The Chow-Witt ring

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Abstract. We define a ring structure on the total Chow-Witt group of any integral smooth scheme over a field of characteristic different from 2.

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Contents

1 Introduction

Let A be a commutative noetherian ring of Krull dimension n and P a projective A-module of rank d. One can ask the following question: does P admit a free factor of rank one? Serre proved a long time ago that the answer is always positive when $d > n$. So in fact the first interesting case is when P is projective of rank equal to the dimension of A. Suppose now that X is an integral smooth scheme over a field k of characteristic not 2. To deal with the above question, Barge and Morel introduced the Chow-Witt groups $\widetilde{CH}^j(X)$ of X (called at that time *groupes de Chow des cycles orientés*, see [BM]) and associated to each vector bundle E of rank n an Euler class $\tilde{c}_n(E)$ in $\widetilde{CH}^n(X)$. It was proved recently that if $X = \mathrm{Spec}(A)$ we have $\tilde{c}_n(P) = 0$ if and only if $P \simeq Q \oplus A$ (see [Mo] for $n \geq 4$, [FS] for $n = 3$ and [BM] or [Fa] for the case $n = 2$). It is therefore important to provide more tools, such as a ring structure and a pull-back for regular embeddings, to compute the Chow-Witt groups and the Euler classes.
To define $\widetilde{CH}^p(X)$ consider the fibre product of the complex in Milnor K-theory

$$0 \to K_p^M(k(X)) \to \bigoplus_{x_1 \in X^{(1)}} K_{p-1}^M(k(x_1)) \to \cdots \to \bigoplus_{x_n \in X^{(n)}} K_{p-n}^M(k(x_n)) \to 0$$

and the Gersten-Witt complex restricted to the fundamental ideals

$$0 \to I^p(k(X)) \to \bigoplus_{x_1 \in X^{(1)}} I^{p-1}(\mathcal{O}_{X,x_1}) \to \cdots \to \bigoplus_{x_n \in X^{(n)}} I^{p-n}(\mathcal{O}_{X,x_n}) \to 0$$

over the quotient complex

$$0 \to I^p/I^{p+1}(k(X)) \to \cdots \to \bigoplus_{x_n \in X^{(n)}} I^{p-n}/I^{p+1-n}(\mathcal{O}_{X,x_n}) \to 0.$$

The group $\widetilde{CH}^p(X)$ is defined as the p-th cohomology group of this fibre product. Roughly speaking, an element of $\widetilde{CH}^p(X)$ is the class of a sum of varieties of codimension p with a quadratic form defined on each variety. We oviously have a map $\widetilde{CH}^p(X) \to CH^p(X)$.
Using the functoriality of the two complexes we see that the Chow-Witt groups satisfy good functorial properties (see [Fa]). For example, we have a pull-back morphism $f^* : \widetilde{CH}^j(X) \to \widetilde{CH}^j(Y)$ associated to each flat morphism $f : Y \to X$ and a push-forward morphism $g_* : \widetilde{CH}^j(Y, L) \to \widetilde{CH}^{j+r}(X)$ associated to each proper morphism $g : Y \to X$, where $r = \dim(X) - \dim(Y)$ and L is a suitable line bundle over Y. Using this functorial behaviour, it is possible to produce a good intersection theory. This is what we do in this paper

using the classical strategy (see for example [Fu] or [Ro]). First we define an exterior product

$$\widetilde{CH}^j(X) \times \widetilde{CH}^i(Y) \to \widetilde{CH}^{i+j}(X \times Y)$$

and then a Gysin-like homomorphism $i^! : \widetilde{CH}^d(X) \to \widetilde{CH}^d(Y)$ associated to a closed embedding $i : Y \to X$ of smooth schemes. The product is then defined as the composition

$$\widetilde{CH}^j(X) \times \widetilde{CH}^i(X) \longrightarrow \widetilde{CH}^{i+j}(X \times X) \xrightarrow{\triangle^!} \widetilde{CH}^{i+j}(X)$$

where $\triangle : X \to X \times X$ is the diagonal embedding. To define the exterior product, we first note that Rost already defined an exterior product on the homology of the complex in Milnor K-theory ([Ro]). Thus it is enough to define an exterior product on the homology of the Gersten-Witt complex and show that both exterior products coincide over the quotient complex. We use the usual product on derived Witt groups ([GN]) and show that this product passes to homology using the Leibnitz rule proved by Balmer (see [Ba2]).

The definition of the Gysin-like map is done by following the ideas of Rost ([Ro]). It uses the deformation to the normal cone to modifiy any closed embedding to a nicer closed embedding and uses also the long exact sequence associated to a triple (Z, X, U) where Z is a closed subset of X and $U = X \setminus Z$.

The product that we obtain has the meaning of intersecting varieties with quadratic forms defined on them. It is therefore not a surprise that the natural map $\widetilde{CH}^{tot}(X) \to CH^{tot}(X)$ turns out to be a ring homomorphism. There is however a surprise: the product that we obtain is a priori neither commutative nor anticommutative. This comes from the fact that the product of triangulated Grothendieck-Witt groups $GW^i \times GW^j \to GW^{i+j}$ does not satisfy any commutativity property.

The organization of this paper is as follows: In section 2, we recall some basic results on triangular Witt groups. This includes the construction of the Gersten-Witt complex, and some results on products and consanguinity. In section 3, we construct the Chow-Witt groups, recall some results and prove some basic facts. The definition of the exterior product takes place in section 4 and the definition of the Gysin-Witt map in section 5. In this part, we also prove the functoriality of this map. Finally we put all the pieces together in section 6 and prove some basic results in section 7.

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1.1 Conventions

All schemes are smooth and integral over a field k of characteristic different from 2, or are localizations of such schemes. For any two schemes X and Y we will always denote by $X \times Y$ the fibre product $X \times_{(k)} Y$.

2 Preliminaries

2.1 Witt groups

We recall here some basic facts on Witt groups of triangulated categories following the exposition of [Ba2]. We suppose that for any triangulated category $\mathcal{C}$ and any objects A, B of $\mathcal{C}$ the group $\mathrm{Hom}(A, B)$ is uniquely 2-divisible. We also suppose that all triangulated categories are essentially small.

Definition 2.1. Let $\mathcal{C}$ be a triangulated category. A duality on $\mathcal{C}$ is a triple (D, δ, ϖ) where $\delta = \pm 1$, $D : \mathcal{C} \to \mathcal{C}$ is a δ-exact contravariant functor and $\varpi : 1 \simeq D^2$ is an isomorphism of functors satisfying $D(\varpi_A) \circ \varpi_{DA} = id_{DA}$ and $T(\varpi_A) = \varpi_{TA}$ for all $A \in \mathcal{C}$. A triangulated category $\mathcal{C}$ with a duality (D, δ, ϖ) is written $(\mathcal{C}, D, \delta, \varpi)$.

Example 2.2. Let X be a regular scheme and $\mathcal{P}(X)$ the category of locally free coherent $\mathcal{O}_X$-modules. Let $D^b(\mathcal{P}(X))$ be the triangulated category of bounded complexes of objects of $\mathcal{P}(X)$. Then the usual duality $^\vee$ on $\mathcal{P}(X)$ defined by $P^\vee = \mathrm{Hom}_{\mathcal{O}_X}(P, \mathcal{O}_X)$ induces a 1-exact duality on $D^b(\mathcal{P}(X))$. We also denote this derived duality by $^\vee$. Moreover, the canonical isomorphism $ev : P \to P^{\vee\vee}$ for any locally free module P induces a canonical isomorphism $\varpi : 1 \to^{\vee\vee}$ in $D^b(\mathcal{P}(X))$. More generally, if L is any invertible module over X, then the duality $\mathrm{Hom}_{\mathcal{O}_X}(\_, L)$ on $\mathcal{P}(X)$ also induces a duality on $D^b(\mathcal{P}(X))$.

Definition 2.3. Let $(\mathcal{C}, D, \delta, \varpi)$ be a triangulated category with duality. For any $i \in \mathbb{Z}$, define $(D^{(i)}, \delta^{(i)}, \varpi^{(i)})$ by $D^{(i)} = T^i \circ D$, $\delta^{(i)} = (-1)^i \delta$ and $\varpi^{(i)} = \delta^i (-1)^{i(i+1)/2} \varpi$. It is easy to check that $(D^{(i)}, \delta^{(i)}, \varpi^{(i)})$ is a duality on $\mathcal{C}$. It is called the i^{th}-shifted duality of (D, δ, ϖ).

Definition 2.4. Let $(\mathcal{C}, D, \delta, \varpi)$ be a triangulated category with duality, $A \in \mathcal{C}$ and $i \in \mathbb{Z}$. A morphism $\varphi : A \to D^{(i)} A$ is i-symmetric if the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\varpi_A^{(i)}"'] & D^{(i)} A \arrow[d, equal] \\ (D^{(i)})^2(A) \arrow[r, "D^{(i)}\varphi"'] & D^{(i)} A. \end{tikzcd}$$

The couple (A, φ) is called an i-symmetric pair.

Definition 2.5. We denote by $Symm^i(\mathcal{C})$ the monoid of isometry classes of i-symmetric pairs, equipped with the orthogonal sum.

DEFINITION 2.6. An i-symmetric form is an i-symmetric pair (A,φ) where φ is an isomorphism.

THEOREM 2.7. *Let $(\mathcal{C},D,\delta,\varpi)$ be a triangulated category with duality and let (A,ϕ) be an i-symmetric pair. Choose an exact triangle containing ϕ*

$$A \xrightarrow{\phi} D^{(i)}A \xrightarrow{\alpha} C \xrightarrow{\beta} TA.$$

Then there exists an $(i+1)$-symmetric isomorphism $\psi : C \to D^{(i+1)}C$ such that the following diagram commutes

$$\begin{tikzcd}
A \arrow[r, "\phi"] \arrow[d, "\varpi^{(i)}"'] & D^{(i)}A \arrow[r, "\alpha"] \arrow[d, equal] & C \arrow[r, "\beta"] \arrow[d, dashed, "\psi"'] & TA \arrow[d, "T\varpi^{(i)}"] \\
D^{(i)}(D^{(i)}A) \arrow[r, "D^{(i)}\phi"'] & D^{(i)}A \arrow[r, "\delta^{(i+1)}D^{(i+1)}\beta"'] & D^{(i+1)}C \arrow[r, "D^{(i+1)}\alpha"'] & T(D^{(i)}(D^{(i)}A))
\end{tikzcd}$$

where the rows are exact triangles and the second one is the dual of the first. Moreover, the $(i+1)$-symmetric form (C,ψ) is unique up to isometry. It is denoted by cone(A,ϕ).

Proof. See [Ba1], Theorem 1.6. □

Example 2.8. Let $A \in \mathcal{C}$. For any i, the morphism $0 : A \to D^{(i)}A$ is symmetric and then $\mathrm{cone}(A,0)$ is well defined.

COROLLARY 2.9. *The above construction gives a well defined homomorphism of monoids $d^i : Symm^{(i)}(\mathcal{C}) \to Symm^{(i+1)}(\mathcal{C})$ such that $d^{i+1}d^i = 0$.*

DEFINITION 2.10. Let $(\mathcal{C},D,\delta,\varpi)$ be a triangulated category with duality. The Witt group $W^i(\mathcal{C})$ is defined as $\mathrm{Ker}(d^i)/\mathrm{Im}(d^{i+1})$. Remark that $\mathrm{Ker}(d^i)$ is just the monoid of isometry classes of i-symmetric forms.

DEFINITION 2.11. Let $(\mathcal{C},D,\delta,\varpi)$ be a triangulated category with duality. The Grothendieck-Witt group $GW^i(\mathcal{C})$ is defined as the quotient of $\mathrm{Ker}(d^i)$ by the submonoid generated by the elements $\mathrm{cone}(A,\phi)-\mathrm{cone}(A,0)$ where $A \in \mathcal{C}$ and ϕ is $(i-1)$-symmetric (0 is also seen as an $(i-1)$-symmetric morphism).

Example 2.12. Let $(D^b(\mathcal{P}(X)),^\vee,1,\varpi)$ be the triangulated category with duality defined in Example 2.2. Its Witt groups are the Witt groups $W^i(X)$ of the scheme X as defined in [Ba1].

2.2 PRODUCTS

Given a pairing $\otimes : \mathcal{C} \times \mathcal{D} \to \mathcal{M}$ of triangulated categories with duality and assuming that this pairing satisfies some nice conditions, the authors of [GN] define a pairing of Witt groups. We briefly recall some definitions (see 1.2 and 1.11 in [GN]):

DEFINITION 2.13. Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{M}$ be triangulated categories. A product between $\mathcal{C}$ and $\mathcal{D}$ with codomain $\mathcal{M}$ is a covariant bi-functor

$$\otimes : \mathcal{C} \times \mathcal{D} \to \mathcal{M}$$

exact in both variables and satisfying the following condition: the functorial isomorphisms $r_{A,B} : A \otimes TB \simeq T(A \otimes B)$ and $l_{A,B} : TA \otimes B \simeq T(A \otimes B)$ make the diagram

$$\begin{tikzcd} TA \otimes TB \arrow[r, "l_{A,TB}"] \arrow[d, "r_{TA,B}"'] & T(A \otimes TB) \arrow[d, "T(r_{A,B})"] \\ T(TA \otimes B) \arrow[r, "T(l_{A,B})"'] & T^2(A \otimes B) \end{tikzcd}$$

skew-commutative.

