theorem referred to in the introduction, i.e., the statement to the effect that any orientable compact connected topological surface of genus greater than 1 does not admit a rational model (Theorem 1.2.1).

Remark 3.2. For sake of completeness let us show how Theorem 3.1 implies that the surfaces $S^2, S^1 \times S^1, \mathbb{P}^2(\mathbb{R})$ and the Klein bottle K admit exactly one rational model. First, this is clear for the orientable surfaces S^2 and $S^1 \times S^1$.

Let X be a rational model of $\mathbb{P}^2(\mathbb{R})$. From Theorem 3.1, we know that X is isomorphic to a rational model of $\mathbb{P}^2(\mathbb{R})$ obtained from $\mathbb{P}^2$ by blowing up at real points only. Therefore, we may assume that X itself is obtained from $\mathbb{P}^2$ by blowing up at real points only. Since $X(\mathbb{R})$ is diffeomorphic to $\mathbb{P}^2(\mathbb{R})$, it follows that X is isomorphic to $\mathbb{P}^2$. Thus any rational model of $\mathbb{P}^2(\mathbb{R})$ is isomorphic to $\mathbb{P}^2$ as a model.

Let X be a rational model of the Klein bottle K. Using Theorem 3.1 one may assume that X is a blowing up of $\mathbb{P}^2$ at real points only. Since $X(\mathbb{R})$ is diffeomorphic to the 2-fold connected sum of $\mathbb{P}^2(\mathbb{R})$, the surface X is a blowing up of $\mathbb{P}^2$ at exactly one real point. It follows that X is isomorphic to $\mathbb{F}$. Therefore, any rational model of the Klein bottle K is isomorphic to $\mathbb{F}$, as a model; compare with [Ma, Theorem 1.3].

One can wonder whether the case where S is a 3-fold connected sum of real projective planes can be treated similarly. The first difficulty is as follows. It is, a priori, not clear why the following two rational models of $\#^3\mathbb{P}^2(\mathbb{R})$ are isomorphic. The first one is obtained from $\mathbb{P}^2$ by blowing up two real points of $\mathbb{P}^2$. The second one is obtained by a successive blow-up of $\mathbb{P}^2$: first blow up $\mathbb{P}^2$ at a real point, and then blow up a real point of the exceptional divisor. In the next section we prove that these two models are isomorphic.

4 The 3-fold connected sum of the real projective plane

We start with a lemma.

Lemma 4.1. *Let P be a real point of $\mathbb{P}^2$, and let $B_P(\mathbb{P}^2)$ be the surface obtained from $\mathbb{P}^2$ by blowing up P. Let E be the exceptional divisor of $B_P(\mathbb{P}^2)$ over P. Let L be any real projective line of $\mathbb{P}^2$ not containing P. Consider L as a curve in $B_P(\mathbb{P}^2)$. Then there is a birational map*

$$f\colon B_P(\mathbb{P}^2) \dashrightarrow B_P(\mathbb{P}^2)$$

whose restriction to the set of real points is an algebraic diffeomorphism such that $f(L(\mathbb{R})) = E(\mathbb{R})$.

Proof. The real algebraic surface $B_P(\mathbb{P}^2)$ is isomorphic to the real Hirzebruch surface $\mathbb{F} = \mathbb{F}_1$, and any isomorphism between them takes the exceptional divisor of $B_P(\mathbb{P}^2)$ to the section at infinity of the conic bundle $\mathbb{F}/\mathbb{P}^1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1})$. The line L in $B_P(\mathbb{P}^2)$ is given by a unique section of $\mathcal{O}_{\mathbb{P}^1}(1)$ over $\mathbb{P}^1$; this section of $\mathcal{O}_{\mathbb{P}^1}(1)$ will also be denoted by L. We denote again by E the section at infinity of $\mathbb{F}$.
We have to show that there is a birational self-map f of $\mathbb{F}$ such that the equality $f(L(\mathbb{R})) = E(\mathbb{R})$ holds. Let R be a nonreal point of L. Let F be the fiber of the conic bundle $\mathbb{F}$ passing through R. The blowing-up of $\mathbb{F}$ at the pair of points R and $\overline{R}$ is a real algebraic surface in which we can contract the strict transforms of F and $\overline{F}$. The real algebraic surface one obtains after these two contractions is again isomorphic to $\mathbb{F}$.
Therefore, we have a birational self-map f of $\mathbb{F}$ whose restriction to the set of real points is an algebraic diffeomorphism. The image, by f, of the strict transform of L in $\mathbb{F}$ has self-intersection -1. Therefore, the image, by f, of the strict transform of L coincides with E. In particular, we have $f(L(\mathbb{R})) = E(\mathbb{R})$. $\square$