DEFINITION 2.14. Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{M}$ be triangulated categories with dualities. Where there is no possible confusion, we drop the subscripts for D, δ and ϖ. A *dualizing pairing between $\mathcal{C}$ and $\mathcal{D}$ with codomain $\mathcal{M}$* is a product $\otimes$ with isomorphisms

$$\eta_{A,B} : DA \otimes DB \simeq D(A \otimes B)$$

natural in A and B which make the following diagrams commute

1.

$$\begin{tikzcd} A \otimes B \arrow[r, "\varpi_A \otimes \varpi_B"] \arrow[d, "\varpi_{A\otimes B}"'] & D^2A \otimes D^2B \arrow[d, "\eta_{DA,DB}"] \\ D^2(A \otimes B) \arrow[r, "D(\eta_{A,B})"'] & D(DA \otimes DB) \end{tikzcd}$$

2.

$$\begin{tikzcd} T(DTA \otimes DB) \arrow[d, "\delta_{\mathcal{C}}\delta_{\mathcal{M}}T(\eta_{TA,B})"'] & DA \otimes DB \arrow[l, "l_{DTA,DB}"'] \arrow[r, "r_{DA,DTB}"] \arrow[d, "\eta_{A,B}"] & T(DA \otimes DTB) \arrow[d, "\delta_{\mathcal{L}}\delta_{\mathcal{M}}T(\eta_{A,TB})"] \\ TD(TA \otimes B) & D(A \otimes B) \arrow[l, "TD(l_{A,B})"] \arrow[r, "TD(r_{A,B})"'] & TD(A \otimes TB). \end{tikzcd}$$

THEOREM 2.15. *Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{M}$ be triangulated categories with duality. Let $\otimes : \mathcal{C} \times \mathcal{D} \to \mathcal{M}$ be a dualizing pairing between $\mathcal{C}$ and $\mathcal{D}$ with codomain $\mathcal{M}$. Then $\otimes$ induces for all $i, j \in \mathbb{Z}$ a pairing*

$$\star : W^i(\mathcal{C}) \times W^j(\mathcal{D}) \to W^{i+j}(\mathcal{M}).$$

Proof. See [GN], Theorem 2.9. □

Example 2.16. Let $(D^b(\mathcal{P}(X)),^\vee,1,\varpi)$ be the triangulated category with duality defined in Example 2.2. The usual tensor product induces a dualizing pairing of triangulated categories and then a product $W^i(X)\times W^j(X)\to W^{i+j}(X)$. Suppose that L and N are invertible modules over X. Then $\mathrm{Hom}_{\mathcal{O}_X}(\_,L)$, $\mathrm{Hom}_{\mathcal{O}_X}(\_,N)$ and $\mathrm{Hom}_{\mathcal{O}_X}(\_,L\otimes N)$ give dualities $^\sharp$, $^\natural$ and $^\flat$ on $D^b(\mathcal{P}(X))$. The tensor product gives a dualizing pairing

$$\otimes:(D^b(\mathcal{P}(X)),^\sharp,1,\varpi)\times(D^b(\mathcal{P}(X)),^\natural,1,\varpi)\to(D^b(\mathcal{P}(X)),^\flat,1,\varpi).$$

2.3 Supports

We briefly recall the notion of triangulated category with supports following [Ba2].

Definition 2.17. Let X be a topological space. A triangulated category defined over X is a pair $(\mathcal{C},\mathrm{Supp})$ where $\mathcal{C}$ is a triangulated category and Supp assigns to each object $A\in\mathcal{C}$ a closed subset $\mathrm{Supp}(A)$ of X such that the following rules are satisfied:

(S1) $\mathrm{Supp}(A)=\emptyset \iff A\simeq 0$.

(S2) $\mathrm{Supp}(A\oplus B)=\mathrm{Supp}(A)\cup\mathrm{Supp}(B)$.

(S3) $\mathrm{Supp}(A)=\mathrm{Supp}(TA)$.

(S4) For every distinguished triangle

$$A\longrightarrow B\longrightarrow C\longrightarrow TA$$

we have $\mathrm{Supp}(C)\subset\mathrm{Supp}(A)\cup\mathrm{Supp}(B)$.

When $\mathcal{I}$ is a saturated triangulated subcategory of $\mathcal{C}$ and S is the multiplicative system of morphisms whose cone is in $\mathcal{I}$, then we can construct a support on the category $S^{-1}\mathcal{C}:=\mathcal{C}/\mathcal{I}$. This is done in [Ba3] when $\mathcal{C}$ has a tensor product. However we will only need some basic facts, so we prove the following lemma:

Lemma 2.18. *let $\mathcal{C}$ be a triangulated category defined over a topological space X. Let $\mathcal{I}$ be a saturated triangulated subcategory of $\mathcal{C}$ and let $Supp(\mathcal{I})=\cup_{A\in\mathcal{I}}Supp(A)$. Suppose that $Supp(A)\subset Supp(\mathcal{I})$ implies $A\in\mathcal{I}$. Let S be the multiplicative system in $\mathcal{C}$ of morphisms f such that $cone(f)\in\mathcal{I}$ and let*

$$\mathcal{I}\longrightarrow\mathcal{C}\longrightarrow\mathcal{C}/\mathcal{I}$$

be the exact sequence of triangulated categories obtained by inverting S. Then $\mathcal{C}/\mathcal{I}$ is a triangulated category defined over $X'=X\setminus Supp(\mathcal{I})$ (with the induced topology).

Proof. We define $\mathrm{Supp}_S(A) := \mathrm{Supp}(A) \cap X'$ for any object $A \in \mathcal{C}/\mathcal{I}$ and show that Supp_S satisfies the properties of Definition 2.17. It is easy to see that the rules (S1), (S2) and (S3) are satisfied. We only have to prove (S4).
First observe that if $s : A \to B$ is a morphism in S and

$$A \xrightarrow{s} B \longrightarrow C \longrightarrow TA$$

is an exact triangle in $\mathcal{C}$ containing s, then $\mathrm{Supp}_S(A) = \mathrm{Supp}_S(B)$ (use (S4) for the category $\mathcal{C}$). This shows that $\mathrm{Supp}_S(A) = \mathrm{Supp}_S(A')$ if $A \simeq A'$ in $\mathcal{C}/\mathcal{I}$. By definition of the triangulation of $\mathcal{C}/\mathcal{I}$, any exact triangle

$$A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow TA$$

in $\mathcal{C}/\mathcal{I}$ is isomorphic to the localization of an exact triangle in $\mathcal{C}$. This shows that $\mathrm{Supp}_S(C) \subset \mathrm{Supp}_S(A) \cup \mathrm{Supp}_S(B)$.

□

Example 2.19. Let $D^b(\mathcal{P}(X))$ be the usual triangulated category. Define the support of an object $P \in D^b(\mathcal{P}(X))$ as the union of the support of all the cohomology groups of P, i.e

$$\mathrm{Supp}(P) = \bigcup_i \mathrm{Supp}(H^i(P)).$$

Then it is easy to see that $(D^b(\mathcal{P}(X)), \mathrm{Supp})$ is a triangulated category with support. Denote by $D^b(\mathcal{P}(X))^{(k)}$ the full subcategory of $D^b(\mathcal{P}(X))$ of objects whose support is of codimension $\geq k$. Then $D^b(\mathcal{P}(X))^{(k)}$ is a saturated triangulated category and the following sequence

$$D^b(\mathcal{P}(X))^{(k)} \to D^b(\mathcal{P}(X)) \to D^b(\mathcal{P}(X))/D^b(\mathcal{P}(X))^{(k)}$$

satisfies the conditions of Lemma 2.18. So $D^b(\mathcal{P}(X))/D^b(\mathcal{P}(X))^{(k)}$ is a triangulated category over $X' = \{x \in X \mid \mathrm{codim}(x) \leq k-1\}$.

The following definitions are also due to Balmer (see [Ba2]):

Definition 2.20. Let $(\mathcal{C}, \mathrm{Supp})$ be a triangulated category over X and assume that $\mathcal{C}$ has a structure of triangulated category with duality $(\mathcal{C}, D, \delta, \varpi)$. Then we say that $\mathcal{C}$ is a triangulated category with duality defined over X if

(S5) $\mathrm{Supp}(A) = \mathrm{Supp}(DA)$ for every object A.

Definition 2.21. Let $(\mathcal{C}, \mathrm{Supp}_{\mathcal{C}})$, $(\mathcal{D}, \mathrm{Supp}_{\mathcal{D}})$ and $(\mathcal{M}, \mathrm{Supp}_{\mathcal{M}})$ be triangulated categories defined over X. Suppose that

$$\otimes : \mathcal{C} \times \mathcal{D} \to \mathcal{M}$$

is a pairing of triangulated categories. The pairing $\otimes$ is defined over X if

(S6) $\mathrm{Supp}_{\mathcal{M}}(A \otimes B) = \mathrm{Supp}_{\mathcal{C}}(A) \cap \mathrm{Supp}_{\mathcal{D}}(B)$.

Example 2.22. The triangulated category $D^b(\mathcal{P}(X))$ with the support defined in Example 2.19 and the pairing of Example 2.16 satisfy the condition (S5) and (S6).

DEFINITION 2.23. The degeneracy locus of a symmetric pair (A,α) is defined to be the support of the cone of α:

$$\mathrm{DegLoc}(\alpha) = \mathrm{Supp}(\mathrm{cone}(\alpha)).$$

DEFINITION 2.24. Let $(\mathcal{C}, \mathrm{Supp})$ be a triangulated category with duality defined over X. The consanguinity of two symmetric pairs α and β is defined to be the following subset of X:

$$\mathrm{Cons}(\alpha,\beta) = (\mathrm{Supp}(\alpha) \cap \mathrm{DegLoc}(\beta)) \cup (\mathrm{DegLoc}(\alpha) \cap \mathrm{Supp}(\beta)).$$

We are now ready to state the Leibnitz formula:

THEOREM 2.25 (Leibnitz formula). *Assume that we have a dualizing pairing $\otimes : \mathcal{C} \times \mathcal{D} \to \mathcal{F}$ of triangulated categories with dualities over X. Let α and β be two symmetric pairs such that $DegLoc(\alpha) \cap DegLoc(\beta) = \emptyset$. Then we have an isometry*

$$\delta_{\mathcal{F}} \cdot d(\alpha \star \beta) = \delta_{\mathcal{C}} \cdot d(\alpha) \star \beta + \delta_{\mathcal{D}} \cdot \alpha \star d(\beta)$$

where $\delta_{\mathcal{C}}, \delta_{\mathcal{D}}, \delta_{\mathcal{F}}$ are the signs involved in the dualities of $\mathcal{C}, \mathcal{D}$ and $\mathcal{F}$.

Proof. See [Ba2], Theorem 5.2. □

3 CHOW-WITT GROUPS

Let $(D^b(\mathcal{P}(X)),^\vee, 1, \varpi)$ be the triangulated category with the usual duality of Example 2.2 and consider its full subcategory $D^b(\mathcal{P}(X))^{(i)}$ of objects with supports of codimension $\geq i$ (here we use the support defined in Example 2.19). Then the duality on $D^b(\mathcal{P}(X))$ induces dualities on $D^b(\mathcal{P}(X))^{(i)}$ for any i ([Ba1]). It is also clear that $D^b(\mathcal{P}(X))^{(i+1)} \subset D^b(\mathcal{P}(X))^{(i)}$ for any i.

DEFINITION 3.1. For all $i \in \mathbb{N}$, denote by $D^b_i(X)$ the triangulated category $D^b(\mathcal{P}(X))^{(i)}/D^b(\mathcal{P}(X))^{(i+1)}$.

Suppose that (A,α) is an i-symmetric form in $D^b_i(X)$. Then there exists an i-symmetric pair (B,β) such that the localization of (B,β) is (A,α) (by localization we mean the map $Symm^i(D^b(\mathcal{P}(X))^{(i)}) \to Symm^i(D^b_i(X))$ induced by the functor $D^b(\mathcal{P}(X))^{(i)} \to D^b_i(X)$). Applying 2.7, we get an $(i+1)$-symmetric form (C,ψ). By construction, $C \in D^b(\mathcal{P}(X))^{(i+1)}$. Localizing this form we get a form (C,ψ) in $W^{i+1}(D^b_{i+1}(X))$. At first sight, this construction depends on some choices but in fact this is not the case (see [Ba1], Corollary 4.16). Hence we get a well defined homomorphism

$$d^i : W^i(D^b_i(X)) \to W^{i+1}(D^b_{i+1}(X)).$$

THEOREM 3.2. *Let X be a regular scheme of dimension n. Then we have a complex*

$$0 \longrightarrow W^0(D_0^b(X)) \xrightarrow{d^0} W^1(D_1^b(X)) \xrightarrow{d^1} \cdots \xrightarrow{d^n} W^n(D_n^b(X)) \longrightarrow 0.$$

Proof. See [BW], Theorem 3.1 and Paragraph 8. □

Let A be a regular local ring. We denote by $W^{fl}(A)$ the Witt group of finite length modules over A (see [QSS] for more informations about Witt groups of finite length modules). The following proposition holds:

PROPOSITION 3.3. *We have isomorphisms*

$$W^i(D_i^b(X)) \simeq \bigoplus_{x\in X^{(i)}} W^{fl}(\mathcal{O}_{X,x}).$$

Proof. See [BW], Theorem 6.1 and Theorem 6.2. □

Remark 3.4. Since we use the isomorphism of the above proposition, we briefly recall how to obtain a symmetric complex from a finite length module. For more details, see [BW] or [Fa], Chapter 3. Choose a point $x \in X^{(i)}$, a finite length $\mathcal{O}_{X,x}$-module M and a symmetric isomorphism $\phi : M \to \mathrm{Ext}^i_{\mathcal{O}_{X,x}}(M, \mathcal{O}_{X,x})$. Let $P_\bullet$ be a resolution of M by locally free coherent $\mathcal{O}_{X,x}$-modules. Then $P_\bullet$ can be chosen of the form

$$0 \longrightarrow P_i \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

Dualizing this complex and shifting i times gives the following diagram

$$\begin{tikzcd}
0 \arrow[r] & P_i \arrow[r] \arrow[d, dashed, "\exists"'] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, dashed, "\exists"'] & M \arrow[r] \arrow[d, "\phi"'] & 0 \\
0 \arrow[r] & P_0^\vee \arrow[r] & \cdots \arrow[r] & P_i^\vee \arrow[r] & \mathrm{Ext}^i_{\mathcal{O}_{X,x}}(M, \mathcal{O}_{X,x}) \arrow[r] & 0.
\end{tikzcd}$$

Using ϕ we get a symmetric morphism $\varphi : P_\bullet \to (P_\bullet)^\vee$. Thus we have constructed an i-symmetric pair in the category $D^b(\mathcal{P}(\mathcal{O}_{X,x}))$ from the pair (M, ϕ). Since $D_i^b(X) \simeq \coprod_{x\in X^{(i)}} D^b(\mathcal{P}(\mathcal{O}_{X,x}))$ ([BW], Proposition 7.1), we can see the pair $(P_\bullet, \varphi)$ as a symmetric pair in $D_i^b(X)$.

DEFINITION 3.5. The complex

$$0 \to W^{fl}(k(X)) \to \bigoplus_{x_1\in X^{(1)}} W^{fl}(\mathcal{O}_{X,x_1}) \to \cdots \to \bigoplus_{x_n\in X^{(n)}} W^{fl}(\mathcal{O}_{X,x_n}) \to 0$$

is called the Gersten-Witt complex of X. We denote it by $C(X, W)$.

This complex is obtained by using the usual duality $^\vee$ on the triangulated category $D^b(\mathcal{P}(X))$ (Example 2.2). For any invertible module L over X, we have a duality derived from the functor $^\sharp = \mathrm{Hom}_{\mathcal{O}_{X,x}}(\_, L)$ and we can apply the same construction to get a Gersten-Witt complex.

DEFINITION 3.6. Let X be a regular scheme and L an invertible $\mathcal{O}_X$-module. We denote by $C(X,W,L)$ the Gersten-Witt complex obtained from the duality $^\sharp$.

THEOREM 3.7. *Let A be a regular local k-algebra and $X = Spec(A)$. Then for any $i > 0$ we have $H^i(C(X,W)) = 0$.*

Proof. See [BGPW], Theorem 6.1. □

Let A be a regular local ring of dimension n. Denote by F the residue field of A. Then any choice of a generator $\xi \in \mathrm{Ext}^n_A(F,A)$ gives an isomorphism $\alpha_\xi : W(F) \to W^{fl}(A)$. Recall that $I(F)$ is the fundamental ideal of $W(F)$. If $n \leq 0$, put $I^n(F) = W(F)$.

DEFINITION 3.8. For any $n \in \mathbb{Z}$ let $I^n_{fl}(A)$ be the image of $I^n(F)$ by α_ξ.

Remark 3.9. It is easily seen that $I^n_{fl}(A)$ does not depend on the choice of the generator $\xi \in \mathrm{Ext}^n_A(F,A)$.

PROPOSITION 3.10. *The differential d of the Gersten-Witt complex satisfies $d(I^m_{fl}(\mathcal{O}_{X,x})) \subset I^{m-1}_{fl}(\mathcal{O}_{X,y})$ for any $m \in \mathbb{Z}$, $x \in X^{(i)}$ and $y \in X^{(i-1)}$.*

Proof. See [Gi3], Theorem 6.4 or [Fa], Theorem 9.2.4. □

DEFINITION 3.11. Let L be an invertible $\mathcal{O}_X$-module. We denote by $C(X,I^d,L)$ the complex

$$0 \to I^d_{fl}(k(X)) \to \bigoplus_{x_1 \in X^{(1)}} I^{d-1}_{fl}(\mathcal{O}_{X,x_1}) \to \cdots \to \bigoplus_{x_n \in X^{(n)}} I^{d-n}_{fl}(\mathcal{O}_{X,x_n}) \to 0.$$

Remark 3.12. In particular, we have $C(X,I^0,L) = C(X,W,L)$.

THEOREM 3.13. *Let A be an essentially smooth local k-algebra. Then for any $i > 0$ we have $H^i(C(X,I^d)) = 0$.*

Proof. See [Gi3], Corollary 7.7. □

Of course, there is an inclusion $C(X,I^{d+1},L) \to C(X,I^d,L)$ and therefore we get a quotient complex.

DEFINITION 3.14. Denote by $C(X,\overline{I}^d)$ the complex $C(X,I^d,L)/C(X,I^{d+1},L)$.

Remark 3.15. For any invertible module L the complexes $C(X,I^d)/C(X,I^{d+1})$ and $C(X,I^d,L)/C(X,I^{d+1},L)$ are *canonically* isomorphic (see [Fa], Corollary E.1.3), so we can drop the L in $C(X,\overline{I}^d)$.

Remark 3.16. The complex $C(X,\overline{I}^d)$ is of the form

$$0 \to I^d_{fl}(k(X))/I^{d+1}_{fl}(k(X)) \to \bigoplus_{x_1\in X^{(1)}} I^{d-1}_{fl}(\mathcal{O}_{X,x_1})/I^d_{fl}(\mathcal{O}_{X,x_1}) \to \ldots .$$

Remark 3.17. As a consequence of Theorem 3.13, we immediately see that $H^i(C(X,\overline{I}^d)) = 0$ for $i > 0$ if $X = \mathrm{Spec}(A)$ where A is an essentially smooth local k-algebra.

Let F be a field and denote by $K^M_i(F)$ the i-th Milnor K-theory group of F. If $i < 0$ it is convenient to put $K^M_i(F) = 0$.

DEFINITION 3.18. Let X be a scheme. Then for any d we have a complex

$$0 \to K^M_d(k(X)) \to \bigoplus_{x_1\in X^{(1)}} K^M_{d-1}(k(x_1)) \to \cdots \to \bigoplus_{x_n\in X^{(n)}} K^M_{d-n}(k(x_n)) \to 0.$$

We denote it by $C(X, K^M_d)$.

Proof. See [Ka], Proposition 1 or [Ro], Paragraph 3. □

We also have the exactness of this complex when X is the spectrum of a smooth local k-algebra:

THEOREM 3.19. *Let A be a smooth local k-algebra. Then for all $i > 0$ we have $H^i(C(X, K^M_d)) = 0$.*

Proof. See [Ro], Theorem 6.1. □

If F is a field, recall that we have a homomorphism due to Milnor

$$s : K^M_j(F) \to I^j(F)/I^{j+1}(F)$$

given by $s(\{a_1, \ldots, a_j\}) =< 1, -a_1 > \otimes \ldots \otimes < 1, -a_j >$. The following is true:

LEMMA 3.20. *The homomorphisms s induce a morphism of complexes*

$$s : C(X, K^M_d) \to C(X,\overline{I}^d).$$

Proof. See [Fa], Proposition 10.2.5. □

DEFINITION 3.21. Let $C(X, G^d, L)$ be the fibre product of $C(X, K^M_d)$ and $C(X, I^d, L)$ over $C(X,\overline{I}^d)$:

$$\begin{tikzcd} C(X,G^d,L) \arrow[r] \arrow[d] & C(X,I^d,L) \arrow[d, "\pi"] \\ C(X,K^M_d) \arrow[r, "s"'] & C(X,\overline{I}^d). \end{tikzcd}$$

Definition 3.22. Let X be a smooth scheme and L an invertible $\mathcal{O}_X$-module. The j-th Chow-Witt group $\widetilde{CH}^j(X,L)$ of X twisted by L is the group $H^j(C(X,G^j,L))$.

Remark 3.23. Denote by $GW^j(D^b_j(X),L)$ the j-th Grothendieck-Witt group of the category $D^b_j(X)$ with the duality derived from $\mathrm{Hom}_{\mathcal{O}_X}(\_,L)$ (see Definition 2.11). It is not hard to see that $C(X,G^j,L)$ is isomorphic to $GW^j(D^b_j(X),L)$ and therefore the complex $C(X,G^j,L)$ is

$$\cdots \longrightarrow C(X,G^j,L)_{j-1} \longrightarrow GW^j(D^b_j(X),L) \xrightarrow{d^j} W^{j+1}(D^b_{j+1}(X),L) \longrightarrow \cdots$$

Hence $\widetilde{CH}^j(X,L)$ is a quotient of $\mathrm{Ker}(d^j)$ and a subquotient of $GW^j(D^b_j(X),L)$.

We also have the exactness of the complex $C(X,G^d,L)$ in the local case:

Theorem 3.24. *Let A be a smooth local k-algebra and $X = Spec(A)$. Then $H^i(C(X,G^j)) = 0$ for all j and all $i > 0$.*

Proof. As $C(X,G^j)$ is the fibre product of the complexes $C(X,K^M_j)$ and $C(X,I^j)$ over $C(X,\overline{I}^j)$, we have an exact sequence of complexes

$$0 \longrightarrow C(X,G^j) \longrightarrow C(X,I^j) \oplus C(X,K^M_j) \longrightarrow C(X,\overline{I}^j) \longrightarrow 0$$

inducing a long exact sequence in cohomology. It follows then from Theorem 3.13 and Theorem 3.19 that $H^i(C(X,G^j)) = 0$ if $i > 1$. For $i = 1$, we have an exact sequence

$$H^0(C(X,I^j)) \oplus H^0(C(X,K^M_j)) \to H^0(C(X,\overline{I}^j)) \to H^1(C(X,G^j)) \to 0.$$

The exact sequence of complexes

$$0 \longrightarrow C(X,I^{j+1}) \longrightarrow C(X,I^j) \longrightarrow C(X,\overline{I}^j) \longrightarrow 0$$

shows that $H^0(C(X,I^j))$ maps onto $H^0(C(X,\overline{I}^j))$. □

Definition 3.25. Let X be a smooth scheme and L an invertible $\mathcal{O}_X$-module. We define the sheaf $\mathcal{G}^j_L$ on X by $\mathcal{G}^j_L(U) = H^0(C(U,G^j,L))$.

We have:

Theorem 3.26. *Let X be a smooth scheme of dimension n. Then for any i we have*

$$H^i_{Zar}(X,\mathcal{G}^j_L) \simeq H^i(C(X,G^j,L)).$$

Proof. Define sheaves $\mathcal{C}_l$ by $\mathcal{C}_l(U) = C(U, G^j, L)_l$ for any $l \geq 0$. It is clear that the $\mathcal{C}_l$ are flasque sheaves. We have a complex of sheaves over X

$$0 \longrightarrow \mathcal{G}_L^j \longrightarrow \mathcal{C}_0 \longrightarrow \mathcal{C}_1 \longrightarrow \cdots \longrightarrow \mathcal{C}_n \longrightarrow 0.$$

Theorem 3.24 shows that this complex is a flasque resolution of $\mathcal{G}_L^j$. Thus the theorem is proved.