Proposition 4.2. *Let S be the* 3*-fold connected sum of* $\mathbb{P}^2(\mathbb{R})$*. Then S admits exactly* 1 *rational model.*

Proof. Fix two real points O_1, O_2 of $\mathbb{P}^2$, and let $B_{O_1,O_2}(\mathbb{P}^2)$ be the real algebraic surface obtained from $\mathbb{P}^2$ by blowing up O_1 and O_2. The surface $B_{O_1,O_2}(\mathbb{P}^2)$ is a rational model of the 3-fold connected sum S of $\mathbb{P}^2(\mathbb{R})$.
Let X be a rational model of S. We prove that X is isomorphic to $B_{O_1,O_2}(\mathbb{P}^2)$, as a model. By Theorem 3.1, we may assume that X is obtained from $\mathbb{P}^2$ by blowing up real points only. Since $X(\mathbb{R})$ is diffeomorphic to a 3-fold connected sum of the real projective plane, the surface X is obtained from $\mathbb{P}^2$ by blowing up twice real points. More precisely, there is a real point P of $\mathbb{P}^2$ and a real point Q of the blow-up $B_P(\mathbb{P}^2)$ of $\mathbb{P}^2$ at P, such that X is isomorphic to the blow-up $B_Q(B_P(\mathbb{P}^2))$ of $B_P(\mathbb{P}^2)$ at Q.
Choose any real projective line L in $\mathbb{P}^2$ not containing P. Then, L is also a real curve in $B_P(\mathbb{P}^2)$. We may assume that $Q \notin L$. By Lemma 4.1, there is a birational map f from $B_P(\mathbb{P}^2)$ into itself whose restriction to the set of real points is an algebraic diffeomorphism, and such that

$$f(L(\mathbb{R})) = E(\mathbb{R}),$$

where E is the exceptional divisor on $B_P(\mathbb{P}^2)$. Let $R = f(Q)$. Then $R \notin E$, and f induces a birational isomorphism

$$\widetilde{f}\colon B_Q(B_P(\mathbb{P}^2)) \longrightarrow B_R(B_P(\mathbb{P}^2))$$

whose restriction to the set of real points is an algebraic diffeomorphism. Since $R \notin E$, the point R is a real point of $\mathbb{P}^2$ distinct from P, and the blow-up $B_R(B_P(\mathbb{P}^2))$ is equal to the blow up $B_{P,R}(\mathbb{P}^2)$ of $\mathbb{P}^2$ at the real points P, R of $\mathbb{P}^2$. It is clear that $B_{P,R}(\mathbb{P}^2)$ is isomorphic to $B_{O_1,O_2}(\mathbb{P}^2)$. It follows that X is isomorphic to $B_{O_1,O_2}(\mathbb{P}^2)$ as rational models of the 3-fold connected sum of $\mathbb{P}^2(\mathbb{R})$. □

Lemma 4.3. *Let S be a nonorientable surface and let X be a rational model of S. Then, there are distinct real points $P_1, \ldots, P_n$ of $\mathbb{P}^2$ such that X is isomorphic to the blowing-up of $\mathbb{P}^2$ at $P_1, \ldots, P_n$, as a model.*

Proof. By Theorem 3.1, we may assume that X is obtained from $\mathbb{P}^2$ by blowing up at real points only. Let

$$X = X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} X_0 = \mathbb{P}^2. \tag{2}$$

be a sequence of blowing ups, where for each $i = 1, \ldots, n$, the map f_i is a blowing up of X_{i-1} at a real point P_i of X_{i-1}.