□

Suppose that $f : X \to Y$ is a flat morphism. Since it preserves codimensions, it induces a morphism of complexes

$$f^* : C(Y, G^j, L) \to C(X, G^j, f^*L)$$

for any $j \in \mathbb{N}$ and any line bundle L over Y ([Fa], Corollary 10.4.2). Hence we have:

THEOREM 3.27. *Let $f : X \to Y$ be a flat morphism and L a line bundle over Y. Then, for any i, j we have homomorphisms*

$$f^* : H^i(C(Y, G^j, L)) \to H^i(C(X, G^j, f^*L)).$$

In particular, if E is a vector bundle over Y and $\pi : E \to Y$ is the projection, we have isomorphisms

$$\pi^* : H^i(C(Y, G^j, L)) \to H^i(C(E, G^j, \pi^*L)).$$

Proof. We have a morphism of complexes $f^* : C(Y, G^j, L) \to C(X, G^j, f^*L)$ which gives the induced homomorphisms in cohomology. For the proof of homotopy invariance, see Corollary 11.3.2 in [Fa]. □

PROPOSITION 3.28. *Let $f : X \to Y$ and $g : Y \to Z$ be flat morphisms. Then $(gf)^* = f^*g^*$.*

Proof. See [Fa], Proposition 3.4.9. □

Suppose that $f : X \to Y$ is a finite morphism with $\dim(Y) - \dim(X) = r$. Consider the morphism of locally ringed spaces $\overline{f} : (X, \mathcal{O}_X) \to (Y, f_*\mathcal{O}_X)$ induced by f. If X is smooth, then $L = \overline{f}^* \mathrm{Ext}^r_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{O}_Y)$ is an invertible module over Y ([Gi2], Corollary 6.6) and we get a morphism of complexes (of degree r)

$$f_* : C(X, G^{j-r}, L \otimes f^*N) \to C(Y, G^j, N)$$

for any invertible module N over Y ([Fa], Corollary 5.3.7).

Proposition 3.29. *Let $f : X \to Y$ be a finite morphism between smooth schemes. Let $dim(Y) - dim(X) = r$ and N be an invertible module over Y. Then the morphism of complexes f_* induces a homomorphism*

$$f_* : H^{i-r}(C(X, G^{j-r}, L \otimes f^*N)) \to H^i(C(Y, G^j, N)).$$

In particular, we have ([Fa], Remark 9.3.5):

Proposition 3.30. *Let $f : X \to Y$ be a closed immersion of codimension r between smooth schemes. Then f induces an isomorphism*

$$f_* : H^{i-r}(C(X, G^{j-r}, L \otimes f^*N)) \to H^i_X(C(Y, G^j, N))$$

for any i, j and any invertible module N over Y.

Important remark 3.31. If $f : X \to Y$ is a closed immersion, then f_* will always be the map with support:

$$f_* : H^{i-r}(C(X, G^{j-r}, L \otimes f^*N)) \to H^i_X(C(Y, G^j, N))$$

The transfer for finite morphisms is functorial ([Fa], proposition 5.3.8):

Proposition 3.32. *Let $f : X \to Y$ and $g : Y \to Z$ be finite morphisms. Then $g_*f_* = (gf)_*$.*

Remark 3.33. Let X be a smooth scheme and D be a smooth effective Cartier divisor on X. Let $i : D \to X$ be the inclusion and $L(D)$ be the line bundle over X associated to D. Then there is a canonical section $s \in L(D)$ (see [Fu], Appendix B.4.5) and an exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{s} L(D) \longrightarrow i_*\mathcal{O}_D \longrightarrow 0.$$

Applying $\mathrm{Hom}_{\mathcal{O}_X}(\_, L(D))$ and shifting, we obtain the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X \arrow[r, "s"] \arrow[d, "\simeq"'] & L(D) \arrow[r] \arrow[d, "\simeq"'] & i_*\mathcal{O}_D \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{O}_X}(L(D), L(D)) \arrow[r, "s"'] & \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, L(D)) \arrow[r] & \mathrm{Ext}^1_{\mathcal{O}_X}(i_*\mathcal{O}_D, L(D)) \arrow[r] & 0
\end{tikzcd}$$

which shows that $\mathrm{Ext}^1_{\mathcal{O}_X}(i_*\mathcal{O}_D, \mathcal{O}_X) \otimes L(D) \simeq i_*\mathcal{O}_D$. Proposition 3.30 shows that we then have an isomorphism

$$i_* : H^{i-1}(C(D, G^{j-1}, i^*L(D))) \to H^i_D(C(X, G^j)).$$

Lemma 3.34. *Let $g : X \to Y$ be a flat morphism and $f : Z \to Y$ a finite morphism. Consider the following fibre product*

$$\begin{array}{ccc} V & \xrightarrow{f'} & X \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Z & \xrightarrow[f]{} & Y. \end{array}$$

Then $(f')_*(g')^* = g^* f_*$.

Proof. See [Fa], Corollary 12.2.8. □

Remark 3.35. Of course, in the above fibre product we suppose that V is also smooth and integral. Such a strong assumption is not necessary in general, but this case is sufficient for our purposes.

Remark 3.36. It is possible to define a map f_* when the morphism f is proper (see [Fa]) but we don't use this fact here.

4 The exterior product

Let X and Y be two schemes. The fibre product $X \times Y$ comes equipped with two projections $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$.

Lemma 4.1. *For any* $i, j \in \mathbb{N}$, *the pairing*

$$\boxtimes : D_i^b(X) \times D_j^b(Y) \to D_{i+j}^b(X \times Y)$$

given by $P \boxtimes Q = p_1^* P \otimes p_2^* Q$ *is a dualizing pairing of triangulated categories with duality.*

Proof. Straight verification. □

Corollary 4.2. *For any* $i, j \in \mathbb{N}$, *the pairing*

$$\boxtimes : D_i^b(X) \times D_j^b(Y) \to D_{i+j}^b(X \times Y)$$

induces a pairing

$$\star : W^i(D_i^b(X)) \times W^j(D_j^b(Y)) \to W^{i+j}(D_{i+j}^b(X \times Y)).$$

Proof. Clear by Theorem 2.15. □

Corollary 4.3. *Let* $\psi \in W^j(D_j^b(Y))$. *Then we have a homomorphism*

$$\mu_\psi : W^i(D_i^b(X)) \to W^{i+j}(D_{i+j}^b(X \times Y))$$

given by $\mu_\psi(\varphi) = \varphi \star \psi$.

Recall that we have isomorphisms $W^i(D_i^b(X)) \simeq \bigoplus_{x \in X^{(i)}} W^{fl}(\mathcal{O}_{X,x})$ (Proposition 3.3).

Definition 4.4. For any $s \in \mathbb{Z}$, denote by $I^s(D^b_i(X))$ the preimage of $\bigoplus_{x \in X^{(i)}} I^s_{fl}(\mathcal{O}_{X,x})$ under the above isomorphism.

Proposition 4.5. *For any $m, p \in \mathbb{N}$ the product*

$$\star : W^i(D^b_i(X)) \times W^j(D^b_j(Y)) \to W^{i+j}(D^b_{i+j}(X \times Y))$$

induces a product

$$\star : I^m(D^b_i(X)) \times I^n(D^b_j(Y)) \to I^{m+n}(D^b_{i+j}(X \times Y)).$$

Proof. Let $x \in X^{(i)}$ and $y \in Y^{(j)}$. It is clear that the product can be computed locally (use [GN], Theorem 3.2). So we can suppose that $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ where A and B are local in x and y respectively. Recall that we have the following diagram

$$\begin{tikzcd} X \times Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d] \\ X \arrow[r] & \mathrm{Spec}(k). \end{tikzcd}$$

Let P be an A-projective resolution of $k(x)$ and Q be a B-projective resolution of $k(y)$. Consider a symmetric form $\rho : k(x) \to \mathrm{Ext}^i_A(k(x), A)$ and a symmetric form $\mu : k(y) \to \mathrm{Ext}^j_B(k(y), B)$. Then $p_1^*(\rho)$ is a symmetric isomorphism supported by the complex $P \otimes_k B$ and $p_2^*(\mu)$ is a symmetric isomorphism supported by the complex $A \otimes_k Q$. The complex $(P \otimes_k B) \otimes_{A \otimes_k B} (A \otimes_k Q)$ (which is isomorphic to $P \otimes_k Q$) has its homology concentrated in degree 0, and this homology is isomorphic to $k(x) \otimes_k k(y)$. Let u be a point of $\mathrm{Spec}(k(x) \otimes_k k(y))$. Then the restriction of $p_1^*\rho \otimes p_2^*\mu$ to u is a finite length module M whose support is on u with a symmetric form

$$M \to \mathrm{Ext}^{i+j}_{(A \otimes B)_u}(M, (A \otimes B)_u).$$

Taking its class in the Witt group, we obtain a $k(u)$-vector space V with a symmetric form $\psi : V \to \mathrm{Ext}^{i+j}_{(A \otimes B)_u}(V, (A \otimes B)_u)$. Now choose a unit $a \in k(x)^\times$. Consider the image a_u of a under the homomorphism $k(x) \to k(u)$. The class of $p_1^*(a\rho) \otimes p_2^*(\mu)$ is the symmetric form

$$a_u\psi : V \to \mathrm{Ext}^{i+j}_{(A \otimes B)_u}(V, (A \otimes B)_u).$$

As the same property holds for any unit $b \in k(y)^\times$, we conclude that

$$p_1^*(< 1, -a_1 > \otimes \ldots \otimes < 1, -a_n > \rho) \otimes p_2^*(< 1, -b_1 > \otimes \ldots \otimes < 1, -b_m > \mu)$$

is equal to $< 1, -(a_1)_u > \otimes \ldots \otimes < 1, -(b_m)_u > \psi$. □

Recall that for any scheme X we have a Gersten-Witt complex (Definition 3.5)

$$C(X,W): \qquad \cdots \longrightarrow W^r(D_r^b(X)) \xrightarrow{d_X^r} W^{r+1}(D_{r+1}^b(X)) \longrightarrow \cdots$$

and a complex $C(X,I^d)$:

$$\cdots \longrightarrow \bigoplus_{x_r\in X^{(r)}} I_{fl}^{d-r}(\mathcal{O}_{X,x_r}) \longrightarrow \bigoplus_{x_{r+1}\in X^{(r+1)}} I_{fl}^{d-r-1}(\mathcal{O}_{X,x_{r+1}}) \longrightarrow \cdots.$$

The above proposition gives:

COROLLARY 4.6. *The product*

$$\star : C(X,W)\times C(Y,W)\to C(X\times Y,W)$$

induces for any $r,s\in N$ a product

$$\star : C(X,I^r)\times C(Y,I^s)\to C(X\times Y,I^{r+s}).$$

Now we investigate the relations between $\star$ and the differentials of the complexes.

PROPOSITION 4.7. *Let $\psi\in W^j(D_j^b(Y))$ be such that $d_Y^j(\psi)=0$. Then the following diagram commutes*

$$\begin{tikzcd}
W^i(D_i^b(X)) \arrow[r, "d_X^i"] \arrow[d, "(-1)^j\mu_\psi"'] & W^{i+1}(D_{i+1}^b(X)) \arrow[d, "\mu_\psi"] \\
W^{i+j}(D_{i+j}^b(X\times Y)) \arrow[r, "d_{X\times Y}^{i+j}"'] & W^{i+j+1}(D_{i+j+1}^b(X\times Y)).
\end{tikzcd}$$

Proof. Let $\varphi\in W^i(D_i^b(X))$. Let $X^{(\geq i+1)}$ be the set of points of X of codimension $\geq i+1$, $Y^{(\geq j+1)}$ the points of Y of codimension $\geq j+1$ and $(X\times Y)^{(\geq i+j+1)}$ the set of points of $X\times Y$ of codimension $\geq i+j+1$. By Lemma 2.18, the triangulated categories $D_i^b(X), D_j^b(Y)$ and $D_{i+j}^b(X\times Y)$ are defined over the topological spaces $X\setminus X^{(\geq i+1)}$, $Y\setminus Y^{(\geq j+1)}$ and $(X\times Y)\setminus(X\times Y)^{(\geq i+j+1)}$. Let $\alpha\in Symm^i(D^b(\mathcal{P}(X))^{(i)})$ and $\beta\in Symm^j(D^b(\mathcal{P}(Y))^{(j)})$ be symmetric pairs representing φ and ψ. By definition, $\mathrm{DegLoc}(\alpha)$ is of codimension $\geq i+1$, $\mathrm{DegLoc}(\beta)$ is of codimension $\geq j+1$ and $d\beta$ is neutral. It is easily seen that $\mathrm{Supp}(dp_1^*\alpha)\cap\mathrm{Supp}(dp_2^*\beta)=\emptyset$ in the topological space $(X\times Y)\setminus(X\times Y)^{(\geq i+j+1)}$. Theorem 2.25 implies that

$$(-1)^{i+j}d(p_1^*\alpha\star p_2^*\beta)=(-1)^i dp_1^*\alpha\star p_2^*\beta+(-1)^j p_1^*\alpha\star dp_2^*\beta.$$

Using Theorem 2.15, we see that we have in $W^{i+j}(D^b_{i+j}(X\times Y))$ the equality

$$(-1)^j d^{i+j}_{X\times Y}(p_1^*\varphi \star p_2^*\psi) = p_1^* d^i_X(\varphi)\star p_2^*\psi.$$

□

The following corollary is obvious.

Corollary 4.8. *Let $\psi \in I^m(D^b_j(Y))$ be such that $d^Y_j(\psi)=0$. Then the following diagram commutes*

$$\begin{tikzcd} I^p(D^b_i(X)) \arrow[r, "d^i_X"] \arrow[d, "(-1)^j\mu_\psi"'] & I^{p-1}(D^b_{i+1}(X)) \arrow[d, "\mu_\psi"] \\ I^{p+m}(D^b_{i+j}(X\times Y)) \arrow[r, "d^{i+j}_{X\times Y}"'] & I^{p+m-1}(D^b_{i+j+1}(X\times Y)). \end{tikzcd}$$

We now have to deal with the complex in Milnor K-theory. Let $C(X,K^M_r)$, $C(Y,K^M_s)$ and $C(X\times Y,K^M_{r+s})$ be the complexes in Milnor K-theory associated to X,Y and $X\times Y$. In [Ro], Rost defines a product

$$\odot : C(X,K^M_r)^i \times C(Y,K^M_s)^j \to C(X\times Y,K^M_{r+s})^{i+j}$$

as follows: Let $u\in (X\times Y)^{(i+j)}$, $x\in X^{(i)}$, $y\in Y^{(j)}$ be such that x and y are the projections of u. Let $\rho=\{a_1,\ldots,a_{r-i}\}\in K^M_{r-i}(k(x))$ and $\mu=\{b_1,\ldots,b_{s-j}\}\in K^M_{s-j}(k(y))$. Then

$$(\rho\odot\mu)_u = l((k(x)\otimes_k k(y))_u)\{(a_1)_u,\ldots,(a_{r-i})_u,(b_1)_u,\ldots,(b_{s-j})_u\}$$

where the $(a_l)_u$ and $(b_t)_u$ are the images of the a_l and b_t under the inclusions $k(x)\to k(u)$ and $k(y)\to k(u)$, and $l((k(x)\otimes_k k(y))_u)$ is the length of the module $k(x)\otimes_k k(y)$ localized in u.

Lemma 4.9. *For any $\rho\in C(X,K^M_r)^i$ and $\mu\in C(Y,K^M_s)^j$ we have*

$$d(\rho\odot\mu) = d(\rho)\odot\mu + (-1)^j\rho\odot d(\mu).$$

Proof. See [Ro], Paragraph 14.4. □

Corollary 4.10. *Let $\mu\in C(Y,K^M_s)^j$ be such that $d\mu=0$. Then the following diagram commutes:*

$$\begin{tikzcd} C(X,K^M_r)^i \arrow[r, "d^i_X"] \arrow[d, "\odot\mu"'] & C(X,K^M_r)^{i+1} \arrow[d, "\odot\mu"] \\ C(X\times Y,K^M_{r+s})^{i+j} \arrow[r, "d^{i+j}_{X\times Y}"'] & C(X\times Y,K^M_{r+s})^{i+j+1}. \end{tikzcd}$$

Proof. Obvious. □

Now we compare the products $\star$ and $\odot$.

PROPOSITION 4.11. *The following diagram commutes:*

$$\begin{tikzcd} C(X,K_r^M)^i\times C(Y,K_s^M)^j \arrow[r, "\odot"] \arrow[d, "s_{(r-i)}\times s_{(s-j)}"'] & C(X\times Y,K_{r+s}^M)^{i+j} \arrow[d, "s_{(r+s-i-j)}"] \\ C(X,\overline{I}^r)^i\times C(Y,\overline{I}^s)^j \arrow[r, "\star"'] & C(X\times Y,\overline{I}^{r+s})^{i+j}. \end{tikzcd}$$

Proof. Let $\{a_1,\ldots,a_{r-i}\}\in K_{r-i}^M(k(x))$ and $\{b_1,\ldots,b_{s-j}\}\in K_{s-j}^M(k(y))$. Let ρ' be a symmetric isomorphism

$$\rho':k(x)\to \mathrm{Ext}^i_{\mathcal{O}_{X,x}}(k(x),\mathcal{O}_{X,x})$$

and μ' a symmetric isomorphism

$$\mu':k(y)\to \mathrm{Ext}^j_{\mathcal{O}_{Y,y}}(k(y),\mathcal{O}_{Y,y}).$$

We then have $\rho:=s_{(r-i)}(\{a_1,\ldots,a_{r-i}\})=<1,-a_1>\otimes\ldots\otimes<1,-a_{r-i}>\rho'$ and $\mu:=s_{(s-j)}(\{b_1,\ldots,b_{s-j}\})=<1,-b_1>\otimes\ldots\otimes<1,-b_{s-j}>\mu'$. Choose a point u in $(X\times Y)^{(i+j)}$ lying over x and y. The proof of Proposition 4.5 shows that

$$(\rho\star\mu)_u=s_{(r+s-i-j)}(\{(a_1)_u,\ldots,(a_{r-i})_u,(b_1)_u,\ldots,(b_{s-j})_u\})\varphi$$

where $\varphi:M\to\mathrm{Ext}^{i+j}_{\mathcal{O}_{X\times Y,u}}(M,\mathcal{O}_{X\times Y,u})$ is a symmetric isomorphism and M is a $k(u)$-vector space. But $\dim_{k(u)}M\equiv l((k(x)\otimes k(y))_u)\pmod 2$ where l denotes the length. So we have in $C(X\times Y,\overline{I}^{r+s})^{i+j}$ the equality

$$(\rho\star\mu)_u=s_{(r+s-i-j)}(\{(a_1)_u,\ldots,(a_{r-i})_u,(b_1)_u,\ldots,(b_{s-j})_u\})l((k(x)\otimes k(y))_u).$$

The right hand term is equal to $s_{(r+s-i-j)}(\{a_1,\ldots,a_{r-i}\}\odot\{b_1,\ldots,b_{s-j}\})$ by definition. □

COROLLARY 4.12. *The products*

$$\star:C(X,I^r)\times C(Y,I^s)\to C(X\times Y,I^{r+s})$$

and

$$\odot:C(X,K_r^M)\times C(Y,K_s^M)\to C(X\times Y,K_{r+s}^M)$$

give a product

$$\diamond:C(X,G^r)\times C(Y,G^s)\to C(X\times Y,G^{r+s}).$$

COROLLARY 4.13. *Let $\mu \in C(Y, G^s)^j$ such that $d_Y^j \mu = 0$. Then μ induces a product*

$$\_ \diamond \mu : H^i(C(X, G^r)) \to H^{i+j}(C(X \times Y, G^{r+s})).$$

Proof. This a direct consequence of Proposition 4.11, Corollary 4.8 and Corollary 4.10. □

Next we have to check that $\_ \diamond \mu$ is well defined on the cohomolgy class of μ.

LEMMA 4.14. *Let $\gamma \in C(Y, G^s)^{j-1}$ and $\mu = d_Y^{j-1}\gamma$. Then $\_ \diamond \mu = 0$.*

Proof. Suppose that α is such that $d_X^i \alpha = 0$. By Corollary 4.8 and Corollary 4.10 we have up to signs $d_{X \times Y}^{i+j-1}(\alpha \diamond \gamma) = \alpha \diamond d^{j-1}\gamma = \alpha \diamond \mu$. So $\alpha \diamond \mu$ is trivial in $H^{i+j}(C(X \times Y, G^{r+s}))$. □

Finally:

THEOREM 4.15. *Let X and Y be smooth schemes. Then for any $i, j, r, s \in \mathbb{N}$ the product*

$$\diamond : C(X, G^r) \times C(Y, G^s) \to C(X \times Y, G^{r+s})$$

induces an exterior product

$$\times : H^i(C(X, G^r)) \times H^j(C(Y, G^s)) \to H^{i+j}(C(X \times Y, G^{r+s})).$$

This exterior product can also be defined with complexes twisted by invertible modules.

THEOREM 4.16. *Let X and Y be smooth schemes. Let L and N be invertible modules over X and Y respectively. For any $i, j, r, s \in \mathbb{N}$, the pairing*

$$\diamond : C(X, G^r, L) \times C(Y, G^s, N) \to C(X \times Y, G^{r+s}, p_1^* L \otimes p_2^* N)$$

induces an exterior product

$$\times : H^i(C(X, G^r, L)) \times H^j(C(Y, G^s, N)) \to H^{i+j}(C(X \times Y, G^{r+s}, p_1^* L \otimes p_2^* N)).$$

Proof. Left to the reader. □

If $i = r$ and $j = s$, we obtain the following corollary:

COROLLARY 4.17. *Let X and Y be smooth schemes. Then for any $i, j \in \mathbb{N}$ the product*

$$\diamond : C(X, G^i) \times C(Y, G^j) \to C(X \times Y, G^{i+j})$$

gives an exterior product

$$\times : \widetilde{CH}^i(X) \times \widetilde{CH}^j(Y) \to \widetilde{CH}^{i+j}(X \times Y).$$

Next we prove some properties of this exterior product:

PROPOSITION 4.18. *The exterior product $\times$ is associative.*

Proof. It clearly suffices to prove that the exterior products $\star$ and $\odot$ are associative. For $\star$ this is clear because of the associativity of the tensor product (up to isomorphism). For the second, see (14.2) in [Ro]. $\square$

Now we deal with the commutativity. Let X and Y be smooth schemes and let $\tau : X \times Y \to Y \times X$ be the flip. We have:

LEMMA 4.19. *Let $\mu \in H^i(C(X, K_r^M))$ and $\eta \in H^j(C(Y, K_s^M))$. Then we have $\tau^*(\eta \odot \mu) = (-1)^{(r-i)(s-j)}(\mu \odot \eta)$.*

Proof. This is clear from the definition. $\square$

LEMMA 4.20. *Let $\mu \in H^i(C(X, I^r))$ and $\eta \in H^j(C(Y, I^s))$. Then we have $\tau^*(\eta \star \mu) = (-1)^{ij}(\mu \star \eta)$.*

Proof. It is clear by the skew-commutativity of the product of Witt groups ([GN], Theorem 3.1). $\square$

Remark 4.21. Of course, the associativity and the anticommutativity of the exterior product are also true for the twisted product of Theorem 4.16.

5 INTERSECTION WITH A SMOOTH SUBSCHEME

5.1 THE GYSIN-WITT MAP

The goal of this section is to define for any closed embedding $i : Y \to X$ of smooth schemes a Gysin-Witt map $i^! : H^r(C(X, G^j)) \to H^r(C(Y, G^j))$. In order to define such a map, we adapt the ideas of Rost ([Ro], Paragraph 11).
First we briefly recall the properties of the deformation to the normal cone. For more details, see [Fu] (Chapter 5) or [Ro] (Chapter 10). Let Y be a closed subscheme of a smooth scheme X. Then there is a smooth scheme $D(X, Y)$, a closed imbedding $j : Y \times \mathbb{A}^1 \hookrightarrow D(X, Y)$ and a flat morphism $\rho : D(X, Y) \to \mathbb{A}^1$ such that the following diagram commutes

$$\begin{tikzcd} Y \times \mathbb{A}^1 \arrow[r, "j"] \arrow[dr, "pr"'] & D(X,Y) \arrow[d, "\rho"] \\ & \mathbb{A}^1 \end{tikzcd}$$

and

(1) $\rho^{-1}(\mathbb{A}^1 - 0) = X \times (\mathbb{A}^1 - 0)$ and the restriction of j is the closed imbedding $i \times Id : Y \times (\mathbb{A}^1 - 0) \hookrightarrow X \times (\mathbb{A}^1 - 0)$.

(2) $\rho^{-1}(0) = N_Y X$, where $N_Y X$ is the normal cone to Y in X and the restriction of j is the embedding as the zero section $s_0 : Y \to N_Y X$.

The scheme $D(X,Y)$ can be obtained as follows: Consider the blow-up M of $X \times \mathbb{A}^1$ along $Y \times 0$ and the blow-up $\tilde{X}$ of $X \times 0$ along $Y \times 0$. Then define $D(X,Y)$ to be $M \setminus \tilde{X}$.
If Y is smooth in a smooth scheme X, then it is locally of complete intersection and $N_Y X$ is a vector bundle over Y of rank $\dim(X) - \dim(Y)$. Moreover, $N_Y X$ is Cartier divisor on $D(X,Y)$. If $\mathbb{A}^1 = \mathrm{Spec}(k[t])$, then the projection $\rho : D(X,Y) \to \mathbb{A}^1$ gives a homomorphism $k[t] \to \mathcal{O}_{D(X,Y)}(D(X,Y))$. We still denote by t the image of t under this homomorphism. We have an exact sequence

$$0 \longrightarrow \mathcal{O}_{D(X,Y)} \xrightarrow{t} \mathcal{O}_{D(X,Y)} \longrightarrow \kappa_* \mathcal{O}_{N_Y X} \longrightarrow 0$$

where $\kappa : N_Y X \to D(X,Y)$ is the inclusion. Remark 3.33 shows that $\mathrm{Ext}^1_{\mathcal{O}_{D(X,Y)}}(\kappa_* \mathcal{O}_{N_Y X}, \mathcal{O}_{D(X,Y)}) \simeq \kappa_* \mathcal{O}_{N_Y X}$ with generator the Koszul complex associated to the global section t.
Let $U = \mathbb{A}^1 - 0$ and consider the form

$$< 1, -t >: \mathcal{O}_U^2 \to \mathcal{O}_U^2$$

in $W^0(D^b(U))$. Now let X be a smooth scheme and consider the projection $\eta : X \times U \to U$. Then $\eta^*(< 1, -t >) \in W^0(D^b(X \times U))$ and we also denote it by $< 1, -t >$. Since the support of this form is $X \times U$, the tensor product gives a functor

$$< 1, -t > \otimes \_ : D_i^b(X \times U) \to D_i^b(X \times U).$$

Using the fact that $< 1, -t >$ is symmetric, we see that this functor is duality preserving (see [GN], Definition 1.8 and Lemma 1.14) and therefore induces for any i a homomorphism

$$< 1, -t > \otimes \_ : W^i(D_i^b(X \times U)) \to W^i(D_i^b(X \times U)).$$

For some sign reasons that will be made clearer in Lemma 5.10, we will in fact consider for any i the homomorphism

$$m_t : W^i(D_i^b(X \times U)) \to W^i(D_i^b(X \times U))$$

defined by $m_t(\alpha) = (-1)^{i+1} < 1, -t > \otimes \alpha$.