To a sequence of blowing-ups as in (2) is associated a forest F of trees. The vertices of F are the centers P_i of the blow-ups f_i. For $i > j$, there is an edge between the points P_i and P_j in F if

- the composition $f_{j+1} \circ \cdots \circ f_{i-1}$ is an isomorphism at a neighborhood of P_i, and
- maps P_i to a point belonging to the exceptional divisor $f_j^{-1}(P_j)$ of P_j in X_j.

Let ℓ be the sum of the lengths of the trees belonging to F. We will show by induction on ℓ that X is isomorphic, as a model, to the blowing-up of $\mathbb{P}^2$ at a finite number of distinct real points of $\mathbb{P}^2$.

This is obvious if $\ell = 0$. If $\ell \neq 0$, let P_j be the root of a tree of nonzero length, and let P_i be the vertex of that tree lying immediately above P_j. By changing the order of the blowing-ups f_i, we may assume that $j = 1$ and $i = 2$.

Choose a real projective line L in $\mathbb{P}^2$ which does not contain any of the roots of the trees of F. By Lemma 4.1, there is a birational map g_1 from $X_1 = B_{P_1}(\mathbb{P}^2)$ into itself whose restriction to the set of real points is an algebraic diffeomorphism and satisfies the condition $g_1(L(\mathbb{R})) = E(\mathbb{R})$, where E is the exceptional divisor of X_1.

Put $X_0' = \mathbb{P}^2$, $X_1' = X_1$, and $f_1' = f_1$. We consider g_1 as a birational map from X_1 into X_1'. Put $P_2' = g_1(P_2)$. Let X_2' be the blowing-up of X_1' at P_2', and let

$$f_2'\colon X_2' \longrightarrow X_1'$$

be the blowing-up morphism. Then, g_1 induces a birational map g_2 from X_2 into X_2' which is an algebraic diffeomorphism on the set of real points.
By iterating this construction, one gets a sequence of blowing ups

$$f_i' \colon X_i' \longrightarrow X_{i-1}' ,$$

where $i = 1, \ldots, n$, and birational morphisms g_i from X_i into X_i' whose restrictions to the sets of real points are algebraic diffeomorphisms. In particular, the rational models $X = X_n$ and $X' = X_n'$ of S are isomorphic.
Let F' be the forest of the trees of centers of X'. Then the sum of the lengths ℓ' of the trees of F' is equal to $\ell - 1$. Indeed, one obtains F' from F by replacing the tree T of F rooted at P_1 by the disjoint union of the tree $T \setminus P_1$ and the tree $\{P_1\}$. This follows from the fact that P_2' does not belong to the exceptional divisor of f_1', and that, no root of the other trees of F belongs to the exceptional divisor of f_1' either. □

As observed in the Introduction, if we are able to prove the n-transitivity of the action of the group $\mathrm{Diff}_{\mathrm{alg}}(\mathbb{P}^2(\mathbb{R}))$ on $\mathbb{P}^2(\mathbb{R})$, then the statement of Theorem 1.2 would follow from Lemma 4.3. However, we did not succeed in proving so. Nevertheless, we will prove the n-transitivity of $\mathrm{Diff}(S^1 \times S^1)$, which is the subject of the next section.
Now that we know that the topological surfaces $S^1, S^1 \times S^1$ and $\#^n\mathbb{P}^2(\mathbb{R})$, for $n = 1, 2, 3$, admit exactly one rational model, one may also wonder whether Lemma 4.3 allows us to tackle the "next" surface, which is the 4-fold connected sum of $\mathbb{P}^2(\mathbb{R})$. We note that Theorem 1.2 and Lemma 4.3 imply that a rational model of such a surface is isomorphic to a surface obtained from $\mathbb{P}^2$ by blowing up 3 distinct real points. However, it it is not clear why the two surfaces of the following type are isomorphic as models. Take three distinct non-collinear real points P_1, P_2, P_3, and three distinct collinear real points Q_1, Q_2, Q_3 of $\mathbb{P}^2$. Then the surfaces $X = B_{P_1,P_2,P_3}(\mathbb{P}^2)$ and $Y = B_{Q_1,Q_2,Q_3}(\mathbb{P}^2)$ are rational models of $\#^4\mathbb{P}^2(\mathbb{R})$ (the 4-fold connected sum of $\mathbb{P}^2(\mathbb{R})$), but it is not clear why they should be isomorphic. One really seems to need some nontrivial algebraic diffeomorphism of $\mathbb{P}^2(\mathbb{R})$, that maps P_i to Q_i for $i = 1, 2, 3$, in order to show that X and Y are isomorphic models. We will come back to this in Section 6 (Lemma 6.1).