Lemma 5.1. *For any $i, j \in \mathbb{N}$ the homomorphism m_t induces a homomorphism*

$$I^j(D_i^b(X \times U)) \to I^{j+1}(D_i^b(X \times U))$$

and the following diagram commutes

$$\begin{array}{ccc} I^j(D_i^b(X\times U)) & \xrightarrow{d^i} & I^{j-1}(D_{i+1}^b(X\times U)) \\ \downarrow{\scriptstyle -m_t} & & \downarrow{\scriptstyle m_t} \\ I^{j+1}(D_i^b(X\times U)) & \xrightarrow[d^i]{} & I^j(D_{i+1}^b(X\times U)). \end{array}$$

Proof. The first assertion is clear. Now $< 1, -t >$ is a global isomorphism and we can use Theorem 2.10 in [GN] (or Theorem 2.25 in the present paper) to see that

$$d^i(< 1, -t > \otimes\alpha) =< 1, -t > \otimes d^i\alpha$$

for any $\alpha \in I^j(D_i^b(X\times U))$. The first term is $(-1)^{i+1}d^i(m_t(\alpha))$ and the second one is $(-1)^{i+2}m_t(d^i\alpha)$. □

Now consider $t \in \mathcal{O}^*_{X\times U}$. For any i and any $x \in X\times U$, we have a multiplication by t:

$$n_t : K_i^M(k(x)) \to K_{i+1}^M(k(x))$$

defined by $n_t(\{a_1,\ldots,a_i\}) = \{t, a_1,\ldots,a_i\}$.

Lemma 5.2. *For any $i, j \in N$ the following diagram commutes*

$$\begin{array}{ccc} C(X\times U, K_j^M)^i & \xrightarrow{d^i} & C(X\times U, K_j^M)^{i+1} \\ \downarrow{\scriptstyle -n_t} & & \downarrow{\scriptstyle n_t} \\ C(X\times U, K_{j+1}^M)^i & \xrightarrow[d^i]{} & C(X\times U, K_{j+1}^M)^{i+1}. \end{array}$$

Proof. See [Ro], Proposition 4.6. □

Corollary-Definition 5.3. *The homomorphisms m_t and n_t induce for any $i, j \in N$ a homomorphism*

$$\{t\} : H^i(C(X\times U, G^j)) \to H^i(C(X\times U, G^{j+1})).$$

We call this homomorphism multiplication by t*.*

Proof. It suffices to show that m_t and n_t give the same operation on the complex $C(X\times U, \overline{I}^j)$. It is straightforward. □

We will need the following lemma:

Lemma 5.4. *Let $f : X \to Y$ be a flat morphism of smooth schemes. Then for any i, j the following diagram commutes*

$$\begin{tikzcd} H^i(C(Y\times U,G^j)) \arrow[r, "\{t\}"] \arrow[d, "(f\times Id)^*"'] & H^i(C(Y\times U,G^{j+1})) \arrow[d, "(f\times Id)^*"] \\ H^i(C(X\times U,G^j)) \arrow[r, "\{t\}"'] & H^i(C(X\times U,G^{j+1})). \end{tikzcd}$$

Proof. First observe that $(f\times Id)^*(<1,-t>) = <1,-t>$ by definition. Then for any $\alpha \in I^r(D_i^b(X\times U))$ we have $(f\times Id)^*(m_t\alpha) = m_t((f\times Id)^*\alpha)$ (use [GN], Theorem 3.4). On the other hand, we have $(f\times Id)^*(n_t(\alpha)) = n_t((f\times Id)^*\alpha)$ for any $\alpha \in K_r^M(k(y))$ ([Ro], Lemma 4.3). Putting this together, we get the conclusion. □

Let $Y\to X$ be a closed embedding of smooth schemes and consider the deformation to the normal cone space $D(X,Y)$. Then N_YX is a Cartier divisor and its complement in $D(X,Y)$ is $X\times U$. We have a long exact sequence associated to this triple ([Fa], Corollary 10.4.9):

$$H^i(C(D(X,Y),G^{j+1})) \to H^i(C(X\times U,G^{j+1})) \xrightarrow{\partial} H^{i+1}_{N_YX}(C(D(X,Y),G^{j+1}))$$

Combining the isomorphism of Proposition 3.30 and the isomorphism

$$\mathcal{O}_{N_YX} \to \overline{\kappa}^*\mathrm{Ext}^1_{\mathcal{O}_{D(X,Y)}}(\kappa_*\mathcal{O}_{N_YX},\mathcal{O}_{D(X,Y)})$$

mapping 1 to the Koszul complex associated to the global section t of $\mathcal{O}_{D(X,Y)}$, we finally get an isomorphism

$$\kappa_* : H^i(C(N_YX,G^j)) \to H^{i+1}_{N_YX}(C(D(X,Y),G^{j+1})).$$

Let $q: N_YX\to Y$ and $\pi: X\times U\to X$ be the projections and consider the following composition:

$$\begin{tikzcd} H^i(C(X,G^j)) \arrow[rr, dashed] \arrow[d, "\pi^*"'] & & H^i(C(Y,G^j)) \\ H^i(C(X\times U,G^j)) \arrow[r, "\{t\}"'] & H^i(C(X\times U,G^{j+1})) \arrow[r, "(\kappa_*)^{-1}\partial"'] & H^i(C(N_YX,G^j)). \arrow[u, "(q^*)^{-1}"'] \end{tikzcd}$$

Definition 5.5. Let Y be a smooth subscheme of a smooth scheme X with inclusion $i: Y\to X$. We denote by $i^!: H^r(C(X,G^j)) \to H^r(C(Y,G^j))$ and call *Gysin-Witt map* the composition $(q^*)^{-1}(\kappa_*)^{-1}\partial\{t\}\pi^*$.

Remark 5.6. Let $i: Y\to X$ be a closed immersion of smooth schemes and let L be an invertible $\mathcal{O}_X$-module. Then we have a twisted version of the Gysin-Witt map:

$$i^! : H^r(C(X,G^j,L)) \to H^r(C(Y,G^j,i^*L)).$$

5.2 Functoriality

The goal of this section is to prove that for any inclusions of smooth schemes $Z \xrightarrow{i} Y \xrightarrow{j} X$ we have $(ji)^! = i^! j^!$. The strategy is not new. We follow the exposition of the sections 11, 12 and 13 in [Ro]. First we prove some lemmas:

LEMMA 5.7. *Let $i : Y \to X$ be a closed immersion and $g : V \to X$ be a flat morphism. Consider the following fibre product*

$$\begin{tikzcd} W \arrow[r, "i'"] \arrow[d, "g'"'] & V \arrow[d, "g"] \\ Y \arrow[r, "i"'] & X. \end{tikzcd}$$

Then we have $(g')^* i^! = (i')^! g^*$.

Proof. Let $D(X,Y)$ be the deformation to the normal cone associated to the inclusion $i : Y \hookrightarrow X$ and $D(V,W)$ be the deformation associated to $i' : W \hookrightarrow V$. Let $U = \mathbb{A}^1 - 0$. Because of the universal properties of blow-ups, we see that g and g' give a morphism $D(g) : D(V,W) \to D(X,Y)$ such that the following diagram commutes:

$$\begin{tikzcd} D(V,W) \arrow[d, "D(g)"'] & V \times U \arrow[l, "\iota'"'] \arrow[d, "g\times 1"] \\ D(X,Y) & X \times U \arrow[l, "\iota"] \end{tikzcd}$$

where ι and ι' are the inclusions of the respective open subsets. We also get a morphism $N(g) : N_W V \to N_Y X$ such that these diagrams commute:

$$\begin{tikzcd} N_W V \arrow[r, "q'"] \arrow[d, "N(g)"'] & W \arrow[d, "g'"] \\ N_Y X \arrow[r, "q"'] & Y \end{tikzcd} \qquad \begin{tikzcd} N_W V \arrow[r, "\kappa'"] \arrow[d, "N(g)"'] & D(V,W) \arrow[d, "D(g)"] \\ N_Y X \arrow[r, "\kappa"'] & D(X,Y). \end{tikzcd}$$

Now use Propositions 3.28 and 3.34, Lemma 5.4, the naturality of the connecting homomorphism ∂ and the diagram

$$\begin{tikzcd} W \arrow[d, "g'"'] & N_W V \arrow[l, "q'"'] \arrow[r, "\kappa'"] \arrow[d, "N(g)"] & D(V,W) \arrow[d, "D(g)"] & V \times U \arrow[l, "\iota'"'] \arrow[r, "\pi'"] \arrow[d, "g\times 1"] & V \arrow[d, "g"] \\ Y & N_Y X \arrow[l, "q"] \arrow[r, "\kappa"'] & D(X,Y) & X \times U \arrow[l, "\iota"] \arrow[r, "\pi"'] & X \end{tikzcd}$$

to conclude (observe that $D(g)$ and $N(g)$ are flat because of [Ro], Remark 10.1).

□

Lemma 5.8. *Let* $Z \xrightarrow{i} Y \xrightarrow{j} X$ *be inclusions of smooth schemes. Then we have inclusions* $a : N_Z Y \to N_Z X$, $c : i^* N_Y X \to N_Y X$ *and isomorphisms* $N_{(i^* N_Y X)}(N_Y X) \simeq N_Z Y \oplus i^* N_Y X \simeq N_{(N_Z Y)}(N_Z X)$.

Proof. The first two assertions are straight computations (see also [Ne]). The relation (2.1) in [Ne] shows that we have canonical isomorphisms

$$N_{(i^* N_Y X)}(N_Y X) \simeq N_Z Y \oplus i^* N_Y X \simeq N_{(N_Z Y)}(N_Z X).$$

□

Lemma 5.9. *Let* $Z \xrightarrow{i} Y \xrightarrow{j} X$ *be inclusions of smooth schemes. Let* $a : N_Z Y \to N_Z X$, $c : i^* N_Y X \to N_Y X$ *be the inclusions and* $q : N_Y X \to Y$, $r : N_Z X \to Z$, $s_1 : N_{(i^* N_Y X)}(N_Y X) \to i^* N_Y X$, $s_2 : N_{(N_Z Y)}(N_Z X) \to N_Z Y$ *the projections. Then we have* $(s_1)^* c^! q^* j^! = (s_2)^* a^! r^* (ji)^!$

Proof. Consider the deformation to the normal cone spaces $D(Y,Z)$ and $D(X,Z)$. Using the universal property of blow-ups, we get a map $D(Y,Z) \to D(X,Z)$ such that the following diagram commutes

$$\begin{tikzcd} N_Z Y \arrow[r, "a"] \arrow[d] & N_Z X \arrow[d] \\ D(Y,Z) \arrow[r] & D(X,Z) \\ Y \times U \arrow[r, "j\times 1"'] \arrow[u] & X \times U \arrow[u] \end{tikzcd}$$

where the top vertical maps are inclusions of the exceptional fiber in the deformation to the normal space and the bottom vertical maps are inclusions of open subsets. It is easy to check that the map $D(Y,Z) \to D(X,Z)$ is a closed immersion. Let $D(X,Y,Z)$ be the deformation to the normal cone space associated to this closed immersion. Using again the universal property of blow-ups, we see that the above diagram gives a sequence

$$D(N_Z X, N_Z Y) \longrightarrow D(X,Y,Z) \longleftarrow D(X,Y) \times U$$

where the first map is a closed immersion and the second one is an open immersion. Consider now the space $D(X,Y,Z)$. We have an open immersion $D(X,Z) \times U \to D(X,Y,Z)$ and a closed immersion (as the special fiber) $N_{D(Y,Z)} D(X,Z) \to D(X,Y,Z)$. In fact, this exceptional fiber is isomorphic to $D(N_Y X, i^* N_Y X)$ (see [Ne], paragraph 3.2). So we get a diagram

$$\begin{tikzcd}
N_Y X \arrow[r, "\kappa"] & D(X,Y) & X \times U \arrow[l, "\iota"'] \arrow[r, "\pi"] & X \\
N_Y X \times U \arrow[u, "\pi"] \arrow[d, "\iota"'] \arrow[r, "\kappa"] & D(X,Y) \times U \arrow[u, "\pi"'] \arrow[d, "\iota"] & X \times U \times U \arrow[l, "\iota"'] \arrow[u, "\pi"'] \arrow[d, "\iota"] \arrow[r, "\pi"] & X \times U \arrow[u, "\pi"'] \arrow[d, "\iota"] \\
D(N_Y X, i^* N_Y X) \arrow[r, "\kappa"] & D(X,Y,Z) & D(X,Z) \times U \arrow[l, "\iota"'] \arrow[r, "\pi"] & D(X,Z) \\
N_{(i^* N_Y X)} N_Y X \arrow[u, "\kappa"] \arrow[r, "\kappa"'] & D(N_Z X, N_Z Y) \arrow[u, "\kappa"'] & N_Z X \times U \arrow[l, "\iota"] \arrow[u, "\kappa"'] \arrow[r, "\pi"'] & N_Z X \arrow[u, "\kappa"']
\end{tikzcd}$$

where all the lines are deformations to the normal cone, the first and fourth columns are also deformations to the normal cone. This diagram is commutative (see [Ne], paragraph 3.2). The maps κ denote inclusions of special fibers, ι denote the inclusions of the complement of these special fibers and π denote the relevant projections. The map $q^* j^!$ is obtained by composing the operations (in cohomology) of the top row and $s_1^* b^!$ is obtained by working with the left column. Similarly, $r^*(ji)^!$ and $s_2^* a^!$ are deduced from the right column and the bottom row. Now all the squares appearing in this diagram are commutative and give commutative diagrams in cohomology (Proposition 3.28, Proposition 3.32, Lemma 3.34 and the naturality of the residual homomorphism ∂). Using this and Lemma 5.4, we get the result.

$\square$

LEMMA 5.10. *Let V, X and W be smooth schemes. Consider the following commutative diagram*

$$\begin{tikzcd}
W \arrow[r, "i"] \arrow[dr, "p'"'] & V \arrow[d, "p"] \\
 & X
\end{tikzcd}$$

where p, p' are flat and i is a closed immersion. Suppose that the composition $N_W V \to W \to X$ is of the same relative dimension as p. Then $i^! p^* = (p')^*$.

Proof. Let $D(V,W)$ be the deformation to the normal cone associated to i and $b : D(V,W) \to V \times \mathbb{A}^1$ be the blow-down map. We have a commutative diagram

$$\begin{tikzcd} W & N_V W \arrow[l, "q"'] \arrow[r, "\kappa"] & D(V,W) \arrow[r, "b"] & V \times \mathbb{A}^1 \arrow[r, "p\times Id"] & X \times \mathbb{A}^1 \\ & & V \times U \arrow[u, "\iota"] \arrow[r, equal] & V \times U \arrow[u, "\iota"] \arrow[r, "p\times Id"'] \arrow[d, "\pi"] & X \times U \arrow[u, "\iota"] \arrow[d, "\pi'"] \\ & & & V \arrow[r, "p"'] & X. \end{tikzcd}$$

By definition, $i^! p^* = (q^*)^{-1}(\kappa_*)^{-1}\partial\{t\}\pi^* p^*$. Using Proposition 3.28, we get $i^! p^* = (q^*)^{-1}(\kappa_*)^{-1}\partial\{t\}(p \times Id)^*(\pi')^*$. By Lemma 5.4, this gives

$$(q^*)^{-1}(\kappa_*)^{-1}\partial\{t\}(p \times Id)^*(\pi')^* = (q^*)^{-1}(\kappa_*)^{-1}\partial(p \times Id)^*\{t\}(\pi')^*.$$

Using Remark 10.1 in [Ro], we see that $f := (p \times Id)b$ is flat because the composition $N_W V \to W \to X$ is of the same relative dimension as p. We have a commutative diagram

$$\begin{tikzcd} H^i(C(X \times \mathbb{A}^1, G^j)) \arrow[r] \arrow[d, "f^*"] & H^i(C(X \times U, G^j)) \arrow[r, "\partial'"] \arrow[d, "(p\times Id)^*"] & H^{i+1}_X(C(X \times \mathbb{A}^1, G^j)) \arrow[r] \arrow[d, "f^*"] & {} \\ H^i(C(D(V,W), G^j)) \arrow[r] & H^i(C(V \times U, G^j)) \arrow[r, "\partial"'] & H^{i+1}_{N_V W}(C(D(V,W), G^j)) \arrow[r] & {} \end{tikzcd}$$

where the first line is the localization long exact sequence associated to the triple $(X \times U, X \times \mathbb{A}^1, X \times 0)$ and the second line is the one associated to the triple $(V \times U, D(V,W), N_V W)$. Then

$$(q^*)^{-1}(\kappa_*)^{-1}\partial(p \times Id)^*\{t\}(\pi')^* = (q^*)^{-1}(\kappa_*)^{-1} f^*\partial'\{t\}(\pi')^*.$$

Consider next the fibre product

$$\begin{tikzcd} N_V W \arrow[r, "\kappa"] \arrow[d, "p'q"'] & D(V,W) \arrow[d, "f"] \\ X \arrow[r, "i_0"'] & X \times \mathbb{A}^1 \end{tikzcd}$$

where $i_0 : X \to X \times \mathbb{A}^1$ is the inclusion in 0. Using Lemma 3.34, we finally find $i^! p^* = (p')^*(i_0)_*^{-1}\partial'\{t\}(\pi')^*$. It remains to show that $(i_0)_*^{-1}\partial'\{t\}(\pi')^* = Id$ to finish the proof. At the level of Milnor K-theory, this is Lemma 4.5 in [Ro]. Thus we only have to prove this result at the level of Witt groups. Let $\alpha \in W^i(D_i^b(X))$ be such that $d\alpha = 0 \in W^{i+1}(D_{i+1}^b(X))$. Now $\mathrm{DegLoc}((\pi')^*\alpha) \cap \mathrm{DegLoc}(< 1, -t >)$ is a closed subset of $X \times \mathbb{A}^1$ of codimension $\geq i + 2$. Therefore we can use 2.25 to compute

$$(-1)^i d(<1,-t> \otimes \alpha) = d(<1,-t>) \otimes \alpha + (-1)^i <1,-t> \otimes d\alpha.$$

By assumption we have $d\alpha = 0$ in $W^{i+1}(D^b_{i+1}(X))$ and then

$$(-1)^i d(<1,-t> \otimes \alpha) = d(<1,-t>) \otimes \alpha = -dt \otimes \alpha$$

in $W^{i+1}(D^b_{i+1}(X))$. By definition of m_t, we find $d(m_t(\alpha)) = dt \otimes \alpha$. The latter is precisely $(i_0)_*\alpha$ (see [GH], Lemma 2.8).

□

Now we have all the tools to prove the following theorem:

Theorem 5.11. *Let* $Z \xrightarrow{i} Y \xrightarrow{j} X$ *be inclusions of smooth schemes. Then* $(ji)^! = i^! j^!$.

Proof. Let $q : N_Y X \to Y$, $p : N_Z Y \to Z$ and $r : N_Z X \to Z$ be the projections. Consider also the projections $s_1 : N_{(i^* N_Y X)}(N_Y X) \to i^* N_Y X$ and $s_2 : N_{(N_Z Y)}(N_Z X) \to N_Z Y$. Denote by $a : N_Z Y \to N_Z X$ and $c : i^* N_Y X \to N_Y X$ the inclusions. We also have a fibre product

$$\begin{tikzcd} i^* N_Y X \arrow[r, "c"] \arrow[d, "q'"'] & N_Y X \arrow[d, "q"] \\ Z \arrow[r, "i"'] & Y. \end{tikzcd}$$

Then

$$(s_1)^*(q')^* i^! j^! = (s_1)^* c^! q^* j^! = (s_2)^* a^! r^* (ji)^! = (s_2)^* p^* (ji)^!$$

where the first equality is due to Lemma 5.7, the second is due to Lemma 5.9 and the third to Lemma 5.10. As $(s_2)^* p^*$ induces an isomorphism in cohomology and $q' s_1 = p s_2$, we get the result. □

6 The ring structure

Let X be a smooth scheme and let $\triangle : X \to X \times X$ be the diagonal inclusion. For any i, j, r, s we have an exterior product (Theorem 4.15)

$$\times : H^i(C(X, G^r)) \times H^j(C(X, G^s)) \to H^{i+j}(C(X \times X, G^{r+s}))$$

and a Gysin-Witt map (Definition 5.5)

$$\triangle^! : H^{i+j}(C(X \times X, G^{r+s})) \to H^{i+j}(C(X, G^{r+s})).$$

Definition 6.1. We denote by $\cdot$ the composition $\triangle^! \circ \times$.

Remark 6.2. If X is a smooth scheme and L, N are invertible $\mathcal{O}_X$-modules, then using Theorem 4.16 and Remark 5.6 we see that there is a product

$$\cdot : H^i(C(X, G^i, L)) \times H^j(C(X, G^j, N)) \to H^{i+j}(C(X, G^{i+j}, L \otimes_{\mathcal{O}_X} N)).$$

Remark 6.3. In particular, we have for any $i, j \in \mathbb{N}$ a product

$$\cdot : H^i(C(X, G^i)) \times H^j(C(X, G^j)) \to H^{i+j}(C(X, G^{i+j}))$$

which by definition is a product $\widetilde{CH}^i(X) \times \widetilde{CH}^j(X) \to \widetilde{CH}^{i+j}(X)$.

Remark 6.4. It is clear from our construction that we also can define a product

$$\cdot : H^i(C(X, K^M_r)) \times H^j(C(X, K^M_s)) \to H^{i+j}(C(X, K^M_{r+s})).$$

This product coincide with the one defined by Rost ([Ro], Chapter 14) and the natural projections $\pi : C(X, G^p) \to C(X, K^M_p)$ give a commutative diagram

$$\begin{tikzcd}
H^i(C(X,G^r)) \times H^j(C(X,G^s)) \arrow[r, "\cdot"] \arrow[d, "\pi\times\pi"'] & H^{i+j}(C(X,G^{r+s})) \arrow[d, "\pi"] \\
H^i(C(X,K^M_r)) \times H^j(C(X,K^M_s)) \arrow[r, "\cdot"'] & H^{i+j}(C(X,K^M_{r+s})).
\end{tikzcd}$$

Remark 6.5. Our technique provides also a product on the cohomology of the Gersten-Witt complex of a scheme. That is, we have a product

$$\cdot : H^i(C(X, W)) \times H^j(C(X, W)) \to H^{i+j}(C(X, W)).$$

Now we prove the associativity of the product we have defined.

Proposition 6.6. *The product $\cdot$ is associative.*

Proof. First note that the exterior product is associative (Proposition 4.18). We consider the following fibre product diagram

$$\begin{tikzcd}
X \arrow[r, "\triangle"] \arrow[d, "\triangle"'] & X \times X \arrow[d, "Id\times\triangle"] \\
X \times X \arrow[r, "\triangle\times Id"'] & X \times X \times X.
\end{tikzcd}$$

We see that $((Id \times \triangle)\triangle)^! = ((\triangle \times Id)\triangle)^!$. Theorem 5.11 shows that we have in fact $\triangle^!(Id \times \triangle)^! = \triangle^!(\triangle \times Id)^!$. Since $(Id \times \triangle)^!$ is clearly $Id \times \triangle^!$ and $(\triangle \times Id)^! = \triangle^! \times Id$, the associativity is proved. $\square$

Remark 6.7. In general, the product does not satisfy any commutativity property. This is due to the fact that $\times$ and $\star$ do not commute with the flip $\tau : X \times X \to X \times X$ (see 4.19 and 4.20). Moreover, the product is not anti-commutative because the signs in 4.19 and 4.20 are not compatible. However, let $\alpha \in \widetilde{CH}^i(X)$ and $\beta \in \widetilde{CH}^j(X)$. Then $\alpha \cdot \beta$ is an element of $\widetilde{CH}^{i+j}(X)$ and is therefore represented by a sum $\sum(P_s, \psi_s) \in \mathrm{Ker}(d^{i+j})$ where

$$d^{i+j} : GW^{i+j}(D^b_{i+j}(X)) \to W^{i+j+1}(D^b_{i+j+1}(X))$$

(see Remark 3.23). Using 4.19 and 4.20, we see that $\beta \cdot \alpha = \sum(P_s, (-1)^{ij}\psi_s)$. For a more precise statement, the reader is referred to Theorem 7.6.

Now remark that there is a canonical class 1_X in $\widetilde{CH}^0(X)$ given by the symmetric form $< 1 >$ in $GW(k(X))$.