5 Algebraic diffeomorphisms of $S^1 \times S^1$ and n-transitivity

The following statement is a variation on classical polynomial interpolation.

Lemma 5.1. *Let m be a positive integer. Let $x_1, \ldots, x_m$ be distinct real numbers, and let $y_1 \ldots, y_m$ be positive real numbers. Then there is a real polynomial p of degree $2m$ that does not have real zeros, and satisfies the condition $p(x_i) = y_i$ for all i.*

Proof. Set

$$p(\zeta) := \sum_{j=1}^{m} \prod_{k \neq j} \frac{(\zeta - x_k)^2}{(x_j - x_k)^2} \cdot y_j.$$

Then p is of degree $2m$, and p does not have real zeros. Furthermore, we have $p(x_i) = y_i$ for all i. $\square$

COROLLARY 5.2. *Let m be a positive integer. Let $x_1, \ldots, x_m$ be distinct real numbers, and let $y_1 \ldots, y_m, z_1, \ldots, z_m$ be positive real numbers. Then there are real polynomials p and q without any real zeros such that* $\operatorname{degree}(p) = \operatorname{degree}(q)$*, and*

$$\frac{p(x_i)}{q(x_i)} = \frac{y_i}{z_i}$$

for all $1 \leq i \leq m$. $\square$

The interest in the rational functions p/q of the above type lies in the following fact.

LEMMA 5.3. *Let p and q be two real polynomials of same degree that do not have any real zeros. Define the rational map $f\colon \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ by*

$$f(x, y) = \left(x, \frac{p(x)}{q(x)} \cdot y \right).$$

Then f is a birational map of $\mathbb{P}^1 \times \mathbb{P}^1$ into itself whose restriction to the set of real points is an algebraic diffeomorphism. $\square$

THEOREM 5.4. *Let n be a natural integer. The group* $\operatorname{Diff}_{\mathrm{alg}}(\mathbb{P}^1 \times \mathbb{P}^1)$ *acts n-transitively on $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$.*

Proof. Choose n distinct real points $P_1, \ldots, P_n$ and n distinct real points $Q_1, \ldots, Q_n$ of $\mathbb{P}^1 \times \mathbb{P}^1$. We need to show that there is a birational map f from $\mathbb{P}^1 \times \mathbb{P}^1$ into itself, whose restriction to $(\mathbb{P}^1 \times \mathbb{P}^1)(\mathbb{R})$ is an algebraic diffeomorphism, such that $f(P_i) = Q_i$, for $i = 1, \ldots, n$.
First of all, we may assume that $P_1, \ldots, P_n, Q_1, \ldots, Q_n$ are contained in the first open quadrant of $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$. In other words, the coordinates of P_i and Q_i are strictly positive real numbers. Moreover, it suffices to prove the statement for the case where $Q_i = (i, i)$ for all i.
By the hypothesis above, there are positive real numbers x_i, y_i such that $P_i = (x_i, y_i)$ for all i. By Corollary 5.2, there are real polynomials p and q without any real zeros such that $\operatorname{degree}(p) = \operatorname{degree}(q)$, and such that the real numbers

$$\frac{p(x_i)}{q(x_i)} \cdot y_i$$

are positive and distinct for all i. Define $f\colon \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ by

$$f(x, y) := \left(x, \frac{p(x)}{q(x)} \cdot y \right).$$

By Lemma 5.3, f is birational, and its restriction to $(\mathbb{P}^1 \times \mathbb{P}^1)(\mathbb{R})$ is an algebraic diffeomorphism. By construction, the points $f(P_i)$ have distinct second coordinates. Therefore, replacing P_i by $f(P_i)$ if necessary, we may assume that the points P_i have distinct second coordinates, which implies that $y_1, \ldots, y_m$ are distinct positive real numbers.