PROPOSITION 6.8. *The class 1_X is a left and right unit for the product $\cdot$.*

Proof. Let $p_2 : X \times X \to X$ be the second projection and consider the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "\triangle"] \arrow[dr, "Id"'] & X \times X \arrow[d, "p_2"] \\ & X. \end{tikzcd}$$

By Lemma 5.10, we see that $\triangle^!(p_2)^* = (Id)^* = Id$. Consider now $\mu \in H^i(C(X, G^j))$. It is clear that $1_X \times \mu = (p_2)^*(\mu)$ and then $1_X \cdot \mu = \mu$. Replacing p_2 by p_1 shows that 1_X is also a right unit. $\square$

Hence we have:

THEOREM 6.9. *Let X be a smooth scheme and let $\widetilde{CH}^*(X)$ be the total Chow-Witt group of X. Then the product $\cdot$ turns $\widetilde{CH}^*(X)$ into a graded associative ring with unit.*

Taking the twists into account, we get the following theorem:

THEOREM 6.10. *Let X be a smooth scheme and let $\bigoplus_{L \in \quad (X)/2} \widetilde{CH}^*(X, L)$ be the total twisted Chow-Witt group of X. Then the product $\cdot$ turns this group into a graded associative ring with unit.*

DEFINITION 6.11. Let X be a smooth scheme. We call *Chow-Witt ring* the ring $\widetilde{CH}^*(X)$ and *twisted Chow-Witt ring* the ring $\bigoplus_{L \in \quad (X)/2} \widetilde{CH}^*(X, L)$.

The following proposition is obvious:

Proposition 6.12. *Let X be a smooth scheme. Then the natural homomorphism $\widetilde{CH}^*(X) \to CH^*(X)$ is a ring homomorphism.*

Remark 6.13. The same methods show that the product of Remark 6.5 gives a graded associative anticommutative ring structure on the total cohomology group $H^*(C(X,W))$ of the Gersten-Witt complex associated to X.

7 Basic properties

We first show that the Chow-Witt ring is a functorial construction.

Definition 7.1. Let X and Y be smooth schemes and $f : X \to Y$ a morphism. Consider the graph morphism $\gamma_f : X \to X \times Y$. We define

$$f^! : \widetilde{CH}^*(Y) \to \widetilde{CH}^*(X)$$

by $f^!(y) = \gamma_f^!(1_X \times y)$ for any $y \in \widetilde{CH}^*(Y)$.

Proposition 7.2. *The map $f^! : \widetilde{CH}^*(Y) \to \widetilde{CH}^*(X)$ is a ring homomorphism.*

Proof. We only have to check that $f^!(y \cdot z) = f^!(y) \cdot f^!(z)$ for any $y, z \in \widetilde{CH}^*(Y)$. Consider the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\gamma_f"] \arrow[d, "\triangle_X"'] & X \times Y \arrow[d, "\triangle_{X\times Y}"] \\ X \times X \arrow[r, "\gamma_f \times \gamma_f"'] & (X \times Y) \times (X \times Y). \end{tikzcd}$$

Theorem 5.11 shows that $\gamma_f^! \triangle_{X\times Y}^! = \triangle_X^! (\gamma_f \times \gamma_f)^!$. Applying this to the cycle $1_X \times y \times 1_X \times z$, we obtain the result. $\square$

Remark 7.3. The proposition shows that $\widetilde{CH}^*(\_)$ is a functor from the category of smooth schemes to the category of rings. It is clear that the homomorphisms $\widetilde{CH}^*(X) \to CH^*(X)$ give a natural transformation $\widetilde{CH}^*(\_) \to CH^*(\_)$.

In the case where $f : X \to Y$ is a flat morphism, we can identify $f^!$ more precisely.

Proposition 7.4. *Let $f : X \to Y$ be a flat morphism. Then $f^! = f^*$.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\gamma_f"] \arrow[dr, "f"'] & X \times Y \arrow[d, "p"] \\ & Y \end{tikzcd}$$

where $p : X \times Y \to Y$ is the projection. Since $N_X(X \times Y)$ is of rank equal to the dimension of Y, we see that the relative dimension of the composition $N_X(X \times Y) \to X \to Y$ is the same as the relative dimension of $p : X \times Y \to Y$. Therefore we can use Lemma 5.10 to get $\gamma_f^! p^* = f^*$. Since $p^*\beta = 1_X \times \beta$ for any cycle on Y, the result is proved. $\square$

Let $Z \subset X$ be a closed subset of pure codimension i. As $D^b_Z(X) \subset D^b(X)^{(i)}$, we have a homomorphism $GW^i_Z(X) \to GW^i(D^b(X)^{(i)})$. Composing with the localization, we obtain a homomorphism $GW^i_Z(X) \to GW^i(D^b_i(X))$. As the composition $GW^i(D^b(X)^{(i)}) \to GW^i(D^b_i(X)) \to W^{i+1}(D^b(X)^{(i+1)})$ is zero (see [Ba1]), we finally obtain a homomorphism (Remark 3.23):

$$\alpha_Z : GW^i_Z(X) \to \widetilde{CH}^i(X).$$

Remark 7.5. Let $f : X \to Y$ be a flat morphism and $Z \subset Y$ be a closed subset of pure codimension i. The definitions of f^* for the Grothendieck-Witt groups and the definition of f^* for the Chow-Witt groups show that the following diagram commutes ([Fa], Theorem 3.2.2 and Corollary 10.4.2):

$$\begin{tikzcd} GW^i_Z(Y) \arrow[r, "\alpha_Z"] \arrow[d, "f^*"'] & \widetilde{CH}^i(Y) \arrow[d, "f^*"] \\ GW^i_{f^{-1}Z}(X) \arrow[r, "\alpha_{(f^{-1}Z)}"'] & \widetilde{CH}^i(X). \end{tikzcd}$$

The next theorem shows that our intersection product is the expected one:

THEOREM 7.6. *Let $Z, T \subset X$ be closed subschemes of respective pure codimension i and j. Suppose that $Z \cap T$ is of pure codimension $i + j$. Then the following diagram commutes*

$$\begin{tikzcd} GW^i_Z(X) \times GW^j_T(X) \arrow[r, "\star"] \arrow[d, "\alpha_Z \times \alpha_T"'] & GW^{i+j}_{Z\cap T}(X) \arrow[d, "\alpha_{Z\cap T}"] \\ \widetilde{CH}^i(X) \times \widetilde{CH}^j(X) \arrow[r, "\cdot"'] & \widetilde{CH}^{i+j}(X). \end{tikzcd}$$

Proof. Let $\gamma \in GW^i_Z(X)$ and $\delta \in GW^j_T(X)$. Consider the deformation to the normal cone space $D(X \times X, X)$ and the blow down map $b : D(X \times X, X) \to X \times X \times \mathbb{A}^1$. We have the following commutative diagram

$$\begin{tikzcd}
X \arrow[rr, equal] & & X \arrow[d, "\triangle"'] & & \\
N_X(X\times X) \arrow[u, "q"] \arrow[rr, "b'"] \arrow[d, "\kappa"'] & & X\times X \arrow[d, "i_0"] & & \\
D(X\times X,X) \arrow[rr, "b"'] & & X\times X\times \mathbb{A}^1 \arrow[rr, "\pi'"] & & X\times X \\
X\times X\times U \arrow[u, "\iota"] \arrow[rr, equal] & & X\times X\times U \arrow[u, "\iota"'] \arrow[urr, "\pi"'] & &
\end{tikzcd} \tag{1}$$

where i_0 is the inclusion in 0, q is the projection and the two bottom squares are fibre products. By definition, we have

$$\alpha_Z(\gamma)\cdot\alpha_T(\delta) = (q^*)^{-1}(\kappa_*)^{-1}\partial\{t\}\pi^*(\alpha_Z(\gamma)\times\alpha_T(\delta)).$$

Let $F = b^{-1}(\pi')^{-1}(p_1^{-1}Z\cap p_2^{-1}T)$ in $D(X\times X,X)$ (where p_1 and p_2 are the projections of $X\times X$ onto X). Observe that $\iota^{-1}F = F\cap(X\times X\times U)$ is non empty and of pure codimension $i+j$ in $X\times X\times U$. Diagram (1) gives

$$F\cap N_X(X\times X) = \kappa^{-1}F = \kappa^{-1}b^{-1}(\pi')^{-1}(p_1^{-1}Z\cap p_2^{-1}T) = q^{-1}(Z\cap T).$$

As $Z\cap T$ is of codimension $i+j$ in X and q is flat, $q^{-1}(Z\cap T)$ is also of codimension $i+j$ in $N_X(X\times X)$ and hence is of codimension $i+j+1$ in $D(X\times X,X)$. Therefore F itself is of pure codimension $i+j$ in $D(X\times X,X)$. By commutativity of the above diagram and Remark 7.5, we have (note that b^* is defined at the level of the Grothendieck-Witt groups, but not at the level of the Chow-Witt groups):

$$\pi^*(\alpha_Z(\gamma)\times\alpha_T(\delta)) = \alpha_{\iota^{-1}F}(\iota^*b^*(\pi')^*(p_1^*\gamma\otimes p_2^*\delta)) = \iota^*\alpha_F(b^*(\pi')^*(p_1^*\gamma\otimes p_2^*\delta)).$$

We have to compute $(\kappa_*)^{-1}\partial\{t\}\pi^*(\alpha_Z(\gamma)\times\alpha_T(\delta))$. By definition of ∂, we have to consider any element $\nu\in C(D(X\times X,X),G^{i+j+1})_{i+j}$ having the property that $\iota^*\nu = \{t\}\pi^*(\alpha_Z(\gamma)\times\alpha_T(\delta))$ and then compute $d_G(\nu)$ where

$$d_G : C(D(X\times X,X),G^{i+j+1})_{i+j}\to C(D(X\times X,X),G^{i+j+1})_{i+j+1}$$

is the differential of the complex $C(D(X\times X,X),G^{i+j+1})$. Consider the commutative diagram

$$\begin{tikzcd}
D(X\times X,X) \arrow[r, "b"] & X\times X\times\mathbb{A}^1 \arrow[r, "pr"] & \mathbb{A}^1 \\
X\times X\times U \arrow[u, "\iota"] \arrow[r, equal] & X\times X\times U \arrow[u, "\iota"'] \arrow[ur, "\eta"'] &
\end{tikzcd}$$

and recall that $N_X(X\times X)$ is the principal Cartier divisor in $D(X\times X, X)$ defined by $f := b^*pr^*(t)$.
Consider the form $b^*(\pi')^*(p_1^*\gamma\otimes p_2^*\delta)$. Its support is F. Localizing at the generic points of F (which are on $X\times X\times U$), we obtain a form ν_0 in $W^{i+j}(D^b_{i+j}(D(X\times X,X)))$. We also obtain an element ν_1 in $\displaystyle\bigoplus_{x\in F^{(0)}} K_0(k(x))$.
The above computation shows that f is a unit in $k(x)$ for any generic point x of F. We get an element

$$\nu := ((-1)^{i+j+1} < 1, -f > \otimes\nu_0, \{f\}\cdot\nu_1) \in C(D(X\times X,X), G^{i+j+1})_{i+j}$$

which satisfy $\iota^*\nu = \{t\}\pi^*(\alpha_Z(\gamma)\times\alpha_T(\delta))$. A straightforward computation (use Theorem 2.25 again) shows that $d_G(\nu) = df\otimes b^*(\pi')^*(p_1^*\gamma\otimes p_2^*\delta)$ in the group $GW^{i+j+1}(D^b_{i+j+1}(D(X\times X,X)))$. But $df = b^*dt$ and

$$b^*dt\otimes b^*(\pi')^*(p_1^*\gamma\otimes p_2^*\delta) = b^*(dt\otimes(\pi')^*(p_1^*\gamma\otimes p_2^*\delta))$$

([GN], Theorem 3.2). Since $dt\otimes(\pi')^*(p_1^*\gamma\otimes p_2^*\delta) = (i_0)_*(p_1^*\gamma\otimes p_2^*\delta)$ ([GH], Lemma 2.8), we finally obtain

$$(\kappa_*)^{-1}\partial\{t\}\pi^*(\alpha_Z(\gamma)\times\alpha_T(\delta)) = \alpha_{F\cap N_X(X\times X)}((b')^*(p_1^*\gamma\otimes p_2^*\delta)).$$

We have a commutative diagram

$$\begin{tikzcd}
N_X(X\times X) \arrow[r, "b'"] \arrow[d, "q"'] & X\times X \\
X. \arrow[ur, "\triangle"'] &
\end{tikzcd}$$

Now $\triangle^{-1}(p_1^{-1}Z\cap p_2^{-1}T) = Z\cap T$ and using the diagram, we see that

$$\alpha_Z(\gamma)\cdot\alpha_T(\beta) = \alpha_{Z\cap T}(\triangle^*(p_1^*\gamma\otimes p_2^*\delta)).$$

Hence it only remains to show that $\triangle^*(p_1^*\gamma\otimes p_2^*\delta) = \gamma\star\delta$ to finish the proof. This is clear by [GN], Theorem 3.2.

$\square$

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