By Corollary 5.2, there are real polynomials p, q without any real zeros such that $\text{degree}(p) = \text{degree}(q)$, and

$$\frac{p(y_i)}{q(y_i)} \cdot x_i = i.$$

Define $f \colon \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ by

$$f(x,y) = \left(\frac{p(y)}{q(y)} \cdot x, y\right).$$

By Lemma 5.3, f is birational and its restriction to the set of real points is an algebraic diffeomorphism. By construction, one has $f(P_i) = (i, y_i)$ for all i. Therefore, we may assume that $P_i = (i, y_i)$ for all i.

Now, again by Corollary 5.2, there are two real polynomials p and q without any real zeros such that $\text{degree}(p) = \text{degree}(q)$, and

$$\frac{p(i)}{q(i)} \cdot y_i = i$$

for all i. Define $f \colon \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ by

$$f(x,y) = \left(x, \frac{p(x)}{q(x)} \cdot y\right).$$

By Lemma 5.3, f is birational, and its restriction to the set of real points is an algebraic diffeomorphism. By construction $f(P_i) = Q_i$ for all i. $\square$

Remark 5.5. One may wonder whether Theorem 5.4 implies that the group $\text{Diff}_{\text{alg}}(\mathbb{P}^2(\mathbb{R}))$ acts n-transitively on $\mathbb{P}^2(\mathbb{R})$. We will explain the implication of Theorem 5.4 in that direction. Let $P_1, \ldots, P_n$ be distinct real points of $\mathbb{P}^2$, and let $Q_1, \ldots, Q_n$ be distinct real points of $\mathbb{P}^2$. Choose a real projective line L in $\mathbb{P}^2$ not containing any of the points $P_1, \ldots, P_n, Q_1, \ldots, Q_n$. Let O_1 and O_2 be distinct real points of L. Identify $\mathbb{P}^1 \times \mathbb{P}^1$ with the surface obtained from $\mathbb{P}^2$ by, first, blowing up O_1, O_2 and, then, contracting the strict transform of L. Denote by E_1 and E_2 the images of the exceptional divisors over O_1 and O_2 in $\mathbb{P}^1 \times \mathbb{P}^1$, respectively. We denote again by $P_1, \ldots, P_n, Q_1, \ldots, Q_n$ the real points of $\mathbb{P}^1 \times \mathbb{P}^1$ that correspond to the real points $P_1, \ldots, P_n, Q_1, \ldots, Q_n$ of $\mathbb{P}^2$.

Now, the construction in the proof of Theorem 5.4 gives rise to a birational map f from $\mathbb{P}^1 \times \mathbb{P}^1$ into itself which is an algebraic diffeomorphism on $(\mathbb{P}^1 \times \mathbb{P}^1)(\mathbb{R})$ and which maps P_i onto Q_i, for $i = 1, \ldots, n$. Moreover, if one carries out

carefully the construction of f, one has that $f(E_1(\mathbb{R})) = E_1(\mathbb{R})$ and $f(E_2(\mathbb{R})) = E_2(\mathbb{R})$ and that the real intersection point O of E_1 and E_2 in $\mathbb{P}^1 \times \mathbb{P}^1$ is a fixed point of f.
Note that one obtains back $\mathbb{P}^2$ from $\mathbb{P}^1 \times \mathbb{P}^1$ by blowing up O and contracting the strict transforms of E_1 and E_2. Therefore, the birational map f of $\mathbb{P}^1 \times \mathbb{P}^1$ into itself induces a birational map g of $\mathbb{P}^2$ into itself. Moreover, $g(P_i) = Q_i$. One may think that g is an algebraic diffeomorphism on $\mathbb{P}^2(\mathbb{R})$. However, the restriction of g to the set of real points is not necessarily an algebraic diffeomorphism! In fact, g is an algebraic diffeomorphism on $\mathbb{P}^2(\mathbb{R}) \setminus \{O_1, O_2\}$. The restriction of g to $\mathbb{P}^2(\mathbb{R}) \setminus \{O_1, O_2\}$ does admit a continuous extension $\widetilde{g}$ to $\mathbb{P}^2(\mathbb{R})$, and $\widetilde{g}$ is obviously a homeomorphism. One may call $\widetilde{g}$ an *algebraic homeomorphism*, but $\widetilde{g}$ is *not necessarily an algebraic diffeomorphism.* It is not difficult to find explicit examples of such algebraic homeomorphisms that are not diffeomorphisms.
That is the reason why we do not claim to have proven n-transitivity of $\mathrm{Diff}_{\mathrm{alg}}(\mathbb{P}^2(\mathbb{R}))$. The only statement about $\mathbb{P}^2(\mathbb{R})$ the above arguments prove is the n-transitivity of the group $\mathrm{Homeo}_{\mathrm{alg}}(\mathbb{P}^2(\mathbb{R}))$ of algebraic homeomorphisms.

6 A nontrivial algebraic diffeomorphism of $\mathbb{P}^2(\mathbb{R})$

The nontrivial diffeomorphisms we have in mind are the following. They have been studied in another recent paper as well [RV].
Let $Q_1, \ldots, Q_6$ be six pairwise distinct complex points of $\mathbb{P}^2$ satisfying the following conditions:

1. the subset $\{Q_1, \cdots, Q_6\}$ is closed under complex conjugation,
2. the subset $\{Q_1, \cdots, Q_6\}$ does not lie on a complex conic,
3. the complex conic passing through any 5 of these six points is nonsingular.

Denote by $C_1, \ldots, C_6$ the nonsingular complex conics one thus obtains. These conics are pairwise complex conjugate. Consider the real Cremona transformation $f = f_Q$ of $\mathbb{P}^2$ defined by first blowing-up $\mathbb{P}^2$ at $Q_1, \ldots, Q_6$ and then contracting the strict transforms of $C_1, \ldots, C_6$. Let $R_1, \ldots, R_6$ denote the points of $\mathbb{P}^2$ that correspond to the contractions of the conics $C_1, \ldots, C_6$.
The restriction to $\mathbb{P}^2(\mathbb{R})$ of the birational map f from $\mathbb{P}^2$ into itself is obviously an algebraic diffeomorphism.
The Cremona transformation f maps a real projective line, not containing any of the points $Q_1, \ldots, Q_6$, to a real rational quintic curve having 6 distinct non-real double points at the points $R_1, \ldots, R_6$. Moreover, it maps a real rational quintic curve in $\mathbb{P}^2$ having double points at $Q_1, \ldots, Q_6$ to a real projective line in $\mathbb{P}^2$ that does not contain any of the points $R_1, \ldots, R_6$.
Observe that the inverse of the Cremona transformation f_Q is the Cremona transformation f_R. It follows that $f = f_Q$ induces a bijection from the set of real rational quintics having double points at $Q_1, \ldots, Q_6$ onto the set of real projective lines in $\mathbb{P}^2$ that do not contain any of $R_1, \ldots, R_6$.

This section is devoted to the proof of following lemma.

Lemma 6.1. *Let n be a natural integer bigger than 1. Let $P_1, \ldots, P_n$ be distinct real points of $\mathbb{P}^2$. Then there is a birational map of $\mathbb{P}^2$ into itself, whose restriction to the set of real points is an algebraic diffeomorphism, such that the image points $f(P_3), \ldots, f(P_n)$ are not contained in the real projective line through $f(P_1)$ and $f(P_2)$.*

Proof. Choose complex points $Q_1, \ldots, Q_6$ of $\mathbb{P}^2$ as above. As observed before, the Cremona transformation $f = f_Q$ induces a bijection from the set of real rational quintic curves having double points at $Q_1, \ldots, Q_6$ onto the set of real projective lines of $\mathbb{P}^2$ not containing any of the above points $R_1, \ldots, R_n$. In particular, there is a real rational quintic curve C in $\mathbb{P}^2$ having 6 nonreal double points at $Q_1, \ldots, Q_6$.

We show that there is a real projectively linear transformation α of $\mathbb{P}^2$ such that $\alpha(C)$ contains P_1 and P_2, and does not contain any of the points $P_3, \ldots, P_n$. The Cremona transformation $f_{\alpha(Q)}$ will then be a birational map of $\mathbb{P}^2$ into itself that has the required properties.

First of all, let us prove that there is $\alpha \in \mathrm{PGL}_3(\mathbb{R})$ such that $P_1, P_2 \in \alpha(C)$. This is easy. Since C is a quintic curve, $C(\mathbb{R})$ is infinite. In particular, C contains two distinct real points. It follows that there is $\alpha \in \mathrm{PGL}_3(\mathbb{R})$ such that $P_1, P_2 \in \alpha(C)$. Replacing C by $\alpha(C)$ if necessary, we may suppose that $P_1, P_2 \in C$.

We need to show that there is $\alpha \in \mathrm{PGL}_3(\mathbb{R})$ such that $\alpha(P_1) = P_1$, $\alpha(P_2) = P_2$ and $\alpha(C)$ does not contain any of the points $P_3, \ldots, P_n$.

To prove the existence of α by contradiction, assume that there is no such automorphism of $\mathbb{P}^2$. Therefore, for all $\alpha \in \mathrm{PGL}_3(\mathbb{R})$ having P_1 and P_2 as fixed points, the image $\alpha(C)$ contains at least one of the points of $P_3, \ldots, P_n$.

Let G be the stabilizer of the pair (P_1, P_2) for the diagonal action of PGL_3 on $\mathbb{P}^2 \times \mathbb{P}^2$. It is easy to see that G is a geometrically irreducible real algebraic group. Let

$$\rho \colon C \times G \longrightarrow \mathbb{P}^2$$

be the morphism defined by $\rho(P, \alpha) = \alpha(P)$. Let

$$X_i := \rho^{-1}(P_i)$$

be the inverse image, where $i = 3, \ldots, n$. Therefore, X_i is a real algebraic subvariety of $C \times G$. By hypothesis, for every $\alpha \in G(\mathbb{R})$, there is an integer i such that $\alpha(C)$ contains P_i. Denoting by p the projection on the second factor from $C \times G$ onto G, this means that

$$\bigcup_{i=3}^{n} p(X_i(\mathbb{R})) = G(\mathbb{R}).$$

Since $G(\mathbb{R})$ is irreducible, there is an integer $i_0 \in [3, n]$ such that the semialgebraic subset $p(X_{i_0}(\mathbb{R}))$ is Zariski dense in $G(\mathbb{R})$. Since G is irreducible and p

is proper, one has $p(X_{i_0}) = G$. In particular, $P_{i_0} \in \alpha(C)$ for all $\alpha \in G(\mathbb{C})$. To put it otherwise, $\alpha^{-1}(P_{i_0}) \in C$ for all $\alpha \in G(\mathbb{C})$, which means that the orbit of P_{i_0} under the action of G is contained in C. In particular, the dimension of the orbit of P_{i_0} is at most one. It follows that P_1, P_2 and P_{i_0} are collinear. Let L be the projective line through P_1, P_2. Then the orbit of P_{i_0} coincides with $L \setminus \{P_1, P_2\}$. It now follows that $L \subseteq C$. This is in contradiction with the fact that C is irreducible. □

7 Proof of Theorem 1.2.2

Let S be a topological surface, either nonorientable or of genus less than 2. We need to show that any two rational models of S are isomorphic. By Remark 3.2, we may assume that S is the n-fold connected sum of $\mathbb{P}^2(\mathbb{R})$, where $n \geq 3$.
Let $O_1, \dots, O_{n-2}$ be fixed pairwise distinct real points of $\mathbb{P}^1 \times \mathbb{P}^1$, and let $B_{n-2}(\mathbb{P}^1 \times \mathbb{P}^1)$ be the surface obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ by blowing up the points $O_1, \dots, O_{n-2}$. It is clear that $B_{n-2}(\mathbb{P}^1 \times \mathbb{P}^1)$ is a rational model of S.
Now, it suffices to show that any rational model of S is isomorphic to $B_{n-2}(\mathbb{P}^1 \times \mathbb{P}^1)$, as a model. Let X be any rational model of S. By Lemma 4.3, we may assume that there are distinct real points $P_1, \dots, P_m$ of $\mathbb{P}^2$ such that X is the surface obtained from $\mathbb{P}^2$ by blowing up $P_1, \dots, P_m$. Since X is a rational model of an n-fold connected sum of $\mathbb{P}^2(\mathbb{R})$, one has $m = n-1$. In particular, $m \geq 2$. By Lemma 6.1, we may assume that the points $P_3, \dots, P_m$ are not contained in the real projective line L through P_1 and P_2.
The blow-up morphism $X \longrightarrow \mathbb{P}^2$ factors through the blow up $\widetilde{\mathbb{P}}^2 = B_{P_1,P_2}(\mathbb{P}^2)$. The strict transform $\widetilde{L}$ of L has self-intersection -1 in $\widetilde{\mathbb{P}}^2$. If we contract $\widetilde{L}$, then we obtain a surface isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Therefore, X is isomorphic to a model obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ by blowing up $m-1 = n-2$ distinct real points of $\mathbb{P}^1 \times \mathbb{P}^1$. It follows from Theorem 5.4 that X is isomorphic to $B_{n-2}(\mathbb{P}^1 \times \mathbb{P}^1)$. □

8 Geometrically rational models

Recall that a nonsingular projective real algebraic surface X is *geometrically rational* if the complex surface $X_{\mathbb{C}} = X \times_{\mathbb{R}} \mathbb{C}$ is rational. Comessatti showed that, if X is a geometrically rational real algebraic surface with $X(\mathbb{R})$ connected, then X is rational; see Theorem IV of [Co1] and the remarks thereafter (see also [Si, Corollary VI.6.5]). Therefore, the main result, namely Theorem 1.2, also applies to geometrically rational models. More precisely, we have the following consequence.

Corollary 8.1. *Let S be a compact connected real two-manifold.*

1. *If S is orientable and the genus of S is greater than* 1*, then S does not admit a geometrically rational real algebraic model.*

2. If S is either nonorientable, or it is diffeomorphic to one of S^2 and $S^1 \times S^1$, then there is exactly one geometrically rational model of S, up to isomorphism. In other words, any two geometrically rational models of S are isomorphic. □

Now, the interesting aspect about geometrically rational real surfaces is that their set of real points can have an arbitrary finite number of connected components. More precisely, Comessati proved the following statement [Co2, p. 263 and further] (see also [Si, Proposition VI.6.1]).

THEOREM 8.2. *Let X be a geometrically rational real algebraic surface such that $X(\mathbb{R})$ is not connected. Then each connected component of $X(\mathbb{R})$ is either nonorientable or diffeomorphic to S^2. Conversely, if S is a nonconnected compact topological surface each of whose connected components is either nonorientable or diffeomorphic to S^2, then there is a geometrically rational real algebraic surface X such that $X(\mathbb{R})$ is diffeomorphic to S.* □

Let S be a nonconnected topological surface. One may wonder whether the geometrically rational model of S whose existence is claimed above, is unique up to isomorphism of models. The answer is negative, as shown by the following example.

EXAMPLE 8.3. Let S be the disjoint union of a real projective plane and 4 copies of S^2. Then, any minimal real Del Pezzo surface of degree 1 is a geometrically rational model of S [Ko2, Theorem 2.2(D)]. Minimal real Del Pezzo surfaces of degree 1 are rigid; this means that any birational map between two minimal real Del Pezzo surfaces of degree 1 is an isomorphism of real algebraic surfaces [Is, Theorem 1.6]. Now, the set of isomorphism classes of minimal real Del Pezzo surfaces of degree 1 is in one-to-one correspondence with an open dense subset of the quotient $\mathbb{P}^2(\mathbb{R})^8/\mathrm{PGL}_3(\mathbb{R})$ for the diagonal action of the group $\mathrm{PGL}_3(\mathbb{R})$. It follows that the topological surface S admits a 8-dimensional continuous family of nonisomorphic geometrically rational models. In particular, the number of nonisomorphic geometrically rational models of S is infinite.

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